J. Eur. Math. Soc. 27, 3771–3792 (2025) DOI 10.4171/JEMS/1440

Spyros Alexakis · Ali Feizmohammadi · Lauri Oksanen

Lorentzian Calderón problem near the Minkowski geometry

Received December 10, 2021; revised November 23, 2022

Abstract. We study a Lorentzian version of the well-known Calderón problem that is concerned with determination of lower order coefficients in a wave equation on a smooth Lorentzian manifold, given the associated Dirichlet-to-Neumann map on its timelike boundary. In the earlier work of the authors it was shown that zeroth order coefficients can be uniquely determined under a two-sided spacetime curvature bound and the additional assumption that there are no conjugate points along null or spacelike geodesics. In this paper we show that uniqueness for the zeroth order coefficient holds for manifolds satisfying a weaker curvature bound and for spacetime perturbations of such manifolds. This relies on a new enhanced optimal unique continuation principle for the wave equation in the exterior regions of double null cones. In particular, we solve the Lorentzian Calderón problem near the Minkowski geometry.

Keywords: inverse problems, wave equation, Lorentzian geometry, pseudo-convexity, unique continuation, boundary control method.

1. Introduction

1.1. Formulation of the problem

We start with the geometric setup and let (M, g) be a Lorentzian manifold of dimension $1 + n$ with signature $(-, +, \dots, +)$. We make the standing assumption that the manifold is of the form

$$M = [-T, T] \times M_0 \tag{1}$$

for some $T > 0$ and a compact connected manifold M_0 with a smooth boundary. The metric g is assumed to be of the form

$$g(t, x) = c(t, x)(-dt^2 + g_0(t, x)), \quad \forall (t, x) \in M, \tag{2}$$

Spyros Alexakis: Department of Mathematics, University of Toronto, Toronto, ON, M5S 2E4, Canada; alexakis@math.toronto.edu

Ali Feizmohammadi: The Fields Institute for Research in Mathematical Sciences, Toronto, ON, M5T 3J1, Canada; ali.feizmohammadi@utoronto.ca

Lauri Oksanen: Department of Mathematics and Statistics, University of Helsinki, 00014 Helsinki, Finland; lauri.oksanen@helsinki.fi

Mathematics Subject Classification 2020: 35R30.

where c is a smooth strictly positive function on M and $g_0(t,\cdot)$ is a family of smooth Riemannian metrics on M_0 that depend smoothly on $t \in [-T,T]$. We remark that if a Lorentzian manifold with timelike boundary admits a global time function then it is isometric to a manifold of the form (1)–(2); see [1, Appendix A] for the details.

Let $V \in C^\infty(M)$. We consider the following wave equation on (M,g), with the zeroth order coefficient given by V:

$$\begin{cases} (\Box + V)u = 0 & \text{on } M^{\text{int}}, \\ u = f & \text{on } \Sigma = (-T,T)\times \partial M_0, \\ u(-T,x) = \partial_t u(-T,x) = 0 & \text{on } M_0. \end{cases} \tag{3}$$

Here, the wave operator $\Box$ is given in local coordinates $(t = x^0,\dots,x^n)$ on M by

$$\Box u = -\sum_{j,k=0}^{n} |\det g|^{-1/2} \frac{\partial}{\partial x^j}\left(|\det g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u\right).$$

It is classical (see for example [22, Theorem 4.1]) that given any $f \in H^1_0(\Sigma)$, equation (3) admits a unique solution u in the energy space

$$C(-T,T;H^1(M_0)) \cap C^1(-T,T;L^2(M_0)). \tag{4}$$

Moreover, $\partial_\nu u|_\Sigma \in L^2(\Sigma)$ where ν is the outward unit normal vector field on Σ.

We define the Dirichlet-to-Neumann map, $\Lambda_V : H^1_0(\Sigma) \to L^2(\Sigma)$, by

$$\Lambda_V f = \partial_\nu u|_\Sigma, \tag{5}$$

where u is the unique solution to (3) with boundary value f on Σ.

We are interested in the inverse problem of determining the coefficient V from the knowledge of the Dirichlet-to-Neumann map Λ_V, or in other words, the question of injectivity of the map $V \mapsto \Lambda_V$. Following [1], we call this the *Lorentzian Calderón problem*.

1.2. Obstruction to uniqueness

There is a natural obstruction to uniqueness for the coefficient V that is due to finite speed of propagation for the wave equation. In order to discuss this obstruction, let us first fix some notation. We say that a piecewise smooth path $\gamma : I \to M$ is *timelike*, *causal* or *spacelike* if for each point on γ, $g(\dot\gamma,\dot\gamma)$ is negative, nonpositive or positive respectively. We call a causal path γ *future-pointing* if $g(\dot\gamma,\partial_t) < 0$ for each point on γ. This can be viewed as a choice for an arrow of time in the manifold. We write $p \le q$ for points $p,q \in M$ if there is a future-pointing piecewise smooth causal path on M from p to q or $p = q$. We write $p \ll q$ if there exists a future-pointing piecewise smooth timelike path on M from p to q. Using these relations, the *causal future* and *past* of a point $p \in M$ are defined via

$$J^+(p) = \{q \in M : p \le q\} \quad \text{and} \quad J^-(p) = \{q \in M : q \le p\}. \tag{6}$$

We also write $I^{\pm}(p)$ for the *chronological future* and *past* of the point p that are defined as in (6) with $\leq$ replaced by $\ll$.

Using the above notation, we note that by finite speed of propagation for the wave equation, the map Λ_V carries no information about the coefficient V on the subset

$$\mathcal{D} = \{p \in M : J^+(p) \cap \Sigma = \emptyset \text{ or } J^-(p) \cap \Sigma = \emptyset\}.$$

We refer the reader to [21, Section 1.1] for the details. In order to remove this obstruction, we will assume throughout this paper that the time interval T is large in comparison to the support of the coefficient V.

1.3. Main results

Without any further assumptions on (M, g), the Lorentzian Calderón problem is wide open. We will solve the Lorentzian Calderón problem for Lorentzian manifolds that satisfy certain geometric assumptions, of the following four types:

- A curvature bound on the manifold.
- An assumption reminiscent of simplicity in Riemannian geometry.
- An assumption on the final time T compared to the support of V.
- A transversality assumption on null geodesics.

The first two assumptions are related to proving our optimal unique continuation result, while the last two conditions are directly related to the Lorentzian Calderón problem.

We begin with the curvature bound. We assume that

(H1) For any point $p \in M$, any spacelike vector $v \in T_pM$, and any null vector $N \in T_pM$ with $g(v, N) = 0$,

$$g(R(N, v)v, N) \leqslant 0,$$

where R stands for the curvature tensor on (M, g) written as a $(1, 3)$-tensor.

This curvature condition is weaker than the curvature assumption imposed in our earlier work [1]; see Section 1.4 for a more detailed comparison. Next, we discuss the simplicity assumption that was also used in our earlier work. Given any $p \in M$, we define the relatively open set

$$\mathscr{E}_p = M \setminus (J^-(p) \cup J^+(p)), \tag{7}$$

and call it the *exterior of the double null cone* emanating from the point p; see Figure 1. We assume that

(H2) For any null geodesic γ and any two points p, q on γ, the only causal path between p and q is along γ. For all $p \in M$, the exponential map $\exp_p$ is a diffeomorphism from the spacelike vectors onto $\mathscr{E}_p$.

Note, for example, that (H2) is satisfied for small perturbations of the Minkowski metric on spacetime cylinders $M = [-T, T] \times \overline{\Omega} \subset \mathbb{R}^{1+n}$ with Ω a bounded domain. On the other hand, (H2) is violated if we take the ultrastatic Lorentzian manifold $M = [-T, T] \times \mathbb{S}^2_+$

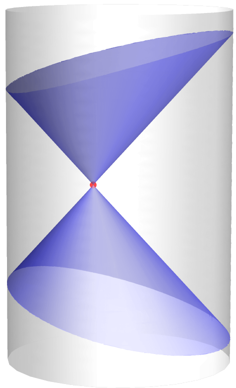

Fig. 1. The schematic for the exterior of the double null cone in the setting of Minkowski geometry in $\mathbb{R}^{1+2}$. The point p is shown in red, Σ is gray and $\partial\mathscr{E}_p \cap M^{\text{int}}$ is shown in blue.

with $\mathbb{S}^2_+$ denoting the surface of the upper unit hemisphere in $\mathbb{R}^3$ with its induced Riemannian metric, assuming that T is larger than the diameter of $\mathbb{S}^2_+$. Indeed, taking any two antipodal points p, q on a great circle in $\mathbb{S}^2_+$, there are two intersecting null geodesics in M whose projection onto $\mathbb{S}^2_+$ goes from p to q.

In addition to (H1) and (H2) we need to make an assumption on the size of the final time T. Roughly speaking, we require that $T > 0$ must be sufficiently large compared to the support of V. Precisely, we assume that

(H3) If $p \in \operatorname{supp} V$, then $\mathscr{E}_p \subset (-T, T) \times M_0$. Moreover, there is $p_0 \in M^{\text{int}} \cap J^-(\operatorname{supp} V)$ such that

$$\mathscr{E}_{p_0} \subset (-T, T) \times M_0 \quad \text{and} \quad \mathscr{E}_{p_0} \cap \mathscr{E}_q = \emptyset,$$

for any $q \in \operatorname{supp} V$.

The fourth condition is needed for controllability theory. We assume that

(H4) All inextendible null geodesics in M must intersect ∂M transversally.

In Section 1.5 we provide some intuition behind the last two assumptions. Our main result can now be stated as follows.

Theorem 1.1. *Let (M, g) be a Lorentzian manifold of the form (1)–(2). For $j = 1, 2$, let $V_j \in C^\infty(M)$. Suppose that assumptions* (H1)–(H4) *are satisfied. Then the following hold:*

(i) *Let Λ_{V_j}, $j = 1, 2$, be defined as in (5) corresponding to the wave equation (3) on (M, g) with $V = V_j$. If $\Lambda_{V_1} f = \Lambda_{V_2} f$ for all $f \in H^1_0(\Sigma)$ then $V_1 = V_2$ on M.*

(ii) *Let $\tilde{g}$ be a smooth Lorentzian metric on M that lies in a sufficiently small $C^2(M)$-neighborhood of g (independent of V_1, V_2). Then* (i) *is valid with (M, g) replaced with $(M, \tilde{g})$.*

As an immediate corollary of the above result, we can solve the Lorentzian Calderón problem near the Minkowski geometry. Recall that the Minkowski metric η on $\mathbb{R}^{1+n}$ is defined via $\eta = -(dt)^2 + (dx^1)^2 + \cdots + (dx^n)^2$, and also that its associated curvature tensor R vanishes identically.

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth strictly convex boundary. For $j = 1, 2$, let $V_j \in C_c^\infty(\mathbb{R} \times \overline{\Omega})$. Let $T > 0$ be sufficiently large so that* (H3) *is satisfied with respect to the Minkowski metric η. Suppose that g is a smooth metric on $M = [-T, T] \times \overline{\Omega}$ that is in a sufficiently small $C^2(M)$-neighborhood of η (independent of V_1, V_2). Let Λ_{V_j} be defined as in (5), corresponding to the wave equation (3) on (M, g) with $V = V_j$. If $\Lambda_{V_1} f = \Lambda_{V_2} f$ for all $f \in H_0^1(\Sigma)$, then $V_1 = V_2$ on M.*

The key tool in proving Theorem 1.1 is a new optimal unique continuation property for the wave equation in exterior regions of double null cones.

Theorem 1.3. *Let (M, g) be a Lorentzian manifold of the form (1)–(2) and assume that* (H1)–(H2) *are satisfied. Let X be a first order linear differential operator with smooth coefficients on M. Then the following unique continuation results hold:*

(i) *Let $p \in M^{\text{int}}$ be such that $\mathscr{E}_p \subset (-T, T) \times M_0$, where $\mathscr{E}_p$ is defined by (7). Let $u \in H^{-s}(M)$ for some $s \geqslant 0$ be a distributional solution to*

$$\Box u + Xu = 0 \quad \text{on } \mathscr{E}_p,$$

where $\Box$ is the wave operator associated to (M, g). Suppose that the traces u and $\partial_\nu u$ both vanish on the set $\Sigma \cap \mathscr{E}_p$. Then $u = 0$ on $\mathscr{E}_p$.

(ii) *Let $\tilde{g}$ be a smooth Lorentzian metric on M that lies in a sufficiently small $C^2(M)$-neighborhood of g (independent of X). Then* (i) *is valid with (M, g) replaced with $(M, \tilde{g})$.*

1.4. Previous literature and comparison with our earlier work

Before reviewing the literature on the Lorentzian Calderón problem, let us make a comparison with related results in the elliptic setting. Recall that the analogous injectivity question for $V \mapsto \Lambda_V$ with Λ_V denoting the Dirichlet-to-Neumann map associated to a Riemannian manifold (M, g) and with $\Box$ replaced by the Laplace–Beltrami operator on (M, g) is the well known Calderón problem. In this elliptic setting, the seminal work [35] proves uniqueness for the coefficient V on Euclidean domains of dimension larger than 2. Uniqueness in the two-dimensional case was proved in [25] for certain classes of smooth V and later in [10] for general smooth V. We also mention that uniqueness is known when the manifold (M, g) and V are both real-analytic [23]. Outside of these categories, uniqueness is only known for certain manifolds (M, g) with a Euclidean direction under additional assumptions [12, 13]. Here, existence of a Euclidean direction essentially means that the components of the Riemannian metric must be independent of one of the coordinates on the manifold. For a review of the literature on the (Riemannian) Calderón problem we refer the reader to the survey article [38].

The results that are related to recovery of lower order coefficients in wave equations can in general be divided into two categories: of time-independent and time-dependent coefficients. Starting with the seminal work [7], there is a rich literature related to the recovery of time-independent coefficients based on the Boundary Control (BC) method. The BC method fundamentally relies on the optimal unique continuation theorem of Tataru [36] (see also the important precursor by Robbiano [28], and related later results by Robbiano and Zuilly [29] and Tataru [37]). This result says that, in the case of the wave equation with analytic in time coefficients, the unique continuation principle holds across any noncharacteristic hypersurface. Nonetheless, the unique continuation principle fails when the coefficients in the equation are only smooth [4]. In line with this, the works by Eskin [14, 15] solve the inverse problem of recovering time-analytic coefficients for the wave equation. We refer the reader to [8, 20] for a review of the results that are based on the BC method.

Outside the category of wave equations with time-analytic coefficients, results are scarce. We mention the works [27, 30, 32] that solve the problem of recovering time-dependent lower order coefficients in the Minkowski spacetime. The approach of Stefanov [32] uses the principle of propagation of singularities for the wave equation to reduce the inverse problem to the study of the injectivity of the light ray transform of the unkown coefficient. The inversion of this transform follows from Fourier analysis in the Minkowski geometry. The reduction step from injectivity of the map $V \mapsto \Lambda_V$ to injectivity of the light ray transform has been generalized to a broad geometric setting [34] but there are few results about injectivity of the light ray transform. Indeed, this transform is known to be injective only in the case of ultrastatic metrics [16] (see also the related earlier work [21]), stationary metrics [17], and in the case of real-analytic spacetimes [33], under certain additional convexity conditions.

In the recent work by the present authors [1], an optimal unique continuation theorem was proved for the wave equation in Lorentzian geometries that satisfy a certain two-sided curvature bound first introduced by Andersson and Howard [5]. This curvature bound requires that there exists some $K \in \mathbb{R}$ such that the following inequality is satisfied at each point $p \in M$ and for any pair of vectors $X, Y \in T_pM$:

$$g(R(X,Y)Y,X) \leqslant K\big(g(X,X)g(Y,Y) - g(X,Y)^2\big), \tag{8}$$

where R stands for the curvature tensor on (M, g).

In [1], we proved that given the curvature bound (8) and under similar assumptions to (H1)–(H2), a unique continuation theorem analogous to Theorem 1.3 is satisfied in the exterior region of double null cones, albeit without the presence of first order terms in the wave operator. As a consequence of this unique continuation principle, together with some controllability theory for the wave equation in rough Sobolev spaces, we were able to show unique determination of zeroth order coefficients in certain geometries where no real-analytic features are present. The result [1] covers perturbations of ultrastatic manifolds with strictly negative spatial sectional curvature. However, the Minkowski spacetime is on the boundary of the spacetimes allowed in [1], and its perturbations are not covered by the theory. Let us also remark that in general the curvature bound (8) is unstable under perturbations.

Our proof of the unique continuation theorem in [1] relied on the notion of spacetime convex functions that can be constructed under two-sided curvature bounds [3, 5], together with Carleman estimates with degenerate weights. Here, by a *spacetime convex function* ϕ, we mean that an inequality of the following form is satisfied at each point $p \in M$ and each $X \in T_p M$:

$$\operatorname{Hess} \phi(X, X) \geqslant \lambda g(X, X) \tag{9}$$

for some $\lambda > 0$. In relation to the latter idea, we mention the earlier works [2, 31] that prove a similar unique continuation result in the Minkowski geometry.

In the present work, we have identified a weaker curvature bound (H1) that replaces the curvature bound (8) of the previous work. Indeed, note that (H1) corresponds to (8) restricted to the case that X is a null vector and Y is a spacelike vector that is normal to X. Thus, (H1) is strictly weaker than (8). This leads to our enhanced unique continuation result (cf. [1, Theorem 2.1]). The weakening of the curvature bound is essentially related to a new comparison result for a Riccati equation (see Proposition 2.3). As a result of assuming only this weaker curvature bound we can solve the Lorentzian Calderón problem for manifolds satisfying (H1)–(H4) as well as for sufficiently small perturbations of such manifolds. In particular, in contrast to [1] we also solve the Lorentzian Calderón problem for spacetime perturbations of the Minkowski geometry.

Let us also mention that the proof of the unique continuation principle in this paper is fundamentally different from the previous work. For each point p on M, we construct a function with strictly pseudo-convex level sets that give a foliation of the exterior of the double null cone $\mathscr{E}_p$. This function can then be used as a Carleman weight to deduce our unique continuation principle. As the function has strictly pseudo-convex level sets, we are also able to allow first order terms in the wave operator. We remark that the level sets of the spacetime convex functions used in [1] are pseudo-convex only in a nonstrict sense.

Unique continuation near the Minkowski geometry was proven independently by Vaibhav Kumar Jena and Arick Shao. Their preprint [19] was posted in arXiv around the same time (17 December 2021) as ours (3 December 2021), and the proofs in these two works are very different. They assume that the curvature of the manifold is small in norm, while we assume the one-sided bound (H1). For example, an ultrastatic manifold whose spatial factor is a Riemannian manifold of arbitrarily large negative curvature is covered by our theory (see [1] for details), but not by [19]. On the other hand, their result has the advantage of being more quantitative in the sense that it comes with a Carleman estimate.

1.5. Sketch of the proof of Theorem 1.1 via our unique continuation principle

As mentioned earlier, our enhanced unique continuation principle (Theorem 1.3) is the key tool in solving the Lorentzian Calderón problem under the weaker curvature bound compared to our earlier work. Indeed, as stated in [1, Section 1.2] any improvements (in terms of geometric assumptions) on unique continuation principles stated on $\mathscr{E}_p$ will automatically lead to improvements on results on the Lorentzian Calderón problem. This is due to the fact that our solution of the Lorentzian Calderón problem in [1] only relies on the unique continuation principle in exterior regions of double null cones together with an exact controllability argument which we will very briefly recall here.

Given a point $p = (t_0, x_0) \in \operatorname{supp} V$, we uniquely recover the value of $V(p)$ from the Dirichlet-to-Neumann map Λ_V by the following procedure. We begin with a point p_0 that is in the sufficiently distant past of p as given in (H3). The significance of p_0 is related to existence of a suitable exact controllability theory on time slices that are in the causal future of p_0. Indeed, writing $p_0 = (t_1, x_1)$ and as shown in [1, Proposition 6.1], given any $s \geqslant 0$ and any $(w_0, w_1) \in H^{-s}(M_0) \times H^{-s-1}(M_0)$ we can find $f \in H^{-s}(\Sigma)$ with compact support in $(-T, t_1) \times \partial M_0$ such that the solution to (3) with boundary data f satisfies $(u|_{t=t_0}, \partial_t u|_{t=t_0}) = (w_0, w_1)$. This allows us to construct Dirichlet boundary values f in (3) that are supported on a neighborhood of $\mathscr{E}_{p_0} \cap \Sigma$ producing focused solutions at the fixed time slice $t = t_0$. Here, by *focused solutions* we mean solutions u to (3) that satisfy

$$u|_{t=t_0} = 0 \quad \text{and} \quad \partial_t u|_{t=t_0} = \omega \delta_{x_0}, \tag{10}$$

where ω is a constant and δ_{x_0} is the delta distribution supported at $x = x_0$. Moreover, these focused solutions can always be *sensed* by measuring the Cauchy data set of solutions to (3) on $\mathscr{E}_p \cap \Sigma$. In order to explain this in more detail, assume that $f \in H^{-(n+1)/2}(\Sigma)$ is chosen such that the solution to (3) with boundary value f satisfies (10). In view of the finite speed of propagation for the wave equation, it follows that

$$u|_{\mathscr{E}_p \cap \Sigma} = \partial_\nu u|_{\mathscr{E}_p \cap \Sigma} = 0. \tag{11}$$

On the other hand, given any solution u to (3) that satisfies (11) with $f \in H^{-(n+1)/2}(\Sigma)$, we may apply our unique continuation principle to deduce that (10) must also be satisfied for some constant ω.

Finally, the knowledge of the focused solutions at p gives us the knowledge of $V(p)$ up to a multiplicative constant that is independent of f. This constant may then be shown to be identically 1 via some additional technical arguments.

We refer the reader to [1, Sections 6–7] for the detailed proof of how unique continuation results in $\mathscr{E}_p$ can be used to solve the Lorentzian Calderón problem. The reader should just swap [1, Theorem 2.1] with our enhanced Theorem 1.3.

1.6. Organization of the paper

In Section 2, we derive a comparison result for a Riccati equation on $\mathbb{R}^n$. This abstract Riccati equation is later utilized in the construction of a function on $\mathscr{E}_p$ with strictly pseudo-convex level sets. In Section 3, we show that conditions (H1)–(H2) are stable after a conformal scaling of the metric. Together with transformation rules for the wave operator under conformal scaling of the metric, we arrive at an equivalent formulation of the unique continuation principle that will be easier to prove; see Theorem 3.1 for the precise formulation. The rest of the paper is devoted to the proof of the latter theorem. In Section 4, we show that under the hypotheses of Theorem 3.1, the Lorentzian distance function to the tip of the null cones gives a foliation of the set $\mathscr{E}_p$ by strictly pseudo-convex hypersurfaces; see Proposition 4.1 for the precise formulation. We show that this foliation can be used together with Hörmander's local unique continuation result across strictly pseudo-convex hypersurfaces to conclude the proof of Theorem 3.1.

2. Comparison result for a Riccati equation on $\mathbb{R}^n$

Let us consider the Minkowski inner product on $\mathbb{R}^n$ with $n \geqslant 2$, defined by

$$\langle v, w \rangle = -v_0 w_0 + \sum_{j=1}^{n-1} v_j w_j .$$

A vector $v \in \mathbb{R}^n$ is *null* if $\langle v, v \rangle = 0$, and a matrix $L \in \mathbb{R}^{n\times n}$ is *symmetric* (with respect to the Minkowski metric) if

$$\langle Lv, w \rangle = \langle v, Lw \rangle \quad \text{for all } v, w \in \mathbb{R}^n.$$

Definition 2.1. A symmetric matrix $L \in \mathbb{R}^{n\times n}$ is *null negative-definite*, written $L \lhd 0$, if

$$\langle Lv, v \rangle < 0 \quad \text{for all nonzero null vectors } v \in \mathbb{R}^n,$$

We also say that L is *null negative semi-definite*, written $L \unlhd 0$, if

$$\langle Lv, v \rangle \leqslant 0 \quad \text{for all null vectors } v \in \mathbb{R}^n.$$

Lemma 2.2. *Let $L \in \mathbb{R}^{n\times n}$ be a symmetric matrix that is null negative semi-definite but not null negative-definite. Then there exists a nonzero null vector $x \in \mathbb{R}^n$ such that $Lx = \lambda x$ for some $\lambda \in \mathbb{R}$.*

Proof. Set $e_0 = (1, 0, \dots, 0)$. Since L is null negative semi-definite but not null negative-definite, there must exist a null vector $x = e_0 + \tilde{x}$ with $\langle e_0, \tilde{x} \rangle = 0$ and $\langle \tilde{x}, \tilde{x} \rangle = 1$ such that

$$\langle Lx, x \rangle = 0.$$

When $n = 2$, there are $c_0, c_1 \in \mathbb{R}$ such that

$$Lx = c_0 e_0 + c_1 \tilde{x}. \tag{12}$$

The claim follows since $c_0 = c_1$ due to

$$0 = \langle Lx, x \rangle = -c_0 + c_1.$$

When $n \geqslant 3$, we proceed by letting y be any vector that satisfies

$$\langle y, y \rangle = 1 \quad \text{and} \quad \langle y, e_0 \rangle = \langle y, \tilde{x} \rangle = 0, \tag{13}$$

and subsequently define, for any $\varepsilon \in (-1, 1)$,

$$z_\varepsilon = e_0 + \sqrt{1-\varepsilon^2}\,\tilde{x} + \varepsilon y.$$

Observe that z_ε is null and also that

$$z_\varepsilon = x + \varepsilon y + O(\varepsilon^2).$$

As $\langle Lx, x\rangle = 0$, we have

$$0 \geqslant \langle Lz_\varepsilon, z_\varepsilon\rangle = 2\varepsilon\langle Lx, y\rangle + O(\varepsilon^2).$$

Since the latter expression must be valid for all y that satisfy (13), we deduce that (12) holds for some $c_0, c_1 \in \mathbb{R}$, and conclude as in the case $n = 2$. ■

Proposition 2.3. *Let $T > 0$. Suppose that $B \in C([0, T]; \mathbb{R}^{n\times n})$ satisfies*

$$B(t) \lhd 0 \quad \textit{for all } t \in [0, T].$$

Let L be the unique symmetric matrix that solves the Riccati equation

$$tL'(t) + L^2(t) - L(t) + t^2 B(t) = 0 \quad \textit{for all } t \in (0, T], \tag{14}$$

with $L(0) = \mathrm{id}$, where id *stands for the identity matrix. Then*

$$-L(t) \lhd 0 \quad \textit{for all } t \in (0, T].$$

Proof. By differentiating (14) we obtain

$$L' + tL'' + L'L + LL' - L' + (t^2B)' = 0.$$

Plugging $t = 0$ in the latter expression, we deduce that

$$L'(0) = 0. \tag{15}$$

Differentiating the equation again and plugging $t = 0$, we also obtain

$$L''(0) = -\tfrac{2}{3}B(0). \tag{16}$$

Together with the fact that $B(0) \lhd 0$, it follows that

$$-L(t) \lhd 0 \quad \text{for all } t \in (0, \delta), \tag{17}$$

for some small $\delta > 0$. To show that $-L \lhd 0$ on the entire interval $(0, T]$, assume that this is not the case. Then, by (17), there is $t_0 \in (0, T)$ such that $-L(t)$ is null negative-definite for all $t \in (0, t_0)$, and $-L(t_0)$ is null negative semi-definite but not null negative-definite. Thus, in view of Lemma 2.2,

$$L(t_0)x_0 = \lambda x_0 \tag{18}$$

for some nonzero null vector x_0 and some $\lambda \in \mathbb{R}$. Next, let us define

$$f(t) = -\langle L(t)x_0, x_0\rangle.$$

Then applying (14) we may write

$$\begin{aligned} t_0 f'(t_0) &= \langle -t_0 L'(t_0)x_0, x_0\rangle \\ &= \langle L(t_0)x_0, L(t_0)x_0\rangle - \langle L(t_0)x_0, x_0\rangle + t_0^2\langle B(t_0)x_0, x_0\rangle. \end{aligned} \tag{19}$$

Note that since x_0 is null and (18) holds, we have

$$\langle L(t_0)x_0, L(t_0)x_0\rangle = 0 \quad \text{and} \quad \langle L(t_0)x_0, x_0\rangle = 0.$$

Thus, we may simplify (19) to obtain

$$t_0 f'(t_0) = t_0^2\langle B(t_0)x_0, x_0\rangle < 0,$$

where we have also used the fact that $B \lhd 0$. On the other hand, $f < 0$ on $(0, t_0)$, since $-L \lhd 0$ on this interval, and also $f(t_0) = 0$. Hence $t_0 f'(t_0) \geqslant 0$, a contradiction. ∎

3. Stability of (H1)–(H2) under conformal scaling and unique continuation

Let us begin by defining a stronger variant of (H1) with a strict inequality:

(H1)′ For any point $p \in M$, any spacelike vector $v \in T_pM$, and any nonzero null vector $N \in T_pM$ with $g(v, N) = 0$, we have

$$g(R(N, v)v, N) < 0.$$

The aim of this section is to show that in order to prove our main unique continuation theorem, Theorem 1.3, it suffices to prove the following alternative theorem:

Theorem 3.1. *Let (M, g) be a Lorentzian manifold of the form (1)–(2) and assume that (H1)′–(H2) are satisfied. Let X be a first order linear differential operator with smooth coefficients on M. Let $p \in M^{\text{int}}$ be such that $\mathscr{E}_p \subset (-T, T) \times M_0$, where $\mathscr{E}_p$ is defined by (7). Let $u \in H^{-s}(M)$ for some $s \geqslant 0$ be a distributional solution to*

$$\Box u + Xu = 0 \quad \text{on } \mathscr{E}_p,$$

where $\Box$ is the wave operator associated to (M, g). Suppose that u and $\partial_\nu u$ both vanish on $\Sigma \cap \mathscr{E}_p$. Then $u = 0$ on $\mathscr{E}_p$.

Observe that, at first sight, Theorem 3.1 is weaker than Theorem 1.3 as we are imposing the stronger geometric assumption (H1)′ here. However, as we will show in the remainder of this section, due to a stability under conformal scaling for (H1)–(H2), Theorem 1.3 follows from Theorem 3.1. To show this, we start with two geometric lemmas. Our first lemma roughly states that conditions (H1)′–(H2) can always be attained from (H1)–(H2) via a conformal scaling of the metric that is close to the constant function.

First, let us fix some notation. We write $R(\cdot,\cdot)\cdot$ for the curvature tensor on (M,g) as a $(1,3)$-tensor that is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any smooth vector fields X, Y, Z. Here, ∇ is the covariant derivative. We also write $R(\cdot,\cdot,\cdot,\cdot)$ for the curvature tensor on (M,g) as a $(0,4)$-tensor with components given by R_{ijkl} with $i,j,k,l = 0,\ldots,n$. Note that given any $p \in M$ and $X, Y \in T_pM$, we have

$$g(R(X,Y)Y,X) = R(X,Y,X,Y) = \sum_{i,j,k,l=0}^{n} R_{ijkl} X^i Y^j X^k Y^l. \tag{20}$$

Given any two symmetric $(0,2)$-tensors A and B, we recall that their *Kulkarni–Nomizu product* $A \owedge B$ is the $(0,4)$-tensor given by the expression

$$\begin{aligned}(A \owedge B)(X_1,X_2,X_3,X_4) &= A(X_1,X_3)\,B(X_2,X_4) + A(X_2,X_4)\,B(X_1,X_3)\\ &\quad - A(X_1,X_4)\,B(X_2,X_3) - A(X_2,X_3)\,B(X_1,X_4).\end{aligned} \tag{21}$$

Given a smooth function ϕ on M, we define the *Hessian* of ϕ, denoted by $\operatorname{Hess}\phi$, as a bilinear form on the tangent space, defined for all $p \in M$ and $X, Y \in T_pM$ via the expression

$$\operatorname{Hess}\phi(X,Y) = g(\nabla_X \nabla\phi, Y). \tag{22}$$

We recall here that $\nabla\phi$ is the gradient of the function ϕ defined in local coordinates $(t = x^0, x^1, \ldots, x^n)$ via

$$\nabla\phi = \sum_{j,k=0}^{n} g^{jk}\, \partial_j\phi\, \partial_k.$$

Finally, and for the sake of brevity, we write t also for the function that maps any point (t,x) in M to its time coordinate t. For instance,

$$\nabla t = -c^{-1}\partial_t, \tag{23}$$

where c is as given by (2). We record here that given any $\phi \in C^\infty(M)$, by compactness there is a constant $C > 0$ such that for any $p \in M$ and $X \in T_pM$,

$$|\operatorname{Hess}\phi(X,X)| \leqslant C|X|^2, \tag{24}$$

where $|X|$ is the length of X with respect to an auxiliary Riemannian metric.

We will consider the following quantitative version of (H1)$'$.

(H1)$''$ There is $\kappa > 0$ such that for any point $p \in M$, any spacelike vector $v \in T_pM$, and any null vector $N \in T_pM$ with $g(v,N) = 0$,

$$R(N,v,N,v) \le -\kappa|N|^2 g(v,v).$$

Lemma 3.2. *Let (M, g) be a Lorentzian manifold of the form* (1)–(2) *that satisfies* (H1)–(H2). *Then in any neighborhood of the zero function there exists $f \in C^\infty(M)$ such that the Lorentzian manifold $(M, e^{2f} g)$ satisfies* (H1)″–(H2).

Proof. There is a constant $C > 0$ such that given any $p \in M$ and any null vector $N \in T_p M$,

$$|N| \leqslant C|g(N, \partial_t)|. \tag{25}$$

Indeed, to prove this, let us define the Riemannian metric

$$h = (dt)^2 + g_0(t, x)$$

on $M = [-T, T] \times M_0$ where g_0 is as in (2). Let $p = (t_0, x_0) \in M$ and $\{e_k\}_{k=0}^n \subset T_{x_0} M_0$ be an orthonormal basis with respect to $g_0(p)$. Writing $N = N^0 \partial_t + \sum_{k=1}^n N^k e_k$, and recalling that N is a null vector, it follows that

$$(N^0)^2 = \sum_{k=1}^n (N^k)^2 \quad \text{and} \quad g(N, \partial_t) = -c(p) N^0,$$

where c is as in (2). Therefore,

$$h(N, N) = (N^0)^2 + \sum_{k=1}^n (N^k)^2 = 2(N^0)^2 = \frac{2}{c(p)^2} g(N, \partial_t)^2.$$

The desired estimate (25) now follows immediately as M is compact and as all norms on finite-dimensional vector spaces are equivalent.

Recalling (24), it follows from (25) that there exists some constant $C_0 > 0$ only depending on (M, g) such that given any $p \in M$ and any null vector $N \in T_p M$,

$$|\mathrm{Hess}\, t(N, N)| \leqslant C_0\, g(N, \partial_t)^2.$$

Fixing any $\lambda > 2C_0$ and defining $\tau(t, x) = e^{\lambda t}$ for all $(t, x) \in M$, it follows that

$$\begin{aligned} \mathrm{Hess}\, \tau(N, N) &= e^{\lambda t}(\lambda\, \mathrm{Hess}\, t(N, N) + \lambda^2\, g(N, \partial_t)^2) \\ &\geqslant \lambda C_0 g(N, \partial_t)^2 \geqslant C_1\, g(N, \nabla \tau)^2, \end{aligned} \tag{26}$$

where $C_1 > 0$ depends on C_0 and λ. Next, let us define

$$\tilde{g} = e^{2f} g \quad \text{with} \quad f = \delta\, \tau, \tag{27}$$

where $\delta \in (0, C_1)$ is sufficiently small so that $(M, \tilde{g})$ satisfies (H2). This is always possible as (H2) is a stable condition with respect to small perturbations of the metric: see [1, Section 3] for the proof.

We claim that $(M, \tilde{g})$ also satisfies (H1)″ when $\delta > 0$ is small enough. It is well known (see e.g. [9, Theorem 1.159]) that under a conformal scaling of the metric, the curvature tensor, when written as a $(0, 4)$-tensor, transforms according to the formula

$$\widetilde{R} = e^{2f} R - e^{2f}\, g \owedge \big(\mathrm{Hess}\, f - df \otimes df + \tfrac{1}{2} g(\nabla f, \nabla f)\, g\big). \tag{28}$$

Here $\widetilde{R}$ is the curvature tensor on $(M, \tilde{g})$. Let $p \in M$, $v \in T_pM$ and nonzero $N \in T_pM$ satisfy

$$g(v, v) > 0, \quad g(N, N) = 0, \quad \text{and} \quad g(N, v) = 0.$$

Using equation (28) together with $R(N, v, N, v) \leqslant 0$, we have

$$\widetilde{R}(N, v, N, v) \leq -\tilde{g}(v, v)(\text{Hess}\, f(N, N) - g(N, \nabla f)^2).$$

Moreover, for small enough $\delta > 0$, the bounds (26) and (25) imply

$$-\text{Hess}\, f(N, N) + g(N, \nabla f)^2 = -\delta\, \text{Hess}\, \tau(N, N) + \delta^2 g(N, \nabla\tau)^2 \leq -\kappa|N|^2$$

for a constant $\kappa > 0$. ∎

Our second geometric lemma roughly states that conditions (H1)″–(H2) are stable under small perturbations of the metric.

Lemma 3.3. *Let (M, g) be a Lorentzian manifold of the form* (1)–(2) *that satisfies* (H1)″–(H2). *Given any smooth Lorentzian metric $\tilde{g}$ in a sufficiently small $C^2(M)$-neighborhood of g, the manifold $(M, \tilde{g})$ also satisfies* (H1)″–(H2) *with (M, g) replaced by $(M, \tilde{g})$.*

Proof. The fact that (H2) remains valid on manifolds $(M, \tilde{g})$, with $\tilde{g}$ a small perturbation of g, follows from [1, Section 3]. We write $\varepsilon > 0$ for the distance in $C^2(M)$ of the metric $\tilde{g}$ from g and write $\widetilde{R}$ for the curvature tensor on $(M, \tilde{g})$.

Our goal is to show that for $\varepsilon > 0$ sufficiently small, any $p \in M$, and any vectors $\widetilde{N} \in T_pM$ and $\tilde{v} \in T_pM$ satisfying

$$|\widetilde{N}| = 1, \quad \tilde{g}(\widetilde{N}, \widetilde{N}) = 0, \quad \tilde{g}(\tilde{v}, \tilde{v}) = 1, \quad \text{and} \quad \tilde{g}(\tilde{v}, \widetilde{N}) = 0,$$

we have

$$\widetilde{R}(\widetilde{N}, \tilde{v}, \widetilde{N}, \tilde{v}) \leq -\kappa/2, \tag{29}$$

where $\kappa > 0$ is as in (H1)″ for (M, g). In view of the symmetries of the curvature tensor, it is enough to show this with $\tilde{v}$ replaced by $\tilde{w} \in \mathbb{R}N + \tilde{v}$ satisfying $|\tilde{w}| \leq C$, where the constant $C > 0$ is given by Lemma 3.4 below. Observe that

$$\tilde{g}(\tilde{w}, \tilde{w}) = 1, \quad \tilde{g}(\tilde{w}, \widetilde{N}) = 0.$$

Let us show that there is $a \in \mathbb{R}$ of size $\mathcal{O}(\varepsilon)$ such that $N = a\partial_t + \widetilde{N}$ satisfies $g(N, N) = 0$. Solving this equation for a gives

$$a = \frac{g(\partial_t, \widetilde{N}) \pm \sqrt{g(\partial_t, \widetilde{N})^2 - 4g(\partial_t, \widetilde{N})g(\widetilde{N}, \widetilde{N})}}{2g(\partial_t, \partial_t)}.$$

We choose the sign that is opposite to the sign of $g(\partial_t, \widetilde{N})$, and $a = \mathcal{O}(\varepsilon)$ follows then from

$$g(\widetilde{N}, \widetilde{N}) = \tilde{g}(\widetilde{N}, \widetilde{N}) + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon).$$

A similar argument shows that there is $b \in \mathbb{R}$ of size $\mathcal{O}(\varepsilon)$ such that $v = b\partial_t + \tilde{w}$ satisfies $g(v, N) = 0$. Moreover, for small enough $\varepsilon > 0$,

$$g(v, v) = \tilde{g}(\tilde{w}, \tilde{w}) + \mathcal{O}(\varepsilon) = 1 + \mathcal{O}(\varepsilon) > 0.$$

Now (H1)$''$ for (M, g) gives, for small enough $\varepsilon > 0$,

$$\widetilde{R}(\widetilde{N}, \tilde{v}, \widetilde{N}, \tilde{v}) = R(N, v, N, v) + \mathcal{O}(\varepsilon) \leqslant -\kappa + \mathcal{O}(\varepsilon) \leqslant -\kappa/2. \qquad \blacksquare$$

Lemma 3.4. *Let (M, g) be a Lorentzian manifold of the form (1)–(2). Then for a sufficiently small $C^1(M)$-neighborhood B of g, there is $C > 0$ such that for all $p \in K$, $\tilde{g} \in B$, and all $N, v \in T_pM$, satisfying*

$$\tilde{g}(N, N) = 0, \quad \tilde{g}(v, v) = 1, \quad \tilde{g}(v, N) = 0, \tag{30}$$

and $N \neq 0$, there are $a \in \mathbb{R}$ and $w \in T_pM$ satisfying

$$v = aN + w, \quad |w| \leq C. \tag{31}$$

Proof. The vector field ∂_t is timelike with respect to any $\tilde{g} \in B$ for sufficiently small B. Thus the restriction of $\tilde{g} \in B$ to the subspace

$$\{w \in T_pM \mid \tilde{g}(w, \partial_t) = 0\} \tag{32}$$

is positive definite for all $p \in M$. Hence there is $\delta > 0$ such that

$$\delta \leq \tilde{g}(w, w) = \frac{\tilde{g}(w, w)}{|w|^2} \tag{33}$$

in the compact set

$$\{(\tilde{g}, w) \mid \tilde{g} \in \overline{B},\ w \in T_pM,\ p \in M,\ \tilde{g}(w, \partial_t) = 0,\ |w| = 1\}. \tag{34}$$

Here $\overline{B}$ is the $C(M)$-closure of B. Due to homogeneity, for all $\tilde{g} \in B$ and all $w \in T_pM$, $p \in M$, satisfying $\tilde{g}(w, \partial_t) = 0$, we have

$$\delta |w|^2 \leq \tilde{g}(w, w). \tag{35}$$

Let $p \in M$ and $\tilde{g} \in B$, and choose coordinates such that $\tilde{g}$ is the Minkowski metric at p, $\partial_t = (1, 0) \in \mathbb{R}^{1+n}$, $N = c(1, 1, 0) \in \mathbb{R}^{1+1+(n-1)}$ for some $c \neq 0$. Write $v = (a', a'', b) \in \mathbb{R}^{1+1+(n-1)}$. Then

$$0 = \tilde{g}(v, N) = c(-a' + a''). \tag{36}$$

Thus $v = aN + w$ where $a = a'/c$ and $w = (0, 0, b)$. Moreover,

$$\tilde{g}(w, w) = 1, \quad \tilde{g}(w, \partial_t) = 0, \tag{37}$$

and the claim follows from (35). $\blacksquare$

Combining Lemmas 3.2 and 3.3 we obtain the following corollary.

Corollary 3.5. *Let* (M, g) *be a Lorentzian manifold of the form* (1)–(2) *that satisfies* (H1)–(H2). *There exists* $f \in C^\infty(M)$ *in a neighborhood of the zero function such that given any smooth Lorentzian metric* $\tilde{g}$ *on* M *that lies in a sufficiently small* $C^2(M)$*-neighborhood of* g*, the manifold* $(M, e^{2f}\tilde{g})$ *satisfies* (H1)′–(H2).

We are now ready to prove that Theorem 3.1 implies Theorem 1.3.

Proof of Theorem 1.3 *via Theorem* 3.1. We assume that (M, g) is as in Theorem 1.3. Let $\tilde{g}$ be in a sufficiently small neighborhood of g in $C^2(M)$, so that the conclusion of Corollary 3.5 holds, that is, there exists $f \in C^\infty(M)$ close to zero such that the manifold $(M, e^{2f}\tilde{g})$ satisfies (H1)′–(H2). Finally, we assume that the point p, the differential operator X and $u \in H^{-s}(M)$, $s > 0$, are as in Theorem 1.3. We aim to show that u vanishes identically on the set $\mathscr{E}_p$ defined by (7) with respect to $(M, \tilde{g})$.

We write $\Box_{e^{2f}\tilde{g}}$ and $\Box_{\tilde{g}}$ for the wave operator on $(M, e^{2f}\tilde{g})$ and $(M, \tilde{g})$ respectively. Then (see e.g. [24])

$$\Box_{e^{2f}\tilde{g}} v = e^{-\frac{n+3}{2}f}(\Box_{\tilde{g}} + q_f)(e^{\frac{n-1}{2}f}\, v) \quad \forall v \in C^\infty(M), \tag{38}$$

where

$$q_f = -e^{-\frac{n-1}{2}f}\Box_{\tilde{g}}\, e^{\frac{n-1}{2}f}.$$

Recall that

$$(\Box_{\tilde{g}} + X)u = 0 \quad \text{on } \mathscr{E}_p.$$

We note that since null geodesics are conformally invariant, the exterior region of the double null cone $\mathscr{E}_p$ is the same set for both manifolds $(M, \tilde{g})$ and $(M, e^{2f}\tilde{g})$. In view of (38), there exists a smooth first order differential operator $\widetilde{X}$ depending on $(M, \tilde{g})$, X and f, such that

$$(\Box_{e^{2f}\tilde{g}} + \widetilde{X})(e^{-\frac{n-1}{2}f}u) = 0 \quad \text{on } \mathscr{E}_p.$$

As the manifold $(M, e^{2f}\tilde{g})$ satisfies (H1)′–(H2), we may apply Theorem 3.1 (with g replaced by $e^{2f}\tilde{g}$, X replaced with $\widetilde{X}$, and u replaced with $e^{-(n-1)/2f}\, u$) to conclude that the distribution u must vanish identically on $\mathscr{E}_p$. ∎

4. Proof of Theorem 3.1 via a strictly pseudo-convex foliation

The aim of this section is to prove the unique continuation principle stated in Theorem 3.1. In order to state our strategy, we first recall that a smooth function ϕ on a Lorentzian manifold (M, g) is said to have a *strictly pseudo-convex level set* at a point $p \in \phi^{-1}(0)$ if

$$\operatorname{Hess}\phi(N, N) > 0 \tag{39}$$

for all nonzero null vectors $N \in T_pM$ that satisfy $g(N, \nabla\phi) = 0$.

Strict pseudo-convexity was used by Hörmander (see e.g. [18, Theorem 28.4.3]) to prove unique continuation for solutions to various pseudo-differential operators. His result applies in particular to the wave equation

$$(\Box + X)u = 0, \tag{40}$$

locally near a point $p \in \phi^{-1}(0)$ across the level set $\phi^{-1}(0)$. More precisely, under the strict pseudo-convexity property above, if a solution u to the latter equation vanishes on the region $\{\phi > 0\}$ in a small neighborhood of p, then if must also vanish in a small neighborhood of p in $\{\phi < 0\}$. The strict pseudo-convexity condition can in this case be interpreted as saying that all the null geodesics that are tangent to $\phi^{-1}(0)$ at p must lie in $\{\phi > 0\}$ away from p.

In order to prove Theorem 3.1 we construct a global foliation of the exterior of the double null cone $\mathscr{E}_p$ by strictly pseudo-convex hypersurfaces. This foliation is going to be given by the level sets of the *Lorentzian distance function*, denoted by r_p below. Precisely, given any point $q \in \mathscr{E}_p$, in view of (H2), we consider the unique unit speed spacelike geodesic, $\gamma : [0, L_q] \to M$, that connects $p = \gamma(0)$ to $q = \gamma(L_q)$. Here, by geodesic we mean that $\nabla_{\dot\gamma}\dot\gamma = 0$ and by unit speed we mean that $g(\dot\gamma, \dot\gamma) = 1$. We define r_p to be the length of γ, that is,

$$r_p(q) = L_q. \tag{41}$$

We note that r_p is a smooth positive function on the open set $\mathscr{E}_p$ (recall from (7) that $\mathscr{E}_p$ does not contain p, nor the null cones emanating from p).

It will be convenient for us to give a coordinate expression for the function $r_p \in C^\infty(\mathscr{E}_p)$. To this end, given any $p \in M^{\text{int}}$ we let $e_0, \dots, e_n$ be an orthonormal basis of T_pM in the sense of [26, Lemma 24, p. 50], that is, for distinct $j, k = 0, \dots, n$,

$$g(e_j, e_k) = 0, \quad g(e_j, e_j) = \varepsilon_j,$$

where $\varepsilon_0 = -1$, and $\varepsilon_j = 1$ for $j = 1, \dots, n$. Then, in the normal coordinate system

$$y = (y^0, \dots, y^n) \mapsto \exp_p(y^j e_j), \tag{42}$$

we have

$$r_p(y) = \left(-(y^0)^2 + (y^1)^2 + \cdots + (y^n)^2\right)^{1/2}. \tag{43}$$

Indeed, consider the local hyperquadric

$$Q = \{\omega \in T_pM : g(\omega, \omega) = 1\}.$$

In the region $\mathscr{E}_p$, we consider the normal polar coordinates $y = r\omega$ with $r > 0$ and $\omega \in Q$, where y is as in (42). Let $q \in \mathscr{E}_p$ and write $q = s\omega$ in polar coordinates. Note that in view of (H2) there is a unique unit speed geodesic segment γ_q connecting p to q. It is well known (see e.g. [26, p. 71]) that the mapping $r \mapsto r\omega$ with $r \in [0, s]$ is the representation of γ_q in polar coordinates. Therefore, $r_p(q)$ defined by (41) is equal to s which is also equal to the right hand side of (43).

For future reference, we also prove that r_p is a distance function, that is,

$$g(\nabla r_p, \nabla r_p) = 1. \tag{44}$$

Gauss's lemma (see e.g. [26, Lemma 1, p. 127]) implies that in polar coordinates, the metric tensor has the orthogonal splitting

$$g = \begin{pmatrix} h(r\omega) & 0 \\ 0 & 1 \end{pmatrix},$$

where the $n \times n$ matrix h corresponds to the ω-coordinates while the last entry corresponds to the r-coordinate. The equality (44) follows immediately since $r_p(y) = r$ for $y = r\omega$.

In the remainder of this section, we prove the following proposition regarding pseudo-convexity of the level sets of r_p, which will be key in proving Theorem 3.1.

Proposition 4.1. *Let* (M, g) *be a Lorentzian manifold of the form* (1)–(2) *that satisfies* (H1)′–(H2). *Let* $p \in M^{\mathrm{int}}$ *and let* r_p *be the Lorentzian distance function to the point* p *defined above. Given any point* $q \in \mathscr{E}_p$*, we have*

$$\operatorname{Hess} r_p(N, N) > 0 \tag{45}$$

for any nonzero $N \in T_q M$ *with* $g(N, \nabla r_p) = g(N, N) = 0$.

Before proving this proposition, let us show how it can be utilized together with the local unique continuation result of Hörmander to conclude the proof of Theorem 3.1.

Proof of Theorem 3.1. Let p and u be as in the hypotheses of Theorem 3.1. First, we apply [1, Lemma 5.2] to conclude that u is smooth on $\mathscr{E}_p$, that is, $u \in C^\infty(\mathscr{E}_p)$. Next, we extend the manifold M_0 to a slightly larger manifold $\widetilde{M}_0$. We write $\widetilde{M} = [-T, T] \times \widetilde{M}_0$ and extend the metric g smoothly to $\widetilde{M}$. We write $\widetilde{\mathscr{E}}_p$ for the exterior of the double null cone on $\widetilde{M}$. We extend u by zero on $\widetilde{\mathscr{E}}_p$ and write $\tilde{u}$ for the extension. As the traces u and $\partial_\nu u$ both vanish on the timelike hypersurface $\mathscr{E}_p \cap \Sigma$, we deduce that $\Box \tilde{u} = 0$ on $\widetilde{\mathscr{E}}_p$, where we recall that $\tilde{u} = 0$ on $\widetilde{\mathscr{E}}_p \setminus M$. We aim to prove that

$$\tilde{u} = 0 \quad \text{on } \widetilde{\mathscr{E}}_p. \tag{46}$$

Let $\tilde{r}_p$ be as defined by (43) but corresponding to the slightly larger set $\widetilde{\mathscr{E}}_p$. By Proposition 4.1 (applied on the larger manifold $\widetilde{M}$) the level sets of $\tilde{r}_p$ are strictly pseudo-convex and give a foliation of $\widetilde{\mathscr{E}}_p$,

$$\widetilde{\mathscr{E}}_p = \bigcup_{s \in (0,a]} \tilde{r}_p^{-1}(s),$$

for some $a > 0$. Let

$$\lambda = \inf\{\tau \in (0, a] : \tilde{u}(y) = 0 \text{ for all } y \in \widetilde{\mathscr{E}}_p \text{ with } \tilde{r}_p(y) \in (\tau, a]\}. \tag{47}$$

In order to prove (46), it suffices to show that $\lambda = 0$. Since $\tilde{u} = 0$ on $\widetilde{\mathscr{E}}_p \setminus M$, λ must be strictly less than a. To get a contradiction, we suppose that $\lambda \in (0, a)$. By smoothness of $\tilde{u}$ on $\widetilde{\mathscr{E}}_p$ we have

$$\tilde{u}(y) = 0 \quad \text{for all } y \in \widetilde{\mathscr{E}}_p \text{ with } \tilde{r}_p(y) \in [\lambda, a].$$

Let $S = \tilde{r}_p^{-1}(\lambda)$. Proposition 4.1 implies that S is a smooth strictly pseudo-convex hypersurface. Applying Hörmander's local unique continuation principle [18, Theorem 28.3.4], it follows that given any point $q \in S$ (away from the boundary $\widetilde{\mathscr{E}}_p \cap \partial \widetilde{M}$), we have

$$\tilde{u} = 0 \quad \text{in a neighborhood of } q.$$

Finally, combining this with the facts that $\tilde{u}$ is continuous on $\widetilde{\mathscr{E}}_p$, that it vanishes on a neighborhood of the boundary $\widetilde{\mathscr{E}}_p \cap \partial \widetilde{M}$ and that S is compact, we conclude that there exists $\varepsilon > 0$ such that

$$\tilde{u}(y) = 0 \quad \text{for all } y \in \widetilde{\mathscr{E}}_p \text{ with } \tilde{r}_p(y) \in (\lambda - \varepsilon, a],$$

contradicting the minimality of λ in (47). ∎

In the remainder of this paper, we aim to prove Proposition 4.1.

4.1. Radial curvature equation

The directional curvature operator is defined by

$$R_v w = R(w, v)v, \quad v, w \in T_x M,\ x \in M,$$

where R is the $(1, 3)$ curvature tensor. We also recall that a function $r : M \to \mathbb{R}$ is called a *distance function* if $g(dr, dr) = 1$. Writing ∂_r for the gradient of r it is straightforward to see that

$$\nabla_{\partial_r} \partial_r = 0.$$

Lemma 4.2. *Let $r : M \to \mathbb{R}$ be a distance function, and consider the shape operator S corresponding to $r^2/2$, defined by*

$$\tfrac{1}{2} \operatorname{Hess} r^2(v, w) = g(Sv, w),$$

for all $v, w \in T_x M$ with $g(v, \partial_r) = g(w, \partial_r) = 0$. In other words, $SX = \nabla_X Y$ for a vector field X with $g(X, Y) = 0$, where Y is the gradient of $r^2/2$. Then S satisfies the radial curvature equation

$$\nabla_Y S - S + S^2 + R_Y = 0 \tag{48}$$

on $Y^\perp = \{X \in T_x M : g(X, Y) = 0\}$.

Proof. First, let us show that $Y^\perp$ is closed with respect to S and R_Y in the sense that $SY^\perp \subset Y^\perp$ and $R_Y Y^\perp \subset Y^\perp$. Writing ∂_r for the gradient of r, we observe that $Y = \frac{1}{2}\nabla r^2 = r\partial_r$. Hence, for any $X \in Y^\perp$,

$$g(SX, Y) = g(\nabla_X Y, Y) = \tfrac{1}{2} X(g(Y, Y)) = \tfrac{1}{2} X(r^2) = rX(r) = g(X, Y) = 0,$$

thus proving that $SX \in Y^{\perp}$. Note also that

$$g(R_Y X, Y) = g(R(X, Y)Y, Y) = 0,$$

by anti-symmetry of R in its last two indices. Thus, $R_Y X \in Y^{\perp}$ as well. Next, we write

$$\begin{aligned}(\nabla_Y S)X + S^2 X &= \nabla_Y(SX) - S\nabla_Y X + S^2 X \\ &= \nabla_Y \nabla_X Y - \nabla_{\nabla_Y X} Y + \nabla_{\nabla_X Y} Y = \nabla_Y \nabla_X Y + \nabla_{[X,Y]} Y.\end{aligned}$$

On the other hand,

$$R_Y X = R(X, Y)Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y.$$

Moreover,

$$\nabla_Y Y = r\nabla_{\partial_r}(r\partial_r) = r\partial_r + r^2 \nabla_{\partial_r}\partial_r = Y.$$

Thus $\nabla_X \nabla_Y Y = SX$ and

$$(\nabla_Y S)X + S^2 X = -R_Y X + SX. \qquad \blacksquare$$

4.2. *Comparison result on* (M, g)

Proof of Proposition 4.1. We will prove inequality (45) at an arbitrary point $q \in \mathscr{E}_p$, by using the Riccati equation (48) associated to the distance function r_p along the radial geodesic segment γ that connects p to q. We write $p = \gamma(0)$ and consider an orthonormal frame $\{\tilde{e}_j(0)\}_{j=0}^{n-1}$ on $\dot{\gamma}(0)^{\perp}$ in the sense that for distinct $j, k = 0, \ldots, n-1$,

$$g(\tilde{e}_j, \tilde{e}_k) = 0, \quad g(\tilde{e}_j, \tilde{e}_j) = \varepsilon_j,$$

where $\varepsilon_0 = -1$, and $\varepsilon_j = 1$ for $j = 1, \ldots, n-1$. For each $j = 0, 1, \ldots, n-1$, we define $\tilde{e}_j(s) \in \dot{\gamma}(s)^{\perp}$ to be the parallel transport of the vector $\tilde{e}_j(0)$ along γ from $\gamma(0)$ to $\gamma(s)$. Note that for all $v \in \dot{\gamma}^{\perp}$,

$$v = \sum_{j=0}^{n-1} \varepsilon_j g(v, \tilde{e}_j)\tilde{e}_j.$$

In particular, the matrix $\tilde{S}$ of a linear map S on $\dot{\gamma}^{\perp}$, defined by $S\tilde{e}_k = \tilde{S}_k^j \tilde{e}_j$, satisfies $\tilde{S}_k^j = \varepsilon_j g(S\tilde{e}_k, \tilde{e}_j)$. Let us now take S as in Lemma 4.2 restricted to the geodesic segment γ and use the abbreviated notation r in place of r_p. Note that the radial curvature equation (48) implies that, on γ,

$$\begin{aligned}r\partial_r \varepsilon_j g(S\tilde{e}_j, \tilde{e}_k) &= \varepsilon_j g((\nabla_{r\partial_r} S)\tilde{e}_j, \tilde{e}_k) \\ &= \varepsilon_j g(S\tilde{e}_j, \tilde{e}_k) - \varepsilon_j g(S^2\tilde{e}_j, \tilde{e}_k) - r^2 \varepsilon_j g(R_{\partial_r}\tilde{e}_j, \tilde{e}_k).\end{aligned}$$

Thus the matrix $\tilde{S}_k^j$ of S satisfies the Riccati equation

$$r\tilde{S}' - \tilde{S} + \tilde{S}^2 + r^2 \tilde{R} = 0, \tag{49}$$

with $\tilde{R}$ the matrix of R_{∂_r} on $\dot{\gamma}^{\perp}$ with respect to the frame $\{\tilde{e}_j\}_{j=0}^{n-1}$ (note that in the flat case, with the metric tensor $\sum_{j=0}^{n} \varepsilon_j (dx^j)^2$, we have $\tilde{S} = \text{id}$). In general, $\tilde{S}(0) = \text{id}$.

Moreover, the curvature bound (H1)$'$ implies that the matrix $\tilde{R}$ is null negative-definite in the sense of Definition 2.1. The proof is complete, thanks to Proposition 2.3 with $L = \tilde{S}$ and $B = \tilde{R}$. ∎

Funding. S.A. acknowledges support from NSERC grant 488916. A.F. was supported by the Fields Institute for Research in Mathematical Sciences.

References

[1] Alexakis, S., Feizmohammadi, A., Oksanen, L.: Lorentzian Calderón problem under curvature bounds. Invent. Math. **229**, 87–138 (2022) Zbl 1491.35446 MR 4438353

[2] Alexakis, S., Shao, A.: Global uniqueness theorems for linear and nonlinear waves. J. Funct. Anal. **269**, 3458–3499 (2015) Zbl 1329.35007 MR 3406859

[3] Alexander, S. B., Bishop, R. L.: Lorentz and semi-Riemannian spaces with Alexandrov curvature bounds. Comm. Anal. Geom. **16**, 251–282 (2008) Zbl 1149.53040 MR 2425468

[4] Alinhac, S.: Non-unicité du problème de Cauchy. Ann. of Math. (2) **117**, 77–108 (1983) Zbl 0516.35018 MR 683803

[5] Andersson, L., Howard, R.: Comparison and rigidity theorems in semi-Riemannian geometry. Comm. Anal. Geom. **6**, 819–877 (1998) Zbl 0963.53038 MR 1664893

[6] Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**, 1024–1065 (1992) Zbl 0786.93009 MR 1178650

[7] Belishev, M. I.: An approach to multidimensional inverse problems for the wave equation. Dokl. Akad. Nauk SSSR **297**, 524–527 (1987) (in Russian) Zbl 0661.35084 MR 924687

[8] Belishev, M. I.: Recent progress in the boundary control method. Inverse Problems **23**, R1–R67 (2007) Zbl 1126.35089 MR 2353313

[9] Besse, A. L.: Einstein manifolds. Ergeb. Math. Grenzgeb. (3) 10, Springer, Berlin (1987) Zbl 0613.53001 MR 867684

[10] Bukhgeim, A. L.: Recovering a potential from Cauchy data in the two-dimensional case. J. Inverse Ill-Posed Probl. **16**, 19–33 (2008) Zbl 1142.30018 MR 2387648

[11] Burq, N., Gérard, P.: Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. C. R. Acad. Sci. Paris Sér. I Math. **325**, 749–752 (1997) Zbl 0906.93008 MR 1483711

[12] Dos Santos Ferreira, D., Kenig, C. E., Salo, M., Uhlmann, G.: Limiting Carleman weights and anisotropic inverse problems. Invent. Math. **178**, 119–171 (2009) Zbl 1181.35327 MR 2534094

[13] Dos Santos Ferreira, D., Kurylev, Y., Lassas, M., Salo, M.: The Calderón problem in transversally anisotropic geometries. J. Eur. Math. Soc. **18**, 2579–2626 (2016) Zbl 1359.35224 MR 3562352

[14] Eskin, G.: Inverse hyperbolic problems with time-dependent coefficients. Comm. Partial Differential Equations **32**, 1737–1758 (2007) Zbl 1153.35081 MR 2372486

[15] Eskin, G.: Inverse problems for general second order hyperbolic equations with time-dependent coefficients. Bull. Math. Sci. **7**, 247–307 (2017) Zbl 1381.35240 MR 3671738

[16] Feizmohammadi, A., Ilmavirta, J., Kian, Y., Oksanen, L.: Recovery of time-dependent coefficients from boundary data for hyperbolic equations. J. Spectr. Theory **11**, 1107–1143 (2021) Zbl 1484.35407 MR 4322032

[17] Feizmohammadi, A., Ilmavirta, J., Oksanen, L.: The light ray transform in stationary and static Lorentzian geometries. J. Geom. Anal. **31**, 3656–3682 (2021) Zbl 1472.53082 MR 4236538

[18] Hörmander, L.: The analysis of linear partial differential operators, IV. Springer, Berlin (1985) Zbl 0612.35001

[19] Jena, V. K., Shao, A.: Control of waves on Lorentzian manifolds with curvature bounds. arXiv:2112.09539 (2021)

[20] Katchalov, A., Kurylev, Y., Lassas, M.: Inverse boundary spectral problems. Chapman & Hall/CRC Monogr. Surveys Pure Appl. Math. 123, Chapman & Hall/CRC, Boca Raton, FL (2001) Zbl 1037.35098 MR 1889089

[21] Kian, Y., Oksanen, L.: Recovery of time-dependent coefficient on Riemannian manifold for hyperbolic equations. Int. Math. Res. Notices **2019**, 5087–5126 Zbl 1459.35398 MR 4001024

[22] Lasiecka, I., Lions, J.-L., Triggiani, R.: Nonhomogeneous boundary value problems for second order hyperbolic operators. J. Math. Pures Appl. (9) **65**, 149–192 (1986) Zbl 0631.35051 MR 867669

[23] Lee, J. M., Uhlmann, G.: Determining anisotropic real-analytic conductivities by boundary measurements. Comm. Pure Appl. Math. **42**, 1097–1112 (1989) Zbl 0702.35036 MR 1029119

[24] Liimatainen, T., Oksanen, L.: Counterexamples to inverse problems for the wave equation. Inverse Probl. Imaging **16**, 467–479 (2022) Zbl 1484.35412 MR 4382875

[25] Nachman, A. I.: Global uniqueness for a two-dimensional inverse boundary value problem. Ann. of Math. (2) **143**, 71–96 (1996) Zbl 0857.35135 MR 1370758

[26] O'Neill, B.: Semi-Riemannian geometry. Pure Appl. Math. 103, Academic Press, New York (1983) Zbl 0531.53051 MR 719023

[27] Ramm, A. G., Sjöstrand, J.: An inverse problem of the wave equation. Math. Z. **206**, 119–130 (1991) Zbl 0697.35164 MR 1086818

[28] Robbiano, L.: Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. Comm. Partial Differential Equations **16**, 789–800 (1991) Zbl 0735.35086 MR 1113107

[29] Robbiano, L., Zuily, C.: Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. Invent. Math. **131**, 493–539 (1998) Zbl 0909.35004 MR 1614547

[30] Salazar, R.: Determination of time-dependent coefficients for a hyperbolic inverse problem. Inverse Problems **29**, article no. 095015, 17 pp. (2013) Zbl 1292.35336 MR 3104931

[31] Shao, A.: On Carleman and observability estimates for wave equations on time-dependent domains. Proc. London Math. Soc. (3) **119**, 998–1064 (2019) Zbl 1437.35441 MR 3964826

[32] Stefanov, P. D.: Uniqueness of the multi-dimensional inverse scattering problem for time dependent potentials. Math. Z. **201**, 541–559 (1989) Zbl 0653.35049 MR 1004174

[33] Stefanov, P.: Support theorems for the light ray transform on analytic Lorentzian manifolds. Proc. Amer. Math. Soc. **145**, 1259–1274 (2017) Zbl 1370.53049 MR 3589324

[34] Stefanov, P., Yang, Y.: The inverse problem for the Dirichlet-to-Neumann map on Lorentzian manifolds. Anal. PDE **11**, 1381–1414 (2018) Zbl 1391.35426 MR 3803714

[35] Sylvester, J., Uhlmann, G.: A global uniqueness theorem for an inverse boundary value problem. Ann. of Math. (2) **125**, 153–169 (1987) Zbl 0625.35078 MR 873380

[36] Tataru, D.: Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. Comm. Partial Differential Equations **20**, 855–884 (1995) Zbl 0846.35021 MR 1326909

[37] Tataru, D.: Unique continuation for operators with partially analytic coefficients. J. Math. Pures Appl. (9) **78**, 505–521 (1999) Zbl 0936.35038 MR 1697040

[38] Uhlmann, G.: Inverse problems: seeing the unseen. Bull. Math. Sci. **4**, 209–279 (2014) Zbl 1325.35286 MR 3228575