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On wavefront sets of global Arthur packets of classical groups: Upper bound

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Abstract. We prove a conjecture of the first named author (2014) on the upper bound Fourier coefficients of automorphic forms in Arthur packets of all classical groups over any number field. This conjecture generalizes the global version of the local tempered L-packet conjecture of Shahidi (1990). Under certain assumption, we also compute the wavefront sets of the unramified unitary dual for split classical groups.

Keywords: endoscopic classification, Arthur packet, Fourier coefficient, unipotent orbit, automorphic form, wavefront set.

1. Introduction

In the classical theory of automorphic forms, Fourier coefficients encode abundant arithmetic information of automorphic forms. In the modern theory of automorphic forms, i.e., the theory of automorphic representations of reductive algebraic groups defined over a number field k (or a global field), Fourier coefficients bridge the connection from harmonic analysis to number theory via automorphic forms. When the reductive group is the general linear group GL_n, by a classical theorem of Piatetski-Shapiro [37] and Shalika [40], every cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k, has a non-zero Whittaker–Fourier coefficient. This fundamental result has been indispensable in the theory, especially the theory of automorphic L-functions. The theorem of Piatetski-Shapiro and Shalika has been extended to the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ in [21] and to the isobaric sum automorphic spectrum of $\mathrm{GL}_n(\mathbb{A})$ in [28].

In general, due to the nature of the discrete spectrum of square-integrable automorphic forms on reductive algebraic groups $\mathrm{G}(\mathbb{A})$, one has to consider more general version of Fourier coefficients, i.e., Fourier coefficients of automorphic forms associated to nilpotent

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orbits in the Lie algebra $\mathfrak{g}$ of G. Such general Fourier coefficients of automorphic forms, including Bessel–Fourier coefficients and Fourier–Jacobi coefficients have been widely used in theory of automorphic L-functions via integral representation method (see [15, 16, 25, 26], for instance), in the automorphic descent method of Ginzburg, Rallis and Soudry to produce special cases of explicit Langlands functorial transfers [17], and in the Gan–Gross–Prasad conjecture on vanishing of the central value of certain automorphic L-functions of symplectic type [12, 14, 26]. More recent applications of such general Fourier coefficients to explicit constructions of endoscopy transfers for classical groups can be found in [20] (and also in [13] for split classical groups).

In this paper, we consider following classical groups defined over k, $\mathrm{G}_n = \mathrm{Sp}_{2n}$, SO_{2n+1}, $\mathrm{SO}_{2n}^{\alpha}$, quasi-split, and U_n, quasi-split or inner forms. We follow the formulation in [18] for the definition of generalized Whittaker–Fourier coefficients of automorphic forms associated to nilpotent orbits, see Section 2 for details. It is well known that nilpotent orbits of the quasi-split classical group G_n are parameterized by symplectic or orthogonal partitions and certain quadratic forms when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}$, and by relevant partitions when $\mathrm{G}_n = \mathrm{U}_n$ (see [11, 35, 43], for instance). For any irreducible automorphic representation π of $\mathrm{G}_n(\mathbb{A})$, let $\mathfrak{n}(\pi)$ be the set of nilpotent orbits providing non-zero generalized Whittaker–Fourier coefficients for π, which is called the *wavefront set* of π, as in [24], for instance. Let $\mathfrak{n}^m(\pi)$ be the subset that consists of maximal elements in $\mathfrak{n}(\pi)$ under the dominance ordering of nilpotent orbits, and denote by $\mathfrak{p}^m(\pi)$ the set of the partitions of type G_n corresponding to nilpotent orbits in $\mathfrak{n}^m(\pi)$.

It is an interesting problem to determine the structure of the set $\mathfrak{n}^m(\pi)$ and equivalently the set $\mathfrak{p}^m(\pi)$ for any given irreducible automorphic representation π of $\mathrm{G}_n(\mathbb{A})$, by means of other invariants of π. When π occurs in the discrete spectrum of square integrable automorphic functions on $\mathrm{G}_n(\mathbb{A})$, the global Arthur parameter attached to π [2, 27, 31] is clearly a fundamental invariant for π. An important conjecture made in [20], which is the natural generalization of the global version of the local tempered L-packet conjecture of Shahidi [38, 39], asserts an intrinsic relation between the structure of the global Arthur parameter of π and the structure of the set $\mathfrak{p}^m(\pi)$. It is well known that the conjecture of Shahidi and its global version (see [23, Section 3] for discussion and proof) have played a fundamental role in the understanding of the local and global Arthur packets for generic Arthur parameters, according to the endoscopic classification of Arthur [2, 27, 31]. It is well expected that the conjecture made in [20] for general global Arthur parameters will be important to the understanding of the structure of general global Arthur packets.

To state the conjecture of [20], for simplicity, we briefly recall the endoscopic classification of the discrete spectrum for $\mathrm{G}_n(\mathbb{A})$ from [2] for $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}$.

The set of global Arthur parameters for the discrete spectrum of G_n is denoted, as in [2], by $\widetilde{\Psi}_2(\mathrm{G}_n)$, the elements of which are of the form

$$\psi := \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r,$$

where ψ_i are pairwise different simple global Arthur parameters of orthogonal type (when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}^{\alpha}$) or symplectic type (when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$), and have the form $\psi_i =$

(τ_i, b_i). The notations are explained in order. Let $\mathcal{A}_{\text{cusp}}(\mathrm{GL}_{a_i})$ be the set of equivalence classes of irreducible cuspidal automorphic representations of $\mathrm{GL}_{a_i}(\mathbb{A})$. We have $\tau_i \in \mathcal{A}_{\text{cusp}}(\mathrm{GL}_{a_i})$ with

$$\sum_{i=1}^{r} a_i b_i = \begin{cases} 2n+1 & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \\ 2n & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n+1} \text{ or } \mathrm{SO}_{2n}^{\alpha}, \end{cases}$$

and the central character of ψ and the central characters of τ_i's satisfy the following constraints:

$$\prod_i \omega_{\tau_i}^{b_i} = \begin{cases} 1 & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n} \text{ or } \mathrm{SO}_{2n+1}, \\ \eta_\alpha & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n}^{\alpha}, \end{cases}$$

following [2, Section 1.4]. More precisely, for each $1 \le i \le r$, $\psi_i = (\tau_i, b_i)$ satisfies the following conditions: if τ_i is of symplectic type (i.e., $L(s, \tau_i, \wedge^2)$ has a pole at $s = 1$), then b_i is even (when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}^{\alpha}$), or odd (when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$); and if τ_i is of orthogonal type (i.e., $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$), then b_i is odd (when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}^{\alpha}$), or even (when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$).

Theorem 1.1 ([2, Theorem 1.5.2]). *For each global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)$, a global Arthur packet $\widetilde{\Pi}_\psi$ is defined. The discrete spectrum of $\mathrm{G}_n(\mathbb{A})$ has the following decomposition:*

$$L^2_{\text{disc}}(\mathrm{G}_n(k)\backslash \mathrm{G}_n(\mathbb{A})) \cong \bigoplus_{\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)} m_\psi \Big(\bigoplus_{\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)} \pi \Big),$$

where $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ denotes the subset of $\widetilde{\Pi}_\psi$ consisting of members which occur in the discrete spectrum, and m_ψ is the discrete multiplicity of Π, which is either 1 or 2.

As in [20], one may call $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ the automorphic L^2-packet attached to ψ. For $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, the structure of the global Arthur parameter ψ deduces constraints on the structure of $\mathfrak{p}^m(\pi)$, which is given by the following conjecture of the first named author. We recall from [20] that for a given global Arthur parameter ψ as above,

$$\underline{p}(\psi) = [(b_1)^{a_1} \cdots (b_r)^{a_r}]$$

is the partition attached to $(\psi, G^\vee(\mathbb{C}))$.

Conjecture 1.2 ([20, Conjecture 4.2 (1) and (2)]). *For any $\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)$, let $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ be the automorphic L^2-packet attached to ψ and $\underline{p}(\psi)$ be the partition attached to $(\psi, \mathrm{G}_n^\vee(\mathbb{C}))$. For any $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, if a partition $\underline{p} \in \mathfrak{p}^m(\pi)$, then*

$$\underline{p} \le \eta_{\mathfrak{g}_n^\vee, \mathfrak{g}_n}(\underline{p}(\psi)).$$

Here $\eta_{\mathfrak{g}_n^\vee, \mathfrak{g}_n}$ denotes the Barbasch–Vogan–Spaltenstein duality map from the partitions for the dual group $\mathrm{G}_n^\vee(\mathbb{C})$ to the partitions for G_n as introduced in [41] and [6], see also [1].

Conjecture 4.2 in [20] consists of two parts: one is the upper-bound conjecture (Conjecture 1.2) and the other is the sharpness conjecture ([20, Conjecture 4.2 (3)], i.e., there exists $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$ such that $\eta_{\mathfrak{g}^\vee,\mathfrak{g}}(\underline{p}(\psi)) \in \mathfrak{p}^m(\pi)$). It is clear that if the global Arthur parameter ψ is generic, then [20, Conjecture 4.2] asserts that the corresponding global Arthur packet $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ contains an automorphic member that is generic, i.e., has a non-zero Whittaker–Fourier coefficient. This is the global version of the local tempered L-packet conjecture of Shahidi [38], and it was proved in [23, Section 3] by using automorphic descent of Ginzburg, Rallis, and Soudry [17]. The goal of this paper is to prove Conjecture 1.2 for general global Arthur parameters. The sharpness conjecture is global in nature and will be fully considered in future projects.

In [22], using the method of *local descents*, we partially prove Conjecture 1.2 for $\mathrm{G}_n = \mathrm{Sp}_{2n}$, namely, for any $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, if a partition $\underline{p} \in \mathfrak{p}^m(\pi)$, then

$$\underline{p} \leq_L \eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}(\underline{p}(\psi)),$$

under the lexicographical order. We refer to [20, Section 4] for more discussion on this conjecture and related topics.

In order to prove Conjecture 1.2, we study the structure of the unramified local components π_v of π and of the set $\mathfrak{p}^m(\pi_v)$ which is defined similarly to $\mathfrak{p}^m(\pi)$. Our discussion reduces the general situation to a special case of strongly negative unramified unitary representations of G_n (see Section 3 for details). In such a special situation, the structure of the wavefront set (Theorem 3.8) can be deduced as a special case from [36, Theorem 0.6].

To be more precise, first, for the Arthur parameter $\psi = \boxplus_{i=1}^r (\tau_i, b_i)$, by [22, Proposition 6.1] (see Proposition 4.1), there exist infinitely many finite places v such that $\mathrm{G}_n(k_v)$ is split, all $\tau_{i,v}$'s have trivial central characters, and hence π_v is the unramified component of an induced representation of the following form

$$\sigma = \times_{i=1}^r v^{\alpha_i} \chi_i(\det_{m_i}) \rtimes \sigma_{\mathrm{sn}},$$

where $0 \leq \alpha_i < 1$, σ_{sn} is a special family of strongly negative representations which have Arthur parameters of the form $\bigoplus_{j=1}^s 1_{W_F'} \otimes S_{2n_j+1}$ (see Section 3 for details), with W_F' being the Weil–Deligne group and $n_1 < n_2 < \cdots < n_s$. It is known that the wavefront set of σ (hence of π_v) is bounded above by the induced orbits once we know the leading orbits for the wavefront set of σ_{sn}. On the other hand, Okada [36, Theorem 0.6] computed the leading orbits in the wavefront set of those unramified representations whose Arthur parameters are trivial when restricted to the Weil–Deligne group.

Theorem 1.3 (Main Theorem). *Conjecture 1.2 holds for any* $\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)$.

We remark that for non-quasi-split even orthogonal groups, once the Arthur classification is carried out (see [7,8] for recent progress in this direction), Conjecture 1.2 can be proved by similar arguments.

In the last part of this paper, we study the wavefront set of the unramified unitary dual for split classical groups $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n}$. Under a conjecture on the wavefront set of negative representations (Conjecture 8.1), we are able to determine the set $\mathfrak{p}^m(\pi)$

for general unramified unitary representations (Theorem 8.2). This provides a reduction towards understanding the wavefront set of the whole unramified unitary dual, which has its own interests.

The structure of this paper is the following. In Section 2, we recall certain twisted Jacquet modules and Fourier coefficients associated to nilpotent orbits, following the formulation in [18]. The structure of unramified unitary dual of $\mathrm{G}_n(F_v)$ was determined by Barbasch in [4] and by Muić and Tadić in [34] with different approaches. In Section 3, we recall from [34] the results on unramified unitary dual for $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n}$. For the split group $\mathrm{G}_n = \mathrm{SO}_{2n}$, we do not need the full classification of the unramified unitary dual as given in [4], instead, we only need a family of unramified unitary representations, see Remark 3.6. In Section 4, we determine, for any given global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)$, the unramified components π_v of any $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$ in terms of the classification data in [34], and prove Theorem 1.3 by means of Theorem 4.2 which is about certain properties of $\underline{p} \in \mathfrak{p}^m(\pi_v)$. Theorem 4.2 is technical and will be proved in Sections 5, 6, and 7 for $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^\alpha$, respectively. In Section 8, we determine the leading orbits in the wavefront set of general unramified unitary representations assuming Conjecture 8.1 for split classical groups $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n}$ (Theorem 8.2). Note that for representations of non-connected groups, we follow [10] for the character expansions at the identity to define the wavefront set.

2. Fourier coefficients associated to nilpotent orbits

In this section, we recall certain twisted Jacquet modules and Fourier coefficients associated to nilpotent orbits, following the formulation of Gomez, Gourevitch and Sahi in [18].

Let G be a reductive group defined over a field F of characteristic zero, and $\mathfrak{g}$ be the Lie algebra of $G = \mathrm{G}(F)$. Given any semi-simple element $s \in \mathfrak{g}$, under the adjoint action, $\mathfrak{g}$ is decomposed into a direct sum of eigenspaces $\mathfrak{g}_i^s$ corresponding to eigenvalues i. The element s is called *rational semi-simple* if all its eigenvalues are in $\mathbb{Q}$. Given a nilpotent element u and a semi-simple element s in $\mathfrak{g}$, the pair (s, u) is called a *Whittaker pair* if s is a rational semi-simple element, and $u \in \mathfrak{g}_{-2}^s$. The element s in a Whittaker pair (s, u) is called a *neutral element* for u if there is a nilpotent element $v \in \mathfrak{g}$ such that (v, s, u) is an $\mathfrak{sl}_2$-triple. A Whittaker pair (s, u) with s being a neutral element is called a *neutral pair*.

Given any Whittaker pair (s, u), define an anti-symmetric form ω_u on $\mathfrak{g} \times \mathfrak{g}$ by

$$\omega_u(X, Y) := \kappa(u, [X, Y]),$$

where κ is the Killing form on $\mathfrak{g}$. For any rational number $r \in \mathbb{Q}$, let $\mathfrak{g}_{\geq r}^s = \bigoplus_{r' \geq r} \mathfrak{g}_{r'}^s$. Let $\mathfrak{u}_s = \mathfrak{g}_{\geq 1}^s$ and let $\mathfrak{n}_{s,u}$ be the radical of $\omega_u|_{\mathfrak{u}_s}$. Then $[\mathfrak{u}_s, \mathfrak{u}_s] \subset \mathfrak{g}_{\geq 2}^s \subset \mathfrak{n}_{s,u}$. For any $X \in \mathfrak{g}$, let $\mathfrak{g}_X$ be the centralizer of X in $\mathfrak{g}$. By [18, Lemma 3.2.6], one has $\mathfrak{n}_{s,u} = \mathfrak{g}_{\geq 2}^s + \mathfrak{g}_1^s \cap \mathfrak{g}_u$. Note that if the Whittaker pair (s, u) comes from an $\mathfrak{sl}_2$-triple (v, s, u), then $\mathfrak{n}_{s,u} = \mathfrak{g}_{\geq 2}^s$. We denote by $N_{s,u} = \exp(\mathfrak{n}_{s,u})$ the corresponding unipotent subgroup of G.

When $F = k_v$ is a non-Archimedean local field, we take $\psi\colon F \to \mathbb{C}^\times$ to be a fixed non-trivial additive character and define a character of $N_{s,u}$ by

$$\psi_u(n) = \psi(\kappa(u, \log(n))).$$

Let σ be an irreducible admissible representation of $\mathrm{G}(F)$. The *twisted Jacquet module* of π associated to a Whittaker pair (s, u) is defined to be $\sigma_{N_{s,u},\psi_u}$. Let $\mathfrak{n}(\sigma)$ be the set of nilpotent orbits $\mathcal{O} \subset \mathfrak{g}$ such that the twisted Jacquet module $\sigma_{N_{s,u},\psi_u}$ is non-zero for some neutral pair (s, u) with $u \in \mathcal{O}$.

When $F = k$ is a number field, let $\mathbb{A}$ be the ring of adeles, and let $\psi\colon F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a fixed non-trivial additive character. Extend the Killing form κ to $\mathfrak{g}(\mathbb{A}) \times \mathfrak{g}(\mathbb{A})$. Define a character of $N_{s,u}(\mathbb{A})$ by

$$\psi_u(n) = \psi(\kappa(u, \log(n))).$$

It is clear from the definition that the character $\psi_u(n)$ is trivial when restricted to the discrete subgroup $N_{s,u}(F)$, and hence can be viewed as a function on

$$[N_{s,u}] := N_{s,u}(F)\backslash N_{s,u}(\mathbb{A}).$$

Let π be an irreducible automorphic representation of $\mathrm{G}(\mathbb{A})$. For any $\phi \in \pi$, the *degenerate Whittaker–Fourier coefficient* of ϕ attached to a Whittaker pair (s, u) is defined to be

$$\mathcal{F}_{s,u}(\phi)(g) := \int_{[N_{s,u}]} \phi(ng)\psi_u^{-1}(n)\,\mathrm{d}n.$$

If (s, u) is a neutral pair, then $\mathcal{F}_{s,u}(\phi)$ is also called a *generalized Whittaker–Fourier coefficient* of ϕ. Define

$$\mathcal{F}_{s,u}(\pi) := \{\mathcal{F}_{s,u}(\phi) \mid \phi \in \pi\},$$

which is called the Fourier coefficient of π. The *wavefront set* $\mathfrak{n}(\pi)$ of π is defined to be the set of nilpotent orbits $\mathcal{O}$ such that $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s, u) with $u \in \mathcal{O}$.

Note that if $\sigma_{N_{s,u}\psi_u}$ or $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s, u) with $u \in \mathcal{O}$, then it is non-zero for any such neutral pair (s, u), since the non-vanishing property of such Whittaker models or Fourier coefficients does not depend on the choices of representatives of $\mathcal{O}$. Moreover, we let $\mathfrak{n}^m(\sigma)$ and $\mathfrak{n}^m(\pi)$ be the sets of maximal elements in the wavefront sets $\mathfrak{n}(\sigma)$ and $\mathfrak{n}(\pi)$, respectively, under the natural ordering of nilpotent orbits (i.e., $\mathcal{O}_1 \le \mathcal{O}_2$ if $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$, the Zariski closure of $\mathcal{O}_2$).

In this paper, we mainly consider classical groups $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^\alpha$ (quasi-split), and U_n (quasi-split or inner forms) and study the sets $\mathfrak{p}(\pi)$ and $\mathfrak{p}(\sigma)$, which are the partitions corresponding to the nilpotent orbits in the wavefront sets of π and σ, respectively. Here π is any irreducible automorphic representation of $\mathrm{G}_n(\mathbb{A})$, which occurs in the discrete spectrum of $\mathrm{G}_n(\mathbb{A})$ as displayed in Theorem 1.1, and σ denotes the unramified local component π_v of π at some finite local place v of the number field k.

3. Unramified unitary dual of split classical groups

We take $F = k_v$ to be a non-Archimedean local field of k. In this section, we recall the classification of the unramified unitary dual of the split classical groups $\mathrm{G}_n = \mathrm{Sp}_{2n}$, $\mathrm{SO}_{2n+1}, \mathrm{O}_{2n}$ over F, which was obtained by Barbasch in [4] and by Muić and Tadić in [34], using different methods. We mainly follow the formulation in [34]. In this paper, we do not need the full classification of the unramified unitary dual of the split group $\mathrm{G}_n = \mathrm{SO}_{2n}$ as given in [4], instead, we only need a family of unramified unitary representations, see Remark 3.6. The classification in [34] starts from classifying two special families of irreducible unramified representations of $\mathrm{G}_n(F)$ that are called *strongly negative* and *negative*. We refer to [32] for definitions of strongly negative and negative representations and for more related discussion on these two families of unramified representations. In the following, we recall from [34] the classification of these two families in terms of *Jordan blocks*. The Muić–Tadić classification also provides the explicit constructions.

A pair (χ, m), where χ is an unramified unitary character of F^* and $m \in \mathbb{Z}_{>0}$, is called a *Jordan block*. When $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{O}_{2n}$, define $\mathrm{Jord}_{\mathrm{sn}}(n)$ to be the collection of all sets of Jordan blocks of the following form:

$$\{(\lambda_0, 2n_1 + 1), \ldots, (\lambda_0, 2n_k + 1), (1_{\mathrm{GL}_1}, 2m_1 + 1), \ldots, (1_{\mathrm{GL}_1}, 2m_l + 1)\}, \tag{3.1}$$

where λ_0 is the unique non-trivial unramified unitary character of F^* of order 2, given by the local Hilbert symbol $(\delta, \cdot)_{F^*}$, with δ being a non-square unit in $\mathcal{O}_F$; k is even, and l is odd when $\mathrm{G}_n = \mathrm{Sp}_{2n}$ and even when $\mathrm{G}_n = \mathrm{O}_{2n}$. There are also the following constraints:

$$0 < n_1 < n_2 < \cdots < n_k, \quad 0 < m_1 < m_2 < \cdots < m_l,$$

and

$$\sum_{i=1}^{k}(2n_i + 1) + \sum_{j=1}^{l}(2m_j + 1) = \begin{cases} 2n + 1 & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \\ 2n & \text{when } \mathrm{G}_n = \mathrm{O}_{2n}. \end{cases}$$

When $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, define $\mathrm{Jord}_{\mathrm{sn}}(n)$ to be the collection of all sets of Jordan blocks of the following form:

$$\{(\lambda_0, 2n_1), \ldots, (\lambda_0, 2n_k), (1_{\mathrm{GL}_1}, 2m_1), \ldots, (1_{\mathrm{GL}_1}, 2m_l)\},$$

where

$$0 \le n_1 < n_2 < \cdots < n_k, \quad 0 \le m_1 < m_2 < \cdots < m_l,$$

both k and l are even and

$$\sum_{i=1}^{k}(2n_i) + \sum_{j=1}^{l}(2m_j) = 2n.$$

For each $\mathrm{Jord} \in \mathrm{Jord}_{\mathrm{sn}}(n)$, we can associate a representation $\sigma(\mathrm{Jord})$, which is the unique irreducible unramified subquotient of the following induced representation.

When $\mathrm{G}_n = \mathrm{Sp}_{2n}$, it is given by

$$\begin{aligned}&\nu^{\frac{n_{k-1}-n_k}{2}}\lambda_0(\det_{n_{k-1}+n_k+1}) \times \nu^{\frac{n_{k-3}-n_{k-2}}{2}}\lambda_0(\det_{n_{k-3}+n_{k-2}+1})\\ &\quad\times\cdots\times\nu^{\frac{n_1-n_2}{2}}\lambda_0(\det_{n_1+n_2+1})\\ &\quad\times\nu^{\frac{m_{l-1}-m_l}{2}}1_{\det_{m_{l-1}+m_l+1}}\times\nu^{\frac{m_{l-3}-m_{l-2}}{2}}1_{\det_{m_{l-3}+m_{l-2}+1}}\\ &\quad\times\cdots\times\nu^{\frac{m_2-m_3}{2}}1_{\det_{m_2+m_3+1}}\rtimes 1_{\mathrm{Sp}_{2m_1}}.\end{aligned}$$

When $\mathrm{G}_n = \mathrm{O}_{2n}$, it is given by

$$\begin{aligned}&\nu^{\frac{n_{k-1}-n_k}{2}}\lambda_0(\det_{n_{k-1}+n_k+1}) \times \nu^{\frac{n_{k-3}-n_{k-2}}{2}}\lambda_0(\det_{n_{k-3}+n_{k-2}+1})\\ &\quad\times\cdots\times\nu^{\frac{n_1-n_2}{2}}\lambda_0(\det_{n_1+n_2+1})\\ &\quad\times\nu^{\frac{m_{l-1}-m_l}{2}}1_{\det_{m_{l-1}+m_l+1}}\times\nu^{\frac{m_{l-3}-m_{l-2}}{2}}1_{\det_{m_{l-3}+m_{l-2}+1}}\\ &\quad\times\cdots\times\nu^{\frac{m_1-m_2}{2}}1_{\det_{m_1+m_2+1}}\rtimes 1_{\mathrm{O}_0}.\end{aligned}$$

When $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, it is given by

$$\begin{aligned}&\nu^{\frac{n_{k-1}-n_k}{2}}\lambda_0(\det_{n_{k-1}+n_k}) \times \nu^{\frac{n_{k-3}-n_{k-2}}{2}}\lambda_0(\det_{n_{k-3}+n_{k-2}})\\ &\quad\times\cdots\times\nu^{\frac{n_1-n_2}{2}}\lambda_0(\det_{n_1+n_2})\\ &\quad\times\nu^{\frac{m_{l-1}-m_l}{2}}1_{\det_{m_{l-1}+m_l}}\times\nu^{\frac{m_{l-3}-m_{l-2}}{2}}1_{\det_{m_{l-3}+m_{l-2}}}\\ &\quad\times\cdots\times\nu^{\frac{m_1-m_2}{2}}1_{\det_{m_1+m_2}}\rtimes 1_{\mathrm{SO}_1}.\end{aligned}$$

Theorem 3.1 ([34, Theorem 5-8]). *Assume that $n > 0$. The map* Jord $\mapsto \sigma$(Jord) *defines a one-to-one correspondence between the set* $\mathrm{Jord}_{\mathrm{sn}}(n)$ *and the set of all irreducible strongly negative unramified representations of* $\mathrm{G}_n(F)$.

The inverse of the map in Theorem 3.1 is denoted by $\sigma \mapsto \mathrm{Jord}(\sigma)$. Based on the classification in Theorem 3.1, irreducible negative unramified representations can be constructed from irreducible strongly negative unramified representations of smaller rank groups as follows.

Theorem 3.2 ([34, Thereom 5-10]). *For any sequence of pairs $(\chi_1, n_1), \ldots, (\chi_t, n_t)$ with χ_i being unramified unitary characters of F^* and $n_i \in \mathbb{Z}_{\geq 1}$, for $1 \leq i \leq t$, and for a strongly negative representation σ_{sn} of $G_{n'}(F)$ with $\sum_{i=1}^t n_i + n' = n$, the unique irreducible unramified subquotient of the induced representation*

$$\chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}) \rtimes \sigma_{\mathrm{sn}}$$

is negative and it is a subrepresentation.

Conversely, any irreducible negative unramified representation σ_{neg} of $\mathrm{G}_n(F)$ *can be obtained from the above construction. The data*

$$(\chi_1, n_1), \ldots, (\chi_t, n_t)$$

and σ_{sn} are unique, up to permutations and taking inverses of χ_i's.

For any irreducible negative unramified representation σ_{neg} with data in Theorem 3.2, we define

$$\mathrm{Jord}(\sigma_{\mathrm{neg}}) = \mathrm{Jord}(\sigma_{\mathrm{sn}}) \cup \{(\chi_i, n_i), (\chi_i^{-1}, n_i) \mid 1 \leq i \leq t\}.$$

By [33, Corollary 3.8], any irreducible negative representation is unitary. In particular, we have the following.

Corollary 3.3. *Any irreducible negative unramified representation of* $\mathrm{G}_n(F)$ *is unitary.*

To describe the general unramified unitary dual, we need to recall the following definition.

Definition 3.4 ([34, Definition 5-13]). Let $\mathcal{M}^{\mathrm{unr}}(n)$ be the set of pairs $(\mathbf{e}, \sigma_{\mathrm{neg}})$, where $\mathbf{e}$ is a multi-set of triples (χ, m, α) with χ being an unramified unitary character of F^*, $m \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$, and σ_{neg} is an irreducible negative unramified representation of $G_{n''}(F)$, having the property that

$$\sum_{(\chi,m)} m \cdot \#\mathbf{e}(\chi, m) + n'' = n$$

with

$$\mathbf{e}(\chi, m) = \{\alpha \mid (\chi, m, \alpha) \in \mathbf{e}\}.$$

Note that $\alpha \in \mathbf{e}(\chi, m)$ is counted with multiplicity.

Let $\mathcal{M}^{u,\mathrm{unr}}(n)$ be the subset of $\mathcal{M}^{\mathrm{unr}}(n)$ consisting of pairs $(\mathbf{e}, \sigma_{\mathrm{neg}})$, which satisfy the following conditions:

(1) If $\chi^2 \neq 1_{\mathrm{GL}_1}$, then $\mathbf{e}(\chi, m) = \mathbf{e}(\chi^{-1}, m)$, and $0 < \alpha < \frac{1}{2}$, for all $\alpha \in \mathbf{e}(\chi, m)$.

(2) If $\chi^2 = 1_{\mathrm{GL}_1}$ and m is even, then $0 < \alpha < \frac{1}{2}$, for all $\alpha \in \mathbf{e}(\chi, m)$, when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{O}_{2n}$; $0 < \alpha < 1$, for all $\alpha \in \mathbf{e}(\chi, m)$, when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$.

(3) If $\chi^2 = 1_{\mathrm{GL}_1}$ and m is odd, then $0 < \alpha < 1$, for all $\alpha \in \mathbf{e}(\chi, m)$, when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{O}_{2n}$; $0 < \alpha < \frac{1}{2}$, for all $\alpha \in \mathbf{e}(\chi, m)$, when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$.

Write elements in $\mathbf{e}(\chi, m)$ as follows:

$$0 < \alpha_1 \leq \cdots \leq \alpha_k \leq \frac{1}{2} < \beta_1 \leq \cdots \leq \beta_l < 1$$

with $k, l \in \mathbb{Z}_{\geq 0}$. They satisfy the following conditions:

(a) If $(\chi, m) \notin \mathrm{Jord}(\sigma_{\mathrm{neg}})$, then $k + l$ is even.

(b) If $k \geq 2$, then $\alpha_{k-1} \neq \frac{1}{2}$.

(c) If $l \geq 2$, then $\beta_1 < \beta_2 < \cdots < \beta_l$.

(d) $\alpha_i + \beta_j \neq 1$ for any $1 \leq i \leq k$, $1 \leq j \leq l$.

(e) If $l \geq 1$, then $\#\{i \mid 1 - \beta_1 < \alpha_i \leq \frac{1}{2}\}$ is even.

(f) If $l \geq 2$, then $\#\{i \mid 1 - \beta_{j+1} < \alpha_i < 1 - \beta_j\}$ is odd for any $1 \leq j \leq l - 1$.

Theorem 3.5 ([34, Theorem 5-14]). *The map*

$$(\mathbf{e}, \sigma_{\mathrm{neg}}) \mapsto \times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}}$$

defines a one-to-one correspondence between the set $\mathcal{M}^{u,\mathrm{unr}}(n)$ and the set of equivalence classes of all irreducible unramified unitary representations of $\mathrm{G}_n(F)$.

Remark 3.6. For $\mathrm{G}_n = \mathrm{SO}_{2n}$, in this paper, we do not need the full classification of the unramified unitary dual as given in [4], instead, we only need the irreducible unramified unitary representations as follows:

$$\sigma = \times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}} \leftrightarrow (\mathbf{e}, \sigma_{\mathrm{neg}}),$$

where $\mathbf{e} = \{(\chi, m, \alpha) \mid \chi$ is an unramified unitary character of F^*, $m \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{R}_{>0}$ and $0 < \alpha < \frac{1}{2}\}$, and σ_{neg} is the unique irreducible negative unramified subrepresentation of the following induced representation:

$$\chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}) \rtimes \sigma_{\mathrm{sn}},$$

with σ_{sn} being the unique irreducible strongly negative unramified constituent of the following induced representation:

$$\begin{aligned}
&\nu^{\frac{n_{k-1}-n_k}{2}}\lambda_0(\det_{n_{k-1}+n_k+1}) \times \nu^{\frac{n_{k-3}-n_{k-2}}{2}}\lambda_0(\det_{n_{k-3}+n_{k-2}+1}) \\
&\quad\times \cdots \times \nu^{\frac{n_1-n_2}{2}}\lambda_0(\det_{n_1+n_2+1}) \\
&\quad\times \nu^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l+1}} \times \nu^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}+1}} \\
&\quad\times \cdots \times \nu^{\frac{m_1-m_2}{2}} 1_{\det_{m_1+m_2+1}} \rtimes 1_{\mathrm{SO}_0}.
\end{aligned}$$

Here, $n_i, m_i \in \mathbb{Z}$, $0 < n_1 < n_2 < \cdots < n_k$, $0 < m_1 < m_2 < \cdots < m_l$, and k, l are even. In this case, we also define

$$\begin{aligned}
\mathrm{Jord}(\sigma_{\mathrm{sn}}) &= \{(\lambda_0, 2n_1), \ldots, (\lambda_0, 2n_k), (1_{\mathrm{GL}_1}, 2m_1), \ldots, (1_{\mathrm{GL}_1}, 2m_l)\}, \\
\mathrm{Jord}(\sigma_{\mathrm{neg}}) &= \mathrm{Jord}(\sigma_{\mathrm{sn}}) \cup \{(\chi_i, n_i), (\chi_i^{-1}, n_i) \mid 1 \le i \le t\},
\end{aligned} \tag{3.2}$$

and define $\mathcal{M}^{u,\mathrm{unr}}(n)$ to be the set of pairs $(\mathbf{e}, \sigma_{\mathrm{neg}})$ as above.

In Sections 4–7, we will mainly consider the following type of unramified unitary representations.

Type I. An irreducible unramified unitary representations of $\mathrm{G}_n(F)$, where $\mathrm{G}_n = \mathrm{Sp}_{2n}$, SO_{2n+1}, SO_{2n}, is called of Type I if it is of the following form:

$$\sigma = \times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}} \leftrightarrow (\mathbf{e}, \sigma_{\mathrm{neg}}), \tag{3.3}$$

where $\mathbf{e}$ is as in Remark 3.6 for $\mathrm{G}_n = \mathrm{SO}_{2n}$, σ_{neg} is the unique irreducible negative unramified subrepresentation of the following induced representation:

$$\chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}) \rtimes \sigma_{\mathrm{sn}},$$

with $\sigma_{\rm sn}$ being the unique irreducible strongly negative unramified constituent of the following induced representation:

$$\begin{aligned}
\mathrm{G}_n = \mathrm{Sp}_{2n}:&\quad \nu^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l+1}} \times \nu^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}+1}} \\
&\quad \times \cdots \times \nu^{\frac{m_2-m_3}{2}} 1_{\det_{m_2+m_3+1}} \rtimes 1_{\mathrm{Sp}_{2m_1}}, \\
\mathrm{G}_n = \mathrm{SO}_{2n+1}:&\quad \nu^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l}} \times \nu^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}}} \\
&\quad \times \cdots \times \nu^{\frac{m_1-m_2}{2}} 1_{\det_{m_1+m_2}} \rtimes 1_{\mathrm{SO}_1}, \\
\mathrm{G}_n = \mathrm{SO}_{2n}:&\quad \nu^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l+1}} \times \nu^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}+1}} \\
&\quad \times \cdots \times \nu^{\frac{m_1-m_2}{2}} 1_{\det_{m_1+m_2+1}} \rtimes 1_{\mathrm{SO}_0}.
\end{aligned} \tag{3.4}$$

Remark 3.7. Assume that $\sigma_{\rm sn}$ is an irreducible strongly negative unramified unitary representation of $\mathrm{G}_n(F)$ as in (3.4). If $\mathrm{G}_n = \mathrm{Sp}_{2n}$ or SO_{2n}, then the local Arthur parameter of $\sigma_{\rm sn}$ is

$$1_{\mathrm{GL}_1} \otimes S_1 \otimes S_{2m_1+1} \oplus \cdots \oplus 1_{\mathrm{GL}_1} \otimes S_1 \otimes S_{2m_l+1}.$$

Here we recall that l is odd when $\mathrm{G}_n = \mathrm{Sp}_{2n}$ and even when $\mathrm{G}_n = \mathrm{SO}_{2n}$. If $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, then the local Arthur parameter of $\sigma_{\rm sn}$ is

$$1_{\mathrm{GL}_1} \otimes S_1 \otimes S_{2m_1} \oplus \cdots \oplus 1_{\mathrm{GL}_1} \otimes S_1 \otimes S_{2m_l},$$

where S_k is the k-th irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. This can be easily obtained from Mœglin's construction of local Arthur packets in [29], or the algorithms given in [3] and [19].

At the end of this section, we recall the following theorem, which is a special case of [36, Theorem 0.6]. We remark that the spherical representations considered in [36, Theorem 0.6] have trivial Arthur parameters on the Weil–Deligne group (see [36, Introduction] for the setting), while general unramified representations have Arthur parameters that are trivial on the subgroup $I_F \times \mathrm{SL}_2(\mathbb{C})$, where I_F is the inertia subgroup of the Weil group W_F.

Theorem 3.8 ([36, Theorem 0.6]). *Let $\sigma_{\rm sn}$ be an irreducible strongly negative unramified unitary representation of $\mathrm{G}_n(F)$ as in (3.4). If $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$, then the set of maximal partitions of the wavefront set of $\sigma_{\rm sn}$ is given by*

$$\mathfrak{p}^m(\sigma_{\rm sn}) = \{\eta_{\mathfrak{g}_n^\vee, \mathfrak{g}_n}([(2m_1+1)\cdots(2m_l+1)])\}.$$

If $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, then the set of maximal partitions of the wavefront set of $\sigma_{\rm sn}$ is given by

$$\mathfrak{p}^m(\sigma_{\rm sn}) = \{\eta_{\mathfrak{g}_n^\vee, \mathfrak{g}_n}([(2m_1)\cdots(2m_l)])\}.$$

4. Arthur parameters and unramified local components

In this section, in terms of the classification of the unramified unitary dual of G_n, we study the structure of the unramified local components $\sigma = \pi_v$ of an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{G}_n(\mathbb{A})$ belonging to an automorphic L^2-packet $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ for an arbitrary global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)$. Then, we prove Theorem 1.3. We first consider the cases of $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^\alpha$ and leave the case of $\mathrm{G}_n = \mathrm{U}_n$ to the end of the section.

4.1. Unramified structure of Arthur parameters

For a given global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G}_n)$, $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ is the corresponding automorphic L^2-packet. It is clear that the irreducible unramified representations, which are the local components of $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, are determined by the local Arthur parameter ψ_v at almost all unramified local places v of k. We fix one of the members, $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, and describe the unramified local component π_v at a finite local place v, where the local Arthur parameter

$$\psi_v = \psi_{1,v} \boxplus \psi_{2,v} \boxplus \cdots \boxplus \psi_{r,v}$$

is unramified, i.e., $\tau_{i,v}$ for $i = 1, 2, \ldots, r$ are all unramified, and $\mathrm{G}_n(k_v)$ is split.

We write $F = k_v$ and first consider the case of $\mathrm{G}_n = \mathrm{Sp}_{2n}$, SO_{2n}^α. Rewrite the global Arthur parameter ψ as follows:

$$\psi = [\boxplus_{i=1}^k (\tau_i, 2b_i)] \boxplus [\boxplus_{j=k+1}^{k+l} (\tau_j, 2b_j + 1)] \boxplus [\boxplus_{s=k+l+1}^{k+l+2t+1} (\tau_s, 2b_s + 1)],$$

where $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2a_i})$ is of symplectic type for $1 \le i \le k$, and $\tau_j \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2a_j})$ and $\tau_s \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2a_s+1})$ are of orthogonal type for $k+1 \le j \le k+l$ and $k+l+1 \le s \le k+l+2t+1$. Define

$$\begin{aligned} I &:= \{1, 2, \ldots, k\}, \\ J &:= \{k+1, k+2, \ldots, k+l\}, \\ S &:= \{k+l+1, k+l+2, \ldots, k+l+2t+1\}. \end{aligned}$$

Let J_1 be the subset of J such that $\omega_{\tau_{j,v}} = 1$, and $J_2 = J \setminus J_1$, that is, for $j \in J_2$, $\omega_{\tau_{j,v}} = \lambda_0$. Let S_1 be the subset of S such that $\omega_{\tau_{s,v}} = 1$, and $S_2 = S \setminus S_1$, that is, for $s \in S_2$, $\omega_{\tau_{s,v}} = \lambda_0$. From the definition of Arthur parameters, we can easily see that $\#\{J_2\} \cup \#\{S_2\}$ is even, which implies that $\#\{J_2\} \cup \#\{S_1\}$ is odd when $\mathrm{G}_n = \mathrm{Sp}_{2n}$ and even when $\mathrm{G}_n = \mathrm{SO}_{2n}^\alpha$. The local unramified Arthur parameter ψ_v has the following structure:

- For $i \in I$,

$$\tau_{i,v} = \times_{q=1}^{a_i} \nu^{\beta_q^i} \chi_q^i \times_{q=1}^{a_i} \nu^{-\beta_q^i} \chi_q^{i,-1},$$

  where $0 \le \beta_q^i < \frac{1}{2}$, for $1 \le q \le a_i$, and χ_q^i's are unramified unitary characters of F^*.
- For $j \in J_1$,

$$\tau_{j,v} = \times_{q=1}^{a_j} \nu^{\beta_q^j} \chi_q^j \times_{q=1}^{a_j} \nu^{-\beta_q^j} \chi_q^{j,-1},$$

  where $0 \le \beta_q^j < \frac{1}{2}$, for $1 \le q \le a_j$, and χ_q^j's are unramified unitary characters of F^*.

- For $j \in J_2$,

$$\tau_{j,v} = \times_{q=1}^{a_j-1} \nu^{\beta_q^j} \chi_q^j \times \lambda_0 \times 1_{GL_1} \times_{q=1}^{a_j-1} \nu^{-\beta_q^j} \chi_q^{j,-1},$$

where $0 \le \beta_q^j < \frac{1}{2}$, for $1 \le q \le a_j$, and χ_q^j's are unramified unitary characters of F^*.

- For $s \in S_1$,

$$\tau_{s,v} = \times_{q=1}^{a_s} \nu^{\beta_q^s} \chi_q^s \times 1_{GL_1} \times_{q=1}^{a_s} \nu^{-\beta_q^s} \chi_q^{s,-1},$$

where $0 \le \beta_q^s < \frac{1}{2}$, for $1 \le q \le a_s$, and χ_q^s's are unramified unitary characters of F^*.

- For $s \in S_2$,

$$\tau_{s,v} = \times_{q=1}^{a_s} \nu^{\beta_q^s} \chi_q^s \times \lambda_0 \times_{q=1}^{a_s} \nu^{-\beta_q^s} \chi_q^{s,-1},$$

where $0 \le \beta_q^s < \frac{1}{2}$, for $1 \le q \le a_s$, and χ_q^s's are unramified unitary characters of F^*.

We define

$$\begin{aligned} \mathrm{Jord}_1 = \{&(\lambda_0, 2b_j + 1), j \in J_2; (\lambda_0, 2b_s + 1), s \in S_2; \\ &(1_{GL_1}, 2b_j + 1), j \in J_2; (1_{GL_1}, 2b_s + 1), s \in S_1\}. \end{aligned}$$

Note that Jord_1 is a multi-set. Let Jord_2 be a set consisting of different Jordan blocks with odd multiplicities in Jord_1. Thus Jord_2 has the form of (3.1) and (3.2). By Theorem 3.1 and Remark 3.6, there is a corresponding irreducible strongly negative unramified representation σ_{sn}. Then we define the following Jordan blocks:

$$\begin{aligned} \mathrm{Jord}_I &= \{(\chi_q^i, 2b_i), (\chi_q^{i,-1}, 2b_i), \ i \in I, \ 1 \le q \le a_i, \ \beta_q^i = 0\}, \\ \mathrm{Jord}_{J_1} &= \{(\chi_q^j, 2b_j + 1), (\chi_q^{j,-1}, 2b_j + 1), \ j \in J_1, \ 1 \le q \le a_j, \ \beta_q^j = 0\}, \\ \mathrm{Jord}_{J_2} &= \{(\chi_q^j, 2b_j + 1), (\chi_q^{j,-1}, 2b_j + 1), \ j \in J_2, \ 1 \le q \le a_j - 1, \ \beta_q^j = 0\}, \\ \mathrm{Jord}_{S_1} &= \{(\chi_q^s, 2b_s + 1), (\chi_q^{s,-1}, 2b_s + 1), \ s \in S_1, \ 1 \le q \le a_s, \ \beta_q^s = 0\}, \\ \mathrm{Jord}_{S_2} &= \{(\chi_q^s, 2b_s + 1), (\chi_q^{s,-1}, 2b_s + 1), \ s \in S_2, \ 1 \le q \le a_s, \ \beta_q^s = 0\}. \end{aligned}$$

Finally, we define

$$\mathrm{Jord}_3 = (\mathrm{Jord}_1 \backslash \mathrm{Jord}_2) \cup \mathrm{Jord}_I \cup \mathrm{Jord}_{J_1} \cup \mathrm{Jord}_{J_2} \cup \mathrm{Jord}_{S_1} \cup \mathrm{Jord}_{S_2}.$$

By Theorem 3.2 and Remark 3.6, corresponding to the data Jord_3 and σ_{sn}, there is an irreducible negative unramified presentation σ_{neg}.

Let

$$\begin{aligned} \mathbf{e}_I &= \{(\chi_q^i, 2b_i, \beta_q^i), \ i \in I, \ 1 \le q \le a_i, \ \beta_q^i > 0\}, \\ \mathbf{e}_{J_1} &= \{(\chi_q^j, 2b_j + 1, \beta_q^j), \ j \in J_1, \ 1 \le q \le a_j, \ \beta_q^j > 0\}, \\ \mathbf{e}_{J_2} &= \{(\chi_q^j, 2b_j + 1, \beta_q^j), \ j \in J_2, \ 1 \le q \le a_j - 1, \ \beta_q^j > 0\}, \\ \mathbf{e}_{S_1} &= \{(\chi_q^s, 2b_s + 1, \beta_q^s), \ s \in S_1, \ 1 \le q \le a_s, \ \beta_q^s > 0\}, \\ \mathbf{e}_{S_2} &= \{(\chi_q^s, 2b_s + 1, \beta_q^s), \ s \in S_2, \ 1 \le q \le a_s, \ \beta_q^s > 0\}. \end{aligned}$$

Then we define

$$\mathbf{e} = \mathbf{e}_I \cup \mathbf{e}_{J_1} \cup \mathbf{e}_{J_2} \cup \mathbf{e}_{S_1} \cup \mathbf{e}_{S_2}.$$

Since the unramified component π_v is unitary, we must have that $(\mathbf{e}, \sigma_{\mathrm{neg}}) \in \mathcal{M}^{u,\mathrm{unr}}(n)$, and π_v is exactly the irreducible unramified unitary representation σ of $\mathrm{G}_n(F)$ which corresponds to $(\mathbf{e}, \sigma_{\mathrm{neg}})$ as in Theorem 3.5 and Remark 3.6.

Now we consider the case of $\mathrm{G}_n = \mathrm{SO}_{2n+1}$. Rewrite the global Arthur parameter ψ as follows:

$$\psi = [\boxplus_{i=1}^{k}(\tau_i, 2b_i + 1)] \boxplus [\boxplus_{j=k+1}^{k+l}(\tau_j, 2b_j)] \boxplus [\boxplus_{s=k+l+1}^{k+l+2t+1}(\tau_s, 2b_s)],$$

where $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2a_i})$ is of symplectic type for $1 \le i \le k$, $\tau_j \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2a_j})$ and $\tau_s \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2a_s+1})$ are of orthogonal type for $k+1 \le j \le k+l$ and $k+l+1 \le s \le k+l+2t+1$. Similarly, we define

$$\begin{aligned} I &:= \{1, 2, \ldots, k\}, \\ J &:= \{k+1, k+2, \ldots, k+l\}, \\ S &:= \{k+l+1, k+l+2, \ldots, k+l+2t+1\}. \end{aligned}$$

Let J_1 be the subset of J such that $\omega_{\tau_{j,v}} = 1$, and $J_2 = J \backslash J_1$, that is, for $j \in J_2$, $\omega_{\tau_{j,v}} = \lambda_0$. Let S_1 be the subset of S such that $\omega_{\tau_{s,v}} = 1$, and $S_2 = S \backslash S_1$, that is, for $s \in S_2$, $\omega_{\tau_{s,v}} = \lambda_0$. The local unramified Arthur parameter ψ_v has the following structure:

- For $i \in I_1$,

  $$\tau_{i,v} = \times_{q=1}^{a_i} \nu^{\beta_q^i} \chi_q^i \times_{q=1}^{a_i} \nu^{-\beta_q^i} \chi_q^{i,-1},$$

  where $0 \le \beta_q^i < \frac{1}{2}$, for $1 \le q \le a_i$, and χ_q^i's are unramified unitary characters of F^*.
- For $j \in J_1$,

  $$\tau_{j,v} = \times_{q=1}^{a_j} \nu^{\beta_q^j} \chi_q^j \times_{q=1}^{a_j} \nu^{-\beta_q^j} \chi_q^{j,-1},$$

  where $0 \le \beta_q^j < \frac{1}{2}$, for $1 \le q \le a_j$, and χ_q^j's are unramified unitary characters of F^*.
- For $j \in J_2$,

  $$\tau_{j,v} = \times_{q=1}^{a_j-1} \nu^{\beta_q^j} \chi_q^j \times \lambda_0 \times 1_{GL_1} \times_{q=1}^{a_j-1} \nu^{-\beta_q^j} \chi_q^{j,-1},$$

  where $0 \le \beta_q^j < \frac{1}{2}$, for $1 \le q \le a_j$, and χ_q^j's are unramified unitary characters of F^*.
- For $s \in S_1$,

  $$\tau_{s,v} = \times_{q=1}^{a_s} \nu^{\beta_q^s} \chi_q^s \times 1_{GL_1} \times_{q=1}^{a_s} \nu^{-\beta_q^s} \chi_q^{s,-1},$$

  where $0 \le \beta_q^s < \frac{1}{2}$, for $1 \le q \le a_s$, and χ_q^s's are unramified unitary characters of F^*.
- For $s \in S_2$,

  $$\tau_{s,v} = \times_{q=1}^{a_s} \nu^{\beta_q^s} \chi_q^s \times \lambda_0 \times_{q=1}^{a_s} \nu^{-\beta_q^s} \chi_q^{s,-1},$$

  where $0 \le \beta_q^s < \frac{1}{2}$, for $1 \le q \le a_s$, and χ_q^s's are unramified unitary characters of F^*.

We define

$$\mathrm{Jord}_1 = \big\{(\lambda_0, 2b_j), j \in J_2; (\lambda_0, 2b_s), s \in S_2; \\ (1_{GL_1}, 2b_j), j \in J_2; (1_{GL_1}, 2b_s), s \in S_1\big\}.$$

Note that Jord_1 is a multi-set. Let Jord_2 be a set consisting of different Jordan blocks with odd multiplicities in Jord_1. Thus Jord_2 has the form of (3.1). By Theorem 3.1, there is a corresponding irreducible strongly negative unramified representation σ_{sn}. Then we define the following Jordan blocks:

$$\begin{aligned}
\mathrm{Jord}_I &= \{(\chi_q^i, 2b_i + 1), (\chi_q^{i,-1}, 2b_i + 1), i \in I, 1 \le q \le a_i, \beta_q^i = 0\},\\
\mathrm{Jord}_{J_1} &= \{(\chi_q^j, 2b_j), (\chi_q^{j,-1}, 2b_j), j \in J_1, 1 \le q \le a_j, \beta_q^j = 0\},\\
\mathrm{Jord}_{J_2} &= \{(\chi_q^j, 2b_j), (\chi_q^{j,-1}, 2b_j), j \in J_2, 1 \le q \le a_j - 1, \beta_q^j = 0\},\\
\mathrm{Jord}_{S_1} &= \{(\chi_q^s, 2b_s), (\chi_q^{s,-1}, 2b_s), s \in S_1, 1 \le q \le a_s, \beta_q^s = 0\},\\
\mathrm{Jord}_{S_2} &= \{(\chi_q^s, 2b_s), (\chi_q^{s,-1}, 2b_s), s \in S_2, 1 \le q \le a_s, \beta_q^s = 0\}.
\end{aligned}$$

Finally, we define

$$\mathrm{Jord}_3 = (\mathrm{Jord}_1 \backslash \mathrm{Jord}_2) \cup \mathrm{Jord}_I \cup \mathrm{Jord}_{J_1} \cup \mathrm{Jord}_{J_2} \cup \mathrm{Jord}_{S_1} \cup \mathrm{Jord}_{S_2}.$$

By Theorem 3.2, corresponding to the data Jord_3 and σ_{sn}, there is an irreducible negative unramified presentation σ_{neg}.

Let

$$\begin{aligned}
\mathbf{e}_I &= \{(\chi_q^i, 2b_i + 1, \beta_q^i), i \in I, 1 \le q \le a_i, \beta_q^i > 0\},\\
\mathbf{e}_{J_1} &= \{(\chi_q^j, 2b_j, \beta_q^j), j \in J_1, 1 \le q \le a_j, \beta_q^j > 0\},\\
\mathbf{e}_{J_2} &= \{(\chi_q^j, 2b_j, \beta_q^j), j \in J_2, 1 \le q \le a_j - 1, \beta_q^j > 0\},\\
\mathbf{e}_{S_1} &= \{(\chi_q^s, 2b_s, \beta_q^s), s \in S_1, 1 \le q \le a_s, \beta_q^s > 0\},\\
\mathbf{e}_{S_2} &= \{(\chi_q^s, 2b_s, \beta_q^s), s \in S_2, 1 \le q \le a_s, \beta_q^s > 0\}.
\end{aligned}$$

Then we define

$$\mathbf{e} = \mathbf{e}_I \cup \mathbf{e}_{J_1} \cup \mathbf{e}_{J_2} \cup \mathbf{e}_{S_1} \cup \mathbf{e}_{S_2}.$$

Since the unramified component π_v is unitary, we must have that $(\mathbf{e}, \sigma_{\mathrm{neg}}) \in \mathcal{M}^{u,\mathrm{unr}}(n)$, and π_v is exactly the irreducible unramified unitary representation σ of $\mathrm{G}_n(F)$ which corresponds to $(\mathbf{e}, \sigma_{\mathrm{neg}})$ as in Theorem 3.5.

4.2. Proof of Theorem 1.3

The following result from [22] is needed for the proof of Theorem 1.3.

Proposition 4.1 ([22, Proposition 6.1]). *For any finitely many non-square elements $\alpha_i \notin k^*/(k^*)^2$, $1 \le i \le t$, there are infinitely many finite places v such that $\alpha_i \in (k_v^*)^2$ for any $1 \le i \le t$.*

Now we are going to prove Theorem 1.3. First we consider the cases of $\mathrm{G}_n = \mathrm{Sp}_{2n}$, SO_{2n+1}, $\mathrm{SO}^{\alpha}_{2n}$. Given any $\psi = \boxplus_{i=1}^{r}(\tau_i, b_i) \in \widetilde{\Psi}_2(\mathrm{G}_n)$, assume that $\{\tau_{i_1}, \ldots, \tau_{i_q}\}$ is a multi-set of all the τ's with non-trivial central characters. Since all τ_{i_j}'s are self-dual, the central characters $\omega_{\tau_{i_j}}$'s are all quadratic characters, which are parameterized by global non-square elements. Assume that $\omega_{\tau_{i_j}} = \chi_{\alpha_{i_j}}$, where $\alpha_{i_j} \in k^*/(k^*)^2$, and $\chi_{\alpha_{i_j}}$ is the quadratic character given by the global Hilbert symbol $(\cdot, \alpha_{i_j})$. Note that $\{\alpha_{i_1}, \ldots, \alpha_{i_q}\}$ is a multi-set. By Proposition 4.1, there are infinitely many finite places v such that α and α_{i_j}'s are all squares in k_v. Therefore, for the given ψ, there are infinitely many finite places v such that $\mathrm{G}_n(k_v)$ split and all $\tau_{i,v}$'s have trivial central characters. From the discussion in Section 3, for any $\pi \in \widetilde{\Pi}_{\psi}(\varepsilon_{\psi})$, there is a finite local place v with such a property that π_v is an irreducible unramified unitary representation of Type I as in (3.3).

We are going to discuss the connection with the classification of Barbasch in [4].

Assume first that $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}^{\alpha}_{2n}$. If σ is an irreducible unramified unitary representation of $\mathrm{G}_n(k_v)$ corresponding to the pair $(\mathbf{e}, \sigma_{\mathrm{neg}}) \in \mathcal{M}^{u,\mathrm{unr}}(n)$, then the orbit $\check{\mathcal{O}}$ corresponding to σ in [4] is given by the following partition:

$$\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{k}(2n_i+1)\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\Big)\Big].$$

When π_v is of Type I as in (3.3), the orbit $\check{\mathcal{O}}$ corresponding to $\sigma = \pi_v$ in [4] is given by the following partition:

$$\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\Big)\Big], \tag{4.1}$$

which turns out to be $\underline{p}(\psi)$ exactly.

Assume now that $\mathrm{G}_n = \mathrm{SO}_{2n+1}$. If σ is an irreducible unramified unitary representation of $\mathrm{G}_n(k_v)$ corresponding to the pair $(\mathbf{e}, \sigma_{\mathrm{neg}}) \in \mathcal{M}^{u,\mathrm{unr}}(n)$, then the orbit $\check{\mathcal{O}}$ corresponding to σ in [4] is given by the following partition:

$$\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{k}(2n_i)\Big)\Big(\prod_{i=1}^{l}(2m_i)\Big)\Big]. \tag{4.2}$$

When π_v is of Type I as in (3.3), the orbit $\check{\mathcal{O}}$ corresponding to $\sigma = \pi_v$ in [4] is given by the following partition:

$$\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{l}(2m_i)\Big)\Big],$$

which turns out to be $\underline{p}(\psi)$ exactly.

We claim that for the cases of $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}^{\alpha}_{2n}$, Theorem 1.3 can be deduced from the following theorem whose proof will be given in the next three sections.

Theorem 4.2. *Let σ be an irreducible unramified unitary representations of $\mathrm{G}_n(k_v)$ of Type I as in (3.3). For any $\underline{p} \in \mathfrak{p}^m(\sigma)$, the following bound*

$$\underline{p} \leq \eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\Big)\Big]$$

holds with the partition on the left-hand side from (4.1) when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$; and the following bound

$$\underline{p} \leq \eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{l}(2m_i)\Big)\Big]$$

holds with the partition on the left-hand side from (4.2) when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$.

For the case when $\mathrm{G}_n = \mathrm{U}_n$ and E/k is a quadratic extension, by similar arguments, for any $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, there is a finite local place v such that $\mathrm{G}_n(E_v) = \mathrm{GL}_n(k_v) \times \mathrm{GL}_n(k_v)$, split, and π_v is unramified. Then, Theorem 1.3 is simply implied by the classification of the unramified unitary dual of GL_n [42] and the result of Mœglin and Waldspurger on the wavefront set of representations of GL_n [30, Section II.2]. Note that for $\mathrm{G}_n = \mathrm{U}_n$, the Barbasch–Vogan–Spaltenstein duality is just the transpose of partitions. We omit the details here.

This completes the proof of Theorem 1.3.

Remark 4.3. We expect that the method of proving Theorem 1.3 in this paper also applies to the inner forms of even orthogonal groups, once the full Arthur classification of the discrete spectrum is carried out (see [7, 8] for recent progress in this direction). The same method can also be applied to the metaplectic double cover of symplectic groups, whose proof will appear elsewhere. Note that for the metaplectic double cover of symplectic groups, the notion of Barbasch–Vogan–Spaltenstein duality has been defined in [5].

5. Proof of Theorem 4.2, $\mathrm{G}_n = \mathrm{Sp}_{2n}$

First, we recall the following general lemma which can be deduced from the argument in [30, Section II.1.3].

Lemma 5.1 ([30, Section II.1.3]). *Let G be a reductive group defined over a non-Archimedean local field F, and let $Q = MN$ be a parabolic subgroup of G. Let δ be an irreducible admissible representation of M. Then*

$$\mathfrak{n}^m(\mathrm{Ind}_Q^G \delta) = \{\mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{g}} \mathcal{O} : \mathcal{O} \in \mathfrak{n}^m(\delta)\},$$

where $\mathfrak{q}$ and $\mathfrak{g}$ are the Lie algebras of Q and G, respectively. For induced nilpotent orbits, see [11, Chapter 7].

Now we prove Theorem 4.2 for the case $\mathrm{G}_n = \mathrm{Sp}_{2n}$. By the assumption of Theorem 4.2, σ is of Type I and is of the form

$$\sigma = \times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}},$$

where σ_{neg} is the unique irreducible negative unramified subrepresentation of the following induced representation:

$$\chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}) \rtimes \sigma_{\mathrm{sn}},$$

with σ_{sn} being the unique strongly negative unramified constituent of the following induced representation:

$$\nu^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l+1}} \times \nu^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}+1}} \times \cdots \times \nu^{\frac{m_2-m_3}{2}} 1_{\det_{m_2+m_3+1}} \rtimes 1_{\mathrm{Sp}_{2m_1}}.$$

Recall that $m_i \in \mathbb{Z}$, $0 < m_1 < m_2 < \cdots < m_l$, and l is odd.

From the properties of representations of general linear groups, it is known that

$$\mathfrak{p}^m(\chi(\det_k)) = \{[1^k]\}$$

for any given character χ and any integer k. By Lemma 5.1, we have

$$\begin{aligned}&\mathfrak{p}^m(\times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \times \chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}))\\ &\quad = \{+_{(\chi,m,\alpha)\in\mathbf{e}}[1^m] + [1^{n_1}] + \cdots + [1^{n_t}]\} = \left\{\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\right]^t\right\}.\end{aligned}$$

By Theorem 3.8, Lemma 5.1, and by [11, Theorem 7.3.3] on formula for induced nilpotent orbits, for any $\underline{p} \in \mathfrak{p}^m(\sigma)$, we have

$$\underline{p} \le \left(2\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\right]^t + \eta_{\mathfrak{so}_{2k+1},\mathfrak{sp}_{2k}}\left(\left[\prod_{j=1}^{l}(2m_j+1)\right]\right)\right)_{\mathrm{Sp}_{2n}},$$

where $2k = (\sum_{i=1}^{l}(2m_i+1)) - 1$.

To prove Theorem 4.2 in this case, it suffices to show the following lemma.

Lemma 5.2. *The following identity*

$$\begin{aligned}&\eta_{\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n}}(\underline{p}(\psi))\\ &\quad = \left(2\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\right]^t + \eta_{\mathfrak{so}_{2k+1},\mathfrak{sp}_{2k}}\left(\left[\prod_{j=1}^{l}(2m_j+1)\right]\right)\right)_{\mathrm{Sp}_{2n}}\end{aligned}$$

holds with $2k = (\sum_{i=1}^{l}(2m_i+1)) - 1$.

Proof. Recall that

$$\underline{p}(\psi) = \Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\prod_{j=1}^{l}(2m_j+1)\Big)\Big]$$

and

$$\eta_{\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n}}(\underline{p}(\psi)) = ((\underline{p}(\psi)^-)_{\mathrm{Sp}_{2n}})^t,$$

where given any partition $\underline{p} = [p_r \cdots p_1]$ with $p_r \geq \cdots \geq p_1$, we have that

$$\underline{p}^- = [p_r \cdots (p_1-1)].$$

On the other hand, we have

$$\begin{aligned}
&\Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \eta_{\mathfrak{so}_{2k+1},\mathfrak{sp}_{2k}}\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]\Big)\Big)_{\mathrm{Sp}_{2n}}\\
&\quad = \Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}}\Big)^t\Big)_{\mathrm{Sp}_{2n}}\\
&\quad = \Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}}\Big)\Big]^t\Big)_{\mathrm{Sp}_{2n}}.
\end{aligned}$$

Given any partition $\underline{p}$ of Sp_{2n}, it is known that $(\underline{p}^{\mathrm{Sp}_{2n}})^t = (\underline{p}^t)_{\mathrm{Sp}_{2n}}$ (see [11, proof of Theorem 6.3.11]). Note that

$$\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}}\Big)\Big]$$

is indeed a symplectic partition. Hence, we have

$$\begin{aligned}
&\Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}}\Big)\Big]^t\Big)_{\mathrm{Sp}_{2n}}\\
&\quad = \Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}}\Big)\Big]^{\mathrm{Sp}_{2n}}\Big)^t.
\end{aligned}$$

Therefore, we only need to show that

$$(\underline{p}(\psi)^-)_{\mathrm{Sp}_{2n}} = \Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}}\Big)\Big]^{\mathrm{Sp}_{2n}}. \tag{5.1}$$

Note that

$$\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^-\Big)_{\mathrm{Sp}_{2k}} = [(2m_l)(2m_{l-1}+2)\cdots(2m_3)(2m_2+2)(2m_1)].$$

We have to rewrite the partition

$$\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\right]$$

as $[k_s^2 k_{s-1}^2 \cdots k_1^2]$ with $k_s \geq k_{s-1} \geq \cdots \geq k_1$. To proceed, we consider the following cases:

(1) $k_1 \geq 2m_1 + 1$;

(2) $k_1 < 2m_1 + 1$.

In each case, for $1 \leq j \leq \frac{l-1}{2}$, we list all the different odd k_i's between $2m_{2j+1} + 1$ and $2m_{2j} + 1$ as

$$2m_{2j+1} + 1 > k_j^1 > k_j^2 > \cdots > k_j^{s_j} > 2m_{2j} + 1.$$

Case (1): $k_1 \geq 2m_1 + 1$. We have

$$\underline{p}(\psi)^- = \left[(k_s^2 k_{s-1}^2 \cdots k_1^2)\Big(\prod_{j=2}^{l}(2m_j + 1)\Big)(2m_1)\right].$$

Then $(\underline{p}(\psi)^-)_{\mathrm{Sp}_{2n}}$ is obtained from $\underline{p}(\psi)^-$ via replacing $(2m_{2j+1} + 1, 2m_{2j} + 1)$ by $(2m_{2j+1}, 2m_{2j} + 2)$, and $k_j^{i,2}$ by $(k_j^i + 1, k_j^i - 1)$ for $1 \leq j \leq \frac{l-1}{2}$, $1 \leq i \leq s_j$. On the other hand, we have

$$\begin{aligned}&\left[k_s^2 \cdots k_1^2\bigg(\bigg(\Big[\prod_{j=1}^{l}(2m_j + 1)\Big]^-\bigg)_{\mathrm{Sp}_{2k}}\bigg)\right]^{\mathrm{Sp}_{2n}}\\ &\quad = [k_s^2 \cdots k_1^2(2m_l)(2m_{l-1} + 2)\cdots(2m_3)(2m_2 + 2)(2m_1)]^{\mathrm{Sp}_{2n}},\end{aligned}$$

which is obtained from

$$[k_s^2 \cdots k_1^2(2m_l)(2m_{l-1} + 2)\cdots(2m_3)(2m_2 + 2)(2m_1)]$$

via replacing $k_j^{i,2}$ by $(k_j^i + 1, k_j^i - 1)$ for $1 \leq j \leq \frac{l-1}{2}$, $1 \leq i \leq s_j$. Hence, we deduce that (5.1) holds.

Case (2): $k_1 < 2m_1 + 1$. We have

$$\underline{p}(\psi)^- = \left[(k_s^2 k_{s-1}^2 \cdots k_2^2)\Big(\prod_{j=1}^{l}(2m_j + 1)\Big)(k_1)(k_1 - 1)\right].$$

To carry out the Sp_{2n}-collapse of $\underline{p}(\psi)^-$, we also need to list all the different odd k_i's between $2m_1 + 1$ and k_1 as

$$2m_1 + 1 > k_0^1 > k_0^2 > \cdots > k_0^{s_0} > k_1.$$

Then $(\underline{p}(\psi)^-)_{\mathrm{Sp}_{2n}}$ is obtained from $\underline{p}(\psi)^-$ via replacing $(2m_{2j+1} + 1, 2m_{2j} + 1)$ by $(2m_{2j+1}, 2m_{2j} + 2)$ and $k_j^{i,2}$ by $(k_j^i + 1, k_j^i - 1)$, for $1 \leq j \leq \frac{l-1}{2}$, $1 \leq i \leq s_j$; and then replacing $(2m_1+1, k_1-1)$ by $(2m_1, k_1)$ if k_1 is even, $(2m_1+1, k_1)$ by $(2m_1, k_1+1)$ if k_1 is odd, and $k_0^{i,2}$ by (k_0^i+1, k_0^i-1) for $1 \le i \le s_0$. On the other hand, we get

$$\left[k_s^2 \cdots k_1^2\left(\left(\left[\prod_{j=1}^{l}(2m_j+1)\right]^{-}\right)_{\mathrm{Sp}_{2k}}\right)\right]^{\mathrm{Sp}_{2n}}$$
$$= [k_s^2 \cdots k_1^2(2m_l)(2m_{l-1}+2)\cdots(2m_3)(2m_2+2)(2m_1)]^{\mathrm{Sp}_{2n}},$$

which is obtained from $[k_s^2 \cdots k_1^2(2m_l)(2m_{l-1}+2)\cdots(2m_3)(2m_2+2)(2m_1)]$ via replacing $k_j^{i,2}$ by (k_j^i+1, k_j^i-1), for $1 \le j \le \frac{l-1}{2}$, $1 \le i \le s_j$; and then replacing k_1^2 by (k_1+1, k_1-1) if k_1 is odd, $k_0^{i,2}$ by (k_0^i+1, k_0^i-1), for $1 \le i \le s_0$. Hence, we deduce that (5.1) still holds.

This completes the proof of the lemma. ∎

The proof of Theorem 4.2 has been completed for $\mathrm{G}_n = \mathrm{Sp}_{2n}$.

6. Proof of Theorem 4.2, $\mathrm{G}_n = \mathrm{SO}_{2n+1}$

By the assumption of Theorem 4.2, σ is of Type I and is of the form

$$\sigma = \times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}},$$

where σ_{neg} is the unique irreducible negative unramified subrepresentation of the following induced representation

$$\chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}) \rtimes \sigma_{\mathrm{sn}},$$

with σ_{sn} being the unique strongly negative unramified constituent of the following induced representation:

$$v^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l}} \times v^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}}} \times \cdots \times v^{\frac{m_1-m_2}{2}} 1_{\det_{m_1+m_2}} \rtimes 1_{\mathrm{SO}_1}.$$

Recall that $m_i \in \mathbb{Z}$, $0 \le m_1 < m_2 < \cdots < m_l$, and l is even.

As in Section 5, by Lemma 5.1, we have

$$\mathfrak{p}^m(\times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \times \chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}))$$
$$= \left\{\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\right)\left(\prod_{i=1}^{t} n_i\right)\right]^t\right\}.$$

By Theorem 3.8, Lemma 5.1, and by [11, Theorem 7.3.3] on formula for induced nilpotent orbits, any $\underline{p} \in \mathfrak{p}^m(\sigma)$ has the following upper bound:

$$\underline{p} \le \left(2\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\right)\left(\prod_{i=1}^{t} n_i\right)\right]^t + \eta_{\mathfrak{sp}_{2k},\mathfrak{so}_{2k+1}}\left(\left[\prod_{j=1}^{l}(2m_j)\right]\right)\right)_{\mathrm{SO}_{2n+1}},$$

where $2k = \sum_{i=1}^{l}(2m_i)$.

To prove Theorem 4.2 in this case, it suffices to show the following lemma.

Lemma 6.1. *The following identity*

$$\eta_{\mathfrak{sp}_{2n},\mathfrak{so}_{2n+1}}(\underline{p}(\psi)) = \Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \eta_{\mathfrak{sp}_{2k},\mathfrak{so}_{2k+1}}\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]\Big)\Big)_{\mathrm{SO}_{2n+1}}$$

holds with $2k = \sum_{i=1}^{l}(2m_i)$.

Proof. Recall that

$$\underline{p}(\psi) = \Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\prod_{j=1}^{l}(2m_j)\Big)\Big]$$

and

$$\eta_{\mathfrak{sp}_{2k},\mathfrak{so}_{2k+1}}(\underline{p}(\psi)) = ((\underline{p}(\psi)^+)_{\mathrm{SO}_{2n+1}})^t,$$

where for any given partition $\underline{p} = [p_r \cdots p_1]$ with $p_r \geq \cdots \geq p_1$, we have

$$\underline{p}^+ = [(p_r + 1)\cdots p_1].$$

On the other hand, we have

$$\begin{aligned}
&\Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \eta_{\mathfrak{sp}_{2k},\mathfrak{so}_{2k+1}}\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]\Big)\Big)_{\mathrm{SO}_{2n+1}} \\
&\quad = \Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Big)_{\mathrm{SO}_{2k+1}}\Big)^t\Big)_{\mathrm{SO}_{2n+1}} \\
&\quad = \Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Big)_{\mathrm{SO}_{2k+1}}\Big)\Big]^t\Big)_{\mathrm{SO}_{2n+1}}.
\end{aligned}$$

Given any partition $\underline{p}$ of SO_{2n+1}, it is known that $(\underline{p}^{\mathrm{SO}_{2n+1}})^t = (\underline{p}^t)_{\mathrm{SO}_{2n+1}}$ (see [11, proof of Theorem 6.3.11]). Note that

$$\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Big)_{\mathrm{SO}_{2k+1}}\Big)\Big]$$

is indeed an orthogonal partition. Hence, we obtain that

$$\begin{aligned}
&\Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Big)_{\mathrm{SO}_{2k+1}}\Big)\Big]^t\Big)_{\mathrm{SO}_{2n+1}} \\
&\quad = \Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\Big(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Big)_{\mathrm{SO}_{2k+1}}\Big)\Big]^{\mathrm{SO}_{2n+1}}\Big)^t.
\end{aligned}$$

Therefore, we only need to show that

$$(\underline{p}(\psi)^+)_{\mathrm{SO}_{2n+1}} = \Bigg[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Bigg(\Bigg(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Bigg)_{\mathrm{SO}_{2k+1}}\Bigg)\Bigg]^{\mathrm{SO}_{2n+1}}. \quad (6.1)$$

Note that the partition

$$\Bigg(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Bigg)_{\mathrm{SO}_{2k+1}}$$

is equal to

$$[(2m_l+1)(2m_{l-1}-1)(2m_{l-2}+1)\cdots(2m_3-1)(2m_2+1)(2m_1-1)1],$$

where we omit the "$(2m_1-1)1$"-term if $2m_1=0$.

We are going to rewrite the partition

$$\Bigg[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Bigg]$$

as $[k_s^2k_{s-1}^2\cdots k_1^2]$ with $k_s\geq k_{s-1}\geq\cdots\geq k_1$. To proceed, we consider the following cases:

(1) $k_s\leq 2m_l$;

(2) $k_s>2m_l$.

In each case, for $1\leq j\leq\frac{l-2}{2}$, we list all the different even k_i's between $2m_{2j+1}$ and $2m_{2j}$ as

$$2m_{2j+1}>k_j^1>k_j^2>\cdots>k_j^{s_j}>2m_{2j}.$$

Case (1): $k_s\leq 2m_l$. We have

$$\underline{p}(\psi)^+ = \Bigg[(k_s^2\cdots k_1^2)(2m_l+1)\prod_{j=1}^{l-1}(2m_j)\Bigg].$$

If $2m_1\neq 0$, to carry out the SO_{2n+1}-collapse of $\underline{p}(\psi)^+$, we also need to list all the different even k_i's between $2m_1$ and 0 as

$$2m_1>k_0^1>k_0^2>\cdots>k_0^{s_0}>0.$$

Then $(\underline{p}(\psi)^+)_{\mathrm{SO}_{2n+1}}$ is obtained from $\underline{p}(\psi)^+$ via replacing $(2m_{2j+1},2m_{2j})$ by $(2m_{2j+1}-1,2m_{2j}+1)$ and $k_j^{i,2}$ by (k_j^i+1,k_j^i-1), for $1\leq j\leq\frac{l-2}{2}$ and $1\leq i\leq s_j$; and then replacing $(2m_1,0)$ by $(2m_1-1,1)$ and $k_0^{i,2}$ by (k_0^i+1,k_0^i-1) with $1\leq i\leq s_0$, if $2m_1\neq 0$. On the other hand, we have

$$\Bigg[k_s^2\cdots k_1^2\Bigg(\Bigg(\Big[\prod_{j=1}^{l}(2m_j)\Big]^+\Bigg)_{\mathrm{SO}_{2k+1}}\Bigg)\Bigg]^{\mathrm{SO}_{2n+1}}$$
$$= [k_s^2\cdots k_1^2(2m_l+1)(2m_{l-1}-1)(2m_{l-2}+1)\cdots(2m_1-1)1]^{\mathrm{SO}_{2n+1}},$$

which is obtained from

$$[k_s^2 \cdots k_1^2 (2m_l+1)(2m_{l-1}-1)(2m_{l-2}+1)\cdots(2m_1-1)1]$$

via replacing $k_j^{i,2}$ by (k_j^i+1, k_j^i-1) for $1 \le j \le \frac{l-2}{2}$, $1 \le i \le s_j$; and then replacing $k_0^{i,2}$ by (k_0^i+1, k_0^i-1), $1 \le i \le s_0$. Hence, (6.1) holds in this case.

Case (2): $k_s > 2m_l$. We have

$$\underline{p}(\psi)^+ = \left[((k_s+1)k_s k_{s-1}^2 \cdots k_1^2)\left(\prod_{j=1}^{l}(2m_j)\right)\right].$$

To carry out the SO_{2n+1}-collapse of $\underline{p}(\psi)^+$, we also need to list all the different even k_i's between k_s and $2m_l$ as

$$k_s > k_l^1 > k_l^2 > \cdots > k_l^{s_l} > 2m_l,$$

and if $2m_1 \neq 0$, list all the different even k_i's between $2m_1$ and 0 as

$$2m_1 > k_0^1 > k_0^2 > \cdots > k_0^{s_0} > 0.$$

Then $(\underline{p}(\psi)^+)_{\mathrm{SO}_{2n+1}}$ is obtained from $\underline{p}(\psi)^+$ via replacing $(2m_{2j+1}, 2m_{2j})$ by $(2m_{2j+1}-1, 2m_{2j}+1)$ and $k_j^{i,2}$ by (k_j^i+1, k_j^i-1) for $1 \le j \le \frac{l-2}{2}$ and $1 \le i \le s_j$; and replacing $(k_s+1, 2m_l)$ by $(k_s, 2m_l+1)$ if k_s is odd and $(k_s, 2m_l)$ by $(k_s-1, 2m_l+1)$ if k_s is even, and $k_l^{i,2}$ by (k_l^i+1, k_l^i-1) with $1 \le i \le s_l$; and finally replacing $(2m_1, 0)$ by $(2m_1-1, 1)$ and $k_0^{i,2}$ by (k_0^i+1, k_0^i-1) for $1 \le i \le s_0$, if $2m_1 \neq 0$. On the other hand, we have

$$\left[k_s^2 \cdots k_1^2 \left(\left(\left[\prod_{j=1}^{l}(2m_j)\right]^+\right)_{\mathrm{SO}_{2k+1}}\right)\right]^{\mathrm{SO}_{2n+1}}$$
$$= [k_s^2 \cdots k_1^2 (2m_l+1)(2m_{l-1}-1)(2m_{l-2}+1)\cdots(2m_1-1)1]^{\mathrm{SO}_{2n+1}},$$

which is obtained from

$$[k_s^2 \cdots k_1^2 (2m_l+1)(2m_{l-1}-1)(2m_{l-2}+1)\cdots(2m_1-1)1]$$

via replacing k_s^2 by (k_s+1, k_s-1) if k_s is even and $k_j^{i,2}$ by (k_j^i+1, k_j^i-1) for $1 \le j \le \frac{l-2}{2}$ and $1 \le i \le s_j$; and replacing $k_l^{i,2}$ by (k_l^i+1, k_l^i-1) for $1 \le i \le s_l$; and finally replacing $k_0^{i,2}$ by (k_0^i+1, k_0^i-1) for $1 \le i \le s_0$. Hence, (6.1) still holds in this case.

This completes the proof of the lemma. ∎

The proof of Theorem 4.2 has been completed for $\mathrm{G}_n = \mathrm{SO}_{2n+1}$.

7. Proof of Theorem 4.2, $\mathrm{G}_n = \mathrm{SO}_{2n}$

By the assumption of Theorem 4.2, σ is of Type I and is of the form

$$\sigma = \times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}},$$

where σ_{neg} is the unique irreducible negative unramified subrepresentation of the following induced representation:

$$\chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t}) \rtimes \sigma_{\mathrm{sn}},$$

with σ_{sn} being the unique strongly negative unramified constituent of the following induced representation:

$$\nu^{\frac{m_{l-1}-m_l}{2}} 1_{\det_{m_{l-1}+m_l+1}} \times \nu^{\frac{m_{l-3}-m_{l-2}}{2}} 1_{\det_{m_{l-3}+m_{l-2}+1}} \times \cdots \times \nu^{\frac{m_1-m_2}{2}} 1_{\det_{m_1+m_2+1}} \rtimes 1_{\mathrm{SO}_0}.$$

Recall that $m_i \in \mathbb{Z}$, $0 < m_1 < m_2 < \cdots < m_l$, and l is even.

As in Sections 5 and 6, by Lemma 5.1, we have

$$\mathfrak{p}^m(\times_{(\chi,m,\alpha)\in\mathbf{e}} v^{\alpha}\chi(\det_m) \times \chi_1(\det_{n_1}) \times \cdots \times \chi_t(\det_{n_t})) = \left\{\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\right]^t\right\}.$$

By Theorem 3.8, Lemma 5.1, and by [11, Theorem 7.3.3] on formula for induced nilpotent orbits, any $\underline{p} \in \mathfrak{p}^m(\sigma)$ has the following upper bound:

$$\underline{p} \le \left(2\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\right]^t + \eta_{\mathfrak{o}_{2k},\mathfrak{o}_{2k}}\left(\left[\prod_{j=1}^{l}(2m_j+1)\right]\right)\right)_{\mathrm{SO}_{2n}}$$

with $2k = \sum_{i=1}^{l}(2m_i+1)$.

To prove Theorem 4.2 in this case, it suffices to show the following lemma.

Lemma 7.1. *The following identity*

$$\eta_{\mathfrak{so}_{2n},\mathfrak{so}_{2n}}(\underline{p}(\psi)) = \left(2\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\right]^t + \eta_{\mathfrak{o}_{2k},\mathfrak{o}_{2k}}\left(\left[\prod_{j=1}^{l}(2m_j+1)\right]\right)\right)_{\mathrm{SO}_{2n}}$$

holds with $2k = \sum_{i=1}^{l}(2m_i+1)$.

Proof. Recall that

$$\underline{p}(\psi) = \left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\prod_{j=1}^{l}(2m_j+1)\Big)\right]$$

and

$$\eta_{\mathfrak{so}_{2n},\mathfrak{so}_{2n}}(\underline{p}(\psi)) = (\underline{p}(\psi)^t)_{\mathrm{SO}_{2n}}.$$

Also recall that given any partition $\underline{p} = [p_r \cdots p_1]$ with $p_r \geq \cdots \geq p_1$, we have

$$\underline{p}^+ = [(p_r + 1) \cdots p_1],$$
$$\underline{p}^- = [p_r \cdots (p_1 - 1)].$$

By [1, Lemma 3.3], given a partition $\underline{p}$ of $2n$, if it is an orthogonal partition or its transpose is a symplectic partition, then $(\underline{p}^t)_{\mathrm{SO}_{2n}} = ((\underline{p}^{+-})_{\mathrm{Sp}_{2n}})^t$. Hence, we obtain that

$$\begin{aligned}
&\left(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \eta_{\mathfrak{so}_{2k},\mathfrak{so}_{2k}}\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]\Big)\right)_{\mathrm{SO}_{2n}}\\
&= \left(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^t\Big)_{\mathrm{SO}_{2k}}\right)_{\mathrm{SO}_{2n}}\\
&= \left(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^{+-}\Big)_{\mathrm{Sp}_{2k}}\Big)^t\right)_{\mathrm{SO}_{2n}}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\prod_{j=1}^{l}(2m_j+1)\Big)\Big]^t\\
&\quad = 2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]\Big)^t
\end{aligned}$$

is a partition of the following form

$$\Big[p_l^1 \cdots p_l^{2m_1+1}\Big(\prod_{j=1}^{l-1} p_j^1 \cdots p_j^{2m_{l+1-j}-2m_{l-j}}\Big)p_0^1 \cdots p_0^{m_0}\Big],$$

where p_l^i with $1 \leq i \leq 2m_1 + 1$, p_{2j}^i with $1 \leq i \leq 2m_{l+1-2j} - 2m_{l-2j}$ and $1 \leq j \leq \frac{l-2}{2}$, and p_0^k with $1 \leq k \leq m_0$ are all even; and p_{2j+1}^i with $1 \leq i \leq 2m_{l-2j} - 2m_{l-2j-1}$ and $0 \leq j \leq \frac{l-2}{2}$ are all odd; and finally,

$$p_l^1 \geq \cdots \geq p_l^{2m_1+1} > p_{l-1}^1, \quad p_j^1 \geq \cdots \geq p_j^{2m_{l+1-j}-2m_{l-j}} > p_{j-1}^1, \quad 1 \leq j \leq l-1,$$

with $p_0^1 \geq \cdots \geq p_0^{m_0}$. Note that

$$\begin{aligned}
\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^{+-}\Big)_{\mathrm{Sp}_{2k}} &= \Big[(2m_l+2)\prod_{j=2}^{l-1}(2m_j+1)(2m_1)\Big]_{\mathrm{Sp}_{2k}}\\
&= \Big[(2m_l+2)\prod_{j=1}^{(l-2)/2}(2m_{2j+1})(2m_{2j}+2)(2m_1)\Big].
\end{aligned}$$

Then the partition

$$2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^{+-}\Big)_{\mathrm{Sp}_{2k}}\Big)^t$$

is equal to the following partition

$$\Big[p_l^1\cdots p_l^{2m_1}(p_l^{2m_1+1}-1)\prod_{j=0}^{(l-2)/2} p_{2j+1}^1\cdots p_{2j+1}^{2m_{l-2j}-2m_{l-2j-1}}$$
$$\times\prod_{j=1}^{(l-2)/2}(p_{2j}^1+1)p_{2j}^2\cdots p_{2j}^{2m_{l+1-2j}-2m_{l-2j}-1}(p_{2j}^{2m_{l+1-2j}-2m_{l-2j}}-1)$$
$$\times(p_0^1+1)p_0^2\cdots p_0^{m_0}\Big].$$

Following the recipe on carrying out the SO_{2n}-collapse [11, Lemma 6.3.8], we obtain that the partition

$$\Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{i=1}^{t} n_i^2\Big)\Big(\prod_{j=1}^{l}(2m_j+1)\Big)\Big]^t\Big)_{\mathrm{SO}_{2n}}$$

is equal to the following partition

$$\Big[(p_l^1\cdots p_l^{2m_1})_{\mathrm{SO}}(p_l^{2m_1+1}-1)\prod_{j=0}^{(l-2)/2} p_{2j+1}^1\cdots p_{2j+1}^{2m_{l-2j}-2m_{l-2j-1}}$$
$$\times\prod_{j=1}^{(l-2)/2}(p_{2j}^1+1)(p_{2j}^2\cdots p_{2j}^{2m_{l+1-2j}-2m_{l-2j}-1})_{\mathrm{SO}}(p_{2j}^{2m_{l+1-2j}-2m_{l-2j}}-1)$$
$$\times(p_0^1+1)(p_0^2\cdots p_0^{m_0})_{\mathrm{SO}}\Big],$$

and the partition

$$\Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \Big(\Big(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]^{+-}\Big)_{\mathrm{Sp}_{2k}}\Big)^t\Big)_{\mathrm{SO}_{2n}}$$

can be written as

$$\Big(p_l^1\cdots p_l^{2m_1}(p_l^{2m_1+1}-1)\prod_{j=0}^{(l-2)/2} p_{2j+1}^1\cdots p_{2j+1}^{2m_{l-2j}-2m_{l-2j-1}}$$
$$\times\prod_{j=1}^{(l-2)/2}(p_{2j}^1+1)p_{2j}^2\cdots p_{2j}^{2m_{l+1-2j}-2m_{l-2j}-1}(p_{2j}^{2m_{l+1-2j}-2m_{l-2j}}-1)$$
$$\times(p_0^1+1)p_0^2\cdots p_0^{m_0}\Big)_{\mathrm{SO}_{2n}},$$

which is equal to

$$\Big[(p_l^1 \cdots p_l^{2m_1})_{\mathrm{SO}}(p_l^{2m_1+1}-1)\prod_{j=0}^{(l-2)/2} p_{2j+1}^1 \cdots p_{2j+1}^{2m_{l-2j}-2m_{l-2j-1}}$$

$$\times \prod_{j=1}^{(l-2)/2}(p_{2j}^1+1)(p_{2j}^2\cdots p_{2j}^{2m_{l+1-2j}-2m_{l-2j}-1})_{\mathrm{SO}}(p_{2j}^{2m_{l+1-2j}-2m_{l-2j}}-1)$$

$$\times (p_0^1+1)(p_0^2\cdots p_0^{m_0})_{\mathrm{SO}}\Big].$$

Hence, we obtain that

$$\eta_{\mathfrak{so}_{2n},\mathfrak{so}_{2n}}(\underline{p}(\psi)) = \left(2\Big[\Big(\prod_{(\chi,m,\alpha)\in \mathrm{e}} m\Big)\Big(\prod_{i=1}^{t} n_i\Big)\Big]^t + \eta_{\mathfrak{so}_{2k},\mathfrak{so}_{2k}}\left(\Big[\prod_{j=1}^{l}(2m_j+1)\Big]\right)\right)_{\mathrm{SO}_{2n}}.$$

This completes the proof of the lemma. ■

The proof of Theorem 4.2 has been completed for $\mathrm{G}_n = \mathrm{SO}_{2n}$.

8. On the wavefront set of unramified unitary representations

In this last section, we study the wavefront set of the unramified unitary representations for split classical groups $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n}$. Under assumptions on the leading orbits in the wavefront set of negative representations, we determine the set $\mathfrak{p}^m(\pi)$ for general unramified unitary representations. This reduction process has its own interests. We remark that for representations of non-connected groups O_{2n}, we follow [10] for the character expansions at the identity to define the wavefront set.

Assume that π is any irreducible unramified unitary representation of $\mathrm{G}_n(F)$ as in Theorem 3.5,

$$\pi = \times_{(\chi,m,\alpha)\in \mathrm{e}} v^{\alpha}\chi(\det_m) \rtimes \sigma_{\mathrm{neg}},$$

where σ_{neg} is a negative representation of $G_{n^*}(F)$, and

$$\mathrm{Jord}(\sigma_{\mathrm{neg}}) = \mathrm{Jord}(\sigma_{\mathrm{sn}}) \cup \{(\chi_i, n_i), (\chi_i^{-1}, n_i) \mid 1 \le i \le t\}.$$

Here $\mathrm{Jord}(\sigma_{\mathrm{sn}})$ is equal to

$$\{(\lambda_0, 2n_1+1), \ldots, (\lambda_0, 2n_k+1), (1_{\mathrm{GL}_1}, 2m_1+1), \ldots, (1_{\mathrm{GL}_1}, 2m_l+1)\},$$

when $G_{n^*} = \mathrm{Sp}_{2n^*}, \mathrm{O}_{2n^*}$, and is equal to

$$\{(\lambda_0, 2n_1), \ldots, (\lambda_0, 2n_k), (1_{\mathrm{GL}_1}, 2m_1), \ldots, (1_{\mathrm{GL}_1}, 2m_l)\},$$

when $G_{n^*} = \mathrm{SO}_{2n^*+1}$, as in Section 3.

We have the following conjecture on the maximal partitions in the wavefront set of negative representations.

Conjecture 8.1. *Given negative representations σ_{neg} as above, we have*

$$\mathfrak{p}^m(\sigma_{\mathrm{neg}}) = \left\{\eta_{\mathfrak{g}_{n^*}^\vee,\mathfrak{g}_{n^*}}\left(\left[\left(\prod_{j=1}^{t} n_j^2\right)\left(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\right)\right]\right)\right\},$$

when $G_{n^*} = \mathrm{Sp}_{2n^*}, \mathrm{O}_{2n^*}$*;*

$$\mathfrak{p}^m(\sigma_{\mathrm{neg}}) = \left\{\eta_{\mathfrak{g}_{n^*}^\vee,\mathfrak{g}_{n^*}}\left(\left[\left(\prod_{j=1}^{t} n_j^2\right)\left(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s)\right)\right]\right)\right\},$$

when $G_{n^*} = \mathrm{SO}_{2n^*+1}$.

Based on Conjecture 8.1, we obtain the explicit description of the maximal partitions in the wavefront set of general irreducible unramified unitary representations π of $\mathrm{G}_n(F)$.

Theorem 8.2. *Assume Conjecture 8.1 is true. For any irreducible unramified unitary representation π of $\mathrm{G}_n(F)$, the maximal partitions in the wavefront set $\mathfrak{p}(\pi)$ are given as follows:*

$$\mathfrak{p}^m(\pi) = \left\{\eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}\left(\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\right)\left(\prod_{j=1}^{t} n_j^2\right)\left(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\right)\right]\right)\right\}$$

when $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{O}_{2n}$*; and*

$$\mathfrak{p}^m(\pi) = \left\{\eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}\left(\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\right)\left(\prod_{j=1}^{t} n_j^2\right)\left(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s)\right)\right]\right)\right\}$$

when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$.

We remark that Ciubotaru, Mason-Brown, and Okada [9] recently computed the maximal orbits in the wavefront set of irreducible Iwahori-spherical representations of split connected reductive p-adic groups with "real infinitesimal characters", which partially proved Conjecture 8.1 and Theorem 8.2. This provides evidence for Conjecture 8.1.

By Lemma 5.1, we have

$$\mathfrak{p}^m(\times_{(\chi,m,\alpha)\in\mathbf{e}} v^\alpha\chi(\det_m)) = \{+_{(\chi,m,\alpha)\in\mathbf{e}}[1^m]\} = \left\{\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\right)\right]^t\right\}.$$

By Lemma 5.1, and by [11, Theorem 7.3.3] on formula for induced nilpotent orbits, we obtain that

$$\mathfrak{p}^m(\pi) = \left\{\left(2\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\right)\right]^t + \underline{p}_{\sigma_{\mathrm{neg}}}\right)_{\mathrm{G}_n} \mid \underline{p}_{\sigma_{\mathrm{neg}}} \in \mathfrak{p}^m(\sigma_{\mathrm{neg}})\right\}.$$

Hence, by the assumption, to prove Theorem 8.2, it suffices to show the following lemma which will be proved case-by-case in the following subsections.

Lemma 8.3. *The following identities hold:*

$$\eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}\Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\Big]\Big)$$
$$=\Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big]^t+\eta_{\mathfrak{g}_{n^*}^\vee,\mathfrak{g}_{n^*}}\Big(\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\Big]\Big)\Big)_{\mathrm{G}_n}$$

when $\mathrm{G}_n=\mathrm{Sp}_{2n},\mathrm{O}_{2n}$*; and*

$$\eta_{\mathfrak{g}_n^\vee,\mathfrak{g}_n}\Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s)\Big)\Big]\Big)$$
$$=\Big(2\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m\Big)\Big]^t+\eta_{\mathfrak{g}_{n^*}^\vee,\mathfrak{g}_{n^*}}\Big(\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s)\Big)\Big]\Big)\Big)_{\mathrm{G}_n}$$

when $\mathrm{G}_n=\mathrm{SO}_{2n+1}$.

8.1. Proof of Lemma 8.3, $\mathrm{G}_n=\mathrm{Sp}_{2n}$

By arguments similar to those in the proof of the Sp_{2n}-case of Lemma 5.2, we only need to show that

$$\Big(\Big[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\Big]^-\Big)_{\mathrm{Sp}_{2n}}$$
$$=\Big(\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\Big[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\Big]^-\Big)_{\mathrm{Sp}_{2n^*}}\Big)^{\mathrm{Sp}_{2n}}. \tag{8.1}$$

For any given partition $\underline{p}=[p_r\cdots p_1]$ with $p_r\geq\cdots\geq p_1$, we recall that $\underline{p}^-=[p_r\cdots(p_1-1)]$. Rewrite the partition $[\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2]$ as $[p_u^2p_{u-1}^2\cdots p_1^2]$ with $p_u\geq p_{u-1}\geq\cdots\geq p_1$; and $[\prod_{i=1}^{t} n_i^2]$ as $[q_v^2q_{v-1}^2\cdots q_1^2]$ with $q_v\geq q_{v-1}\geq\cdots\geq q_1$. And rewrite

$$\Big[\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\Big]$$

as $[\prod_{w=1}^{l+k}(2r_w+1)]$ with $r_{l+k}\geq r_{l+k-1}\geq\cdots\geq r_1>0$. Then, (8.1) becomes

$$\Big(\Big[\Big(\prod_{i=1}^{u} p_1^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\Big]^-\Big)_{\mathrm{Sp}_{2n}}$$
$$=\Big(\Big(\prod_{i=1}^{u} p_1^2\Big)\Big(\Big[\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\Big]^-\Big)_{\mathrm{Sp}_{2n^*}}\Big)^{\mathrm{Sp}_{2n}}. \tag{8.2}$$

To proceed, we consider the following cases:

(1) When $q_1 \geq 2r_1+1$, we have (a) $p_1 \geq 2r_1+1$ and (b) $p_1 < 2r_1+1$.

(2) When $q_1 < 2r_1+1$, we have (a) $p_1 \geq q_1$, and (b) $p_1 < q_1$.

In each case, for $1 \leq z \leq \frac{l+k-1}{2}$, if $2r_{2z+1}+1 > 2r_{2z}+1$, we list all the different odd p_i's, q_j's between $2r_{2z+1}+1$ and $2r_{2z}+1$ as

$$2r_{2z+1}+1 > p_z^1 > p_z^2 > \cdots > p_z^{x_z} > 2r_{2z}+1,$$
$$2r_{2z+1}+1 > q_z^1 > q_z^2 > \cdots > q_z^{y_z} > 2r_{2z}+1.$$

Case (1-a): $q_1 \geq 2r_1+1$, $p_1 \geq 2r_1+1$. We have

$$\begin{aligned}&\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^-\right)_{\mathrm{Sp}_{2n}}\\ &\quad=\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n}}.\end{aligned}$$

The collapse

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-1}{2}$, $1 \leq i \leq x_z$, and $1 \leq j \leq y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$. On the other hand, we have

$$\begin{aligned}&\left(\left(\prod_{i=1}^{u} p_1^2\right)\left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^-\right)_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}\\ &\quad=\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}.\end{aligned} \tag{8.3}$$

Then

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}$$

can be obtained from

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-1}{2}$ and $1 \leq j \leq y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}\right]$$

via replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \leq z \leq \frac{l+s-1}{2}$ and $1 \leq i \leq x_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$. Hence, (8.2) holds in this case.

Case (1-b): $q_1 \geq 2r_1+1$, $p_1 < 2r_1+1$. We have

$$\begin{aligned}
&\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^{-}\right)_{\mathrm{Sp}_{2n}}\\
&\quad=\left[\left(\prod_{i=2}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)p_1(p_1-1)\right]_{\mathrm{Sp}_{2n}}.
\end{aligned}$$

To carry out the Sp_{2n}-collapse, we also need to list all the different odd p_i's, q_j's between $2r_1+1$ and p_1 as

$$\begin{aligned}
&2r_1+1 > p_0^1 > p_0^2 > \cdots > p_0^{x_0} > p_1,\\
&2r_1+1 > q_0^1 > q_0^2 > \cdots > q_0^{y_0} > p_1.
\end{aligned}$$

Then

$$\left[\left(\prod_{i=2}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)p_1(p_1-1)\right]_{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\left[\left(\prod_{i=2}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)p_1(p_1-1)\right]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-1}{2}$, $1 \leq i \leq x_z$, and $1 \leq j \leq y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and replacing $(2r_1+1, p_1-1)$ by $(2r_1, p_1)$ if p_1 is even and $(2r_1+1, p_1)$ by $(2r_1, p_1+1)$ if p_1 is odd; and finally replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \leq i \leq x_0$ and $1 \leq j \leq y_0$.

On the other hand, as in case (1-a), we still have (8.3). Then the partition

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}$$

can be obtained from

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-1}{2}$ and $1 \leq j \leq y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=2}^{l+k}(2r_w+1)(2r_1)\right]_{\mathrm{Sp}_{2n^*}}\right]$$

via replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \leq z \leq \frac{l+s-1}{2}$ and $1 \leq i \leq x_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and then replacing p_1^2 by (p_1+1, p_1-1) if p_1 is odd, $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) for $1 \leq i \leq x_0$ and $1 \leq j \leq y_0$. Hence, (8.2) holds in this case.

Case (2-a): $q_1 < 2r_1+1$, $p_1 \geq q_1$. We have

$$\begin{aligned}&\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^{-}\right)_{\mathrm{Sp}_{2n}}\\&\quad=\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]_{\mathrm{Sp}_{2n}}.\end{aligned}$$

To carry out the Sp_{2n}-collapse, we also need to list all the different odd p_i's, q_j's between $2r_1+1$ and q_1 as

$$\begin{aligned}2r_1+1 &> p_0^1 > p_0^2 > \cdots > p_0^{x_0} > q_1,\\2r_1+1 &> q_0^1 > q_0^2 > \cdots > q_0^{y_0} > q_1.\end{aligned}$$

Then

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]_{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-1}{2}$, $1 \leq i \leq x_z$, and $1 \leq j \leq y_z$, whenever

$2r_{2z+1}+1 > 2r_{2z}+1$; and replacing $(2r_1+1, q_1-1)$ by $(2r_1, q_1)$ if q_1 is even, $(2r_1+1, q_1)$ by $(2r_1, q_1+1)$ if q_1 is odd; and finally replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) for $1 \le i \le x_0$ and $1 \le j \le y_0$. On the other hand, we have

$$\begin{aligned}&\left(\left(\prod_{i=1}^{u} p_1^2\right)\left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^-\right)_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}\\ &\quad = \left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}. \qquad (8.4)\end{aligned}$$

Then

$$\left[\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]_{\mathrm{Sp}_{2n^*}}$$

can be obtained from

$$\left[\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1) for $1 \le z \le \frac{l+s-1}{2}$ and $1 \le j \le y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and then replacing $(2r_1+1, q_1-1)$ by $(2r_1, q_1)$ if q_1 is even and $(2r_1+1, q_1)$ by $(2r_1, q_1+1)$ if q_1 is odd, and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) for $1 \le j \le y_0$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]_{\mathrm{Sp}_{2n^*}}\right)^{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=2}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\right]_{\mathrm{Sp}_{2n^*}}\right]$$

via replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \le z \le \frac{l+s-1}{2}$ and $1 \le i \le x_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and then replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) for $1 \le i \le x_0$. Hence, (8.2) holds in this case.

Case (2-b): $q_1 < 2r_1+1$, $p_1 < q_1$. We have

$$\begin{aligned}&\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^-\right)_{\mathrm{Sp}_{2n}}\\ &\quad = \left[\left(\prod_{i=2}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)p_1(p_1-1)\right]_{\mathrm{Sp}_{2n}}.\end{aligned}$$

To carry out the Sp_{2n}-collapse, we also need to list all the different odd p_i's, q_j's between $2r_1+1$ and p_1 as

$$2r_1+1 > p_0^1 > p_0^2 > \cdots > p_0^{x_0} > p_1,$$
$$2r_1+1 > q_0^1 > q_0^2 > \cdots > q_0^{y_0} > p_1.$$

Then

$$\Big[\Big(\prod_{i=2}^{u} p_1^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)p_1(p_1-1)\Big]_{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\Big[\Big(\prod_{i=2}^{u} p_1^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)p_1(p_1-1)\Big]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-1}{2}$, $1 \le i \le x_z$, and $1 \le j \le y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and then replacing $(2r_1+1, p_1-1)$ by $(2r_1, p_1)$ if p_1 is even, $(2r_1+1, p_1)$ by $(2r_1, p_1+1)$ if p_1 is odd, and also $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) for $1 \le i \le x_0$ and $1 \le j \le y_0$.

On the other hand, we still have (8.4). Then the partition

$$\Big[\Big(\prod_{j=2}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\Big]_{\mathrm{Sp}_{2n^*}}$$

can be obtained from

$$\Big[\Big(\prod_{j=2}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\Big]$$

via replacing $(2r_{2z+1}+1, 2r_{2z}+1)$ by $(2r_{2z+1}, 2r_{2z}+2)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1) for $1 \le z \le \frac{l+s-1}{2}$ and $1 \le j \le y_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and then replacing $(2r_1+1, q_1-1)$ by $(2r_1, q_1)$ if q_1 is even, $(2r_1+1, q_1)$ by $(2r_1, q_1+1)$ if q_1 is odd, and also $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) if $q_0^j \neq q_1$ and $1 \le j \le y_0$. And the partition

$$\Big(\Big(\prod_{i=1}^{u} p_1^2\Big)\Big[\Big(\prod_{j=2}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\Big]_{\mathrm{Sp}_{2n^*}}\Big)^{\mathrm{Sp}_{2n}}$$

can be obtained from

$$\Big[\Big(\prod_{i=1}^{u} p_1^2\Big)\Big[\Big(\prod_{j=2}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)q_1(q_1-1)\Big]_{\mathrm{Sp}_{2n^*}}\Big]$$

via replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \le z \le \frac{l+s-1}{2}$ and $1 \le i \le x_z$, whenever $2r_{2z+1}+1 > 2r_{2z}+1$; and then replacing p_1^2 by (p_1+1, p_1-1) if p_1 is odd, $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) for $1 \le i \le x_0$. Hence, (8.1) holds in this case.

The proof of Lemma 8.3 has been completed for $\mathrm{G}_n = \mathrm{Sp}_{2n}$.

8.2. Proof of Lemma 8.3, $G_n = \mathrm{SO}_{2n+1}$

By similar arguments as in the proof of the SO_{2n+1}-case of Lemma 5.2, we only need to show that

$$\left(\left[\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\right)\left(\prod_{j=1}^{t} n_j^2\right)\left(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s)\right)\right]^+\right)_{\mathrm{SO}_{2n+1}}$$
$$= \left(\left(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\right)\left(\left[\left(\prod_{j=1}^{t} n_j^2\right)\left(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s)\right)\right]^+\right)_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}. \tag{8.5}$$

For any given partition $\underline{p} = [p_r \cdots p_1]$ with $p_r \geq \cdots \geq p_1$, recall that $\underline{p}^+ = [(p_r+1) \cdots p_1]$. Rewrite the partition $[\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2]$ as $[p_u^2 p_{u-1}^2 \cdots p_1^2]$ with $p_u \geq p_{u-1} \geq \cdots \geq p_1$; and the partition $[\prod_{i=1}^{t} n_i^2]$ as $[q_v^2 q_{v-1}^2 \cdots q_1^2]$ with $q_v \geq q_{v-1} \geq \cdots \geq q_1$. And rewrite the partition $[(\prod_{i=1}^{l}(2m_i)\prod_{s=1}^{k}(2n_s))]$ as $[\prod_{w=1}^{l+k}(2r_w)]$ with $r_{l+k} \geq r_{l+k-1} \geq \cdots \geq r_1 \geq 0$. Then, (8.5) becomes

$$\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n+1}}$$
$$= \left(\left(\prod_{i=1}^{u} p_1^2\right)\left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}. \tag{8.6}$$

To proceed, we consider the following cases:

(1) When $q_v \leq 2r_{l+k}$, we have (a) $p_u \leq 2r_{l+k}$ and (b) $p_u > 2r_{l+k}$.

(2) When $q_v > 2r_{l+k}$, we have (a) $p_u \leq q_v$ and (b) $p_u > q_v$.

In each case, for $1 \leq z \leq \frac{l+k-2}{2}$, if $2r_{2z+1} > 2r_{2z}$, we list all the different even p_i's, q_j's between $2r_{2z+1}$ and $2r_{2z}$ as

$$2r_{2z+1} > p_z^1 > p_z^2 > \cdots > p_z^{x_z} > 2r_{2z},$$
$$2r_{2z+1} > q_z^1 > q_z^2 > \cdots > q_z^{y_z} > 2r_{2z}.$$

If $2r_1 \neq 0$, we also list all the different even p_i's, q_j's between $2r_1$ and 0 as

$$2r_1 > p_0^1 > p_0^2 > \cdots > p_0^{x_0} > 0,$$
$$2r_1 > q_0^1 > q_0^2 > \cdots > q_0^{y_0} > 0.$$

Case (1-a): $q_v \leq 2r_{l+k}$, $p_u \leq 2r_{l+k}$. We have

$$\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n+1}}$$
$$= \left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n+1}}.$$

The collapse

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\left(2r_{l+k}+1\right)\prod_{w=1}^{l+k-1}(2r_w)\right]$$

via replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-2}{2}$, $1 \le i \le x_z$, and $1 \le j \le y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and then replacing $(2r_1, 0)$ by $(2r_1-1, 1)$, $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \le i \le x_0$ and $1 \le j \le y_0$, if $2r_1 \ne 0$.

On the other hand, we have

$$\begin{aligned}&\left(\left(\prod_{i=1}^{u} p_1^2\right)\left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}\\ &\quad=\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}. \qquad (8.7)\end{aligned}$$

Then

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}$$

can be obtained from

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]$$

via replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-2}{2}$ and $1 \le j \le y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and then replacing $(2r_1, 0)$ by $(2r_1-1, 1)$ and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) for $1 \le j \le y_0$, if $2r_1 \ne 0$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right]$$

via replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \le z \le \frac{l+s-2}{2}$ and $1 \le i \le x_z$, whenever $2r_{2z+1} > 2r_{2z}$; and then replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) for $1 \le i \le x_0$, if $2r_1 \ne 0$. Hence, (8.6) holds in this case.

Case (1-b): $q_v \leq 2r_{l+k}$, $p_u > 2r_{l+k}$. We have

$$\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n+1}}$$
$$= \left[(p_u+1)p_u\left(\prod_{i=1}^{u-1} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n+1}}.$$

To carry out the SO_{2n+1}-collapse, we also need to list all the different even p_i's between p_u and $2r_{l+k}$ as

$$p_u > p_{l+k}^1 > p_{l+k}^2 > \cdots > p_{l+k}^{x_{l+k}} > 2r_{l+k}.$$

The collapse

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]$$

via replacing $(p_u, 2r_{l+k})$ by $(p_u-1, 2r_{l+k}+1)$ if p_u is even, $(p_u+1, 2r_{l+k})$ by $(p_u, 2r_{l+k}+1)$ if p_u is odd, and $p_{l+k}^{i,2}$ by $(p_{l+k}^i+1, p_{l+k}^i-1)$, for $1 \leq i \leq x_{l+k}$; and replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-2}{2}$, $1 \leq i \leq x_z$, and $1 \leq j \leq y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $(2r_1, 0)$ by $(2r_1-1, 1)$, $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \leq i \leq x_0$ and $1 \leq j \leq y_0$, if $2r_1 \neq 0$.

On the other hand, we still have (8.7). We obtain the partition

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}$$

from

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]$$

via replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \leq z \leq \frac{l+s-2}{2}$ and $1 \leq j \leq y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and replacing $(2r_1, 0)$ by $(2r_1-1, 1)$ and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \leq j \leq y_0$, if $2r_1 \neq 0$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(\prod_{j=1}^{v} q_j^2\right)(2r_{l+k}+1)\prod_{w=1}^{l+k-1}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right]$$

via replacing p_u^2 by (p_u+1, p_u-1) if p_u is even, $p_{l+k}^{i,2}$ by $(p_{l+k}^i+1, p_{l+k}^i-1)$, for $1 \le i \le x_{l+k}$; and replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), for $1 \le z \le \frac{l+s-2}{2}$, $1 \le i \le x_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), $1 \le i \le x_0$, if $2r_1 \neq 0$. Hence, (8.6) holds in this case.

Case (2-a): $q_v > 2r_{l+k}$, $p_u \le q_v$. We have

$$\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n+1}}$$
$$=\left[\left(\prod_{i=1}^{u} p_1^2\right)(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n+1}}.$$

To carry out the SO_{2n+1}-collapse, we also need to list all the different even p_i's and q_j's between q_v and $2r_{l+k}$ as

$$q_v > p_{l+k}^1 > p_{l+k}^2 > \cdots > p_{l+k}^{x_{l+k}} > 2r_{l+k},$$
$$q_v > q_{l+k}^1 > q_{l+k}^2 > \cdots > q_{l+k}^{y_{l+k}} > 2r_{l+k}.$$

The collapse

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left[\left(\prod_{i=1}^{u} p_1^2\right)(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]$$

via replacing $(q_v, 2r_{l+k})$ by $(q_v-1, 2r_{l+k}+1)$ if q_v is even, $(q_v+1, 2r_{l+k})$ by $(q_v, 2r_{l+k}+1)$ if q_v is odd, $p_{l+k}^{i,2}$ by $(p_{l+k}^i+1, p_{l+k}^i-1)$, and $q_{l+k}^{j,2}$ by $(q_{l+k}^j+1, q_{l+k}^j-1)$, for $1 \le i \le x_{l+k}$ and $1 \le j \le y_{l+k}$; and replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-2}{2}$, $1 \le i \le x_z$, and $1 \le j \le y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $(2r_1, 0)$ by $(2r_1-1, 1)$, $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \le i \le x_0$ and $1 \le j \le y_0$, if $2r_1 \neq 0$.

On the other hand, we have

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}$$
$$= \left(\left(\prod_{i=1}^{u} p_1^2\right)\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}. \quad (8.8)$$

Then

$$\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}$$

can be obtained from

$$\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]$$

via replacing $(q_v, 2r_{l+k})$ by $(q_v-1, 2r_{l+k}+1)$ if q_v is even, $(q_v+1, 2r_{l+k})$ by $(q_v, 2r_{l+k}+1)$ if q_v is odd, and $q_{l+k}^{j,2}$ by $(q_{l+k}^j+1, q_{l+k}^j-1)$, for $1 \le j \le y_{l+k}$; and replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-2}{2}$ and $1 \le j \le y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $(2r_1, 0)$ by $(2r_1-1, 1)$ and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1) for $1 \le j \le y_0$, if $2r_1 \ne 0$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left(\prod_{i=1}^{u} p_1^2\right)\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}$$

via replacing p_u^2 by (p_u+1, p_u-1) if p_u is even, and $p_{l+k}^{i,2}$ by $(p_{l+k}^i+1, p_{l+k}^i-1)$, for $1 \le i \le x_{l+k}$; and replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \le z \le \frac{l+s-2}{2}$ and $1 \le i \le x_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1) for $1 \le i \le x_0$, if $2r_1 \ne 0$. Hence, (8.6) still holds in this case.

Case (2-b): $q_v > 2r_{l+k}$, $p_u > q_v$. We have

$$\left(\left[\left(\prod_{i=1}^{u} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]^+\right)_{\mathrm{SO}_{2n+1}}$$
$$= \left[(p_u+1)p_u\left(\prod_{i=1}^{u-1} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n+1}}.$$

To carry out the SO_{2n+1}-collapse, we also need to list all the different even p_i's and q_j's between p_u and $2r_{l+k}$ as

$$p_u > p_{l+k}^1 > p_{l+k}^2 > \cdots > p_{l+k}^{x_{l+k}} > 2r_{l+k},$$
$$p_u > q_{l+k}^1 > q_{l+k}^2 > \cdots > q_{l+k}^{y_{l+k}} > 2r_{l+k}.$$

The collapse $[(p_u+1)p_u(\prod_{i=1}^{u-1} p_1^2)(\prod_{j=1}^{v} q_j^2)\prod_{w=1}^{l+k}(2r_w)]_{\mathrm{SO}_{2n+1}}$ can be obtained from

$$\left[(p_u+1)p_u\left(\prod_{i=1}^{u-1} p_1^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]$$

via replacing $(p_u, 2r_{l+k})$ by $(p_u-1, 2r_{l+k}+1)$ if p_u is even, $(p_u+1, 2r_{l+k})$ by $(p_u, 2r_{l+k}+1)$ if p_u is odd, $p_{l+k}^{i,2}$ by $(p_{l+k}^i+1, p_{l+k}^i-1)$, and $q_{l+k}^{j,2}$ by $(q_{l+k}^j+1, q_{l+k}^j-1)$, for $1 \le i \le x_{l+k}$ and $1 \le j \le y_{l+k}$; and replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$, $p_z^{i,2}$ by (p_z^i+1, p_z^i-1), and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-2}{2}$, $1 \le i \le x_z$, and $1 \le j \le y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $(2r_1, 0)$ by $(2r_1-1, 1)$, $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \le i \le x_0$ and $1 \le j \le y_0$, if $2r_1 \neq 0$.

On the other hand, we still have (8.8). We obtain the partition

$$\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}$$

from

$$\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]$$

via replacing $(q_v, 2r_{l+k})$ by $(q_v-1, 2r_{l+k}+1)$ if q_v is even, $(q_v+1, 2r_{l+k})$ by $(q_v, 2r_{l+k}+1)$ if q_v is odd, and $q_{l+k}^{j,2}$ by $(q_{l+k}^j+1, q_{l+k}^j-1)$ if $q_{l+k}^j \neq q_v$, for $1 \le j \le y_{l+k}$; and replacing $(2r_{2z+1}, 2r_{2z})$ by $(2r_{2z+1}-1, 2r_{2z}+1)$ and $q_z^{j,2}$ by (q_z^j+1, q_z^j-1), for $1 \le z \le \frac{l+s-2}{2}$ and $1 \le j \le y_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $(2r_1, 0)$ by $(2r_1-1, 1)$ and $q_0^{j,2}$ by (q_0^j+1, q_0^j-1), for $1 \le j \le y_0$, if $2r_1 \neq 0$. And the partition

$$\left(\left(\prod_{i=1}^{u} p_1^2\right)\left[(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}\right)^{\mathrm{SO}_{2n+1}}$$

can be obtained from

$$\left(\prod_{i=1}^{u} p_1^2\right)\left[\left(q_v+1)q_v\left(\prod_{j=1}^{v-1} q_j^2\right)\prod_{w=1}^{l+k}(2r_w)\right]_{\mathrm{SO}_{2n^*+1}}$$

via replacing p_u^2 by (p_u+1, p_u-1) if p_u is even and $p_{l+k}^{i,2}$ by $(p_{l+k}^i+1, p_{l+k}^i-1)$, for $1 \le i \le x_{l+k}$; and replacing $p_z^{i,2}$ by (p_z^i+1, p_z^i-1) for $1 \le z \le \frac{l+s-2}{2}$, $1 \le i \le x_z$, whenever $2r_{2z+1} > 2r_{2z}$; and finally replacing $p_0^{i,2}$ by (p_0^i+1, p_0^i-1), for $1 \le i \le x_0$, if $2r_1 \neq 0$. Hence, (8.6) holds in this case.

The proof of Lemma 8.3 has been completed for $\mathrm{G}_n = \mathrm{SO}_{2n+1}$.

8.3. Proof of Lemma 8.3, $\mathrm{G}_n = \mathrm{O}_{2n}$

By similar arguments as in the proof of the O_{2n}-case of Lemma 5.2, we only need to show that

$$\left(\left[\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\right]^t\right)_{\mathrm{O}_{2n}} \tag{8.9}$$
$$= \left(\Big(\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2\Big)^t + \left(\left[\Big(\prod_{j=1}^{t} n_j^2\Big)\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\right]^t\right)_{\mathrm{O}_{2n^*}}\right)_{\mathrm{O}_{2n}}.$$

For any given partition $\underline{p} = [p_r \cdots p_1]$ with $p_r \geq \cdots \geq p_1$, recall that $\underline{p}^+ = [(p_r+1) \cdots p_1]$ and $\underline{p}^- = [p_r \cdots (p_1-1)]$. Rewrite the partition $[\prod_{(\chi,m,\alpha)\in\mathbf{e}} m^2]$ as $[p_u^2 p_{u-1}^2 \cdots p_1^2]$ with $p_u \geq p_{u-1} \geq \cdots \geq p_1$; and the partition $[\prod_{i=1}^{t} n_i^2]$ as $[q_v^2 q_{v-1}^2 \cdots q_1^2]$ with $q_v \geq q_{v-1} \geq \cdots \geq q_1$. And rewrite the partition

$$\left[\Big(\prod_{i=1}^{l}(2m_i+1)\prod_{s=1}^{k}(2n_s+1)\Big)\right]$$

as $[\prod_{w=1}^{l+k}(2r_w+1)]$ with $r_{l+k} \geq r_{l+k-1} \geq \cdots \geq r_1 > 0$. Then, (8.9) becomes

$$\left(\left[\Big(\prod_{i=1}^{u} p_i^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\right]^t\right)_{\mathrm{O}_{2n}}$$
$$= \left(\Big(\prod_{i=1}^{u} p_i^2\Big)^t + \left(\left[\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\right]^t\right)_{\mathrm{O}_{2n^*}}\right)_{\mathrm{O}_{2n}}.$$

As in the proof of the O_{2n}-case of Lemma 5.2, it is easy to see that

$$\left[\Big(\prod_{i=1}^{u} p_i^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\right]^t$$
$$= \left[\Big(\prod_{i=1}^{u} p_i^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\right]^t + \left[\Big(\prod_{w=1}^{l+k}(2r_w+1)\Big)\right]^t$$

is a partition of the following form

$$\left[p_{l+k}^1 \cdots p_{l+k}^{2r_1+1}\Big(\prod_{j=1}^{l+k-1} p_j^1 \cdots p_j^{2r_{l+k+1-j}-2r_{l+k-j}}\Big)p_0^1 \cdots p_0^{r_0}\right],$$

where p_{l+k}^i with $1 \leq i \leq 2r_1+1$, p_{2j}^i with $1 \leq i \leq 2r_{l+k+1-2j} - 2r_{l+k-2j}$ and $1 \leq j \leq \frac{l+k-2}{2}$, and p_0^k with $1 \leq k \leq r_0$, are all even; and p_{2j+1}^i with $1 \leq i \leq 2r_{l+k-2j} - 2r_{l+k-2j-1}$ and $0 \leq j \leq \frac{l+k-2}{2}$ are all odd; and finally $p_{l+k}^1 \geq \cdots \geq p_{l+k}^{2r_1+1}$, $p_j^1 \geq \cdots \geq p_j^{2r_{l+k+1-j}-2r_{l+k-j}} \geq p_{j-1}^1$ with $1 \leq j \leq l+k-1$, and $p_0^1 \geq \cdots \geq p_0^{r_0} > 0$.

Following the recipe on carrying out the O_{2n}-collapse [11, Lemma 6.3.8], we obtain that

$$\left(\left[\left(\prod_{i=1}^{u} p_i^2\right)\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^t\right)_{O_{2n}}$$

is equal to

$$\begin{aligned}\Bigg[&(p_{l+k}^1\cdots p_{l+k}^{2r_1})_O(p_{l+k}^{2r_1+1}-1)\prod_{j=0}^{(l+k-2)/2} p_{2j+1}^1\cdots p_{2j+1}^{2r_{l+k-2j}-2r_{l+k-2j-1}}\\ &\times\prod_{j=1}^{(l+k-2)/2}(p_{2j}^1+1)(p_{2j}^2\cdots p_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_O\\ &\times(p_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}}-1)(p_0^1+1)(p_0^2\cdots p_0^{r_0})_O\Bigg].\end{aligned} \tag{8.10}$$

Similarly, we have

$$\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^t=\left[\left(\prod_{j=1}^{v} q_j^2\right)\right]^t+\left[\prod_{w=1}^{l+k}(2r_w+1)\right]^t$$

is a partition of the following form

$$\left[q_{l+k}^1\cdots q_{l+k}^{2r_1+1}\left(\prod_{j=1}^{l+k-1} q_j^1\cdots q_j^{2r_{l+k+1-j}-2r_{l+k-j}}\right)q_0^1\cdots q_0^{r_0}\right],$$

where q_{l+k}^i with $1\le i\le 2r_1+1$, q_{2j}^i with $1\le i\le 2r_{l+k+1-2j}-2r_{l+k-2j}$ and $1\le j\le\frac{l+k-2}{2}$, and q_0^k with $1\le k\le r_0$, are all even; and q_{2j+1}^i with $1\le i\le 2r_{l+k-2j}-2r_{l+k-2j-1}$ and $0\le j\le\frac{l+k-2}{2}$ are all odd; and finally $q_{l+k}^1\ge\cdots\ge q_{l+k}^{2r_1+1}$, $q_j^1\ge\cdots\ge q_j^{2r_{l+k+1-j}-2r_{l+k-j}}\ge q_{j-1}^1$ with $1\le j\le l+k-1$, and $q_0^1\ge\cdots\ge q_0^{r_0}\ge 0$. (Note that we are adding 0's at the end of the partition if necessary.)

Following the recipe on carrying out the O_{2n^*}-collapse [11, Lemma 6.3.8], we obtain that the partition

$$\left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^t\right)_{O_{2n^*}}$$

is equal to

$$\begin{aligned}\Bigg[&(q_{l+k}^1\cdots q_{l+k}^{2r_1})_O(q_{l+k}^{2r_1+1}-1)\prod_{j=0}^{(l+k-2)/2} q_{2j+1}^1\cdots q_{2j+1}^{2r_{l+k-2j}-2r_{l+k-2j-1}}\\ &\times\prod_{j=1}^{(l+k-2)/2}(q_{2j}^1+1)(q_{2j}^2\cdots q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_O\\ &\times(q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}}-1)(q_0^1+1)(q_0^2\cdots q_0^{r_0})_O\Bigg].\end{aligned}$$

Without loss of generality and adding 0's if necessary, we can assume that $(\prod_{i=1}^{u} p_i^2)^t$ is an even partition of the following form:

$$\left[s_{l+k}^1 \cdots s_{l+k}^{2r_1+1}\left(\prod_{j=1}^{l+k-1} s_j^1 \cdots s_j^{2r_{l+k+1-j}-2r_{l+k-j}}\right)s_0^1 \cdots s_0^{r_0}\right].$$

Then the partition

$$\left(\prod_{i=1}^{u} p_i^2\right)^t + \left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^t\right)_{\mathrm{O}_{2n^*}}$$

is equal to

$$\begin{aligned}\Big[&((s_{l+k}^1 \cdots s_{l+k}^{2r_1}) + (q_{l+k}^1 \cdots q_{l+k}^{2r_1})_{\mathrm{O}})(t_{l+k}^{2r_1+1}-1)\\ &\times \prod_{j=0}^{(l+k-2)/2} t_{2j+1}^1 \cdots t_{2j+1}^{2r_{l+k-2j}-2r_{l+k-2j-1}}\\ &\times \prod_{j=1}^{(l+k-2)/2} (t_{2j}^1+1)((s_{2j}^2 \cdots s_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})\\ &\quad + (q_{2j}^2 \cdots q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}})\\ &\quad \times (t_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}}-1)(t_0^1+1)((s_0^2 \cdots s_0^{r_0}) + (q_0^2 \cdots q_0^{r_0})_{\mathrm{O}})\Big],\end{aligned}$$

where all the t-terms are the summation of the corresponding q-terms and s-terms. It is clear that the t-terms are exactly the corresponding p-terms in (8.10).

Now, following the recipe on carrying out the O_{2n^*}-collapse [11, Lemma 6.3.8], we obtain that the partition

$$\left(\left(\prod_{i=1}^{u} p_i^2\right)^t + \left(\left[\left(\prod_{j=1}^{v} q_j^2\right)\prod_{w=1}^{l+k}(2r_w+1)\right]^t\right)_{\mathrm{O}_{2n^*}}\right)_{\mathrm{O}_{2n}}$$

is equal to

$$\begin{aligned}\Big[&((s_{l+k}^1 \cdots s_{l+k}^{2r_1}) + (q_{l+k}^1 \cdots q_{l+k}^{2r_1})_{\mathrm{O}})_{\mathrm{O}}(t_{l+k}^{2r_1+1}-1)\\ &\times \prod_{j=0}^{(l+k-2)/2} t_{2j+1}^1 \cdots t_{2j+1}^{2r_{l+k-2j}-2r_{l+k-2j-1}}\\ &\times \prod_{j=1}^{(l+k-2)/2} (t_{2j}^1+1)((s_{2j}^2 \cdots s_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})\\ &\quad + (q_{2j}^2 \cdots q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}})_{\mathrm{O}}\\ &\quad \times (t_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}}-1)(t_0^1+1)((s_0^2 \cdots s_0^{r_0}) + (q_0^2 \cdots q_0^{r_0})_{\mathrm{O}})_{\mathrm{O}}\Big].\end{aligned}$$

Since s-terms are all even, by [1, Lemma 3.1], we have

$$
\begin{aligned}
&((s_{l+k}^1 \cdots s_{l+k}^{2r_1}) + (q_{l+k}^1 \cdots q_{l+k}^{2r_1})_{\mathrm{O}})_{\mathrm{O}} \\
&\quad = (s_{l+k}^1 \cdots s_{l+k}^{2r_1})_{\mathrm{O}} + (q_{l+k}^1 \cdots q_{l+k}^{2r_1})_{\mathrm{O}} = ((s_{l+k}^1 \cdots s_{l+k}^{2r_1}) + (q_{l+k}^1 \cdots q_{l+k}^{2r_1}))_{\mathrm{O}} \\
&\quad = (p_{l+k}^1 \cdots p_{l+k}^{2r_1})_{\mathrm{O}}, \\
&((s_{2j}^2 \cdots s_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1}) + (q_{2j}^2 \cdots q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}})_{\mathrm{O}} \\
&\quad = (s_{2j}^2 \cdots s_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}} + (q_{2j}^2 \cdots q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}} \\
&\quad = ((s_{2j}^2 \cdots s_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1}) + (q_{2j}^2 \cdots q_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1}))_{\mathrm{O}} \\
&\quad = (p_{2j}^2 \cdots p_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}}, \\
&((s_0^2 \cdots s_0^{r_0}) + (q_0^2 \cdots q_0^{r_0})_{\mathrm{O}})_{\mathrm{O}} \\
&\quad = (s_0^2 \cdots s_0^{r_0})_{O} + (q_0^2 \cdots q_0^{r_0})_{\mathrm{O}} = ((s_0^2 \cdots s_0^{r_0}) + (q_0^2 \cdots q_0^{r_0}))_{\mathrm{O}} = (p_0^2 \cdots p_0^{r_0})_{\mathrm{O}}.
\end{aligned}
$$

Hence, the partition

$$
\Bigg(\Big(\prod_{i=1}^{u} p_i^2\Big)^t + \Bigg(\Bigg[\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\Bigg]^t\Bigg)_{O_{2n^*}}\Bigg)_{\mathrm{O}_{2n}}
$$

is equal to

$$
\begin{aligned}
&\Bigg[(p_{l+k}^1 \cdots p_{l+k}^{2r_1})_{\mathrm{O}}(p_{l+k}^{2r_1+1}-1)\prod_{j=0}^{(l+k-2)/2} p_{2j+1}^1 \cdots p_{2j+1}^{2r_{l+k-2j}-2r_{l+k-2j-1}} \\
&\quad \times \prod_{j=1}^{(l+k-2)/2}(p_{2j}^1+1)(p_{2j}^2 \cdots p_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}-1})_{\mathrm{O}} \\
&\quad \times (p_{2j}^{2r_{l+k+1-2j}-2r_{l+k-2j}}-1)(p_0^1+1)(p_0^2 \cdots p_0^{r_0})_{\mathrm{O}}\Bigg],
\end{aligned}
$$

which is exactly equal to

$$
\Bigg(\Bigg[\Big(\prod_{i=1}^{u} p_i^2\Big)\Big(\prod_{j=1}^{v} q_j^2\Big)\prod_{w=1}^{l+k}(2r_w+1)\Bigg]^t\Bigg)_{\mathrm{O}_{2n}}
$$

by (8.10). Therefore, we have shown (8.9), hence complete the proof of $\mathrm{G}_n = \mathrm{O}_{2n}$-case of Lemma 8.3.

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