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On the geometry of the free factor graph for Aut(F_N)

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Abstract. Let Φ be a pseudo-Anosov diffeomorphism of a compact (possibly non-orientable) surface Σ with one boundary component. We show that if $b \in \pi_1(\Sigma)$ is the boundary word, $\phi \in \mathrm{Aut}(\pi_1(\Sigma))$ is a representative of Φ fixing b, and ad_b denotes conjugation by b, then the orbits of $\langle \phi, \mathrm{ad}_b \rangle \cong \mathbb{Z}^2$ in the graph of free factors of $\pi_1(\Sigma)$ are quasi-isometrically embedded. It follows that for $N \geq 2$ the free factor graph for $\mathrm{Aut}(F_N)$ is not hyperbolic, in contrast to the $\mathrm{Out}(F_N)$ case.

Dedicated to Slava Grigorchuk on his 70th birthday

1. Introduction

The set of nontrivial proper free factors of a free group F_N can be ordered by inclusion and the geometric realization of the resulting poset is the *free factor complex* for $\mathrm{Aut}(F_N)$. This complex was introduced by Hatcher and Vogtmann [12] who proved, in analogy with the Solomon–Tits theorem for the Tits building associated to $\mathrm{GL}(N, \mathbb{Z})$, that this complex has the homotopy type of a wedge of spheres of dimension $N - 2$. Our focus here is on the large-scale geometry of the complex rather than its topology. This geometry is captured entirely by the 1-skeleton, that is, the *free factor graph* $\mathcal{AF}_N$, metrized as a length space with edges of length 1. Thus, the vertices of $\mathcal{AF}_N$ are the nontrivial, proper free factors of F_N, and there is an edge joining A to B if $A < B$ or $B < A$. (When $N = 2$, this definition is modified – see Section 2.)

There is a natural action of $\mathrm{Aut}(F_N)$ on $\mathcal{AF}_N$. The quotient of $\mathcal{AF}_N$ by the subgroup of inner automorphisms is called the *free factor graph for* $\mathrm{Out}(F_N)$. This graph, which is denoted by $\mathcal{OF}_N$, has emerged in recent years as a pivotal object in the study of $\mathrm{Out}(F_N)$. Much of its importance derives from the following fundamental result of Bestvina and Feighn [3].

Theorem A (Bestvina–Feighn [3]). *The free factor graph $\mathcal{OF}_N$ is Gromov-hyperbolic. The fully irreducible elements of* $\mathrm{Out}(F_N)$ *act as loxodromic isometries of $\mathcal{OF}_N$ (i.e., have quasi-isometrically embedded orbits) while every other element has a finite orbit.*

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The natural map $\mathrm{Out}(F_N) \to \mathrm{Isom}(\mathcal{OF}_N)$ is an isomorphism [2], so Theorem A describes all possible actions of cyclic groups on $\mathcal{OF}_N$. As the graph is hyperbolic, we also know that $\mathbb{Z}^2$ cannot act with quasi-isometric orbits. In fact, results in the literature [2, 11] tell us something much stronger.

Theorem B. *If $r \geq 2$, then every faithful action of $\mathbb{Z}^r$ by isometries on $\mathcal{OF}_N$ has a finite orbit.*

Proof. The natural map $\mathrm{Out}(F_N) \to \mathrm{Isom}(\mathcal{OF}_N)$ is an isomorphism [2], so it suffices to prove this theorem for abelian subgroups of $\mathrm{Out}(F_N)$. The centralizers in $\mathrm{Out}(F_N)$ of fully irreducible elements are virtually cyclic, so an abelian subgroup of rank greater than 1 contains no fully irreducible elements. Handel and Mosher [11] proved that if a finitely generated subgroup $G < \mathrm{Out}(F_N)$ contains no fully irreducible elements, then G has a finite orbit in $\mathcal{OF}_N$ (a fact that remains true even if G is not finitely generated [14]). ■

The purpose of this article is to prove results concerning $\mathcal{AF}_N$ that contrast sharply with Theorems A and B. We shall construct actions of $\mathbb{Z}^2$ on $\mathcal{AF}_N$ such that the orbits are free and metrically undistorted. We shall also establish a criterion that tells us certain inner automorphisms act as isometries with quasi-isometrically embedded orbits, just as all fully irreducible automorphisms do. For the moment, however, we are unable to offer a concise classification of all the isometries that have such orbits. (The natural map $\mathrm{Aut}(F_N) \to \mathrm{Isom}(\mathcal{AF}_N)$ is an isomorphism, so, as in the Out case, there ought to be a classification in purely algebraic terms.)

The abelian subgroups $\mathbb{Z}^2 \hookrightarrow \mathrm{Aut}(F_N)$ that we shall focus on are constructed using pseudo-Anosov diffeomorphisms of compact surfaces with one boundary component. Let Σ be such a surface with Euler characteristic $1 - N$. Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of Σ, that is, π_0 of the subgroup of $\mathrm{Diff}(\Sigma)$ that fixes $\partial\Sigma$ pointwise. Then by fixing a basepoint on $\partial\Sigma$, we obtain an identification $\pi_1\Sigma = F_N$ and a monomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Aut}(F_N)$. The Dehn twist in the boundary of Σ is central in $\mathrm{Mod}(\Sigma)$. We are interested in the $\mathbb{Z}^2$ subgroup that this twist generates with any pseudo-Anosov element of $\mathrm{Mod}(\Sigma)$. The Dehn twist acts on $\pi_1\Sigma = F_N$ as an inner automorphism ad_b, where $b \in F_N$ is the boundary loop.

Theorem C. *Let Σ be a compact surface with one boundary component, fix an identification $\pi_1\Sigma = F_N$, let $b \in F_N$ be the homotopy class of the boundary loop, let $\phi \in \mathrm{Aut}(F_N)$ be the automorphism induced by a pseudo-Anosov element of $\mathrm{Mod}(\Sigma)$, let $\Lambda = \langle \phi, \mathrm{ad}_b \rangle < \mathrm{Aut}(F_N)$, and note that $\Lambda \cong \mathbb{Z}^2$. Then, for every $A \in \mathcal{AF}_N$, the orbit map $\lambda \mapsto \lambda(A)$ defines a quasi-isometric embedding $\Lambda \cong \mathbb{Z}^2 \hookrightarrow \mathcal{AF}_N$.*

Corollary D. *For $N \geq 2$, the free factor graph $\mathcal{AF}_N$ is not Gromov-hyperbolic.*

These results should be compared with the work of Hamenstädt [10], who constructed similar quasi-flats in spotted disc complexes of handlebodies and in certain sphere complexes.

A crucial property of the boundary element $b \in F_N$ is that it is *filling* in the sense that it is not contained in any proper free factor of F_N. Given any filling element b, we analyse the action of ad_b on $\mathcal{AF}_N$ with the aim of showing that the orbits of $\langle \mathrm{ad}_b \rangle$ give quasi-isometric embeddings of $\mathbb{Z}$. The key tool in this analysis is the notion of *b-reduced decomposition* for elements $w \in F_N$, which we introduce in Section 3. Using these decompositions, we define an integer-valued invariant $[A]_b$ of a free factor A, and we use this to define a Lipschitz retraction of $\mathcal{AF}_N$ onto the $\langle \mathrm{ad}_b \rangle$-orbit of a vertex V; roughly speaking, this retraction sends the vertex A to $\mathrm{ad}_b^n(V)$ if $n = [A]_b$.

Theorem E. *If an element $b \in F_N$ is not contained in any proper free factor, then there is a Lipschitz retraction of $\mathcal{AF}_N$ onto each orbit of $\langle \mathrm{ad}_b \rangle$. In particular, these orbits are quasi-isometrically embedded.*

In Section 2, we recall classical results of Whitehead and others describing the structure of elements $b \in F_N$ that are not contained in any proper free factor. This structure controls the behaviour of the function $A \mapsto [A]_b$ that is used in Section 3 to prove Theorem E. With Theorem E in hand, one feels that Theorem C ought to follow easily, since orbits of the fully irreducible automorphism ϕ project to quasi-geodesics in $\mathcal{OF}_N$, but the details are slightly delicate; this is explained in Section 4.

The fact that we cannot offer a concise classification of the isometries of $\mathcal{AF}_N$ is symptomatic of the fact that the large-scale geometry of $\mathcal{AF}_N$ is poorly understood. In Section 5, we highlight some of the begging questions in this regard.

2. Background

We assume that the reader is familiar with the rudiments of the theory of free-group automorphisms.

We write F_N to denote the free group on N generators. Let $\mathcal{S}$ be a basis of F_N. Every element $w \in F_N$ is represented by a unique reduced word in the letters $\mathcal{S} \cup \mathcal{S}^{-1}$; the length of this word is denoted by $|w|_{\mathcal{S}}$. We write $=$ to denote equality in F_N and $\equiv$ to denote equality as words. A subgroup $A \leq F_N$ is a *free factor* if it is generated by a subset of a basis, and is a *proper free factor* if $A \neq F_N$.

The natural definition of $\mathcal{AF}_N$ that we gave in the introduction is unsatisfactory in the case $N = 2$, since there are no edges. We remedy this by decreeing that a pair of vertices $\langle u \rangle$ and $\langle v \rangle$ are to be connected by an edge if and only if $\{u, v\}$ is a basis for F_2. Then $\mathcal{OF}_2$ is defined to be the quotient by the action of the inner automorphisms. With these conventions, $\mathcal{OF}_2$ is the Farey graph with the standard action of $\mathrm{Out}(F_2) \cong \mathrm{GL}(2, \mathbb{Z})$.

2.1. Whitehead graphs, word length, and the cut-vertex lemma

An element of F_N is *primitive* if it belongs to some basis, and is *simple* if it belongs to some proper free factor. Every element w has a *cyclic reduction* w_0, obtained by writing $w \equiv w_1 w_0 w_1^{-1}$ with w_1 as long as possible. An element is *cyclically reduced* (with respect to the basis $\mathcal{S}$) if $w = w_0$. We use $\mathrm{Wh}_{\mathcal{S}}(w)$ to denote the cyclic Whitehead graph of w. This is the graph with vertex set $\mathcal{S} \cup \mathcal{S}^{-1}$ that has an edge from x to y^{-1} if xy is a subword of the cyclic reduction w_0 or if x is the last letter of w_0 and y is the first letter of w_0. Importantly for us, the cyclic Whitehead graph keeps track of the turns crossed by the axis of the element w in the Cayley tree determined by $\mathcal{S}$.

A vertex v of a graph is a *cut vertex* if the full subgraph spanned by vertices not equal to v is disconnected. The famous lemma below was stated by Whitehead for primitive elements [21], and the generalization to simple elements was observed by various authors (see, e.g., [15, Proposition 49] or [13, 16, 20]). (For readers who pursue the references: Whitehead uses the word *simple* differently to modern authors – his *simple sets* are subsets of bases.)

Lemma 2.1 (Whitehead's cut-vertex lemma [21]). *Let $N \geq 2$. If w is a simple element of F_N, then for any basis $\mathcal{S}$ of F_N the cyclic Whitehead graph* $\mathrm{Wh}_{\mathcal{S}}(w)$ *contains a cut vertex.*

If the Whitehead graph $\mathrm{Wh}_{\mathcal{S}}(w)$ contains a cut vertex that is not isolated, then there is a *Whitehead automorphism* that reduces the cyclic length of w. Using this, one can prove the following standard proposition, which can be seen as a partial converse to Whitehead's lemma.

Proposition 2.2. *Let $N \geq 2$. If $w \in F_N$ is not contained in a proper free factor and $|w|_{\mathcal{S}} \leq |\phi(w)|_{\mathcal{S}}$ for all $\phi \in \mathrm{Aut}(F_N)$, then* $\mathrm{Wh}_{\mathcal{S}}(w)$ *contains no cut vertex.*

Note that any graph with at least 3 vertices that is disconnected contains a cut vertex. As we are working with $N \geq 2$, all of the Whitehead graphs we consider have at least 4 vertices, and so those without cut vertices are connected.

3. Orbits in $\mathcal{AF}_N$ of filling inner automorphisms

The goal of this section is to prove Theorem E: If $b \in F_N$ is not contained in a proper free factor, then there is a Lipschitz retraction of $\mathcal{AF}_N$ onto each $\langle \mathrm{ad}_b \rangle$-orbit. These retractions will be constructed using *b-reduced decompositions*.

3.1. b-reduced decompositions and a cancellation lemma

Definition 3.1 (b-reduced decomposition). Fix a basis $\mathcal{S}$ of F_N. Given a cyclically reduced word b and a reduced word w representing an element of F_N, the *b-reduced decomposition* of w (with respect to $\mathcal{S}$) is the decomposition

$$w \equiv b^k w_b b^{-k}$$

(equality as reduced words) such that $|k|$ is maximal. Define $[w]_b^{\mathcal{S}} := k \in \mathbb{Z}$. Given a subgroup $A \subset F_N$, define

$$[A]_b^{\mathcal{S}} := \sup\{[a]_b^{\mathcal{S}} : a \in A\},$$

allowing $[A]_b^{\mathcal{S}} = \infty$.

We shall drop the superscript $\mathcal{S}$ from $[w]_b^{\mathcal{S}}$ and $[A]_b^{\mathcal{S}}$ when there is no danger of ambiguity.

Remark 3.2 (Geometric interpretation). Consider the action of F_N on the Cayley tree associated to $\mathcal{S}$. As b is cyclically reduced, it acts as translation through a distance $|b|$ in an axis X_b that passes through the basepoint 1 and contains each vertex b^i. Roughly speaking, $[a]_b$ records (with sign) the time that the path from 1 to the axis X_a of a spends on X_b. More precisely, the orthogonal projection of X_a onto X_b is contained in a minimal interval of the form $[b^i, b^j]$ and $[a]_b$ is i if $i > 0$, is j if $j < 0$, and is 0 if $i \leq 0 \leq j$ – see the proof of Lemma 3.3. Note that the projection of X_a to X_b is a point if $X_a \cap X_b = \emptyset$ and equals $X_a \cap X_b$ otherwise.

The minimal invariant subtree T_A for a subgroup $A \leq F_N$ is the union of the axes X_a with $a \in A$. If $T_A \cap X_b$ is compact, as it will be if A is finitely generated and $A \cap \langle b \rangle = 1$, then either it is empty, in which case the projection of T_A is a point and $[A]_b = [a]_b$ for all $a \in A$, or else $T_A \cap X_b$ is contained in a minimal interval of the form $[b^I, b^J]$ and from the geometric description of $[a]_b$ we have $I \leq [A]_b \leq J$. These observations lead to the philosophically important approximation

$$[A]_b \approx \frac{\operatorname{dist}(\pi_{X_b}(T_A), 1)}{|b|},$$

where π_{X_b} denotes orthogonal projection to X_b and the constants implicit in the approximation depend on the length of $T_A \cap X_b$.

Lemma 3.3 (Cancellation lemma). *Suppose that $b \in F_N$ is not contained in a proper free factor and let $\mathcal{S}$ be a basis of F_N that minimizes $|b|_{\mathcal{S}}$. Suppose that $a \in F_N$ is contained in a proper free factor and $[a]_b^{\mathcal{S}} = 0$. Then the first and last $|b|_{\mathcal{S}} + 1$ letters of $w \equiv b^3 \cdot a \cdot b^{-3}$ remain after reducing w. In particular, $[b^3ab^{-3}]_b^{\mathcal{S}} \geq 1$.*

Proof. As $\mathcal{S}$ minimizes $|b|_{\mathcal{S}}$, in particular b is cyclically reduced. Let T be the Cayley tree for F_N with respect to the basis $\mathcal{S}$. As b is cyclically reduced, the axis X_b of b passes through the identity vertex $1 \in T$. Let X_a be the axis for a. Let p be the shortest path from 1 to X_a. The intersection of p with X_b is of length at most $|b|_{\mathcal{S}} - 1$, as the reduced word representing a starts with the word labelling p and ends with its inverse and $[a]_b = 0$. Furthermore, the intersection of X_a with X_b can have length at most $|b|_{\mathcal{S}}$: To see this, note that the cyclic Whitehead graph $\mathrm{Wh}_{\mathcal{S}}(b)$ does not contain a cut vertex by Proposition 2.2, whereas $\mathrm{Wh}_{\mathcal{S}}(a)$ does by the cut-vertex lemma, so X_a cannot contain all the turns (pairs of consecutive labels from $\mathcal{S} \cup \mathcal{S}^{-1}$) that appear along X_b.

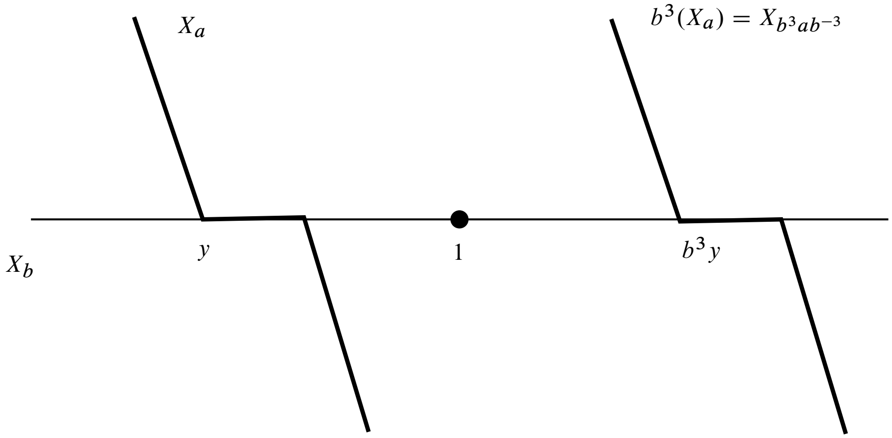

Figure 1. The proof of the cancellation lemma.

As in Figure 1, let y be the furthest point along X_b in the b^{-1} direction that is contained in $p \cup X_a$. By the above, $d(1, y) \leq 2|b|_{\mathcal{S}} - 1$. It follows that the distance from 1 to b^3y (in the positive direction along X_b) is at least $|b|_{\mathcal{S}} + 1$. This implies that the path from 1 to the axis of b^3ab^{-3} contains the interval $[1, b^3y]$. Again, as the path from 1 to the axis gives the initial and terminal (in reverse) subwords of b^3ab^{-3}, the result follows. ∎

Remark 3.4. Consider the orbit $\mathcal{O} = \{\langle b^k ab^{-k}\rangle : k \in \mathbb{Z}\}$ of a free factor $\langle a\rangle$ generated by a primitive element under the action of ad_b. It is not hard to see that the map $f : \mathcal{O} \to \mathbb{Z}$ given by $f(w) = [w]_b$ is injective outside of $f^{-1}(0)$. And under the assumptions of the above lemma, $|f^{-1}(0)| \leq 3$.

Proposition 3.5. *Suppose $b \in F_N$ is cyclically reduced with respect to $\mathcal{S}$ and that the cyclic Whitehead graph $\mathrm{Wh}_{\mathcal{S}}(b)$ is connected with no cut vertex. Let T be the Cayley tree with respect to $\mathcal{S}$ and let X_b be the axis of b in T. If $A \leq F_N$ is a proper free factor and T_A is the minimal subtree of A in T, then*

(1) $|T_A \cap X_b| \leq |b|_{\mathcal{S}}$.

(2) *For any two elements $a_1, a_2 \in A$,*

$$|[a_1]_b^{\mathcal{S}} - [a_2]_b^{\mathcal{S}}| \leq 1.$$

In particular, $[A]_b^{\mathcal{S}} \in \mathbb{Z}$.

Proof. If $T_A \cap X_b = \emptyset$, then X_a has the same projection to X_b for all $a \in A$. In this case, part (1) is trivial and $[a_1]_b^{\mathcal{S}} = [a_2]_b^{\mathcal{S}}$ for all $a_1, a_2, \in A$ by the geometric interpretation (Remark 3.2). Otherwise, the projection T_A to X_b is a compact interval equal to $T_A \cap X_b$. As $T_A \cap X_b$ is compact, there exists $a \in A$ such that the axis of a contains $T_A \cap X_b$ by [17, Lemma 4.3]. As every element of A is simple, as in the preceding proof this would contradict the cut-vertex lemma if $|T_A \cap X_b| > |b|_{\mathcal{S}}$. This proves (1). For (2), it then suffices to show that if $a_1, a_2 \in F_N$ are elements of F_N and the projections of X_{a_1} and X_{a_2} to X_b are contained in an interval of length at most $|b|_{\mathcal{S}}$, then $|[a_1]_b^{\mathcal{S}} - [a_2]_b^{\mathcal{S}}| \leq 1$. This again follows from the geometric interpretation: $T_A \cap X_b$ is contained in $(b^{i-1}, b^{i+1}]$ for some $i < 0$, or contained in $[b^{i-1}, b^{i+1})$ for some $i \geq 0$. In the first case, one can check that $[a]_b^{\mathcal{S}} \in \{i, i+1\}$ for all $a \in A$, and in the second $[a]_b^{\mathcal{S}} \in \{i-1, i\}$ for all $a \in A$. ∎

Theorem 3.6. *Suppose that $b \in F_N$ is not contained in a proper free factor and that $\mathcal{S}$ is a basis that minimizes $|b|_{\mathcal{S}}$. Then the map $\mathcal{AF} \to \mathbb{Z}$ given by $A \mapsto [A]_b^{\mathcal{S}}$ is 1-Lipschitz if $N \geq 3$ and is 2-Lipschitz if $N = 2$.*

Proof. Proposition 2.2 tells us that Proposition 3.5 applies to b, and Proposition 3.5 (2) assures us that if $A < B$ then $|[A]_b^{\mathcal{S}} - [B]_b^{\mathcal{S}}| \leq 1$; this covers the case $N = 3$. For the case $N = 2$, a different argument is needed, because $\mathcal{AF}_N$ is defined differently. In this case, the vertex $\langle a_1 \rangle$ is adjacent to $\langle a_2 \rangle$ if $\{a_1, a_2\}$ is a basis. As in Proposition 3.5 (1), we know that the projection $\pi_{X_b}(X_{a_i})$ of each axis to X_b has length at most $|b|_{\mathcal{S}}$. We claim that, furthermore, if the projections are disjoint then the arc connecting $\pi_{X_b}(X_{a_1})$ to $\pi_{X_b}(X_{a_2})$ is length at most $|b|_{\mathcal{S}}$. This claim clearly gives a uniform bound on the difference between $[a_1]_b$ and $[a_2]_b$, and with a similar case-by-case analysis as in Proposition 3.5, one can check that $[a_1]_b$ and $[a_2]_b$ differ by at most 2.

To prove the claim, if X_{a_1} and X_{a_2} do not intersect, then the bridge between them is contained in the axis for $a_1 a_2$ (see [6, Figure 4]). It follows that if $\pi_{X_b}(X_{a_1})$ and $\pi_{X_b}(X_{a_2})$ do not intersect, then the arc between the projections is contained in $\pi_{X_b}(X_{a_1 a_2})$. Since $a_1 a_2$ is primitive, Proposition 3.5 (1) tells us that $\pi_{X_b}(X_{a_1 a_2})$ has length at most $|b|_{\mathcal{S}}$, and we have proved the claim. ∎

3.2. Change of basis

Proposition 3.7. *Let $b \in F_N$ be an element that is not contained in a proper free factor. Let $\mathcal{S}$ and $\mathcal{T}$ be bases of F_N which minimize the word length of b. Then there exists a constant K such that for all $A \in \mathcal{AF}_N$ we have*

$$[A]_b^{\mathcal{T}} - K \leq [A]_b^{\mathcal{S}} \leq [A]_b^{\mathcal{T}} + K.$$

Proof. Let $T^{\mathcal{S}}$ and $T^{\mathcal{T}}$ be the Cayley trees of F_N given by $\mathcal{S}$ and $\mathcal{T}$, respectively. Let $f : T^{\mathcal{S}} \to T^{\mathcal{T}}$ be the map that sends each edge of $T^{\mathcal{S}}$ to the geodesic in $T^{\mathcal{T}}$ with the same endpoints. Let $1_{\mathcal{S}}$ and $1_{\mathcal{T}}$ be the basepoints of the respective Cayley trees and use $g_{\mathcal{S}} = g.1_{\mathcal{S}}$ and $g_{\mathcal{T}} = g.1_{\mathcal{T}}$ to denote the vertices in the trees given by $g \in F_N$. If $p = [1_{\mathcal{S}}, g_{\mathcal{S}}]$ is a path in $T^{\mathcal{S}}$ from $1_{\mathcal{S}}$ to the axis of an element $a \in F_N$, then the *bounded backtracking property* (see [5, 9]) implies that $g_{\mathcal{T}} = f(g_{\mathcal{S}})$ lies at distance at most C from the axis of a in $T^{\mathcal{T}}$, where C is the bounded backtracking constant. Therefore, as $|b|_{\mathcal{S}} = |b|_{\mathcal{T}}$, the proposition reduces to showing that for any path p based at $1_{\mathcal{S}}$, the reduction of $f(p)$ spends approximately the same amount of time travelling along the axis of b in $T^{\mathcal{T}}$ as the path p does along the axis of b in $T^{\mathcal{S}}$ (in the same direction). This is reasonably standard: Decompose such a path p as $p = p_1 p_2 p_3$, where $p_1 = [1_{\mathcal{S}}, b_{\mathcal{S}}^k]$ for some k, the path p_2 is a segment of length $\leq |b|_{\mathcal{S}}$, and p_3 is a path disjoint from the axis $X_b^{\mathcal{S}} \subset T^{\mathcal{S}}$ except for its initial endpoint. As f is bi-Lipschitz on the vertex set, p_3 can further be decomposed as $p_3' p_3''$, where $f(p_3'')$ is disjoint from $X_b^{\mathcal{T}}$ and p_3' is of uniformly bounded length. Then the intersection of the reduction of $f(p)$ with $X_b^{\mathcal{T}}$ in $T_{\mathcal{T}}$ is $[1_{\mathcal{T}}, b_{\mathcal{T}}^k]$, up to cancellation/concatenation with a subpath of $f(p_2 p_3')$, which is of uniformly bounded length. ■

4. Building quasi-flats

In this section, we will build quasi-flats in $\mathcal{AF}_N$. The first proposition is quite technical: Exceptionally, in the proof of this proposition, we will assume that the reader is familiar with the *stable train tracks* of [4].

Proposition 4.1. *Let Σ be a compact surface with one boundary component, and fix a basepoint on the boundary giving an identification $\pi_1\Sigma = F_N$. Let $b \in F_N$ be the homotopy class of the boundary loop, let $\phi_0 \in \mathrm{Aut}(F_N)$ be the automorphism induced by a pseudo-Anosov element of* Mod(Σ)*, and let $[\phi_0]$ be its outer automorphism class. Then, there exists a basis $\mathcal{S}$ of F_N and a representative ϕ of $[\phi_0]$ fixing b, such that:*

(1) *The basis $\mathcal{S}$ minimizes $|b|_{\mathcal{S}}$.*

(2) *For all $A \in \mathcal{AF}_N$, there exists a constant M_A such that*

$$|[\phi^r \mathrm{ad}_b^k(A)]_b - [\mathrm{ad}_b^k(A)]_b| \leq M_A$$

for all $k, r \in \mathbb{Z}$.

(3) *ϕ is the unique representative of $[\phi_0]$ that fixes b and acts on the Gromov boundary ∂F_N so that the endpoints of the infinite words b^∞ and $b^{-\infty}$ are both non-attracting fixed points.*

Proof. In [4, Sections 1–3], it is shown that an automorphism ϕ_0 as above has a *stable train track* representative $f: G \to G$ on a graph G with exactly one indivisible Nielsen path (iNP) ρ which, as the automorphism is geometric, forms a loop representing b based at a point $x \in G$ that is fixed by f. Furthermore, ρ crosses every edge in G exactly twice (see [4, Lemma 3.9] and the discussion in [4, Section 4]).

We identify $\pi_1(G, x)$ with F_N via a choice of maximal tree T in G together with an orientation of the edges in $G \setminus T$. Each edge-loop based at x determines a word in the letters $\mathcal{S}^{\pm 1}$ whose length is the number of edges of $G \setminus T$ crossed by the loop. In particular, $|b|_{\mathcal{S}} = 2N$. Since b is the boundary of a surface of Euler characteristic $1 - N$, this tells us that $\mathcal{S}$ minimizes the length of b. We define ϕ to be the automorphism induced by $f_*: \pi_1(G, x) \to \pi_1(G, x)$.

We now prove the second part with $r \geq 0$. There exists a legal loop l in G based at x that crosses some edge exactly once (this is a variant of Francaviglia and Martino's *sausage lemma* [8, Lemma 3.14]). Any such loop represents a primitive element in F_N. As in [4, Section 2], we lift f to a map $\tilde{f}\colon \tilde{G} \to \tilde{G}$ on the universal cover that represents ϕ (i.e., $\tilde{f}(gy) = \phi(g)\tilde{f}(y)$ for all $y \in \tilde{G}$). The map $\tilde{f}$ fixes a lift $\tilde{x}$ of our basepoint $x \in G$, and the axis of b (acting as a deck transformation of $\tilde{G}$) contains $\tilde{x}$. As the loop l is based at x, its axis $\mathrm{Axis}_{\tilde{G}}(l)$ in $\tilde{G}$ also contains $\tilde{x}$. As l is legal and $\tilde{x}$ is fixed by $\tilde{f}$, the axis of $\phi^r(l)$ also contains $\tilde{x}$ for all $r \geq 0$. It follows that iterated images of l are cyclically reduced, so that $[\phi^r(l)]_b = 0$ for all $r \geq 0$. For $k \in \mathbb{Z}$, the axis of $\mathrm{ad}_b^k(l)$ is also legal and contains the point $b^k\tilde{x}$, so that $k - 1 \leq [\mathrm{ad}_b^k(l)]_b \leq k + 1$. As $b^k\tilde{x}$ is also fixed under $\tilde{f}$, this implies that $k - 1 \leq [\phi^r \mathrm{ad}_b^k(l)]_b \leq k + 1$ for all $k \in \mathbb{Z}$ and $r \geq 0$. As the map $\mathcal{AF}_N \to \mathbb{Z}$ induced by $[A] \mapsto [A]_b$ is 2-Lipschitz, we have

$$
\begin{aligned}
|[\phi^r \mathrm{ad}_b^k(A)]_b - [\mathrm{ad}_b^k(A)]_b| &\leq |[\phi^r \mathrm{ad}_b^k(A)]_b - [\phi^r \mathrm{ad}_b^k(l)]_b| + |[\phi^r \mathrm{ad}_b^k(l)]_b - [\mathrm{ad}_b^k(l)]_b| \\
&\quad + |[\mathrm{ad}_b^k(l)]_b - [\mathrm{ad}_b^k(A)]_b| \\
&\leq 2d_{\mathcal{AF}_N}(A, \langle l \rangle) + 2 + 2d_{\mathcal{AF}_N}(A, \langle l \rangle) := C_A,
\end{aligned}
$$

for all $r \geq 0$ and $k \in \mathbb{Z}$.

Note that any possible choices for representatives of $[\phi_0]$ fixing b differ by a power of ad_b. If ϕ is the representative chosen above, then as $\tilde{f}(b^n\tilde{x}) = b^n\tilde{x}$ for all $n \in \mathbb{Z}$, both ends of the axis of b in $\tilde{G}$ are non-attracting under the action of ϕ on ∂F_N. For $k \neq 0$, one end of the axis will be attracting and the other will be repelling under the action of $\mathrm{ad}_b^k\phi$ on the boundary. This establishes (3).

To deal with the case where $r < 0$ in (2), we run the entire argument again for $[\phi_0]^{-1}$. This provides a representative ψ of $[\phi_0]^{-1}$ fixing b, such that for all $A \in \mathcal{AF}_N$ there exists a constant C'_A such that $|[\psi^r \mathrm{ad}_b^k(A)]_b - [\mathrm{ad}_b^k(A)]_b| \leq C'_A$ for all $k \in \mathbb{Z}$ and $r \geq 0$. Note that although you might be working with a different basis $\mathcal{S}'$ on which b is minimal length, the projections $\mathcal{AF}_N \to \mathbb{Z}$ are coarsely equivalent by Proposition 3.7. Furthermore, this is the unique representative of $[\phi_0]^{-1}$ fixing b such that b^∞ and $b^{-\infty}$ are non-attracting under $\partial\psi$. This implies that $\psi = \phi^{-1}$, so we can finish part (2) by taking $M_A := \max\{C_A, C'_A\}$. ∎

We finally have all of the tools that we need to prove Theorem C, which we restate for the reader's convenience.

Theorem 4.2. *Let Σ be a compact surface with one boundary component, fix an identification $\pi_1\Sigma = F_N$, let $b \in F_N$ be the homotopy class of the boundary loop, let $\phi \in \mathrm{Aut}(F_N)$ be the automorphism induced by a pseudo-Anosov element of* $\mathrm{Mod}(\Sigma)$*, let $\Lambda = \langle \phi, \mathrm{ad}_b \rangle < \mathrm{Aut}(F_N)$, and note that $\Lambda \cong \mathbb{Z}^2$. Then, for every $A \in \mathcal{AF}_N$, the orbit map $\lambda \mapsto \lambda(A)$ defines a quasi-isometric embedding $\Lambda \cong \mathbb{Z}^2 \hookrightarrow \mathcal{AF}_N$.*

Proof. Note that it is enough to prove the result for one orbit. We may also replace ϕ with $\mathrm{ad}_b^k \phi$ without changing Λ, so we can assume that ϕ and the basis $\mathcal{S}$ are as described in Proposition 4.1. Let $s \in \mathcal{S}$ and let $A = \langle s \rangle$. Then $k - 1 \leq [\mathrm{ad}_b^k(A)]_b^{\mathcal{S}} \leq k + 1$ for all $k \in \mathbb{Z}$. We define a map $\mathcal{OF}_N \to \mathbb{Z}$ as follows: For each conjugacy class $[B]$ of a free factor, we pick a closest point $[\phi]^j([A])$ in the $[\phi]$-orbit of $[A]$. This assignment $[B] \mapsto j$ is Lipschitz [3], and by composing it with the natural projection $\mathcal{AF}_N \to \mathcal{OF}_N$, we obtain a Lipschitz map $R\colon \mathcal{AF}_N \to \mathbb{Z}$ which is invariant under $\mathrm{Inn}(F_N)$. It follows from Theorem 3.6 that the map $[\cdot]_b^{\mathcal{S}} \times R\colon \mathcal{AF}_N \to \mathbb{Z}^2$ is also Lipschitz, and we claim that $B \mapsto \phi^{R(B)} \mathrm{ad}_b^{[B]_b}(A)$ gives a Lipschitz retraction of $\mathcal{AF}_N$ onto the A-orbit of $\Lambda = \langle \phi, \mathrm{ad}_b \rangle$. To see this, note that $R(\phi^j \mathrm{ad}_b^k(A)) = j$ for all $j, k \in \mathbb{Z}$, and

$$k - 1 - M_A \leq [\phi^j \mathrm{ad}_b^k(A)]_b \leq k + 1 + M_A,$$

where M_A is the constant provided by Proposition 4.1. ■

Corollary 4.3. *For $N \geq 2$, the* Aut *free factor complex $\mathcal{AF}_N$ is not Gromov-hyperbolic.*

Proof. This follows immediately from the theorem and the fact that for every $N \geq 2$ there is a compact surface with one boundary component that has Euler characteristic $1 - N$ and admits pseudo-Anosov diffeomorphisms. For N even, this is an orientable surface of genus $N/2$, while for N odd we must take the connected sum of an orientable surface of genus $(N-1)/2$ with a projective plane [18]. ■

5. Problem list

We end the paper with a list of open problems.

Our results extend the list of automorphisms that are known to have undistorted orbits in $\mathcal{AF}_N$: Bestvina and Feighn [3] proved that this is true for fully irreducible automorphisms, and we have added inner automorphisms ad_b given by filling elements.

Question 1. Which cyclic subgroups of $\mathrm{Aut}(F_N)$ have undistorted orbits in $\mathcal{AF}_N$? Is every orbit of a cyclic subgroup either finite or undistorted?

One can ask about other abelian subgroups of $\mathrm{Aut}(F_N)$, or more ambitiously, about more general quasi-flats.

Question 2. Which abelian subgroups of $\mathrm{Aut}(F_N)$ have undistorted orbits in $\mathcal{AF}_N$?

Question 3. Are there any quasi-flats of rank 3, that is (possibly non-equivariant) quasi-isometric embeddings of $\mathbb{Z}^3$ into $\mathcal{AF}_N$?

We find it unlikely that there should be an action of $\mathbb{Z}^3$ with quasi-isometric orbits. The image of any $\mathbb{Z}^3 \cong \Lambda < \mathrm{Aut}(F_N)$ in $\mathrm{Out}(F_N)$ contains a copy of $\mathbb{Z}^2$; as we saw in the introduction, this implies that Λ has a finite orbit in $\mathcal{OF}_N$, and therefore a subgroup of finite index in Λ fixes some $[A] \in \mathcal{OF}_N$. So if there were a free action of $\mathbb{Z}^3$ on $\mathcal{AF}_N$ with quasi-isometrically embedded orbits, then there would be such an orbit in the fibre over a point in $\mathcal{OF}_N$. It does seem possible, however, that there are $\mathbb{Z}^2$ subgroups with undistorted orbits in such a fibre: Candidates can be obtained by replacing the pseudo-Anosov ϕ in Theorem C with a partial pseudo-Anosov supported on a subsurface containing the boundary of Σ. In this case, $\langle \phi, \mathrm{ad}_b \rangle \cong \mathbb{Z}^2$ does not fix an obvious free factor, but does fix the conjugacy class of a free factor. It is not clear whether the orbits of such subgroups are undistorted or not.

The quasi-flats in $\mathcal{AF}_N$ constructed in this paper come in families where each pair of quasi-flats coarsely share a common line: If b is the boundary word for a surface Σ, then independent pseudo-Anosov homeomorphisms of Σ will give distinct quasi-flats, each pair of which will have bounded neighbourhoods whose intersection is a quasi-line that is a bounded neighbourhood of an $\langle \mathrm{ad}_b \rangle$-orbit. This observation allows us to construct many more non-equivariant quasi-isometric embeddings $\mathbb{Z}^2 \hookrightarrow \mathcal{AF}_N$ by piecing together half-flats and parallel strips from the various equivariant flats coarsely containing $\langle \mathrm{ad}_b \rangle$-orbits. This pattern of intersections shows that the families of quasi-flats that we currently have cannot be used to define a relative hyperbolic structure on $\mathcal{AF}_N$. (We refer to [19] for the definition of such a structure.) But our lack of knowledge about quasi-flats in general means that we cannot answer the following question.

Question 4. Is $\mathcal{AF}_N$ relatively hyperbolic? If not, is it thick in the sense of [1]?

We have seen that if a compact surface Σ has one boundary component, then every isomorphism $\pi_1(\Sigma) \cong F_N$ gives rise to a family of quasi-flats that coarsely intersect in a quasi-line. If such families account for all quasi-flats in $\mathcal{AF}_N$, then, in the spirit of Dowdall and Taylor's *co-surface graph* [7], one might hope to obtain a hyperbolic space by coning them off.

Definition 5.1 (An Aut version of the co-surface graph). Let $\mathcal{AC}_N$ be the graph obtained from $\mathcal{AF}_N$ by adding a new edge between two points A and B if there exists a surface Σ with one boundary component such that both A and B are tethered subsurface subgroups of Σ (subsurfaces with an embedded arc to the basepoint, which is on the boundary).

Dowdall and Taylor proved that the co-surface graph of $\mathrm{Out}(F_N)$ is hyperbolic. Following their lead, we can ask the same question about the Aut version.

Question 5. Is $\mathcal{AC}_N$ Gromov-hyperbolic?

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