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Substitutional subshifts and growth of groups

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Abstract. We show how to use symbolic dynamics of Schreier graphs to embed the Grigorchuk group into a simple torsion group of intermediate growth and to construct a continuum of growth types of simple torsion groups.

To Rostyslav Ivanovych on the occasion of his 70th birthday

1. Introduction

The paper is a continuation of [15], where first examples of simple groups of intermediate growth were constructed. It has two main goals: illustrating flexibility of the techniques of [15, 16] and describing a different approach for defining and studying groups introduced there. We use explicit symbolic substitutional systems and Schreier graphs rather than generate groups by homeomorphisms of the Cantor set. The difference is mostly formal (as there are standard ways of transforming one type of definitions into the other), but one approach may be more convenient in some situations than the other.

As the first illustration, we show in Section 3 how to embed the Grigorchuk group [8] into a simple torsion group of intermediate growth. A procedure of constructing a finitely generated simple group from an expansive minimal action on a Cantor set is described in [16]. The standard action of the Grigorchuk group on the Cantor set is equicontinuous (hence not expansive) which is related to the fact that the Grigorchuk group is residually finite. However, we can easily transform an action into an expansive one by "exploding" points. This is, for example, the way how equicontinuous action of $\mathbb{Z}$ on the circle by irrational rotation is transformed into an expansive *Denjoy system* (see [7, 17]). One can perform the same trick with the Grigorchuk group and get a simple finitely generated group containing the Grigorchuk group and sharing with it many finiteness properties.

We describe this construction in a symbolic way by describing how to construct the Schreier graphs of the new group by substitutions. In fact, the Schreier graphs of the virtually simple group containing the Grigorchuk group will coincide with the graphs of the

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action of the Grigorchuk group on the Cantor set, except that one of the generators (usually denoted by a in the literature) will be split into three elements with disjoint supports, so that each edge labeled by a for the Grigorchuk group will be relabeled by one of three different letters a_0, a_1, a_2.

We give a direct proof (Theorem 3.5) that the new group is virtually simple just by studying the Schreier graphs. The fact that the group is torsion and of intermediate growth will follow from the results of [15].

L. Bartholdi and A. Erschler have proved in [1] that every group of locally subexponential growth can be embedded into a 2-generated group of sub-exponential growth. It would be interesting to know if every such group can be embedded into a simple group of intermediate growth.

Section 4 contains our second result, showing that there is a continuum of pairwise different growth types among finitely generated simple groups (Theorem 4.4). We construct a family of groups G_α generated by 6 elements such that each group in the family has a simple subgroup of index 16, and there is a continuum of growth types of groups in the family. The idea of the proof is similar to the analogous result of R. Grigorchuk in [10] for the class of finitely generated groups. But it has to use a different approach to defining groups, since the approach in [10] is based on defining group actions on rooted trees, which necessarily leads to residually finite groups.

Our result is the first explicit example of uncountably many growth types of simple groups. It was shown in [14] that there is a continuum of pairwise non-quasi-isometric finitely generated groups that are simple (among many other properties). Examples of infinite sets of pairwise non-quasi-isometric simple finitely presented groups are constructed in [5, 18].

2. Groups defined by Schreier graphs

2.1. Graphs of actions

Let S be a finite set. An *S-labeled directed graph* Γ is given by the *set of vertices* V, the *set of edges (arrows)* E, the *source and range* maps $\mathsf{s}, \mathsf{r} : E \to V$, and a *labeling map* $\lambda : E \to S$. We will usually drop the word "directed" when talking about S-labeled graphs.

The graph is *perfectly labeled* if for every $v \in V$ and every $h \in S$ there exists a unique $e_1 \in E$ such that $\mathsf{s}(e_1) = v$ and $\lambda(e_1) = h$, and a unique $e_2 \in E$ such that $\mathsf{r}(e_2) = v$ and $\lambda(e_2) = h$.

Note that a *morphism* between two S-labeled graphs $\Gamma_1 = (V_1, E_1, \mathsf{s}, \mathsf{r}, \lambda)$ and $\Gamma_2 = (V_2, E_2, \mathsf{s}, \mathsf{r}, \lambda)$ is a pair of maps $\phi : V_1 \to V_2, \phi : E_1 \to E_2$ such that $\mathsf{s}(\phi(e)) = \phi(\mathsf{s}(e))$, $\mathsf{r}(\phi(e)) = \phi(\mathsf{r}(e))$, and $\lambda(\phi(e)) = \lambda(e)$ for every $e \in E_1$. The morphism is an *isomorphism* if the maps $\phi : V_1 \to V_2$ and $\phi : E_1 \to E_2$ are bijections. A morphism is a *covering* if it is surjective, and for every $v \in V_1$ the map ϕ induces a bijection from $\mathsf{s}^{-1}(v)$ to $\mathsf{s}^{-1}(\phi(v))$ and from $\mathsf{r}^{-1}(v)$ to $\mathsf{r}^{-1}(\phi(v))$.

Note that if Γ_1 and Γ_2 are perfectly labeled and (weakly) connected, then every morphism between them is a covering. Moreover, for any two perfectly labeled connected graphs Γ_1 and Γ_2 and every pair of vertices $v_i \in \Gamma_i$, there exists at most one morphism $\phi : \Gamma_1 \to \Gamma_2$ such that $\phi(v_1) = v_2$. All graphs considered in our paper will be connected.

All our actions are from the left.

If Γ is perfectly labeled, then every $h \in S$ defines a permutation $V \to V$ by the condition $h(v) = \mathsf{r}(e)$, where e is the unique edge such that $\mathsf{s}(e) = v$ and $\lambda(e) = h$. The group generated by these permutations is called the *group defined by* Γ. We usually identify the elements of S with the corresponding permutations of V. Conversely, for every group generated by a finite set S of permutations of a set V, we can define the corresponding *graph of the action* with the set of vertices V, set of edges $E = S \times V$, and maps $\mathsf{s}(s, v) = v$, $\mathsf{r}(s, v) = s(v)$, $\lambda(s, v) = s$.

For a graph Γ, the combinatorial distance between two vertices v_1, v_2 is the smallest number of edges in a sequence $e_1, e_2, \dots, e_n$ such that $v_1 \in \{\mathsf{s}(e_1), \mathsf{r}(e_1)\}$, $v_2 \in \{\mathsf{s}(e_n), \mathsf{r}(e_n)\}$, and $\{\mathsf{s}(e_i), \mathsf{r}(e_i)\} \cap \{\mathsf{s}(e_{i+1}), \mathsf{r}(e_{i+1})\} \neq \emptyset$. In other words, it is the length of the shortest path connecting v_1 to v_2 if we disregard the orientation of the edges.

A *rooted graph* is a graph with a marked vertex called the *root*. A morphism of two rooted graphs is a morphism mapping the root to the root. We have already mentioned that two rooted perfectly labeled graphs have at most one morphism between them.

Let $\mathcal{G}_S$ be the set of *rooted* perfectly S-labeled connected graphs. Denote by $B_v(R)$ the ball of radius R with center v. Introduce a metric on $\mathcal{G}_S$ by defining the distance between two rooted graphs (Γ_1, v_1), (Γ_2, v_2) to be 2^{-R}, where R is the largest radius such that the rooted graphs $(B_{v_1}(R), v_1)$ and $(B_{v_2}(R), v_2)$ are isomorphic (as rooted S-labeled directed graphs). This metric induces a natural topology on $\mathcal{G}_S$. An equivalent definition of $\mathcal{G}_S$ is as the Chabauty space of all subgroups of the free group F_S generated by S. Namely, a rooted perfectly S-labeled graph (Γ, v) is in a bijective correspondence with the subgroup of F_S consisting of products $s_1 s_2 \cdots s_n \in F_S$ such that the corresponding product $s_1 s_2 \cdots s_n$ of permutations of V fixes v. (The graph (Γ, v) is reconstructed as the *Schreier graph* of the cosets modulo the subgroup.) The topology on $\mathcal{G}_S$ is then the same as the topology on the space of subgroups of F_S induced from the direct product topology on the set 2^{F_S} of subsets of F_S (see, e.g., [6]).

Suppose that G is the group defined by a perfectly S-labeled connected graph Γ. If Δ is another perfectly S-labeled connected graph such that there is a morphism $\phi : \Gamma \to \Delta$, then the action of G on Γ pushes forward by ϕ to an action on Δ. In particular, the group generated by Δ is a quotient of the group generated by Γ (and the corresponding epimorphism is induced by the tautological map on S).

The hull of a perfectly labeled graph Γ is the closure of the set $\{(\Gamma, v) : v \in V\}$ in the space $\mathcal{G}_S$, where V is the set of vertices of Γ. It is easy to see that if a graph Δ is an element of the hull of a graph Γ, then the group defined by Δ is a quotient of the group defined by Γ (also with the epimorphism induced by the tautological map on S). In particular, if Γ_1 and Γ_2 have the same hulls, then they define the same groups.

Another easy observation, which we will often use, is that if $g \in G$ is an element of length n (with respect to the generating set S) of the group defined by Γ, then the image $g(v)$ of a vertex $v \in \Gamma$ belongs to $B_v(n)$ and depends only on the isomorphism class of the rooted graph $(B_v(n), v)$. Namely, if $g = h_1 h_2 \cdots h_n$ for $h_i \in S \cup S^{-1}$, then $g(v)$ is the end of the unique sequence $v = v_{n+1}, v_n, v_{n-1}, \ldots, v_1$ of vertices of $(B_v(n), v)$, where for every i, if $h_i \in S$, then v_i is the end of the arrow labeled by h_i starting in v_{i+1}, and if $h_i \in S^{-1}$, then v_i is the beginning of the arrow labeled by h_i^{-1} ending in v_{i+1}.

2.2. Linear graphs

Most of the graphs defining group in our paper will be of a very special form.

We say that a bi-infinite sequence $w = (A_n)_{n\in\mathbb{Z}}$ of non-empty subsets of S is *admissible* if $A_n \cap A_{n+1} = \emptyset$ for every $n \in \mathbb{Z}$. We imagine such a sequence as a graph with the set of vertices $\mathbb{Z}$ in which for every n there are $|A_n|$ edges connecting n to $n+1$ labeled by the elements of A_n. We also add loops at n labeled by the elements of $S \setminus (A_n \cup A_{n-1})$. Then the graph Γ_w associated with the sequence w is perfectly S-labeled. Note that all the edges are not directed in our case. (More precisely, we assume that $h = h^{-1}$ for every $h \in S$, so that each edge e is equal to its inverse.)

We also consider finite or one-sided infinite sequences satisfying the same condition $A_n \cap A_{n+1} = \emptyset$. We usually call such sequences *words* or *segments* (imagining them in the latter case as subgraphs of a graph Γ_w defined by an infinite sequence $w = (A_n)_{n\in\mathbb{Z}}$).

For a segment $I = (A_1, A_2, \ldots, A_n)$, the segment I^{-1} is the segment $(A_n, \ldots, A_2, A_1)$ obtained by writing the word in the opposite order. A *sub-segment* (or a *subword*) of I is a segment of the form $(A_i, A_{i+1}, \ldots, A_{i+k})$.

For a set $\mathcal{W}$ of finite or infinite (one-sided or two-sided) admissible sequences, we define the *subshift generated by* $\mathcal{W}$ as the set $\mathcal{S}_{\mathcal{W}}$ of all sequences $(A_n)_{n\in\mathbb{Z}}$ such that every finite segment $(A_i, A_{i+1}, \ldots, A_{i+k})$ is a segment of an element of $\mathcal{W}$.

Suppose that w is an admissible sequence. Then the action of an element $h \in S$ on the set of vertices $\mathbb{Z}$ of Γ_w is given by

$$h(n) = \begin{cases} n+1 & \text{if } h \in A_n, \\ n-1 & \text{if } h \in A_{n-1}, \\ n & \text{otherwise.} \end{cases}$$

Note that the element h has order 2 if $h \in \bigcup_{n\in\mathbb{Z}} A_n$ and 1 otherwise. Denote by G_w the group defined by the corresponding perfectly labeled graph, that is, the group of permutations of $\mathbb{Z}$ generated by the permutations defined by the above formula. It is a subgroup of the *wobbling group* on $\mathbb{Z}$, see [12], that is, the group of permutations $\alpha : \mathbb{Z} \to \mathbb{Z}$ such that $\sup|\alpha(n) - n| < \infty$.

We assume that Γ_w is a rooted graph with the root $0 \in \mathbb{Z}$. The hull of Γ_w can be described as the set of rooted graphs Γ_u for all sequences u belonging to the subshift generated by w and by the sequence w^{-1} written in the opposite direction, that is, to the set of all sequences u such that every finite subword of u is a subword of w or of w^{-1}. Equivalently, it is the closure of the union of the orbits of w and w^{-1} under the two-sided shift. If the sets of finite subwords of w and of w^{-1} are equal, then the hull is equal to the subshift generated by w. (This will be the case in all our examples.)

Example 2.1. Take $S = \{a, b\}$, and let $w = (\cdots\{a\}\{b\}\{a\}\{b\}\cdots)$. Then G_w is the infinite dihedral group. The subshift generated by w has two elements.

Example 2.2. Consider $S = \{a, b, c\}$ and the Markov chain $w = (\{s_n\})_{n\in\mathbb{Z}}$, where s_i is chosen at random with probability 1/2 from the two possibilities in $S \setminus \{s_{i-1}\}$. Then G_w is isomorphic with probability 1 to the free product $C_2 * C_2 * C_2$, since every element of the free product will appear as a subword of w with probability 1. In other words, with probability 1 the subshift generated by w is the set of all sequences in the Markov chain.

On the other hand, the group G_w is amenable for a wide class of sequences w.

Definition 2.3. Note that a sequence $(a_n)_{n\in\mathbb{Z}}$ is *repetitive* if for every finite segment $(a_i, a_{i+1}, \ldots, a_{i+n})$ there exists a constant C_n such that for every $k \in \mathbb{Z}$ there exists $j \in \mathbb{Z}$ such that $|j - k| < C_n$ and $(a_i, a_{i+1}, \ldots, a_{i+n}) = (a_j, a_{j+1}, \ldots, a_{j+n})$. We say that it is *linearly repetitive* if there exists a constant L such that $C_n \le Ln$ for all $n \ge 1$.

In other words, a sequence is repetitive if every finite subword appears in it infinitely often with bounded gaps between consecutive appearances. If the length of the gaps is bounded by a linear function of the length of the subword, then the sequence is called linearly repetitive.

A subshift (i.e., a closed shift-invariant subset of the space of sequences) is called *minimal* if the orbit of each of its elements is dense. It is a classical fact that a sequence is repetitive if and only if it generates a *minimal subshift*.

We say that a subshift $\mathcal{S}_w$ is linearly repetitive, if w is linearly repetitive. It is easy to check that this does not depend on the choice of a particular sequence w generating the subshift.

The following is a direct corollary of the main result of [12].

Theorem 2.4. *If $w = (A_n)_{n\in\mathbb{Z}} \in (2^S)^{\mathbb{Z}}$ is a repetitive admissible sequence, then G_w is amenable.*

2.3. Substitutions

We will construct admissible sequences of non-empty subsets of S using the following construction.

We start with a set X_0 of *initial segments*, a set K of segments called *connectors*, and define inductively sets X_n of the nth *generation segments* by representing each element

$x \in X_n$ as a concatenation $x = x_1^{\varepsilon_1} e_1 x_2^{\varepsilon_2} e_2 \cdots e_{k-1} x_k^{\varepsilon_k}$, where each x_i is an element of a set X_m for $m < n$ (where m may depend on x_i), $e_i \in K$, and ε_i is either nothing or -1, denoting the operation of writing a sequence in the opposite direction. The number k and the elements x_i, e_i, ε_i depend on x. A representation of x as a concatenation of previously defined segments is called sometimes a *substitution*.

The subshift generated by the sequence $(X_n)_{n \geq 0}$ of sets of segments is defined as the set of bi-infinite graphs Γ_w such that every segment of Γ_w is a sub-segment of a segment belonging to X_n for some n. If the subshift is minimal (i.e., if every graph Γ_w belonging to it is repetitive), then the group defined by any element Γ_w of the subshift does not depend on the choice of Γ_w.

Example 2.5. Let $S = \{a, b, c, d\}$. Choose a single initial segment $\{a\}$. Let the set of connectors be $K = \{\{b, c\}, \{b, d\}, \{c, d\}\}$. Define the sets $X_n = \{I_n\}$ consisting of single segments I_n by the following substitution rule:

$$I_{n+1} = \begin{cases} I_n\{b, c\}I_n & \text{if } n \equiv 1 \pmod 3, \\ I_n\{b, d\}I_n & \text{if } n \equiv 2 \pmod 3, \\ I_n\{c, d\}I_n & \text{if } n \equiv 3 \pmod 3. \end{cases}$$

The group defined by this sequence (i.e., by any graph Γ such that every segment of Γ is a sub-segment of one of the segments I_n) is the Grigorchuk group [8]. See more about this example in Section 3.

Example 2.6. Take $S = \{a_i, b_i, c_i, d_i : i = 0, 1, 2, 3\}$, the initial segments I_0, I_1 consisting of single vertices (i.e., empty words), connectors $e_0 = \{a_0, b_0, c_0\}$, $e_1 = \{a_1, b_2, c_1\}$, $e_2 = \{a_2, b_2, c_2\}$, and e_{3k+i} for $k > 0$ equal to $\{b_i, c_i\}$ for $k \equiv 0 \pmod 3$, $\{b_i, d_i\}$ for $k \equiv 1 \pmod 3$, $\{c_i, d_i\}$ for $k \equiv 2 \pmod 3$. Define the segments I_n for $n \geq 2$ by the substitution:

$$I_n = I_{n-2}^{-1} e_{n-2} I_{n-1}^{-1}.$$

The group defined by these segments is the group of intermediate growth analyzed in [15, Section 8].

The following two propositions are classical (see, e.g., [4, Proposition 1.4.6]).

Proposition 2.7. *Let K be a finite set of segments, and let $(X_n)_{n \geq 0}$ be a sequence of finite sets of segments such that each segment $x \in X_{n+1}$ is of the form $x_1 e_1 x_2 e_2 \cdots e_{m_x - 1} x_{m_x}$ for $x_i \in X_n$ and $e_i \in K$. Suppose that for every n there exists $m \geq n$ such that every segment $z \in X_m$ contains every segment $x \in X_n$. Then the subshift $\mathcal{S}$ defined by the set $\bigcup_n X_n$ is minimal.*

Proof. If I is any sub-segment of an element of $\mathcal{S}$, then, by definition of $\mathcal{S}$, it is a sub-segment of a segment $x \in X_n$ for some n. Let $m \geq n$ be such that every segment $y \in X_m$

contains x. By definition, every element of $\mathcal{S}$ is of the form $\cdots e_{-1}x_0e_1x_1e_2\cdots$ for some sequences $x_i \in X_m$ and $e_i \in K$. Then the distance between any two consecutive copies of x in every $w \in \mathcal{S}$ will be less than twice the maximum of length of elements of X_m plus the maximum of lengths of elements of K. It follows that every $w \in \mathcal{S}$ is repetitive. ■

Proposition 2.8. *Suppose that in the conditions of Proposition 2.7, the numbers $|m-n|$ and m_x are uniformly bounded. Then the subshift generated by the union of the sets X_n is linearly repetitive.*

Proof. Note that after replacing X_n by the $X_n \cup X_nK$ (where K is the set of connectors), we may assume that the segments in X_{n+1} are obtained just by concatenation of the elements of X_n.

After passing to a subsequence of $(X_n)_{n\geq 0}$, we may assume that $m = n+1$. Let L_n be the maximal length of an element of X_n. Then the length of every element of X_{n+1} is at least L_n and at most CL_n for some fixed C not depending on n, since $|m-n|$ and m_x are uniformly bounded. It follows that the ratio of lengths of any two elements of $X_{n+1} \cup X_n$ belongs to an interval (C_2^{-1}, C_2) for some constant $C_2 > 1$.

Let v be a sub-segment of an element of the subshift $\mathcal{S}$ generated by the sequence $(X_n)_{n\geq 0}$. Let n be the smallest number such that v is a sub-segment of a segment x_1x_2 for $x_1, x_2 \in X_n$ such that x_1x_2 is a segment of an element of $\mathcal{S}$. Then the length of v is not greater than $2L_n$. On the other hand, the length of v cannot be smaller than the length of the shortest element of X_{n-1}. It follows that the ratio of the lengths of v and any element of X_n belongs to an interval (C_3^{-1}, C_3) for a constant $C_3 > 1$ not depending on v and n. Then the second condition of the proposition implies that the gaps between isomorphic copies of v in an element of $\mathcal{S}$ have length bounded above by $C_4|v|$ for a constant C_4 not depending on v. ■

2.4. Groups of intermediate growth

If G is a group generated by a finite set S, then the associated *growth function* is the number $\gamma_G(n) = \gamma_{G,S}(n)$ of elements of G that can be written as products $s_1s_2\cdots s_k$ for $s_i \in S \cup S^{-1}$ and $k \leq n$. The growth *rate* is the equivalence class of $\gamma_{G,S}(n)$ with respect to the equivalence relation identifying two non-decreasing functions f_1, f_2 if there exists a constant $C > 1$ such that

$$f_1(n) \leq f_2(Cn) \quad \text{and} \quad f_2(n) \leq f_1(Cn)$$

for all $n \geq 1$. The growth rate does not depend on the choice of the generating set.

Theorem 2.9. *Let $\mathcal{S} \subset (2^S)^{\mathbb{Z}}$ be a linearly repetitive infinite subshift consisting of admissible sequences and containing three sequences $(B_n)_{n\in\mathbb{Z}}$, $(C_n)_{n\in\mathbb{Z}}$, $(D_n)_{n\in\mathbb{Z}}$ such that $B_n = B_{-n} = C_n = C_{-n} = D_n = D_{-n}$ for all $n \geq 1$ and $B_0 = \{c,d\}, C_0 = \{b,d\}, D_0 = \{b,c\}$ for some pairwise distinct $b, c, d \in S$.*

Then $G_{\mathcal{S}}$ is an infinite torsion group and its growth function $\gamma(R)$ is bounded from above by e^{Cn^α} for some $C > 1$ and $0 < \alpha < 1$.

Proof. The proof is the same as the proof of Theorem 6.6 of [15], where only linear repetitivity and the structure of the graphs $(B_n)_{n\in\mathbb{Z}}$, $(C_n)_{n\in\mathbb{Z}}$, and $(D_n)_{n\in\mathbb{Z}}$ and their smallest common covering graph Ξ are used. The growth estimate, improved compared to the original one, is proved in [3]. ∎

Example 2.10. Theorem 2.9 is applicable to the first Grigorchuk group, defined in Example 2.5, since the corresponding graphs are linearly repetitive by Proposition 2.8, and the corresponding subshift contains $I_\infty^{-1}\{b,c\}I_\infty$, $I_\infty^{-1}\{d,b\}I_\infty$, and $I_\infty^{-1}\{c,d\}I_\infty$, where $I_\infty = \{a\}\{b,c\}\{a\}\{d,b\}\{a\}\{b,c\}\{a\}\cdots$ is the inductive limit of the segments I_n with respect to the embeddings of I_n to the suffix of I_{n+1}. We see that Theorem 2.9 is a generalization of the main result of [9].

Example 2.11. The same arguments show that the group defined in Example 2.6 is also of intermediate growth.

3. Embedding Grigorchuk group into a simple group

The *first Grigorchuk group* $\mathcal{G}$ is the group generated by the transformations of $\{0,1\}^\infty$ generated by the following recurrently defined permutations a, b, c, d:

$$\begin{aligned} a(0w) &= 1w, & a(1w) &= 0w,\\ b(0w) &= 0a(w), & b(1w) &= 1c(w),\\ c(0w) &= 0a(w), & b(1w) &= 1d(w),\\ d(0w) &= 0w, & d(1w) &= 1b(w). \end{aligned}$$

See an equivalent definition in [8].

The following recurrent description of the orbital graphs of the action of $\mathcal{G}$ on $\{0,1\}^\infty$ is well known (see [2]). The precise description of the space of orbital graphs is from [19].

Proposition 3.1. *Let $\mathcal{S}$ be the subshift generated by the following segments:*

$$I_1 = \{a\}, \quad I_{n+1} = I_n e_n I_n,$$

where

$$e_n = \begin{cases} \{b,c\} & \text{for } n \equiv 1 \pmod 3,\\ \{b,d\} & \text{for } n \equiv 2 \pmod 3,\\ \{c,d\} & \text{for } n \equiv 3 \pmod 3. \end{cases}$$

Then the group defined by $\mathcal{S}$ is isomorphic to the Grigorchuk group $\mathcal{G}$ (for the same generators a, b, c, d).

There is a surjective equivariant continuous map $\Phi : \mathcal{S} \to \{0, 1\}^\infty$, which is one-to-one except for the set of graphs isomorphic (as non-rooted graphs) to one of the three graphs $I_\infty^{-1} e_n I_\infty$, where I_∞ is the direct limit of the segments I_n with respect to the embedding of I_n into the left half of I_{n+1}. The map Φ maps all three rooted graphs $I_\infty^{-1} e_n I_\infty$ to $111\cdots$, so that Φ is three-to-one on the exceptional set.

The orbital graph of $111\cdots$ is one-ended chain described by the sequence I_∞.

We see that Example 2.5 defines the orbital graphs of the Grigorchuk group. We refer the readers to [19] for the proof of the proposition. Here, we will only describe the map Φ by labeling the vertices of I_n. Define I_1 as the graph $1 \overset{a}{\text{———}} 0$. Define then

$$I_{n+1} = (I_n 1) \overset{e_n}{\text{———}} (I_n 0)^{-1},$$

where $I_n 1$ and $I_n 0$ are obtained from I_n by appending to the end of the names of their vertices symbols 1 and 0, respectively. As usual, I^{-1} denotes reverting the orientation of a segment I (and keeping the labels of the vertices and edges).

The new vertex-labeled graphs I_n are isomorphic to the graphs I_n from the proposition. (Note that I_n in the proposition are symmetric.)

It is checked by induction that the left endpoint of I_n is $\underbrace{11\cdots1}_{n \text{ times}}$, its right endpoint is $\underbrace{11\cdots1}_{n-1 \text{ times}}0$, and if v and u are two edges of I_n connected by an edge labeled by a generator s, then $s(vw) = uw$ for all $w \in \{0, 1\}^\infty$. The map Φ is then the limit of the described labeling.

The Grigorchuk group also acts on the set $\{0, 1\}^*$ of finite words (using the same definition as for the infinite sequences). The action preserves the structure of a rooted tree on $\{0, 1\}^*$ (where a vertex v is connected to vx for every $x \in \{0, 1\}$). This implies that the Grigorchuk group is residually finite.

Dynamically, the fact that the Grigorchuk group is residually finite corresponds to the fact that its action on $\{0, 1\}^\infty$ is equicontinuous. On the other hand, expansive actions can be used to construct simple groups (see [16]). A standard trick to force expansivity is "exploding" points of an orbit. For example, this is the way expansive Denjoy homeomorphisms of the Cantor set are obtained from equicontinuous actions by irrational rotations on the circle (see [7, 17]).

We can perform the same trick with the Grigorchuk group. For example, we can split the point $000\cdots$ in two by separating the sequences with an odd and an even number of leading symbols 0 into two neighborhoods of the two copies of $000\cdots$. After we propagate this split along the $\mathcal{G}$-orbit of $000\cdots$, the Grigorchuk group will act on the new Cantor set expansively. Then the corresponding alternating full group of the action, as defined in [16], will be simple and finitely generated.

More explicitly, the new Cantor set $\mathcal{X}$ will be the set of all right-infinite sequences $x_1 x_2 \cdots$ over the alphabet $\{0_0, 0_1, 1\}$ such that each length 2 subword $x_i x_{i+1}$ belongs to the set

$$\{0_0 0_1,\ 0_1 0_0,\ 0_1 1,\ 1 0_0,\ 1 0_1,\ 11\}.$$

Here, the symbol 0_0 "predicts" that there will be an even number of zeros (including itself) before the first 1, while 0_1 "predicts" that the number of zeros will be odd.

The operation of erasing indices is then a continuous surjective map $\mathcal{X} \to \{0,1\}^\infty$, which is one-to-one except for the sequences $w \in \{0,1\}^\infty$ containing only finitely many symbols 1, which have two preimages.

The action of the Grigorchuk group on $\{0,1\}^\infty$ naturally lifts to the action on $\mathcal{X}$ given by the rules

$$a(10_x w) = 0_{1-x}0_x w, \quad a(11w) = 0_1 1w, \quad a(0_x w) = 1w,$$

and

$$\begin{aligned} b(0_{1-x}0_x w) &= 0_1 a(0_x w), & b(0_1 10_x w) &= 0_x a(10_x w), \\ b(0_1 11w) &= 0_0 a(11w), & b(1w) &= 1c(w), \\ c(0_{1-x}0_x w) &= 0_1 a(0_x w), & c(0_1 10_x w) &= 0_x a(10_x w), \\ c(0_1 11w) &= 0_0 a(11w), & c(1w) &= 1d(w), \\ d(0_x w) &= 0_x w, & d(1w) &= 1b(w). \end{aligned}$$

One can show that this action is expansive. The topological full group of the action is generated by b, c, d, and the restrictions of a to three subsets of $\mathcal{X}$ corresponding to the three possible values of the *second* coordinate of a sequence $w \in \mathcal{X}$:

$$10_0 w \overset{a_0}{\longleftrightarrow} 0_1 0_0 w, \quad 10_1 w \overset{a_1}{\longleftrightarrow} 0_0 0_1 w, \quad 11w \overset{a_2}{\longleftrightarrow} 0_1 1w.$$

Then, by [13, 16], the derived subgroup of the full group $\langle b, c, d, a_0, a_1, a_2 \rangle$ is simple. It has finite index in the full group, since the full group is generated by finitely many elements of order 2.

Instead of proving the above statements, we will describe the full group $\widehat{\mathcal{G}}$ anew as a group defined by its orbital graphs, and prove that it is virtually simple directly, using only the structure of its Schreier graphs (see Figure 1).

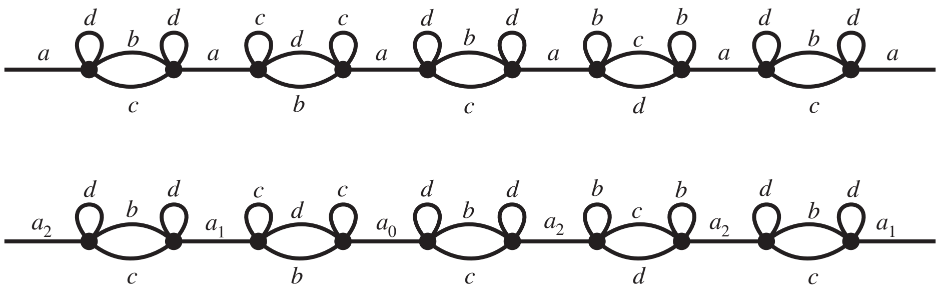

Figure 1. Orbital graphs of $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

The generating set of our group $\widehat{\mathcal{G}}$ will be $S = \{a_0, a_1, a_2, b, c, d\}$. Consider the following initial segments:

$$I_1 = \{a_2\}\{b,c\}\{a_0\},$$
$$J_1 = \{a_2\}\{b,c\}\{a_1\},$$

and for $n \geq 1$ define

$$I_{n+1} = J_n e_n J_n^{-1},$$
$$J_{n+1} = J_n e_n I_n^{-1},$$

where the connector e_n is equal to $\{b,c\}$, $\{d,b\}$, $\{c,d\}$ if $n \equiv 0,1,2 \pmod 3$, respectively. Note that I_n is symmetric for $n \geq 2$.

For example, we have

$$I_2 = \{a_2\}\{b,c\}\{a_1\}\{d,b\}\{a_1\}\{b,c\}\{a_2\}, \quad J_2 = \{a_2\}\{b,c\}\{a_1\}\{d,b\}\{a_0\}\{b,c\}\{a_2\}.$$

We see that I_n and J_n for $n \geq 0$ start and end with $\{a_2\}\{b,c\}$ and $\{b,c\}\{a_2\}$, and that each of the edges $\{a_0\}$ and $\{a_1\}$ is surrounded by $\{d,b\}$ on one side and $\{b,c\}$ on the other.

Let $\mathcal{S}$ be the subshift generated by the set of segments I_n, J_n, and let $\widehat{\mathcal{G}}$ be the group it defines. Note that a_0, a_1, a_2 commute (they are never neighbors) and that the subgroup $\langle a,b,c,d\rangle \leq \widehat{\mathcal{G}}$, where $a = a_0a_1a_2$, has the same orbital graphs as the Grigorchuk group, hence it is isomorphic to it.

For every n, each orbital graph Γ_w is obtained by connecting infinitely many copies of I_n and J_n by connectors e_0, e_1, e_2 in some order. (Here, and in the sequel, by a "copy" of an interval I we mean an isomorphic copy of I or I^{-1}.) It follows from the description of I_1 and J_1 that the decomposition into copies of I_1 and J_1 is unique. This in turn implies that the decomposition of Γ_w into the copies of I_n and J_n is unique for every n.

Let H_n, for $n \geq 2$, be the subgroup of elements $g \in \widehat{\mathcal{G}}$ satisfying the following conditions:

(1) Each copy of I_n and J_n in Γ_w is g-invariant.

(2) The element g commutes with isomorphisms between the copies of I_n and with isomorphisms between copies of J_n (in both orientations).

(3) The action of g on a copy of I_n leaves both halves J_{n-1}, J_{n-1}^{-1} of I_n invariant.

Denote by S_{J_n} the symmetric group on the set of vertices of J_n. It follows from the definition that H_n is naturally identified with a subgroup of $S_{J_n} \times S_{J_{n-1}}$. Denote by A_{J_n} the corresponding alternating subgroups.

Proposition 3.2. *The group H_n is equal to $S_{J_n} \times S_{J_{n-1}}$ for every $n \geq 2$.*

Proof. Denote by H_{I_n} and H_{J_n} the intersections of H_n with $\{1\} \times S_{J_{n-1}}$ and $S_{J_n} \times \{1\}$, respectively.

It is easy to see that the element a_0 and the commutators $[b, a_0], [c, a_0], [d, a_0]$ belong to H_{J_2}. It follows that H_{J_2} contains the symmetric group on the four vertices of the sub-segment $\{d, b\}\{a_0\}\{b, c\}$ of J_2. Taking commutators of the elements of this symmetric group with b, c, a_1, a_2, and taking into account that $a_0 \in H_{J_2}$, we conclude that $H_{J_2} = S_{J_2} \times \{1\}$.

We have $a_1 \in H_2$. Since the projection of H_2 onto S_{J_2} is surjective, it follows that H_{I_2} contains the action of a_1 on the vertices of I_2. Taking commutators with b, c, a_2, we conclude that the elements of H_{I_2} can permute the vertices of one half $\{a_2\}\{b, c\}\{a_1\}$ of I_2 by any permutation, that is, that $H_{I_2} = S_{J_1}$. This finishes the proof for $n = 2$.

Suppose that we know that the proposition holds for n, and let us prove it for $n + 1$. Note that the right end of J_n and the left end of J_n^{-1} are adjacent in Γ_w to $\{e_n\}$ only. Copies of I_n^{-1} appear in Γ_w only inside J_{n+1}, and their right end (the left end of I_n^{-1}, respectively) is adjacent only to e_n. Their left ends (the right ends of I_n^{-1}) are the right ends of J_{n+1}, hence they are always adjacent to e_{n+1}. One of the letters b, c, d, write it as x, belongs to e_n but does not belong to e_{n+1}. Then $H_{I_n} \leq H_{J_{n+1}}$ and $[x, H_{I_n}] \subset H_{J_{n+1}}$. Taking commutators of H_{J_n} with the group generated by $[x, H_{I_n}]$, we show that the alternating group of permutations of the vertices of J_{n+1} is contained in $H_{J_{n+1}}$. But H_{I_n} contains odd permutations, hence $H_{J_{n+1}} = S_{J_{n+1}}$. It follows that $H_{I_{n+1}}$ contains the group S_{J_n}, hence $H_{n+1} = S_{J_{n+1}} \times S_{J_n}$. ∎

We obviously have $H_n < H_{n+1}$. Let H_∞ be the union of the subgroups H_n.

Proposition 3.3. *The derived subgroup H'_∞ of H_∞ is simple and has index 2 in H_∞.*

Proof. The derived subgroup of H_n is the direct product $A_{J_n} \times A_{J_{n-1}}$ of alternating groups of permutations of the vertices of J_n and J_{n-1}. It follows that H'_∞ is the union of the subgroups $A_{J_n} \times A_{J_{n-1}}$. Let $g \in H'_n$ be an arbitrary non-trivial element. Let $(g_1, g_2) \in A_{J_n} \times A_{J_{n-1}}$ be the corresponding element of the direct product of alternating groups. Let (h_1, h_2) be the element of $A_{J_{n+1}} \times A_{J_n}$ representing $g \in H_{n+1}$. The set of vertices of J_{n+1} is a union of a set in a bijection with the set of vertices of J_n and a set in a bijection with the set of vertices of I_n. The permutation h_1 acts by g_1 on the first set and by g_2 on the second one. The permutation h_2 acts by copies of g_1 on two halves of the set of vertices of I_{n+1}. Consequently, for all $m \geq n + 2$ both coordinates of g in the direct product $A_{J_m} \times A_{J_{m-1}}$ are non-trivial. Consequently, the normal closure of g in H'_m is equal to H'_m. Hence, the normal closure of g in H'_∞ is the whole derived subgroup H'_∞.

If $(k_1, k_2) \in (\mathbb{Z}/2\mathbb{Z})^2$ is the parity of an element g of $S_{J_n} \times S_{J_{n-1}} \cong H_n$, then the parity of g as an element of $S_{J_{n+1}} \times S_{J_n} \cong H_{n+1}$ is (k_1, k_1), since the permutation in the second coordinate $S_{J_{n-1}}$ is copied twice as a permutation of the second half I_n of J_{n+1}. It follows that an element of $S_{J_n} \times S_{J_{n-1}} \cong H_n$ belongs to H'_∞ if and only if its first coordinate is an even permutation. Consequently, $[H_\infty : H'_\infty] = 2$. An element of H_∞ not belonging to H'_∞ is, for example, a_2. ∎

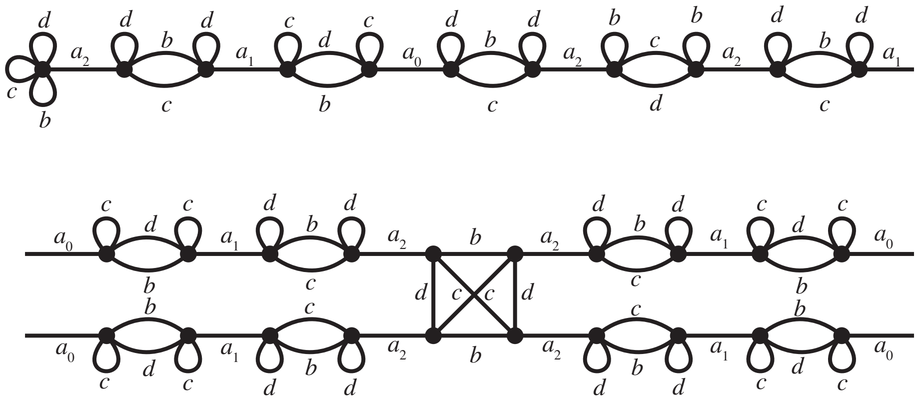

Figure 2. Graphs J_∞ and Ξ.

Denote by J_∞ the inductive limit of the segments J_n with respect to the embedding of J_n to the left part of J_{n+1}. Then J_∞ is a right-infinite chain. Let ξ be its left endpoint. The subshift $\mathcal{S}$ contains the graphs $\Lambda_n = J_\infty^{-1} e_n J_\infty$ for all $n = 0, 1, 2$, since $J_k^{-1} e_n J_k$ are sub-segments of I_{n+1} for all $k < n$. The smallest common cover of the graphs Λ_n, for $n = 0, 1, 2$, is the graph Ξ obtained by taking four copies of J_∞ and connecting the copies of ξ together by a Cayley graph of the Klein four-group $\{1, b, c, d\}$. The four copies of J_∞ in Ξ can be denoted (J_∞, x) for $x \in \{1, b, c, d\}$ so that the left end of J_∞ in the copy (J_∞, x) is connected by an edge labeled by y to the left end of (J_∞, xy). See Figure 2, where J_∞ is shown above Ξ.

Since Ξ covers the graphs Λ_n, the group $\widehat{\mathcal{G}}$ naturally acts on the set of vertices of Ξ, that is, Ξ is a Schreier graph of $\widehat{\mathcal{G}}$. In fact, Ξ is the Schreier graph of $\widehat{\mathcal{G}}$ with respect to the intersection of the stabilizers of the copies of the left end of J_∞ in the Schreier graphs Λ_n.

Proposition 3.4. *An element $g \in \widehat{\mathcal{G}}$ belongs to H_∞ if and only if it leaves invariant the subsets (J_∞, x) of Ξ.*

Proof. If $g \in H_\infty$, then there exists n such that $g \in H_n$. But then g leaves the copies of the segment J_n, I_n invariant, hence it leaves also invariant the subsets (J_∞, x) of Ξ which are unions of such segments.

Conversely, suppose that $g \in \widehat{\mathcal{G}}$ leaves the subsets (J_∞, x) invariant. Note that each of the segments I_n, J_n is of the form $J_{n-2} \cdots J_{n-2}^{-1}$. It follows that in the decomposition $\cdots X_1 e_{k_1} X_2 e_{k_2} \cdots$ of Γ_w into the copies X_i of the segments I_n, J_n, the connectors e_{k_i} are surrounded by $J_{n-2}^{-1} e_{k_i} J_{n-2}$. If the length of g is shorter than the length of J_{n-2}, then any path describing the action of g on $X = J_{n-2}^{-1} e_{k_i} J_{n-2}$ (if it is completely inside X) can be lifted to a path in Ξ so that e_{k_i} is covered by the Cayley graph of $\{1, b, c, d\}$ (i.e., by the central part connecting the copies of J_∞). Since g leaves the parts (J_∞, h) invariant, the path will start and end in one half (J_{n-2} or J_{n-2}^{-1}) of X. It follows that the points of Γ_w do not cross the connectors e_{k_i} under the action of g. Moreover, the action of g on each copy X_i of the segments I_n, J_n will depend only on the isomorphism class of X_i. It follows that $g \in H_\infty$. ∎

Theorem 3.5. *The group $\widehat{\mathscr{G}}$ is virtually simple.*

Proof. Let $A = \langle H_\infty'^{\widehat{\mathscr{G}}} \rangle$ be the normal closure of H_∞' in $\widehat{\mathscr{G}}$. We will prove that it is a simple subgroup of finite index.

Let us prove first that A has finite index. For a vertex $v \in J_\infty$ and an element $g \in \widehat{\mathscr{G}}$, denote by $\tau_g(v)$ the element of $\{1, b, c, d\}$ such that the lift of the action of $\mathscr{G}$ to Ξ moves $(v, 1)$ to $(g(v), \tau_g(v))$. We have $\tau_g(v) = 1$ for all but finitely many vertices of J_∞. We have

$$\tau_{g_1 g_2}(v) = \tau_{g_1}(g_2(v))\tau_{g_2}(v) \tag{3.1}$$

for all $g_1, g_2 \in \widehat{\mathscr{G}}$ and all vertices v.

Denote by $\phi(g)$ the product of the values of $\tau_g(v)$ for all vertices of J_∞. It follows from (3.1) and the fact that $\{1, b, c, d\}$ is commutative, that $\phi : \widehat{\mathscr{G}} \to \{1, b, c, d\}$ is a homomorphism. It is also easy to check that $\phi(b) = b, \phi(c) = c$, and $\phi(d) = d$, so ϕ is an epimorphism.

By Proposition 3.4, an element $g \in \widehat{\mathscr{G}}$ belongs to H_∞ if and only if $\tau_g(v) = 1$ for all vertices v of J_∞. In particular, $H_\infty < \ker\phi$.

It is checked directly that $\tau_{[a_2,b]}(v)$ is equal to 1 for all vertices v except for two of them (the left endpoint ξ of J_∞ and $ba_2(\xi)$), where it is equal to b. Similarly, $\phi_{[a_2,c]}$ is equal to c on two vertices and to 1 everywhere else. The group H_∞' is 2-transitive on J_∞, that is, for any two pairs (v_1, v_2) and (w_1, w_2) of different points of J_∞, there exists $h \in H_\infty'$ such that $h(v_1) = w_1$ and $h(v_2) = w_2$. (This follows from the description of H_∞' given above and the fact that the alternating groups A_n are 2-transitive for $n \geq 4$.) Consequently, every element of $\ker\phi$ can be written as a product of an element of H_∞ and elements of the form $[a_2, b]^h$ and $[a_2, c]^h$ for $h \in H_\infty'$. Note that $[a_2, b], [a_2, c] \in A$, since $a_2 \in H_\infty'$. As $[H_\infty : H_\infty'] = 2$, this implies that A is a subgroup of index at most 2 in $\ker\phi$. Consequently, A is a subgroup of finite index in $\widehat{\mathscr{G}}$.

Let us show that A is simple. Suppose that $N \lhd A$ is a non-trivial proper normal subgroup of A. Let $g \in N \setminus \{1\}$. The element g moves a vertex v of $J_\infty^{-1} e_1 J_\infty$. There exists n such that both v and $g(v)$ belong to the central segment $J_n^{-1} e_1 J_n$ of $J_\infty^{-1} e_1 J_\infty$, and the distance from v to the boundary of $J_n^{-1} e_1 J_n$ is greater than the length of g. We can find an isomorphic copy of $J_n^{-1} e_1 J_n$ inside J_m for some large m and an element $h \in H_m'$ moving v but fixing pointwise all elements of the $|g|$-neighborhood of the boundary of J_m. Then the commutator $[h, g]$ is non-trivial and belongs to H_m'. Since H_∞' is simple, it follows that $N \geq H_\infty'$.

It remains to show that the normal closure of H_∞' in $\widehat{\mathscr{G}}$ is equal to the normal closure of H_∞' in A. The group H_∞' is generated by the set of permutations $g \in \widehat{\mathscr{G}}$ such that the $\langle g \rangle$-orbits are of lengths 1 and 3, and g preserves the sets of vertices of the copies (J_∞, x) of J_∞ in Ξ. Moreover, for any D, we can take as a generating set a subset C of this set such that for every $g \in C$ every two $\langle g \rangle$-orbits of length 3 are on distance at least D from each other.

It follows from the arguments from the proof that A is a subgroup of finite index in $\widehat{\mathcal{G}}$ that a conjugate of an element of C by an element of $\widehat{\mathcal{G}}$ is equal to a conjugate by an element of A. Consequently, the group generated by ${H'_\infty}^A$ is equal to the group generated by ${H'_\infty}^{\widehat{\mathcal{G}}}$. ■

The intersection of $\mathcal{G}$ with the simple finite index subgroup of $\widehat{\mathcal{G}}$ has finite index in $\mathcal{G}$. It is known that every subgroup of finite index in $\mathcal{G}$ has a subgroup isomorphic to $\mathcal{G}$ (see, e.g., [11, Section 12]). Consequently, the simple finite index subgroup of $\widehat{\mathcal{G}}$ contains an isomorphic copy of the Grigorchuk group. So, we have embedded the Grigorchuk group into a simple finitely generated torsion group of intermediate growth $\widehat{\mathcal{G}}$. Indeed, the group $\widehat{\mathcal{G}}$ is torsion and of intermediate growth by Theorem 2.9.

4. A continuum of growth types

We construct in this section, using the techniques of defining groups by their Schreier graphs, a family of virtually simple groups containing continuum of pairwise different growth types.

Our set of generators will be $S = \{a_0, a_1, a_2, x, y, b, c, d\}$. We will use the same notation

$$e_1 = \{b, c\}, \quad e_2 = \{d, b\}, \quad e_3 = \{c, d\}$$

as for the Grigorchuk group and the virtually simple group $\widehat{\mathcal{G}}$.

Start with

$$I_0 = \{a_0\}\{a_1\}, \quad J_0 = \{a_0\}\{a_2\}.$$

Let $\alpha = (\alpha_0, \alpha_1, \dots)$ be a sequence of symbols σ or non-trivial elements of the free monoid $\{x, y\}^*$. Define the sequence of pairs of segments, associated with α, by the following rule.

If $\alpha_n = \sigma$, then

$$I_{n+1} = J_n e_n J_n^{-1},$$
$$J_{n+1} = J_n\{x\}I_n\{y\}I_n\{y\}J_n^{-1}.$$

If $\alpha_n = t_1 t_2 \cdots t_m \in \{x, y\}^*$, then

$$I_{n+1} = J_n e_n J_n^{-1},$$
$$J_{n+1} = J_n e_n I_n\{t_1\}I_n\{t_2\}I_n\{t_3\}\cdots\{t_m\}J_n^{-1}.$$

Let $\mathcal{S}_\alpha$ be the subshift defined by the set of $\{I_n, J_n\}_{n \geq 0}$, and let G_α be the group defined by $\mathcal{S}_\alpha$.

Note that since every two connectors e_n are disjoint, the subgroup $\{1, b, c, d\}$ of G_α is isomorphic to the Klein four-group for every α.

Proposition 4.1. *The group G_α is virtually simple for every sequence α. More precisely, the derived subgroup G'_α is simple and $G_\alpha/G'_\alpha \cong (\mathbb{Z}/2\mathbb{Z})^4$.*

Proof. For every $n \geq 0$, every element of the subshift $\mathcal{S}_\alpha$ is obtained by concatenating isomorphic copies of the segments I_n, J_n using the connectors e_1, e_2, e_3, $\{x\}$, and $\{y\}$. It is easy to see that the decomposition into a concatenation of the copies of the segments I_0 and J_0 (and connectors between them) is unique. Since the segment I_{n+1} is the only segment of the form $J_n e_n J_n^{-1}$, it follows by induction that the decomposition into copies of I_n and J_n is unique for every $n \geq 0$.

Each segment I_n consists of two copies of J_{n-1}. Therefore, we get a canonical partition of each graph $\Gamma \in \mathcal{S}_\alpha$ into isomorphic copies of J_n and J_{n-1}.

Similarly to Section 3, denote for $n \geq 1$ by H_n the subgroup of all elements $g \in G_\alpha$ preserving the set of vertices of each copy of J_n and J_{n-1} of the partition and commuting with all isomorphisms between the copies of J_n and between the copies of J_{n-1} (in both orientations). The group H_n is naturally identified with a subgroup of the direct product $S_{J_n} \times S_{J_{n-1}}$ of symmetric groups on the sets of vertices of J_n and J_{n-1}. Denote by H_{J_n} and H_{I_n} the intersections of H_n with $S_{J_n} \times \{1\}$ and $\{1\} \times S_{J_{n-1}}$.

Note that $a_1, [a_0, a_1]$ generate a subgroup of G_α isomorphic to S_3, preserving the vertices of each copy of I_0, acting trivially on the copies of J_0, and commuting with isomorphisms between the copies of I_0. Similarly, $a_2, [a_0, a_2]$ generate a subgroup of G_α acting as the full symmetric group on the sets of vertices of the copies of J_0.

Let us prove at first that H_n contains the direct product $A_{J_n} \times A_{J_{n-1}}$ of the groups of even permutations of the sets of vertices of J_n and J_{n-1}, that is, that H_{J_n} contains the alternating group on J_n and H_{I_n} contains the alternating group on J_{n-1}. Let us prove this by induction. Both the inductive step and the base case $n = 1$ will be proved at the same time.

Let $v_1, v_2, \ldots, v_m$ be the vertices of I_n listed in the order they appear in the segment. Then H_n contains the permutation $h = (v_1, v_2)(v_{m-1}, v_m)$ for $n \geq 1$ or the permutation (v_1, v_2) for $n = 0$, acting identically on the copies of J_n. Consider the commutator $[x, h]$ if $\alpha_n = \sigma$ or the commutator $[b, h]$ if $\alpha_n \neq \sigma$. This commutator will be a cycle of length 3 on the set of vertices of J_{n+1} and will act identically on the vertices of I_{n+1}. Conjugating it by elements of H_{J_n}, by elements of H_{I_n}, and by x, y, b, c, d, we will get enough 3-cycles to generate $A_{J_{n+1}} \leq H_{J_{n+1}}$. The group $H_{I_{n+1}}$ will contain H_{J_n} by the inductive assumption (or by the fact that $\langle a_2, [a_0, a_2]\rangle$ is S_3). This finishes the inductive argument.

Let $h_1 \in S_{J_n}, h_2 \in S_{J_{n+1}}$ be arbitrary permutations. Since there are even numbers of copies of J_n and J_{n-1} in J_{n+1}, they will induce an even permutation of the set of vertices of J_{n+1}. By the same argument, they will induce an even permutation of the set of vertices of J_{n+2}. It follows that the corresponding permutation belongs to $A_{J_{n+2}} \times A_{J_{n+1}} \leq H_{n+2}$. Consequently, $S_{J_n} \times S_{J_{n-1}}$ is contained in G_α, hence H_n coincides with the whole group $S_{J_n} \times S_{J_{n-1}}$.

Denote by H_∞ the union of the groups H_n. Note that we have shown that H_∞ is isomorphic to the direct limit of the direct products of the alternating groups $A_{J_n} \times A_{J_{n-1}}$, which implies that H_∞ is perfect.

Let J_∞ be the limit of J_n with respect to the embedding of J_n to the left end of J_{n+1}. It follows from the recursions defining the segments I_n and J_n that the subshift $\mathcal{S}_\alpha$ contains the graphs $J_\infty^{-1} z J_\infty$ for each $z \in \{\{x\}, \{y\}, e_1, e_2, e_3\}$, since the corresponding finite segments $J_n^{-1} z J_n$ are sub-segments of I_m and J_m for all $m \geq n + 3$. The same arguments as in the proof of Proposition 3.4 show that an element $g \in G_\alpha$ belongs to H_∞ if and only if it leaves invariant the sets of vertices of J_∞ and J_∞^{-1} in the graphs $J_\infty^{-1} z J_\infty$ for every $z \in \{\{x\}, \{y\}, e_1, e_2, e_3\}$. Note that x, y, b, c, d generate a group K isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, as they commute with each other. Similarly to the proof of Theorem 3.5, define $\tau_g(v)$ for a vertex v of J_∞ to be the product $h_1 h_2 h_3 \in K$ of the elements $h_1 \in \langle x \rangle$, $h_2 \in \langle y \rangle$, $h_3 \in \{1, b, c, d\}$, where h_1 is equal to x if v is moved by g from J_∞ to J_∞^{-1} in $J_\infty^{-1}\{x\}J_\infty$ and to 1 otherwise, h_2 is equal to y if v is moved by g from J_∞ to J_∞^{-1} in $J_\infty^{-1}\{y\}J_\infty$ and to 1 otherwise, and h_3 describes to which branch (J_∞, h) the vertex v is moved by g in the graph Ξ obtained by gluing the copies of J_∞ together along the endpoint of J_∞ by the Cayley graph of $\{1, b, c, d\}$.

Then $g \in H_\infty$ if and only if τ_g is constant 1. Let $\phi(g)$ be the product of the values of $\tau_g(v)$ over all vertices v of J_∞. Then, by the same arguments as in the proof of Theorem 3.5, $\phi : G_\alpha \to K$ is an epimorphism and its kernel is equal to the normal closure of H_∞ in G_α (we are in a better situation here than for Theorem 3.5, since $H'_\infty = H_\infty$ now).

The same argument as in the proof of Theorem 3.5 shows that for every non-trivial element $g \in \ker\phi$ the normal closure of g in $\ker\phi$ contains H_∞ and that the normal closure of H_∞ in $\ker\phi$ and in G_α coincide, which proves that $\ker\phi$ is simple. Note that this implies that $\ker\phi = G'_\alpha$, since K is commutative. ∎

Proposition 4.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence such that $\alpha_n = \sigma$ for all n big enough. Then there exist $C > 1$ and $0 < \alpha < 1$ such that the growth of G_α satisfies $\gamma_{G_\alpha}(n) \leq \exp(Cn^\alpha)$ for all $n \geq 1$.*

For every finite sequence $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of symbols σ and elements of $\{x, y\}^*$ and for every $R \geq 1$, there exists n such that for every sequence

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, \underbrace{\sigma, \sigma, \ldots, \sigma}_{n \text{ times}}, \alpha_{k+n+1}, \alpha_{k+n+2}, \ldots)$$

the ball of radius R in the Cayley graph of G_α is isomorphic to the ball of radius R of the group $G_{(\alpha_1, \alpha_2, \ldots, \alpha_k, \sigma, \sigma, \sigma, \ldots)}$.

Proof. The first statement follows directly from Theorem 2.9.

Let us prove the second statement. For every $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and R, there exists n such that the segment I_m, J_m defined by $(\alpha_1, \alpha_2, \ldots, \alpha_k, \sigma, \sigma, \ldots)$ for $m \geq k + n$ has length more than R.

Note that the set of segments of length R in I_{m+1} and J_{m+1} do not depend then on the symbol α_m. Consequently, for every

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, \underbrace{\sigma, \sigma, \ldots, \sigma}_{n \text{ times}}, \alpha_{k+n+1}, \alpha_{k+n+2}, \ldots),$$

the set of segments of length R in the graphs belonging to $\mathcal{S}_\alpha$ coincides with the set of segments of length R in the graphs belonging to $\mathcal{S}_{\alpha'}$ for $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_k, \sigma, \sigma, \ldots)$. It follows that an equality $g_1 = g_2$ for products of length $\leq R$ of the generators $x, y, b, c, d, a_0, a_1, a_2$ are true or false in G_α and $G_{\alpha'}$ at the same time, that is, the groups G_α and $G_{\alpha'}$ have isomorphic balls of radius R in their Cayley graphs. ∎

Proposition 4.3. *For every finite sequence* $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1})$*, there exists a natural number* M *such that for every* $N \geq 1$ *there exists a word* $w \in \{x, y\}^*$ *such that for every infinite sequence* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, w, \alpha_{k+1}, \alpha_{k+2}, \ldots)$*, the growth of the group* G_α *satisfies*

$$\gamma_{G_\alpha}(Mn) \geq 2^n$$

for all $n = 1, \ldots, N$.

Proof. Consider the segment I_k defined by $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1})$, and let g be the product of the labels along a simple path from the left to the right endpoints of I_k, so that g has length equal to the length of I_k and moves it left end to the right end.

Let $w \in \{x, y\}^*$ be a word containing all words $v \in \{x, y\}^*$ of length $\leq N$ as subwords. For every word $v = t_1 t_2 \cdots t_n \in \{x, y\}^n$, consider the corresponding element $g_v = t_n g \cdot t_{n-1} g \cdots t_1 g$ of G_α for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, w, \alpha_{k+1}, \alpha_{k+2}, \ldots)$.

Since w contains $v = t_1 t_2 \cdots t_n \in \{x, y\}^n$, the segment I_{k+1} will contain the subsegment $Z_v = I_k\{t_1\} I_k\{t_2\} \cdots I_k\{t_n\}$. The element $g_v = t_n g \cdot t_{n-1} g \cdots t_1 g$ will move the leftmost vertex of Z_v to its rightmost vertex, and the length of g_v is equal to the length of Z_v. Any other element $g_{v'}$ for $v' \in \{x, y\}^*$ of length $\leq n$ will move the left end of Z_v to another vertex, since any geodesic path connecting the ends of Z_v must have length n and contain exactly the same edges $t_1, t_2, \ldots, t_n$ on the same places as in the geodesic path corresponding to g_v. It follows that all elements g_v for $v \in \{x, y\}^n$, for $n \leq N$, are pairwise different. There are 2^n of them, and their length is $n(|I_k| + 1)$. It follows that $M = |I_k| + 1$ and the chosen w will satisfy the conditions of the proposition. ∎

Theorem 4.4. *There exists a continuum of pairwise different growth types of simple finitely generated groups.*

Proof. After we proved Propositions 4.2 and 4.3, the proof of the theorem is similar to the proof of Theorem 7.2 of [10].

Choose a sequence $C_1, C_2, \ldots$ of positive integers converging to infinity. We will define for every sequence $\rho = (r_1, r_2, \ldots) \in \{0, 1\}^\infty$ a sequence α_ρ such that there is a continuum of different growth types among the groups G_{α_ρ}. The sequence α_ρ will be the limit of finite sequences $\alpha_{(r_1,r_2,\ldots,r_n)}$ so that $\alpha_{(r_1,r_2,\ldots,r_{n+1})}$ is a continuation of $\alpha_{(r_1,r_2,\ldots,r_n)}$. At the same time, we will define a sequence $R_1, R_2, \ldots$ of positive integers.

Using Propositions 4.2 and 4.3, we can find R_1, $\alpha_0 = (\sigma, \sigma, \ldots, \sigma)$, and $\alpha_1 = (w)$ for $w \in \{x, y\}^*$, such that if $\gamma_0(n)$ is the growth of G_α for any α beginning with α_0, and $\gamma_1(n)$ is the growth of G_α for any α beginning with α_1, then we have

$$\gamma_0(C_1 R_1) \leq \gamma_1(R_1).$$

Suppose that we have defined R_n and the sequences α_v for all sequences $v \in \{0,1\}^n$ of length n. Then by Propositions 4.2 and 4.3, there exists a number k, a word $w \in \{x, y\}^*$, and a number $R_{n+1} > R_n$ such that if γ_0 is the growth function of any group G_α, where α starts with α_v for $v \in \{0,1\}^n$ followed by $\underbrace{\sigma, \sigma, \ldots, \sigma}_{k \text{ times}}$, and γ_1 is the growth function of any group $G_{\alpha'}$ for α' starting with α_u for $u \in \{0,1\}^n$ followed by w, then we have

$$\gamma_0(C_{n+1} R_{n+1}) \leq \gamma_1(R_{n+1}).$$

This will give us an inductive definition of the groups G_{α_ρ} for $\rho \in \{0,1\}^\infty$ and of the sequence R_n. Suppose now that $\rho_1, \rho_2 \in \{0,1\}^\infty$ are two sequences which differ in infinitely many coordinates. Suppose that the growth functions γ_i of $G_{\alpha_{\rho_i}}$ are equivalent. Then there exists $C > 1$ such that $\gamma_2(R) < \gamma_1(CR)$ and $\gamma_1(R) < \gamma_2(CR)$ for all $R \geq 1$. Since ρ_1 and ρ_2 are different in infinitely many coordinates, there exists n such that $C_n > C$ and the nth coordinates of ρ_1 and ρ_2 are different. Then we will have either $\gamma_1(C_n R_n) \leq \gamma_2(R_n) < \gamma_1(CR_n)$ or $\gamma_2(C_n R_n) \leq \gamma_1(R_n) < \gamma_2(CR_n)$, which is a contradiction.

It follows that the sets of groups with equivalent growth functions in the set of groups $\{G_{\alpha_\rho} : \rho \in \{0,1\}^\infty\}$ are at most countable, hence we have a continuum of pairwise different growth types in this set. Each group in this set is virtually simple by Proposition 4.1. Since a finite index subgroup has the same growth type as the group, it follows that there is a continuum of different growth types of simple finitely generated groups. ∎

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