Groups Geom. Dyn. 19 (2025), 567–596
DOI 10.4171/GGD/888

Branch groups with infinite rigid kernel

Alejandra Garrido and Zoran Šunić

Abstract. A theoretical framework is established for explicitly calculating rigid kernels of self-similar regular branch groups. This is applied to a new infinite family of branch groups in order to provide the first examples of self-similar, branch groups with infinite rigid kernel. The groups are analogues of the Hanoi Towers group on 3 pegs, based on the standard actions of finite dihedral groups on regular polygons with odd numbers of vertices, and the rigid kernel is an infinite Cartesian power of the cyclic group of order 2, except for the original Hanoi group. The proofs rely on a symbolic-dynamical approach, related to finitely constrained groups.

To Slava Grigorchuk, in gratitude and acknowledgement of the many ramifications of his work

1. Introduction

A group G with a faithful action on a level-homogeneous rooted tree T is a *branch group* if it acts level-transitively, and for all n, the rigid stabiliser $\mathsf{Rst}_G(n) = \prod_{v \in \text{level } n} \mathsf{Rst}_G(v)$ has finite index in G, where $\mathsf{Rst}_G(v)$ consists of all elements of G that are only supported on the subtree rooted at v.

Since the appearance in the 1980s of the first finitely generated examples, branch groups have been recognised as an important class of groups. This is not only because of the many examples in the class with interesting properties: finitely generated infinite torsion groups, groups of intermediate word growth, of non-uniformly exponential word growth, amenable but not elementary amenable, etc. [9, 10, 20], but also because the subgroup structure of branch groups forces them to appear as cases in classifications. For instance, by a result of Wilson ([18], see also [11]), all residually finite just infinite groups (infinite groups all of whose proper quotients are finite) are either just infinite branch groups, virtually direct powers of non-abelian hereditarily just infinite groups, or virtually abelian. This classification also applies in the case of just infinite profinite groups (all quotients by closed normal subgroups are finite) [19]. Profinite branch groups also arise as compact-open subgroups of *locally decomposable* groups, one of five types into which

Mathematics Subject Classification 2020: 20E05 (primary); 20E18, 22C05 (secondary).
Keywords: branch group, congruence subgroup problem, finitely constrained group, self-similar group, tree shift.

one can separate the groups that are totally disconnected, locally compact, compactly generated, topologically simple and non-discrete [2].

If G is a branch group, it must be residually finite, so it embeds densely into its profinite completion $\widehat{G}$. The action of G on the rooted tree T gives another completion, with respect to the topology generated by the stabilisers $\mathsf{St}_G(n)$ of levels of the tree; this completion, denoted $\overline{G}$, is in fact the closure of G in the permutation topology of $\mathsf{Aut}(T)$, the (profinite) group of all automorphisms of T. The rigid level stabilisers, $\mathsf{Rst}_G(n), n \in \mathbb{N}$, being of finite index, also yield a profinite topology on G, and G also embeds densely into the completion $\widetilde{G}$ with respect to this topology. Since $\mathsf{Rst}_G(n) \leq \mathsf{St}_G(n)$ for every $n \in \mathbb{N}$, and they are all of finite index in G, there are surjective morphisms

$$\widehat{G} \to \widetilde{G} \to \overline{G},$$

and the *congruence subgroup problem* asks what are the kernels: the *congruence kernel* $\ker(\widehat{G} \to \overline{G})$, the *branch kernel* $\ker(\widehat{G} \to \widetilde{G})$ and the *rigid kernel* $\ker(\widetilde{G} \to \overline{G})$.

The term "congruence" is used by analogy with the classical congruence subgroup problem for $\mathrm{SL}_n(\mathbb{Z})$, from which these questions take inspiration. The problem of determining the congruence, branch and rigid kernels for a branch group G was first posed in [1], where the first systematic study of this problem was undertaken. It was determined in [5] that the congruence subgroup problem is independent of the branch action of G.

There are now various examples of branch groups in the literature that show different behaviours for the various kernels. For example, Grigorchuk groups and GGS groups all have trivial congruence kernel (we say that they have the *congruence subgroup property*) [4, 11]. Pervova constructed in [14] the first examples of branch groups with non-trivial (in fact, infinite) congruence kernel. The rigid kernel of these groups is trivial. The first example of a branch group shown to have a non-trivial rigid kernel is $H^{(3)}$, the Hanoi Towers group on 3 pegs [7]. It was determined in [1] and, later, using a different approach, in [16], that the rigid kernel is the Klein 4-group. An infinite family of branch groups with non-trivial rigid kernel is constructed by Skipper in [16], but the rigid kernel for these examples is not determined and the groups come without self-similar actions (they are constructed as subgroups of some self-similar groups).

In this paper, we provide the first known examples of self-similar, branch groups with infinite rigid kernel. They are an infinite family of "dihedral generalisations" of the Hanoi Towers group $H^{(3)}$, one for each odd $d \geq 3$, acting on the d-regular rooted tree. The rigid kernel is determined explicitly; it is isomorphic to a Cartesian product of cyclic groups of order 2. The product is infinite for $d \geq 5$ and, of course, of rank 2 if $d = 3$.

Main examples: Hanoihedral groups. Let $d = 2k + 1$ be an odd integer, with $d \geq 3$ ($k \geq 1$), and $X = \{0, 1, \dots, d-1\}$. For $i = 0, \dots, d-1$, let μ_i be the involution in $\mathsf{Sym}(X)$ given by

$$\mu_i = (i-1\ i+1)(i-2\ i+2)\cdots(i-k\ i+k),$$

where the entries in the inversion pairs are considered modulo d (for instance, for $d = 5$, $\mu_4 = (35)(26) = (03)(12)$). Note that μ_i can be interpreted as the mirror symmetry of the regular d-gon with vertices $0, 1, \dots, d-1$ with respect to the axis through vertex i. The group generated by all μ_i, $i = 0, \dots, d-1$, is the dihedral group $D(d)$, the group of symmetries of the regular d-gon. For $i, j = 0, \dots, d-1$, we have $\mu_i \mu_j = \rho^{2(i-j)}$, where $\rho = (012 \cdots d-1)$ is the rotation of the d-gon by $2\pi/d$. We also have $\mu_i \rho \mu_i = \rho^{-1}$ and $\mu_i \rho = \rho \mu_{i-1}$ (indices modulo d). The dihedral group $D(d)$ consists of d mirror symmetries, $\mu_0, \dots, \mu_{d-1}$, and d rotations, $1, \rho, \rho^2, \dots, \rho^{d-1}$. The rotations form a subgroup of index 2 in $D(d)$, which is also the commutator subgroup of $D(d)$.

For $i = 0, \dots, d-1$, let a_i be the automorphism of the d-ary rooted tree X^* given by

$$a_i = \mu_i(1, \dots, 1, a_i, 1, \dots, 1),$$

where the only non-trivial section appears in coordinate i (the only coordinate not moved by μ_i). Let D be the self-similar group

$$D = \langle a_0, \dots, a_{d-1} \rangle.$$

Note that we defined one group for every odd integer $d \geq 3$, but this is not reflected in our notation for D or its generators (we do not index them by d) lest the notation become cumbersome.

The definition of the generators of D fits the general construction of generators for "Hanoi-like groups" from a set of permutations, mentioned in [8, Example 4]. Namely, the Hanoi Towers group $H^{(d)}$, $d \geq 3$, is generated by the automorphisms corresponding to all transpositions (ij), the group D is generated by the automorphisms corresponding to all mirror permutations μ_i, while the examples in [16] by the automorphisms corresponding to all d cycles of length $d-1$ obtained from the cycle $(01 \cdots d-1)$ by removing one letter. More generally, all of these examples may be seen as variations inspired by the groups constructed by P. Neumann [12, Section 5]. However, the end result is very different, as P. Neumann's examples are just infinite groups with very simple branching structure and trivial congruence kernel, while the group D has a non-trivial rigid kernel.

The following hold for these examples. Here, $X * D'$ denotes $\prod_{v \in X} \delta_x(D') \leq \mathsf{Aut}(X^*)$, where $\delta_x(g)$ is the element of $\mathsf{Aut}(X^*)$ that acts as $g \in \mathsf{Aut}(X^*)$ on the subtree rooted at $x \in X$ and fixes the rest of X^*. See Section 2 for more definitions.

Theorem 1. *Let d be an odd integer, with $d \geq 3$. Then:*

(1) *D is a self-similar, self-replicating, contracting, regular branch group, branching over its commutator subgroup D'.*

(2) $D/\mathsf{St}_D(1) = D(d)$.

(3) *The group D/D' is the elementary abelian 2-group of rank d and*

$$|D' : X * D'| = d \cdot 2^{(d-1)(d-2)}.$$

(4) *The rigid level stabiliser* R_n *of level* n *is* $X^n * D'$. *For* $n \geq 1$, *we have*

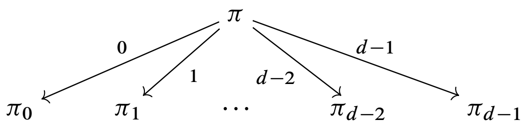
$$|D : R_n| = 2^{(d-2)d^n+2} \cdot d^{\frac{d^n-1}{d-1}}.$$

(5) *The rigid kernel of* D *is an elementary abelian* 2*-group*

$$A \times \prod_{X^* \setminus \{\varepsilon\}} B,$$

where A *has rank* $(d-1)(d-2)$ *and* B *has rank* $(d-1)(d-3)$. *In particular, the rigid kernel is a Klein-*4 *group when* $d = 3$ *and isomorphic to* $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ *when* $d > 3$.

(6) *The closure* $\overline{D}$ *is a finitely constrained group defined by the patterns of size* 2 *that can be described as follows. A pattern of size* 2



is an allowed pattern if and only if the permutation $\pi\pi_0\pi_1 \cdots \pi_{d-2}\pi_{d-1}$ *is a rotation in* $D(d)$ *(i.e., the number of mirror symmetries among* $\pi, \pi_0, \ldots, \pi_{d-1}$ *is even).*

(7) *The closure* $\overline{D}$ *is a regular branch group branching over* $\overline{D}_1$, *the stabiliser of level* 1. *We have, for* $n \geq 1$,

$$|D : \mathsf{St}_D(n)| = |\overline{D} : \overline{\mathsf{St}_D}(n)| = 2^{d^{n-1}} \cdot d^{\frac{d^n-1}{d-1}},$$

and the Hausdorff dimension of $\overline{D}$ *is*

$$1 - \frac{1}{d} \cdot \frac{\log 2}{\log 2d}.$$

Notice that since the rigid kernel is elementary abelian, the branch completion $\widetilde{D}$ of D is not a branch group, because these do not contain non-trivial virtually abelian normal subgroups. In particular, the rigid kernel will be in the kernel of any branch action of $\widetilde{D}$.

In order to make our calculations, we also develop a theoretical framework for finding the structure of rigid kernels. As for most of the studied examples of branch groups, we concentrate on the case of self-similar groups (see Section 2 for definitions) because in that case we can exploit the symbolic-dynamical results obtained in [13, 17] for *finitely constrained groups*. In particular, we obtain the following criterion for determining whether or not the rigid kernel is trivial.

Theorem 2. *Let* $G \leq \mathsf{Aut}(X^*)$ *be a self-similar, level-transitive, regular branch group, with maximal branching subgroup* K. *The following are equivalent:*

(1) *G has trivial rigid kernel.*

(2) *G branches over some level stabiliser.*

(3) *$K \geq \mathsf{St}_G(n)$ for some $n \geq 1$.*

If G is in addition self-replicating, then $\overline{G}$ is a finitely constrained group and the above items are also equivalent to

(4) $|X * G : \mathsf{St}_G(1)| = |X * \overline{G} : \overline{\mathsf{St}_G(1)}|$.

Example 1.1. It is known that, for $G = H^{(3)}$, the Hanoi Towers group on 3 pegs, we have

$$|G \times G \times G : G_1| = 2^5 \quad \text{and} \quad |\overline{G} \times \overline{G} \times \overline{G} : \overline{\mathsf{St}_G}(1)| = 2.$$

Thus, $H^{(3)}$ has a non-trivial rigid kernel.

The value of the former index was indicated in [8], while the value of the latter can be easily inferred from the description of the closure $\overline{H^{(3)}}$ as a finitely constrained group, announced in [6]. These indices were mentioned and used in [17] in the context of the calculation of the Hausdorff dimension of $\overline{H^{(3)}}$ (which happened to be $1 - 1/3 \log_6 2$), along with a remark that relates the fact that these indices are different to the fact that $H^{(3)}$ is not branching over any level stabiliser, which is to say, in the terminology of [1], that $H^{(3)}$ has non-trivial rigid kernel. The present text grew out of an attempt to elucidate and formalise that remark.

Under the assumption that the group is self-similar and regular branch, we obtain various results on the size of the rigid kernel, the most explicit and simple to state of which is as follows.

Theorem 3. *Let G be a self-similar, level-transitive regular branch group with maximal branching subgroup K. Suppose that $\mathsf{St}_G(2)K \geq \mathsf{St}_G(1)$.*

(1) *The rigid kernel of G is trivial if and only if $\mathsf{St}_G(2) = \mathsf{Triv}_G(2)$.*

(2) *Suppose moreover that G/K is in a class of groups that is closed under subgroups, quotients and direct products, in which all short exact sequences that are in the class split as direct products (e.g., elementary abelian groups). Then the rigid kernel is*

$$\Gamma = \frac{\mathsf{St}_G(2)}{\mathsf{Triv}_G(2)} \times \prod_{X^* \setminus \{\varepsilon\}} \frac{\mathsf{St}_G(2) \cap K}{\mathsf{Triv}_G(2)},$$

and it is infinite if and only if $\mathsf{St}_G(2) \cap K/\mathsf{Triv}_G(2)$ is non-trivial.

Section 2 contains definitions and necessary prerequisites and the proofs of the above theorems. Section 3 is devoted to the Hanoihedral groups and proving the items in Theorem 1.

2. Symbolic portraits and branch completions

2.1. Tree shifts

Let A be a set. Symbolic dynamics has for a long time concerned the study of one-sided sub-shifts: closed subsets of $A^{\mathbb{N}}$ with the product topology (where A is discrete) that are invariant under the shift map $\cdot_1 : A^{\mathbb{N}} \to A^{\mathbb{N}}$, $f_1(n) = f(1+n)$. This shift map extends to a right action of $\mathbb{N}$ on $A^{\mathbb{N}}$.

Now, $\mathbb{N}$ is the free monoid on a single generator: the set of all finite strings over the alphabet $\{1\}$ with operation concatenation. This of course generalises to the free monoid over an alphabet X (which we always take to be finite): the set X^* of all finite strings over X. The right Cayley graph of X^* is an infinite, regular, rooted tree whose vertices of level n are the words X^n over X of length n.

A way of visualising an element of A^{X^*} is as a *tree portrait*: every vertex of the right Cayley graph of X^* is decorated by a value in A.

Passing from $\mathbb{N}$ to X^*, we obtain the following.

Definition 2.1. A subset $S \subseteq A^{X^*}$ is *self-similar over* A if it is invariant under the shifts $\cdot_x : A^{X^*} \to A^{X^*}$, for $x \in X$, defined by $f \mapsto f_x$, where $f_x(u) = f(xu)$. If the shifts are onto $S \subseteq A^{X^*}$, then S is *self-replicating over* A.

Extending these maps to all of X^* yields a right action of X^* on A^{X^*}.

A *tree shift over the alphabet* A is a closed subset $S \subseteq A^{X^*}$ that is self-similar over A.

Tree shifts have been studied in [3, 13].

Suppose now that A is a finite group. Then A^{X^*} is a compact totally disconnected (i.e., profinite) group, whose basic open neighbourhoods of the identity are the cylinder sets

$$\mathsf{Triv}^A(n) := \{f \in A^{X^*} : f(u) = e_A \text{ for all } u \in X^{<n}\}, \quad \text{where } X^{<n} := \bigcup_{i=0}^{n-1} X^i,\ n \geq 1.$$

The subspace topology of $G \leq A^{X^*}$ has $\mathsf{Triv}^A_G(n) := G \cap \mathsf{Triv}^A(n)$, $n \geq 1$ as basic identity neighbourhoods.

Notation. For an element $g \in A^{X^*}$ and $v \in X^*$, we write $\delta_v(g)$ to mean the element $f \in A^{X^*}$ such that:

(1) $f_v = g$.

(2) $f(u)$ is trivial for all prefixes u of v.

(3) f_w is trivial for all $w \in X^*$ that is not comparable with v.

Given a subgroup $H \leq A^{X^*}$ and $n \in \mathbb{N}$, define $X^n * H := \prod_{v \in X^n} \delta_v(H) \leq \mathsf{Triv}^A(n)$.

Definition 2.2. A subgroup $G \leq A^{X^*}$ that is self-similar over A is said to *branch symbolically over* $H \leq G$ if $H \geq X * H$. In particular,

$$\mathsf{Triv}^A_G(1) \geq \mathsf{Triv}^A_H(1) \geq X * H \geq X * \mathsf{Triv}^A_H(1) = \mathsf{Triv}^A_H(2).$$

The most obvious example of a group A to take is $A = \mathsf{Sym}(X)$. Then the full shift $\mathsf{Sym}(X)^{X^*}$ can be viewed as the group $\mathsf{Aut}(X^*)$ of rooted tree automorphisms

$$\mathsf{Aut}(X^*) = \varprojlim_{n\geq 1} \mathsf{Aut}(X^{\leq n}) = \varprojlim_{n\geq 1} \overbrace{\mathsf{Sym}(X) \wr \cdots \wr \mathsf{Sym}(X)}^{n},$$

where

$$\mathsf{Aut}(X^{\leq n}) = \overbrace{\mathsf{Sym}(X) \wr \cdots \wr \mathsf{Sym}(X)}^{n}$$

is the automorphism group of the finite subtree of X^* consisting of words of length at most n. Multiplication and inversion of portraits are the operations in an infinitely iterated wreath product: if $f, g \in \mathsf{Sym}(X)^{X^*}$, then

$$\begin{aligned} fg(\varepsilon) &= f(\varepsilon)g(\varepsilon) \\ fg(x) &= f(g(\varepsilon)(x))g(x), \quad x \in X \\ &\cdots \\ fg(x_1x_2\cdots x_n) &= f(y_1y_2\cdots y_n)g(x_1x_2\cdots x_n), \\ &\qquad \text{where } y_i = g(x_1x_2\cdots x_{i-1})(x_i),\ x_i \in X. \end{aligned}$$

The element $y_1\cdots y_n \in X^*$ is the image of $x_1\cdots x_n \in X^*$ under the action of g. Writing $g[u]$ for the image of $u \in X^*$ under $g \in \mathsf{Aut}(X^*)$, the last line above becomes

$$fg(u) = f(g[u])g(u) \in \mathsf{Sym}(X).$$

In this setting, $\mathsf{Triv}^A(n) = \mathsf{St}(n) = \bigcap_{v\in X^n} \mathsf{St}(v)$, the stabiliser of all vertices of length at most n, and $\mathsf{Aut}(X^*)/\mathsf{St}(n) \cong \mathsf{Aut}(X^{\leq n})$. The definition of multiplication ensures that the shift maps $\cdot_x : \mathsf{Aut}(X^*) \to \mathsf{Aut}(X^*)$ are partial endomorphisms (endomorphisms when restricted to $\mathsf{St}(x)$).

When $A = \mathsf{Sym}(X)$, we do not mention A in the definition of self-similar or branching symbolically.

Definition 2.3. A *self-similar group* is a subgroup G of $\mathsf{Aut}(X^*)$ that is self-similar over $\mathsf{Sym}(X)$, that is, invariant under the right action of X^* on $\mathsf{Aut}(X^*)$ given by the shift maps $\cdot_w : \mathsf{Aut}(X^*) \to \mathsf{Aut}(X^*)$, $f_w(u) = f(wu)$, for $w \in X^*$. The group G is *self-replicating* if it is self-replicating over $\mathsf{Sym}(X)$.

A *closed self-similar group* is a closed subgroup of $\mathsf{Aut}(X^*)$ that is also self-similar (i.e., a subgroup of $\mathsf{Aut}(X^*)$ that is also a tree-sub-shift).

A self-similar group $G \leq \mathsf{Aut}(X^*)$ *branches over* $K \leq G$ if $X * K \leq K$. It is a *regular branch* group if it branches over some finite index normal subgroup.

Given a self-similar group $G \leq \mathsf{Aut}(X^*)$, its closure $\overline{G} \leq \mathsf{Aut}(X^*)$ is a closed self-similar group, called its *congruence completion*. It can also be seen as the inverse limit

$$\overline{G} = \varprojlim_{n\in\mathbb{N}} G/\mathsf{St}_G(n)$$

of the inverse system of groups $\{G/\mathsf{St}_G(n), n \in \mathbb{N}\}$ and canonical morphisms $\{s_{m,n} : G/\mathsf{St}_G(m) \to G/\mathsf{St}_G(n), m \geq n \in \mathbb{N}\}$, where $\mathsf{St}_G(n) = \mathsf{St}(n) \cap G$.

Many of the most studied examples of self-similar groups are regular branch. For such groups, it makes sense to consider the larger tree shift space $(\mathsf{Sym}(X) \times G/K)^{X^*}$ where now $A = \mathsf{Sym}(X) \times G/K \geq G/(\mathsf{St}(1) \cap K)$. Note that we can keep the previous wreath product structure, by letting all factors G/K act trivially, so that $(\mathsf{Sym}(X) \times G/K)^{X^*}$ becomes the group $S_K := \mathsf{Aut}(X^*) \ltimes (G/K)^{X^*}$. Recalling that we denoted by $g[u]$ the image of vertex $u \in X^*$ under $g \in \mathsf{Aut}(X^*)$, multiplication in S_K is defined by

$$(f, h)(g, l) = (fg, h^g l),$$

where $h^g(u) = h(g[u]) \in G/K$ and multiplication in G/K^{X^*} is performed componentwise. Again, this definition ensures that all shift maps $\cdot_x : S_K \to S_K$ for $x \in X$ are partial endomorphisms.

For this case, we will denote $\mathsf{Triv}^{\mathsf{Sym}(X)\times G/K}(n)$ simply by $\mathsf{Triv}^K(n)$. Having identified $(\mathsf{Sym}(X) \times G/K)^{X^*}$ with $S_K = \mathsf{Aut}(X^*) \ltimes (G/K)^{X^*}$, these subgroups become

$$\mathsf{Triv}^K(n) = \{f \in S_K : f(u) = (\mathrm{id}, K) \text{ for all } u \in X^{<n}\}.$$

Given any self-similar group $G \leq \mathsf{Aut}(X^*)$ that branches over some finite index normal subgroup K, there is a natural embedding

$$\theta_K : G \hookrightarrow S_K, \quad g \mapsto (g(u), (g_u K)_{u \in X^*}),$$

where $g_u K$ is, for $u \in X^*$, the image modulo K of the u-shift $g_u \in G$ of $g \in G$. By the definition of multiplication in S_K, the map θ_K is a homomorphism. Recalling that we have identified $S_K = \mathsf{Aut}(X^*) \ltimes (G/K)^{X^*}$ with $(\mathsf{Sym}(X) \times G/K)^{X^*}$, the element $\theta_K(g)$ can be identified with $(g_u(\mathsf{St}(1) \cap K))_{u \in X^*}$, because of the wreath product structure we have given to $(\mathsf{Sym}(X) \times G/K)^{X^*}$ and because we are viewing $G/(\mathsf{St}(1) \cap K)$ as a subgroup of $G/K \times G/\mathsf{St}_G(1) \leq G/K \times \mathsf{Sym}(X)$.

Notation. We will identify G with its image $\theta_K(G) \leq S_K$ whenever this is convenient. Abusing notation, $\mathsf{Triv}_G^K(n)$ will denote $\mathsf{Triv}_{\Theta_K(G)}^K(n) = \theta_K^{-1}(\mathsf{Triv}^K(n)) = \mathsf{St}_G(n) \cap X^{n-1} * K$.

Definition 2.4. The closure $\widetilde{G_K}$ of G in S_K is a profinite group that is self-similar over $\mathsf{Sym}(X) \times G/K$ and branches symbolically over the closure of K in S_K. It is called the *K-symbolic completion* of G. It can also be seen as the inverse limit

$$\widetilde{G_K} = \varprojlim_{n \in \mathbb{N}} G/\mathsf{Triv}_G^K(n)$$

of the inverse system of groups $\{G/\mathsf{Triv}_G^K(n) : n \geq 1 \in \mathbb{N}\}$ and canonical homomorphisms $\{t_{m,n} : G/\mathsf{Triv}_G^K(m) \to G/\mathsf{Triv}_G^K(n) : m \geq n \geq 1 \in \mathbb{N}\}$.

Definition 2.5. Given a group $G \leq \mathsf{Aut}(X^*)$, one can also consider, for a vertex $v \in X^*$, the rigid stabiliser of v in G:

$$\mathsf{Rst}_G(v) = \{g \in G : g(u) = \mathrm{id} \in \mathsf{Sym}(X) \text{ for all } X^* \setminus vX^*\}$$

the subgroup that fixes all vertices outside the subtree vX^* with root v. Note that if $vX^* \cap uX^* = \emptyset$, then $\mathsf{Rst}_G(v)$ and $\mathsf{Rst}_G(u)$ commute. In particular, the subgroup generated by all $\mathsf{Rst}_G(v)$ for $v \in X^n$ is the direct product

$$\mathsf{Rst}_G(n) = \prod_{v \in X^n} \mathsf{Rst}_G(v)$$

and called the *rigid level stabiliser* of level n in G.

If G is regular branch over K, then $X * K \leq \mathsf{Rst}_G(1)$ has finite index in G. Continuing inductively, $X^n * K \leq \mathsf{Rst}_G(n)$ all have finite index in G, for $n \geq 1$. This last condition, along with a transitive action on X^n for each n, ensures that G is a *branch group*. The completion $\varprojlim_n G/\mathsf{Rst}_G(n)$ is called the *branch completion* of G.

In general, since $\mathsf{Triv}_G^K(n) = \mathsf{St}_G(n) \cap X^{n-1} * K \leq \mathsf{Rst}_G(n-1)$, the K-symbolic completion of G maps onto the branch completion. For a certain choice of K, these two completions actually coincide.

The subgroup $M := \bigcap_{u \in X^*} (\mathsf{Rst}_G(u))_u$ is the unique maximal branching subgroup of G. If G is transitive on all levels of X^*, then, according to [1, Corollary 1.6], there exists $n \geq 0$ such that $X^m * M \geq \mathsf{Rst}_G(m+n)$ for all $m \geq 0$. In particular,

$$\begin{aligned}\mathsf{Triv}_G^M(m+1) = \mathsf{St}_G(m+1) \cap X^m * M &\geq \mathsf{St}_G(m+1) \cap \mathsf{Rst}_G(m+n) \\ &\geq \mathsf{Rst}_G(m+n+1).\end{aligned}$$

Thus, $\{\mathsf{Triv}_G^M(n) : n \geq 1 \in \mathbb{N}\}$ and $\{\mathsf{Rst}_G(n) : n \geq 1 \in \mathbb{N}\}$ generate the same profinite topology, so the M-symbolic and branch completions of G coincide.

Remark. In the remainder, we shall always assume that $G \leq \mathsf{Aut}(X^*)$ is a self-similar, level-transitive, regular branch group, so that we can identify the M-symbolic with the branch completion of G and omit the M notation, writing $\mathsf{Triv}_G(n)$ instead of $\mathsf{Triv}_G^M(n)$.

2.2. Completions and kernels

Let $G \leq \mathsf{Aut}(X^*)$ be a self-similar, level-transitive, regular branch group with maximal branching subgroup K, so that there exists $k \in \mathbb{N}$ such that $\mathsf{Triv}_G(n) \geq \mathsf{Rst}_G(n+k) \geq \mathsf{Triv}_G(n+k)$ for every $n \in \mathbb{N}$. For every $m \geq n \geq 1 \in \mathbb{N}$, the following square of canonical morphisms commutes:

$$\begin{array}{ccc} G/\mathsf{Triv}_G(m) & \xrightarrow{\psi_m} & G/\mathsf{St}_G(m) \\ \big\downarrow{\scriptstyle t_{m,n}} & & \big\downarrow{\scriptstyle s_{m,n}} \\ G/\mathsf{Triv}_G(n) & \xrightarrow{\psi_n} & G/\mathsf{St}_G(n) \end{array}$$

This gives a unique morphism $\psi : \widetilde{G} = \varprojlim_n G/\mathsf{Triv}_G(n) \to \overline{G} = \varprojlim_n G/\mathsf{St}_G(n)$ defined by $\psi((g_n)_n) = (\psi_n(g_n))_n$, whose kernel is

$$\ker \psi = \varprojlim_n \ker \psi_n = \varprojlim_n \mathsf{St}_G(n)/\mathsf{Triv}_G(n).$$

This is the inverse limit of the inverse system whose maps $r_{m,n} : \mathsf{St}_G(m)/\mathsf{Triv}_G(m) \to \mathsf{St}_G(n)/\mathsf{Triv}_G(n)$ are simply the restrictions of the $t_{m,n}$ for all $m \geq n \geq 1 \in \mathbb{N}$.

Definition 2.6. With notation as above, $\ker \psi$ is the *rigid (or symbolic) kernel* of G.

Since G is residually finite, it embeds into its profinite completion $\widehat{G}$, the inverse limit of the inverse system $\{G/N : N \trianglelefteq_f G\}$ of finite quotients of G, and canonical maps $q_{M,N} : G/M \to G/N$ for $M \leq N \trianglelefteq_f G$.

Since each $\mathsf{Triv}_G(n)$ and $\mathsf{St}_G(n)$ are normal subgroups of finite index in G, restricting to $\widehat{G}$ the projection maps to, respectively, $\prod_n G/\mathsf{Triv}_G(n)$ and $\prod_n G/\mathsf{St}_G(n)$ gives surjective morphisms $\eta : \widehat{G} \to \widetilde{G}$ and $\theta : \widehat{G} \to \overline{G}$. Moreover, $\theta = \psi \circ \eta$.

This gives two further kernels: the *branch kernel*

$$\ker \eta = \varprojlim_{N \trianglelefteq_f \mathsf{Triv}_G(n), n} \mathsf{Triv}_G(n)/N$$

and the *congruence kernel*

$$\ker \psi \circ \eta = \varprojlim_{N \trianglelefteq_f \mathsf{St}_G(n), n} \mathsf{St}_G(n)/N.$$

These completions and their kernels are important to understand the structure of groups acting on rooted trees, as they give insight into their finite quotients. By analogy with arithmetic groups, groups for which the congruence kernel is trivial are said to have the *congruence subgroup property*.

Examples include many of the first discovered self-similar groups (Grigorchuk, Gupta–Sidki, etc.). The first examples of groups with non-trivial congruence kernels were constructed by Pervova in [14], where she also showed that they have trivial rigid kernel. It was not until [1], where a systematic study of the congruence subgroup property for branch groups was undertaken, that an example (the Hanoi Towers group) was shown to have non-trivial branch and rigid kernel. Further examples followed in [16]. The computations to find the rigid kernel were technical in these cases. The advantage of looking at tree shifts and portraits is that it clarifies what these kernels are.

The map ψ is precisely the restriction to $\widetilde{G}$ of the canonical epimorphism $S_M = \mathsf{Aut}(X^*) \ltimes (G/M)^{X^*} \twoheadrightarrow \mathsf{Aut}(X^*)$. The kernel of ψ therefore consists of all tree portraits in $\widetilde{G}$ that have trivial $\mathsf{Sym}(X)$ part in every vertex.

2.3. Finitely constrained groups

We briefly return to the general setting of shifts of A^{X^*} (closed subsets that are invariant under shift maps), where A is any set. A *pattern* is a function from $X^{<n}$ to A for some

$n \geq 1$, which is the *size* of the pattern. We say that a pattern of size $n \geq 1$ *appears* in some $f \in A^{X^*}$ if there exists $u \in X^*$ such that $f_u|_{X^{<n}}$ is the specified pattern. We will also say that this is *the pattern of size n at u of f*.

It is well known in symbolic dynamics that any shift can be defined by declaring some collection of *forbidden patterns* which do not appear in any element of the shift. If a finite collection of forbidden patterns suffices for this definition, the shift is called a *shift of finite type*.

In the case where A is a group, one can analogously define and study *groups of finite type* or *finitely constrained groups*: subgroups of A^{X^*} that are simultaneously shifts of finite type. These groups are characterised in the following theorem [13].

Theorem 2.7 ([13]). *Let $\Gamma \leq A^{X^*}$, where A is a group that acts on X, and $n \geq 1$. The following are equivalent:*

(1) *Γ is a finitely constrained group defined by patterns of size n.*
(2) *Γ is closed, self-similar over A, and branches symbolically over* $\mathsf{Triv}_\Gamma(n-1)$.
(3) *Γ is the closure of a group $G \leq A^{X^*}$ that is self-similar over A and branches symbolically over* $\mathsf{Triv}_G(n-1)$.

Observe that if $G \leq \mathsf{Aut}(X^*)$ is a self-similar, regular branch group, with maximal branching subgroup K, then

$$\mathsf{Triv}_G^K(2) = \mathsf{St}_G(2) \cap X * K = X * (\mathsf{St}_G(1) \cap K) = X * \mathsf{Triv}_G^K(1).$$

In other words, G branches over $\mathsf{Triv}_G^K(1)$, so, after making the identifications explained in the previous section, we obtain the following corollary from Theorem 2.7.

Corollary 2.8. *If $G \leq \mathsf{Aut}(X^*)$ is a self-similar, regular branch group, with maximal branching subgroup K, then the symbolic (branch) completion $\widetilde{G_K}$ is a finitely constrained group defined by patterns of size* 2.

Moreover, if G is a self-similar, regular branch group, with maximal branching subgroup K, then the kernel R_K of the canonical "forgetful" map $\Psi : S_K = \mathsf{Aut}(X^*) \ltimes (G/K)^{X^*} \twoheadrightarrow \mathsf{Aut}(X^*)$ is precisely $R_K = \mathrm{id} \ltimes (G/K)^{X^*}$, which is evidently a finitely constrained group defined by patterns of size 1. The rigid kernel of G is therefore $\widetilde{G_K} \cap R_K$, the intersection of two closed groups that are self-similar over $\mathsf{Sym}(X) \times G/K$, and it branches symbolically over $\mathsf{Triv}^K_{R_K \cap \widetilde{G_K}}(1)$. Applying Theorem 2.7, we conclude the following corollary.

Corollary 2.9. *If $G \leq \mathsf{Aut}(X^*)$ is a self-similar, regular branch group, with maximal branching subgroup K, then the rigid kernel* $\ker : \widetilde{G_K} \to \overline{G}$ *is a finitely constrained group defined by patterns of size* 2.

Corollary 2.8 also has consequences for the structure of $\overline{G}$ as a tree shift. It is natural to wonder whether it is also a finitely constrained group. A priori, it is only a *sofic tree shift*. For our purposes (see [3, Section 2] for equivalent definitions), given two finite alphabets A, B, a sofic tree shift in B^{X^*} is the image of a tree shift of finite type in A^{X^*} under a continuous map that commutes with the shift action of X^*.

The canonical map $\Psi : S_K = \mathsf{Aut}(X^*) \ltimes (G/K)^{X^*} \to \mathsf{Aut}(X^*), (f,h) \mapsto f$ is indeed continuous, and it commutes with the shift action of X^* on S_K and $\mathsf{Aut}(X^*)$ because

$$\Psi((f, h)_u)(v) = f(uv) = \Psi(f, h)_u(v) \in \mathsf{Sym}(X)$$

holds for all $(f,h) \in S_K$ and $u, v \in X^*$. The map $\psi : \widetilde{G} \to \overline{G}$ is the restriction of Ψ to the tree shift of finite type $\widetilde{G}$, so $\overline{G}$ is indeed a sofic tree shift.

The following theorem of Penland and Šunić shows that, in many cases, groups that are sofic tree shifts are already finitely constrained groups.

Theorem 2.10 ([13, Theorem A]). *Let A be a finite group acting on X and $G \le A^{X^*}$. If the normaliser of G in A^{X^*} contains a level-transitive subgroup that is self-replicating over A, then G is a sofic tree shift group if and only if G is a finitely constrained group.*

This together with Corollary 2.8 allows us to conclude the following.

Corollary 2.11. *Let $G \le \mathsf{Aut}(X^*)$ be a self-replicating, level-transitive, regular branch group. Then the closure $\overline{G}$ of G in $\mathsf{Aut}(X^*)$ is a finitely constrained group.*

Note that the above says nothing about the size of defining patterns of $\overline{G}$, just that they are bounded.

Theorem 2.12 (Theorem 2). *Let $G \le \mathsf{Aut}(X^*)$ be a self-similar, level-transitive, regular branch group, with maximal branching subgroup K. The following are equivalent:*

(1) *G has trivial rigid kernel.*

(2) *G branches over some level stabiliser.*

(3) *$K \ge \mathsf{St}_G(n)$ for some $n \ge 1$.*

If G is in addition self-replicating, then $\overline{G}$ is a finitely constrained group and the above items are also equivalent to

(4) $|X * G : \mathsf{St}_G(1)] = |X * \overline{G} : \overline{\mathsf{St}_G(1)}]$.

Proof. Let $G \le \mathsf{Aut}(X^*)$ be a self-similar, level-transitive, regular branch group, with maximal branching subgroup K.

(1) $\Leftrightarrow$ (2). The rigid kernel of G is trivial if and only if $\{\mathsf{Triv}_G(n) : n \in \mathbb{N}\}$ and $\{\mathsf{St}_G(n) : n \in \mathbb{N}\}$ generate the same topology on G, which holds if and only if for every $n \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that $\mathsf{Triv}_G(n) \ge \mathsf{St}_G(m)$. This is equivalent to the existence of some $s \in \mathbb{N}$ such that $X * \mathsf{St}_G(s) = \mathsf{St}_G(s + 1)$. To wit, if $\mathsf{Triv}_G(n) \ge \mathsf{St}_G(m)$,

then, since G branches over $\mathsf{Triv}_G(1)$ and K is the maximal branching subgroup, we have $\mathsf{St}_G(m) \leq K$ and thus $X * \mathsf{St}_G(m) \leq \mathsf{St}(m+1) \cap (X * K) \leq \mathsf{St}_G(m+1)$, so $X * \mathsf{St}_G(m) = \mathsf{St}_G(m+1)$. Conversely, if $X * \mathsf{St}_G(s) = \mathsf{St}_G(s+1)$, then G branches over $\mathsf{St}_G(s)$ and therefore $\mathsf{St}_G(s) \leq K$, so $\mathsf{St}_G(s) = \mathsf{St}_K(s)$ and $\mathsf{St}_G(n+s) = X^n * \mathsf{St}_K(s) \leq X^n * \mathsf{St}_K(1) = \mathsf{Triv}_G(n+1)$ for every $n \in \mathbb{N}$.

(2) $\Leftrightarrow$ (3). If K is the maximal branching subgroup, then $K \geq \mathsf{St}_G(n)$ for some $n \geq 1$ if and only if $\mathsf{St}_G(n+1) \geq X * \mathsf{St}_G(n)$.

If G is also assumed to be self-replicating, then Corollary 2.11 says that $\overline{G}$ is a finitely constrained group. In particular, by Theorem 2.7, there exists some $n \geq 0$ such that $\overline{\mathsf{St}_G(n+1)} = X * \overline{\mathsf{St}_G(n)}$. This implies that

$$\begin{aligned}|X * G : X * \mathsf{St}_G(n)| &= |X * \overline{G} : X * \overline{\mathsf{St}_G(n)}| = |X * \overline{G} : \overline{\mathsf{St}_G(n+1)}| \\ &= |X * \overline{G} : \overline{\mathsf{St}_G(1)}||\overline{\mathsf{St}_G(1)} : \overline{\mathsf{St}_G(n+1)}| \\ &= |X * \overline{G} : \overline{\mathsf{St}_G(1)}||\mathsf{St}_G(1) : \mathsf{St}_G(n+1)|.\end{aligned}$$

At the same time,

$$\begin{aligned}|X * G : X * \mathsf{St}_G(n)| &= \frac{|X * G : \mathsf{St}_G(n+1)|}{|X * \mathsf{St}_G(n) : \mathsf{St}_G(n+1)|} \\ &= \frac{|X * G : \mathsf{St}_G(1)| \cdot |\mathsf{St}_G(n) : \mathsf{St}_G(n+1)|}{|X * \mathsf{St}_G(n) : \mathsf{St}_G(n+1)|}.\end{aligned}$$

Putting these together, $|X * G : \mathsf{St}_G(1)| = |X * \overline{G} : \overline{\mathsf{St}_G(1)}| \cdot |X * \mathsf{St}_G(n) : \mathsf{St}_G(n+1)|$; therefore, $|X * G : \mathsf{St}_G(1)| = |X * \overline{G} : \overline{\mathsf{St}_G(1)}|$ if and only if $X * \mathsf{St}_G(n) \leq \mathsf{St}_G(n)$. This shows that items (2) and (4) are equivalent. ∎

Note that the above result is not "quantitative". Namely, when the two indices in condition (4) are not equal, we have no information on the rigid kernel other than that it is non-trivial.

2.4. The rigid kernel

For the rest of this section, fix $G \leq \mathsf{Aut}(X^*)$, a self-similar, level-transitive, regular branch group, with maximal branching subgroup K. We will drop the K notation from now on, so $\widetilde{G} = \widetilde{G_K}$ and $R = R_K$, etc.

As stated above, the rigid kernel $\ker(\widetilde{G} \to \overline{G})$ is $\widetilde{G} \cap R$ where $R = \mathrm{id} \ltimes (G/K)^{X^*}$, and it is a finitely constrained group defined by patterns of size 2, by Corollary 2.9.

Notice in particular that the rigid kernel inherits from R any property of G/K that is inherited by subgroups of Cartesian products (for instance, and this will be important for our examples, being elementary abelian).

The patterns of size 2 defining $\widetilde{G} \cap R$ are in

$$\mathsf{St}_G(2)/\mathsf{Triv}_G(2) \cong \mathsf{St}_G(2)(X * K)/(X * K).$$

Once a root pattern is fixed, the patterns of size 2 at each $x \in X$ must be chosen so that their value at x (which must be in $\{\mathrm{id}\} \times G/K$) matches the value at x of the root pattern of size 2. This means that the pattern of size 2 at $x \in X$ is in a coset of $(\mathsf{St}_G(2) \cap \mathsf{Triv}_G(1))/\mathsf{Triv}_G(2)$ inside $\mathsf{St}_G(2)/\mathsf{Triv}_G(2)$ or, equivalently, in a coset of $(\mathsf{St}_G(2)(X * K) \cap K)/(X * K)$ in $\mathsf{St}_G(2)(X * K)/(X * K)$. Note that $K/(X * K)$ is not trivial in general, so there may still be non-trivial elements in $(\mathsf{St}_G(2)(X * K) \cap K)/(X * K)$.

The same argument holds for all $v \in X^* \setminus \{\varepsilon\}$, so $\widetilde{G} \cap R$ is the inverse limit of the extensions

$$X^{n-2} * \frac{\mathsf{St}_G(2) \cap K}{\mathsf{Triv}_G(2)} \to \mathsf{St}_G(n)/\mathsf{Triv}_G(n) \to \mathsf{St}_G(n-1)/\mathsf{Triv}_G(n-1) \quad n \geq 3.$$

Lemma 2.13. *Let X be the inverse limit of the inverse system $\{X_i, \phi_{j,i} : X_j \to X_i \mid j \geq i \geq 1 \in \mathbb{N}\}$ of finite groups and homomorphisms. Then X is also the inverse limit of the surjective inverse system $\{Y_i, \psi_{j,i} : j \geq i \geq 1\}$, where $Y_i = \bigcap_{j \geq i} \operatorname{im} \phi_{j,i}$ and $\psi_{j,i} = \phi|_{Y_j}$.*

Proof. A straightforward exercise in the definitions of inverse limit. ∎

Theorem 2.14. *Let G be a self-similar, level-transitive, regular branch group with maximal branching subgroup K.*

(1) *The rigid kernel of G is finite if and only if*

$$\bigcap_{n \geq m} (\mathsf{St}_G(n)\mathsf{Triv}_G(m)) \cap K \leq \mathsf{Triv}_G(m) \quad \textit{for every } m \geq 1.$$

(2) *If the system $\{\mathsf{St}_G(m)/\mathsf{Triv}_G(m), r_{n,m}\}$ is surjective, then the rigid kernel is finite if and only if $\mathsf{St}_G(2) \cap K \leq \mathsf{Triv}_G(2)$, in which case the rigid kernel is isomorphic to $\mathsf{St}_G(1)/\mathsf{Triv}_G(1)$.*

Proof. For each $i \geq 1$, define $H_i := \bigcap_{k \geq i} \mathsf{St}_G(k)\mathsf{Triv}_G(i)/\mathsf{Triv}_G(i)$. By Lemma 2.13, we can replace each $\mathsf{St}_G(i)/\mathsf{Triv}_G(i)$ by H_i and each $r_{j,i}$ by its restriction to H_j (which we still denote by $r_{j,i}$). The rigid kernel is finite if and only if there exists $i \geq 1$ such that $r_{j,i}$ is an isomorphism for each $j \geq i$, which occurs if and only if the kernel of $r_{j,i}$ is trivial. This is equivalent to $\mathsf{Triv}_G(i) \cap \bigcap_{k \geq j} \mathsf{St}_G(k)\mathsf{Triv}_G(j) = \mathsf{Triv}_G(j)$, which occurs if and only if the left-hand side is contained in the right-hand side, as the other containment holds always. Now,

$$\begin{aligned}
&\mathsf{Triv}_G(i) \cap \bigcap_{k \geq j} \mathsf{St}_G(k)\mathsf{Triv}_G(j) \\
&\quad = X^{i-1} * (K \cap \mathsf{St}_G(1)) \cap \bigcap_{k \geq j} (G \cap X^{i-1} * (\mathsf{St}_G(k-i+1)\mathsf{Triv}_G(j-i+1))) \\
&\quad = X^{i-1} * \Big(K \cap \bigcap_{k \geq j} \mathsf{St}_G(k-i+1)\mathsf{Triv}_G(j-i+1)\Big)
\end{aligned}$$

and similarly, $\mathsf{Triv}_G(j) = X^{i-1} * \mathsf{Triv}_G(j-i+1)$. So the rigid kernel is finite if and only if there exists $i \geq 1$ such that

$$X^{i-1} * \Big(K \cap \bigcap_{k \geq j} \mathsf{St}_G(k-i+1)\mathsf{Triv}_G(j-i+1)\Big) \leq X^{i-1} * \mathsf{Triv}_G(j-i+1)$$

for every $j \geq i$. But this occurs exactly when

$$\Big(K \cap \bigcap_{k \geq j} \mathsf{St}_G(k-i+1)\mathsf{Triv}_G(j-i+1)\Big) \leq \mathsf{Triv}_G(j-i+1).$$

This makes $j-i$ the only variable, so that the rigid kernel is finite if and only if

$$\Big(K \cap \bigcap_{n \geq m} \mathsf{St}_G(n)\mathsf{Triv}_G(m)\Big) \leq \mathsf{Triv}_G(m) \quad \text{for every } m \geq 1. \tag{2.1}$$

If the system $\{\mathsf{St}_G(m)/\mathsf{Triv}_G(m), r_{n,m}\}$ is surjective, then $\mathsf{St}_G(n)\mathsf{Triv}_G(m) = \mathsf{St}_G(m)$ for every $n \geq m \geq 1$, so (2.1) becomes $K \cap \mathsf{St}_G(m) \leq \mathsf{Triv}_G(m)$ for every $m \geq 1$. In particular, $K \cap \mathsf{St}_G(2) = \mathsf{Triv}_G(2)$. Now, suppose that the inverse system is surjective and that $K \cap \mathsf{St}_G(2) = \mathsf{Triv}_G(2)$. Then, for each $m \geq 2$, the kernel of $r_{m,m-1}$ is

$$\frac{\mathsf{St}_G(m) \cap \mathsf{Triv}_G(m-1)}{\mathsf{Triv}_G(m)} = \frac{X^{m-2} * (\mathsf{St}_G(2) \cap K)}{\mathsf{Triv}_G(m)} = \frac{X^{m-2} * \mathsf{Triv}_G(2)}{\mathsf{Triv}_G(m)} = \frac{\mathsf{Triv}_G(m)}{\mathsf{Triv}_G(m)}$$

where the second-to-last equality holds because $K \cap \mathsf{St}_G(2) = \mathsf{Triv}_G(2)$. This means that $r_{m,m-1}$ is an isomorphism, for each $m \geq 2$, and therefore the rigid kernel is isomorphic to $\mathrm{im}(r_{m,1}) = \mathsf{St}_G(1)/\mathsf{Triv}_G(1)$, which is finite, as required. ∎

Lemma 2.15. *The inverse system* $\{\mathsf{St}_G(n)/\mathsf{Triv}_G(n), r_{m,n} : m \geq n \geq 1 \in \mathbb{N}\}$ *is surjective if and only if* $\mathsf{St}_G(2)K \geq \mathsf{St}_G(1)$.

Proof. For every $m \geq n \geq 1$, the image of $r_{m,n}$ is $\mathsf{St}_G(m)\mathsf{Triv}_G(n)/\mathsf{Triv}_G(n)$. Observe that

$$\begin{aligned}\mathsf{St}_G(n+1)\mathsf{Triv}_G(n) &= (G \cap X^{n-1} * \mathsf{St}_G(2))(X^{n-1} * \mathsf{Triv}_G(1)) \\ &= G \cap X^{n-1} * (\mathsf{St}_G(2)\mathsf{Triv}_G(1)) \\ &= G \cap X^{n-1} * (\mathsf{St}_G(1) \cap \mathsf{St}_G(2)K) \\ &\leq G \cap X^{n-1} * \mathsf{St}_G(1) = \mathsf{St}_G(n)\end{aligned} \tag{2.2}$$

where the third equality follows from $\mathsf{Triv}_G(1) = \mathsf{St}_G(1) \cap K$. If the inverse system is surjective, then, in particular, $\mathsf{St}_G(2)\mathsf{Triv}_G(1) = \mathsf{St}_G(1) \cap \mathsf{St}_G(2)K = \mathsf{St}_G(1)$, so $\mathsf{St}_G(2)K \geq \mathsf{St}_G(1)$. Conversely, if $\mathsf{St}_G(2)K \geq \mathsf{St}_G(1)$, then we obtain equalities in (2.2). Inductively, this implies that $\mathsf{St}_G(m)\mathsf{Triv}_G(n) = \mathsf{St}_G(n)$ for every $m \geq n \geq 1$. ∎

Theorem 2.16 (Theorem 3). *Let G be a self-similar, level-transitive, regular branch group with maximal branching subgroup K. Suppose that* $\mathsf{St}_G(2)K \geq \mathsf{St}_G(1)$.

(1) *The rigid kernel of* G *is trivial if and only if* $\mathsf{St}_G(2) = \mathsf{Triv}_G(2)$.

(2) *Suppose moreover that* G/K *is in a class of groups that is closed under subgroups, quotients and direct products, in which all short exact sequences that are in the class split as direct products (e.g., elementary abelian groups). Then the rigid kernel is*

$$\Gamma = \frac{\mathsf{St}_G(2)}{\mathsf{Triv}_G(2)} \times \prod_{X^*\setminus\{\varepsilon\}} \frac{\mathsf{St}_G(2) \cap K}{\mathsf{Triv}_G(2)},$$

and it is infinite if and only if $\mathsf{St}_G(2) \cap K/\mathsf{Triv}_G(2)$ *is non-trivial.*

Proof. According to Lemma 2.15, the inverse system $\{\mathsf{St}_G(i)/\mathsf{Triv}_G(i), r_{j,i}\}$ is surjective.

(1) The rigid kernel is trivial if and only if $\mathsf{St}_G(n) = \mathsf{Triv}_G(n)$ for every $n \geq 1$. Conversely, suppose that $\mathsf{St}_G(2) = \mathsf{Triv}_G(2)$. Then $\mathsf{St}_G(2) \cap K = \mathsf{Triv}_G(2)$, so Theorem 2.14 implies that the rigid kernel is isomorphic to $\mathsf{St}_G(1)/\mathsf{Triv}_G(1)$, which is trivial because it is the image of the trivial quotient $\mathsf{St}_G(2)/\mathsf{Triv}_G(2)$.

(2) For every $n \geq 2$, we have the following short exact sequence:

$$1 \to \frac{\mathsf{St}_G(n) \cap \mathsf{Triv}_G(n-1)}{\mathsf{Triv}_G(n)} \to \frac{\mathsf{St}_G(n)}{\mathsf{Triv}_G(n)} \to \frac{\mathsf{St}_G(n-1)}{\mathsf{Triv}_G(n-1)} \to 1,$$

where the leftmost non-trivial term is equal to $X^{n-2} * (\mathsf{St}_G(2) \cap K/\mathsf{Triv}_G(2))$. Since $\mathsf{St}_G(n)/\mathsf{Triv}_G(n) \leq \prod_{X^{<n}} G/K$, the assumption on G/K implies that the above short exact sequence splits as a direct product. Inductively, this means that

$$\frac{\mathsf{St}_G(n)}{\mathsf{Triv}_G(n)} = \frac{\mathsf{St}_G(2)}{\mathsf{Triv}_G(2)} \times \prod_{\varepsilon \neq x \in X^{<n}} x * \frac{\mathsf{St}_G(2) \cap K}{\mathsf{Triv}_G(2)}$$

for every $n \geq 2$. Taking the inverse limit gives the result. ■

For all self-similar examples in the literature where the congruence subgroup problem has been considered, the inverse system $\{\mathsf{St}_G(n)/\mathsf{Triv}_G(n), r_{m,n} : m \geq n \geq 1 \in \mathbb{N}\}$ turns out not to be surjective, except for the Hanoi Towers group $H^{(3)}$, as will be seen in the next section. In all the other cases, the rigid kernel is trivial anyway because $K \geq \mathsf{St}_G(n)$ for some n. It would be interesting to have examples of self-similar branch groups with non-trivial rigid kernel and such that the inverse system $\{\mathsf{St}_G(n)/\mathsf{Triv}_G(n), r_{m,n} : m \geq n \geq 1 \in \mathbb{N}\}$ is not surjective.

3. An infinite family of Hanoi-like groups

We analyse, for odd d, the group $D = \langle a_0, \ldots, a_{d-1}\rangle$ and prove the items in Theorem 1.

Notation. All arithmetic operations on the indices are done modulo d.

3.1. Element decomposition and the contracting property

The following are straightforward to verify.

Lemma 3.1. *With the notation of the introduction, and $x \in X$, we have:*

$$\begin{aligned} \mu_i(x) &= 2i - x & \rho^i(x) &= i + x \\ \mu_i \mu_j &= \rho^{2(i-j)} & \rho^i \rho^j &= \rho^{i+j} \\ \mu_i \rho^j &= \mu_{i-j/2} & \rho^j \mu_i &= \mu_{i+j/2} \\ \mu_i \rho^j \mu_i &= \rho^{-j} & \rho^{-j} \mu_i \rho^j &= \mu_{i-j} \end{aligned}$$

$$\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1} = \begin{cases} \rho^{2(i_m - i_{m-1} + i_{m-2} - i_{m-3} + \cdots + i_2 - i_1)}, & m \text{ even}, \\ \mu_{i_m - i_{m-1} + i_{m-2} - i_{m-3} + \cdots - i_2 + i_1}, & m \text{ odd}. \end{cases}$$

$$\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1}(x) = \begin{cases} 2(i_m - i_{m-1} + i_{m-2} - i_{m-3} + \cdots + i_2 - i_1) + x, & m \text{ even}, \\ 2(i_m - i_{m-1} + i_{m-2} - i_{m-3} + \cdots - i_2 + i_1) - x, & m \text{ odd}. \end{cases}$$

Lemma 3.2. *Let $g = a_{i_m} a_{i_{m-1}} \cdots a_{i_2} a_{i_1}$. The root permutation of g is*

$$g_{(\varepsilon)} = \mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1} = \begin{cases} \rho^{2(i_m - i_{m-1} + i_{m-2} - i_{m-3} + \cdots + i_2 - i_1)}, & m \text{ even}, \\ \mu_{i_m - i_{m-1} + i_{m-2} - i_{m-3} + \cdots - i_2 + i_1}, & m \text{ odd}. \end{cases}$$

The following table gives the position to which each a_{i_j} is contributed in the first level decomposition of g:

a_{i_j}	*contributed to position*
a_{i_1}	i_1
a_{i_2}	$2i_1 - i_2$
a_{i_3}	$2i_1 - 2i_2 + i_3$
a_{i_4}	$2i_1 - 2i_2 + 2i_3 - i_4$
$\cdots$	

If a_{i_k} and a_{i_j}, with $k > j$ happen to be contributed to the same position, then a_{i_k} appears to the left of a_{i_j} (just as in g) in the corresponding section. More formally,

$$\begin{aligned} g &= \prod_{j=m}^{1} a_{i_j} = \prod_{j=m}^{1} \mu_{i_j}\delta_{i_j}(a_{i_j}) = \prod_{j=m}^{1} \mu_{i_j} \cdot \prod_{j=m}^{1} \delta_{\mu_{i_1}\mu_{i_2}\cdots\mu_{i_{j-1}}(i_j)}(a_{i_j}) \\ &= \prod_{j=m}^{1} \mu_{i_j} \cdot \prod_{j=m}^{1} \delta_{2(i_1 - i_2 + i_3 - i_4 + \cdots + (-1)^j i_{j-1}) + (-1)^{j+1} i_j}(a_{i_j}), \end{aligned}$$

where all the products should be written from left to right with decreasing indices.

Lemma 3.3. *The group D is contracting, with nucleus $\{1, a_0, \ldots, a_{d-1}\}$. Moreover, for all $g \in D$ and $x \in X$,*

$$|g_x| \leq \frac{1}{2}(|g| + 1),$$

where the length function is with respect to the generating set $S = \{a_0, \ldots, a_{d-1}\}$. In particular, for all $g \in D$ with $|g| \geq 2$, we have

$$|g_x| < |g|.$$

3.2. The commutator subgroup D'

For a word W over $S = \{a_0, \ldots, a_{d-1}\}$ and $j = 0, \ldots, d-1$, let $\exp_j(W)$ be the number of occurrences of the letter a_j in W.

Lemma 3.4. *If the word W represents the identity in D, then, for $j = 1, \ldots, d-1$,*

$$\exp_j(W) \equiv_2 0.$$

Proof. By induction on $|W|$. The only word of length up to 1 that represents the identity is the empty word. This establishes the base of the induction. Let the claim be true for all words of length $\leq m$, for some $m \geq 1$, and let W be a word of length $m + 1$ representing the identity in D. If two consecutive letters in W are equal, we may remove them without affecting the parity of the exponents or the group element represented by W, obtain a shorter word, and apply the induction hypothesis. Otherwise, because of the contraction, the first level sections of W can be represented by words $W_0, \ldots, W_{d-1}$ of length $\leq m$. Since the sections of a trivial element are trivial, each of the word $W_0, \ldots, W_{d-1}$ represents the identity. Therefore, by the induction hypothesis, for $j = 0, \ldots, d-1$,

$$\exp_j(W) = \sum_{i=0}^{d-1} \exp_j(W_i) \equiv_2 0. \qquad \blacksquare$$

The last result shows that the exponents modulo 2 are well defined at the level of group elements by setting $\exp_j(g) = \exp_j(W)$, for $g \in D$ and $j = 0, \ldots, d-1$, where W is any word over S representing g.

Lemma 3.5. *With notation as above, the following statements hold:*

(1) *The map* $\exp : D \to (\mathbb{Z}/2\mathbb{Z})^d$ *by* $\exp(g) = (\exp_0(g), \ldots, \exp_{d-1}(g))$ *is a surjective homomorphism.*

(2) *The kernel of* $\exp$ *is the commutator of D.*

(3) *A word W over S represents an element in the commutator D' if and only if* $\exp_j(W) \equiv_2 0$, *for $j = 0, \ldots, d-1$.*

(4) $|D : D'| = 2^d$ *and D/D' is elementary abelian* 2*-group of rank d, generated by (the image) of S.*

(5) $|D : \mathsf{St}_D(1)D'| = 2$ *and* $\mathsf{St}_D(1)D'/D'$ *is the kernel of the augmentation map* $\mathrm{aug} : D/D' \to \mathbb{Z}/2\mathbb{Z}$ *defined by* $a_0^{\alpha_0} \cdots a_{d-1}^{\alpha_{d-1}} \mapsto \sum_{i=0}^{d-1} \alpha_i$.

(6) $|\mathsf{St}_D(1) : \mathsf{St}_D(1) \cap D'| = |D'\mathsf{St}_D(1) : D'| = 2^{d-1}$.

Proof. Since the parity of $\exp_j$ does not depend on the representatives, we have $\exp_j(gh) \equiv_2 \exp_j(g) + \exp_j(h)$, which shows that exp is a homomorphism. Surjectivity is clear, since $\exp_j(a_j) = 1$.

Let K be the kernel of exp. By definition of exp, the element g is in K if and only if, for $j = 0, \dots, d-1$, we have $\exp_j(g) \equiv_2 0$. In other words, an element g represented by a word W over S is in K if and only if every generator appears an even number of times in W.

Since the image of exp is abelian, we have $D' \leq K$. On the other hand, since every generator of D has order 2, every element of D represented by a word in which every generator appears an even number of times is in the commutator D'. Thus, $K \leq D'$.

Since $D' = K$ and exp is surjective, we have $D/D' = D/K = (\mathbb{Z}/2\mathbb{Z})^d$. Note that, for $j = 0, \dots, d-1$, the vector $\exp(a_j)$ is the jth standard basis vector of $(\mathbb{Z}/2\mathbb{Z})^d$.

As $D/\mathsf{St}_D(1) \cong D(d)$ and d is odd, the abelianisation of $D(d)$ has order 2 and is isomorphic to $D/D'\mathsf{St}_D(1)$, where only the parity of the number of reflections matters.

The last item follows from the previous two. ■

Lemma 3.6. *The group D branches over its commutator subgroup D'.*

Proof. For $i \neq j$, the indices i, $2i - j$, and $3i - 2j$ are distinct, and from Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned} a_{2i-j}a_i a_j a_i &= \delta_{3i-2j}(a_i)\, \delta_{2i-j}(a_{2i-j}a_j)\, \delta_i(a_i) \\ (a_{2i-j}a_i a_j a_i)^2 &= \delta_{2i-j}([a_{2i-j}, a_j]). \end{aligned}$$

If $i = 0$, then

$$(a_{2i-j}a_i a_j a_i) = \delta_{-2j}(a_i)\delta_{-j}(a_{-j}a_j)\delta_0(a_0) = (a_0, *, *, \dots, *), \tag{3.1}$$

where $*$ represents elements that are not important; while if $i \neq 0$,

$$(a_{2i-j}a_i a_j a_i)^{a_{i/2}} = \delta_0(a_i)\delta_{j-i}(a_{2i-j}a_j)\delta_{2(j-i)}(a_i) = (a_i, *, *, \dots, *). \tag{3.2}$$

If $2i - j = 0$, set $h = 1$; otherwise, $h = a_{i-j/2}$. Then

$$((a_{2i-j}a_i a_j a_i)^2)^h = ([a_{2i-j}, a_j], 1, 1, \dots, 1). \tag{3.3}$$

By Lemma 3.5, the elements $((a_{2i-j}a_i a_j a_i)^2)^h$ are all in D', and letting i, j run through $\{0, \dots, d-1\}$ in (3.3) yields that D' contains the set $\{\delta_0([a_i, a_j]) : i \neq j \in \{0, \dots, d-1\}\}$. Conjugating this set by appropriate products of elements (3.1) and (3.2), we obtain that $D' \geq 0 * D'$. Since D is transitive on X, we can conjugate by elements that permute X to get $D' \geq X * D'$. ■

3.3. The first level stabiliser $\mathsf{St}_D(1)$

While it is possible to analyse the first level stabiliser directly, we find it suitable to use graph homology.

For a simple, connected, undirected graph $\Gamma = (V, E)$, the edge space is the vector space of dimension $|E|$ (the coordinates are indexed by E) over the field with 2 elements. Fix a vertex v_0 in V. Each walk w starting at v_0 in Γ is represented in the edge space by the vector $(\#_e(w))_{e\in E}$, where $\#_e(w)$ is the number of times the edge e is used, in either direction, in the walk w (note that only the parity plays a role). The cycle space of Γ, based at v_0, is the subspace of the edge space spanned by the representations of the closed walks in Γ, starting and ending at v_0 (the cycle space does not depend on v_0, but we prefer to have a base point anyway).

Let T be a spanning tree of Γ and E' be the set of edges not on the tree T. The number of edges in E' is $|E| - |V| + 1$. For each edge $e \in E'$ pick, arbitrarily, one endpoint to be the origin $o(e)$ and the other to be the terminus $t(e)$, and let p_e be unique path in T from v_0 to $o(e)$, followed by the edge e to $t(e)$, followed by the unique path in T from $t(e)$ to v_o. The cycle space has dimension $|E| - |V| + 1$ and the representatives of p_e, for $e \in E'$, in the edge space form a basis of the cycle space.

The cycle space can also be described through vertex conditions as follows. A vector $(x_e)_{e\in E}$ is in the cycle space if and only if, for every vertex v, we have $\sum_{e\sim v} x_e = 0$, where the summation is taken over all edges incident with v (all edges with either $o(e) = v$ or $t(e) = v$). Each of these $|V|$ vertex conditions is implied by the other vertex conditions, and any $|V| - 1$ of them are linearly independent. Note that the vertex conditions simply state that $(x_e)_{e\in E}$ represents a closed walk, based at v_0, such that each edge e in E is used, up to parity, x_e times, if and only if, for every vertex v, the number of distinct edges incident with v that are used an odd number of times is even (so that the number of entrances and exits from v can be matched).

In our situation, the graph Γ is the left Schreier graph of the action of D on its quotient $D(d)$, with respect to the generating set $S = \{a_0, \dots, a_{d-1}\}$ of D. The graph Γ is isomorphic to the complete bipartite graph $K_{d,d}$. The d rotation vertices ρ_j, $j = 0, \dots, d-1$ are connected by a total of d^2 edges to the d mirror symmetry vertices μ_j, $j = 0, \dots, d-1$. For every edge e, we declare the rotation vertex incident to e to be the origin, and the mirror symmetry vertex incident to e to be the terminus.

For a fixed rotation vertex ρ_j, there are d edges, labelled by the generators in S, using ρ^j as origin and connecting ρ^j to the appropriate mirror symmetry. Since $\mu_i\rho^j = \mu_{i-j/2}$, the edge labelled by a_i with origin ρ^j looks like

$$\rho^j \quad \bullet \xrightarrow{\;a_i\;} \bullet\ \mu_{i-j/2}. \tag{3.4}$$

Similarly, for a fixed mirror symmetry vertex μ_j, there are d edges, labelled by the generators in S, using μ_j as terminus and connecting μ_j to the appropriate rotation. Since

$\mu_i \mu_j = \rho^{2(i-j)}$, the edge labelled by a_i with terminus μ_j looks like

$$\rho^{2(i-j)} \bullet \overset{a_i}{\longrightarrow} \bullet \quad \mu_j.$$

We interpret $(X * D)/(X * D') = X * (D/D') \cong (\mathbb{Z}/2\mathbb{Z})^{d^2}$ as the edge space of Γ as follows. We take the trivial rotation $1 = \rho^0$ to be the base of Γ. A word $W = a_{i_m} \cdots a_{i_1}$ over S is identified with the corresponding walk in Γ starting at 1 (the first step is along the edge labelled by a_{i_1}, the next along the edge labelled by a_{i_2}, and so on). When we pass along the edge e in (3.4) in the direction from the origin ρ^j, the letter a_i is contributed to position $\rho^{-j}(i) = i - j$ in the decomposition of $g = a_{i_m} \cdots a_{i_1}$. When we pass along the same edge in the direction from the terminus, the letter a_i is contributed to position $\mu_{i-j/2}^{-1}(i) = \mu_{i-j/2}(i) = i - j$. Thus, a_i is contributed to the same position, $i - j$, regardless of the direction in which we pass the edge e. This means that we can identify $(X * D)/(X * D') \cong (\mathbb{Z}/2\mathbb{Z})^{d^2}$ with the edge space of Γ by observing that, for the edge e in (3.4), a word $W = a_{i_m} \cdots a_{i_1}$ and $g \in D$ represented by W, we have $\#_e(W) = \exp_i(g_{i-j})$. In other words, the parity of the letter a_i in the $(i-j)$th component of the decomposition of g, taken as an element of $(X * D)/(X * D') \cong (\mathbb{Z}/2\mathbb{Z})^{d^2}$, keeps track of the parity of $\#_e$.

The first level stabiliser $\mathsf{St}_D(1)$ consists of all group elements represented by words which are closed walks based at 1 in Γ. Thus, the image of $\mathsf{St}_D(1)$ in the edge space $(X * D)/(X * D')$ is precisely the cycle space of Γ.

Proposition 3.7. *The following statements about the first level stabiliser* $\mathsf{St}_D(1)$ *of* D *are true.*

(1) *An element* $a_{i_m} a_{i_{m-1}} \cdots a_1$ *belongs to the first level stabiliser* $\mathsf{St}_D(1)$ *if and only if* m *is even and*

$$i_1 + i_3 + \cdots \equiv_d i_2 + i_4 + \cdots.$$

(2) *The first level stabiliser* $\mathsf{St}_D(1)$ *is generated by the elements*

$$a_j a_{j+i} a_i a_0 = \delta_j(a_j)\, \delta_{j-i}(a_{j+i})\, \delta_{-i}(a_i)\, \delta_0(a_0),$$

for $i, j = 1, \ldots, d-1$.

(3) *The elements of the first level stabiliser can be described as follows. An element* $(g_0, \ldots, g_{d-1})$ *of* $X * D$ *is an element of* $\mathsf{St}_D(1)$ *if and only if, for* $j = 0, \ldots, d-1$,

$$\sum_{i=0}^{d-1} \exp_{j+i}(g_i) \equiv_2 \sum_{i=0}^{d-1} \exp_{j-i}(g_i) \equiv_2 0.$$

Each of these $2d$ *parity conditions is implied by the other* $2d-1$.

(4) $|X * D : \mathsf{St}_D(1)| = 2^{2d-1}$ *and* $|\mathsf{St}_D(1) : X * D'| = 2^{(d-1)^2}$*. Both* $(X * D)/\mathsf{St}_D(1)$ *and* $\mathsf{St}_D(1)/(X * D')$ *are elementary abelian 2-groups, of ranks* $2d - 1$ *and* $(d-1)^2$*, respectively.*

Proof. (1) The root permutation of $a_{i_m} \cdots a_{i_1}$ is $\mu_{i_m} \cdots \mu_{i_1}$ and $a_{i_m} \cdots a_{i_1}$ represents an element in $\mathsf{St}_D(1)$ if and only if $\mu_{i_m} \cdots \mu_{i_1} = 1$, which happens precisely when m is even (so that we have a rotation) and $i_m - i_{m-1} + \cdots + i_2 - i_1 \equiv_d 0$ (so that the rotation is trivial).

(2) Let T be the spanning tree of Γ consisting of all edges incident with the trivial rotation 1 together with all edges incident with μ_0. For $i, j = 1, \ldots, d-1$, the cycles

$$1 \xrightarrow{\ a_0\ } \mu_0 \xrightarrow{\ a_i\ } \rho^{2i} \xrightarrow{\ a_{i+j}\ } \mu_j \xrightarrow{\ a_j\ } 1$$

form a basis of the cycle space (the third edge is the one not in T, and it connects the non-trivial rotation vertex ρ^{2i} to the mirror symmetry vertex μ_j, not equal to μ_0). Thus, the elements $a_j a_{i+j} a_i a_0$ generate $\mathsf{St}_D(1)$.

(3) In our situation, we have $2d$ vertex conditions, one for every vertex in the Cayley graph of $D(d)$.

Fix a vertex ρ^j representing a rotation. Its incident edges are a_{j+i}, for $i = 0, \ldots, d-1$. The edge a_{j+i} is contributed to position $\rho^{-j}(j+i) = i$. The vertex condition at ρ^j then reads

$$\sum_{i=0}^{d-1} \exp_{j+i}(g_i) \equiv_2 0.$$

Fix a vertex $\mu_{j/2}$ representing a mirror symmetry. Its incident edges are a_{j-i}, for $i = 0, \ldots, d-1$. The edge a_{j-i} is contributed to position $\mu_{j/2}^{-1}(j-i) = \mu_{j/2}(j-i) = i$. The vertex condition at $\mu_{j/2}$ then reads

$$\sum_{i=0}^{d-1} \exp_{j-i}(g_i) \equiv_2 0.$$

(4) The dimension of the cycle space $\mathsf{St}_D(1)/(X * D')$ is $(d-1)^2$ since we have $|E| - |V| + 1 = d^2 - 2d + 1 = (d-1)^2$. The dimension of the complementary space $(X * D)/\mathsf{St}_D(1)$ is $|V| - 1 = 2d - 1$. ■

Proposition 3.8. *The rigid level stabiliser* $R_n = \mathsf{Rst}_D(n)$ *of level* n *in* D *is* $X^n * D'$*. For* $n \geq 1$*, we have*

$$|D : R_n| = 2^{(d-2)d^n+2} \cdot d^{\frac{d^n-1}{d-1}}.$$

Proof. We have already seen in Lemma 3.6 that $x * D' \leq D'$ and therefore $x * D' \leq \mathsf{Rst}_D(x)$ for every $x \in X$. This implies that $X * D' \leq R_1$. Inductively, $X^n * D' \leq X^{n-1} * D' \leq D$ for every $n \geq 1$ and so $X^n * D' \leq R_n$ for every $n \geq 1$.

To show the opposite containment, consider an element $g = \delta_x(g_x) \in \mathsf{Rst}_D(x)$ for some $x \in X$. Then, since $g \in \mathsf{St}_D(1)$, it must satisfy the equations in Proposition 3.7 (3), which reduce to $\exp_j(g_x) \equiv_2 0$ for all $j \in \{0, \dots, d-1\}$. By Lemma 3.5, $g_x \in D'$, so $\mathsf{Rst}_{D'}(x) \le \mathsf{Rst}_D(x) \le x * D'$ for every $x \in X$ and $X * D' = R_1 = \mathsf{Rst}_{D'}(1)$.

From this we obtain that $X^2 * D' = X * \mathsf{Rst}_{D'}(1) = R_2$ and inductively that $X^n * D' = R_n$ for all $n \ge 1$.

By the above, Proposition 3.7 (4), and since $D/\mathsf{St}_D(1) \cong D(d)$, we have

$$|D : R_1| = |D : \mathsf{St}_D(1)||\mathsf{St}_D(1) : (X * D')| = 2d \cdot 2^{(d-1)^2}.$$

This in turn implies that

$$|D' : R_1| = |D' : (X * D')| = \frac{|D : R_1|}{|D : D'|} = d2^{(d-1)^2+1-d} = d \cdot 2^{(d-1)(d-2)}.$$

Since $|D : R_1| = |D : D'||D' : R_1|$ and $R_i = X * R_{i-1}$, so $R_i = X^i * D'$ for all i, we have

$$|D : R_n| = |D : R_1| \times \prod_{i=1}^{n-1} |R_i : R_{i+1}| = |D : D'| \times \prod_{i=0}^{n-1} |D' : X * D'|^{d^i} = 2^s \cdot d^t,$$

where $t = \sum_{i=0}^{n-1} d^i = (d^n - 1)/(d-1)$ and $s = d + (d-1)(d-2)\sum_{i=0}^{n-1} d^i = d^n(d-2) + 2$. ∎

3.4. The rigid kernel of D

We first prove that the hypotheses of Theorem 3 hold for D. According to Proposition 3.8, the maximal branching subgroup of D is D', and we know from Lemma 3.5 that D/D' is an elementary abelian 2-group of rank d. This means that D satisfies the hypotheses of the second item of the theorem. It only remains to see that $\mathsf{St}_D(2)D' \ge \mathsf{St}_D(1)$ and find $\mathsf{St}_D(2)/\mathsf{Triv}_D(2)$.

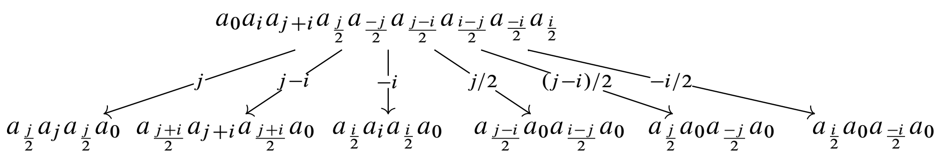
Lemma 3.9. *The following items hold:*

(1) $(\mathsf{St}_D(2)(X * D'))/(X * D') \cong \mathsf{St}_D(2)/\mathsf{Triv}_D(2)$.

(2) *The quotient* $(\mathsf{St}_D(2)(X * D'))/(X * D')$ *is an elementary abelian 2-group of rank* $(d-2)(d-1)$ *with a basis given by the images modulo* $(X * D')$ *of*

$$a_j a_{j+i} a_i a_0 \cdot a_{\frac{j}{2}} a_0 a_{\frac{-j}{2}} a_0 \cdot a_{\frac{j-i}{2}} a_0 a_{\frac{i-j}{2}} a_0 \cdot a_{\frac{-i}{2}} a_0 a_{\frac{i}{2}} a_0, \quad j, i \in \{1, \dots, d-1\}, i \ne -j. \tag{3.5}$$

(3) *The patterns of the corresponding generators of* $\mathsf{St}_D(2)/\mathsf{Triv}_D(2)$ *are*



(4) $\mathsf{St}_D(2)D' = \mathsf{St}_D(1)D'$ *and* $\{\mathsf{St}_D(n)/\mathsf{Triv}_G(n), r_{m,n}\}$ *is a surjective inverse system.*

Proof. (1) Since $\mathsf{Triv}_D(2) = X * \mathsf{Triv}_D(1) = X * (\mathsf{St}_D(1) \cap D')$ and $\mathsf{St}_D(2) = \mathsf{St}_D(1) \cap (X * \mathsf{St}_D(1))$, we have

$$\frac{\mathsf{St}_D(2)}{\mathsf{Triv}_D(2)} = \frac{\mathsf{St}_D(1) \cap X * \mathsf{St}_D(1)}{X * \mathsf{St}_D(1) \cap X * D'} \cong \frac{\mathsf{St}_D(1) \cap (X * \mathsf{St}_D(1)D')}{X * D'} = \frac{\mathsf{St}_D(2)(X * D')}{X * D'}.$$

(2) From the above, we must work out $\mathsf{St}_D(1) \cap (X * \mathsf{St}_D(1)D')/(X * D')$. Recall from Proposition 3.7 (2) that $\mathsf{St}_D(1)/(X * D')$ is generated by $(d-1)^2$ elements:

$$a_j a_{j+i} a_i a_0 = \delta_j(a_j)\delta_{j-i}(a_{j+i})\delta_{-i}(a_i)\delta_0(a_0) \quad \text{for } i, j \in \{1, \ldots, d-1\}.$$

Recall from Lemma 3.5 that $\mathsf{St}_D(1)D'/D'$ is the kernel of the augmentation map $\mathrm{aug} : D/D' \to \mathbb{Z}/2\mathbb{Z}$ defined by $a_0^{\alpha_0} \cdots a_{d-1}^{\alpha_{d-1}} \mapsto \sum_{i=0}^{d-1} \alpha_i$. Thus, $\mathsf{St}_D(1) \cap (X * \mathsf{St}_D(1)D')/(X * D')$ is the kernel of the map

$$\mathrm{Aug} : \mathsf{St}_D(1)/(X * D') \to (\mathbb{Z}/2\mathbb{Z})^d, \quad (x_0, \ldots, x_{d-1}) \mapsto (\mathrm{aug}(x_0), \ldots, \mathrm{aug}(x_{d-1})).$$

The parity conditions given in Proposition 3.7 (3) when added all together imply that $(x_0, \ldots, x_{d-1}) \in \mathsf{St}_D(1)/(X * D')$ satisfies the equation $\sum_{i=0}^{d-1} \mathrm{aug}(x_i) \equiv_2 0$ and therefore the image of Aug has dimension at most $d-1$ in $\mathbb{Z}/2\mathbb{Z}^d$. Now note that, as d is odd, $2j$ ranges through all of $\{1, \ldots, d-1\}$ with $j \in \{1, \ldots, d-1\}$. So the $d-1$ images

$$\mathrm{Aug}(a_j a_0 a_{-j} a_0) = \mathrm{Aug}(\delta_j(a_j a_{-j})\delta_{2j}(a_0)\delta_0(a_0)) = \delta_{2j}(1)\delta_0(1)$$

for $j \in \{1, \ldots, d-1\}$ are linearly independent, and so they form a basis for the image of Aug. Consequently, the kernel of Aug is an elementary abelian 2-group of rank $(d-1)^2 - (d-1) = (d-2)(d-1)$.

By expressing the images under Aug of the generators $\{a_j a_{j+i} a_i a_0 : i, j = 1, \ldots, d-1\}$ of $\mathsf{St}_D(1)$ in terms of the above basis elements, we obtain that $\mathrm{Aug}(a_j a_{j+i} a_i a_0) = \mathrm{Aug}(a_{-i} a_{-i-j} a_{-j} a_0)$ and so the $(d-1)(d-2)$ relations

$$\begin{aligned}\mathrm{Aug}(a_j a_{j+i} a_i a_0) = \mathrm{Aug}(a_{j/2} a_0 a_{-j/2} a_0) + \mathrm{Aug}(a_{(j-i)/2} a_0 a_{(i-j)/2} a_0) \\ + \mathrm{Aug}(a_{-i/2} a_0 a_{i/2} a_0),\end{aligned}$$

for $i, j \in \{1, \ldots, d-1\}, i \neq -j$ form a basis for the kernel of Aug. This basis consists of the images, modulo $X * D'$, of

$$a_0 a_i a_{j+i} a_j \cdot a_{j/2} a_0 a_{-j/2} a_0 \cdot a_{-i/2} a_0 a_{i/2} a_0 \cdot a_{(j-i)/2} a_0 a_{(i-j)/2} a_0, \quad j, i = 1, \ldots, d-1, i \neq -j.$$

(3) Using that $a_j a_{j+i} a_i a_0 = \delta_j(a_j)\delta_{j-i}(a_{j+i})\delta_{-i}(a_i)\delta_0(a_0)$, it is straightforward to see that an element g of the form shown in (3.5) has the following non-trivial sections: $g_j = a_j a_0$, $g_{-i} = a_i a_0$, $g_{j-i} = a_{j+i} a_0$, and $g_k = a_k a_{-k}$ for $k = j/2, -i/2, (j-i)/2$.

In general, these sections are not in $\mathsf{St}_D(1)$, as needed for g to be in $\mathsf{St}_D(2)$; they are only in $\mathsf{St}_D(1)D'$. However, since $X * D' \leq \mathsf{Triv}_D(1)$, we can find appropriate elements of D to multiply g and obtain an element of $\mathsf{St}_D(2)$ with the same pattern at the root. For example, we can use the following elements with appropriate values of $k \in \{1, \dots, d-1\}$

$$a_k a_0 \cdot a_0[a_k, a_{k/2}]a_0 = a_{k/2}a_k a_{k/2}a_0 \in \mathsf{St}_D(1),$$
$$a_k a_{-k}[a_{-k}, a_0] = a_k a_0 a_{-k} a_0 \in \mathsf{St}_D(1)$$

to obtain the patterns in the statement.

(4) Choosing $j = i$ in (3.5) and taking the image modulo D', we obtain the generators $\{a_j a_0 D' : j = 1, \dots, d-1\}$ which are exactly the generators of $\mathsf{St}_D(1)D'/D'$. Thus, $\mathsf{St}_D(2)D' \geq \mathsf{St}_D(1)$, and Lemma 2.15 yields that the inverse system $\{\mathsf{St}_D(n)/\mathsf{Triv}_D(n) : n \geq 1\}$ is surjective. ■

In particular, by the first part of Theorem 3, D has non-trivial rigid kernel. It only remains to find the structure of $\mathsf{St}_D(2) \cap D'/\mathsf{Triv}_D(2)$ to conclude, using the second item of the theorem, that the rigid kernel of D is an infinite Cartesian product of finite groups.

Lemma 3.10. *The quotient* $\mathsf{St}_D(2) \cap D'/\mathsf{Triv}_D(2)$ *is an elementary abelian* 2*-group of rank* $(d-1)(d-3)$*. Writing* $[j, i] := a_j a_{j+i} a_i a_0 \in \mathsf{St}_D(1)$ *for* $j, i \in \{1, \dots, d-1\}$*, the following patterns of size* 2 *generate* $\mathsf{St}_D(2) \cap D'/\mathsf{Triv}_D(2)$*:*

$$\begin{aligned}&\delta_0([j/2, -j/2] \cdot [(j-i)/2, -(j-i)/2] \cdot [-i/2, i/2] \cdot [j, i])\\&\quad \cdot \delta_j([j/2, j/2])\delta_{j-i}([(j+i)/2, (j+i)/2])\delta_{-i}([i/2, i/2])\delta_{(j-i)/2}\\&\quad \cdot ([(j-i)/2, -(j-i)/2]) \cdot \delta_{\pm j/2}([\mp j/4, \mp j/4])\delta_{\pm i/2}([\pm i/2, \pm i/2])\\&\quad \cdot \delta_{\pm(j+i)/2}([(j+i)/4, (j+i)/4]) \cdot \delta_{\pm(j-i)/4}([(j-i)/4, -(j-i)/4])\\&\quad \cdot \delta_{\pm(j+i)/4}([(j+i)/4, -(j+i)/4])\end{aligned}$$

for $j, i \in \{1, \dots, d-1\}, i \neq \pm j$*.*

Proof. As in the proof of Lemma 3.9, we will work out the quotient $\mathsf{St}_D(2)(X * D') \cap D'/X * D'$, as it is isomorphic to $\mathsf{St}_D(2) \cap D'/\mathsf{Triv}_D(2)$. As $\mathsf{St}_D(2)(X * D') \cap D' = \mathsf{St}_D(2)(X * D') \cap \mathsf{St}_D(1) \cap D'$, we consider Aug from Lemma 3.9 and another map whose kernel is $\mathsf{St}_D(1) \cap D'/(X * D')$.

Consider $g = (g_0, \dots, g_{d-1}) \in \mathsf{St}_D(1)$. By Lemma 3.5, $g \in D'$ if and only if each generator a_j of D appears an even number of times in g. Now, by the definition of the a_j, any appearance of a_j in g gives exactly one appearance of a_j in some g_i. So $g \in D'$ if and only if, for every $j \in X$, the number of appearances of a_j across all g_i is even ($\forall j \in X : \sum_{i=0}^{d-1} \exp_j(g_i) \equiv_2 0$, where $\exp_i$ is as in Lemma 3.4). In other words, $\mathsf{St}_D(1) \cap D'$ is the kernel of the map

$$\mathrm{Exp} : \mathsf{St}_D(1) \cap (X * D) \to (\mathbb{Z}/2\mathbb{Z})^d,$$
$$(g_0, \dots, g_{d-1}) \mapsto \left(\sum_{i=0}^{d-1} \exp_0(g_i), \dots, \sum_{i=0}^{d-1} \exp_{d-1}(g_i) \right).$$

By Lemma 3.9, $\mathsf{St}_D(2)(X * D')/(X * D')$ is the kernel of the map Aug, so

$$\frac{(\mathsf{St}_D(2)(X * D') \cap \mathsf{St}_D(1) \cap D')}{(X * D')}$$

is the kernel of the map

$$\begin{aligned}\mathrm{Aug} \times \mathrm{Exp} : \mathsf{St}_D(1)/(X * D') &\to (\mathbb{Z}/2\mathbb{Z})^{2d} \\ \mathbf{g} = (g_0, \ldots, g_{d-1}) &\mapsto (\mathrm{Aug}(\mathbf{g}), \mathrm{Exp}(\mathbf{g})) \\ &= \Bigg((\mathrm{aug}(g_0), \ldots, \mathrm{aug}(g_{d-1})), \\ &\qquad \Bigg(\sum_{i=0}^{d-1} \exp_0(g_i), \ldots, \sum_{i=0}^{d-1} \exp_{d-1}(g_i) \Bigg) \Bigg).\end{aligned}$$

The $2d$ parity conditions given in Proposition 3.7 (3) imply that the sum of entries in each of the images of Aug and Exp must be 0 modulo 2. Thus, the image of Aug × Exp has dimension at most $2d - 2$.

To improve readability, we write $[j, i] := a_j a_{j+i} a_i a_0$, δ_n for the nth canonical basis vector in $(\mathbb{Z}/2\mathbb{Z})^d$ and for the zero vector in $(\mathbb{Z}/2\mathbb{Z})^d$. Notice that the elements

$$\begin{cases} (\mathrm{Aug} \times \mathrm{Exp})([j, j]) = (\delta_j \delta_{-j},\, \delta_0 \delta_{2j}), & j \in \{1, \ldots, d-1\}, \\ (\mathrm{Aug} \times \mathrm{Exp})([j, -j]) = (\delta_0 \delta_{2j},\, \delta_j \delta_{-j}), & j \in \{1, \ldots, d-1\} \end{cases} \tag{3.6}$$

are linearly independent, as d is odd. Again because d is odd, these elements span the image of Aug × Exp:

$$\begin{aligned}(\mathrm{Aug} \times \mathrm{Exp})([2j, -2j][j, j][-j, -j]) &= (\delta_0 \delta_{4j}, -) \\ (\mathrm{Aug} \times \mathrm{Exp})([2j, 2j][j, -j][-j, j]) &= (-, \delta_0 \delta_{4j})\end{aligned}$$

for $j \in \{1, \ldots, d-1\}$. So the image of Aug × Exp is indeed of rank $(2d - 2)$, and therefore the kernel has rank $(d-1)^2 - (2d-2) = (d-1)(d-3)$.

For $j, i \in \{1, \ldots, d-1\}, i \neq \pm j$, we have the following $(d-1)(d-3)$ relations:

$$\begin{aligned}(\mathrm{Aug} \times \mathrm{Exp})([j, i]) &= (\delta_0 \delta_j \delta_{j-i} \delta_{-i},\, \delta_0 \delta_j \delta_i \delta_{j+i}) \\ &= (\delta_0 \delta_j, -) + (\delta_0 \delta_{j-i}, -) + (\delta_0 \delta_{-i}, -) + (-, \delta_0 \delta_j) \\ &\quad + (-, \delta_0 \delta_i) + (-, \delta_0 \delta_{j+i}),\end{aligned}$$

which yield generators of $\mathsf{St}_D(2)(X * D') \cap D'/X * D'$, the images modulo $X * D'$ of

$$\begin{aligned}&[j, i] \cdot [j/2, -j/2][j/4, j/4][-j/4, -j/4] \cdot [i/2, -i/2][i/4, i/4][-i/4, -i/4] \\ &\quad \cdot [(j-i)/2, (i-j)/2][(j-i)/4, (j-i)/4][(i-j)/4, (i-j)/4] \\ &\quad \cdot [j/2, j/2][-j/4, j/4][-j/4, j/4] \cdot [i/2, i/2][-i/4, i/4][i/4, -i/4] \\ &\quad \cdot [(j+i)/2, (j+i)/2][(j+i)/4, (-j-i)/4][(-j-i)/4, (j+i)/4]\end{aligned}$$

for $j, i \in \{1, \dots, d-1\}, i \neq \pm j$. Using the same trick as in the proof of Lemma 3.9, we can multiply each of the above elements by a suitably chosen element of $(X * D')$ to ensure that the patterns at each $x \in X$ are in $\mathsf{St}_D(1)$. This gives the patterns in the statement. ■

Notice that for $d = 3$, the quotient $\mathsf{St}_D(2) \cap D'/\mathsf{Triv}_D(2)$ is trivial and we have only $(d-1)(d-2) = 2$ generators of $\mathsf{St}_D(2)$ modulo $X * D'$, namely $a_1a_2a_1a_0 \cdot a_2a_0a_1a_0 \cdot a_1a_0a_2a_0 \equiv (a_0a_2, a_0a_2, a_0a_2)$ and $a_2a_1a_2a_0 \cdot a_1a_0a_2a_0 \cdot a_2a_0a_1a_0 \equiv (a_0a_1, a_0a_1, a_0a_1)$. Therefore, the rigid kernel is isomorphic to the Klein-4 group, generated by these elements. This agrees with the conclusions of [1, 15].

Corollary 3.11. *The rigid kernel of D is isomorphic to a Cartesian product of cyclic groups of order 2. If $d = 3$, the rank is 2; if $d \geq 5$, the rank is infinite.*

3.5. Congruence completion and Hausdorff dimension of D

The Hausdorff dimension of a closed subset of A^{X^*} is a way of measuring its "relative size". If $A = F \leq \mathsf{Sym}(X)$, $d = |X|$, and $G \leq F^{X^*} = \varprojlim_n F \wr \overset{n}{\cdots} \wr F$ is a closed subgroup, its Hausdorff dimension is

$$\mathrm{hdim}(G) = \liminf_{n\to\infty} \frac{\log_{|F|} |G : \mathsf{St}_G(n)|}{\log_{|F|} F \wr \overset{n}{\cdots} \wr F} = \liminf_{n\to\infty} \frac{(d-1)\log_{|F|} |G : \mathsf{St}_G(n)|}{d^n - 1}. \tag{3.7}$$

In performing these calculations for $\overline{D}$, the appropriate choice of $F \leq \mathsf{Sym}(X)$ to take is $D(d)$, so $|F| = 2d$ in the formulas above.

Proposition 3.12. *We have, for $n \geq 1$,*

$$|D : \mathsf{St}_D(n)| = |\overline{D} : \overline{\mathsf{St}_D(n)}| = 2^{d^{n-1}} \cdot d^{\frac{d^n-1}{d-1}},$$

and the Hausdorff dimension of $\overline{D}$ is

$$\mathrm{hdim}(\overline{D}) = 1 - \frac{1}{d} \cdot \frac{\log 2}{\log 2d}.$$

Proof. The first equality is a general fact about profinite closures. We only need to show that the sizes of $D/\mathsf{St}_D(n)$ are as claimed. For this, we use that

$$|D : \mathsf{St}_D(n)| = \frac{|D : \mathsf{Rst}_D(n)|}{|\mathsf{St}_D(n) : \mathsf{Triv}_D(n)| \cdot |\mathsf{Triv}_D(n) : \mathsf{Rst}_D(n)|}.$$

The numerator has already been determined in Proposition 3.8. For the denominator, we have, firstly

$$|\mathsf{Triv}_D(n) : \mathsf{Rst}_D(n)| = |X^{n-1} * (\mathsf{St}_D(1) \cap D') : X^n * D'| = |\mathsf{St}_D(1) \cap D' : X * D'|^{d^{n-1}}.$$

Since $|\mathsf{St}_D(1) : X * D'| = 2^{(d-1)^2}$ by Proposition 3.7 (4) and $|\mathsf{St}_D(1) : \mathsf{St}_D(1) \cap D'| = 2^{d-1}$ by Lemma 3.5, we have that $|\mathsf{St}_D(1) \cap D' : X * D'| = 2^{(d-1)(d-2)}$.

Now, from the proof of Theorem 3, we extract that

$$\begin{aligned}|\mathsf{St}_D(n) : \mathsf{Triv}_D(n)| &= \left| X^{n-2} * \left(\frac{\mathsf{St}_D(2) \cap D'}{\mathsf{Triv}_D(2)} \right) \right| \cdot |\mathsf{St}_D(n-1) : \mathsf{Triv}_D(n-1)| \\ &= |\mathsf{St}_D(2) \cap D' : \mathsf{Triv}_D(2)|^a \cdot |\mathsf{St}_D(1) : \mathsf{Triv}_D(1)|,\end{aligned}$$

where $a = \sum_{i=0}^{n-2} d^i = (d^{n-1} - 1)/(d-1)$. Using Lemma 3.10 and that $|\mathsf{St}_D(1) : \mathsf{Triv}_D(1)| = 2^{d-1}$, we conclude that

$$|\mathsf{St}_D(n) : \mathsf{Triv}_D(n)| = (2^{(d-1)(d-3)})^a \cdot 2^{d-1} = 2^{(d-3)(d^{n-1}-1)+d-1}.$$

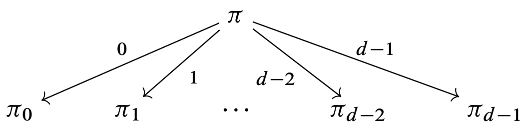
Putting all of the above together, we get that $|D : \mathsf{St}_D(n)| = (2^s d^t)/(2^u \cdot 2^v)$, where $t = (d^n - 1)/(d-1)$, $s = d^n(d-2) + 2$, $u = (d-3)(d^{n-1} - 1) + d - 1$, $v = (d-1)(d-2)d^{n-1}$, so $s - u - v = d^{n-1}$ and we get the first part of the statement.

For the Hausdorff dimension, the result follows from (3.7) and

$$\begin{aligned}\log_{2d}(|\overline{D} : \mathsf{St}_{\overline{D}}(n)|) = \log_{2d}(|D : \mathsf{St}_D(n)|) &= \log_{2d}\left(2^{d^{n-1}} \cdot d^{\frac{d^n-1}{d-1}}\right) \\ &= \frac{d^n - 1}{d-1} - \frac{d^{n-1} - 1}{d-1} \log_{2d}(2).\end{aligned}$$

■

Corollary 3.13. (1) *The closure $\overline{D}$ is a finitely constrained group defined by the patterns of size 2 that can be described as follows. A pattern of size 2*



is an allowed pattern if and only if the permutation $\pi\pi_0\pi_1 \cdots \pi_{d-2}\pi_{d-1} \in D(d)$ is a rotation in $D(d)$ (the number of mirror symmetries among $\pi, \pi_0, \dots, \pi_{d-1}$ is even).

(2) *The closure $\overline{D}$ is a regular branch group branching over $\overline{\mathsf{St}_D(1)} = \mathsf{St}_{\overline{D}}(1)$, the stabiliser of level 1. Moreover,*

$$[X * D : \mathsf{St}_D(1)] = 2^{2d-1} \quad \text{and} \quad [X * \overline{D} : \overline{\mathsf{St}_D(1)}] = 2,$$

so we recover, using Theorem 2, the known fact that D has non-trivial rigid kernel.

Proof. (1) First note that all elements of $D/\mathsf{St}_D(2)$ satisfy the constraints in the statement, because each generator a_i of D contributes a single reflection μ_i to the pattern at the root and the same reflection μ_i to the pattern at exactly one vertex of level 1. Since D is self-similar, all patterns in $\overline{D}$ of size n must also satisfy these constraints. The fact that all allowed patterns in the statement of size n arise in $\overline{D}$ follows from the fact that $|\overline{D} : \mathrm{St}_{\overline{D}}(n)|$ as calculated in Proposition 3.12 is exactly the number of allowed patterns.

(2) The first part of the statement follows from Theorem 2.7 and the fact that $\overline{D}$ is finitely constrained, defined by patterns of size 2. For the second part, the first equality is proved in Proposition 3.7, while the second follows from the constraint of index 2 in the allowed patterns of size 2. ∎

Acknowledgements. Both authors are grateful to the anonymous referee for carefully reading a previous version of this article, suggesting many improvements and correcting mistakes.

Funding. The first author received financial support from Spain's Ministry of Science and Innovation (PID2020-114032GB-I00) and the Severo Ochoa Programme for Centres of Excellence in Research and Development (CEX2019-000904-S).

References

[1] L. Bartholdi, O. Siegenthaler, and P. Zalesskii, The congruence subgroup problem for branch groups. *Israel J. Math.* **187** (2012), no. 1, 419–450 Zbl 1271.20031 MR 2891709

[2] P.-E. Caprace, C. D. Reid, and G. A. Willis, Locally normal subgroups of totally disconnected groups. Part II: compactly generated simple groups. *Forum Math. Sigma* **5** (2017), article no. e12 Zbl 1401.22003 MR 3659769

[3] T. Ceccherini-Silberstein, M. Coornaert, F. Fiorenzi, and Z. Šunić, Cellular automata between sofic tree shifts. *Theoret. Comput. Sci.* **506** (2013), 79–101 Zbl 1301.68185 MR 3106617

[4] G. A. Fernández-Alcober, A. Garrido, and J. Uria-Albizuri, On the congruence subgroup property for GGS-groups. *Proc. Amer. Math. Soc.* **145** (2017), no. 8, 3311–3322 Zbl 1431.20017 MR 3652785

[5] A. Garrido, On the congruence subgroup problem for branch groups. *Israel J. Math.* **216** (2016), no. 1, 1–13 Zbl 1380.20028 MR 3556961

[6] R. Grigorchuk, V. Nekrashevych, and Z. Šunić, Hanoi towers group on 3 pegs and its pro-finite closure. *Oberwolfach Rep.* **25** (2006), 15–17

[7] R. Grigorchuk and Z. Šunik̸, Asymptotic aspects of Schreier graphs and Hanoi Towers groups. *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 8, 545–550 Zbl 1135.20016 MR 2217913

[8] R. Grigorchuk and Z. Šunić, Self-similarity and branching in group theory. In C. M. Campbell, M. R. Quick, E. F. Robertson, G. C. Smith (eds.), *Groups St. Andrews 2005 Volume 1*, pp. 36–95, London Math. Soc. Lecture Note Ser. 339, Cambridge University Press, Cambridge, 2007 Zbl 1185.20044 MR 2327317

[9] R. I. Grigorchuk, Bernside's problem on periodic groups (in Russian). *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 1, 53–54 English translation: *Funct. Anal. Appl.* **14** (1980) no. 1, 41–43 Zbl 0595.20029 MR 0565099

[10] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 5, 939–985 English translation: *Math. USSR-Izv.* **25** (1985), no. 2, 259–300 Zbl 0583.20023 MR 0764305

[11] R. I. Grigorchuk, Just infinite branch groups. In M. du Sautoy, D. Segal, and A. Shalev (eds.), *New horizons in pro-p groups*, pp. 121–179, Progr. Math. 184, Birkhäuser, Boston, MA, 2000 Zbl 0982.20024 MR 1765119

[12] P. M. Neumann, Some questions of Edjvet and Pride about infinite groups. *Illinois J. Math.* **30** (1986), no. 2, 301–316 Zbl 0598.20029 MR 0840129

[13] A. Penland and Z. Šunić, A language hierarchy and Kitchens-type theorem for self-similar groups. *J. Algebra* **537** (2019), 173–196 Zbl 1485.20068 MR 3990041

[14] E. Pervova, Profinite completions of some groups acting on trees. *J. Algebra* **310** (2007), no. 2, 858–879 Zbl 1129.20020 MR 2308183

[15] R. Skipper, A constructive proof that the Hanoi towers group has non-trivial rigid kernel. *Topology Proc.* **53** (2019), 1–14 Zbl 1479.20020 MR 3764118

[16] R. Skipper, The congruence subgroup problem for a family of branch groups. *Internat. J. Algebra Comput.* **30** (2020), no. 2, 397–418 Zbl 1452.20023 MR 4077417

[17] Z. Šunić, Hausdorff dimension in a family of self-similar groups. *Geom. Dedicata* **124** (2007), 213–236 Zbl 1169.20015 MR 2318546

[18] J. S. Wilson, Groups with every proper quotient finite. *Proc. Cambridge Philos. Soc.* **69** (1971), no. 3, 373–391 Zbl 0216.08803 MR 0274575

[19] J. S. Wilson, On just infinite abstract and profinite groups. In M. du Sautoy, D. Segal, and A. Shalev (eds.), *New horizons in pro-p groups*, pp. 181–203, Progr. Math. 184, Birkhäuser, Boston, MA, 2000 Zbl 0981.20021 MR 1765120

[20] J. S. Wilson, On exponential growth and uniformly exponential growth for groups. *Invent. Math.* **155** (2004), no. 2, 287–303 Zbl 1065.20054 MR 2031429

Received 1 October 2023.

Alejandra Garrido
Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid; Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain; a_garrido@ucm.es, alejandra.garrido@icmat.es

Zoran Šunić
Department of Mathematics, Hofstra University, Hempstead, NY 11549, USA; Ss. Cyril and Methodius University in Skopje, 1000 Skopje, North Macedonia; zoran.sunic@hofstra.edu