Groups Geom. Dyn. 19 (2025), 637–645
DOI 10.4171/GGD/891

Intersection-saturated groups without free subgroups

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Abstract. A group G is said to be intersection-saturated if for every strictly positive integer n and every map $c: \mathcal{P}(\{1,\dots,n\}) \setminus \emptyset \to \{0,1\}$, one can find subgroups $H_1,\dots,H_n \le G$ such that for every non-empty subset $I \subseteq \{1,\dots,n\}$, the intersection $\bigcap_{i\in I} H_i$ is finitely generated if and only if $c(I) = 0$. We obtain a new criterion for a group to be intersection-saturated based on the existence of arbitrarily high direct powers of a subgroup admitting an automorphism with a non-finitely generated set of fixed points. We use this criterion to find new examples of intersection-saturated groups, including Thompson's groups and the Grigorchuk group. In particular, this proves the existence of finitely presented intersection-saturated groups without non-abelian free subgroups, thus answering a question of Delgado, Roy and Ventura.

To Slava Grigorchuk, on the occasion of his 70th birthday

1. Introduction

It is a classical result of Howson [7] that the intersection of finitely many finitely generated subgroups of a non-abelian free group is again finitely generated. This, however, does not hold for all groups (groups for which it does are said to possess the *Howson property*). Recently, Delgado, Roy and Ventura [4] introduced a new notion, *intersection configurations*, that can be seen as a generalisation of these considerations.

Definition 1.1 (Cf. [4]). Let $n \in \mathbb{N}_{\ge 1}$ be a strictly positive integer.

(1) An *n-configuration* is a map $c: \mathcal{P}(\{1,\dots,n\}) \setminus \{\emptyset\} \to \{0,1\}$.

(2) An n-configuration c is said to be *realisable* in a group G if there exist subgroups $H_1,\dots,H_n \le G$ such that for all non-empty subset $I \subseteq \{1,\dots,n\}$, the subgroup $\bigcap_{i\in I} H_i$ is finitely generated if and only if $c(I) = 0$.

Thus, a group G has the Howson property if and only if the 2-configuration given by $c(\{1\}) = c(\{2\}) = 0$ and $c(\{1,2\}) = 1$ is not realisable in G.

It is natural to ask which groups satisfy the property that all configurations are realisable. Such groups were called *intersection-saturated* by Delgado, Roy and Ventura.

Mathematics Subject Classification 2020: 20E07.
Keywords: intersection-saturated groups, Thompson's groups, Grigorchuk group, branch groups.

Definition 1.2 ([4, Definition 3.7]). A group G is *intersection-saturated* if for every $n \in \mathbb{N}_{\geq 1}$, every n-configuration is realisable in G.

In [4], Delgado, Roy and Ventura produced examples of intersection-saturated groups, all containing $F_2 \times \mathbb{Z}^m$ for some $m \in \mathbb{N}$, where F_2 denotes the free groups on two generators. They asked [4, Question 7.2] if there exists a finitely presented intersection-saturated group not containing such a subgroup.

In the present note, we introduce a new technique to find intersection-saturated groups, which allows us to answer this question positively. More precisely, we obtain the following criterion.

Theorem A. *Let G be a group. Suppose that for every $n \in \mathbb{N}_{\geq 1}$, there exists a finitely generated group K_n and an automorphism $f_n \colon K_n \to K_n$ such that*

(I) *the subgroup of fixed points of f_n is not finitely generated,*

(II) *G contains a subgroup isomorphic to K_n^n, where K_n^n denotes the direct product of K_n with itself n times.*

Then, G is intersection-saturated.

We then apply this criterion to obtain several new examples of intersection-saturated groups, including Thompson's groups (Corollary 3.1) and the Grigorchuk group (Corollary 3.3). These examples allow us to answer the question of Delgado, Roy and Ventura.

Corollary B. *There exist finitely presented intersection-saturated groups without non-abelian free subgroups.*

In fact, we go one step further and prove that there are finitely presented intersection-saturated groups that are amenable.

Corollary C. *There exist finitely presented amenable intersection-saturated groups.*

2. Proof of Theorem A

Let us first fix the notation that we will use for the rest of this note.

Notation 2.1. For $n \in \mathbb{N}_{\geq 1}$, we will write $[n] = \{1, \ldots, n\}$.

Notation 2.2. If G is a group and $n \in \mathbb{N}_{\geq 1}$, we will denote by G^n the direct product of n copies of G. For any $g \in G^n$ and $i \in [n]$, we will denote by g_i the ith component of g, so that $g = (g_1, g_2, \ldots, g_n)$.

Notation 2.3. Let G be a group, let $n \in \mathbb{N}$ be any integer and let $H_1, \dots, H_n \leq G$ be a collection of subgroups. For $\emptyset \neq I \in \mathcal{P}([n])$, we will write

$$H_I = \bigcap_{i \in I} H_i.$$

Let us now introduce the following construction, which will serve as a basis for the proof of the main result.

Lemma 2.4. *Let G be a finitely generated group and suppose that there exists an automorphism f of G such that the subgroup of fixed points of f is not finitely generated. Then, for any $n \in \mathbb{N}_{\geq 1}$, there exist finitely generated subgroups $H_1, \dots, H_n \leq G^n$ such that H_I is finitely generated for any proper non-empty subset of $[n]$ but $H_{[n]}$ is not finitely generated.*

Proof. If $n = 1$, then it suffices to take any non-finitely generated subgroup of G, such as the set of fixed points of the automorphism f. Thus, let us now assume that $n > 1$. For each $1 \leq i \leq n-1$, we define H_i by

$$H_i = \{g \in G^n \mid g_i = g_{i+1}\} \leq G^n,$$

and we define H_n by

$$H_n = \{g \in G^n \mid g_n = f(g_1)\} \leq G^n.$$

Let $I \subset [n]$ be a proper subset. We have

$$H_I = \{g \in G^n \mid g_i = g_{i+1} \; \forall i \in I \setminus \{n\} \text{ and } g_n = f(g_1) \text{ if } n \in I\}.$$

If we write $k = |I| < n$, it is not hard to see that H_I is isomorphic to G^{n-k}, since the coordinates corresponding to elements of I are uniquely determined by the $n-k$ other coordinates, which have no restrictions placed upon them. As G is finitely generated, it follows that H_I must also be finitely generated.

Now, let us see that $H_{[n]}$ is not finitely generated. We have

$$\begin{aligned} H_{[n]} &= \{g \in G^n \mid g_1 = g_2 = \cdots = g_n = f(g_1)\} \\ &= \{(g, g, \dots, g) \in G^n \mid g = f(g)\}. \end{aligned}$$

Thus, we see that $H_{[n]}$ is isomorphic to the subgroup of fixed points of f in G, which is not finitely generated by assumption. ∎

Using the previous lemma, we can realise all configurations taking value 1 at most once in direct products, as the next lemma shows.

Lemma 2.5. *Let G be a finitely generated group with an automorphism f such that the subgroup of fixed points of f is not finitely generated. Then, for all $n \in \mathbb{N}_{\geq 1}$, every n-configuration $c\colon \mathcal{P}([n]) \setminus \{\emptyset\} \to \{0, 1\}$ such that $|c^{-1}(1)| \leq 1$ is realisable in G^n.*

Proof. Let us fix some $n \in \mathbb{N}_{\geq 1}$ and a configuration c taking the value 1 at most once. If c takes only the value 0, then one can simply set $H_1 = H_2 = \cdots = H_n = 1$. Let us now assume that there exists some non-empty set $I_1 \subseteq [n]$ such that $c(I_1) = 1$. By our assumptions on c, this set must be unique. Let us write $I_1 = \{j_1, \ldots, j_{n_1}\}$, where $n_1 = |I_1| \leq n$.

By Lemma 2.4, there exist subgroups $H_{j_1}, \ldots, H_{j_{n_1}} \leq G^{n_1} \leq G^n$ such that H_J is finitely generated for all proper subsets $J \subset I_1$ but H_{I_1} is not finitely generated. Note that we have chosen here an arbitrary embedding of G^{n_1} in G^n. For $i \notin I_1$, let us define $H_i = 1$. Then, for any $J \subseteq \{1, 2, \ldots, n\}$, we have one of the following three cases:

(1) $J = I_1$, in which case H_J is not finitely generated,

(2) $J \subsetneq I_1$, in which case H_J is finitely generated,

(3) $J \not\subseteq I_i$, in which case $H_J = 1$ and is thus finitely generated.

Thus, as desired, H_J is finitely generated if and only if $c(J) = 0$. ∎

To pass from configuration maps with at most one non-zero value to arbitrary configuration maps, let us recall the notion of the *join* of two configurations, as defined by Delgado, Roy and Ventura [4].

Definition 2.6 ([4, Definition 3.3]). Let $n \in \mathbb{N}_{\geq 1}$ and let c_1, c_2 be two n-configurations. Their *join* is the n-configuration $c_1 \wedge c_2 \colon \mathcal{P}(\{1, 2, \ldots, n\}) \setminus \{\emptyset\} \to \{0, 1\}$ defined by

$$c_1 \wedge c_2(I) = \begin{cases} 0 & \text{if } c_1(I) = 0 \text{ and } c_2(I) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Let us now see that the join of two n-configurations realisable in groups G_1 and G_2, respectively, is always realisable in the direct product $G_1 \times G_2$.

Lemma 2.7. *Let $n \in \mathbb{N}_{\geq 1}$ and let c_1, c_2 be two n-configurations. Let G_1, G_2 be two groups such that c_1 and c_2 are realisable configurations in G_1 and G_2, respectively. Then, $c_1 \wedge c_2$ is realisable in $G_1 \times G_2$.*

Proof. Let $H_1, \ldots, H_n \leq G_1$ and $K_1, \ldots, K_n \leq G_2$ be subgroups realising the configurations c_1 and c_2, respectively. For all $i \in [n]$, let $L_i = H_i \times K_i \leq G_1 \times G_2$. Then, we claim that $L_1, \ldots, L_n$ are subgroups realising the configuration $c_1 \wedge c_2$. Indeed, let $I \subseteq [n]$ be a non-empty subset. Then, $L_I = H_I \times K_I$. If both H_I and K_I are finitely generated, then L_I is also finitely generated, but if one of H_I or K_I is not finitely generated, then L_I cannot be either. This shows that $L_1, \ldots, L_n$ exactly realises $c_1 \wedge c_2$. ∎

We are now ready to prove Theorem A, which we restate for the convenience of the reader.

Theorem A. *Let G be a group. Suppose that for every $n \in \mathbb{N}_{\geq 1}$, there exists a finitely generated group K_n and an automorphism $f_n \colon K_n \to K_n$ such that*

(I) *the subgroup of fixed points of* f_n *is not finitely generated,*

(II) G *contains a subgroup isomorphic to* K_n^n.

Then, G *is intersection-saturated.*

Proof. Let us fix $n \in \mathbb{N}_{\geq 1}$ and an n-configuration c. Let $k = |c^{-1}(1)|$, and let $I_1, \dots, I_k \in \mathcal{P}([n]) \setminus \{\emptyset\}$ be an enumeration of the sets mapped to 1 by c. For every $1 \leq i \leq k$, we define the n-configuration $c_i \colon \mathcal{P}([n]) \setminus \{\emptyset\} \to \{0, 1\}$ by $c_i(J) = 1$ if and only if $J = I_i$. It is obvious that $c = c_1 \wedge c_2 \wedge \dots \wedge c_k$ (note that the join is associative, so that this expression is well defined).

By Lemma 2.5, for every $1 \leq i \leq k$, the configuration c_i is realisable in K_{kn}^n. Applying Lemma 2.7 inductively, we then conclude that the configuration $c = c_1 \wedge \dots \wedge c_k$ is realisable in K_{kn}^{kn}. Since K_{kn}^{kn} embeds in G by assumption, we conclude that the configuration c is realisable in G. ∎

3. New examples of intersection-saturated groups

We will now apply Theorem A to find new examples of intersection-saturated groups. Our first application is Thompson's groups. We refer the reader to [2] for an introduction to these groups.

Corollary 3.1. *Thompson's groups* F, T *and* V *are intersection-saturated.*

Proof. Since $F \leq T \leq V$, it suffices to prove the result for F. By [6, Corollary 22], the restricted wreath product $\mathbb{Z} \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z} \rtimes \mathbb{Z}$ is contained in F. Let $g \in \bigoplus_{\mathbb{Z}} \mathbb{Z}$ be any non-trivial element, and let $C(g)$ denote the centraliser of g in $\mathbb{Z} \wr \mathbb{Z}$. Since $\bigoplus_{\mathbb{Z}} \mathbb{Z}$ is abelian, it is clear that $\bigoplus_{\mathbb{Z}} \mathbb{Z} \leq C(g)$, and since the action of $\mathbb{Z}$ on $\bigoplus_{\mathbb{Z}} \mathbb{Z}$ has no fixed point except for the identity, we have in fact $\bigoplus_{\mathbb{Z}} \mathbb{Z} = C(g)$.

Let $f_g \colon \mathbb{Z} \wr \mathbb{Z} \to \mathbb{Z} \wr \mathbb{Z}$ denote conjugation by g. Its set of fixed points is $C(g) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$, which is not finitely generated. It is known that for every $n \in \mathbb{N}$, F contains a subgroup isomorphic to F^n (this follows directly, e.g., from [2, Lemma 4.4]). Thus, F contains also a subgroup isomorphic to $(\mathbb{Z} \wr \mathbb{Z})^n$ for every $n \in \mathbb{N}$, and we conclude by Theorem A that F is intersection-saturated. ∎

Since Thompson's group F is finitely presented and does not contain a non-abelian free subgroup [2, Corollary 4.9], we immediately obtain the following corollary, which answers [4, Question 7.2].

Corollary B. *There exist finitely presented intersection-saturated groups without non-abelian free subgroups.*

Since the amenability of Thompson's group F is famously an open question, Corollary B still leaves open the question of the existence of finitely presented amenable intersection-saturated groups. We will answer this question – thanks to our second application of Theorem A, which is about branch groups. We refer the reader to [1] for the definition and an introduction to these groups.

Theorem 3.2. *Let G be a finitely generated branch group. If G contains an element $\alpha \in G$ whose centraliser $C_G(\alpha)$ is not finitely generated, then G is intersection-saturated.*

Proof. It follows from the definition of a branch group (see, e.g., [1, Definition 1.1]) that for every $n \in \mathbb{N}$, there exists a subgroup $K_n \leq G$ such that

(1) K_n has exactly k_n distinct conjugates, for some $n \leq k_n < \infty$,

(2) for every $g \in G$, either $gK_ng^{-1} = K_n$ or $[K_n, gK_ng^{-1}] = K_n \cap gK_ng^{-1} = 1$,

(3) the subgroup $H_n = \langle \{gK_ng^{-1} \mid g \in G\} \rangle \cong K_n^{k_n}$ is normal and of finite index in G.

We note that since G is finitely generated, condition (3) implies that $K_n^{k_n}$, and thus K_n, must also be finitely generated. Therefore, to apply Theorem A, it suffices to find for every $n \in \mathbb{N}$ an automorphism $f_n \colon K_n \to K_n$ whose set of fixed points is not finitely generated.

Let us now fix some $n \in \mathbb{N}$. By condition (1), there exists a bijection $\phi \colon [k_n] \to \{gK_ng^{-1} \mid g \in G\}$ between the set $[k_n]$ and the set of conjugates of K_n. Since G acts by conjugation on the set of conjugates of K_n, we can pull this action back through ϕ to obtain an action of G on $[k_n]$. Let $l = |[k_n]/\langle\alpha\rangle|$ be the number of orbits in $[k_n]$ under the action of the subgroup of G generated by α, and for every $i \in [k_n]$, let $o(i) = |\langle\alpha\rangle \cdot i|$ be the size of the orbit of i under the action of the subgroup generated by α. Let $\{j_1, \ldots, j_l\} \subseteq [k_n]$ be a set containing exactly one representative of each orbit under the action of $\langle\alpha\rangle$. We claim that $C_{H_n}(\alpha)$, the centraliser of α in H_n, is isomorphic to

$$L = \prod_{i=1}^{l} C_{\phi(j_i)}(\alpha^{o(j_i)}).$$

To see this, let us define a homomorphism $\psi \colon L \to H_n$ by setting

$$\psi(h_{j_i}) = \prod_{m=0}^{o(j_i)-1} \alpha^m h_{j_i} \alpha^{-m} \in \prod_{m=0}^{o(j_i)-1} \phi(\alpha^m \cdot j_i)$$

for $h_{j_i} \in C_{\phi(j_i)}(\alpha^{o(j_i)}) \leq \phi(j_i)$ and then defining $\psi(h_{j_1} \cdots h_{j_l}) = \psi(h_{j_1}) \cdots \psi(h_{j_l})$. Using the fact that by condition (2), distinct conjugates commute and intersect trivially, it is easy to check that ψ is an injective homomorphism. Furthermore, for $h_{j_i} \in C_{\phi(j_i)}(\alpha^{o(j_i)})$, we have

$$\alpha\psi(h_{j_i})\alpha^{-1} = \prod_{m=1}^{o(j_i)} \alpha^m h_{j_i} \alpha^{-m} = \prod_{m=0}^{o(j_i)-1} \alpha^m h_{j_i} \alpha^{-m} = \psi(h_{j_i}),$$

where we have used the fact that $\alpha^{o(j_i)} h_{j_i} \alpha^{-o(j_i)} = h_{j_i}$ since $h_{j_i} \in C_{\phi(j_i)}(\alpha^{o(j_i)})$. It follows that the image of ψ is contained in $C_{H_n}(\alpha)$.

Now, let $h = h_1 \cdots h_{k_n} \in C_{H_n}(\alpha)$ be any element of H_n centralising α. We have

$$\alpha h \alpha^{-1} = (\alpha h_1 \alpha^{-1}) \cdots (\alpha h_{k_n} \alpha^{-1}) = h_1 \cdots h_{k_n}.$$

Since, for any $j \in [k_n]$, we have $\alpha h_j \alpha^{-1} \in \phi(\alpha \cdot j)$, we must have $\alpha h_j \alpha^{-1} = h_{\alpha \cdot j}$ for all $j \in [k_n]$, which implies that $\alpha^m \cdot h_j \alpha^{-m} = h_{\alpha^m \cdot j}$. In particular, for every $i \in [l]$, we have $\alpha^{o(j_i)} h_{j_i} \alpha^{-o(j_i)} = h_{\alpha^{o(j_i)} \cdot j_i} = h_{j_i}$, which means that $h_{j_i} \in C_{\phi(j_i)}(\alpha^{o(j_i)})$. Since every $j \in [k_n]$ can be written as $\alpha^m \cdot h_{j_i}$ for some $i \in [l]$ and some $m \in \{0, \ldots, o(j_i) - 1\}$, we conclude from all this that $h = \psi(h_{j_1} \cdots h_{j_l})$. This shows that $\psi(L) = C_{H_n}(\alpha)$ and thus finishes showing that $C_{H_n}(\alpha) \cong L$.

By assumption, $C_G(\alpha)$ is not finitely generated. Since H_n is of finite index in G, $C_{H_n}(\alpha) = C_G(\alpha) \cap H_n$ must be of finite index in $C_G(\alpha)$ and thus cannot be finitely generated. This implies that there exists some $i_0 \in [l]$ such that $C_{\phi(j_{i_0})}(\alpha^{o(j_{i_0})})$ is not finitely generated. Indeed, we have just seen that $C_{H_n}(\alpha) \cong \prod_{i=1}^{l} C_{\phi(j_i)}(\alpha^{o(j_i)})$, so if all $C_{\phi(j_i)}(\alpha^{o(j_i)})$ were finitely generated, $C_{H_n}(\alpha)$ would be as well. Notice that by construction, $\alpha^{o(j_{i_0})}$ normalises $\phi(j_{i_0})$, so that conjugation by $\alpha^{o(j_{i_0})}$ is an automorphism of $\phi(j_{i_0})$ whose set of fixed points, $C_{\phi(j_{i_0})}(\alpha^{o(j_{i_0})})$, is not finitely generated. Since $\phi(j_{i_0})$ is by definition a conjugate of K_n, and thus isomorphic to it, we have just proven the existence of an automorphism $f_n \colon K_n \to K_n$ whose set of fixed points is not finitely generated. This finishes proving that the assumptions of Theorem A are satisfied by G and thus that G is intersection-saturated. ■

As a corollary, we get that the Grigorchuk group is intersection-saturated (we refer the reader to [3] for an introduction to this group).

Corollary 3.3. *The Grigorchuk group is intersection-saturated.*

Proof. The Grigorchuk group is a finitely generated branch group (see, e.g., [1, Proposition 1.25]), and by a theorem of Rozhkov [8, Theorem 1], it admits elements whose centralisers are not finitely generated. Thus, by Theorem 3.2, it is intersection-saturated. ■

Although we consider this to be out of the scope of the current article, we believe that it should be fairly straightforward to adapt Rozhkov's arguments in [8] to show that all finitely generated branch *spinal groups* possess elements whose centralisers are not finitely generated and thus are intersection-saturated (see [1] for the definition of spinal groups). We do not currently know, however, this must be the case for all finitely generated branch groups.

Question 3.4. Do all finitely generated branch groups possess an element whose centraliser is not finitely generated?

If the answer to the above question is negative, could there exist finitely generated branch groups that are not intersection-saturated?

Question 3.5. Are all finitely generated branch groups intersection-saturated?

Let us finally conclude this note by using Corollary 3.3 to show that there exist finitely presented amenable intersection-saturated groups.

Corollary C. *There exist finitely presented amenable intersection-saturated groups.*

Proof. By a result of Grigorchuk ([5], Theorem 1), there exists a finitely presented amenable group $\overline{\Gamma}$ containing the Grigorchuk group Γ as a subgroup. Since Γ is intersection-saturated by Corollary 3.3, $\overline{\Gamma}$ is intersection-saturated. ∎

Acknowledgements. The author would like to thank Jordi Delgado and Enric Ventura for bringing the question to his attention and for their valuable comments and suggestions on a previous version of this text. He is also grateful to Corentin Bodart for noticing a mistake in an earlier version.

Funding. The author acknowledges the support of the Leverhulme Trust Research Project Grant RPG-2022-025.

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Received 21 December 2023.

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