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# Groups elementarily equivalent to a finitely generated free metabelian group

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**Abstract.** We describe groups elementarily equivalent to a free metabelian group with $n$ generators. We also explore an exponentiation that naturally occurs in metabelian groups.

*Dedicated to Slava Grigorchuk on the occasion of his birthday*

## 1. Introduction

In this paper, we describe all groups elementarily equivalent to a free metabelian group $G$ of finite rank $> 1$. To do this, we first prove that $G$ is regularly bi-interpretable with $\mathbb{Z}$. Then following [4] we show that groups elementarily equivalent to $G$ are precisely the non-standard versions $G(\widetilde{\mathbb{Z}})$ of $G$, where $\widetilde{\mathbb{Z}} \equiv \mathbb{Z}$, and describe their algebraic structure. Along the way, we prove that the set of all bases of $G$ is absolutely definable in $G$. For this, we provide a new characterization of bases of $G$ in terms of the normal forms and their coordinate functions which is interesting in its own right. Furthermore, this regular bi-interpretation of $G$ with $\mathbb{Z}$ is strong and injective, which gives many interesting model-theoretic properties of $G$. Thus, $G$ is rich, that is, the first-order logic over $G$ is as expressive as the weak second-order logic over $G$ (see [9]), $G$ admits elimination of imaginaries and the projective logical geometries over $G$ and $\mathbb{Z}$ form equivalent categories (see [5]), etc. An important part of our characterization of the algebraic structure of non-standard models $G(\widetilde{\mathbb{Z}})$ comes from the theory of exponential groups (see [1, 17, 18]). It turns out that every non-standard group $G(\widetilde{\mathbb{Z}})$ is an exponential $\widetilde{\mathbb{Z}}$-group, and we can describe the $\widetilde{\mathbb{Z}}$-exponentiation in $G(\widetilde{\mathbb{Z}})$.

## 2. Interpretability and bi-interpretability

One can use the model-theoretic notion of interpretability and bi-interpretability to study structures elementarily equivalent to a given one. In this paper, we are going to do this for

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free metabelian groups. We recall here some precise definitions and several known facts that may not be very familiar to algebraists.

A *language* (or *a signature*) $L$ is a triple $(Fun, Pr, C)$, where $Fun = \{f, \ldots\}$ is a set of functional symbols $f$ coming together with their arities $n_f \in \mathbb{N}$, $Pr$ is a set of relation (or predicate) symbols $Pr = \{P, \ldots\}$ coming together with their arities $n_P \in \mathbb{N}$, and a set of constant symbols $C = \{c, \ldots\}$. Sometimes we write $f(x_1, \ldots, x_n)$ or $P(x_1, \ldots, x_n)$ to show that $n_f = n$ or $n_P = n$. Usually we denote variables by small letters $x, y, z, a, b, u, v, \ldots$, while the same symbols with bars $\overline{x}, \overline{y}, \ldots$ denote tuples of the corresponding variables, say $\overline{x} = (x_1, \ldots, x_n)$. In this paper, we always assume, if not said otherwise, that the languages we consider are finite. The following languages appear frequently throughout the text: the language of groups $\{\cdot, ^{-1}, 1\}$, where 1 is the constant symbol for the identity element, $\cdot$ is the binary multiplication symbol, and $^{-1}$ is the symbol of inversion, and the language of unitary rings $\{+, \cdot, 0, 1\}$ with the standard symbols for addition, multiplication, the additive identity 0, and the unity 1.

A structure in the language $L$ (an $L$-structure) with the base set $A$ is sometimes denoted by $\mathbb{A} = \langle A; L\rangle$ or simply by $\mathbb{A} = \langle A; f, \ldots, P, \ldots, c, \ldots\rangle$. For a given structure $\mathbb{A}$ by $L(\mathbb{A})$, we denote the language of $\mathbb{A}$. When the language $L$ is clear from the context, we follow the standard algebraic practice and denote the structure $\mathbb{A} = \langle A; L\rangle$ simply by $A$.

Let $\mathbb{B} = \langle B; L\rangle$ be a structure. A subset $A \subseteq B^n$ is called *definable* in $\mathbb{B}$ if there is a formula $\varphi(x_1, \ldots, x_n)$ (without parameters) in $L(\mathbb{B})$ such that $A = \{(b_1, \ldots, b_n) \in B^n \mid \mathbb{B} \models \varphi(b_1, \ldots, b_n)\}$. In this case, we denote $A$ by $\varphi(B^n)$ or $\varphi(\mathbb{B})$ and say that $\varphi$ *defines* $A$ in $\mathbb{B}$. Similarly, an operation $f$ on the subset $A$ is definable in $\mathbb{B}$ if its graph is definable in $\mathbb{B}$. A constant $c$ is definable if the relation $x = c$ is definable. An $n$-ary predicate $P(x_1, \ldots, x_n)$ is definable in $\mathbb{B}$ if the set $\{(b_1, \ldots, b_n) \in \mathbb{B}^n \mid P(b_1, \ldots, b_n) \text{ is true}\}$ is definable in $\mathbb{B}$.

**Definition 2.1.** An algebraic structure $\mathbb{A} = \langle A; f, \ldots, P, \ldots, c, \ldots\rangle$ is absolutely interpretable (or 0-interpretable) in a structure $\mathbb{B}$ if there is a subset $A^* \subseteq B^n$ definable in $\mathbb{B}$, an equivalence relation $\sim$ on $A^*$ definable in $\mathbb{B}$, operations $f^*, \ldots$, predicates $P^*, \ldots$, and constants $c^*, \ldots$, on the quotient set $A^*/{\sim}$ all interpretable in $\mathbb{B}$ such that the structure $\mathbb{A}^* = \langle A^*/{\sim}; f^*, \ldots, P^*, \ldots, c^*, \ldots\rangle$ is isomorphic to $\mathbb{A}$.

More formally, an interpretation of $\mathbb{A}$ in $\mathbb{B}$ is described by the following set of formulas in the language $L(\mathbb{B})$:

$$\Gamma = \{U_\Gamma(\overline{x}), E_\Gamma(\overline{x}_1, \overline{x}_2), Q_\Gamma(\overline{x}_1, \ldots, \overline{x}_{t_Q}) \mid Q \in L(\mathbb{A})\}$$

(here $\overline{x}$ and $\overline{x}_i$ are $n$-tuples of variables) which interpret $\mathbb{A}$ in $\mathbb{B}$ (as in Definition 2.1). Namely, $U_\Gamma$ defines in $\mathbb{B}$ a subset $A_\Gamma = U_\Gamma(B^n) \subseteq B^n$, $E_\Gamma$ defines in $\mathbb{B}$ an equivalence relation $\sim_\Gamma$ on $A_\Gamma$, and the formulas $Q_\Gamma$ define functions $f_\Gamma$, predicates $P_\Gamma$, and constants $c_\Gamma$ that interpret the corresponding symbols from $L(\mathbb{A})$ on the quotient set $A_\Gamma/\sim_\Gamma$ in such a way that the $L$-structure $\Gamma(\mathbb{B}) = \langle A_\Gamma/\sim_\Gamma; f_\Gamma, \ldots, P_\Gamma, \ldots, c_\Gamma, \ldots\rangle$ is isomorphic to $\mathbb{A}$.

Note that we interpret a constant $c \in L(\mathbb{A})$ in the structure $\Gamma(\mathbb{B})$ by the $\sim_\Gamma$-equivalence class of some tuple $\bar{b}_c \in A_\Gamma$ defined in $\mathbb{B}$ by the formula $Q_c$. We write $\mathbb{A} \simeq \Gamma(\mathbb{B})$ if $\Gamma$ interprets $\mathbb{A}$ in $\mathbb{B}$ as described above and refer to $\Gamma$ as an *interpretation code* or just *code*. The number $n$ is called the dimension of $\Gamma$, denoted by $n = \dim \Gamma$. By $\mu_\Gamma$, we denote a surjective map $A_\Gamma \to \mathbb{A}$ (here $\mathbb{A} = \langle A; L(\mathbb{A})\rangle$) that gives rise to an isomorphism $\bar{\mu}_\Gamma : \Gamma(\mathbb{B}) \to \mathbb{A}$. We refer to this map $\mu_\Gamma$ as *the coordinate map* of the interpretation $\Gamma$. When the formula $E_\Gamma$ defines the identity relation $(x_1 = x_1') \wedge \cdots \wedge (x_n = x_n')$, the surjection $\mu_\Gamma$ is injective, in which case, $\Gamma(\mathbb{B})$ is called an *injective interpretation*. Finally, notation $\mathbb{A} \overset{\Gamma}{\rightsquigarrow} \mathbb{B}$ means that $\mathbb{A}$ is interpretable in $\mathbb{B}$ by the code $\Gamma$.

More generally, the formulas that interpret $\mathbb{A}$ in $\mathbb{B}$ may contain elements from $\mathbb{B}$ that are not in the language $L(\mathbb{B})$, that is, some parameters, say $p_1, \ldots, p_k \in B$. In this case, we assume that all the formulas from the code $\Gamma$ have a tuple of extra variables $\bar{y} = (y_1, \ldots, y_k)$ for parameters in $\mathbb{B}$:

$$\Gamma = \{U_\Gamma(\bar{x}, \bar{y}), E_\Gamma(\bar{x}_1, \bar{x}_2, \bar{y}), Q_\Gamma(\bar{x}_1, \ldots, \bar{x}_{t_Q}, \bar{y}) \mid Q \in L(\mathbb{A})\}, \tag{2.1}$$

so that after the assignment $y_1 \to p_1, \ldots, y_k \to p_k$, the code interprets $\mathbb{A}$ in $\mathbb{B}$. In this event, we write $\mathbb{A} \simeq \Gamma(\mathbb{B}, \bar{p})$ (here $\bar{p} = (p_1, \ldots, p_k)$) and say that $\mathbb{A}$ is interpretable in $\mathbb{B}$ by the code $\Gamma$ with parameters $\bar{p}$. In the case, when $\bar{p} = \emptyset$, one gets again the absolute interpretability. Sometimes, it is convenient to consider interpretations $\mathbb{A} \simeq \Gamma(\mathbb{B}, \bar{p})$ together with their coordinate maps, that is, as triples $(\Gamma, \bar{p}, \mu_\Gamma)$.

We say that a structure $\mathbb{A}$ is interpreted in a given structure $\mathbb{B}$ *uniformly* with respect to a subset $D \subseteq B^k$ if there is a code $\Gamma$ such that $\mathbb{A} \simeq \Gamma(\mathbb{B}, \bar{p})$ for every tuple of parameters $\bar{p} \in D$. If $\mathbb{A}$ is interpreted in $\mathbb{B}$ uniformly with respect to a 0-definable subset $D \subseteq B^k$, then we say that $\mathbb{A}$ is *regularly interpretable* in $\mathbb{B}$ and write in this case $\mathbb{A} \simeq \Gamma(\mathbb{B}, \varphi)$, provided $D$ is defined by $\varphi$ in $\mathbb{B}$. This notion appeared first in [15, Section 1.1] or [14], and it is similar to the notion of *interpretability with definable parameters* [7, Remark 5, p. 215]. Note that the absolute interpretability is a particular case of the regular interpretability where the set $D$ is empty.

It is known that the relation $\mathbb{A} \rightsquigarrow \mathbb{B}$ is transitive on algebraic structures (see, e.g., [4,7]). The proof of this fact is based on the notion of $\Gamma$-translation and composition of codes, which we present now.

Let $\Gamma$ be the code (2.1). Then for any formula $\varphi(x_1, \ldots, x_m)$ in the language $L(\mathbb{A})$, there is a formula $\varphi_\Gamma(\bar{x}_1, \ldots, \bar{x}_m, \bar{y})$ in the language $L(\mathbb{B})$, the $\Gamma$-translation of $\varphi$, such that if $\mathbb{A} \simeq \Gamma(\mathbb{B}, \bar{p})$, then for any coordinate map $\mu_\Gamma \colon A_\Gamma \to A$, one has

$$\mathbb{A} \models \varphi(a_1, \ldots, a_m) \iff \mathbb{B} \models \varphi_\Gamma(\mu_\Gamma^{-1}(a_1), \ldots, \mu_\Gamma^{-1}(a_m), \bar{p})$$

for any elements $a_i \in A$ (see [4,7]). Here $\mu_\Gamma^{-1}(a_i)$ means an arbitrary preimage of $a_i$ under $\mu_\Gamma$. Furthermore, for any elements $\bar{b}_i \in B^n$ if $\mathbb{B} \models \varphi_\Gamma(\bar{b}_1, \ldots, \bar{b}_m, \bar{p})$, then $\bar{b}_i \in \mu_\Gamma^{-1}(a_i)$ for some $a_i \in A$ with $\mathbb{A} \models \varphi(a_1, \ldots, a_m)$.

**Definition 2.2.** Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be algebraic structures. Consider codes

$$\Gamma = \{U_\Gamma(\bar{x}, \bar{y}), E_\Gamma(\bar{x}, \bar{x}', \bar{y}), Q_\Gamma(\bar{x}_1, \ldots, \bar{x}_{t_Q}, \bar{y}) \mid Q \in L(\mathbb{A})\}$$

and

$$\Delta = \{U_\Delta(\bar{u},\bar{z}), E_\Delta(\bar{u},\bar{u}',\bar{z}), Q_\Delta(\bar{u}_1,\dots,\bar{u}_{t_Q},\bar{z}) \mid Q \in L(\mathbb{B})\}$$

which consist of $L(\mathbb{C})$-formulas. Then the *composition* of the codes $\Gamma$ and $\Delta$ is the code

$$\Gamma \circ \Delta = \{U_{\Gamma\circ\Delta}, E_{\Gamma\circ\Delta}, Q_{\Gamma\circ\Delta} \mid Q \in L(\mathbb{A})\}$$
$$= \{(U_\Gamma)_\Delta, (E_\Gamma)_\Delta, (Q_\Gamma)_\Delta \mid Q \in L(\mathbb{A})\}.$$

The following is an important technical result on the transitivity of interpretations.

**Lemma 2.3** ([4]). *Let $\mathbb{A} = \langle A; L(\mathbb{A})\rangle, \mathbb{B} = \langle B; L(\mathbb{B})\rangle$, and $\mathbb{C} = \langle C; L(\mathbb{C})\rangle$ be algebraic structures and $\Gamma, \Delta$ be codes as above. If $\mathbb{A} \overset{\Gamma}{\rightsquigarrow} \mathbb{B}$ and $\mathbb{B} \overset{\Delta}{\rightsquigarrow} \mathbb{C}$, then $\mathbb{A} \overset{\Gamma\circ\Delta}{\rightsquigarrow} \mathbb{C}$.*

*Furthermore, the following conditions hold:*

(1) *If $\bar{p}, \bar{q}$ are parameters and $\mu_\Gamma, \mu_\Delta$ are coordinate maps of interpretations $\Gamma, \Delta$, then $(\bar{\bar{p}}, \bar{q})$, where $\bar{\bar{p}} \in \mu_\Delta^{-1}(\bar{p})$, are parameters for $\Gamma \circ \Delta$.*

(2) *$\mu_\Gamma \circ \mu_\Delta = \mu_\Gamma \circ \mu_\Delta^n\big|_{U_{\Gamma\circ\Delta}(\mathbb{C},(\bar{\bar{p}},\bar{q}))}$ is a coordinate map of the interpretation $\mathbb{A} \simeq \Gamma \circ \Delta(\mathbb{C}, (\bar{\bar{p}}, \bar{q}))$ and any coordinate map $\mu_{\Gamma\circ\Delta}\colon U_{\Gamma\circ\Delta}(\mathbb{C}, (\bar{\bar{p}}, \bar{q})) \to A$ has a form $\mu_{\Gamma 1} \circ \mu_\Delta$ for a suitable coordinate map $\mu_{\Gamma 1}$ of the interpretation $\mathbb{A} \simeq \Gamma(\mathbb{B}, \bar{p})$, provided $\mu_\Delta$ is fixed.*

Observe that composition of absolute (regular) interpretations is absolute (regular).

Now we discuss a very strong version of mutual interpretability of two structures, so-called *bi-interpretability*.

**Definition 2.4.** Algebraic structures $\mathbb{A}$ and $\mathbb{B}$ are called *strongly bi-interpretable* (with parameters) in each other, if

(1) there exists an interpretation $(\Gamma, \bar{p}, \mu_\Gamma)$ of $\mathbb{A}$ into $\mathbb{B}$ and an interpretation $(\Delta, \bar{q}, \mu_\Delta)$ of $\mathbb{B}$ into $\mathbb{A}$, so the algebraic structures $\Gamma \circ \Delta(\mathbb{A}, (\bar{\bar{p}}, \bar{q}))$ and $\Delta \circ \Gamma(\mathbb{B}, (\bar{\bar{q}}, \bar{p}))$ are uniquely defined and $\Gamma \circ \Delta(\mathbb{A}, (\bar{\bar{p}}, \bar{q}))$ is isomorphic to $\mathbb{A}$, while $\Delta \circ \Gamma(\mathbb{B}, (\bar{\bar{q}}, \bar{p}))$ is isomorphic to $\mathbb{B}$;

(2) the composition $\mu_\Gamma \circ \mu_\Delta\colon U_{\Gamma\circ\Delta}(\mathbb{A}, (\bar{\bar{p}}, \bar{q})) \to A$ is definable in $\mathbb{A}$ and the composition $\mu_\Delta \circ \mu_\Gamma\colon U_{\Delta\circ\Gamma}(\mathbb{B}, (\bar{\bar{q}}, \bar{p})) \to B$ is definable in $\mathbb{B}$.

In this case, we additionally say that $\mathbb{A}$ and $\mathbb{B}$ are *strongly injectively bi-interpretable*, if the interpretations $\Gamma$ and $\Delta$ are injective.

Note that there is another slightly different notion of *bi-interpretation*, which for contrast we sometimes call a *weak bi-interpretation*, where in the above definition condition (2) that requires definability of the maps $\mu_\Gamma \circ \mu_\Delta$ and $\mu_\Delta \circ \mu_\Gamma$ is replaced by a weaker one that requires definability of some coordinate maps $A_{\Gamma\circ\Delta} \to A$ and $B_{\Delta\circ\Gamma} \to B$. Often, authors do not even mention the difference, implicitly assuming either one or another. To be precise, we endorse these two notions explicitly. Observe that the bi-interpretation defined in the books [7, 9] is weak, but in the paper [2], it is strong. There are many interesting applications of strong bi-interpretations which we cannot derive from the weak ones.

Two algebraic structures $\mathbb{A}$ and $\mathbb{B}$ are called *0-bi-interpretable* or *absolutely bi-*interpretable (strongly) in each other if in the definition above the tuples of parameters $p$ and $q$ are empty.

Unfortunately, 0-bi-interpretability is rather rare. Indeed, if $\mathbb{A}$ and $\mathbb{B}$ are 0-bi-interpretable in each other, then their groups of automorphisms are isomorphic [7]. Bi-interpretability with parameters occurs much more often, but it gives much less for applications; in particular, it is not applicable for first-order classification problems (addition of constants changes the language).

Fortunately, there is a notion of regular bi-interpretability, which is less restrictive, occurs more often, and which enjoys many properties of 0-bi-interpretability.

**Definition 2.5.** Algebraic structures $\mathbb{A}$ and $\mathbb{B}$ are called *regularly bi-interpretable*, if

(1) there exist a regular interpretation $(\Gamma, \varphi)$ of $\mathbb{A}$ in $\mathbb{B}$ and a regular interpretation $(\Delta, \psi)$ of $\mathbb{B}$ in $\mathbb{A}$;

(2) there exists formula $\theta_{\mathbb{A}}(\bar{u}, x, \bar{r})$ in $L(\mathbb{A})$, where $|\bar{u}| = \dim \Gamma \cdot \dim \Delta$, $|\bar{r}| = \dim_{\text{par}} \Gamma \circ \Delta$, such that for any tuple $\bar{r}_0 \in \varphi_\Delta \wedge \psi(\mathbb{A})$ the formula $\theta_{\mathbb{A}}(\bar{u}, x, \bar{r}_0)$ defines some coordinate map $U_{\Gamma\circ\Delta}(\mathbb{A}, \bar{r}_0) \to A$;

(3) there exists formula $\theta_{\mathbb{B}}(\bar{u}, x, \bar{t})$ in $L(\mathbb{B})$, where $|\bar{u}| = \dim \Gamma \cdot \dim \Delta$, $|\bar{t}| = \dim_{\text{par}} \Delta \circ \Gamma$, such that for any tuple $\bar{t}_0 \in \psi_\Gamma \wedge \varphi(\mathbb{B})$ the formula $\theta_{\mathbb{B}}(\bar{u}, x, \bar{t}_0)$ defines some coordinate map $U_{\Delta\circ\Gamma}(\mathbb{B}, \bar{t}_0) \to B$.

Regular bi-interpretation is called *injective* if both $(\Gamma, \varphi)$ and $(\Delta, \psi)$ are injective.

**Definition 2.6.** We say that $\mathbb{A}$ and $\mathbb{B}$ are *strongly regularly bi-interpretable*, if they are regularly bi-interpretable, that is, (1)–(3) hold, and additionally

(4) for any pair of parameters $(\bar{p}, \bar{q})$, $\bar{p} \in \varphi(\mathbb{B})$, $\bar{q} \in \psi(\mathbb{A})$, there exists a pair of coordinate maps $(\mu_\Gamma, \mu_\Delta)$ for interpretations $(\Gamma, \bar{p})$ and $(\Delta, \bar{q})$, such that for any $\bar{r}_0 = (\bar{\bar{p}}, \bar{q})$, $\bar{\bar{p}} \in \mu_\Delta^{-1}(\bar{p})$, and $\bar{t}_0 = (\bar{\bar{q}}, \bar{p})$, $\bar{\bar{q}} \in \mu_\Gamma^{-1}(\bar{q})$, the coordinate maps $\mu_\Gamma \circ \mu_\Delta : U_{\Gamma\circ\Delta}(\mathbb{A}, \bar{r}_0) \to A$ and $\mu_\Delta \circ \mu_\Gamma : U_{\Delta\circ\Gamma}(\mathbb{B}, \bar{t}_0) \to B$ are defined in $\mathbb{A}$ and $\mathbb{B}$ correspondingly by the formulas $\theta_{\mathbb{A}}(\bar{u}, x, \bar{r}_0)$ and $\theta_{\mathbb{B}}(\bar{u}, x, \bar{t}_0)$.

**Theorem 2.7** ([4, 12]). *Let $\mathbb{A}$ and $\mathbb{B}$ be regularly bi-interpretable in each other, so $\mathbb{A} \cong \Gamma(\mathbb{B}, \varphi)$ and $\mathbb{B} \cong \Delta(\mathbb{A}, \psi)$. Then*

(1) *for any $\widetilde{\mathbb{B}} \equiv \mathbb{B}$, the code $(\Gamma, \varphi)$ regularly interprets a structure $\widetilde{\mathbb{A}} \simeq \Gamma(\widetilde{\mathbb{B}}, \varphi)$ in $\widetilde{\mathbb{B}}$ such that $\widetilde{\mathbb{A}} \equiv \mathbb{A}$;*

(2) *every $L(\mathbb{A})$-structure $\widetilde{\mathbb{A}}$ elementarily equivalent to $\mathbb{A}$ is isomorphic to $\Gamma(\widetilde{\mathbb{B}}, \varphi)$ for some $\widetilde{\mathbb{B}} \equiv \mathbb{B}$;*

(3) *for any $\mathbb{B}_1 \equiv \mathbb{B} \equiv \mathbb{B}_2$, one has*

$$\Gamma(\mathbb{B}_1, \varphi) \cong \Gamma(\mathbb{B}_2, \varphi) \iff \mathbb{B}_1 \cong \mathbb{B}_2.$$

In fact, one does not need bi-interpretability of $\mathbb{A}$ and $\mathbb{B}$ in Theorem 2.7; it suffices to have a weaker condition that $\mathbb{A}$ is regularly invertibly interpretable in $\mathbb{B}$ (see [4, 12]). However, for other applications, bi-interpretability is required.

Elimination of imaginaries plays an important part in model theory (see [7] for details). This involves imaginaries *without parameters*.

**Definition 2.8.** We say that an algebraic structure $\mathbb{B}$ has *uniform elimination of imaginaries with parameters*, if the algebraic structure $\mathbb{B}_B$, obtained from $\mathbb{B}$ by adding all elements of $\mathbb{B}$ to the language as constants, has uniform elimination of imaginaries without parameters.

**Theorem 2.9** ([4]). *Let $\mathbb{A}$ and $\mathbb{B}$ be strongly bi-interpretable with parameters in each other, so $\mathbb{A} \simeq \Gamma(\mathbb{B}, \overline{p})$ and $\mathbb{B} \simeq \Delta(\mathbb{A}, \overline{q})$, and $\Delta$ is injective. If $\mathbb{B}$ has uniform elimination of imaginaries with parameters, then $\mathbb{A}$ has uniform elimination of imaginaries with parameters.*

Note that $\mathbb{Z}$ has uniform elimination of imaginaries (with or without parameters). Hence, all structures strongly injectively bi-interpretable with $\mathbb{Z}$ (with parameters) enjoy uniform elimination of imaginaries with parameters.

## 3. Bi-interpretation of a free metabelian group with $\mathbb{Z}$

It was announced in [10] and proved in [9] that a free metabelian group $G$ of finite rank $n \geq 2$ is prime, atomic, homogeneous, and QFA (quasi finitely axiomatizable). Furthermore, it was shown in [9] that $G$ is rich which implies a host of model-theoretic results for $G$. This was done by proving the existence of bi-interpretation of $G$ with $\mathbb{Z}$. Here we improve on our bi-interpretation from [9] and provide new applications of this in the next section.

Throughout this section, we denote by $G$ a free metabelian group of rank $n \geq 2$.

### 3.1. Preliminaries for metabelian groups

In this section, we introduce notation and describe some results that we need in the sequel.

Let $G' = [G, G]$ be the commutant of $G$ and $G_m$ the $m$th term of the lower central series of $G$. For a subset $A \subseteq G$ denote by $\langle A \rangle$ the subgroup generated by $A \subseteq G$ and by $C_G(A)$ the centralizer of $A$ in $G$. If $g, h \in G$, then $[g, h] = g^{-1}h^{-1}gh$ is the commutator of $g$ and $h$, and $g^h = h^{-1}gh$ is the conjugate of $g$ by $h$. The maximal root of an element $g \in G$ is an element $g_0 \in G$ such that $g_0$ is not a proper power in $G$ and $g \in \langle g_0 \rangle$. Note that the maximal roots of the elements in $G$ exist and they are unique.

We will be using the following standard commutator identities that hold in every group for any elements $a, b, c$:

$$[a,b]^{-1} = [b,a], \qquad [a^{-1},b] = [b,a]^{a^{-1}}, \tag{3.1}$$

$$[ab,c] = [a,c]^b[b,c], \quad [a,bc] = [a,c][a,b]^c. \tag{3.2}$$

In [11], Mal'cev obtained a description of centralizers of elements in free solvable groups; in particular, in free metabelian groups $G$, centralizers are as follows.

**Lemma 3.1.** *Let $g \in G, g \neq 1$. Then*

(1) *if $g \in G'$, then $C_G(g) = G'$;*

(2) *if $g \notin G'$, then $C_G(g) = \langle g_0 \rangle$, where $g_0$ is the unique maximal root of $g$.*

Let $v \in G \setminus G'$. Define a map $\lambda_v : G' \to G'$ such that for $c \in G'$ $\lambda_v(c) = [v,c]$. Then the map $\lambda_v$ is a homomorphism. Indeed, using the second commutator identity in (3.2), one has for $c_1, c_2 \in G'$

$$[v,c_1c_2] = [v,c_2][v,c_1]^{c_2} = [v,c_2][v,c_1] = [v,c_1][v,c_2], \tag{3.3}$$

as claimed. Similarly, in the notation above, the map $\mu_v : c \to [c,v]$ is a homomorphism $\mu_v : G' \to G'$.

Let $v \in G \setminus G'$ and $d \in G'$. Then for any $k \in \mathbb{Z}$, there exists $c \in G'$ such that

$$(vd)^k = v^k d^k [c,v].$$

We prove first, by induction on $k$, that

$$d^k v = vd^k[c,v]$$

for some $c \in G'$. Indeed, for $k = 1$, one has the standard equality $dv = vd[d,v]$. Now

$$\begin{aligned} d^{k+1}v = d^k dv = d^k vd[d,v] &= vd^k[c_1,v]d[d,v] \\ &= vd^{k+1}[c_1,v][d,v] = vd^{k+1}[c_1d,v], \end{aligned}$$

the last equality comes from property (3.3), that the map $\mu_v$ is a homomorphism on $G'$.

Now one can finish the claim by induction on $k$ as follows (here elements $c_i \in G'$ appear as the result of application of the induction step and the claim above):

$$\begin{aligned} (vd)^{k+1} &= (vd)^k vd = v^k d^k [c_2,v]vd = v^k d^k v[c_2,v][[c_2,v],v]d \\ &= v^k vd^k[c_3,v][c_2,v][[c_2,v],v]d = v^{k+1}d^{k+1}[c_3c_2[c_2,v],v] \\ &= v^{k+1}d^{k+1}[c,v], \end{aligned}$$

where the second to last equality comes again from property (3.3), and $c = c_3c_2[c_2,v]$. This proves the claim.

**Lemma 3.2** ([9, Lemma 4.23]). *Let $d \in G'$. If for any $v \in G \setminus G'$, there exists $c \in G'$ such that $d = [c,v]$, then $d = 1$.*

### 3.2. Normal forms of elements

Fix a finite set of generators $X = \{x_1, \dots, x_n\}$.

We denote by $^{-}$: $G \to G/G'$ the canonical epimorphism $g \to \overline{g} = gG'$ from the group $G$ onto its abelianization $\overline{G} = G/G'$. Put $a_1 = \overline{x}_1, \dots, a_n = \overline{x}_n$. The group $G$ acts by conjugation on $G'$, which gives an action of the abelianization $\overline{G}$ on $G'$. This action extends linearly to an action of the group ring $\mathbb{Z}\overline{G}$ on $G'$ and turns $G'$ into a $\mathbb{Z}\overline{G}$-module. Since $a_1, \dots, a_n$ generate $\overline{G}$, then the identity map $a_i \to a_i, i = 1, \dots, n$, extends to an epimorphism $\eta : A \to \mathbb{Z}\overline{G}$ from the ring of Laurent polynomials $A = \mathbb{Z}[a_1, a_1^{-1}, \dots, a_n, a_n^{-1}]$ onto $\mathbb{Z}\overline{G}$, which provides an action of $A$ onto $G'$ and turns $G'$ into $A$-module. If $a_1, \dots, a_n$ forms a basis of $G'$, then $\eta$ is an isomorphism, and we can identify $\mathbb{Z}\overline{G}$ with $A$. For the action of $A$ on $G'$, we use exponential notation, that is, for $u \in G'$ and $a \in A$, we denote by $u^a$ the result of the action of $a$ on $u$. We use this notation throughout the paper.

Using the standard commutator identities, which hold in every group, one can write every element $u \in G'$ in the form

$$u = \Pi_{1 \le j < i \le n} [x_i, x_j]^{Q_{i,j}}, \tag{3.4}$$

where $Q_{i,j}$ are Laurent polynomials from $A$. Hence, the set of commutators

$$C_X = \{[x_i, x_j] \mid 1 \le j < i \le n\}$$

generates $G'$ as an $A$-module. Note that $G$, as well as any metabelian group, satisfies the Jacobi identity, that is, for every $u, v, w \in G$,

$$[u, v, w][v, w, u][w, u, v] = 1. \tag{3.5}$$

In particular, for $u = x_i, v = x_j, w = x_k$, one gets (in the module notation)

$$[x_i, x_j]^{a_k - 1} [x_j, x_k]^{a_i - 1} [x_k, x_i]^{a_j - 1} = 1,$$

hence

$$[x_i, x_j]^{a_k - 1} = [x_k, x_j]^{a_i - 1} [x_k, x_i]^{1 - a_j}, \tag{3.6}$$

so $C_X$ is not a free generating set of the module $G'$. However, there are nice normal forms of elements of the module $G'$ (see [9, 13]). To deal with normal forms in $G$, we need a few preliminary results.

**Remark 3.3.** Let $1 \ne \overline{g} \in \overline{G}$ and $\delta \in \mathbb{Z}$. Then $\overline{g} - 1$ divides $\overline{g}^{\delta} - 1$ in the ring $\mathbb{Z}(\overline{G})$, that is, the element $\frac{\overline{g}^{\delta} - 1}{\overline{g} - 1}$ is uniquely defined in $\mathbb{Z}(\overline{G})$. Indeed, if $\delta > 0$, then

$$\overline{g}^{\delta} - 1 = (\overline{g} - 1)(\overline{g}^{\delta - 1} + \dots + \overline{g} + 1).$$

If $\delta < 0$, then $\overline{g}^{\delta} = (\overline{g}^{-1})^{|\delta|}$ and the formula above applies.

The following generalizes equality (3.6).

**Lemma 3.4.** *Let* $1 \le j < i < k \le n$ *and* $\delta \in \mathbb{Z}$. *Then*

$$[x_i, x_j]^{a_k^\delta - 1} = [x_k, x_j]^{(a_i-1)\frac{a_k^\delta-1}{a_k-1}} [x_k, x_i]^{(1-a_j)\frac{a_k^\delta-1}{a_k-1}}. \tag{3.7}$$

*Proof.* Note that

$$[x_i, x_j]^{a_k^\delta - 1} = [x_i, x_j]^{(a_k-1)\frac{a_k^\delta-1}{a_k-1}}.$$

Now we apply (3.6) to $[x_i, x_j]^{(a_k-1)}$ and multiply the result (in the module $G'$) by $\frac{a_k^\delta-1}{a_k-1}$. ∎

**Lemma 3.5.** *Let* $z, g \in G \setminus G'$ *and* $\gamma, \delta \in \mathbb{Z}$, *then*

(1) $[z, g^\delta] = [z, g]^{\frac{\bar{g}^\delta - 1}{\bar{g}-1}}$;

(2) $[z^\gamma, g^\delta] = [z, g]^{(\frac{\bar{z}^\gamma-1}{\bar{z}-1})(\frac{\bar{g}^\delta-1}{\bar{g}-1})}$.

*Proof.* The Jacobi identity

$$[z, g^\delta, g][g^\delta, g, z][g, z, g^\delta] = 1$$

reduces to

$$[z, g^\delta, g] = [g, z, g^\delta]^{-1},$$

since $[g^\delta, g, z] = 1$. After rewriting it in the module notation, we get

$$[z, g^\delta]^{\bar{g}-1} = [z, g]^{\bar{g}^\delta - 1}.$$

This proves (1). (2) follows from (1). ∎

**Proposition 3.6.** *Let* $X = \{x_1, \dots, x_n\}$ *be a generating set of* $G$ *and* $a_1 = \bar{x}_1, \dots, a_n = \bar{x}_n$. *Then every element* $u \in G'$ *can be presented as the following product:*

$$u = \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1, \dots, a_i)}, \tag{3.8}$$

*where* $\beta_{ij}(a_1, \dots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \dots, a_i, a_i^{-1}]$.

*Proof.* We showed above that every element $u \in G'$ can be written in the form (3.4):

$$u = \Pi_{1 \le j < i \le n} [x_i, x_j]^{Q_{i,j}},$$

where $Q_{i,j}$ are Laurent polynomials from $A$. Note that this can be done algorithmically. Now we describe a collecting process that transforms products in the form (3.4) to products of the form (3.8), which we term *collected forms*. Since $G'$ is commutative, it suffices to show how to collect an element $[x_i, x_j]^Q$, where $Q \in A$. Similarly, since $Q$ is a sum of the type $\Sigma_i \gamma_i M_i$, where $M_i \in \overline{G}$ and $\gamma_i \in \mathbb{Z}$, it suffices to collect $[x_i, x_j]^M$, where $M \in \overline{G}$. Decompose $M$ into a product $M = M_1 a_k^\delta M_2$, where $M_1 \in \langle a_1, \dots, a_i \rangle$, $k > i$,

$\delta \in \mathbb{Z}$, and $M_2 \in \langle a_{k+1}, \ldots, a_n \rangle$. Note that $[x_i, x_j]^{M_1}$ is collected. To collect $[x_i, x_j]^{a_k^\delta}$, write it as

$$[x_i, x_j]^{a_k^\delta} = [x_i, x_j]^{a_k^\delta - 1}[x_i, x_j]$$

and apply (3.7) from Lemma 3.4. This results in a collected form $w$, where

$$w = \Pi_{1 \le j < i \le k}[x_i, x_j]^{f_{ij}(a_1,\ldots,a_i)},$$

for some $f_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}]$. Note that

$$w^{M_1} = \Pi_{1 \le j < i \le k}[x_i, x_j]^{f_{ij}(a_1,\ldots,a_i)M_1}$$

is also collected. Since $[x_i, x_j]^M = (w^{M_1})^{M_2}$, the argument above shows that to collect $[x_i, x_j]^M$, it suffices to collect elements of the type $[x_i, x_j]^{M_2}$, where $1 \le j < i \le k$. Now we can repeat the collecting process above. This shows that every element $u \in G'$ can be written in the form (3.8). ∎

**Corollary 3.7.** *Let $X = \{x_1, \ldots, x_n\}$ be a generating set of $G$. Then every element $g \in G$ can be presented as the following product:*

$$g = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \Pi_{1 \le j < i \le n}[x_i, x_j]^{\beta_{ij}(a_1,\ldots,a_i)}, \tag{3.9}$$

*where $\gamma_i \in \mathbb{Z}, 1 \le i \le n$, $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}]$.*

**Proposition 3.8.** *Let $X = \{x_1, \ldots, x_n\}$ be a basis of $G$ as a free metabelian group. Then every element $u \in G'$ can be uniquely presented as the following product:*

$$u = \Pi_{1 \le j < i \le n}[x_i, x_j]^{\beta_{ij}(a_1,\ldots,a_i)}, \tag{3.10}$$

*where $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}]$.*

*Proof.* Observe that the images of $X$ in $\overline{G}$ form a basis of $\overline{G}$, hence $\mathbb{Z}\overline{G} \simeq A$. By Proposition 3.6, every element $u \in G'$ has some decomposition of the form (3.10). To prove uniqueness of the form (3.8), we use induction on $n$. Let

$$u = \Pi_{1 \le j < i \le n}[x_i, x_j]^{\beta_{ij}(a_1,\ldots,a_i)},$$

where $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}]$. Assume that $u = 1$. We need to show that $\beta_{ij} = 0$ for all $1 \le j < i \le n$.

For $n = 2$, the $\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}]$-module $G'$ is free with basis $[x_2, x_1]$ (see [3]), so the result follows.

For $n > 2$, consider the canonical epimorphism $\mu_n : G \to H = G/ncl(x_n)$, where $ncl(x_n)$ is the normal closure of $x_n$ in $G$, so $\mu_n(x_i) = x_i, 1 \le i < n$, $\mu_n(x_n) = 1$. Note that $H$ is a free metabelian group of rank $n - 1$. Clearly,

$$\mu_n(u) = \Pi_{1 \le j < i \le n-1}[x_i, x_j]^{\beta_{ij}(a_1,\ldots,a_i)}.$$

Hence, by induction, $\beta_{ij} = 0$ for $1 \leq j < i \leq n-1$. Therefore,

$$u = [x_n, x_1]^{\beta_{n1}(a_1,\ldots,a_n)} \cdots [x_n, x_{n-1}]^{\beta_{n,n-1}(a_1,\ldots,a_n)}.$$

For an integer $N > 0$, consider a homomorphism $\lambda_N : G \to K = \langle x_2, \ldots, x_n \rangle \leq G$ such that $\lambda_N(x_1) = x_2^N$, $\lambda_N(x_i) = x_i$ for $2 \leq i \leq n$. Clearly, $\lambda_N$ induces the corresponding endomorphism on $\overline{G}$, hence on the ring $A$ and on the $A$-module $G'$. We continue to denote it by $\lambda_N$. Note that $K$ is a free metabelian group or rank $n-1$.

If $\overline{g} = a_1^{\delta_1} \cdots a_n^{\delta_n}$, then $\lambda_N(\overline{g}) = a_2^{\delta_1 N + \delta_2} a_3^{\delta_3} \cdots a_n^{\delta_n}$. One can choose a large enough $N$ such that $\lambda_n$ is injective on all the monomials that occur in $\beta_{nj}$, $j = 1, \ldots, n-1$. This implies that if $\beta_{nj} \neq 0$, then $\lambda_N(\beta_{ij}) \neq 0$. Now

$$\lambda_N(u) = [x_n, x_2^N]^{\lambda_N(\beta_{n1})} [x_n, x_2]^{\lambda_n(\beta_{n2})} \cdots [x_n, x_{n-1}]^{\lambda_N(\beta_{n,n-1})}.$$

By Lemma 3.5, $[x_n, x_2^N] = [x_n, x_2]^{\frac{a_2^N - 1}{a_2 - 1}}$. By induction, we get

$$\frac{a_2^N - 1}{a_2 - 1} \lambda_N(\beta_{n1}) = 0, \quad \lambda_n(\beta_{n2}) = 0, \ldots, \lambda_N(\beta_{n,n-1}) = 0.$$

Hence, $\beta_{n1} = 0, \ldots, \beta_{n,n-1} = 0$. This proves uniqueness. ∎

**Corollary 3.9.** *Let $X = \{x_1, \ldots, x_n\}$ be a basis of $G$ as a free metabelian group. Then every element $g \in G$ can be uniquely presented as the following product, termed the* normal form *of $g$ relative to $X$:*

$$g = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \Pi_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}(a_1,\ldots,a_i)}, \tag{3.11}$$

*where $\gamma_i \in \mathbb{Z}$, $1 \leq i \leq n$, $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}]$.*

Our next task is to describe the multiplication in $G$ in terms of normal forms. We need some notation. Let $X = \{x_1, \ldots, x_n\}$ be a finite subset of $G$, order $X$ as $x_1 < \cdots < x_n$, and form a tuple $x = (x_1, \ldots, x_n)$. Similarly, order the set $C_X = \{[x_i, x_j] \mid 1 \leq j < i \leq n\}$, say by introducing the lexicographical order on pairs of indices $(i, j)$, and form a tuple $c_x = ([x_2, x_1], [x_3, x_1], \ldots, [x_n, x_{n-1}])$. Denote by $\widetilde{x}$ the concatenation $x \cdot c_x = (x_1, \ldots, x_n, [x_2, x_1], \ldots, [x_n, x_{n-1}])$ of $x$ and $c_x$. If $X$ is a basis of $G$ as a free metabelian group, then $\widetilde{x}$ is termed a *normal form basis*, or a *module basis* of $G$, and its length $n + n(n-1)/2$ is denoted by $\dim(G)$.

For $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, put

$$x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n},$$

and for polynomials

$$\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}],$$

where $a_1 = \overline{x}_1, \ldots, a_n = \overline{x}_n$, form a tuple $\beta = (\beta_{2,1}, \ldots, \beta_{n,n-1})$ and put

$$c_x^\beta = [x_2, x_1]^{\beta_{2,1}}, \ldots, [x_n, x_{n-1}]^{\beta_{n,n-1}}.$$

If $g \in G$ and $g = x^\gamma c_x^\beta$, then the tuple $t_{\tilde{x}}(g) = \gamma \cdot \beta$ is called the tuple of *coordinates* (or $\tilde{x}$-coordinates) of $g$ with respect to $\tilde{x}$. We write $t_{\tilde{x}}(g) = (t_1(g), \ldots, t_d(g))$, where $d = \dim(G)$. For $g, h \in G$, the multiplication in $G$ completely determines the coordinates $t_{\tilde{x}}(gh)$ of the product $gh$ in terms of the coordinates $t_{\tilde{x}}(g)$ and $t_{\tilde{x}}(h)$. Hence, in this sense, $t_i(gh)$ can be viewed as a function of $t_{\tilde{x}}(g)$ and $t_{\tilde{x}}(h)$. Our next goal is to describe these functions.

Denote by $\varepsilon_\ell(z)$ the function $\mathbb{Z} \to \mathbb{Z}\overline{G}$ defined by $\varepsilon_\ell(z) = \frac{a_\ell^z - 1}{a_\ell - 1}$, $\ell = 1, \ldots, n$. Let $\mathcal{F}$ be the set of all formal expressions obtained from variables $V = \{z_1, z_2, \ldots\}$, symbols 0 and 1, and functions $\varepsilon_1, \ldots, \varepsilon_n$ (in the variables from $V$) by finitely many operations of addition $+$ and multiplication $\cdot$. Note that every function $\varepsilon_\ell$, and hence every expression $f(z_1, \ldots, z_n) \in \mathcal{F}$, naturally defines a function $f : R^n \to R$ in every integral domain $R$ (we denote the expression and the corresponding function by the same symbol). Since $R$ is associative, commutative, and unitary, every such function $f$ can be presented in the form

$$f = p(z_1, \ldots, z_m, \varepsilon_{\ell_1}(y_1), \ldots, \varepsilon_{\ell_s}(y_s)), \tag{3.12}$$

where $p = p(z_1, \ldots, z_m, u_1, \ldots, u_s)$ is a polynomial with integer coefficients and where each variable $u_i$ is replaced by the function $\varepsilon_{\ell_i}(y_i)$ with $1 \le \ell_i \le n$ and $y_i \in V$. We are going to prove now that the coordinate functions $t_i(gh)$ are defined by some functions from $\mathcal{F}$ *uniformly* in the set $X = \{x_1, \ldots, x_n\}$, that is, for every $i$, $1 \le i \le d$, there is a function $f_i \in \mathcal{F}$ (we may assume $f_i$ is in the form (3.12)) such that for every $n$-element subset $X = \{x_1, \ldots, x_n\} \subseteq G$ ($n$ is fixed upfront) if $t_{\tilde{x}}(g)$ and $t_{\tilde{x}}(h)$ are coordinates of some elements $g, h \in G$ with respect to $\tilde{x}$ (which may not be a module basis), then $t_i(gh) = f_i(t_{\tilde{x}}(g), t_{\tilde{x}}(h))$. To prove this, we need two technical results.

**Lemma 3.10.** *Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of elements of $G$. Then for any $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n \in \mathbb{Z}$,*

$$x_1^{\gamma_1} \cdots x_n^{\gamma_n} x_1^{\delta_1} \cdots x_n^{\delta_n} = x_1^{\gamma_1 + \delta_1} \cdots x_n^{\gamma_n + \delta_n} \overline{\Pi},$$

*where*

$$\overline{\Pi} = \Pi_{j<i} [x_i, x_j]^{\frac{(a_i^{\gamma_i} - 1)(a_j^{\gamma_j} - 1)}{(a_i - 1)(a_j - 1)} a_{j+1}^{\delta_{j+1}} \cdots a_n^{\delta_n}}.$$

*Proof.* In the product

$$x_1^{\gamma_1} \cdots x_n^{\gamma_n} x_1^{\delta_1} \cdots x_n^{\delta_n},$$

we move every $x_j^{\delta_j}$ to the left, using the formulas

$$x_i^{\gamma_i} x_j^{\delta_j} = x_j^{\delta_j} x_i^{\gamma_i} [x_i^{\gamma_i}, x_j^{\delta_j}],$$

and also the formulas

$$[x_i, x_s]^Q x_j^{\delta_j} = x_j^{\delta_j} [x_i, x_s]^{Q a_j^{\delta_j}}$$

whenever $x_j^{\delta_j}$ meets a commutator $[x_i, x_s]^Q$, $Q \in A$, on its immediate left. We do it until $x_j^{\delta_j}$ reaches $x_j^{\gamma_j}$. When all $x_j^{\delta_j}$ moved to the left, the result will be

$$x_1^{\gamma_1+\delta_1} \cdots x_n^{\gamma_n+\delta_n} \overline{\Pi} = \Pi_{j<i} [x_i^{\gamma_i}, x_j^{\delta_j}]^{a_{j+1}^{\delta_{j+1}} \cdots a_n^{\delta_n}}.$$

Now the result follows from Lemma 3.5. ∎

**Lemma 3.11.** *Let $X = \{x_1, \ldots, x_n\}$ be a set of elements of $G$ and elements $g, h \in G$ be given as products*

$$g = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)}$$

*and*

$$h = x_1^{\delta_1} \cdots x_n^{\delta_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\nu_{ij}(a_1, \ldots, a_i)}.$$

*Then*

$$gh = x_1^{\gamma_1+\delta_1} \cdots x_n^{\gamma_n+\delta_n} \overline{\Pi}\ \Pi_{1 \le j < i \le n} [x_i, x_j]^{a_1^{\delta_1} \cdots a_n^{\delta_n} \beta_{ij}(a_1, \ldots, a_i) + \nu_{ij}(a_1, \ldots, a_i)}, \tag{3.13}$$

*where $\overline{\Pi}$ is defined in Lemma 3.10.*

*Proof.* To prove this, we use the same argument as in Lemma 3.10 and the result itself from Lemma 3.10. ∎

Now we are ready to describe the coordinate functions $t_1(gh), \ldots, t_d(gh)$.

**Proposition 3.12.** *For every $1 \le j < i \le n$, there is a function $f_{ij} \in \mathcal{F}$ (we may assume $f_{ij}$ is in the form (3.12)) such that for any subset $X = \{x_1, \ldots, x_n\}$ of $G$ and for any $g, h \in G$ given as products*

$$g = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)}$$

*and*

$$h = x_1^{\delta_1} \cdots x_n^{\delta_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\nu_{ij}(a_1, \ldots, a_i)},$$

*the following equality holds:*

$$gh = x_1^{\gamma_1+\delta_1} \cdots x_n^{\gamma_n+\delta_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{f_{ij}(t_{\tilde{x}}(g), t_{\tilde{x}}(h))}, \tag{3.14}$$

*where $t_{\tilde{x}}(g)$ and $t_{\tilde{x}}(h)$ are the coordinates of elements $g, h \in G$ with respect to the tuple $\tilde{x}$.*

*Proof.* We take the product $gh$ in the form (3.13). The product there that belongs to $G'$ is not in the normal form yet. We apply inductively equation (3.7) from Lemma 3.4 to this product and bring it to the normal form (3.14). ∎

Now we are ready to give a description of a basis of $G$ as a free metabelian group.

**Theorem 3.13.** *A set $Z = \{z_1, \dots, z_n\} \subset G$ forms a basis of $G$ if and only if every element $g \in G$ has a unique representation as the following product:*

$$g = z_1^{\gamma_1} \cdots z_n^{\gamma_n} \Pi_{1 \le j < i \le n} [z_i, z_j]^{\beta_{ij}(\overline{z}_1, \dots, \overline{z}_i)},$$

*where $\beta_{ij}(\overline{z}_1, \dots, \overline{z}_i) \in \mathbb{Z}[\overline{z}_1, \overline{z}_1^{-1}, \dots, \overline{z}_i, \overline{z}_i^{-1}]$.*

*Proof.* Let $X = \{x_1, \dots, x_n\}$ be a basis of $G$ as a free metabelian group. Then a map $x_1 \to z_1, \dots, x_n \to z_n$ extends to a bijection

$$\mu : x_1^{\gamma_1} \cdots x_n^{\gamma_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1, \dots, a_i)} \to z_1^{\gamma_1} \cdots z_n^{\gamma_n} \Pi_{1 \le j < i \le n} [z_i, z_j]^{\beta_{ij}(\overline{z}_1, \dots, \overline{z}_i)}$$

on $G$. This bijection is a homomorphism since by Proposition 3.12 the multiplication in $G$ is given by the same functions $f_i \in \mathcal{F}$ in terms of coordinates $t_{\overline{x}}$ and $t_{\overline{z}}$. ■

The following statement follows from the normal forms of elements in $G'$.

**Proposition 3.14** ([9, Proposition 4.4]). *The group $G'$ has the structure of a free module over $\mathbb{Z}[a_1, a_1^{-1}, a_2, a_2^{-1}]$ with the basis*

$$\{[x_i, x_j]^{a_3^{\delta_3} \cdots a_j^{\delta_j}}\}$$

*for all $1 \le i < j \le n$, $\delta_3, \dots, \delta_j \in \mathbb{Z}^{j-2}$.*

Now we describe briefly the method of Fox derivatives that we use in the sequent. Let $F_n$ be the free group of rank $n$ with the basis $\{z_1, \dots, z_n\}$, let $\pi : F_n \to \overline{g}$. A partial *Fox derivative* associated with $z_i$ is the linear map $D_i : \mathbb{Z}(F_n) \to \mathbb{Z}(F_n)$ satisfying the conditions

$$D_i(z_i) = 1, \quad D_i(z_j) = 0, \quad i \neq j$$

and

$$D_i(uv) = D_i(u) + uD_i(v)$$

for all $u, v \in F_n$. The main identity is $D_1(w)(z_1 - 1) + \cdots + D_n(w)(z_n - 1) = w - 1$. The derivative $D_i$ induces a linear map $d_i : \mathbb{Z}G \to \mathbb{Z}\overline{G}$. We briefly explain the details. One can compute

$$D_i([u^{-1}, v^{-1}]) = (1 - uvu^{-1})D_i(u^{-1}) + (u - uvu^{-1}v^{-1})D_i(v)$$

for all $u, v \in F_n$. It follows that for $\pi' : \mathbb{Z}F_n \to \mathbb{Z}\overline{G}$, for all $w \in F_n''$, $w \in \ker \pi'$. Hence, $D_i$ induces a linear map $d_i : \mathbb{Z}G \to \mathbb{Z}\overline{G}$ (that we will also call Fox derivative).

From the definition, we have

$$d_i(x_i) = 1, \quad d_i(x_j) = 0, \quad i \neq j,$$
$$d_i(uv) = d_i(u) + (u\pi)d_i(v)$$

for all $u, v \in G$. The main identity is $d_1(w)(a_1 - 1) + \cdots + d_n(w)(a_n - 1) = w\pi - 1$, where $w$ is an arbitrary element of $G$.

It can be verified that for $w \in G'$ and $a \in \overline{G}$, $d_i(w^a) = a^{-1} d_i(w)$. Also note that for $w \in G'$ and $u \in G$, we have

$$d_i(wu) = d_i(w) + d_i(u).$$

If

$$a = \alpha_1 a_1 + \cdots + \alpha_k a_k \in \mathbb{Z}\overline{G}$$

with $\alpha_i \in \mathbb{Z}, a_i \in \overline{G}$, we denote by

$$a_{\text{inv}} = \alpha_1 {a_1}^{-1} + \cdots + \alpha_k {a_k}^{-1}.$$

Then

$$d_i(w^a) = a_{\text{inv}} d_i(w).$$

For $u \in G, \alpha \in \mathbb{Z}$,

$$d_i(u^\alpha) = \frac{u^\alpha - 1}{u - 1} d_i(u).$$

### 3.3. $\mathbb{Z}$ is absolutely interpretable in $G$

Let $A$ and $B$ be abelian groups, and let $f : A \times A \to B$ be a bilinear map between them. We associate with such $f$ a two-sorted structure $(A, B; f)$ (here $A$ and $B$ are groups, and $f$ is the predicate for the graph of $f$). The map $f$ is said to be *non-degenerate* if for $a \in A$ $f(a, A) = 0$ if and only if $a = 0$, and similarly, $f(A, a) = 0$ implies $a = 0$. The map $f$ is called *full* if $B$ is generated by $f(A, A)$. An associative commutative unitary ring $R$ is a *ring of scalars* of $f$ if there exist faithful actions of $R$ on $A$ and $B$, which turn $A$ and $B$ into $R$-modules and such that $f$ is $R$-bilinear with respect to these actions. There is a canonical embedding of $R$ into $\operatorname{End}(A)$. $R$ is termed the *largest* ring of scalars of $f$ if for any other ring of scalars $R'$ of $f$, one has $R' \leq R$ when viewed as subrings of $\operatorname{End}(A)$. If $f$ is full and non-degenerate, then the maximal ring of scalars of $f$ exists, and it is unique [16]; we denote it by $R(f)$. Moreover, it was shown in [16] that if $f : A \times A \to B$ is a full non-degenerate bilinear map between finitely generated abelian groups $A$ and $B$, then the largest ring of scalars $R(f)$ of $f$ and its actions on $A$ and $B$ are absolutely interpretable in the structure $(A, B; f)$. Here, the actions of $R(f)$ on $A$ and on $B$ are absolutely interpretable in $(A, B; f)$ if the two-sorted structures $(A, R(f); s_A(x, y, z))$ and $(B, R(f); s_B(x, y, z))$, where $A, B$ are groups, $R(f)$ is a ring, and $s_A(x, y, z)$ and $s_B(x, y, z)$ are predicates that define the scalar multiplications of $R(f)$ on $A$ and $B$, respectively, are absolutely interpretable in $(A, B; f)$. We use these facts to describe an absolute interpretation of $\mathbb{Z}$ in $G$.

Note that every verbal subgroup of $G$ has finite verbal width [20], hence it is absolutely definable in $G$ (see, e.g., [6]). It follows that all the terms of the lower central series of $G$, in particular, the commutant $G'$ and $G_3 = [G, G']$, are absolutely definable in $G$. Hence, the free nilpotent group $G/G_3$ is absolutely interpretable in $G$, as well as the bilinear map

$$f_G : G/G' \times G/G' \to G'/G_3, \tag{3.15}$$

the commutation in $G/G_3$, defined by $(xG', yG') \mapsto [x, y]$. The map $f_G$ is non-degenerate and full, while the abelian groups $G/G'$ and $G'/G_3$ are finitely generated, hence by the result mentioned above, there is a largest ring of scalars $R(f_G)$ of $f_G$, such that $R(f_G)$, and its actions on $G/G'$ and $G'/G_3$ are absolutely definable in the structure $(G/G', G'/G_3; f_G)$, hence in $G$. To get an absolute interpretation of $\mathbb{Z}$ in $G$, it suffices to note that, as was shown in [6] and earlier by other means in [19], $R(f_G) \simeq \mathbb{Z}$. Thus, $\mathbb{Z}$ and its actions on $G/G'$ and $G'/G_3$ are absolutely interpretable in $G$. We denote this interpretation of $\mathbb{Z}$ in $G$ by $\mathbb{Z}^*$. We remark that this interpretation of $\mathbb{Z}$ in $G$ is never injective, because the interpretation of the group $G/G_3$ in $G$ is based on a non-trivial equivalence relation mod $G_3$ in $G$. We will describe in Section 3.5 another, not absolute but regular, interpretation of $\mathbb{Z}$ in $G$ which is injective.

Now, we may use in our formulas expressions of the type $y = x^m \bmod G'$ for $x, y \in G \setminus G'$, as well as $p^m = q \bmod G_3$ for $p, q \in G'$, and $m \in \mathbb{Z}$, viewing them as notation for the corresponding formulas of group theory language which come from the interpretations of $\mathbb{Z}^*$ and its actions on $G/G'$ and $G'/G_3$. More precisely, the interpretation $\mathbb{Z}^*$ is given by a definable in $G$ subset $U^* \subseteq G^k$ together with a definable in $G$ equivalence relation $\sim$ on $U^*$ and formulas $\psi_+(\bar{x}, \bar{y}, \bar{z})$, $\psi_\circ(\bar{x}, \bar{y}, \bar{z})$ with $k$-tuples of variables $\bar{x}, \bar{y}, \bar{z}$, which define binary operations on the factor-set $U^*/\sim$ (denoted by $+$ and $\circ$), and the structure $\langle U^*/\sim; +, \circ\rangle$ is a ring that is isomorphic to $\mathbb{Z}$. For $m \in \mathbb{Z}$, by $m^*$ we denote the image of $m$ in $\mathbb{Z}^*$ under the isomorphism $\mathbb{Z} \to \mathbb{Z}^*$. Furthermore, as we mentioned, the exponentiation by $\mathbb{Z}^*$ on $G/G'$ and on $G'/G_3$ is also 0-interpretable, which means that there are formulas in the group language, say $\mathrm{expnil}_1(u, v, \bar{x})$ and $\mathrm{expnil}_2(u, v, \bar{x})$, such that for $g, h \in G$ and $m \in \mathbb{Z}$, one has $g^m = h (\bmod G')$ if and only if $\mathrm{expnil}_1(g, h, m^*)$ holds in $G$ and also for elements $p, q \in G'$ $p^m = q (\bmod G_3)$ if and only if $\mathrm{expnil}_2(p, q, m^*)$ holds in $G$.

### 3.4. Interpretation of $\mathbb{Z}$-exponentiation in $G$

Now, in the notation above, we construct a formula $\exp(u, v, \bar{x})$ of the group language, where $\bar{x}$ is a $k$-tuple of variables, such that for $g, h \in G$ and $m \in \mathbb{Z}$, the following holds:

$$g = h^m \iff G \models \exp(g, h, m^*).$$

To construct the formula $\exp(u, v, \bar{x})$, we consider two cases, for each of them build the corresponding formula $\exp_i(u, v, \bar{x})$, and then use them to build $\exp(u, v, \bar{x})$.

**Case 1.** Let $g \in G \setminus G'$. In Section 3.3, we described a formula $\mathrm{expnil}_1(u, v, \bar{x})$ of group language such that for $g, h \in G$ and $m \in \mathbb{Z}$, one has

$$g^m = h (\bmod G') \iff G \models \mathrm{expnil}_1(g, h, m^*).$$

Now put

$$\exp_1(u, v, \bar{x}) = ([u, v] = 1 \wedge \mathrm{expnil}_1(u, v, \bar{x})).$$

Then the formula $\exp_1(g, h, m^*)$ holds in $G$ on elements $g, h \in G$ and $m^* \in \mathbb{Z}^*$ if and only if $h = g^m (\mathrm{mod}\, G')$ and $h \in C_G(g)$. Since the centralizer $C_G(g)$ is cyclic, there is only one such $h$, and in this case, $h = g^m$.

**Case 2.** Let $1 \neq g \in G'$. Then as was shown in Section 3.1, for any $w \in G \setminus G'$ and every $m \in \mathbb{Z}$, there exists $c \in G'$ such that the following equality holds:

$$(wg)^m = w^m g^m [c, w]. \tag{3.16}$$

Consider the following condition on elements $g, h \in G', m \in \mathbb{Z}$:

$$C_2(g, h, m) = \forall w (w \in G \setminus G' \to \exists c (c \in G' \wedge ((wg)^m = w^m h[c, w]))). \tag{3.17}$$

Equation (3.16) shows that $h = g^m$ satisfies $C_2(g, h, m)$. We claim that $h = g^m$ is the only element in $G'$ that satisfies $C_2(g, h, m)$ in $G$. Indeed, suppose $C_2(g, h, m)$ holds in $G$ for some $h \in G'$. Then for any $w \in G \setminus G'$, there exists $c_1 \in G'$ such that

$$(wg)^m = w^m h[c_1, w].$$

Then $w^m g^m [c, w] = w^m h[c_1, w]$, so

$$h^{-1} g^m = [c, w][c_1, w]^{-1} = [c, w][c_1^{-1}, w] = [cc_1^{-1}, w].$$

Now by Lemma 3.2, one gets $h^{-1} g^m = 1$, so $h = g^m$, as claimed. To finish the proof, it suffices to show that the condition $C_2(g, h, m)$ can be defined by some formula $\exp_2(u, v, \bar{x})$ of group theory in $G$. Note that in $C_2(g, h, m)$, the elements $w$ and $wg$ are in $G \setminus G'$, hence we can use the formula $\exp_1(u, v, \bar{x})$ to write down the equality $(wg)^m = w^m uh[c, w]$, and then the whole formula $\exp_2(u, v, \bar{x})$.

Finally, the formula

$$\exp(u, v, \bar{x}) = (u \notin G' \to \exp_1(u, v, \bar{x})) \wedge (u \in G' \to \exp_2(u, v, \bar{x}))$$

defines $\mathbb{Z}$-exponentiation on the whole group $G$.

### 3.5. Regular injective interpretation of $\mathbb{Z}$ in $G$

Let $\exp(u, v, \bar{x})$ be the formula from Section 3.4. Then for every $g \neq 1$, formula $\exp(g, v, \bar{x})$ from Section 3.4 defines a bijection $\lambda_g : \mathbb{Z}^* \to \langle g \rangle$ defined by $m \to g^m$, $m \in \mathbb{Z}^*$. This bijection allows one to transfer the operations of addition $+$ and multiplication $\circ$ in the ring $\mathbb{Z}^*$ defined in $G$ by the formulas $\psi_+(\bar{x}, \bar{y}, \bar{z})$ and $\psi_\circ(\bar{x}, \bar{y}, \bar{z})$ (see Section 3.3) from the set $\mathbb{Z}^*$ to the set $\langle g \rangle$. The resulting definable operations $+_g$ and $\circ_g$ give an interpretation $\mathbb{Z}_g^* = \langle \langle g \rangle; +_g, \circ_g \rangle$ of the ring $\mathbb{Z}$ on the cyclic subgroup $\langle g \rangle$ in $G$ with the coordinate map defined by $\mu_g : g^m \to m$. This interpretation is uniform in $g$, that is, it has the same formulas for every $1 \neq g \in G$; therefore, since the condition $g \neq 1$ is definable in $G$, the interpretations $\mathbb{Z}_g^*$ give a regular interpretation of $\mathbb{Z}$ in $G$, which is injective by construction.

### 3.6. Absolute and regular interpretations of $\mathbb{Z}[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}]$ in $G$

We proved above that the ring $\mathbb{Z}$ is absolutely interpretable in $G$ as $\mathbb{Z}^*$ (Section 3.3) and also it is regularly injectively interpretable in $G$ via the interpretations $\mathbb{Z}_g^*$, $1 \neq g \in G$ (Section 3.5). By transitivity of interpretations to show that the ring $R = \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}]$ is absolutely (or regularly injectively) interpretable in $G$, it suffices to show that $R$ is absolutely interpretable in $\mathbb{Z}$ and then compose this interpretation with the interpretations $\mathbb{Z}^*$ and $\mathbb{Z}_g^*$.

To see that $R$ is absolutely interpretable in $\mathbb{Z}$, note that $R$ is computable (since the word problem in $R$ is decidable), hence there exists a computable injective function $\nu : R \to \mathbb{N}$ such that the set $\nu(R)$ is computable in $\mathbb{N}$ and the images under the map $\nu$ of the ring operations of the ring $R$ are computable in $\mathbb{N}$, that is, the following operations on $\nu(R)$ (here $Q_i \in R$, $i = 1, 2, 3$) are computable in $\mathbb{N}$:

$$k_1 \oplus k_2 = k_3 \Longleftrightarrow \bigwedge_{i=1}^{3} (k_i = \nu(Q_i)) \wedge (Q_1 + Q_2 = Q_3),$$

$$k_1 \odot k_2 = k_3 \Longleftrightarrow \bigwedge_{i=1}^{3} (k_i = \nu(Q_i)) \wedge (Q_1 \cdot Q_2 = Q_3).$$

This is a general fact on computable algebraic structures. Nevertheless, it is convenient to give a sketch of a particular such enumeration $\nu : R \to \mathbb{N}$. Every polynomial $P \in \mathbb{Z}[a_1, \ldots, a_n]$ can be uniquely presented as an integer linear combination of pairwise distinct monomials on commuting variables $a_1, \ldots, a_n$:

$$P = \Sigma_{i=1}^{d} \gamma_i a_1^{\alpha_{i1}} \cdots a_n^{\alpha_{in}},$$

where $0 \neq \gamma_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{N}$. Hence, the polynomial $P$ is uniquely presented by a tuple

$$u = (\gamma_1, \alpha_{11}, \ldots, \alpha_{1n}, \ldots, \gamma_d, \alpha_{d1}, \ldots, \alpha_{dn}). \tag{3.18}$$

If $Q \in R$, then $Q$ can be uniquely presented in the form $Q = \frac{P}{\overline{a}^{\overline{\beta}}}$ for some $P \in \mathbb{Z}[a_1, \ldots, a_n]$ and some monomial $\overline{a}^{\overline{\beta}} = a_1^{\beta_1} \cdots a_n^{\beta_n}$, $\beta_i \in \mathbb{N}$, such that $\gcd(P, \overline{a}^{\overline{\beta}}) = 1$. It follows that $Q$ can be uniquely presented by a pair of tuples $(u_Q, v_Q)$, where $u_Q = u, v_Q = \overline{\beta}$. Fix an arbitrary computable bijection

$$\tau : \bigcup_{i \in \mathbb{N}} \mathbb{Z}^i \to \mathbb{N}$$

which enumerates all finite tuples of integers. Then $Q$ is uniquely presented by the pair $(\tau(u_Q), \tau(v_Q)) \in \mathbb{N}^2$, and the set of all such pairs is a computable subset of $\mathbb{N}^2$. For a fixed computable bijection $\tau_2 : \mathbb{N}^2 \to \mathbb{N}$, put

$$\nu(Q) = \tau_2((\tau(u_Q), \tau(v_Q))).$$

By construction, the subset $\nu(R)$ is computable in $\mathbb{N}$, and given a number $k \in \nu(R)$, one can algorithmically find the corresponding Laurent polynomial $Q$ such that $k = \nu(Q)$.

Then it is easy to see that the operations $\oplus$ and $\odot$ on $\nu(R)$ are computable in $\mathbb{N}$. To finish the proof, it suffices to note that all the computable operations or predicates on $\mathbb{N}$ are definable in $\mathbb{N}$, and $\mathbb{N}$ is definable in $\mathbb{Z}$ (every non-negative integer is a sum of squares of four integers). This shows that $R$ is absolutely interpretable in $\mathbb{Z}$. Hence, it is absolutely interpretable in $G$ via $\mathbb{Z}^*$ and we denote this interpretation by $R^*$. For $Q \in R$ by $\nu^*(Q) \in R^*$, we denote the image of $\nu(Q)$ under the isomorphism $\mathbb{Z} \to \mathbb{Z}^*$. Similarly, $R$ is regularly injectively interpretable in $G$ via the interpretations $\mathbb{Z}_g^*$, $g \neq 1$, which we denote by $R_g^*$. The image of $\nu(Q)$ in $R_g^*$ under the isomorphism $\mathbb{Z} \to \mathbb{Z}_g^*$ we denote by $\nu_g^*(Q)$. Finally, we need to mention the coordinate maps of these interpretations. Let $\overline{a} = (a_1, \ldots, a_n)$ be an arbitrary fixed basis of $G$, then the map $\mu_{\overline{a}} : R^* \to R$ that maps $\nu^*(Q) \in R^* \to Q \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}]$ is the coordinate map for the interpretation $R^*$, and the map $\mu_{\overline{a}} : R_g^* \to R$ that maps $\nu_g^*(Q) \in R_g^* \to Q \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}]$ is the coordinate map of $R_g^*$.

### 3.7. Interpretation of $\mathbb{Z}\overline{G}$-module $G'$ in $G$

In this section, for a fixed basis $\overline{a} = (a_1, \ldots, a_n)$ of $G$, we interpret in $G$ the action of the ring $R = \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}]$ on $G'$. More precisely, we describe an interpretation of the $R$-module $G'$ (viewed as a two-sorted structure $G'_R = (G', R; s)$ where $G'$ is an abelian group, $R$ is a ring, and $s$ is the predicate for scalar multiplication or $R$ on $G'$, see Section 3.3) in $G$ with parameters $\overline{a}$. In fact, we interpret $G'_R$ in $G$ as $(G'_R)^* = (G', R^*; s^*)$, where $R^*$ is the interpretation of $R$ in $G$ from Section 3.6, and $s^*$ is the predicate for the action of $R^*$ on $G'$. We need parameters $\overline{a}$ to interpret $s^*$. This interpretation is uniform in $\overline{a}$ (the same formulas work for other bases $\overline{b}$ of $G$). A similar argument gives an injective interpretation of the module $G'_R$ in $G$ as $(G', R_g^*; s^*)$ in $G$ with parameters $\overline{a}$ and $g$.

To interpret the module $G'_R = (G', R; s)$ in $G$ for a given basis $\overline{a}$ of $G$, we interpret the subgroup $G'$ as $G'$ (which is a definable subgroup of $G$) and the ring $R$ by $R^*$, so it suffices to show how to interpret the predicate $s$ in $G$. We need two preliminary results.

For a tuple $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $m \leq n$, denote by $\lambda_{\overline{\alpha}}$ the homomorphism $\lambda_{\overline{\alpha}} : \mathbb{Z}[a_1, \ldots, a_n] \to \mathbb{Z}[a_{m+1}, \ldots, a_n]$ such that $a_i \to \alpha_i, i = 1, \ldots, m$. The kernel $I_{\overline{\alpha}}$ of $\lambda_{\overline{\alpha}}$ is the ideal generated in $\mathbb{Z}[a_1, \ldots, a_n]$ by $\{a_1 - \alpha_1, \ldots, a_m - \alpha_m\}$. Notice that for every polynomial $P = P(a_1, \ldots, a_m) \in \mathbb{Z}[a_1, \ldots, a_n]$, one has $\lambda_{\overline{\alpha}}(P) = P(\alpha_1, \ldots, \alpha_m)$, so

$$P(a_1, \ldots, a_m) = P(\alpha_1, \ldots, \alpha_m) + \Sigma_{i=1}^m (a_i - \alpha_i) f_i,$$

for some $f_i \in \mathbb{Z}[a_1, \ldots, a_n]$.

Let $A$ and $B$ be rings and $\Lambda$ a set of homomorphisms from $A$ into $B$. Recall that $A$ is discriminated into $B$ by a set $\Lambda$ if for any finite subset $A_0 \subseteq A$, there is a homomorphism $\lambda \in \Lambda$ which is injective on $A_0$.

The following result is known, but we need the proof itself.

**Claim 3.15.** *The ring* $\mathbb{Z}[a_1, \ldots, a_n]$ *is discriminated into* $\mathbb{Z}$ *by the set of homomorphisms* $\{\lambda_{\bar{\alpha}} \mid \bar{\alpha} \in \mathbb{Z}^n\}$.

*Proof.* Since $\mathbb{Z}[a_1, \ldots, a_n]$ is an integral domain it suffices to show $\Lambda$ separates $\mathbb{Z}[a_1, \ldots, a_n]$ into $\mathbb{Z}$, that is, for every non-zero polynomial $Q \in \mathbb{Z}[a_1, \ldots, a_n]$, there exists $\lambda \in \Lambda$ such that $\lambda(Q) \neq 0$. Indeed, let $A_0 = \{P_1, \ldots, P_t\}$ with $P_i \neq P_j$ for $1 \leq j < i \leq t$. Put $Q_{ij} = P_i - P_j$ and $Q = \Pi_{1 \leq j < i \leq t} Q_{ij}$. Then $Q \neq 0$. If for some $\lambda \in \Lambda$ $\lambda(Q) \neq 0$, then $\lambda$ is injective on $A_0$.

Now we prove by induction on $n$ that $\Lambda$ separates $\mathbb{Z}[a_1, \ldots, a_n]$ into $Z$. If $P \in \mathbb{Z}[a_1]$, then $\lambda_{\alpha_1}$ for each sufficiently large $\alpha_1$ separates $P$ into $\mathbb{Z}$. If $P \in \mathbb{Z}[a_1, \ldots, a_n]$, then for some $m \in \mathbb{N}$

$$P = Q_m a_n^m + Q_{m-1} a_n^{m-1} + \cdots + Q_1 a_n + Q_0,$$

where $Q_i \in \mathbb{Z}[a_1, \ldots, a_{n-1}]$ and $Q_m \neq 0$. By induction, there is $\bar{\beta} = (\beta_1, \ldots, \beta_{n-1}) \in \mathbb{Z}^{n-1}$ such that the homomorphism $\lambda_{\bar{\beta}}$ discriminates $Q_m$ into $\mathbb{Z}$. Then

$$\lambda_{\bar{\beta}}(Q_m) a_n^m + \lambda_{\bar{\beta}}(Q_{m-1}) a_n^{m-1} + \cdots + \lambda_{\bar{\beta}}(Q_1) a_n + \lambda_{\bar{\beta}}(Q_0)$$

is a non-zero polynomial in $\mathbb{Z}[a_n]$. Now one can separate this polynomial into $\mathbb{Z}$ by sending $a_n$ to a large enough integer $\alpha_n$, as above. This proves the claim. ∎

For $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, denote by $(G')^{I_{\bar{\alpha}}}$ the submodule of the module $G'$ obtained from $G'$ by the action of the ideal $I_{\bar{\alpha}}$. The group $(G')^{I_{\bar{\alpha}}}$ is an abelian subgroup of $G$ generated by the set $\{g^Q \mid g \in G', Q \in I_{\bar{\alpha}}\}$, hence by the set $\{g^{a_i - \alpha_i} \mid g \in G', i = 1, \ldots, m\}$.

**Claim 3.16.** *For any basis* $(a_1, \ldots, a_n)$ *of* $G$ *and any tuple* $(\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$*, the subgroup* $(G')^{I_{\bar{\alpha}}} \leq G'$ *is definable in* $G$ *uniformly in* $(a_1, \ldots, a_n)$ *and* $(\alpha_1, \ldots, \alpha_m)$*. More precisely, let* $\mathbb{Z}^*$ *be* 0*-interpretation of* $\mathbb{Z}$ *in* $G$ *from Section* 3.3*. Then there is a formula* $\varphi(y, y_1, \ldots, y_n, \bar{z}_1, \ldots, \bar{z}_m)$ *of group theory such that for any basis* $(a_1, \ldots, a_n)$ *of* $G$ *and any tuple* $(\bar{k}_1, \ldots, \bar{k}_m) \in (\mathbb{Z}^*)^n$*, the formula* $\varphi(y, x_1, \ldots, x_n, \bar{k}_1, \ldots, \bar{k}_m)$ *defines in* $G$ *the subgroup* $(G')^{I_{\bar{\alpha}}}$*, where* $\alpha_i = \bar{k}_i \in \mathbb{Z}^*, i = 1, \ldots, m$.

Indeed, the abelian group $(G')^{I_{\bar{\alpha}}}$ is generated by the set $\{g^{a_i - \alpha_i} \mid g \in G', i = 1, \ldots, m\}$. It follows that every element $u \in (G')^{I_{\bar{\alpha}}}$ can be presented as a product

$$u = g_1^{a_1 - \alpha_1} \cdots g_m^{a_m - \alpha_m},$$

for some $g_1, \ldots, g_m \in G'$, or, equivalently, in the form

$$u = g_1^{a_1} g_1^{-\alpha_1} \cdots g_m^{a_m} g_m^{-\alpha_m}, \tag{3.19}$$

where $g_i^{a_i}$ is a conjugation of $g_i$ by $a_i$, and $g_i^{-\alpha_i}$ is the standard exponentiation of $g_i$ by the integer $-\alpha_i$, $i = 1, \ldots, m$. It was shown that there exists a formula $\exp_2(u, v, \bar{z})$ such that for any $g, h \in G'$ and $\alpha = \bar{k} \in \mathbb{Z}^*$, the formula $\exp_2(g, h, \bar{k})$ holds in $G$ if and only if $g = h^\alpha$. Using formula $\exp_2(u, v, \overline{z})$ and definability of the commutant $G'$ in $G$ (see Section 3.3), one can write down condition (3.19) by a group theory formula uniformly in $(a_1, \dots, a_n)$ and $(\alpha_1, \dots, \alpha_m)$, as claimed.

**Lemma 3.17** ([9, Lemma 4.24]). *Let $g, h \in G'$ and $P \in \mathbb{Z}[a_1, \dots, a_m]$, $m \leq n$. Then $g^P = h$ if and only if the following condition holds:*

$$\forall \alpha_1, \dots, \alpha_m \in \mathbb{Z}(g^{P(\alpha_1, \dots, \alpha_m)} = h \bmod (G')^{I_{\overline{\alpha}}}). \tag{3.20}$$

**Corollary 3.18.** *Let $g, h \in G'$ and*

$$Q = P(a_1, \dots, a_m)(a_1^{k_1} \cdots a_m^{k_m})^{-1} \in \mathbb{Z}[a_1, a_1^{-1}, \dots, a_n, a_n^{-1}].$$

*Then $g^Q = h$ if and only if*

$$\exists f \in G' \, \forall \overline{\alpha} \in \mathbb{Z}^m (g^{P(\overline{\alpha})} = f \bmod (G')^{I_{\overline{\alpha}}}) \wedge (h^{\alpha_1^{k_1} \cdots \alpha_m^{k_m}} = f \bmod (G')^{I_{\overline{\alpha}}}).$$

**Lemma 3.19.** *The following are true in $G$:*

(1) *There is a formula $E(x, y, \overline{u}, \overline{v})$ of group language such that for any basis $\overline{a}$ of $G$ for any $g, h \in G'$ and any $Q \in \mathbb{Z}[a_1, \dots, a_m]$, one has*

$$G \models E(g, h, \nu^*(Q), \overline{a}) \Longleftrightarrow g^Q = h.$$

(2) *There is a formula $D(x, y, u, \overline{v}, z)$ of group language such that for any basis $\overline{a}$ of $G$, for any $1 \neq f \in G$, for any $g, h \in G'$, and any $Q \in \mathbb{Z}[a_1, \dots, a_m]$, one has*

$$G \models D(g, h, \nu_f^*(Q), \overline{a}, f) \Longleftrightarrow g^Q = h.$$

*Proof.* We prove (1); the argument for (2) is similar. In view of Lemma 3.17 and Corollary 3.18, it suffices to show that condition (3.20) can be written by a formula of group language. By Claim 3.16, the subgroup $(G')^{I_{\overline{\alpha}}} \leq G'$ is definable in $G$ uniformly in $\overline{a} = (a_1, \dots, a_n)$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$, hence the relation $u = v \bmod (G')^{I_{\overline{\alpha}}}$ is definable in $G$ uniformly in $\overline{a}$ and $\overline{\alpha}$. In Section 3.4, we showed that the exponentiation in $G$ by elements from $\mathbb{Z}^*$ is definable in $G$ by the formula $\exp(u, v, \overline{x})$. To finish the proof, it suffices to show that there is a formula $M(\overline{u}, \overline{u}_1, \dots, \overline{u}_m, \overline{w})$ of group language such that for any $P \in \mathbb{Z}[a_1, a_1^{-1}, \dots, a_m, a_m^{-1}]$, $\alpha_1, \dots, \alpha_m \in \mathbb{Z}$, and $\beta \in \mathbb{Z}$, one has

$$G \models M(\nu^*(P), \alpha_1^*, \dots, \alpha_m^*, \beta^*) \Longleftrightarrow P(\alpha_1^*, \dots, \alpha_m^*) = \beta^*. \tag{3.21}$$

Note that given $\nu^*(P), \alpha_1^*, \dots, \alpha_m^*$, one can compute in $\mathbb{Z}^*$ the value $P(\alpha_1^*, \dots, \alpha_m^*)$. Indeed, from $\nu^*(P)$, one can recover by formulas in $\mathbb{Z}^*$ the tuple

$$u^* = (\gamma_1^*, \alpha_{11}^*, \dots \alpha_{1n}^*, \dots, \gamma_d^*, \alpha_{d1}^*, \dots, \alpha_{dn}^*)$$

from (3.18) in Section 3.6 that describes in $\mathbb{Z}^*$ the polynomial $P$ (since $\mathbb{Z}^* \simeq \mathbb{Z}$, one can use the same formulas that recover $P$ from $u$ in $\mathbb{Z}$). Having $u^*$ and $\alpha_1^*, \dots, \alpha_m^*$, one can compute in $\mathbb{Z}^*$ the number $P(\alpha_1^*, \dots, \alpha_m^*)$. Since every computable predicate in $\mathbb{Z}^*$ is definable in $\mathbb{Z}^*$, there exists a formula in the ring language that defines the predicate $P(\alpha_1^*, \dots, \alpha_m^*) = \beta^*$ in $\mathbb{Z}^*$. But $\mathbb{Z}^*$ is absolutely interpretable in $G$, hence there is the required formula $M(\bar{u}, \bar{u}_1, \dots, \bar{u}_m, \bar{w})$ of group language which satisfies (3.21). This proves (1). ■

**Proposition 3.20.** *Let $\bar{a}$ be a basis of $G$. Then the following are true:*

(1) *The module $G'_R = (G', R; s)$ is interpretable in $G$ with parameters $\bar{a}$ as $(G', R^*; s^*)$, where $R^*$ is the interpretation of $R$ in $G$ from Section 3.6, and $s^*$ is the predicate for the action of $R^*$ on $G'$ defined by the formula $E(x, y, \bar{u}, \bar{v})$ from Lemma 3.19. This interpretation is uniform in bases $\bar{a}$.*

(2) *The module $G'_R = (G', R; s)$ is injectively interpretable in $G$ with parameters $\bar{a}$ and $1 \neq g \in G$ as $(G', R_g^*; s^*)$, where $R_g^*$ is the interpretation of $R$ in $G$ from Section 3.6, and $s^*$ is the predicate for the action of $R^*$ on $G'$ defined by the formula $D(x, y, u, \bar{v}, z)$ from Lemma 3.19. This interpretation is uniform in bases $\bar{a}$ and $1 \neq g \in G$.*

*Proof.* As we mentioned above, $G'$ is absolutely definable in $G$, the ring $R$ is interpretable in $G$ via interpretations $R^*$ and $R_g^*$, where $g \in G, g \neq 1$. By Lemma 3.19, the predicates $s^*$ from $(G', R^*; s^*)$ and $(G', R_g^*; s^*)$ are definable in $G$ by formulas $E(x, y, \bar{u}, \bar{v})$ and $D(x, y, u, \bar{v}, z)$ with parameters $\bar{a}$ and $\bar{a}, g$ uniformly in these parameters. This proves the theorem. ■

### 3.8. Regular bi-interpretability of $\mathbb{Z}$ and $G$

**Theorem 3.21.** *The following hold in $G$:*

(1) *The set of all free bases $x = (x_1, \dots, x_n)$ of $G$ is absolutely definable in $G$.*

(2) *There is a formula $F(u, w, \bar{v}, \bar{y}, \bar{z})$, where $\bar{v}$ and $\bar{y}$ are tuples of variables of length $n$, and $\bar{z}$ is a tuple of variables of length $n(n-1)/2$ such that for any $h \in G$, any $g \in G, g \neq 1$, any basis $x = (x_1, \dots, x_n)$ of $G$, any tuple $\gamma \in \mathbb{Z}^n$, and any tuple $\beta = (\beta_{2,1}, \dots, \beta_{n,n-1})$, where $\beta_{i,j} \in \mathbb{Z}[a_1, a_1^{-1}, \dots, a_i, a_i^{-1}]$, $a_i = \bar{x_i}$, the following equivalence holds:*

$$G \models F(h, g, x, \gamma^*, \nu(\beta)^*) \iff h = x^\gamma c_x^\beta,$$

*that is, $t_{\bar{x}}(h) = \gamma \cdot \beta$ (see Section 3.2). Here $m \to m^*$ is the isomorphism $\mathbb{Z} \to \mathbb{Z}_g^*$.*

*Proof.* It follows from Theorem 3.13 and interpretation of $\mathbb{Z}$-exponentiation on $G$ (Section 3.4) and interpretation of $\mathbb{Z}(\bar{G})$-exponentiation on $G'$ (Section 3.7). ■

**Proposition 3.22.** *The group $G$ is absolutely and injectively interpretable in $\mathbb{Z}$.*

*Proof.* Fix a basis $\overline{x} = (x_1, \ldots, x_n)$ in $G$. Then any element $g \in G$ can be written in the canonical form (relative to $\overline{x}$)

$$g = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \Pi_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)}, \tag{3.22}$$

where $a_i = x_i G' \in G/G'$, $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}]$. Hence, $g$ can be uniquely represented by a tuple of integers

$$t(g) = (\gamma_1, \ldots, \gamma_n, \nu(\beta_{21}), \ldots, \nu(\beta_{n,n-1})), \tag{3.23}$$

which are the coordinates of $g$ in the normal form relative to the basis $\overline{x}$ (here $\nu(\beta_{ij})$ is the code of the polynomial $\beta_{ij}$ under the computable bijection $\nu$ from Section 3.6).

The set $S_G = \{t(g) \mid g \in G\}$ is clearly definable in $\mathbb{Z}^{n+n(n-1)/2}$. For a tuple $\overline{s} \in S_G$, denote by $\mu_{\overline{x}}(\overline{s})$ the unique element $g \in G$ with coordinates $t(g)$ with respect to $\overline{x}$. This gives a bijection $\mu : S_G \to G$.

Observe that the multiplication and inversion in $G$ (as operations on the canonical forms of elements) are computable, since the word problem in $G$ is decidable. It follows that the multiplication and inversion in $G$ when elements $g$ of $G$ are given through their codes $\nu_G(g)$ are computable in $\mathbb{N}$; therefore, their graphs are computably enumerable, and hence definable in arithmetic $\mathbb{N}$, as well as in $\mathbb{Z}$. This gives an interpretation $\Gamma$ of $G$ in $\mathbb{Z}$, $G \simeq \Gamma(\mathbb{Z})$, with the coordinate map $\mu_{\overline{x}}$. ∎

**Theorem 3.23.** *Every free metabelian group $G$ of finite rank $\geq 2$ is regularly strongly and injectively bi-interpretable with $\mathbb{Z}$.*

*Proof.* By Proposition 3.22, $G$ is absolutely and injectively interpretable in $\mathbb{Z}$ by some code $\Gamma$, that is, $G \simeq \Gamma(\mathbb{Z})$, and with the coordinate map $\mu_x : \Gamma(\mathbb{Z}) \to G$, which depends on a choice of a basis $x$ of $G$. On the other hand, $\mathbb{Z}$ is injectively and regularly interpretable as $\mathbb{Z}_g^* = \langle\langle g\rangle; +_g, \circ_g\rangle$, with a parameter $g \in G, g \neq 1$ and the coordinate map $\mu_g : g^m \to m$ (see Section 3.5). This gives a regular injective interpretation $\mathbb{Z} \simeq \Delta(G, g)$. It follows that $G \simeq \Gamma \circ \Delta(G, g)$ and the coordinate map $\Gamma \circ \Delta(G, g) \to G$ is precisely the map defined by the formula $F(u, g, x, \overline{y}, \overline{z})$, where $g \in G, g \neq 1$, $x$ is a tuple of parameters (a basis of $G$) that occur in the formula $F(u, w, \overline{v}, \overline{y}, \overline{z})$ from Theorem 3.21 when $g$ is substituted for $w$ and $x$ is substituted for variables $\overline{v}$. However, the parameters $x$ are not part of the interpretation $\mathbb{Z} \simeq \Delta(G, g)$; to fix this, we add $x$ to the set of parameters in $\Delta$, so now $\mathbb{Z} \simeq \Delta(G, g, x)$. Note that the parameters $x$ do not occur in any formulas in $\Delta$ (this is allowed). By Theorem 3.21, the set of all bases of $G$ is absolutely definable in $G$, so the interpretation $\mathbb{Z} \simeq \Delta(G, g, x)$ is regular and injective, and the formula $F(u, g, x, \overline{y}, \overline{z})$ defines the coordinate map $\Gamma \circ \Delta(G, g) \to G$ for parameters $g, x$. In the other direction, we have $\mathbb{Z} \simeq \Gamma \circ \Delta(\mathbb{Z})$, and the corresponding coordinate map $\Gamma \circ \Delta(\mathbb{Z}) \to \mathbb{Z}$ is defined by $(g^*)^m \to m$, where $g^*$ is the image of $g$ under the isomorphism $\mu_x^{-1} : G \to \Gamma(Z)$. The parameter $g^*$ is part of the interpretation $\mathbb{Z}_g \simeq \Delta(\Gamma(\mathbb{Z}), g^*)$. Since $\Gamma(\mathbb{Z})$ is computable, the function $(g^*)^m \to m$ is computable in $\mathbb{Z}$, hence definable in $\mathbb{Z}$ with parameter $g^*$. This proves the theorem. ∎

**Corollary 3.24.** *Every free metabelian group of finite rank $\geq 2$ has uniform elimination of imaginaries with parameters.*

## 4. Groups elementarily equivalent to a free metabelian group

In this section, we will describe groups elementarily equivalent to a free metabelian group $G$ of rank $n \geq 2$ with basis $x_1, \ldots, x_n$.

Since $G$ is regularly bi-interpretable with $\mathbb{Z}$, we can use Theorem 2.7 with $G = \mathbb{A}$ and $\mathbb{Z} = \mathbb{B}$. Then in the notation above, $G = \Gamma(\mathbb{Z})$ and $\mathbb{Z} = \Delta(G, \bar{g})$. If $H \equiv G$, then the same formulas give the interpretation $H = \Gamma(\tilde{\mathbb{Z}})$, where $\tilde{\mathbb{Z}} \equiv \mathbb{Z}$. We call $\tilde{\mathbb{N}}$, a structure elementarily equivalent to $\mathbb{N}$, a model of arithmetic. Notice that $\mathbb{N}$ and $\mathbb{Z}$ are absolutely bi-interpretable. A ring $\tilde{\mathbb{Z}}$ (resp. $\tilde{\mathbb{N}}$) is called a non-standard model if it is not isomorphic to $\mathbb{Z}$ (resp. $\mathbb{N}$) (see [8, 12]).

First, we notice that $H$ is a so-called exponential group with exponents in $\tilde{\mathbb{Z}}$. Let us recall the four axioms of exponential groups from [17]. Let $A$ be an arbitrary associative ring with identity and $\Gamma$ a group. Fix an action of the ring $A$ on $\Gamma$, that is, a map $\Gamma \times A \to \Gamma$. The result of the action of $\alpha \in A$ on $g \in \Gamma$ is written as $g^\alpha$. Consider the following axioms:

(1) $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$.

(2) $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$.

(3) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$.

(4) $[g, h] = 1 \Rightarrow (gh)^\alpha = g^\alpha h^\alpha$.

**Definition 4.1.** Groups with $A$-actions satisfying axioms (1)–(4) are called *$A$-exponential groups*.

These axioms can be written by first-order formulas in $G$ and $H$. This implies the following lemma.

**Lemma 4.2.** *The group $H$ is $\tilde{Z}$-exponential group.*

Our main goal now is to describe the structure of $H$. We know that $G$ can be represented as a pair $\mathbb{Z}\bar{G}$ and a module $G'_{\mathbb{Z}\bar{G}}$ with the action of $\mathbb{Z}\bar{G}$ on $G'_{\mathbb{Z}\bar{G}}$ interpretable in $G$ by Section 3.7.

We have $G \to_\Gamma \mathbb{Z} \to_\Delta G$, where the interpretation $\Gamma(\mathbb{Z})$ is via normal forms; therefore, $H \to_\Gamma \tilde{\mathbb{Z}} \to_\Delta H$. The element $g = x_1^{\gamma_1} \cdots x_n^{\gamma_n} u \in G$, where

$$u = \Pi_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)},$$

where $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}] \leq \mathbb{Z}\bar{G}$ is interpreted as a tuple of elements in $\mathbb{Z}$, $g \to (\gamma_i, \ldots, \gamma_n, \bar{\beta}_{11}, \ldots, \bar{\beta}_{n-1,n})$, where $\beta_{ij}$ are tuples.

For $H$, instead of $\mathbb{Z}\overline{G}$ we will have a non-standard Laurent polynomial ring $\widetilde{\mathbb{Z}}\overline{G}_{NS}$ described in [12]. This is a ring elementarily equivalent to $\mathbb{Z}\overline{G}$. More precisely, regular bi-interpretation of $G$ with $\mathbb{Z}$ induces the regular bi-interpretation of $\mathbb{Z}\overline{G}$ with $\mathbb{Z}$, $\mathbb{Z}\overline{G} = \Gamma_1(\mathbb{Z})$. Then $\widetilde{\mathbb{Z}}\overline{G}_{NS} = \Gamma_1(\widetilde{\mathbb{Z}})$. The same formula as in the standard case says that for $h \in H$,

$$h = x_1^{\widetilde{\gamma}_1} \cdots x_n^{\widetilde{\gamma}_n} u, \quad u = \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1,\dots,a_i)}, \tag{4.1}$$

where $\widetilde{\gamma}_i \in \widetilde{\mathbb{Z}}$, $\beta_{ij}(a_1, \dots, a_i) \in \widetilde{\mathbb{Z}}\overline{G}_{NS}$. It is interpreted as a non-standard tuple of tuples of elements in $\widetilde{\mathbb{Z}}$:

$$g \to (\widetilde{\gamma}_i, \dots, \widetilde{\gamma}_n, \overline{\beta}_{11}, \dots, \overline{\beta}_{n-1,n}).$$

There is a formula connecting $P[a_1, \dots, a_n] \in \widetilde{\mathbb{Z}}\overline{G}_{NS}$ and its evaluation $P[\alpha_1, \dots, \alpha_n]$, where $\alpha_1, \dots, \alpha_n \in \widetilde{\mathbb{Z}}$. Lemma 3.17 gives the interpretation of the action of the standard Laurent polynomial ring $\mathbb{Z}\overline{G}$ on $G'$ and, therefore, the interpretation of the action of the non-standard Laurent polynomial ring $\widetilde{\mathbb{Z}}\overline{G}_{NS}$ on $H'$, where $H'$ is the $\widetilde{\mathbb{Z}}\overline{G}_{NS}$ module generated by $\{[x_i, x_j]\}$.

Lemma 3.5 implies the following.

**Lemma 4.3.** *For any $x, z \in H$ and $\delta, \gamma \in \widetilde{\mathbb{Z}}$,*

$$[z^\gamma, x^\delta] = [z, x]^{\left(\frac{\overline{z}^\gamma - 1}{\overline{z} - 1}\right)\left(\frac{\overline{x}^\delta - 1}{\overline{x} - 1}\right)},$$

*where $\frac{(\overline{x}^\delta - 1)}{(\overline{x} - 1)}, \frac{(\overline{z}^\gamma - 1)}{(\overline{z} - 1)} \in \widetilde{\mathbb{Z}}\overline{G}_{NS}$.*

Denote by $a_i$ the image of $x_i \in H$ in $\widetilde{\mathbb{Z}}\overline{G}_{NS}$.

**Theorem 4.4.** *If $H$ is a group elementarily equivalent to $G$, then $H$ has the following structure:*

(1) *Elements $h \in H$ have the normal form*

$$h = x_1^{\widetilde{\gamma}_1} \cdots x_n^{\widetilde{\gamma}_n} u, \quad u = \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1,\dots,a_i)},$$

*where $\widetilde{\gamma}_i \in \widetilde{\mathbb{Z}}$, $\beta_{ij}(a_1, \dots, a_i) \in \widetilde{\mathbb{Z}}\overline{G}_{NS}$.*

(2) *$H'$ is a module over $\widetilde{\mathbb{Z}}\overline{G}_{NS}$ with generators $\{[x_i, x_j]\}$.*

(3) *Multiplication in $H$ is defined as follows. If $g, h \in H$ are given by their normal forms*

$$g = x_1^{\widetilde{\gamma}_1} \cdots x_n^{\widetilde{\gamma}_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1,\dots,a_i)}$$

*and*

$$h = x_1^{\widetilde{\delta}_1} \cdots x_n^{\widetilde{\delta}_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{\nu_{ij}(a_1,\dots,a_i)},$$

*then*

$$gh = x_1^{\widetilde{\gamma}_1 + \widetilde{\delta}_1} \cdots x_n^{\widetilde{\gamma}_n + \widetilde{\delta}_n} \Pi_{1 \le j < i \le n} [x_i, x_j]^{f_{ij}(t(g), t(h))}, \tag{4.2}$$

*where $f_{ij} \in \mathcal{F}$ are non-standard functions corresponding to functions from Proposition 3.12, and $t(g)$ and $t(h)$ are coordinates in $\widetilde{\mathbb{Z}}\overline{G}_{NS}$ of elements $g, h \in H$ in the base $x \cdot y$.*

*Proof.* The functions $\varepsilon_\ell(z)$ and all functions in $\mathcal{F}$ are defined in the ring $\widetilde{\mathbb{Z}}\overline{G}_{NS}$ by the same formulas as in $\mathbb{Z}\overline{G}$. ■

**Lemma 4.5.** *If $x, y \in H - H'$ and $b$ is the image of $y$ in $\widetilde{Z}\overline{G}_{NS}$, then we have the formula for the $\delta$-commutator*

$$y^{-\delta}x^{-\delta}(xy)^{\delta} = [x, y]^{-f(a,b)},$$

*where $f(a, b)$ is a non-standard polynomial such that*

$$\begin{aligned}(a-1)f(a,b) &= \frac{(a^\delta b^\delta - 1)}{(ab-1)} + \frac{(1-b^\delta)}{(b-1)} \\ &= b^{\delta-1}(a^{\delta-1}-1) + b^{\delta-2}(a^{\delta-2}-1) + \cdots + b(a-1).\end{aligned}$$

*Proof.* We can write that $y^{-\delta}x^{-\delta}(xy)^{\delta}$ is in the module generated by the commutator $[x, y]$, and we wish to see what is the non-standard polynomial $f(a, b)$.

We have $[x, y^{-\delta}x^{-\delta}(xy)^{\delta}] = [x, y]^{f(a,b)(a-1)}$. At the same time,

$$\begin{aligned}[x, y^{-\delta}x^{-\delta}(xy)^{\delta}] &= [x, (xy)^{\delta}][x, y^{-\delta}x^{-\delta}]^{(ab)^{\delta}} \\ &= [x, xy]^{\frac{((ab)^{\delta}-1)}{(ab-1)}}[x, y^{-\delta}]^{a^{-\delta}(ab)^{\delta}} \\ &= [x, y]^{\frac{(a^\delta b^\delta - 1)}{(ab-1)} + \frac{(1-b^\delta)}{(b-1)}}.\end{aligned}$$

Polynomial

$$\frac{(a^\delta b^\delta - 1)}{(ab-1)} + \frac{(1-b^\delta)}{(b-1)} = b^{\delta-1}(a^{\delta-1}-1) + b^{\delta-2}(a^{\delta-2}-1) + \cdots + b(a-1)$$

is divisible by $(a-1)$ in the standard ring of Laurent polynomials; therefore, it is divisible by $(a-1)$ in the non-standard ring. Since the rings are integral domains, $f(a, b)$ is the result of this division. ■

## 5. $A$-metabelian groups

The question about varieties of exponential groups was discussed in [1]. Let $\Gamma$ be an arbitrary exponential group with exponents in $A$. We set

$$(\Gamma, \Gamma)_A = \langle (g, h)_\alpha = h^{-\alpha}g^{-\alpha}(gh)^{\alpha}, g, h \in \Gamma, \alpha \in A\rangle_A.$$

The $A$-subgroup $(\Gamma, \Gamma)_A$ is called the $A$-commutant of the group $\Gamma$. By [1], a free abelian $A$-group with base $X$ is a free $A$-module and is $A$-isomorphic to the factor-group of the free $A$-group with base $X$ by its $A$-commutant. The $A$-commutant is called the first $A$-commutant and denoted by $\Gamma^{(1,A)}$. The $A$-commutant of $\Gamma^{(1,A)}$ is the second $A$-commutant $\Gamma^{(2,A)}$. Then $\Gamma$ is called in [1] $n$-step $A$-solvable group if $\Gamma^{(n,A)} = 1$. Clearly, $n$-step $A$-solvable group is $n$-solvable $A$-group. If $\Gamma^{(2,A)} = 1$, we call $\Gamma$ an $A$-metabelian group. Notice that $H$ that is elementarily equivalent to a free metabelian group is $\widetilde{\mathbb{Z}}$-metabelian because $\alpha$-commutators belong to $H'$ and commute in $H$.

A discretely ordered ring is an ordered ring in which there is no element between 0 and 1. Let $A$ be a discretely ordered ring and $K$ be a multiplicative $A$-module with generators $a_1, \ldots, a_n$. Consider a group algebra $A(K)$. Let $R$ be the $A$ algebra generated by $A(K)$ and for all positive $\delta \in A$, by series $\frac{(a^\delta - 1)}{(a-1)} = \Sigma_{0 \le \alpha < \delta} a^\alpha$, and $\Sigma_{0 \le \alpha < \delta} b^\alpha \frac{a^\alpha - 1}{a-1}$, where $a, b \in K$.

We define an $A$-metabelian exponential group $M$ with generators $x_1, \ldots, x_n$ by the following axioms:

(1) $M$ is an $A$-metabelian group.

(2) The $A$-commutant $M'$ is an $R$-module.

(3) For any $z, x \in M$ and $\delta \in A$, $[z, x^\delta] = [z, x]^{\frac{(a^\delta - 1)}{(a-1)}}$.

(4) For any $x, y \in M$ and $\delta \in A$, $y^{-\delta} x^{-\delta} (xy)^\delta = [x, y]^{f(a,b)}$, where

$$f(a, b) = [(a^\delta b^\delta - 1)/(ab - 1) + (1 - b^\delta)/(b - 1)]/(1 - a) \in R.$$

Let now $M$ be a free group with generators $x_1, \ldots, x_n$ in the category of $A$-metabelian exponential groups.

**Lemma 5.1.** *The group $M'$ is an $R$-module generated by elements $[x_i, x_j]$. If $u$ belongs to $M'$, then it can be uniquely written as*

$$u = \Pi_{1 \le j < i \le n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)},$$

*where $\beta_{ij}(a_1, \ldots, a_i) \in R$.*

*Proof.* Consider a $\delta$-commutator $y^{-\delta} x^{-\delta} (xy)^\delta$. We have, using identities (1) and (2),

$$\begin{aligned} [x, y^{-\delta} x^{-\delta} (xy)^\delta] &= [x, (xy)^\delta][x, y^{-\delta} x^{-\delta}]^{(ab)^\delta} \\ &= [x, xy]^{\frac{((ab)^\delta - 1)}{(ab-1)}} [x, y^{-\delta}]^{a^{-\delta} (ab)^\delta} \\ &= [x, y]^{\frac{(a^\delta b^\delta - 1)}{(ab-1)} + \frac{(1 - b^\delta)}{(b-1)}}. \end{aligned}$$

Every commutator can be represented as $[x_i^\beta, (x_{i_1} \cdots x_{i_t})^\alpha, x_{j_1}^{\alpha_1}, \ldots, x_{j_k}^{\alpha_k}]$. We can assume that $i \ge j_1 \ge j_2 \cdots \ge j_k$, otherwise use the Jacobi identity. If $i$ is greater than or equal to all $i_1, \ldots, i_t$, then the representation from Lemma 4.3 gives elements $[x_i, x_{i_1} \cdots x_{i_t}]^{f(a_{i_1}, \ldots, a_{i_t})}$. Bringing the commutator $[x_i, x_{i_1} \cdots x_{i_t}]$ to normal form and acting by $f(a_{i_1}, \ldots, a_{i_t})$ gives elements in normal form.

Suppose now some of $i_1, \dots, i_t$ is greater than $i$. Consider a general example

$$\begin{aligned}[x_1, (x_2x_3)^\delta] &= [x_1, (x_2x_3)]^{\frac{((a_2a_3)^\delta - 1)}{(a_2a_3-1)}} \\ &= [x_1, x_2x_3]^\delta \Pi_{0\le\alpha<\delta}[x_1, x_2x_3, (x_2x_3)^\alpha] \\ &= [x_1, x_2x_3]^\delta \Pi_{0\le\alpha<\delta}[x_1, x_2x_3, x_3^\alpha]^{x_2^\alpha} \Pi_{0\le\alpha<\delta}[x_1, x_2x_3, x_2^\alpha] \\ &= [x_1, x_2x_3]^\delta [x_1, x_2x_3, x_3]^{\Sigma_{0\le\alpha<\delta}\frac{a_2^\alpha(a_3^\alpha-1)}{(a_3-1)}} \\ &\quad \cdot [x_1, x_2x_3, x_2]^{\Sigma_{0\le\alpha<\delta}\frac{(a_2^\alpha-1)}{(a_2-1)}}.\end{aligned}$$

Using the Jacobi identity, we rewrite $[x_1, x_2x_3, x_3] = [x_3, x_2x_3]^{a_1-1}[x_3, x_1]^{-a_2a_3}$. This finally gives a representation of the commutator $[x_1, (x_2x_3)^\delta]$ in the normal form. A general case can be treated similarly.

To prove uniqueness of normal forms, we need the analogue of Fox derivatives from $A(M)$ to $R$. We define a partial Fox derivative as a linear mapping $d_i : A(M) \to R$ satisfying the properties of $d_i$ from Section 3.1 and

$$d_i(g^\delta) = \frac{g^\delta - 1}{g - 1} d_i(g) = d_i(g)\Sigma_{0\le\alpha<\delta} g^\alpha. \tag{5.1}$$

A consequence of these is an equality:

$$Dg^{-\delta} = -g^{-\delta} Dg^\delta.$$

One can also compute for $f(a, b) \in R$

$$d_i([x, y]^{f(a,b)}) = f(a, b)_{\text{inv}} d_i([x, y]). \tag{5.2}$$

The uniqueness of the normal form can be proved by using Fox derivatives and the homomorphisms $\varepsilon_I$, where $I \subseteq \{1, \dots, n\}$ and $x_i\varepsilon_j = x_i$ if $i \in I$ and $x_i\varepsilon_j = 1$ if $i \notin I$ (as it is done in [13] for normal forms in a free metabelian group). For example, we have $u\varepsilon_{\{1,2\}} = [x_2, x_1]^{\beta_{12}(a_1,a_2)}$, hence $\beta_{12}(a_2, a_1)$ is defined uniquely. Multiply $u$ by $[x_2, x_1]^{-\beta_{21}(a_1,a_2)}$ to get $u'$.

Then $u'\varepsilon_{\{1,2,3\}} = [x_3, x_1]^{\beta_{31}(a_1,a_2,a_3)}[x_3, x_2]^{\beta_{32}(a_1,a_2,a_3)}$.

Then $d_1(u'\varepsilon_{\{1,2,3\}}) = \beta_{31(\text{inv})}(a_1, a_2, a_3)(a_3 - 1)a_1^{-1}a_3^{-1}$,

$$d_2(u'\varepsilon_{\{1,2,3\}}) = \beta_{32(\text{inv})}(a_1, a_2, a_3)(a_3 - 1)a_2^{-1}a_3^{-1}.$$

This allows us to compute uniquely $\beta_{31}(a_1, a_2, a_3)$ and $\beta_{32}(a_1, a_2, a_3)$, respectively, and so on. ■

**Theorem 5.2.** *If $H \equiv G$, where $G$ is a free metabelian group, then $H$ contains a free $\widetilde{\mathbb{Z}}$-metabelian exponential group as a subgroup.*

*Proof.* We know that $H$ is $\widetilde{\mathbb{Z}}$-exponential group for some $\widetilde{\mathbb{Z}} \equiv \mathbb{Z}$. And $H'$ is a $\widetilde{\mathbb{Z}}\overline{G}_{NS}$-module. Let $M$ be a free $\widetilde{\mathbb{Z}}$-metabelian exponential group defined above. Then normal forms of elements in $M$ are exactly normal forms (4.1) of elements in $H$; therefore, $M$ is a subgroup of $H$. ■

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