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On the prehistory of growth of groups

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Abstract. The subject of growth of groups has been active in the former Soviet Union since the early 1950s and in the West since 1968, when articles of Švarc and Milnor have been published independently. The main purpose of this note is to quote a few articles showing that, before 1968 and at least retrospectively, growth of groups has already played some role in various subjects.

The fundamental notions in the theory of the growth of groups can be defined as follows. Let Γ be a finitely generated group and S a finite generating set of Γ. The *word length* function $\ell_S : \Gamma \to \mathbf{N}$ is defined by $\ell_S(\gamma) = \min\{k \geq 0 \mid \gamma \in (S \cup S^{-1})^k\}$. Let $\sigma(\Gamma, S; k)$ denote the cardinal of the sphere $\{\gamma \in \Gamma \mid \ell_S(\gamma) = k\}$ and $\beta(\Gamma, S; k)$ denote the cardinal of the ball $B(\Gamma, S; k) = \{\gamma \in \Gamma \mid \ell_S(\gamma) \leq k\}$. It is straightforward to check that $\sigma(\Gamma, S; k) \leq |S \cup S^{-1}|(|S \cup S^{-1}| - 1)^{k-1}$ for all $k \geq 1$; in particular, $\beta(\Gamma, S; k) \leq a^k$ for a constant a and for all $k \geq 0$. If T is another finite generating set of Γ, there exists a constant $c > 0$ such that $B(\Gamma, S; k) \subset B(\Gamma, T, ck)$ and $B(\Gamma, T; k) \subset B(\Gamma, S, ck)$, hence $\beta(\Gamma, S; k) \leq \beta(\Gamma, T, ck)$ and $\beta(\Gamma, T; k) \leq \beta(\Gamma, S, ck)$ for all $k \geq 0$; it follows that the next definitions do not depend on the choice of S. The group Γ is said to be of *exponential growth* if there exists a constant $b > 1$ such that $\beta(\Gamma, S; k) \geq b^k$ for all $k \geq 0$, of *subexponential growth* otherwise, of *polynomial growth* if there exist constants $c > 0$ and $d \geq 0$ such that $\beta(\Gamma, S; k) \leq ck^d$ for all $k \geq 0$, and of *intermediate growth* if it is of subexponential growth and not of polynomial growth.

Here are some of the most basic questions: What are the groups in each class, exponential, polynomial, and intermediate growth? For a given group Γ, what are the more accurate properties of the functions $\beta(\Gamma, S; k)$? What are the functions which are of the form $\beta(\cdots; k)$? What are the implications of the theory with other subjects?

In this paper, we first describe shortly the very beginning of the subject, and we mention a few of the most spectacular later results. However, our main focus is on the results published before 1968, and for some even before 1955, which can be seen retrospectively as showing how notions of group growth have been used early for various purposes.

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1. The beginning, Švarc, Milnor, and a few others (1955–1968)

The notion of growth for finitely generated groups appears in articles published by Efremovich and Švarc in the early 1950s and independently by Milnor in 1968 [Efre–53, Svar–55, Miln–68a, Miln–68b]. (Švarc left the Soviet Union in 1989, and now his name is rather written Schwarz.) Very soon after his first paper, Milnor in [Miln–68c] called attention to the fact that [Svar–55] "contains many ideas utilized in [3]" (where [3] is our [Miln–68b]).

Before 1968, the paper [Svar–55], written by Švarc during his undergraduate years, was essentially ignored outside the former Soviet Union. Concerning the period between 1955 and 1968, we quote two extracts from [Avez–76, Definition I.6] and [Grom–93, Item 0.5].

After having defined exponential growth and non-exponential growth, Avez writes: "This notion is due to V. Arnold (oral communication, 1965), Švarc [Svar–55], and Milnor [Miln–68b]. Finite extensions of finitely generated nilpotent groups are the only known examples of groups of non-exponential growth [Wolf–68]."

In what he calls some random historical remarks, Gromov writes: "The ideas of the growth of balls, Folner sets and sets of conjugacy classes in groups (especially in fundamental groups of manifolds of negative curvature, see [Marg–67, Marg–69]) were quite popular in the sixties among ergodic theorists in Moskow and Leningrad. (Much of these ideas I learned at the time from A. Vershik, D. Kazhdan and G. Margulis.) Then the geometers took a part in the story and related the growth to curvature. The first results here for non-negative curvature are due to A. Švarc [Svar–55]. Similar results were obtained independently by J. Milnor [Miln–68b]."

For a description of the results of Efremovich and Švarc, we quote the following lines from [Svar–08]. "My first serious work was inspired by Efremovich's remark that the 'volume invariant' of universal covering of a compact manifold is a topological invariant of the manifold. (If two compact manifolds are homeomorphic, then the natural homeomorphism between universal coverings is uniformly continuous. Efremovich proved that under certain conditions the growth of the volume of a ball with radius tending to infinity is an invariant of uniformly continuous homeomorphisms.) I proved that the volume invariant of universal covering can be expressed in terms of the fundamental group of the original manifold; in modern language it is determined by the growth of the fundamental group. I also gave estimates for volume invariants of manifolds with non-positive and with negative curvature. Thirteen years later J. Milnor published a paper containing the same results with the only difference that Milnor was able to use in his proofs some theorems derived after the appearance of my paper. At the moment of writing his first paper in this direction Milnor did not know about my work, but his second paper contained corresponding references. The notion of growth of a group (volume invariant of a group in my terminology) was studied later in numerous papers (one should mention, in particular, the results by Gromov and Grigorchuk). A new interesting field — geometric group theory — was born from these papers."

There is a short description of the work and life of Efremovich in [Efremovich].

2. Some later developments, Gromov (1981), Grigorchuk (1983), and growth beyond finitely generated groups

The importance of the subject of group growth was largely recognized with results of Gromov and Grigorchuk.

A finitely generated group Γ which has a nilpotent subgroup N of finite index has polynomial growth [Wolf–68]. More precisely, let

$$C^1(N) = N \supset C^2(N) = [N,N] \supset \cdots \supset C^{j+1}(N) = [N, C^j(N)] \supset \cdots$$

be the descending central series of N, let r be its class, namely the smallest integer such that $C^{r+1}(N) = \{e\}$, and let $h(N) = \sum_{j=1}^{r} \mathrm{rank}(C^j(N)/C^{j+1}(N))$ be its Hirsch length. Then there exist two constants C_1, C_2 such that $C_1 k^{h(N)} \le \beta(\Gamma, S; k) \le C_2 k^{h(N)}$ for all $k \ge 0$; this is due independently to Guivarc'h [Guiv–71, Guiv–73], Brian Hartley (unpublished), and Bass [Bass–72]. Moreover, the limit $c = \lim_{k\to\infty} \frac{\beta(\Gamma,S;k)}{k^{h(N)}}$ exists [Pans–83]. It is a long-standing conjecture that $\beta(\Gamma, S; k) = ck^{h(N)} + o(k^{h(N)-1})$; some progress in this direction can be found in [Stol–98, BrLD–13].

Gromov showed the converse of Wolf's theorem: any finitely generated group of polynomial growth has a nilpotent subgroup of finite index [Grom–81]. The paper is sometimes cited as the starting point of geometric group theory. There are now several proofs and various refinements of this fundamental theorem; see [LMTT–23] and references there.

It took some time to discover groups of intermediate growth. For example, a finitely generated solvable group has either polynomial or exponential growth (established in [Miln–68c, Wolf–68], see also [Rose–74]), and the same holds for a finitely generated linear group (as a consequence of [Tits–72]). It was Grigorchuk who showed the existence of finitely generated groups of intermediate growth [Grig–83, Grig–84]. These papers are at the origin of new domains for group theory: branch groups and self-similar groups.

The volume growth of balls in the universal covering of a compact Riemannian manifold M is equivalent to the volume growth of the fundamental group $\pi_1(M)$, so that growth of groups is of interest in differential geometry. An early result in this direction appears already in [Miln–68b]: if a compact manifold M can be given a Riemannian structure with negative sectional curvature, then the fundamental group $\pi_1(M)$ is of exponential growth. More recently, motivated by results of Jørgensen and Thurston establishing that the set of volumes of hyperbolic 3-manifolds, a subset of $\mathbf{R}_+$, is well ordered of ordinal type ω^ω, Fujiwara and Sela showed the following in [FuSe–23]: let Γ be a finitely generated group and let $\mathrm{Exp}(\Gamma)$ be the set of positive real numbers which are exponential growth rates of Γ, that is, which are of the form $e(\Gamma, S) \doteqdot \limsup_{k\to\infty} \beta(\Gamma, S; k)^{1/k}$ for some finite generating set S of Γ (in fact, $e(\Gamma, S) = \lim_{k\to\infty} \beta(\Gamma, S; k)^{1/k} = \inf_{k\to\infty} \beta(\Gamma, S; k)^{1/k}$ because $\beta(\Gamma, S; k+\ell) \le \beta(\Gamma, S; k)\beta(\Gamma, S; \ell)$ for all $k, \ell \ge 0$ and also $e(\Gamma, S) = \lim_{k\to\infty} \sigma(\Gamma, S; k)^{1/k}$ when Γ is infinite); if Γ is a non-elementary hyperbolic group, then $\mathrm{Exp}(\Gamma)$ is a well-ordered set, hence in particular it has a minimum; moreover, for every $r \in \mathrm{Exp}(\Gamma)$, there are finitely many equivalence classes of generating sets S such that $e(\Gamma, S) = r$ (two generating sets are equivalent if one is the image of the other by an automorphism of Γ).

Growth of groups extends naturally to the setting of locally compact groups. In particular, Guivarc'h and Jenkins [Guiv–73, Jenk–73] have characterized connected Lie groups of polynomial growth as those of type (R), that is, as those for which $\mathrm{ad}(x)$ has purely imaginary eigenvalues for all x in the Lie algebra of the group (this is considerably easier to prove than Gromov's characterization of finitely generated groups of polynomial growth). More generally, Losert has characterized locally compact groups (not necessarily Lie groups) of polynomial growth in a series of papers, the first being [Lose–87]. Breuillard has obtained nice results on the geometry of locally compact groups of polynomial growth and the shape of large balls [Breu–14]. The world of compactly generated locally compact groups of intermediate growth is more mysterious, even if we know a few examples of such groups which are "not too close to discrete groups" [Corn–19].

From the 1960s, the point of view of growth was not restricted to groups. In particular, finitely generated algebras give rise to growth functions; their growth rates are called Gelfand–Kirillov dimensions because of the first papers where this notion appears; see [GeKi–66, KrLe–00]. Given a field k, a finitely generated group Γ is of polynomial growth if and only if the Gelfand–Kirillov dimension of its group algebra $k[\Gamma]$ is finite.

Several reviews of the subject of group growth have appeared, of which we mention [GrHa–97, Grig–14M], and there is a nice exposition of the theory in the book by Mann [Mann–12].

3. Three open problems: The growth of finitely presented groups, the gap conjecture, and the search for groups with small stretched exponential growth rates

It is still unknown whether there exists any finitely presented group of intermediate growth. This is one of the major open problems of the subject.

The following problem is almost 30 years old and fascinating. Let Γ be a group generated by a finite set S. Suppose that the growth of Γ is strictly dominated by $e^{\sqrt{k}}$; that is, suppose that $\beta(\Gamma, S; k) \leq ce^{c\sqrt{k}}$ for some constant $c > 0$ and for all $k \geq 0$, and that there does not exist an integer $N > 0$ such that $e^{\sqrt{k}} \leq N\beta(\Gamma, S; Nk)$ for all $k \geq 0$. Does it follow that Γ is virtually nilpotent and therefore of polynomial growth? The *gap conjecture* is that the answer is positive; this would be a strong reinforcement of Gromov's theorem. The answer is positive if G is assumed to be residually a finite p-group for some prime p [Grig–89], and more generally if G is residually nilpotent (equivalently if G is residually a finite nilpotent group) [LuMa–91]; see [Grig–14g] for a fuller discussion of the question. But it is known that a finitely generated group of intermediate growth need not be residually finite [Ersc–04] (those of polynomial growth are residually finite by [Grom–81]).

It has long been known that there exists a function b such that any group Γ generated by a finite set S with growth function strictly dominated by b is a group of polynomial growth and therefore a virtually nilpotent group; see [Grom–81, Question, p. 72] for a hint on this, [DrWi–84, Theorem 7.6] for something more explicit, and [ShTa–10, Corollary 1.10] for $b(k) = R^{c(\log\log k)^c}$ for some sufficiently small constant c and for $R > \frac{1}{c}$. The gap conjecture claims that this holds with $b(k) = e^{\sqrt{k}}$.

There is related ongoing research of small rates of the following kind. Let Γ be a group generated by a finite generating set S, say a group of intermediate growth to avoid trivialities. The *stretched exponential growth rates* are defined by $\rho_-(\Gamma) = \liminf_{k\to\infty} \frac{\log\log\beta(\Gamma,S;k)}{\log k}$ and $\rho_+(\Gamma) = \limsup_{k\to\infty} \frac{\log\log\beta(\Gamma,S;k)}{\log k}$; they do not depend on the choice of S. The problem is to find groups of intermediate growth with ρ_+ as small as possible (of course not smaller than $\frac{1}{2}$ if the gap conjecture is true). Let λ_0 be the positive root of the polynomial $X^3 - X^2 - 2X - 4$ and set $\alpha_0 = \frac{\log 2}{\log\lambda_0}$; we have $\lambda_0 \sim 2.4675$ and $\alpha_0 \sim 0.7674$. For any α_-, α_+ with $\alpha_0 \le \alpha_- \le \alpha_+ < 1$, there exists a finitely generated group of intermediate growth Γ such that $\rho_-(\Gamma) = \alpha_-$ and $\rho_+(\Gamma) = \alpha_+$ [Brie–14, BaEr–14]. For the first Grigorchuk group, that of [Grig–83], there have been several estimates before the final result: $\rho_-(\Gamma) = \rho_+(\Gamma) = \alpha_0$ [ErZh–20]. At the time of writing, there is not any known example of group Γ with $\rho_+(\Gamma) < \alpha_0$; but several experts believe that they will be found.

In the following sections, we discuss older results which show some flavour of group growth.

4. Carl Friedrich Gauss and the growth of $\mathbf{Z}^2$ (1834)

The free abelian group of rank two, $\mathbf{Z}^2$, can be seen as the lattice of integer points in the Euclidean plane; this has been so even before the concept of group was made precise. Consider the length function on $\mathbf{Z}^2$ given by the Euclidean norm and the growth of $\mathbf{Z}^2$ as the function R defined by

$$R(t) = |\{(a,b) \in \mathbf{Z}^2 \mid a^2 + b^2 \le t\}| \quad \text{for all } t \ge 0,$$

that is, $R(t)$ is the number of points of $\mathbf{Z}^2$ in the disc of radius $\sqrt{t}$ centred at the origin. The function $R(t)$ is interesting in number theory, more precisely in the study of integers which are sums of two squares; but we like to view $R(t)$ also as a function describing the growth of $\mathbf{Z}^2$. The *circle problem* is to estimate the difference $R(t) - \pi t$ for large t. In 1843, Gauss showed that

$$|R(t) - \pi t| \le 2\pi(1 + \sqrt{2t}) = O(\sqrt{t}).$$

In [Gauss, pp. 271 and 280], Gauss wrote thirty values of $R(k)$, including $R(10000) = 31417$ and $R(100000) = 314197$.

After Gauss, it has been shown that $|R(t) - \pi t| = O(t^\alpha)$ for values $\alpha < \frac{1}{2}$, in particular for $\alpha = \frac{1}{3}$ (Sierpinski, 1906, see [Sier–88, Page 385]). Considerable effort has been devoted to improve α (see [BeKZ–18]); for example, $|R(t) - \pi t| = O(t^{\alpha+\varepsilon})$ for $\alpha = \frac{517}{1648} \sim 0.31371$ and for all $\varepsilon > 0$ [BoWa]. It is conjectured that the estimate holds for every $\alpha > \frac{1}{4}$; in 1915, Landau and Hardy showed that $\frac{1}{4}$ is a lower bound for the best α.

For k a non-negative integer, set $r_2(k) = |\{(a, b) \in \mathbf{Z}^2 \mid a^2 + b^2 = k\}|$, so that $R(k) = \sum_{j=0}^{k} r_2(j)$. Values of $r_2(k)$ and $R(k)$ for small k and relevant references are given in [OEIS, A004018 and A057655]. The growth series $\sum_{k=0}^{\infty} r_2(k) z^k$ is $(\theta_3(z))^2$, where θ_3 is the third Jacobi theta function [CoSl–99, Chapter IV, Section 5].

5. Word lengths, spheres, and balls

Word lengths, spheres, and balls can be found in the literature much before the theory of group growth. For example, $\ell_S(\gamma)$ appears as the "exponent of the substitution γ" in [Poin–82, p. 11], the paper in which Poincaré shows a presentation of a Fuchsian group in terms of one of its fundamental polygons in the hyperbolic plane. The word metric on Γ, defined by $d_S(\gamma, \gamma') = \ell_S(\gamma^{-1}\gamma)$, has been used systematically by Dehn in his first paper on decision problems in group theory; see [Dehn–11] and [DeSt–87, pp. 130 and 143]. Spheres and balls, noted respectively Γ_k and $\bigcup_{j=0}^{k} \Gamma_j$, appear in [ArKr–63], where the authors establish equidistribution in the unit sphere of Euclidean space $\mathbf{R}^3$ of the points of the orbit of a semigroup generated by two appropriate rotations.

6. Wacław Sierpinski (1946), Georgiĭ Adel'son–Vel'skiĭ and Yuli Anatoljevitch Šreĭder (1957), Joseph Rosenblatt (1974), and the supramenability of groups of subexponential growth

In the 1929 paper which created the subject of amenability [Neum–29], John von Neumann considers actions of a group Γ on a set X given with a non-empty subset E. Such an action is *amenable* if there exists a finitely additive positive measure μ on X normalized by $\mu(E) = 1$ and invariant by Γ (the measure need not be finite, except of course when $E = X$). The group Γ itself is *amenable* (eine messbare Gruppe in [Neum–29]) if every action of Γ on every set X given with $E = X$ is amenable, and this holds as soon as the left action of Γ on itself is amenable (with $E = X = \Gamma$). The group Γ is *supramenable* (a terminology due to Rosenblatt [Rose–74]) if every action of Γ on a set X given with any subset $E \neq \emptyset$ is amenable, and this holds as soon as the left action of Γ on itself, with any subset E, is amenable. The Γ-set E has a *paradoxical decomposition* if there exists a partition of E in disjoint sets $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ and elements $g_1, \ldots, g_k, h_1, \ldots, h_\ell$ in Γ such that E is equal to both the disjoint unions $\bigsqcup_{i=1}^{k} g_i A_i$ and $\bigsqcup_{j=1}^{\ell} h_j B_j$. A paradoxical decomposition of E is an obstruction to the existence of μ as above [Neum–29, p. 82]. Remarkably, Tarski has shown that it is the only obstruction: either E has a paradoxical decomposition or there exists a Γ-invariant finitely additive positive measure μ on X normalized by $\mu(E) = 1$ [Tars–36].

For example, the action on $X = \mathbf{R}^d$ of the isometry group of the Euclidean space $\mathbf{R}^d$ given with the unit ball E is amenable when $d = 1$ and $d = 2$ and is not when $d \geq 3$. In dimension 3, Hausdorff, Banach, and Tarski have obtained famous results which express non-amenability in a spectacular way: the action of the rotation group SO(3) on the unit ball in $\mathbf{R}^3$ is non-amenable; see [Haus–14, Appendix to Chapter X, p. 469]. Moreover, two bounded subsets A and B of $\mathbf{R}^3$ with non-empty interiors are equidecomposable; this means that there exist partitions $A = \bigsqcup_{i=1}^{k} A_i$, $B = \bigsqcup_{i=1}^{k} B_i$, and isometries $g_1, \dots, g_k$ of $\mathbf{R}^3$ such that $g_1 A_1 = B_1, \dots, g_k A_k = B_k$; see [BaTa–24].

In [Sier–46], Sierpinski saw that any finitely generated subgroup of the isometry group of $\mathbf{R}$ is of subexponential growth (indeed of polynomial growth), and that this implies that the action of this isometry group on $\mathbf{R}$ is not paradoxical. The argument shows essentially that the isometry group of $\mathbf{R}$ is supramenable and much more (see below).

In [AdSr–57, Theorem 2], it is shown that a finitely generated group of subexponential growth is amenable.

Later, Rosenblatt showed much more. He introduced the terminology "supramenable", and he defined a group to be *exponentially bounded* if all its finitely generated subgroups are of subexponential growth. He showed that exponentially bounded groups are supramenable. Moreover, a finitely generated solvable group either has a nilpotent subgroup of finite index, and thus is of polynomial growth and supramenable, or is not supramenable and contains a free semigroup on two generators, and thus is of exponential growth [Rose–74]. Rosenblatt also conjectured that an amenable group which has no subsemigroup free on two generators should be supramenable, but this was disproved by examples of Grigorchuk: for each prime p, there exists a finitely generated amenable p-group which is not supramenable [Grig–87].

It is unknown whether there exist finitely generated groups of exponential growth which are supramenable.

We reproduce now Sierpinski's argument, cast in the more general situation of a group Γ acting on a set X (instead of the affine group of $\mathbf{R}$ acting on $\mathbf{R}$) and a non-empty subset E of X. Suppose that there exists a paradoxical decomposition of E: there exist as above subsets $A_1, \dots, A_k, B_1, \dots, B_\ell$ of E and a subset S of elements $g_1, \dots, g_k, h_1, \dots, h_\ell$ of Γ (not necessarily distinct from each other) such that

$$E = \left(\bigsqcup_{i=1}^{k} A_i \right) \sqcup \left(\bigsqcup_{j=1}^{\ell} B_j \right) = \bigsqcup_{i=1}^{k} g_i A_i = \bigsqcup_{j=1}^{\ell} h_j B_j .$$

The following argument shows that the subgroup of Γ generated by S has exponential growth.

Set $A = \bigsqcup_{i=1}^{k} A_i$, $B = \bigsqcup_{j=1}^{\ell} B_j$. Define bijections $\varphi \colon E \to A$ and $\psi \colon E \to B$ by $\varphi(x) = g_i^{-1}(x)$ when $x \in g_i A_i$ and $\psi(x) = h_j^{-1}(x)$ when $x \in h_j B_j$. Choose $x_0 \in E$. Observe first that $\varphi(x_0) \neq \psi(x_0)$, because A and B are disjoint, then that $\varphi\varphi(x_0), \varphi\psi(x_0), \psi\varphi(x_0), \psi\psi(x_0)$ are also distinct, because φ and ψ are injective and $A \cap B = \emptyset$, and so on. This shows that, for any positive integer n, the 2^n words of length n

in φ and ψ are maps $E \to E$ with distinct values at x_0. For any of these words, say χ, the value $\chi(x_0)$ is of the form $s_1^{-1}s_2^{-1}\cdots s_n^{-1}(x_0)$, for $s_1, s_2, \dots, s_n \in S$. It follows that the subgroup of Γ generated by S has at least 2^n distinct elements γ of word length $\ell_S(\gamma) \leq n$, and this ends the argument.

Sierpinski's argument shows that a group Γ which can act on a pair $X \supset E$ such that E has a paradoxical decomposition has a finitely generated subgroup of exponential growth. By contraposition, it follows that a finitely generated group of subexponential growth is supramenable, a result much stronger than the one in [AdSr–57], and a proof much more direct than the one in [Rose–74].

7. Hans Ulrich Krause and finitely generated abelian groups with isomorphic Cayley graphs (1953)

In his thesis [Krau–53, Satz 16.1], Krause shows that two finitely generated abelian groups have isomorphic Cayley graphs with respect to well-chosen generating sets if and only if the two following conditions are satisfied: (i) the two groups have the same rank, and (ii) their torsion groups have the same order. In the proof, it is shown that the rank of an abelian group Γ generated by a finite set S is the polynomial growth rate $\limsup_{k\to\infty}\left(\frac{\log\beta(\Gamma,S;k)}{\log k}\right)$. (In fact, this lim sup is a limit.)

8. Jacques Dixmier and polynomial growth of nilpotent connected Lie groups (1960, 1966)

Lemma 3 of [Dixm–60] is the following. Let G be a nilpotent connected Lie group, μ a Haar measure on G, and S a compact subset of G. Then there exists an integer N (which depends on G but not on S) such that $\mu(S^k) = O(k^N)$ when $k \to \infty$; in the particular case of a generating compact subset S, this means that G is a group of polynomial growth.

The lemma is used by Dixmier in the proof of the following result. Consider a locally compact group G, the group algebra $L^1(G)$, and the two-sided ideal I of those elements $f \in L^1(G)$ such that, for every irreducible unitary representation π of G, the operator $\pi(f)$ is of finite rank. If G is a nilpotent connected Lie group, then I is dense in $L^1(G)$. (The same property of I was established earlier for semisimple Lie groups by Harish-Chandra.)

Polynomial growth has been established later for solvable connected Lie groups of type (R) in [Dixm–66].

9. Henry Dye and orbital equivalence (1963)

The following is established by [Dye–63, Theorem 1]. Let Γ be a finitely generated group, generated by a finite subset F. The notation of Dye is $h_1 = |F|$ and $h_k = |F^k \setminus F^{k-1}|$

for $k \geq 2$. If
$$\inf_{k\geq 1} \frac{h_{2k}}{h_1 + \cdots + h_k} = 0,$$
then Γ is approximately finite. In particular, finitely generated groups of polynomial growth are approximately finite.

To define approximate finiteness, consider actions of countable groups on non-atomic standard probability spaces by measure preserving transformations. Two such actions of Γ_1 on X_1 and Γ_2 on X_2 are orbit equivalent if there exists a measure preserving Borel isomorphism $f\colon X_1 \to X_2$ such that $f(\Gamma_1 x)$ coincides with the orbit $\Gamma_2 f(x)$ for almost all x in X_1. Consider some ergodic measure preserving action of the infinite cyclic group $\mathbf{Z}$ on a non-atomic standard probability space; a basic example is the Bernoulli shift action β of $\mathbf{Z}$ on $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{Z}}$. A countable group Γ is approximately finite in the sense of Dye if, for every ergodic measure preserving action α of Γ on a non-atomic probability space X, the actions α and β are orbit equivalent.

It is now known that an infinite countable group is approximately finite if and only if it is amenable [OrWe–80, Hjor–05].

10. Grigoriĭ Margulis, Anosov flows, and the growth of fundamental groups (1967)

On a compact Riemannian smooth manifold M, an Anosov flow is a smooth flow $\Phi = \{\Phi_t\}_{t\in\mathbf{R}}$ which satisfies the following conditions. There exists a Φ-invariant continuous splitting $TM = E^u \oplus E^T \oplus E^s$ of the tangent bundle of M, where the three terms are, respectively, the unstable (or expanding) subbundle of TM, the line bundle tangent to Φ, and the stable (or contracting) subbundle of TM, and there exist constants $\nu > 0$, $c > 0$, such that
$$\|(\Phi_t)_*(v)\| \geq ce^{\nu t}\|v\| \quad \text{and} \quad \|(\Phi_{-t})_*(v)\| \leq ce^{-\nu t}\|v\| \quad \text{for all } v \in E^u \text{ and } t \geq 0$$
and
$$\|(\Phi_t)_*(v)\| \leq ce^{-\nu t}\|v\| \quad \text{and} \quad \|(\Phi_{-t})_*(v)\| \geq ce^{\nu t}\|v\| \quad \text{for all } v \in E^s \text{ and } t \geq 0$$
(the two inequalities for $(\Phi_{-t})_*(v)$ follow from the two inequalities for $(\Phi_t)_*(v)$, see [AnSi–67, p. 121]).

In one of his first published papers, Margulis shows that if a 3-dimensional manifold M has an Anosov flow, then the fundamental group of M has exponential growth [Marg–67]. This has been generalized to manifolds of higher dimensions and Anosov flows with one of the subbundles E^u, E^s of rank one [PlTh–72].

For the contrast, let us quote the following result of Franks. On a compact Riemannian smooth manifold M, a $\mathcal{C}^1$ map $f\colon M \to M$ is expanding if there are constants $\lambda > 1$ and

$c > 0$ such that $\|Tf^m v\| \geq c\lambda^m \|v\|$ for all $v \in TM$ and $m \geq 1$. Here is the result: if a compact manifold admits an expanding self-map, then its fundamental group has polynomial growth [Fran–70, Theorem 8.3].

11. Harry Kesten and recurrent random walks on groups (1967)

Let Γ be a finitely generated group. A symmetric probability measure μ on Γ such that $\{\gamma \in \Gamma \mid \mu(\gamma) > 0\}$ is a finite generating set gives rise to a random walk on Γ. The group Γ is recurrent if there exists such a measure such that the associated random walk is recurrent (equivalently if for any such measure the associated random walk is recurrent). It is a classical theorem of Pólya that the simple random walk on $\mathbf{Z}^d$ is recurrent if $d \leq 2$ and transient if $d \geq 3$ [Poly–21], and it has been known since at least 1962 that an infinite finitely generated abelian group is recurrent if and only if it is a finite extension of $\mathbf{Z}$ or a finite extension of $\mathbf{Z}^2$ [Dudl–62].

In his thesis [Kest–59], Kesten introduced the subject of random walks on countable groups (not necessarily abelian groups, as in the work of Pólya and a few others). Later, he revisited the subject once in [Kest–67]. His work is put in historical perspective in [SCZh–21]. Kesten asked how the existence of a recurrent random walk on a group is related to its growth type; in particular, he conjectured that a finitely generated group of exponential growth does not have any recurrent random walk [Kest–67, Conjecture 4]. The conjecture was made more precise (the growth of a recurrent group is at most quadratic) and generalized to second countable locally compact groups; see the introduction of [GuRa–12]. For discrete groups, the final result is due to Varopoulos (1986): a finitely generated group is recurrent if and only if it is of at most quadratic growth, if and only if it is either finite, or a finite extension of $\mathbf{Z}$, or a finite extension of $\mathbf{Z}^2$; see [VaSC–92]. (A finitely generated group Γ is of at most quadratic growth if there exists a constant $C > 0$ such that $\beta(\Gamma, S; k) \leq Ck^2$ for all $k \geq 1$.)

12. Generating functions

To encode a sequence $(a_k)_{k\geq 0}$ of integral numbers, several types of series or functions can be used, and the best choice depends on the subject. One choice is the *ordinary generating function* of the sequence $(a_k)_{k\geq 0}$:

$$\Sigma(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbf{Z}[\![z]\!].$$

When $\Sigma(z)$ converges for z small enough and whenever possible, we like to identify the "simple function of analysis" of which $\Sigma(z)$ is the Taylor series at the origin. There is a rich source of examples and theorems on these generating functions in the book [FlSe–09].

An early example occurs in a letter of Euler to Goldbach dated September 4, 1751. The letter is reproduced partly in [FlSe–09, Section I.1] and in full in [Euler, Letter 154, pp. 489–491]. (In [Knut–97, Section 1.2.9], Knuth mentions three still earlier appearances of generating functions by de Moivre, Stirling, and Euler in connection with numbers of partitions of integers.) In his letter, Euler considers the number c_k of decompositions of a convex $(k+2)$-gon in triangles; set moreover $c_0 = 1$. The generating function of $(c_k)_{k \geq 0}$

$$\sum_{k \geq 0} c_k z^k = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4k-2)}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot \quad (k+1)} z^k = \frac{1 - \sqrt{1-4z}}{2z}$$

is algebraic. (Euler does not consider our c_0, and writes a instead of our z, so that his series is rather $1 + 2a + 5a^2 + 14a^3 + 42a^5 + \cdots = \frac{1-2a-\sqrt{1-4a}}{2aa}$.) The numbers c_k are now known as Catalan numbers and are often written in terms of binomial coefficients: $c_k = \frac{1}{k+1}\binom{2k}{k}$. For 214 different kinds of objects that are counted using Catalan numbers and for a historical survey, see [Stan–15].

The simplest sequences are those which satisfy a linear recurrence relation; they correspond exactly to rational generating functions. More precisely, consider a positive integer d and complex numbers $q_1, q_2, \ldots, q_d$ with $q_d \neq 0$. Set $Q(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_d z^d = \prod_{j=1}^{e} (1 - \gamma_j z)^{d_j}$, where $\gamma_1, \ldots, \gamma_e \in \mathbf{C}$ are distinct complex numbers and $d_1, \ldots, d_e$ their multiplicities; note that $\sum_{j=1}^{e} d_j = d$. Then, for a sequence $(a_k)_{k \geq 0}$, the following conditions are equivalent:

- (R1) $\sum_{k \geq 0} a_k z^k = \frac{P(z)}{Q(z)}$ for some polynomial $P(z)$ of degree less than d,
- (R2) $a_{k+d} + q_1 a_{k+d-1} + q_2 a_{k+d-2} + \cdots + q_d a_k = 0$ for all $k \geq 0$,
- (R3) $a_k = \sum_{j=1}^{e} P_j(k) \gamma_j^k$ for all $k \geq 0$, for some polynomials $P_j(z)$ of degree less than d_j (with $j = 1, \ldots, e$).

For this, and for variations (when $\deg P \geq d$ or when $Q(z) = (1-z)^d$), see [Stan–96, Chapter 0].

The Fibonacci sequence $(F_k)_{k \geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots)$ is a notorious example:

- (1) generating function $\sum_{k=0}^{\infty} F_k z^k = \frac{z}{1-z-z^2}$,
- (2) linear recursion $F_{k+2} - F_{k+1} - F_k = 0$ for all $k \geq 0$, and
- (3) Binet's formula $F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$, published by Binet in 1845 but already in [Ber–1728, Section 7] and [Eul–1767, p. 128].

13. Growth series for finitely generated groups

Let Γ be a finitely generated group and S a finite generating set of Γ. For $k \geq 0$, recall that $\sigma(\Gamma, S; k)$ is the cardinal of the sphere of radius k and $\beta(\Gamma, S; k)$ is the cardinal of the ball of radius k in Γ. The *growth series* of the pair (Γ, S) is the generating function

$$\Sigma(\Gamma, S; z) = \sum_{k=0}^{\infty} \sigma(\Gamma, S; k) z^k = \sum_{\gamma \in \Gamma} z^{\ell_S(\gamma)} \in \mathbf{Z}[\![z]\!].$$

The radius of convergence of this series is strictly positive and is $\frac{1}{e(G,S)}$, where $e(G, S) = \lim_{k\to\infty} \sigma(\Gamma, S; k)^{1/k}$ is the exponential growth rate of the pair (G, S). It is sometimes better to consider

$$B(\Gamma, S; z) = \sum_{k=0}^{\infty} \beta(\Gamma, S; k) z^k = \frac{\Sigma(\Gamma, S; z)}{1 - z}.$$

For example, for the infinite cyclic group $\Gamma = \mathbf{Z}$ generated by $S = \{1\}$, we have

$$\Sigma(\mathbf{Z}, \{1\}; z) = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + 2z^5 + \cdots = \frac{1+z}{1-z}. \tag{13.1}$$

More generally, for the free abelian group $\mathbf{Z}^n$ generated by a basis S_n, we have

$$\Sigma(\mathbf{Z}^n, S_n; z) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{n} \binom{n}{\ell} \binom{k+n-\ell-1}{k-\ell} \right) z^k = \left(\frac{1+z}{1-z} \right)^n.$$

The infinite sum simplifies to $\Sigma_{k=0}^{\infty} 4kz^k$, $\Sigma_{k=1}^{\infty}(4k^2+2)z^k$, $\Sigma_{k=1}^{\infty} \frac{8k(k^2+2)}{3} z^k$, when $n = 2, 3, 4$, respectively [OEIS, A008574, A005899, and A008412].

Growth series have been studied for several other classes of groups. For a Coxeter system (Γ, S) with S finite, the growth series $\Sigma(\Gamma, S; z)$ is a rational function (see [Bour–68, chap. IV, § 1, exerc. 26 and § 4, exerc. 10]). This function has interesting values; for example, its value at 1 is rational and is the inverse of the Euler–Poincaré characteristic of the group Γ [Serr–71, Proposition 17, p. 112].

For a Gromov hyperbolic group Γ and an arbitrary generating set S, Gromov has shown that $\Sigma(\Gamma, S; z)$ is a rational function [Grom–87, Corollary 5.2.A′]. This generalizes a result of Cannon [Cann–84, Theorem 7]. There are a few other finitely generated groups for which $\Sigma(\Gamma, S; z)$ is known to be rational for all S: virtually abelian groups ([Bens–83], quoted again below) and the Heisenberg group $\left(\begin{smallmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{smallmatrix}\right)$ [DuSh–19].

There are some groups Γ with generating sets S such that $\Sigma(\Gamma, S; z)$ is an irrational algebraic function [Parr–92]. The growth series of a pair (Γ, S) can also be transcendental; it is always transcendental when Γ is of intermediate growth [Mann–12, Chapter 15]. Stoll showed that there are 2-step nilpotent groups Γ with two finite generating sets S, T such that $\Sigma(\Gamma, S; z)$ is rational and $\Sigma(\Gamma, T; z)$ transcendental [Stol–96].

The first finitely generated groups of intermediate growth of which the asymptotics of the growth series is precisely known appear in [BaEr–12].

14. Hilbert series

Consider again the group $\Gamma = \mathbf{Z}^n$ for some $n \geq 1$ and an arbitrary finite generating set S. As already stated above, there exists a polynomial $P \in \mathbf{Z}[z]$ such that

$$\Sigma(\mathbf{Z}^n, S; z) = \frac{P(z)}{(1-z)^n}.$$

Here is one way to show this. The group algebra $\mathbf{C}[\Gamma]$, with linear basis $(\delta_\gamma)_{\gamma\in\Gamma}$ and multiplication defined by $\delta_\gamma\delta_{\gamma'} = \delta_{\gamma\gamma'}$, has a filtration $\mathbf{C}[\Gamma] = \bigcup_{k\geq 0} B_k$ where B_k is the linear subspace generated by $\{\delta_\gamma \mid \ell_S(\gamma) \leq k\}$; set moreover $B_{-1} = \{0\}$. The associated graded algebra $A = \bigoplus_{k\geq 0}(B_k/B_{k-1})$ is commutative and generated by a finite set of elements of degree 1. It is a theorem of Hilbert that the Hilbert series

$$\sum_{k\geq 0} \dim_{\mathbf{C}}(B_k/B_{k-1})z^k = \Sigma(\mathbf{Z}, S; z)$$

of such an algebra is rational of the form $\frac{P(z)}{(1-z)^n}$; for a proof, see, for example, [AtMa–69, Theorem 11.1]. The observation that the growth series of (Γ, S) is the Hilbert series of an appropriate graded algebra, and thus in particular a rational function of the form $\frac{P(z)}{(1-z)^n}$, is due to several authors, including Wagreich, quoted in [Bens–83]; see also [Bill–84]. More generally, when Γ is a virtually abelian finitely generated group and S an arbitrary finite generating set, the series $\Sigma(\Gamma, S; z)$ is a rational function of which all poles are roots of unity [Bens–83]; but we do not know whether this can be proved by using a (non-commutative!) graded algebra, as used in the proof sketched above for abelian groups.

The "theorem of Hilbert" refers to [Hilb–90, Theorem IV, p. 512]. In fact, Hilbert shows that the series satisfies a condition like (R3) of our Section 12, rather than (R1). It was already standard in this time to write "Hilbert series" which are rational functions for the dimensions of the homogeneous components of a graded algebra. I am grateful to Hanspeter Kraft for showing me that this can be found in the work of Sylvester on the theory of invariants, around 1880 (see, e.g., papers 38, 40, and 59 in [Sylvester]) and also to Michel Brion for showing me an even earlier work [Cayl–56, No 28] where Cayley uses generating functions which are in fact Hilbert series, products of terms of the form $(1-x^k)^{a_k}$ (with a_k positive or negative), to discuss covariant algebras and the fact they are not (or they are ...) finitely generated.

Hilbert series are also called Poincaré series, especially when they encode dimensions of homology spaces; see [Babe–86].

15. Eugène Ehrhart and the number of integral points in the multiples of a polytope (1962)

Consider a Euclidean space V of dimension n, with scalar product denoted by $\langle \cdot \mid \cdot \rangle$, a lattice Γ in V, that is, a subgroup of V isomorphic to $\mathbf{Z}^n$ generated by a basis of V, and a polytope P which is the convex hull of a finite subset of Γ; for each non-negative integer k, let $E_P(k)$ denote the number of points in $kP \cap \Gamma$. In 1962, Ehrhart published a note on the numbers $E_P(k)$ and the series $\sum_{k=0}^{\infty} E_P(k)z^k$ [Ehrh–62, Brio–95]. For a polytope of non-empty interior, this series is a growth series of the group $\Gamma \approx \mathbf{Z}^n$ for an appropriate choice of generating set.

For the lattice $\mathbf{Z}^n$ in $\mathbf{R}^n$ and the convex hull $P = \mathrm{Conv}(\pm e_1, \dots, \pm e_n)$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbf{R}^n$, we have, with the notation of Section 13,

$$\sum_{k=0}^{\infty} E_P(k)z^k = B(\mathbf{Z}^n, \mathbf{Z}^n \cap P; z) = \frac{1}{1-z}\left(\frac{1+z}{1-z}\right)^n.$$

Other cases are studied from this point of view in [BaHV–99]. For example, when $V = \left\{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \sum_{j=1}^{n+1} x_i = 0\right\}$, $\Gamma = \mathbf{Z}^{n+1} \cap V \approx \mathbf{Z}^n$, and P is the convex hull of $\{\pm(e_i - e_j) \mid 1 \le i < j \le n+1\}$,

$$\sum_{k=0}^{\infty} E_P(k)z^k = B(\Gamma, \Gamma \cap P, z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n} \binom{n}{k}^2 z^j = \frac{1}{1-z} P_n\left(\frac{1+z}{1-z}\right),$$

where P_n is the Legendre polynomial of degree n.

16. Theta functions

Consider again a Euclidean vector space V of dimension n and a lattice Γ in V. For elements of Γ, consider no longer the word length as above, but rather the norm $\Gamma \to \mathbf{R}_+$, $x \mapsto \|x\| = \sqrt{\langle x \mid x\rangle}$. The *theta function* of Γ is defined by $\Theta_\Gamma(\tau) = \sum_{x \in \Gamma} e^{i\pi\tau\|x\|^2}$, so that Θ_Γ is a holomorphic function on the upper half-plane $\{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$. When Γ is an integral lattice, namely when $\langle x \mid y\rangle \in \mathbf{Z}$ for all $x, y \in \Gamma$, the theta series is alternatively viewed as a power series in $q = e^{i\pi\tau}$:

$$\Theta_\Gamma(q) = \sum_{x \in \Gamma} q^{\|x\|^2} = \sum_{r=0}^{\infty} |\{x \in \Gamma \mid \langle x \mid x\rangle = r\}| q^r.$$

For example, when $\Gamma = \mathbf{Z}$ is embedded the standard way in the real line $V = \mathbf{R}$, the series is

$$\Theta_{\mathbf{Z}}(q) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots = \theta_3(q), \qquad (16.1)$$

where θ_3 is as above the third Jacobi theta function. More generally, for $\mathbf{Z}^n$ embedded the standard way in the standard Euclidean space $\mathbf{R}^n$, we have $\Theta_{\mathbf{Z}^n}(q) = (\theta_3(q))^n$ [CoSl–99, Chapter IV, Section 5].

Nonzero coefficients of the series in formula (13.1) for $\Sigma(\mathbf{Z}, \{1\}; z)$ are the same as nonzero coefficients of the series in formula (16.1) for $\Theta_{\mathbf{Z}}(q)$; is this more than a meaningless coincidence? It is tempting to speculate that theta functions could be of some interest for other groups than lattices in Euclidean spaces.

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