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Universality for free fermions and the local Weyl law for semiclassical Schrödinger operators

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Abstract. We study local asymptotics for the spectral projector associated to a Schrödinger operator $-\hbar^2\Delta + V$ on $\mathbb{R}^n$ in the semiclassical limit as $\hbar \to 0$. We prove local uniform convergence of the rescaled integral kernel of this projector towards a universal model, inside the classically allowed region as well as on its boundary. This implies universality of microscopic fluctuations for the corresponding free fermions (determinantal) point processes, both in the bulk and around regular boundary points. Our results apply to a general class of smooth potentials in arbitrary dimension $n \geq 1$. These results are complemented by studying both macroscopic and mesoscopic fluctuations of the point process. We obtain tail bounds for macroscopic linear statistics and, provided $n \geq 2$, a central limit theorem for both macroscopic and mesoscopic linear statistics in the bulk.

Keywords: semiclassical analysis, determinantal point processes, Weyl law, Schrödinger operators.

1. Introduction

Consider a semiclassical Schrödinger operator on $L^2(\mathbb{R}^n)$,

$$H_\hbar = -\hbar^2\Delta + V,$$

where Δ is the standard (negative) Laplacian, $\hbar > 0$ plays the role of the Planck constant, and the potential $V: \mathbb{R}^n \to \mathbb{R}$ is locally integrable, bounded from below, and confining (that is, $V(x) \to +\infty$ as $x \to \infty$). The operator $H_\hbar$ is then essentially self-adjoint with domain $\mathrm{H}^2(\mathbb{R}^n) \cap \{u \in L^2, Vu \in L^2\}$ and has compact resolvent: it admits a non-decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of eigenvalues tending to $+\infty$, and an associated Hilbert basis $(v_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions

$$H_\hbar v_k = \lambda_k v_k, \quad \langle v_k, v_\ell\rangle_{L^2} = \delta_{k=\ell}, \quad k, \ell \in \mathbb{N}.$$

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Note that these eigenvalues and eigenfunctions depend on $\hbar$. We refer to the book [49] for background on the spectral theory of Schrödinger operators. Using the eigenfunctions $(v_k)_{k\in\mathbb{N}}$, one can build a *Slater determinant* with N particles associated to $H_\hbar$, which is the following normalised element of $L^2(\mathbb{R}^{n\times N})$:

$$\Psi_N(\mathbf{x}) = \frac{1}{\sqrt{N!}} \det_{N\times N}[v_k(x_j)], \quad \mathbf{x} \in \mathbb{R}^{n\times N}.$$

Physically, Ψ_N represents a zero temperature state[1] of a system of N non-interacting spinless fermions subject to the Hamiltonian $H_\hbar$. Using the usual probabilistic interpretation of quantum mechanics, Ψ_N gives rise to an N-point process (a probability measure $\mathbb{P}_N$ on $\mathbb{R}^{n\times N}$) whose density with respect to the Lebesgue measure is

$$\mathbb{P}_N[\mathrm{d}\mathbf{x}] = |\Psi_N(x)|^2 = \frac{1}{N!}|\det_{N\times N}[v_k(x_j)]|^2. \tag{1.1}$$

The purpose of this article is to show that in a thermodynamical limit where both $N \to +\infty$ and $\hbar \to 0$, the properties of $\mathbb{P}_N$ are identical for a large class of potentials V. We use methods from semiclassical analysis and the fact that $\mathbb{P}_N$ is a *determinantal point process*.

Realisations $\{x_j\}_{j=1}^N \in \mathbb{R}^{n\times N}$ of $\mathbb{P}_N$ are usually referred to as free fermions or non-interacting Fermi gases [71] since the N particles are only subject to the external potential V and the Pauli exclusion principle. In the large N limit, this induces non-trivial spatial correlations, named *quantum fluctuations*. Using cold atoms in optical traps, it is now possible to experimentally simulate such non-interacting Fermi gases in a general potential and study these quantum fluctuations [11, 44]. This has led to a significant interest in the statistical physics literature regarding the theoretical description and universality of these local fluctuations. These predictions rely on standard methods from many-body physics such as *local density approximations*, and on *random matrix theory*. Indeed, for $V : x \mapsto x^2$ in dimension 1, $\mathbb{P}_N$ corresponds to the law of the eigenvalues of the Gaussian unitary ensemble (GUE), the most studied model of random matrices. We refer to the reviews [19, 29] and Section 1.2 for some background on these results.

Except for a few specific cases, like the harmonic oscillator, the eigenvalues and eigenfunctions $(\lambda_k, v_k)_{k\in\mathbb{N}}$ of the Schrödinger operator $H_\hbar$ is neither explicit nor given by induction formulas. The main novelty of this paper is to apply semiclassical methods to study (non-interacting) Fermi gases for a general class of potential V, a problem which is open in the mathematical literature since [71]. This requires extending, beyond the standard framework, both semiclassical estimates for the spectral projectors of Schrödinger operators and estimates for determinantal processes with general reproducing kernels. The greatest source of difficulty in this problem lies in the fact that the ground state of such free fermions are *gapless*: one has $\lambda_{N+1} - \lambda_N \to 0$ in the asymptotic regime considered.

[1] Uniqueness of the zero temperature state is ensured when $\lambda_N < \lambda_{N+1}$, a condition which we will enforce throughout this article.

Hence, one can only hope to describe the state Ψ_N up to $\mathcal{O}(\hbar)$. In contrast, for gapped systems where $\lambda_{N+1} - \lambda_N$ is bounded from below, one can use perturbation theory to describe the state Ψ_N up to any polynomial precision in $\hbar$. Such cases with recent activity are the Szegő or Bergman projectors (see notably [9, 10, 20] for a fermionic perspective).

1.1. Main results

1.1.1. Setting and assumptions. The function Ψ_N is non-ambiguously defined when $\lambda_N < \lambda_{N+1}$. To enforce this condition and to introduce the asymptotic setting considered, we let $\mu \in \mathbb{R}$, $\hbar > 0$, $V \in L^1_{\mathrm{loc}}$ with $V \to +\infty$ as $x \to \pm\infty$, and let

$$\Pi_{\hbar,\mu} = \mathbb{1}_{(-\infty,\mu]}(H_\hbar)$$

be the spectral projector of $H_\hbar$ on the interval $(-\infty, \mu]$. We also denote by $\Pi_{\hbar,\mu}$ the integral kernel of the previous operator,

$$\Pi_{\hbar,\mu}\colon (x, y) \mapsto \sum_{\lambda_k \le \mu} v_k(x) v_k(y).$$

Then, letting $N = \mathrm{Rank}(\Pi_{\hbar,\mu})$, the probability density associated with the state Ψ_N in (1.1) is

$$\mathbb{P}_N[\mathrm{d}\mathbf{x}] = \frac{1}{N!} \det_{N\times N}[\Pi_{\hbar,\mu}(x_i, x_j)]. \tag{1.2}$$

A precise physical description of $\mathbb{P}_N[\mathrm{d}\mathbf{x}]$ is the joint probability of positions of a system of non-interacting fermions, subject to the one-body Hamiltonian $H_\hbar$, connected to a reservoir with chemical potential μ, and at zero temperature equilibrium. The parameters we tune are μ and $\hbar$, then N is determined by $N = \mathrm{Rank}(\Pi_{\hbar,\mu}) = \max(j \in \mathbb{N}, \lambda_j \le \mu)$ so that automatically $\lambda_N < \lambda_{N+1}$.

Since $\Pi_{\hbar,\mu}$ is an orthogonal projection, the random measure $\mathrm{X} := \sum_{j=1}^N \delta_{x_j}$, where the configuration $\{x_j\}_{j=1}^N$ follows $\mathbb{P}_N$, is a determinantal point process with kernel $\Pi_{\hbar,\mu}$. This means that all the correlation functions of X are given by determinants of the kernel $\Pi_{\hbar,\mu}$; we refer to Appendix A.3 for background on determinantal point processes.

Let us now specify the hypothesis on V that we use throughout this article.

Definition 1.1. We say that

(H) A couple $(\mu, V) \in \mathbb{R} \times L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$ satisfies (H) if there exists $M \in \mathbb{R}$ with $\min V < \mu < M$ such that $\{V \le M\}$ is compact and $V \in C^\infty(\{V < M\}, \mathbb{R})$.

The scope of this article is the following asymptotic regime: (μ, V) satisfies (H) are fixed, and $\hbar \to 0$. In this regime, one has in particular $N < +\infty$ for every fixed $\hbar > 0$, and $N \to +\infty$ as $\hbar \to 0$. In fact, if (μ, V) satisfies (H), one can prove the Weyl law:

$$\lim_{\hbar\to 0} \hbar^n N = \frac{\omega_n Z}{(2\pi)^n}, \quad Z := \int_{\mathbb{R}^n} (\mu - V(x))_+^{n/2} \mathrm{d}x, \tag{1.3}$$

where $\omega_n := \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ denotes the volume of the unit Euclidean ball in $\mathbb{R}^n$ and $(y)_+ = \max\{0, y\}$ for $y \in \mathbb{R}$; see Proposition 2.10.

1.1.2. Pointwise Weyl law and macroscopic fluctuations. A stronger form of (1.3) is that the density of states (proportion of particles by unit volume) is given by

$$\varrho(x) := Z^{-1}(\mu - V(x))_+^{n/2}, \quad x \in \mathbb{R}^n.$$

Our first result is a probabilistic interpretation of this *pointwise Weyl law*. Let us denote by d_W the Kantorovich or Wasserstein$_1$ distance on the space on probability measures on $\mathbb{R}^n$:

$$\mathrm{d}_W(\nu_1, \nu_2) = \sup\left\{\int f\,\mathrm{d}(\nu_1 - \nu_2) : f \in \mathrm{Lip}_1(\mathbb{R}^n)\right\}.$$

This provides a natural metric on the space of probability measures which is stronger than weak convergence. Recall that X denotes the determinantal point process on $\mathbb{R}^n$ associated with the operator $\Pi_{\hbar,\mu} = \mathbb{1}_{-\hbar^2\Delta+V\leq\mu}$ and that $N = \mathrm{tr}(\Pi_{\hbar,\mu})$ is the particle number.

Theorem I.1. *Let* (μ, V) *satisfy* (H). *There exists a constant* $c > 0$ *such that for any* $\delta \in (0, 1]$ *and for any* $\hbar \in (0, 1]$*,*

$$\mathbb{P}_N[\mathrm{d}_W(N^{-1}\mathrm{X}, \varrho) \geq \delta] \leq C_\epsilon \exp(-cN\delta^2).$$

Theorem I.1 is a law of large numbers for the random probability measure $N^{-1}\mathrm{X}$, with an exponential rate of convergence.

The set $\{V \leq \mu\}$ which supports ϱ is called the *droplet*. The droplet consists of a *bulk* $\{V < \mu\}$ and an *edge* $\{V = \mu\}$. The real random variables

$$\mathrm{X}(f) = \sum_{j=1}^{N} f(x_j),$$

for $f : \mathbb{R}^n \to \mathbb{R}$, are called *linear statistics*.

Our next theorems describe the fluctuations of linear statistics in the bulk. They exhibit Gaussian-like tails and, if $n \geq 2$, the normalised linear statistics converge in distribution to a Gaussian, as a form of central limit theorem (CLT).

Theorem I.2. *Let* (μ, V) *satisfy* (H) *and let* $f \in C_c^\infty(\{V < \mu\}, \mathbb{R})$. *There exist* $c > 0$*,* $\epsilon > 0$ *such that for all* $\hbar \in (0, 1]$ *and all* $t \leq \epsilon\sqrt{\hbar N}$*,*

$$\mathbb{P}_N[|\mathrm{X}(f) - \mathbb{E}\mathrm{X}(f)| \geq \sqrt{\hbar N}t] \leq 2e^{-ct^2}.$$

This shows that the particles' fluctuations are much smaller than in the independent case, a phenomenon related to rigidity and hyperuniformity [89] of the particles. For free fermions, this rigidity is entirely due to the "repulsion" coming from the exclusion principle.

We expect (Conjecture 1 below) that the variance of $\mathrm{X}(f)$ is always of order $N\hbar$ for a sufficiently smooth function f, so that Theorem I.2 captures the typical size of the fluctuations. In particular, in dimension 2 or more, the variance of $\mathrm{X}(f)$ diverges, and it is a general feature of determinantal point processes (see Corollary A.12) that divergence of the variance implies a CLT.

Theorem I.3. *Let $n \geq 2$, let (μ, V) satisfy (H) and let $f \in C(\mathbb{R}^n, \mathbb{R})$ with at most exponential growth such that $f \in H^1(\Omega)$ on an open set $\Omega \subset \{V \leq \mu\}$ with $\int_\Omega |\nabla f|^2 > 0$. It holds in distribution as $\hbar \to 0$, or equivalently $N \to \infty$,*

$$\frac{\mathrm{X}(f) - \mathbb{E}\mathrm{X}(f)}{\sqrt{\operatorname{var} \mathrm{X}(f)}} \Rightarrow \mathcal{N}_{0,1}.$$

Even if $f \in C_c^\infty(\{V < \mu\}, \mathbb{R})$, it is not clear that $\operatorname{var} \mathrm{X}(f)$ does not oscillate because of Remark 1.2 below, and its behaviour is expected to depend strongly on the properties of the Newtonian dynamics associated with the potential V. The proofs of Theorems I.2 and I.3 rely on an upper bound and lower bound, respectively, for the variance, but these bounds differ by a factor $\hbar$ (see (3.13) and (5.28)): we are able to show that

$$c\hbar^{2-n} \leq \operatorname{var} \mathrm{X}(f) = -\frac{1}{2} \operatorname{tr}([\Pi_\hbar, f]^2) \leq C\hbar^{1-n}.$$

This prompts the following conjecture.

Conjecture 1. *Let $n \geq 2$ and let $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ be nonzero. There exist $0 < c < C$ such that*

$$c\hbar^{1-n} \leq \operatorname{var} \mathrm{X}(f) = -\frac{1}{2} \operatorname{tr}([\Pi_\hbar, f]^2) \leq C\hbar^{1-n}.$$

The constants c and C presumably correspond to (weighted) $H^{1/2}$ Sobolev norms, as in the case of the free Laplacian and as in the mesoscopic CLT theorem (Theorem III.1 below). In the physics literature, certain examples of counting statistics (non-smooth test functions) have been considered in [82].

In dimension $n = 1$, we showed in [32] that under generic conditions on the potential V, for $f \in C^\infty(\mathbb{R}, \mathbb{R})$ with at most polynomial growth, without any normalisation, the statistic $\mathrm{X}(f)$ obeys a CLT as $\hbar \to 0$ and the variance converges to a weighted $H^{1/2}$ square-norm. This is similar to the random matrix cases treated in the one cut case, e.g., in [14, 57, 63]. It is however important to note that the situation is different for free fermions in the multi-cut regime. The method of [32] is different from the techniques developed in this article and relies on the complete integrability of one-dimensional Schrödinger systems (WKB approximations).

1.1.3. Universal limit at microscopic scales. The method of proof of Theorems I.1, I.2 and I.3 relies on the asymptotics of the integral kernel of $\Pi_\hbar = \mathbb{1}_{(-\infty,\mu]}(H_\hbar)$ near the diagonal. At the *microscopic scale*, that is, when zooming to the typical distance between particles, this kernel converges to a universal limit, which does not depend on V but only on the dimension n and on whether we are in the bulk $\{V < \mu\}$ or at the edge $\{V = \mu\}$ of the droplet. This universal limit implies convergence in distribution of the point process at this microscopic scale.

Given $x_0 \in \mathbb{R}^n$ and $\mathcal{U} \in \mathrm{SO}_n$, we define the rescaled kernel obtained by zooming around x_0 at scale ϵ as

$$K_{x_0,\epsilon}\colon (x, y) \mapsto \epsilon^n \Pi_{\hbar,\mu}(x_0 + \epsilon\mathcal{U}^* x, x_0 + \epsilon\mathcal{U}^* y). \tag{1.4}$$

This is the kernel of the determinantal point process

$$T^*_{x_0,\epsilon}\mathrm{X} = \sum_{j=1}^{N} \delta_{\epsilon^{-1}\mathcal{U}(x_j - x_0)}. \tag{1.5}$$

Theorem II.1. *Let (μ, V) satisfy (H) and $\hbar \in (0,1]$. Let $x_0 \in \{V < \mu\}$, let $\epsilon := \frac{2\pi\omega_n^{-1/n}\hbar}{\sqrt{\mu - V(x_0)}}$, let $\mathcal{U} \in \mathrm{SO}_n$ and set $K_{x_0,\epsilon}$ as in formula (1.4). For any compact sets $\mathcal{A} \Subset \{V < \mu\}$ and $\mathcal{K} \Subset \mathbb{R}^{2n}$, there exists a constant $C > 0$ such that*

$$\sup_{x_0 \in \mathcal{A}} \sup_{(x,y)\in\mathcal{K}} |K_{x_0,\epsilon}(x,y) - K_{\mathrm{bulk}}(x,y)| \le C\hbar,$$

where the bulk kernel is $K_{\mathrm{bulk}} := \mathbb{1}_{(-\infty, 4\pi^2\omega_n^{-2/n}]}(-\Delta)$; see the explicit formula (1.6) below.

Remark 1.2. This result is optimal in the sense that one cannot in general obtain asymptotics for $K_{x_0,\epsilon}$ beyond $\mathcal{O}(\hbar)$. There is no asymptotic expansion in powers of $\hbar$ and the error is known to oscillate. However, our methods can be used to obtain a stronger mode of convergence: using the ellipticity of the operator, it holds for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$,

$$\sup_{x_0 \in \mathcal{A}} \sup_{(x,y)\in\mathcal{K}} |\partial_x^\alpha \partial_y^\beta K_{x_0,\epsilon}(x,y) - \partial_x^\alpha \partial_y^\beta K_{\mathrm{bulk}}(x,y)| = \mathcal{O}(\hbar).$$

Only local C^0-convergence is required for our applications to determinantal point processes.

Theorem II.2. *Let (μ, V) satisfy (H) and $\hbar \in (0,1]$. Let $x_0 \in \{V = \mu\} \cap \{\nabla V \neq 0\}$, let $\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}$ and $\mathcal{U} \in \mathrm{SO}_n$ such that $\mathcal{U}(\nabla V(x_0)) = |\nabla V(x_0)|\mathrm{e}_1$. Let $K_{x_0,\epsilon}$ be as in (1.4). For any compact sets $\mathcal{A} \Subset \{V = \mu; \nabla V \neq 0\}$ and $\mathcal{K} \Subset \mathbb{R}^{2n}$, there exists a constant $C > 0$ such that*

$$\sup_{x_0 \in \mathcal{A}} \sup_{(x,y)\in\mathcal{K}} |K_{x_0,\epsilon}(x,y) - K_{\mathrm{edge}}(x,y)| \le C\hbar^{1/3},$$

where the edge kernel is $K_{\mathrm{edge}} := \mathbb{1}_{(-\infty,0]}(-\Delta + x_1)$; see the explicit formula (1.7) below.

Theorems II.1 and II.2 directly imply universality of the point process obtained by zooming at microscopic scales either in the bulk or at the edge of the droplet (see Proposition A.14).

Theorem II.3. *Let (μ, V) satisfy (H).*

1. *If $V(x_0) < \mu$ and $\epsilon = 2\pi\hbar\frac{\omega_n^{-1/n}}{\sqrt{\mu - V(x_0)}}$, then for any $\mathcal{U} \in \mathrm{SO}_n$, the point process $T^*_{x_0,\epsilon}\mathrm{X}$ given by (1.5) converges in distribution as $\hbar \to 0$ to the determinantal point process associated with K_{bulk}.*
2. *If $V(x_0) = \mu$, $\nabla V(x_0) \neq 0$ and $\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}$, then for any $\mathcal{U} \in \mathrm{SO}(n)$ satisfying $\mathcal{U}(\nabla V(x_0)) = |\nabla V(x_0)|\mathrm{e}_1$, the point process $T^*_{x_0,\epsilon}\mathrm{X}$ given by (1.5) converges in distribution, as $\hbar \to 0$, to the determinantal point process associated with K_{edge}.*

Let us now explain the heuristics behind these results and define the limit objects K_{bulk} and K_{edge}. Following Theorem I.1 and formula (1.3), the typical inter-particle distance around a point x_0 in the bulk is approximately $\epsilon = \hbar c_n(Z\rho(x_0))^{-1/n}$, where $c_n = 2\pi\omega_n^{-1/n}$. After zooming at scale ϵ, the potential of the Schrödinger operator is close to a constant,

$$\hbar^2\epsilon^{-2}\Delta + V(x_0+\epsilon x) - \mu \simeq c_n^2(Z\rho(x_0))^{2/n}(-\Delta + c_n^{-2}).$$

Hence, the natural candidate for the scaling limit is

$$K_{\text{bulk}} = \mathbb{1}_{(-\infty,c_n^{-2}]}(-\Delta)\colon (x,y) \mapsto \frac{\mathrm{J}_{n/2}(c_n^{-1}|x-y|)}{\sqrt{\omega_n}|x-y|^{n/2}}, \tag{1.6}$$

where $(\mathrm{J}_\nu)_{\nu>0}$ are Bessel functions of the first kind, cf. (A.2). The normalisation is such that the determinantal point process with kernel K_{bulk} is translation and rotation invariant on $\mathbb{R}^n$ with density 1.

At the boundary of the droplet, the density of states ϱ vanishes. Assuming that $\vec{\gamma} := \nabla V(x_0) \neq 0$, if we zoom at scale ϵ and apply an orthogonal matrix $\mathcal{U}$, we obtain the approximation

$$-\epsilon^{-2}\hbar^2\Delta + V(x_0+\epsilon\mathcal{U}^*x) - \mu \simeq \epsilon^{-2}\hbar^2(-\Delta + x_1)$$

provided

$$\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}, \quad \mathcal{U}V(x_0) = \|\nabla V(x_0)\|e_1.$$

Hence, at the edge the typical inter-particle distance is much larger than in the bulk and given by $\hbar^{2/3}|\nabla V(x_0)|^{-1/3}$. Then, up to the rotation $\mathcal{U}$, the natural candidate for the scaling limit is now

$$\begin{aligned} K_{\text{edge}} &= \mathbb{1}_{(-\infty,0]}(-\Delta + x_1)\colon \\ (x,y) &\mapsto \int_0^{+\infty} \mathrm{Ai}(x_1+s)\,\mathrm{Ai}(y_1+s)\frac{\mathrm{J}_{(n-1)/2}(\sqrt{s}|x^\perp - y^\perp|)}{(2\pi|x^\perp - y^\perp|)^{(n-1)/2}}s^{(n-1)/2}\mathrm{d}s, \end{aligned} \tag{1.7}$$

where Ai denotes the standard *Airy function* (cf. Appendix A.5), and if $n \geq 2$, we decompose $x = (x_1, x^\perp) \in \mathbb{R}\times\mathbb{R}^{n-1}$ and similarly for y.

The bulk and edge point processes from Theorem II.3 are the natural generalisations of the Sine and Airy point processes in higher dimension $n \geq 2$. The relevance of the determinantal point processes associated with K_{bulk} has first been recognised in [75, 90] under the name "Fermi-sphere" processes. The edge processes associated with K_{edge} were first obtained in [26, 27], and the kernel asymptotics were proven in the case of the multi-dimensional harmonic oscillator in [47].

1.1.4. Gaussian field at mesoscopic scales. One can interpolate between Theorems I.1, I.2 and I.3, on the one hand, and Theorem II.3, on the other hand, by considering mesoscopic scales, that is, the behaviour of the rescaled point process $T^*_{\epsilon,x_0}\mathrm{X}$ for $\hbar \ll \epsilon \ll 1$. In this case, we obtain convergence of the linear statistics to a Gaussian field with $H^{1/2}$ covariance.

Theorem III.1. *Suppose that $n \geq 2$, let (μ, V) satisfy (H) and let $x_0 \in \{V < \mu\}$. We consider a mesoscopic scale $\epsilon\colon [0,1] \to [0,1]$ such that $\hbar^{1-\beta} \leq \epsilon(\hbar) \leq \hbar^{\beta}$ for some $\beta \in (0,1)$. Let*

$$\delta(\hbar) = \frac{\hbar}{\epsilon(\hbar)} \frac{1}{\sqrt{\mu - V(x_0)}} \quad \textit{and} \quad \sigma_n^2 = \frac{\omega_{n-1}}{(2\pi)^n}.$$

We define the random distribution

$$\mathrm{X}_{\hbar,\epsilon} := \sigma_n^{-1} \delta(\hbar)^{(n-1)/2} (T_{x_0,\epsilon}^* \mathrm{X} - \mathbb{E}[T_{x_0,\epsilon}^* \mathrm{X}]).$$

Then for any $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$, we have

$$\lim_{\hbar \to 0} \mathbb{E}[e^{\mathrm{X}_{\hbar,\epsilon}(g)}] = e^{-\Sigma^2(g)/2}, \quad \textit{where } \Sigma^2(g) = \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 |\xi| \mathrm{d}\xi.$$

In particular, $\mathrm{X}_{\hbar,\epsilon}$ converges in the sense of finite-dimensional distributions as $\hbar \to 0$ to a (centred) Gaussian field G *on $\mathbb{R}^n$ with correlation kernel*

$$\mathbb{E}\mathrm{G}(f)\mathrm{G}(g) = \int \widehat{f}(\xi) \overline{\widehat{g}(\xi)} |\xi| \mathrm{d}\xi, \quad f, g \in H^{1/2}(\mathbb{R}^n).$$

The variance Σ^2 is the square of the $H^{1/2}$ seminorm, and it satisfies the following scaling invariance property:

$$\Sigma^2(T_{\epsilon,x_0} f) = \epsilon^{(1-n)/2} \Sigma^2(f),$$

where $T_{\epsilon,x_0} f = f(x_0 + \epsilon \mathcal{U} \cdot)$ for $x_0 \in \mathbb{R}^n$, $\epsilon > 0$, $\mathcal{U} \in \mathrm{SO}_n$ and $f \in H^{1/2}(\mathbb{R}^n)$. In particular, the Gaussian field G has the following invariance property: $\epsilon^{(n-1)/2} T_{\epsilon,x_0}^* G \overset{\text{law}}{=} G$.

The proof of Theorem III.1 relies on Corollary A.14 which is valid for general determinantal processes, and it boils down again to obtaining the asymptotics of the variance

$$\operatorname{var} \mathrm{X}_{\hbar,\epsilon}(f) = -\frac{1}{2} \sigma_n^{-2} \delta(\hbar)^{n-1} \operatorname{tr}([K_{x_0,\epsilon}, f]^2).$$

These asymptotics are involved and cannot be directly deduced from Theorem II.1. In the same way, the upper bound on the macroscopic variance leading to Theorem I.2. However, all our results are derived using the same method based on a (semiclassical) expression for the quantum propagator $e^{\mathrm{i}tH_\hbar/\hbar}$.

Again, the situation in dimension 1 is special as $\operatorname{var} T_{\epsilon,x_0}^* \mathrm{X}(f)$ is bounded for a smooth function f. Nonetheless, by analogy with the known case of the harmonic oscillator $V\colon x \mapsto x^2$ (cf. [64]), we expect that Theorem III.1 holds in full generality.

Remark 1.3. In the formulation of Theorem III.1, we have exactly centred the random variable $\mathrm{X}_{\hbar,\eta}$, but its expectation $\mathbb{E}[T_{\epsilon,x_0}^* \mathrm{X}]$ admits a simple equivalent. Indeed, if $\epsilon_0 = \frac{c_n \hbar}{\sqrt{\mu - V(x_0)}}$ with c_n as in (1.6) denotes the microscopic scale at x_0, using (1.4) with $\mathcal{U} = \mathrm{I}$, we can rewrite for $f \in C_c(\mathbb{R}^n, \mathbb{R})$,

$$\mathbb{E}[T_{\epsilon,x_0}^* \mathrm{X}] = c_n^{-n} \delta(\hbar)^{-n} \int f(x) K_{\epsilon_0, x_0 + \epsilon x}(0,0) \mathrm{d}x.$$

As a consequence of the uniformity of Theorem II.1 with respect to x_0, we obtain

$$\mathbb{E}[T^*_{\epsilon,x_0}\mathrm{X}] = c_n^{-n}\delta(\hbar)^{-n}\int f(x)\mathrm{d}x + \mathcal{O}(\eta^{-n}\hbar).$$

In general, we can only replace $\mathbb{E}[T^*_{\epsilon,x_0}\mathrm{X}]$ by its leading contribution in Theorem III.1 when the scale is small enough, that is, if $\epsilon(\hbar) \ll \hbar^{(n-1)/(n+1)}$.

1.2. Discussion and related works

1.2.1. Hermitian random matrices. The free fermions point processes studied in this article are basic examples of quantum gases which are exactly solvable due to their determinantal structure.[2] In fact, they were introduced by Macchi [71] as the first instances of determinantal point processes. So far, there has been only little rigorous progress on the statistical properties of free fermions, with the exception of the one-dimensional harmonic oscillator which corresponds to the eigenvalues of the Gaussian unitary ensemble [85, Section 2]. The GUE is a central model in random matrix theory which has been extensively studied and Theorems I.1, II.1, II.2, II.3 and III.1 are well known in this context, e.g., [40]. In particular, in dimension $n = 1$, the microscopic limits K_{bulk} and K_{edge} from (1.6) and (1.7) are the celebrated Sine and Airy kernels

$$K_{\mathrm{bulk}}(x,y) = \frac{\sin(\pi|x-y|)}{\pi|x-y|},$$
$$K_{\mathrm{edge}}(x,y) = \int_0^{+\infty} \mathrm{Ai}(x+s)\,\mathrm{Ai}(y+s)\mathrm{d}s = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}.$$

The Sine and Airy point processes arise in a wide range of other contexts (as originally surmised by Wigner) and have been extensively studied. In particular, they exhibit well-known integrable structures, related, for instance, to the Tracy–Widom distribution and the Kardar–Parisi–Zhang equation, e.g., [25].

One-dimensional free fermions and the eigenvalues of Hermitian random matrices fall in the same universality class and there is a substantial body of works on the fluctuations of random matrices. The closest context being that of unitary-invariant ensembles which are also determinantal processes whose correlation kernels are expressed in terms of orthogonal polynomials [31]. Universal asymptotics for these *Christoffel–Darboux* kernels are known under very general assumptions and can be obtained by several different methods; cf. the surveys [35, 69]. In fact, there are exact mappings between the ground state of one-dimensional free fermions trapped by specific potentials with $\hbar = N^{-1}$ and the three classical unitary-invariant ensembles. The following table collects the weight $\prod_{j=1}^N w_N(\lambda_j)$ of the random matrix ensemble together with the corresponding change of variables and Schrödinger eigensystem.

[2] In a physical context, this *integrable structure* of the N-particle density function $\mathbb{P}_N$ arises from applying Wick's theorem; see e.g., [70].

Hermite	
$w_N(\lambda)$	$e^{-\lambda^2}$
mapping	$x_j = \lambda_j N^{-1/2}$
eigensystem	$\left(-\frac{1}{N^2}\Delta + x^2\right) H_{N,n}(x) = 2\frac{n}{N} H_{N,n}(x)$
Laguerre	
$w_N(\lambda)$	$\lambda^{N\alpha} e^{-\lambda} \mathbb{1}_{\lambda>0}, \alpha \geq 0$
mapping	$x_j = \sqrt{\lambda_j N^{-1}}, x_j \in \mathbb{R}_+$
eigensystem	$\left(-\frac{1}{N^2}\Delta + x^2 + \left(\alpha^2 - \frac{1}{4N^2}\right)x^{-2}\right) L_{N,n}(x) = 2\left(\frac{2n+1}{N} + \alpha\right) L_{N,n}(x)$
Jacobi	
$w_N(\lambda)$	$(1-\lambda)^{N\alpha}(1+\lambda)^{N\beta} \mathbb{1}_{\lambda\in(-1,1)}, \alpha, \beta \geq 0$
mapping	$x_j = \arccos(\lambda_j), x_j \in [0, \pi]$
eigensystem	$\left(-\frac{1}{N^2}\Delta + \frac{(\alpha\cos\frac{x}{2})^2 + (\beta\sin\frac{x}{2})^2 - \frac{1}{4N^2}}{(\sin x)^2}\right) J_{N,n}(x) = \left(\frac{2n+1}{2N} + \alpha + \beta\right)^2 J_{N,n}(x)$

There is also a correspondence between free fermions on the unit circle and the circular unitary ensembles [24]. One can realise Gaussian/circular β-ensembles for any $\beta > 0$ by considering the ground state of certain interacting Fermi gases known as Calogero–Sutherland systems [39, 82] (these are the only cases where such exact mappings exist). This suggests that more generally, β-ensembles fall in the universality class of certain interacting one-dimensional Fermi gases.

Let us also mention that there is an extensive body of work on *rigidity* and CLT for eigenvalues of random matrices. For instance, the counterpart of Theorem III.1 in dimension 1 is well known for unitary-invariant ensembles ($\beta = 2$) and general β-ensembles [8, 64].

1.2.2. Free fermions on $\mathbb{R}^n$. Recently, there has been a significant activity in theoretical physics on the statistical properties of Fermi gases in general dimensions. Let us emphasise again the results from [26–28] which have been an inspiration for our work. By explicit asymptotic calculations based on short-time expansion of the quantum propagator (this approach is similar to ours, albeit non-rigorous), Dean et al. established universality of microscopic fluctuations for free Fermi gases in arbitrary dimension at the bulk and (regular) edge points. Then, [62,81] explore other *universal local behaviours* around a singularity of the external potential, respectively around an interior point where the density of state vanishes. We refer to the review [29] for applications of these results, some further perspectives on the connection with random matrices and a discussion of the positive temperature regime. The article [30] studies the influence of impurities (modelled by delta function potentials) inside a Fermi gas. It is worth mentioning [45] on large hole probabilities for Fermi gases confined by rotation-invariant potentials. Finally, there are also explicit results on number variance fluctuations, cf. [82] and the *perspectives* below.

1.2.3. Other universality classes for fermionic systems. In general dimension, a (classical) statistical mechanics model to compare our results is the Coulomb gas, also known as one-component plasma or Jellium. Coulomb gases arise in the description of several different physical phenomena, such as superconductivity, the (fractional) quantum Hall effect or the eigenvalues of random normal matrices [76]. The prototypical model in this class is the Ginibre ensemble which is also determinantal [3]. In fact, the (infinite) Ginibre point process can also be interpreted as the ground state of a (free) quantum system subject to a strong magnetic Laplacian which forces the particles to lie in the lowest Landau level [38, 65].

Such fermionic systems can be generalised on $\mathbb{C}^n$ or over an integrable compact Kähler manifold for any $n \geq 1$. In contrast to the Euclidean case considered here, the kernel of such determinantal point process associated with a magnetic Laplacian is gapped. Hence, they fall in a different universality class. Notably, these Bargmann kernels (linked with Berezin–Toeplitz quantisation associated with the Kähler structure) have semiclassical expansion up to $\mathcal{O}(\hbar^\infty)$ which renders asymptotics possible using methods from perturbation theory. This has been used to study scaling limits in the bulk [9, 10] and at the edge in the orthogonal polynomial case [48]. From a probabilistic perspective, another significant difference is that these Ginibre-like processes exhibit exponential decay of (spatial) correlations. Concretely, to compare with the Ginibre kernel $|K_{\mathrm{Gin}}(x-y)| = e^{-|x-y|^2/2}$, the universal limit

$$|K_{\mathrm{bulk}}(x,y)|^2 = \frac{\mathrm{J}_1^2(2\sqrt{\pi}|x-y|)}{\pi|x-y|^2}$$

decays like $|x-y|^{-3}$ in dimension 2 as $|x-y| \to \infty$. For Coulomb gases, besides the long-range interactions, this (super)-exponential decay of correlations is a consequence of a *screening* phenomenon.

Besides in the determinantal case, the description of the thermodynamical limit of Coulomb gases on $\mathbb{R}^n$ for $n \geq 2$ is still a major open problem [76], though recent progress has been achieved [4, 66]. For two-dimensional Coulomb gases, mesoscopic and macroscopic fluctuations have been studied in [6, 7, 67]. In particular, a CLT for smooth linear statistics holds under general conditions and the limit can be described in terms of the Gaussian free field (H^1-noise, in contrast to the $H^{1/2}$-noise which arises from Theorem III.1). We also refer to [2, 73] for proofs in the Ginibre ensemble and normal matrix models and further on the relationship to the GFF. In dimension $n \geq 3$, there has been progress in the Hierarchical case [21, 43] and for the true model under a "no phase transition" assumption [77]. Hence, Theorems I.3 and III.1 are among the few CLTs valid for a class of correlated statistical mechanics models in arbitrary dimension. An analogous CLT was obtained in [5] for another family of determinantal processes on the n-dimensional hypercube, called multivariate orthogonal ensembles, and interesting applications to numerical quadrature are discussed.

1.2.4. Semiclassical projector asymptotics. In dimension 1, universal properties of free fermions associated with Schrödinger operators in the semiclassical limit are heuristically described in the textbook [88, Chapter 3.3], as well as in [34], based on Wentzel–Kramers–Brillouin (WKB) approximations. In higher dimensions, there is no exact mapping between free fermions and random matrix models, and furthermore the WKB method is less powerful (the individual eigenfunctions of $H_\hbar$ do not admit asymptotic formulas). We refer to [26, 27] for a physical derivation of, e.g., the edge scaling limit, based on a short-time expansion of the heat kernel associated with the Schrödinger operator: $s \mapsto \exp(-sH_\hbar)$. This approach can be made rigorous (using the Tauberian theorem of Hardy–Littlewood–Karamata), but yields non-sharp kernel asymptotics (only $O(\hbar^{1/2})$ in Theorem II.1). Instead, we rely on the description of the quantum propagator $t \mapsto e^{\mathrm{i}tH_\hbar/\hbar}$ (see Section 2.3), which is known to yield the sharpest forms of the Weyl law.

Semiclassical techniques devoted to the study of spectral projectors are mostly developed in the homogeneous case of the Laplace–Beltrami operator over a compact manifold and the Laplace operator on a domain of $\mathbb{R}^n$ with Neumann or Dirichlet boundary conditions; we refer to the recent review [55]. The cornerstone is to use a semiclassical Fourier transform to write a spectral function $f(H_\hbar)$ of the operator as

$$f(H_\hbar) = \frac{1}{\hbar}\int \exp\Big(\mathbf{i}\frac{t}{\hbar}H_\hbar\Big)\hat{f}\Big(\frac{t}{\hbar}\Big)\mathrm{d}t.$$

The operator $\exp(\mathbf{i}\frac{t}{\hbar}H_\hbar)$ has a physical interpretation: it captures the time evolution of a quantum system subject to $H_\hbar$, and admits an approximation as an integral operator for t in a (fixed) neighbourhood $[-\tau, \tau]$ of 0; cf. Proposition 2.11. Even if f is not smooth, one can hope that

$$f(H_\hbar) \approx \frac{1}{\hbar}\int \exp\Big(\mathbf{i}\frac{t}{\hbar}H_\hbar\Big)\hat{f}\Big(\frac{t}{\hbar}\Big)\mathbb{1}_{|t|\leq\nu}\mathrm{d}t,$$

then perform computations using the right-hand side. The last step consists in recovering $f(H_\hbar)$ from the asymptotics of this frequency cutoff, which involves *Fourier Tauberian theorems*.

In the context of the Laplace–Beltrami operator over a smooth compact manifold, the equivalent of Theorem II.1 is well known, at least on the diagonal $x = y$. A major part of the literature is devoted to the improvements on the $\mathcal{O}(\hbar)$ remainder, which depends on dynamical assumptions on the geodesic flow. Recent developments include the study of kernel asymptotics at mesoscopic scales around the diagonal, [17, 18, 61].

Using the Tauberian method, the Weyl law for semiclassical Schrödinger operator (1.3) was obtained in [50], with a remainder $\mathcal{O}(\hbar)$, under the assumption that $\nabla V \neq 0$ along the whole edge. Asymptotics on the diagonal in the bulk and at a non-degenerate edge (that is, Theorems II.1 and II.2 at $x = y = 0$) have been computed in [60] using the preliminary results of [22], under a growth condition of V at infinity. In this article's context, the study of the kernel on the diagonal translates to asymptotics for the expectation of linear statistics of the Fermi gas.

Quantities associated with the semiclassical spectral projector for harmonic oscillators were studied in detail in [46], with an $O(\hbar^{1/2})$ error term (see notably Proposition 1.8 therein). The upper bound on the variance which leads to Theorem I.2 (cf. (5.28)) is similar to the bound in trace norm obtained in [41], which also holds under hypothesis (H). Remarkably, many recent advances in the semiclassical literature about the off-diagonal asymptotics of spectral projectors [17, 18, 46, 47] are also motivated by probabilistic models (and notably the behaviour of the nodal sets of random linear combinations of eigenfunctions).

1.2.5. A few perspectives. Even within the scope of semiclassical Schrödinger operators, the asymptotic picture is far from complete. For instance, to complement our results, it is of interest to obtain a large deviation principle for the empirical measure and determine how the rate function depends on the potential. An analogous question can be raised about the asymptotics of the variance in Theorem I.3 in relation to Conjecture 1. In particular, it is relevant to investigate in which case the normalised variance converges to an $H^{1/2}$ norm or whether there can be oscillations. In dimension 1 where the variance of smooth linear statistics is bounded, the issue of a mesoscopic CLT is still open.

For fermions point processes, the equivalent of Theorem III.1 when f is the indicator of a smooth set ("How many particles are in this box"), is a topical question since the number variance is a common measure of the entanglement entropy of subsystems, e.g., [15, 16, 68]. The gapless case (such as semiclassical Schrödinger operators) is linked to a conjecture by Widom concerning pseudodifferential operators with rough symbols [83, 91], while the asymptotics of the variance of linear statistics performed in Section 5 relies on the high regularity of f, the case of an indicator function falls outside the scope of this article. This problem has been extensively studied recently in the context of random matrix theory based on so-called Fisher–Hartwig asymptotics and it relates optimal particles' rigidity and Gaussian multiplicative chaos, e.g., [23, 42, 82]

In a gapped system (integer quantum hall states linked with Berezin–Toeplitz quantisation), the so-called *area law* holds and macroscopic CLTs have been rigorously established in [20, 37].

Finally, one can also mention the "hard edge" model of a semiclassical Laplacian $-\hbar^2\Delta_\Omega$ with Dirichlet or Neumann boundary conditions on a relatively compact open set Ω. The behaviour of the point process near the boundary of Ω is presumed to be very different from the "soft edge" case of Theorem II.2. A complete description of the off-diagonal kernel at the relevant scale is an ongoing problem, see for instance the recent upper bounds on the size of the off-diagonal kernel [56].

1.3. Organisation and notations

In Section 2, we describe the basic semiclassical techniques and objects that are used throughout this article. We begin with *Agmon estimates*, which allow us to prove exponential decay for eigenfunctions of $H_\hbar = -\hbar^2\Delta + V$ in the "forbidden region" where the value of V is greater than the eigenvalue. Then, we introduce (semiclassical) pseudodifferential operators and Fourier integral operators, which are used to obtain asymptotics for the integral kernel of spectral functions of $H_\hbar$.

Section 3 is devoted to the proof of Theorem II.1 and its corollaries (Theorems I.1 and I.2, and the first part of Theorem II.3). Theorem II.1 is proved using the stationary phase method and Tauberian techniques. Its probabilistic consequences make use of fine properties of determinantal point processes.

In Section 4, we modify the previous arguments to study the edge of the droplet, proving Theorem II.2 and the second part of Theorem II.3. The main difference is the treatment of rapidly oscillating integrals, whose *phase* is degenerate compared to Section 3.

Finally, we study mesoscopic fluctuations in Section 5 by proving Theorems I.3 and III.1. Again, we carefully study the oscillating integrals cut-off at short time, to obtain asymptotics for the variance of the (rescaled) point process; then we use concentration inequalities and properties of determinantal point processes (notably the fact that the variance diverges in dimensions $n \geq 2$) to conclude the proof of the CLT.

For completeness, we review in Appendix A.1 several notations and basic facts which are instrumental to prove our results. We also derive several general estimates on determinantal point processes and oscillatory integrals, which we believe may be of independent interest, such as Proposition A.11, Corollary A.12, and Propositions A.18, A.19.

- Section A.1 presents our Hilbert space setup and various operator topologies that we use.
- The usual Laplacian Δ on $L^2(\mathbb{R}^n)$ plays a crucial role in our analysis. Section A.2 reviews the basic spectral theory for Δ, based on the Fourier transform, as well as standard elliptic regularity estimates.
- Section A.3 provides an introduction to the theory of determinantal point process. We give a general definition (independent of the concept of *correlation kernel*) and review the notion of weak convergence used in Theorem II.3. We also revisit a classical CLT of Soshnikov [86] by obtaining new bounds for the Laplace functional of a general determinantal point process which might be of independent interest (cf. Proposition A.11).
- Section A.4 gives several versions of the stationary phase lemma, which are tuned to obtain the asymptotics of oscillatory integrals which arise in our proofs. Depending on the regularity of the integrand, we obtain different estimates for the remainder which might be of independent interest.
- Section A.5 reviews basic facts about the Airy function including its integral representation, asymptotics and its relationship to the spectral theory of operator (1.7).

In the rest of this article, we use the following conventions:

- We denote by $x \cdot \xi$ the Euclidean inner product for $x, \xi \in \mathbb{R}^n$ and $|x| = \sqrt{x \cdot x}$ for $x \in \mathbb{R}^n$.
- We write $A \Subset \mathbb{R}^n$ to denote a compact subset A of $\mathbb{R}^n$.
- We denote by $B^n_{x,r} = \{z \in \mathbb{R}^n : |x - z| \leq r\}$ the Euclidean ball of radius r around x in $\mathbb{R}^n$ and $\omega_n = |B^n_{0,1}|$.

- We denote by $\mathrm{d}x$ the Lebesgue measure on $\mathbb{R}^n$ and by $\langle \phi, \varphi \rangle = \int \phi(x)\varphi(x)\mathrm{d}x$ the usual inner product on $L^2(\mathbb{R}^n)$. We denote $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$.
- For $\phi \in \mathcal{S}(\mathbb{R}^n)$, we define its Fourier transform by

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}\xi\cdot x}\phi(x)\frac{\mathrm{d}x}{(2\pi)^{n/2}}, \quad \xi \in \mathbb{R}^n.$$

  With this convention, the Fourier transform extends to a unitary operator $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ and we write $\widehat{\phi} = \mathcal{F}\phi$ for $\phi \in L^2(\mathbb{R}^n)$, as well as $\phi \in \mathcal{S}'$, where $\mathcal{S}'$ denotes the space of Schwartz distributions, dual to the space of Schwartz functions $\mathcal{S}$.
- Given $V\colon \mathbb{R}^n \to \mathbb{R}$ satisfying (H), $\hbar \in (0,1]$ and $\mu \in \mathbb{R}$, we denote

$$H_\hbar = -\hbar^2\Delta + V \quad \text{and} \quad \Pi_{\hbar,\mu} = \mathbb{1}_{(-\infty,\mu]}(H_\hbar).$$

  Without loss of generality, we can always assume that $V \geq 0$ and $\mu \in (0, M)$.
- If an operator A acting on $L^2(\mathbb{R}^n)$ admits an integral kernel, we also denote by A its integral kernel. Given $f\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{U} \in \mathrm{SO}_n$, $\epsilon \in (0,1]$ and $x_0 \in \{V < M\}$, provided that $f(H_\hbar)$ admits an integral kernel, we write

$$K^f_{x_0,\epsilon}\colon (x,y) \mapsto \epsilon^n f(H_\hbar)(x_0 + \epsilon\mathcal{U}^*x, x_0 + \epsilon\mathcal{U}^*y). \tag{1.8}$$

  In particular, according to (1.4), we simply have $K_{x_0,\epsilon} = K^{\mathbb{1}[0,\mu]}_{x_0,\epsilon}$. We also let

$$K^f_{x_0,0} = f(-\Delta + V(x_0)).$$

- Given a multi-index $\alpha \in \mathbb{N}_0^n$ and $f \in S(\mathbb{R}^n)$, we denote $\partial_x^\alpha f = \partial_{x_n}^{\alpha_n} \cdots \partial_{x_1}^{\alpha_1} f$ and define the norms for $k \in \mathbb{N}_0$,

$$\|f\|_{C^k(\Omega)} = \sup_{x\in\Omega} \sum_{\alpha:|\alpha|\leq k} |\partial_x^\alpha f(x)|,$$

  where $\Omega \subset \mathbb{R}^n$ is any open set and the sum is over all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$. The definition of this norm extends to general functions. We also set $\|\cdot\|_{C^k} = \|\cdot\|_{C^k(\mathbb{R}^n)}$.
- We use the notation $\partial_x = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$.
- $C, c > 0$ are constants which (vary from line to line) only depend on the dimension $n \in \mathbb{N}$ and the potential V.
- $C_\alpha, c_\alpha > 0$ are constants which depend only on $n \in \mathbb{N}$, V, the parameter α and vary from line to line.
- If an element A of a Banach space depends on a parameter $\eta > 0$, we write for $k \in \mathbb{N}_0$, $A = \mathcal{O}(\eta^k)$ if there exists a constant C_k such that $\|A\| \leq C_k\eta^k$ as $\eta \to 0$. We further write $A = \mathcal{O}(\eta^\infty)$ when $A = \mathcal{O}(\eta^k)$ for all $k \in \mathbb{N}_0$. In the cases where there is ambiguity about the used norm, we indicate it as a subscript of $\mathcal{O}$.
- We denote the commutator of two operators A, B on $L^2(\mathbb{R}^n)$ by $[A, B] = AB - BA$.

2. Semiclassical techniques for Schrödinger operators

In this section, we cover basic techniques from semiclassical analysis tailored to the study of Schrödinger operators of the form $H_\hbar = -\hbar^2\Delta + V$, in the semiclassical limit $\hbar \to 0$. Some of these techniques rely on stronger assumptions than (H). The following ancillary hypothesis will be useful in this section.

From now on, let M be as in (H) and we assume (without loss of generality) that $V \geq 0$.

Definition 2.1. We say that a potential $V \in C^\infty(\mathbb{R}^n, \mathbb{R}_+)$ satisfies (H′) if

(H′) for every $k \geq 0$, there exists a constant $C_k > 0$ such that, for all $x \in \mathbb{R}^n$ outside of a compact set, for every $|\alpha| = k$,

$$|\partial^\alpha V(x)| \leq C_k |x|^2,$$

and if there exists $c > 0$ such that, outside of a compact set,

$$c|x|^2 \leq V(x).$$

This section is devoted to the foundational techniques and results that allow us to study the spectrum of $H_\hbar$ as $\hbar \to 0$. Section 2.1 is relatively elementary: we use operator bracketing and *Agmon estimates* to prove that, when (μ, V) satisfies (H), the spectrum of $H_\hbar$ below μ consists of a finite number of eigenvalues, and that the eigenfunctions tend very quickly to zero in the forbidden region $\{V > \mu\}$. This is used in Section 2.2 to prove a *replacement principle*; we can change V on $\{V > M\}$ so that it satisfies (H′). Then, in Section 2.3, we introduce semiclassical pseudodifferential operators and Fourier integral operators. They are a priori suited to the stronger assumption (H′), but thanks to the replacement principle, we can relax the conditions on V in the forbidden region. Then, under (H), Proposition 2.12 expresses rapidly oscillating functions of $H_\hbar$ of the form

$$e^{\mathrm{i}tH_\hbar/\hbar}\vartheta(H_\hbar), \quad t \in [-\tau, \tau],$$

for a (fixed) small $\tau > 0$ and $\vartheta \in C_c^\infty(\mathbb{R})$ supported in $(-\infty, M)$, as an integral operator up to a small remainder in C^∞-kernel topology. This will be our main tool to prove the results of Section 1.1.

2.1. Spectral theory and Agmon estimates

Lemma 2.2. *Let (μ, V) satisfy (H). There exist $M \geq \mu$ and $C_M > 0$ such that for all $\hbar \in (0, 1]$,*

$$\mathrm{Rank}(\mathbb{1}_{H_\hbar \leq M}) \leq C_M \hbar^{-n}.$$

Hence, $H_\hbar$ has discrete spectrum on $[-\infty, M]$ consisting of finite rank eigenvalues.

Moreover, if V satisfies (H′), there exists $C > 0$ such that for all $\hbar \in (0, 1]$ and $\mu > 0$,

$$\mathrm{Rank}(\mathbb{1}_{H_\hbar \leq \mu}) \leq C\hbar^{-n}\mu^n.$$

Proof. We proceed as in [72, Theorem XIII.81]. We start with an upper bound for the Neumann Laplacian $\Delta_{\mathcal{N}}$ on a hypercube of size 1:

$$\mathrm{Rank}(\mathbb{1}_{-\Delta_{\mathcal{N}}\leq\Lambda}) \leq C_n(\Lambda+1)^{n/2}$$

valid for $\Lambda \geq 0$; if $\Lambda < 0$, the left-hand side is zero.

Now, for each $j \in \mathbb{Z}^n$, let V_j denote the infimum of V on the hypercube Ω_j centred at j with size 1, and let $\Delta_{\mathcal{N},j}$ denote the Neumann Laplacian on Ω_j. Then, writing

$$L^2(\mathbb{R}^n) = \overline{\bigoplus_{j\in\mathbb{Z}^d} L^2(\Omega_j)}^{L^2},$$

the following inequality holds in the sense of quadratic forms:

$$H_\hbar = -\hbar^2\Delta + V \geq \bigoplus_{j\in\mathbb{Z}^d}(-\hbar^2\Delta_{\mathcal{N},j} + V_j).$$

By (H), there is only a finite number of indices j such that $V_j \leq M$. Hence, the previous bound implies that

$$\mathrm{Rank}(\mathbb{1}_{-\hbar^2\Delta+V\leq M}) \leq C_M\,\mathrm{Rank}(\mathbb{1}_{-\Delta_{\mathcal{N}}\leq\hbar^{-2}M}),$$

which shows that

$$\mathrm{Rank}(\mathbb{1}_{-\hbar^2\Delta+V\leq M}) \leq C\hbar^{-n}.$$

If moreover V satisfies (H′), then the number of indices j such that $V_j \leq \mu$ grows like $\mu^{n/2}$, and $\mathrm{Rank}(\mathbb{1}_{-\Delta_{\mathcal{N}}\leq\hbar^{-2}\mu}) \leq C\hbar^{-n}\mu^{n/2}$ for a universal constant $C > 0$. This concludes the proof. ∎

According to Lemma 2.2, the spectrum of $-\hbar^2\Delta + V$ below M consists of a finite number of finite rank eigenvalues. Even though V is not regular on $\{V > M\}$, there are robust tools to study the decay properties of any eigenfunction with eigenvalue less than M on this region.

Proposition 2.3. *Let (μ, V) satisfy (H), let $\delta > 0$ be such that $(\mu+\delta, V)$ satisfies (H), and define*

$$\mathrm{f}_\delta\colon \mathbb{R}^n \ni x \mapsto \delta\,\mathrm{dist}(x, \{V \leq \mu+\delta\}).$$

Let v be a normalised eigenfunction of $H_\hbar = -\hbar^2\Delta + V$ with eigenvalue $\leq \mu$. Then

$$\|e^{\mathrm{f}_\delta/\hbar}v\|_{L^2} \leq 1 + \frac{2\mu}{\delta}.$$

Proof. The proof is based on the celebrated Agmon estimates [1]. Let $R > 0$ and let

$$\mathrm{f}_\delta^R\colon x \mapsto \min(\mathrm{f}_\delta(x), R).$$

First notice that f_δ^R is bounded, Lipschitz, and such that $|\nabla \mathrm{f}_\delta^R| = \delta\mathbb{1}_{\Omega_R}$ a.e., where

$$\Omega_R = \left\{x \in \mathbb{R}^n,\ \mathrm{dist}(x, \{V \leq \mu+\delta\}) \leq \frac{R}{\delta}\right\}.$$

Integrating by parts, for any $u \in H^1(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\langle u, e^{\mathrm{f}_\delta^R/\hbar} H_\hbar e^{-\mathrm{f}_\delta^R/\hbar} u\rangle &= \langle u H_\hbar u\rangle - \|u \nabla \mathrm{f}_\delta^R\|_{L^2}^2 \\ &= \langle u, (H_\hbar - \delta^2 \mathbb{1}_{\Omega_R}) u\rangle.\end{aligned}$$

Take $u = v e^{\mathrm{f}_\delta^R/\hbar}$, where v is a normalised eigenfunction of $H_\hbar$ with eigenvalue $\lambda \leq \mu$. In this case, both $v, u \in H^1(\mathbb{R}^n)$, and the last equation reads

$$\lambda \int e^{2\mathrm{f}_\delta^R/\hbar} |v|^2 = \hbar^2 \int |\nabla u|^2 + \int e^{2\mathrm{f}_\delta^R/\hbar} (V - \delta^2 \mathbb{1}_{\Omega_R}) |v|^2.$$

In particular, if $\lambda \leq \mu$, the eigenfunction v satisfies

$$\int e^{2\mathrm{f}_\delta^R/\hbar} (V - \delta^2 - \mu) |v|^2 \leq 0.$$

We decompose this integral above into two parts: $\{V \leq \mu + \delta\}$, where $f_\delta^R = 0$, and its complement set where $V - \delta^2 - \mu \geq \frac{\delta}{2}$. We obtain

$$\int_{\{V > \mu + \delta\}} e^{2\mathrm{f}_\delta^R/\hbar} |v|^2 \leq \frac{2}{\delta} \int_{\{V \leq \mu + \delta\}} (\mu + \delta^2 - V) |v|^2 \leq \frac{2\mu}{\delta},$$

where we used that $V \geq 1$ and $\int_{\{V \leq \mu + \delta\}} |v|^2 \leq 1$. This bound does not depend on R, so that by monotone convergence, we conclude that

$$\int_{\{V > \mu + \delta\}} e^{2\mathrm{f}_\delta/\hbar} |v|^2 \leq 2\frac{\mu}{\delta}.$$

Using again $f_\delta^R = 0$ on $\{V \leq \mu + \delta\}$ and $\mathrm{f}_\delta = 0$ on $\{V \leq \mu + \delta\}$. ∎

This implies, in particular, the uniform decay for the integral kernel for (compactly supported) spectral functions of $H_\hbar$.

2.2. *Replacement principle*

Hypothesis (H′) is much more suited to semiclassical analysis. One can *replace*, to some extent, a potential satisfying (H) with one satisfying (H′), as follows.

Definition 2.4. Let (μ, V_1) satisfy (H), let V_2 satisfy (H′). Let $M > \mu$, and suppose that $V_2 = V_1$ on $\{V_1 \leq M\}$ and $\{V_2 > M\} = \{V_1 > M\}$. Then, we say that V_2 replaces V_1 up to M.

Proposition 2.5. *Let V_2 replace V_1 up to M and recall that the trace-norm $\|\cdot\|_{\mathrm{J}^1}$ is defined in Proposition A.1. Let us denote $H_{j;\hbar} = -\hbar^2\Delta + V_j$ for $j \in \{1, 2\}$. Then for any $f \in C_c^\infty(\mathbb{R})$ with support in $[0, M)$, we have as $\hbar \to 0$,*

$$f(H_{1;\hbar}) = f(H_{2;\hbar}) + \mathcal{O}_{\mathrm{J}^1}(\hbar^\infty).$$

Proof. It suffices to prove that there are constants $C, c > 0$ such that

$$\|f(H_{1;\hbar}) - f(H_{2;\hbar})\|_{L^2\to L^2} \leq Ce^{-c/\hbar} \tag{2.1}$$

since by Lemma 2.2, both $f(H_{j;\hbar})$ have rank $\mathcal{O}(\hbar^{-n})$. In addition, it suffices to prove the claim in the case where f takes non-negative values (by linearity).

We rely again on an Agmon estimate. Let $\epsilon > 0$ be small and $\phi \in C^\infty(\mathbb{R}^n, [0,1])$ be a cutoff with $\phi = 0$ on $\{V_j \leq M - \frac{\epsilon}{2}\}$ and $\phi = 2c$ on $\{V_j \geq M - \frac{\epsilon}{4}\}$, where $c = c_\epsilon > 0$ is small enough such that $|\nabla\phi|^2 \leq \frac{\epsilon}{4}$. Then, proceeding exactly as in the proof of Proposition 2.3, for any normalised eigenfunction v of $H_{j;\hbar} = -\hbar^2\Delta + V_j$ (for either $j \in \{1,2\}$) with eigenvalue λ,

$$\int e^{2\phi/\hbar}(V_j - |\nabla\phi|^2 - \lambda)|v|^2 \leq 0.$$

Let us split this integral into two parts: on $\{V_j \leq M - \frac{\epsilon}{2}\}$, one has $e^{2\phi/\hbar} = 1$ and $|\nabla\phi| = 0$; on $\{V_j \geq M - \frac{\epsilon}{2}\}$, one has $V_j - |\nabla\phi|^2 - \lambda \geq \frac{\epsilon}{4}$ provided that $\lambda \leq M - \epsilon$ and $\phi = 2c$. We obtain

$$e^{4c/\hbar}\int_{\{V_j\geq M-\epsilon/4\}}|v|^2 \leq \int_{\{V_j\geq M-\epsilon/2\}} e^{2\phi/\hbar}|v|^2 \leq \frac{4}{\epsilon}\int_{\{V_j<M-\epsilon/2\}}(\lambda - V_j)|v|^2 \leq \frac{4M}{\epsilon}.$$

This yields the following uniform control: for every $\hbar \in (0,1]$, $j \in \{1,2\}$ and every normalised eigenfunction v of either $H_{j;\hbar}$ with eigenvalue $\leq M - \epsilon$,

$$\int_{\{V_j\geq M-\epsilon/4\}}|v|^2 \leq Ce^{-4c/\hbar}.$$

Moreover, using the eigenvalue equation, one has

$$\begin{aligned}\int_{\{M-\epsilon/4\leq V_j\leq M\}}|\Delta v|^2 &\leq 2\frac{\lambda^2}{\hbar^4}\int_{\{M-\epsilon/4\leq V_j\leq M\}}|v|^2 + 2\int_{\{M-\epsilon/4\leq V_j\leq M\}}V_j^2|v|^2\\ &\leq Ce^{-3c/\hbar}.\end{aligned}$$

Let $\chi \in C^\infty(\mathbb{R}^n, [0,1])$ be equal to 1 on $\{V \leq M - \frac{\epsilon}{4}\}$ and to 0 on $\{V \geq M\}$. We claim that if v is a normalised eigenfunction of either $H_{j;\hbar}$, then χv is an almost eigenfunction of both $H_{j;\hbar}$ for $j \in \{1,2\}$ in the sense that

$$\|H_{j;\hbar}(\chi v) - \lambda\chi v\|_{L^2} \leq Ce^{-2c/\hbar}. \tag{2.2}$$

First observe that

$$H_{j;\hbar}(\chi v) = \lambda\chi v + \hbar^2(v\Delta\chi - 2\nabla\chi\cdot\nabla v)$$

for $j \in \{1,2\}$ since $V_1 = V_2$ on $\mathrm{supp}(\chi)$ so that

$$\|H_{j;\hbar}(\chi v) - \lambda\chi v\|_{L^2} \leq C\hbar^2\Big(\int_{\{V\geq M-\epsilon/4\}}|v|^2 + \int|\nabla\chi\cdot\nabla v|^2\Big).$$

Second, by an integration by parts and then by the Cauchy–Schwarz inequality,

$$\begin{aligned}\Big(\int |\nabla\chi\cdot\nabla v|^2\Big)^2 &\le 2\Big(2\int v\Delta\chi(\nabla\chi\cdot\nabla v)\Big)^2+2\Big(\int v\Delta v|\nabla\chi|^2\Big)^2\\ &\le C\Big(\int_{\{V\ge M-\epsilon/4\}}|v|^2\Big)\Big(\int|\nabla\chi\cdot\nabla v|^2+\int_{\{M-\epsilon/4\le V\le M\}}|\Delta v|^2\Big)\\ &\le Ce^{-4c/\hbar}\Big(\int|\nabla\chi\cdot\nabla v|^2+Ce^{-3c/\hbar}\Big).\end{aligned}$$

This computation shows that $\|H_{j;\hbar}(\chi v)-\lambda\chi v\|_{L^2}\le C\hbar^2e^{-3c/\hbar}$, which proves (2.2). Using the resolvent identity and the fact that $\|(H_{j;\hbar}-z)^{-1}\|\le\frac{1}{|\mathrm{Im}(z)|}$, this bound implies that for all $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\|((H_{1;\hbar}-z)^{-1}-(H_{2;\hbar}-z)^{-1})\chi v\|_{L^2}\le\frac{2C}{|\mathrm{Im}(z)|^2}e^{-2c/\hbar}.$$

Since in addition

$$\|(H_{j;\hbar}-z)^{-1}(1-\chi)v\|_{L^2}\le\frac{1}{|\mathrm{Im}(z)|}\Big(\int_{\{V_j\ge M-\epsilon/4\}}|v|^2\Big)^{1/2}\le\frac{C}{|\mathrm{Im}(z)|}e^{-2c/\hbar},$$

we conclude that for any normalised eigenfunction v of either $H_{j;\hbar}$ with eigenvalue $\le M-\epsilon$ and $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\|((H_{1;\hbar}-z)^{-1}-(H_{2;\hbar}-z)^{-1})v\|_{L^2}\le\frac{C}{|\mathrm{Im}(z)|^2}e^{-2c/\hbar}. \tag{2.3}$$

Now, if $f\in C_c^\infty(\mathbb{R},\mathbb{R}_+)$ with support in $[0,M)$, using the spectral resolution

$$f(H_{j;\hbar})\colon (x,y)\mapsto\sum_{\lambda\in\sigma(H_{j;\hbar})}f(\lambda)v_\lambda(x)v_\lambda(y),$$

where (λ,v_λ) are normalised eigenpairs of $H_{j;\hbar}$, we verify that

$$\begin{aligned}&\|((H_{1;\hbar}-z)^{-1}-(H_{2;\hbar}-z)^{-1})f(H_{j;\hbar})\|_{L^2\to L^2}\\ &\quad\le\max_{\lambda\le M-\epsilon}\|((H_{1;\hbar}-z)^{-1}-(H_{2;\hbar}-z)^{-1})v_\lambda\|_{L^2}\sum_{\lambda\in\sigma(H_{j;\hbar})}f(\lambda),\end{aligned}$$

where we used that $\mathrm{supp}(f)\subset[0,M-\epsilon]$ for $\epsilon>0$ small enough. By Lemma 2.2 and formula (2.3), this implies that

$$\|((H_{1;\hbar}-z)^{-1}-(H_{2;\hbar}-z)^{-1})f(H_{j;\hbar})\|_{L^2\to L^2}\le\frac{C}{|\mathrm{Im}(z)|^2}e^{-c/\hbar}.$$

Using the Helffer–Sjöstrand formula [52], it follows that

$$\|(f(H_{1;\hbar})-f(H_{2;\hbar}))f(H_{j;\hbar})\|_{L^2\to L^2}\le Ce^{-c/\hbar}.$$

This shows that $\|f(H_{1;\hbar})^2-f(H_{2;\hbar})^2\|_{L^2\to L^2}\le 2Ce^{-c/\hbar}$ and it proves (2.1) (upon replacing f^2 by f). ∎

Let us mention that the conclusion of Proposition 2.5 breaks down without regularity hypotheses on f. In particular, in the case where $f = \mathbb{1}_{(-\infty,\mu]}$, which is the spectral function of interest, individual eigenfunctions of $H_{1;\hbar}$ are $O(e^{-c/\hbar})$-quasimodes of $H_{2;\hbar}$ but their energy might shift from just below μ to just above it. Hence, without further assumptions, the trace norm of $\mathbb{1}_{(-\infty,\mu]}(H_{1;\hbar}) - \mathbb{1}_{(-\infty,\mu]}(H_{2;\hbar})$ can be as large as $\hbar^{-n}$ and the operator norm can fail to tend to zero.

2.3. *Pseudodifferential operators*

If the potential V satisfies (H′), then the operator $H_\hbar = -\hbar^2\Delta + V$ is an example of a *Weyl pseudodifferential operator*. Informally speaking, $H_\hbar$ is the quantum equivalent of the classical energy

$$\mathbb{R}^{2n} \ni (x,\xi) \mapsto V(x) + |\xi|^2 \in \mathbb{R},$$

using the quantisation rule $\xi \leftrightarrow -\mathbf{i}\hbar\nabla$. We write

$$H_\hbar = \mathrm{Op}_\hbar((x,\xi) \mapsto V(x) + |\xi|^2).$$

Weyl pseudodifferential operators generalise differential operators with smooth coefficients. The advantage of this setting is that this class of operators is, up to a small error, preserved by smooth functional calculus: if $P_\hbar$ is a Weyl pseudodifferential operator and $f \in C^\infty(\mathbb{R})$ has good properties at infinity, then $f(P_\hbar)$ is also a Weyl pseudodifferential operator.

Pseudodifferential operators are associated with *symbols*, which are defined as follows.

Definition 2.6. Let $k,n \in \mathbb{N}$, $m \in \mathbb{Z}$. The Fréchet space $S^m(\mathbb{R}^k,\mathbb{R}^n)$ is defined as the set of smooth functions $a\colon \mathbb{R}^k \times \mathbb{R}^n \times (0,1] \to \mathbb{C}$ such that, for any multi-index $\alpha \in \mathbb{N}_0^{k+n}$, there exists $C_\alpha > 0$ such that, for all $(x,\xi,\hbar) \in \mathbb{R}^{k+n} \times (0,1]$, one has

$$|\partial^\alpha_{x,\xi} a(x,\xi;\hbar)| \le C_\alpha (1+|\xi|^2)^{m/2}(1+|x|^2)^{m/2}.$$

The optimal constants C_α are the seminorms on S^m.

An element $a \in S^m(\mathbb{R}^k \times \mathbb{R}^n)$ is said to be *elliptic* when there exists $c > 0$ such that, for all $(x,\xi,\hbar) \in \mathbb{R}^{k+n} \times (0,1]$,

$$|a(x,\xi;\hbar)| \ge c(1+|\xi|^2)^{m/2}(1+|x|^2)^{m/2}.$$

The symbol classes defined here are particular cases of the symbol classes considered in the textbook [33]. Indeed, the function

$$(x,\xi) \mapsto (1+|\xi|^2)^{m/2}(1+|x|^2)^{m/2}$$

is an order function as in [33, Definition 7.4].

A natural element of these symbol classes is the harmonic oscillator $x^2+\xi^2$ which belongs to S^2. Note that, contrary to *microlocal* symbol classes, the small parameter in our semiclassical techniques is $\hbar > 0$ and not $|\xi|^{-1}$; in particular, we do not impose that differentiating the symbol with respect to ξ improves its decay.

For instance, if V satisfies (H′), then $(x,\xi) \mapsto V(x) + |\xi|^2 \in S^2(\mathbb{R}^n, \mathbb{R}^n)$ is elliptic near infinity.

Definition 2.7. An element $a \in S^m(\mathbb{R}^k, \mathbb{R}^n)$ is called a *classical symbol* if there exists a sequence $(a_k)_{k\geq 0}$ of smooth functions in $\mathbb{R}^k \times \mathbb{R}^n$ such that for every $k \geq 0$, a_k is an $\hbar$-independent element of S^m and, for every $\ell \in \mathbb{N}_0$,

$$\hbar^{-\ell-1}\Big(a - \sum_{k\leq \ell} \hbar^k a_k\Big) \in S^m.$$

The principal symbol of a is then defined as a_0.

Definition 2.8. Let $a \in S^m(\mathbb{R}^n, \mathbb{R}^n)$. The Weyl quantisation $\mathrm{Op}_\hbar(a)$ of a is the family of integral operators with distribution-valued kernels

$$\mathbb{R}^{2n} \ni (x,y) \mapsto \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{\mathrm{i}}{\hbar}((x-y)\cdot\xi)} a\Big(\frac{x+y}{2}, \xi; \hbar\Big)\mathrm{d}\xi.$$

This is a family (indexed by $\hbar \in (0,1]$) of operators on $L^2(\mathbb{R}^n)$ with dense domain. If a is real-valued, then $\mathrm{Op}_\hbar(a)$ is symmetric.

Weyl pseudodifferential operators obey an exact (smooth) functional calculus.

Proposition 2.9 ([33, Theorem 8.7]). *Let $\vartheta \in C_c^\infty(\mathbb{R},\mathbb{R})$. Let $P_\hbar = \mathrm{Op}_\hbar(p)$ be a family of self-adjoint pseudodifferential operators with symbol $p \in S^m(\mathbb{R}^n)$ such that $p + \mathbf{i}$ is elliptic and $m > 0$. Then*

$$\vartheta(P_\hbar) = \mathrm{Op}_\hbar(a),$$

where $\mathrm{Op}_\hbar(a)$ is a self-adjoint pseudodifferential operator with symbol $a \in \mathcal{S}$. Moreover, if p is classical, then a is also classical and its principal part is $a_0 = \vartheta(p)$.

If $p \in S^m$ is real-valued and elliptic near infinity (if, for instance, $p = |\xi|^2 + V$ with V satisfying (H′)), then $p + \mathbf{i}$ is elliptic.

We emphasise that this representation of spectral functions of pseudodifferential operators is *exact*. An explicit induction formula for the symbol a (giving, in particular, the condition on the support of a) can be found in [33, after Theorem 8.7]. Specifically, Schrödinger operators are elliptic in the sense of the last proposition with $m = 2$.

As an illustration of the versatility of this functional calculus and the “replacement principle”, let us prove a weak form of the *Weyl law*, that is, obtain formula (1.3).

Proposition 2.10. *Let (μ, V) satisfy (H). Recall that ω_n denotes the volume of the unit Euclidean ball in $\mathbb{R}^n$ and that $\Pi_{\hbar,\mu} = \mathbb{1}_{(-\infty,\mu]}(H_\hbar)$. Then, as $\hbar \to 0$,*

$$N = \mathrm{tr}(\Pi_{\hbar,\mu}) \sim (2\pi\hbar)^{-n}\omega_n \int (\mu - V(x))_+^{n/2}\mathrm{d}x.$$

Proof. Let V_2 replace V_1 up to M. Then, by combining Propositions 2.5 and 2.9, for any $\vartheta \in C_c^\infty(\mathbb{R}, [0,1])$ supported inside $(0, \frac{M+\mu}{2}]$, one has

$$\begin{aligned}\operatorname{tr}(\vartheta(H_\hbar)) &= \operatorname{tr}(\vartheta(H_{2;\hbar})) + O(\hbar^\infty)\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} a(x,\xi;\hbar)\mathrm{d}\xi\mathrm{d}x + \mathcal{O}_\vartheta(\hbar^\infty)\\ &= \frac{1}{(2\pi\hbar)^n}\int \vartheta(V(x)+|\xi|^2)\mathrm{d}x\mathrm{d}\xi + \mathcal{O}_\vartheta(\hbar^{-n+1}),\end{aligned} \tag{2.4}$$

where we used that a is a classical symbol with compact support on $\mathbb{R}^{2n}$ and principal part $a_0(x,\xi) = \vartheta(V_2(x) + |\xi|^2)$.

The spectrum of the operator $H_\hbar$ lies in $[\min V, \infty)$, so that if $\vartheta_+ \geq \mathbb{1}_{[\min(V),\mu]} \geq \vartheta_-$, then $\vartheta_+(H_\hbar) \geq \Pi_{\hbar,\mu} \geq \vartheta_-(H_\hbar)$ as operators.

Hence, by (2.4), taking the limit as $\hbar \to 0$ and then the infimum over all ϑ_+ larger than $\mathbb{1}_{[\min(V),\mu]}$, we obtain

$$\limsup_{\hbar\to 0}(2\pi\hbar)^n \operatorname{tr}(\Pi_{\hbar,\mu}) \leq \int_{\mathbb{R}^{2n}} \mathbb{1}_{|\xi|\leq(\mu-V(x))_+^{1/2}}\mathrm{d}\xi\mathrm{d}x = \omega_n \int (\mu - V(x))_+^{n/2}\mathrm{d}x.$$

Similarly, by taking a supremum over $\vartheta_- \leq \mathbb{1}_{[\min(V),\mu]}$, we obtain the other inequality. Here, it is very important that for every $E \in \mathbb{R}$, the set $\{V(x) + |\xi|^2 = E\}$ has zero measure; indeed, outside of the measure zero set $\{\xi = 0\}$, the symbol $V(x) + |\xi|^2$ has no critical points. ∎

If ϑ is not smooth, then $\vartheta(P_\hbar)$ cannot be written as a pseudodifferential operator in a satisfactory way. However, by a Fourier transform, writing

$$\vartheta(\lambda) = \frac{1}{2\pi\hbar}\int e^{\mathrm{i}t\lambda/\hbar}\widehat{\vartheta}\Big(\frac{\lambda}{\hbar}\Big)\mathrm{d}t,$$

the crucial step to obtain approximation for $\vartheta(P_\hbar)$ is the study of the operator $f_{t;\hbar}(H_\hbar)$, where $f_{t;\hbar}(\lambda) = e^{\mathrm{i}t\lambda/\hbar}\chi(\lambda)$ oscillates at frequency $\mathcal{O}(\hbar^{-1})$ and $\chi \in C_c^\infty(\mathbb{R})$ is introduced for technical reasons. In this case, $f_{t;\hbar}(H_\hbar)$ is an approximation of the propagator associated with the Schrödinger operator $H_\hbar$. It can be expressed as an integral operator with an oscillating phase, but one must replace $e^{\mathrm{i}((x-y)\cdot\xi)/\hbar}$ in Definition 2.7 by the solution of an order 1 partial differential equation given in (2.7) and called the Hamilton–Jacobi equation.[3]

Proposition 2.11 ([33, Chapter 10, notably (10.2), (10.5) and (10.8)]). *Let $m > 0$. Let $p \in S^m(\mathbb{R}^n, \mathbb{R}^n)$ be independent of $\hbar$ and such that $p + \mathbf{i}$ is elliptic. Let $P_\hbar = \mathrm{Op}_\hbar(p)$. Given $\vartheta \in C_c^\infty(\mathbb{R}, \mathbb{R}_+)$, there exist $\tau > 0$ and a classical symbol $a \in S^0(\mathbb{R}^{2n+1}, \mathbb{R}^n)$ such that*

$$e^{\mathrm{i}tP_\hbar/\hbar}\vartheta(P_\hbar) = I_{\hbar,t}^{\phi,a} + \mathcal{O}_{\mathrm{J}^1}(\hbar^\infty) \quad \textit{uniformly for } t \in [-\tau,\tau],$$

[3] Equation (2.7) corresponds to the classical (Lagrangian) dynamics associated with the symbol $p(x,\xi)$ of $P_\hbar = \mathrm{Op}_\hbar(p)$.

where $I_{\hbar,t}^{\phi,a}$ *is a (non-self-adjoint) integral operator with kernel*

$$I_{\hbar,t}^{\phi,a}\colon (x,y) \mapsto \frac{1}{(2\pi\hbar)^n}\int e^{\frac{i}{\hbar}(\phi(t,x,\xi)-y\cdot\xi)} a(t,x,y,\xi;\hbar)\mathrm{d}\xi. \tag{2.5}$$

Here,

- $(x,y,\xi)\mapsto a(t,x,y,\xi;\hbar)$ *has compact support* $\mathcal{K}\Subset\mathbb{R}^{3n}$ *for all* $t\in[-\tau,\tau]$ *and* $\hbar\in(0,1]$.
- $\mathcal{K}$ *is a small* τ*-neighbourhood of* $\{(x,x,\xi),\vartheta(p(x,\xi))>0\}$.
- *The principal part of* a *at* $t=0$ *satisfies, on the diagonal,*

$$a_0(0,x,x,\xi)=\vartheta(p(x,\xi)). \tag{2.6}$$

- *There exists a compact* $\mathcal{K}_\tau\Subset\mathbb{R}^{2n+1}$ *containing* $[-\tau,\tau]\times\mathcal{K}$ *such that* $\phi\colon\mathcal{K}_\tau\to\mathbb{R}$ *is the (unique) solution of the initial value problem*

$$\phi(0,x,\xi)=x\cdot\xi,\quad \partial_t\phi(t,x,\xi)=p(x,\partial_x\phi(t,x,\xi)). \tag{2.7}$$

Notice the similarities with Proposition 2.9: at $t=0$, one has $I_{\hbar,0}^{\phi,a}=\mathrm{Op}_\hbar(a)$.

The compact set $\mathcal{K}_\tau$ and the solution ϕ of the Hamilton–Jacobi equation can be obtained from $\mathcal{K}$, p, $\phi|_{t=0}$ by the method of characteristics (see [36, Section 3.2.4, Theorem 2] for a general statement concerning first-order PDEs and [36, Section 3.2.5 (c), Example 6] for an application to Hamilton–Jacobi equations).

Our goal is to apply Propositions 2.9 and 2.11 to obtain pointwise estimates for the spectral projector $\Pi_\hbar=\mathbb{1}_{[0,\mu]}(H_\hbar)$, but there are two small obstacles. First, $H_\hbar=-\hbar^2\Delta+V$ is a pseudodifferential operator, with elliptic symbol in S^2, only when V satisfies (H′). Second, we need to improve the control of the remainder in Proposition 2.11, from J^1-norm to local C^∞-norm. This relies on elliptic estimates (in our case, Proposition A.6).

The next proposition will be our main input to prove the results of Section 1.1.

Proposition 2.12. *Assume that* (μ,V) *satisfies* (H) *and let* $\vartheta\in C_c^\infty((-\infty,M),\mathbb{R}_+)$ *with* $M>\mu$ *as in* (H). *There exists* $\tau>0$ *such that for* $t\in[-\tau,\tau]$,

$$\vartheta(H_\hbar)e^{\mathrm{i}tH_\hbar/\hbar}=I_{\hbar,t}^{\phi,a}+R_{\hbar,t}$$

with $I_{\hbar,t}^{\phi,a}$ *as in* (2.5) *(under the same assumptions for the classical symbol* $a\in S^0$*) and the error term satisfies* $\|R_{\hbar,t}\|_{\mathrm{J}^1}=\mathcal{O}(\hbar^\infty)$ *uniformly. Moreover, for any* $\mathcal{K}\Subset\{V<M\}^2$ *and every multi-indices* $\alpha,\beta\in\mathbb{N}_0^n$,

$$\max_{(x,y)\in\mathcal{K}}\sup_{t\in[-\tau,\tau]}\sup_{\hbar\in(0,1]}|\partial_x^\alpha\partial_y^\beta R_{\hbar,t}(x,y)|=\mathcal{O}_{\alpha,\beta}(\hbar^\infty). \tag{2.8}$$

Proof. Let V_2 replace V_1 up to M and assume that $V_1,V_2\geq 0$. By Proposition 2.5, it holds for any $\vartheta\in C_c^\infty([0,M),\mathbb{R}_+)$,

$$\|\vartheta(H_{1;\hbar})-\vartheta(H_{2;\hbar})\|_{\mathrm{J}^1}=\mathcal{O}(\hbar^\infty).$$

Next, let $\varkappa \in C_c^\infty([0,M),\mathbb{R})$ such that $\varkappa(\lambda)=\lambda$ on the support of ϑ. Then for $j\in\{1,2\}$,

$$e^{\mathrm{i}tH_{j;\hbar}/\hbar}\vartheta(H_{j;\hbar}) = e^{\mathrm{i}t\varkappa(H_{j;\hbar})/\hbar}\vartheta(H_{j;\hbar}).$$

Hence, since $\lambda \mapsto e^{\mathrm{i}\lambda}$ is 1-Lipschitz, by using again Proposition 2.5,

$$\|\vartheta(H_{1;\hbar})e^{\mathrm{i}tH_{1;\hbar}/\hbar} - \vartheta(H_{2;\hbar})e^{\mathrm{i}tH_{2;\hbar}/\hbar}\|_{\mathrm{J}^1} = \mathcal{O}(\hbar^\infty),$$

We can deduce from this trace norm estimate, a (local) C^∞ control for the kernel is valid in $\{V<M\}$ in the following way. Letting (λ_k^1,ϕ_k^1) and (λ_k^2,ϕ_k^2) be respective spectral resolutions of $H_{1;\hbar}$ and $H_{2;\hbar}$ below M, one has for $j\in\{1,2\}$,

$$\vartheta(H_{j;\hbar})e^{\mathrm{i}tH_{j;\hbar}/\hbar}(x,y) = \sum_{\lambda_k^j\le M}\vartheta(\lambda_k^j)e^{\mathrm{i}t\lambda_k^j}\phi_k^j(x)\phi_k^j(y).$$

Let $\vartheta_m(\lambda)=\lambda^{2m}\vartheta(\lambda)$ for $\lambda\ge 0$ and $m\in\mathbb{N}_0$. In particular, for $(x,y)\in\{V<M\}^2$, it holds for every $m\in\mathbb{N}$,

$$\vartheta_m(H_{j;\hbar})e^{\mathrm{i}tH_{j;\hbar}/\hbar}(x,y) = (-\hbar^2\Delta_x+V)^m(-\hbar^2\Delta_y+V)^m\sum_{\lambda_k^j\le M}\vartheta(\lambda_k^j)e^{\mathrm{i}t\lambda_k^j}\phi_k^j(x)\phi_k^j(y).$$

Using Proposition A.6 and the following Remark A.7, for any $\mathcal{K}\Subset\{V<M\}^2$ and every $k\ge 0$, there exists C_k such that with $m=k+\lfloor\frac{n}{4}\rfloor+1$,

$$\begin{aligned}
&\|\vartheta(H_{1;\hbar})e^{\mathrm{i}tH_{1;\hbar}/\hbar} - \vartheta(H_{2;\hbar})e^{\mathrm{i}tH_{2;\hbar}/\hbar}\|_{C^{2k}(\mathcal{K})}\\
&\quad\le C_k\hbar^{-2k-n}\|(-\hbar^2\Delta+V)_x^m(-\hbar^2\Delta+V)_y^m\\
&\qquad\qquad\times[\vartheta(H_{1;\hbar})e^{\mathrm{i}tH_{1;\hbar}/\hbar} - \vartheta(H_{2;\hbar})e^{\mathrm{i}tH_{2;\hbar}/\hbar}]\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}\\
&\quad\le C_k\hbar^{-2k-n}\|\vartheta_m(H_{1;\hbar})e^{\mathrm{i}tH_{1;\hbar}/\hbar} - \vartheta_m(H_{2;\hbar})e^{\mathrm{i}tH_{2;\hbar}/\hbar}\|_{\mathrm{J}^1}
\end{aligned}$$

since the L^2-norm of a kernel operator is equal to the Hilbert–Schmidt norm of the corresponding operator, which is smaller than its trace norm (cf. Proposition A.1).

Hence, provided that $\operatorname{supp}\vartheta\subset[0,M)$, the conclusions of Proposition 2.11 applied to the symbol $p(x,\xi)=|\xi|^2+V(x)$ also hold if V satisfies (H). It remains to prove the claim under hypothesis (H′) in which case we can use ellipticity.

Let $\chi\colon\mathbb{R}\to[0,1]$ be compactly supported inside $(0,M)$ and such that $\chi=1$ on the support of ϑ. Given $m\in\mathbb{N}_0$, let

$$\chi_m\colon\lambda\mapsto\lambda^{-m}\chi(\lambda)\quad\text{and}\quad\vartheta_m\colon\lambda\mapsto\lambda^{2m}\vartheta(\lambda).$$

Then, for every $m\in\mathbb{N}$,

$$\chi_m(H_\hbar)e^{\mathrm{i}tH_\hbar/\hbar}\vartheta_m(H_\hbar)\chi_m(H_\hbar) = e^{\mathrm{i}tH_\hbar/\hbar}\vartheta(H_\hbar).$$

Let us now apply Proposition 2.11 to $e^{\mathrm{i}tH_\hbar/\hbar}\vartheta_m(H_\hbar)$. Under hypothesis (H′), for every $m\in\mathbb{N}_0$, there exists a classical symbol a_m such that

$$e^{\mathrm{i}tH_\hbar/\hbar}\vartheta_m(H_\hbar) = I_{t,\hbar}^{\phi,a_m} + R_{t,\hbar}^m,$$

where, since all ϑ_m have the same support,

$$\|R^m_{t,\hbar}\|_{\mathrm{J}^1} = \mathcal{O}_m(\hbar^\infty) \quad \text{uniformly for } t \in [-\tau, \tau].$$

In particular,

$$e^{\mathrm{i}tH_\hbar/\hbar}\vartheta(H_\hbar) = \chi_m(H_\hbar) I^{\phi,a_m}_{t,\hbar}\chi_m(H_\hbar) + \widetilde{R}^m_{t,\hbar}, \quad \widetilde{R}^m_{t,\hbar} := \chi_m(H_\hbar) R^m_{t,\hbar}\chi_m(H_\hbar).$$

Let $\mathcal{K} \Subset \{V < M\}^2$ and let $\varkappa \in C^\infty_c(\mathbb{R}^{2n}, \mathbb{R}_+)$ be a cutoff equal to 1 on $\mathcal{K}$. Applying Proposition A.6, we have for every $k, m \in \mathbb{N}$,

$$\|\widetilde{R}^m_{\hbar,t}\|_{C^{2k}(\mathcal{K})} \le C_k \hbar^{-2k-n}\|(-\hbar^2\Delta + V)_x^{k+\lfloor n/4\rfloor+1}(-\hbar^2\Delta + V)_y^{k+\lfloor n/4\rfloor+1}[\widetilde{R}^m_{\hbar,t}\varkappa]\|_{L^2},$$

Setting $m = k + \lfloor \frac{n}{4}\rfloor + 1$, by definitions, one has

$$\|(-\hbar^2\Delta + V)_x^{k+\lfloor n/4\rfloor+1}(-\hbar^2\Delta + V)_y^{k+\lfloor n/4\rfloor+1}[\widetilde{R}_{\hbar,t}\varkappa]\|_{L^2} \le C\|R_{\hbar,t}\|_{L^2}.$$

Again, the L^2-norm of the kernel of $R_{\hbar,t}$ is controlled by the J^1-norm of the corresponding operator; to conclude,

$$\|\widetilde{R}_{\hbar,t}\|_{C^{2k}(\mathcal{K})} = \mathcal{O}(\hbar^\infty).$$

It remains to show that for $t \in [-\tau, \tau]$,

$$\chi_m(H_\hbar) I^{\phi,a_m}_{\hbar,t}\chi_m(H_\hbar) = I^{\phi,a}_{\hbar,t} + \mathcal{O}(\hbar^\infty)$$

for a compactly supported symbol a which satisfies (2.6) and that the error is controlled as in (2.8).

Observe that as $\chi_m \in C^\infty_c$, by Proposition 2.9, there exists a classical symbol $b_m \in S^0$ so that

$$\chi_m(H_\hbar) = \mathrm{Op}_\hbar(b_m)$$

and thus, for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned}&\chi_m(H_\hbar) I^{\phi,a_m}_{\hbar,t}\chi_m(H_\hbar)(x, y)\\ &\quad = \frac{1}{(2\pi\hbar)^{3n}}\int e^{\frac{\mathrm{i}}{\hbar}\Psi(t,x_1,x_2,y,\xi_1,\xi_2,\xi)}A(t, x_1, x_2, y, \xi_1, \xi_2, \xi)\mathrm{d}x_1\mathrm{d}\xi_1\mathrm{d}x_2\mathrm{d}\xi_2\mathrm{d}\xi,\end{aligned}$$

where

$$\begin{aligned}\Psi(t, x_1, x_2, y, \xi_1, \xi_2, \xi) &= (x - x_1)\cdot\xi_1 + \phi(t, x_1, \xi) - x_2\cdot\xi + (x_2 - y)\cdot\xi_2,\\ A(t, x_1, x_2, y, \xi_1, \xi_2, \xi) &= b_m\Big(\frac{x + x_1}{2}, \xi_1; \hbar\Big)a_m(t, x_1, x_2, \xi; \hbar)b_m\Big(\frac{x_2 + y}{2}, \xi_2; \hbar\Big).\end{aligned}$$

For any $t \in [-\tau, \tau]$, $(x_1, x_2, \xi) \mapsto a_m(t, x_1, x_2, \xi; \hbar)$ has compact support. Hence, for $(x, y) \in \mathcal{K}$, both variables $\frac{x+x_1}{2}$ and $\frac{x_2+y}{2}$ lie in a compact subset of $\mathbb{R}^n$, and one can use the fact that $a_m \in \mathcal{S}$ has rapid decay in ξ. This allows us to localise the integral over ξ_1, ξ_2 to a fixed compact, up to an error whose $C^k(\mathcal{K})$-norm is $\mathcal{O}_k(\hbar^\infty)$ for every $k \in \mathbb{N}_0$.

Hence, for fixed x, y, ξ, one can apply the stationary phase lemma to the previous integral. One can easily check that the only critical point of the oscillating phase is

$$(x_1, \xi_1, x_2, \xi_2) = (x, \partial_x \phi(t, x, \xi), y, \xi)$$

and the Hessian is non-degenerate at this point, with determinant 1. By Proposition A.15, this implies that for $(x, y) \in \mathcal{K}$ and $t \in [-\tau, \tau]$,

$$\begin{aligned}
&\chi_m(H_\hbar) I_{\hbar,t}^{\phi,a_m} \chi_m(H_\hbar)(x, y) \\
&\quad = \frac{1}{(2\pi\hbar)^n} \int e^{\frac{i}{\hbar}(\phi(t,x,\xi)-y\cdot\xi)} a(t, x, y, \xi; \hbar) \mathrm{d}\xi + R_{\hbar,t}(t, x, y),
\end{aligned}$$

where

- $a \in S^0$ is a classical symbol such that for all $t \in [-\tau, \tau]$, $(x, y, \xi) \mapsto a(t, x, y, \xi; \hbar)$ has a given compact support.
- For every $k \in \mathbb{N}_0$,

$$\sup_{t\in[-\tau,\tau]} \|R_{\hbar,t}(t, x, y)\|_{C^k(\mathcal{K})} = \mathcal{O}_k(\hbar^\infty).$$

- The principal part of a satisfies at $t = 0$ on the diagonal,

$$a_0(0, x, x, \xi) = b_{m,0}(x, \xi) a_{m,0}(0, x, x, \xi) b_{m,0}(x, \xi).$$

  Here we used the equations for the critical point (in particular, $x_1 = x_2 = x$ on the diagonal) and that $\xi_1 = \partial_x \phi(0, x, \xi) = \xi$ by the Hamilton–Jacobi equation (2.7). Since $b_{m,0}(x, \xi) = \chi_m(p(x, \xi))$, $a_{m,0}(t, x, x, \xi) = \vartheta_m(p(x, \xi))$ (cf. Proposition 2.11) and, by construction, $\chi_m^2 \vartheta_m = \vartheta$, we obtain that

$$a_0(0, x, x, \xi) = \vartheta(p(x, \xi)), \quad p(x, \xi) = |\xi|^2 + V(x).$$

This concludes the proof. ∎

Recall that the free fermion point process, denoted by X, is the determinantal point process associated with the operator $\Pi_{\hbar,\mu} = \mathbb{1}_{(-\infty,\mu]}(H_\hbar)$, and $N = \operatorname{tr} \Pi_{\hbar,\mu}$. For fixed $\hbar > 0$, the probability measure $N^{-1}\mathbb{E}\mathrm{X}$ admits a density with respect to the Lebesgue measure on $\mathbb{R}^n$, since it can be expressed using the first N eigenfunctions $(v_j)_{1\le j\le N}$ of $H_\hbar$ by

$$N^{-1}\mathbb{E}\mathrm{X} = \frac{1}{N} \sum_{k=1}^{N} |v_k(x)|^2 \mathrm{d}x,$$

where each term of the sum belongs to $L^1(\mathbb{R}^n, \mathbb{R})$ by definition. The *intensity function* is the density of this probability measure,

$$\rho_N(x) := N^{-1} \Pi_{\hbar,\mu}(x, x), \quad x \in \mathbb{R}^n. \tag{2.9}$$

As a simple consequence of Proposition 2.12, we obtain (locally) uniform bounds for the intensity of this point process.

Proposition 2.13. *Assume that* (μ, V) *satisfies* (H) *and let* $M > \mu$ *be as in* (H)*. For any compact* $\mathcal{K} \Subset \{V < M\}$*, there exists a constant* C *(depending on* $\mathcal{K}$ *and* μ*) such that for all* $\hbar \in (0, 1]$,

$$\max_{\mathcal{K}} \rho_N \le C.$$

Proof. Recall that $V \ge 0$ and let $\vartheta \in C_c^\infty(\mathbb{R}, [0, 1])$ be such that $\operatorname{supp}(\vartheta) \subset (-\infty, M)$ and $\mathbb{1}_{[0,\mu]} \le \vartheta$. Then, as the spectrum of $H_\hbar$ lies in $[0, \infty)$,

$$\Pi_{\hbar,\mu} = \mathbb{1}_{\{H_\hbar \le \mu\}} \le \vartheta(H_\hbar)$$

as operators, so their kernels can be compared pointwise on the diagonal. Moreover, by Proposition 2.12 with $t = 0$, we have $\vartheta(H_\hbar) = I^{\psi,a}_{\hbar,0} + \mathcal{O}(\hbar^\infty)$ where the kernel of the error is controlled locally uniformly inside $\{V < M\}$. By (2.5), this implies that for any $\mathcal{K} \Subset \{V < M\}$,

$$\Pi_{\hbar,\mu}(x, x) \le \frac{1}{(2\pi\hbar)^n} \int a(0, x, x, \xi; \hbar) \mathrm{d}\xi + \mathcal{O}(\hbar^\infty)$$

uniformly for $x \in \mathcal{K}$, where $(x, \xi) \mapsto a(0, x, x, \xi; \hbar)$ is uniformly bounded with compact support (independently of $\hbar \in (0, 1]$). In particular, there exists $C > 0$ such that, for all $x \in \mathcal{K}$ and all $\hbar \in (0, 1]$,

$$\Pi_{\hbar,\mu}(x, x) \le C\hbar^{-n}.$$

Recalling from Lemma 2.10 that

$$N \sim (2\pi\hbar)^{-n}\omega_n \int (\mu - V(x))_+^{n/2} \mathrm{d}x,$$

this completes the proof. ∎

Remark 2.14. Since V is not supposed to be regular on $\{V > M\}$, the ellipticity techniques used to control the L^∞-norm from the L^2-norm do not work there. If V is C^∞ everywhere, on the other hand, it is possible to obtain pointwise equivalents of Proposition 2.3, see, for instance, [51, Proposition 5.5].

To conclude this section and illustrate the methods used in this article, we apply Proposition 2.12 with $t = 0$ to derive the *microscopic asymptotics* of the kernel of the operator $\chi(H_\hbar)$ for a fixed smooth spectral function χ.

Proposition 2.15. *Assume that* (μ, V) *satisfies* (H) *and let* $\chi \in C_c^\infty((-\infty, M), \mathbb{R}_+)$ *with* $M > \mu$ *as in* (H)*. Using notation* (1.8)*, for any compact sets* $\mathcal{A} \subset \{V < M\}$ *and* $\mathcal{K} \subset \mathbb{R}^{2n}$*, it holds*

$$\max_{x_0 \in \mathcal{A}} \max_{(x,y) \in \mathcal{K}} |K^\chi_{x_0,\hbar}(x, y) - K^\chi_{x_0,0}(x, y)| = \mathcal{O}(\hbar).$$

Proof. By Proposition 2.12 with $t = 0$, one can write

$$K^\chi_{x_0,\hbar}(x, y; x_0, \hbar) = \frac{1}{(2\pi)^n} \int e^{\mathrm{i}(x-y)\cdot\xi} a(x_0 + \hbar x, x_0 + \hbar y, \xi; \hbar) \mathrm{d}\xi + R(x, y; x_0, \hbar),$$

where the error $R(x, y; x_0, \hbar)$ and all its derivatives are $\mathcal{O}(\hbar^\infty)$ uniformly for $x_0 \in \mathcal{A}$ and $(x, y) \in \mathcal{K}$. Note that in this integral, the phase is independent of the parameter $\hbar$ and the symbol of a satisfies

$$a(x_0, x_0, \xi; \hbar) = \chi(V(x_0) + |\xi|^2) + \mathcal{O}(\hbar).$$

Upon identifying the kernel of the operator $K^\chi_{x_0,0}$ as the leading term, this concludes the proof. ■

3. The spectral projector in the bulk at microscopic scale

This section is devoted to the proof of Theorem II.1 and its consequences.

Our starting point consists in writing compactly supported spectral functions of the operator $H_\hbar = -\hbar^2\Delta + V$ via a semiclassical Fourier transform,

$$f(H_\hbar) = \frac{\chi(H_\hbar)}{\sqrt{2\pi}\hbar} \int \widehat{g}(t) e^{\mathrm{i}tH_\hbar/\hbar} \mathrm{d}t,$$

where $f : \lambda \mapsto \chi(\lambda)g(\hbar^{-1}\lambda)$. Using the fact that $\chi(H_\hbar)e^{\mathrm{i}tH_\hbar/\hbar}$ can be well approximated by integral operators for a short time $t \in [-\tau, \tau]$, see Proposition 2.12, up to an $\mathcal{O}(\hbar^\infty)$ error, provided the support of $\widehat{g}$ lies inside $[-\tau, \tau]$, the integral kernel of this operator has the form

$$\frac{\sqrt{2\pi}}{(2\pi\hbar)^{n+1}} \int e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x,\xi) - y\cdot\xi)} a(x, y, \xi, t; \hbar) \mathrm{d}\xi,$$

where a is a classical symbol and ϕ satisfies the following Hamilton–Jacobi differential equation:

$$\begin{cases} \partial_t \phi = V(x) + |\partial_x \phi|^2, \\ \phi|_{t=0} = x \cdot \xi. \end{cases} \tag{3.1}$$

One can then obtain the asymptotics of such integrals by applying the stationary phase method as $\hbar \to 0$ using the properties of the phase ϕ and of the symbol a for small t. In particular, we study in Section 3.1 general properties of ϕ that will be useful in the rest of this article.

The approach described above cannot be directly applied to the spectral function of interest $\mathbb{1}_{(-\infty,\mu]}$, which is not of the form $\lambda \mapsto \chi(\lambda)g(\hbar^{-1}\lambda)$ where χ and $\widehat{g}$ have compact support. The idea is to regularise this function by applying a frequency cutoff $\rho_\hbar$ on scale $\hbar^{-1}$ and consider instead $f_{\hbar,\mu} = \vartheta \cdot (\mathbb{1}_{(-\infty,\mu]} * \rho_\hbar)$, where $\vartheta \in C_c^\infty(\mathbb{R}, \mathbb{R}_+)$ is equal to 1 on $[0, \mu]$. In Section 3.2, we perform the first step of the proof of Theorem II.1 which consists in obtaining the asymptotics of the regularised kernel associated with such $f_{\hbar,\mu}$. Then, in Section 3.3, we recover $\mathbb{1}_{(-\infty,\mu)}(H_\hbar)$ from its frequency cutoff using the Tauberian theorem of Hörmander, concluding the proof of Theorem II.1. Finally, the probabilistic consequences of Theorem II.1 for free fermions processes are discussed in Section 3.4.

3.1. Study of a phase function

Proposition 3.1. *Let (μ, V) satisfy (H) and let $M > \mu$ be as in (H). Let $\mathcal{K} \Subset \{V < M\} \times \mathbb{R}^n$. Let $\mathcal{K}' \Subset \mathbb{R}^{2n+1}$ be a neighbourhood of $\{0\} \times \mathcal{K}$, let $\phi\colon \mathcal{K}' \to \mathbb{R}$ solve (3.1) and let*

$$\Psi\colon (t, x, \xi) \mapsto \phi(t, x, \xi) - x \cdot \xi.$$

There exist $\tau > 0$, $\eta \in C^\infty([-\tau, \tau] \times \mathcal{K}, \mathbb{R}^n)$ with $\det \frac{\mathrm{d}\eta}{\mathrm{d}\xi} \neq 0$ and $g \in C^\infty([-\tau, \tau] \times \mathbb{R}^n, \mathbb{R})$ such that for all $(t, x, \xi) \in [-\tau, \tau] \times \mathcal{K}$,

$$\Psi(t, x, \xi) = t(|\eta(t, x, \xi)|^2 + g(t, x) + V(x)),$$

and g has the following Taylor expansion as $t \to 0$:

$$g(t, x) = t^2 \frac{|\nabla V(x)|^2}{12} + \mathcal{O}(t^4).$$

Proof. Since ϕ solves the Hamilton–Jacobi equation (3.1), it is smooth on $\mathcal{K}'$ and, as $t \to 0$,

$$\phi = x \cdot \xi + t(V(x) + |\xi|^2) + \mathcal{O}(t^2). \tag{3.2}$$

In particular,

$$\left.\frac{\Psi(t, x, \xi)}{t}\right|_{t=0} = V(x) + |\xi|^2$$

is a Morse function of ξ. This property is preserved for $t \in [-\tau, \tau] \times \mathcal{K}$ if $\tau > 0$ is small enough, that is, $\frac{\Psi}{t}$ admits exactly one non-degenerate critical point with respect to ξ, which is a (global) minimum. By the Morse lemma, there exists a smooth change of variables $\xi \mapsto \eta$, which is smooth in (t, x) such that

$$\frac{\Psi(t, x, \xi)}{t} = |\eta(t, x, \xi)|^2 + g(t, x) + V(x).$$

Using (3.1), we can iterate the Taylor expansion of ϕ as $t \to 0$. We obtain

$$\begin{aligned}
\phi(t, x, \xi) &= x \cdot \xi + t(V(x) + |\xi|^2) + t^2 \xi \cdot \nabla V(x) + \frac{t^3}{3} |\nabla V(x)|^2 \\
&\quad + \frac{2}{3} t^3 \langle \xi, \mathrm{Hess}(V)(x) \xi \rangle + \mathcal{O}((|\xi| + t) t^4) \\
&= x \cdot \xi + t\Big(\Big\langle \xi + \frac{t}{2} \nabla V(x), \mathrm{I} + \frac{2t^2}{3} \mathrm{Hess}(V)(x), \xi + \frac{t}{2} \nabla V(x) \Big\rangle \\
&\qquad + V(x) + \frac{|\nabla V(x)|^2}{12} t^2 \Big) + \mathcal{O}((|\xi| + t) t^4).
\end{aligned}$$

Note that in this expansion, all terms of even power in t are odd with respect to ξ, which explains the error term. This yields as $t \to 0$,

$$\begin{aligned}
\eta(t, x, \xi) &= \Big(\mathrm{Id} + \frac{t^2}{3} \mathrm{Hess}(V)(x)\Big)\Big(\xi + \frac{t}{2} \nabla V(x)\Big) + \mathcal{O}(|\xi| t^4), \\
g(t, x) &= \frac{|\nabla V(x)|^2}{12} t^2 + \mathcal{O}(t^4).
\end{aligned} \tag{3.3}$$

This completes the proof. ∎

3.2. *Convergence of regularised kernels*

In this section, we obtain asymptotics for a family of regularisations of the operator $\Pi_\hbar$ which are essentially obtained by smoothing the function $\mathbb{1}_{(-\infty,\mu]}$ using a frequency cutoff at scale $\hbar^{-1}$. To define this approximation, we introduce the following parameters.

Notations 3.1. Let (μ_0, V) satisfy (H), without loss of generality $V \geq 0$, and let $M > 0$ be as in (H). Let

- $\varkappa \in C_c^\infty(\mathbb{R}, [0,1])$, where $\varkappa = 1$ on $[0, M]$.
- $\vartheta \in C_c^\infty((\mu_0, M), \mathbb{R}_+)$, where μ_0 can be fixed at will later on.
- $\tau > 0$ is a small parameter such that for $t \in [-\tau, \tau]$, one can
  - apply Proposition 2.12;
  - apply Proposition 3.1 after fixing a large compact set $\mathcal{K} \Subset \mathbb{R}^{2n}$ which contains $\{(x,\xi), \vartheta(V(x) + |\xi|^2) > 0\}$.
- $\rho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ is even with $\int_\mathbb{R} \rho(\lambda)\mathrm{d}\lambda = 1$; $\hat{\rho}$ is supported on $[-\tau, \tau]$.
- $\rho_\hbar = \hbar^{-1}\rho(\hbar^{-1}\cdot)$ for $\hbar \in (0, 1]$.

In the sequel, we treat μ as a parameter, and we also let $\varkappa_\mu := \varkappa \mathbb{1}_{[0,\mu]}$ and $\vartheta_\mu := \vartheta \mathbb{1}_{[0,\mu]}$ for $\mu_0 \leq \mu < M$. We will consider spectral functions of $H_\hbar$ of the form

$$f_{\hbar,\mu}\colon \sigma \mapsto \vartheta(\sigma) \int \varkappa_\mu(\lambda)\rho_\hbar(\lambda - \sigma)\mathrm{d}\lambda.$$

By Proposition 2.12 applied to the operator $\vartheta(H_\hbar)e^{\mathrm{i}tH_\hbar/\hbar}$, it holds

$$f_{\hbar,\mu}(H_\hbar) = \frac{1}{\sqrt{2\pi}\hbar} \int \varkappa_\mu(\lambda)(I_{\hbar,t}^{\phi,a} + R_{\hbar,t})e^{-\mathrm{i}t\lambda/\hbar}\hat{\rho}(t)\mathrm{d}\lambda\mathrm{d}t. \tag{3.4}$$

The role of $\varkappa$ is to limit the integral above to a compact set in $\lambda \in \mathbb{R}$. In particular, the operator $f_{\hbar,\mu}(H_\hbar)$ has an integral kernel of the form

$$\begin{aligned} f_{\hbar,\mu}(H_\hbar)\colon (x,y) \mapsto &\frac{\sqrt{2\pi}}{(2\pi\hbar)^{n+1}} \int e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x,\xi) - \xi\cdot y - t\lambda)} a(x,y,\xi,t;\hbar)\hat{\rho}(t)\varkappa_\mu(\lambda)\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}t \\ &+ R_\hbar(x,y), \end{aligned} \tag{3.5}$$

where $\xi \mapsto a(x,y,\xi,t;\hbar)$ has compact support for $t \in [-\tau,\tau]$ and the error $R_\hbar$, as well as all its derivatives, are $\mathcal{O}(\hbar^\infty)$ and controlled uniformly for $\mu \in \mathbb{R}_+$ and locally uniformly for $x, y \in \{V < M\}$.

The goal of this section is to apply the stationary phase method to the integral kernel (3.5) in order to obtain the following asymptotics for the rescaled kernel $K_{x_0,\hbar}^{f_{\hbar,\mu}}$ of $f_{\hbar,\mu}(H_\hbar)$, as defined in (1.8).

Proposition 3.2. *Let $\mathcal{K} \Subset \mathbb{R}^{2n}$, let $f_{\hbar,\mu}$ be as in Notations 3.1 and let*

$$\mathcal{A} \Subset \{(x_0,\mu) \in \mathbb{R}^n \times \mathbb{R}_+ : V(x_0) \leq \mu_0 < \mu \leq M\}.$$

Then, it holds uniformly for $(x_0,\mu) \in \mathcal{A}$ and $(x,y) \in \mathcal{K}$,

$$K_{x_0,\hbar}^{f_{\hbar,\mu}}(x,y) = K_{x_0,0}^{\vartheta_\mu}(x,y) + \mathcal{O}(\hbar).$$

Proof. By formula (3.5), it suffices to obtain the asymptotics for $(x, y) \in \mathcal{K}$ of the rescaled kernel

$$(x, y) \mapsto \frac{1}{(2\pi)^{n+1/2}\hbar} \int e^{\frac{i}{\hbar}(\phi(t,x_0+\hbar x,\xi)-(x_0+\hbar y)\cdot\xi - t\lambda)} \times a(x_0 + \hbar x, x_0 + \hbar y, \xi, t; \hbar)\widehat{\rho}(t)\varkappa_\mu(\lambda)\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}t,$$

where $a \in S^0$ is a classical symbol whose principal part is given by (2.6) at time $t = 0$ and $p(x, \xi) = |\xi|^2 + V(x)$. Since the phase ϕ is smooth for $t \in [-\tau, \tau]$ and Proposition 3.1 applies, by a Taylor expansion and making a spherical change of variables $\eta(\xi) = r(\xi)\omega(\xi)$, where $(r, \omega) \in \mathbb{R}_+ \times S^{n-1}$, there exists a classical symbol $b \in S^0$ such that for $(x, y) \in \mathcal{K}$,

$$\begin{aligned} &\sqrt{2\pi}e^{\frac{i}{\hbar}(\phi(t,x_0+\hbar x,\xi)-(x_0+\hbar y)\cdot\xi-t\lambda)}a(x_0 + \hbar x, x_0 + \hbar y, \xi, t; \hbar)\widehat{\rho}(t)\mathrm{d}\xi \\ &\quad = e^{\frac{i}{\hbar}\psi(t,r,\lambda)}b(x, y, r\omega, t; \hbar)r^{n-1}\mathrm{d}r\mathrm{d}\omega \end{aligned} \tag{3.6}$$

and

$$\psi(t, r, \lambda) = t(r^2 + g(t, x_0) + V(x_0) - \lambda).$$

Since $\widehat{\rho}$ and a have compact supports, the function $(r, t) \mapsto b(x, y, r, \omega, t; \hbar)$ also has compact support. Moreover, since $\widehat{\rho}(0) = \frac{1}{\sqrt{2\pi}}$ and $\partial_x\phi(0, x, \xi) = \xi$ (by (3.1) and the change of variable (3.3)), the principal part of b satisfies at $t = 0$,

$$b_0(x, y, \eta, 0) = e^{i(x-y)\cdot\eta}\vartheta(r^2 + V(x_0)). \tag{3.7}$$

In particular, r is bounded away from 0 in the previous integral since ϑ is supported inside (μ_0, M) and $V(x_0) < \mu_0$.

This implies that uniformly for $(x, y) \in \mathcal{K}$,

$$K^{f_{\hbar,\mu}}_{x_0,\hbar}(x, y) = \frac{1}{(2\pi)^{n+1}\hbar} \int e^{\frac{i}{\hbar}\psi(t,r,\omega,\lambda)}b(x, y, r\omega, t; \hbar)\varkappa_\mu(\lambda)r^{n-1}\mathrm{d}t\mathrm{d}\lambda\mathrm{d}\omega\mathrm{d}r + \mathcal{O}(\hbar^\infty).$$

We apply the stationary phase method to the previous integral in the variables $(r, t) \in \mathbb{R} \times [-\tau, \tau]$ for a fixed $(\lambda, \omega) \in \mathbb{R}^*_+ \times S^{n-1}$. By (3.2), the equations for the critical point(s) are

$$\begin{cases} \dfrac{\partial\psi}{\partial r} = 2rt = 0, \\ \dfrac{\partial\psi}{\partial t} = V(x_0) + r^2 - \lambda + O(t) = 0. \end{cases} \tag{3.8}$$

These equations have the following consequences:

- Since r is bounded away from 0 (otherwise $b = 0$), any critical point satisfies $t = 0$.
- If $\lambda \le V(x_0)$, there are no critical points of the phase near the support of the symbol b; by Proposition A.16, the integral is $\mathcal{O}(\hbar^\infty)$ with the required uniformity.
- If $\lambda > V(x_0)$, there is a unique critical point given by $r_\star(\lambda) := \sqrt{\lambda - V(x_0)}$, and we assume that $r_\star$ is bounded away from 0.

We also verify that $\psi|_{t=0} = 0$ and the Hessian of the phase at the critical point, Hess $\psi = \begin{pmatrix} * & 2r_\star \\ 2r_\star & 0 \end{pmatrix}$, is not degenerate. Hence, by applying Proposition A.15, we obtain uniformly for $(x, y) \in \mathcal{K}$,

$$K^{f_{\hbar,\mu}}_{x_0,\hbar}(x, y) = \frac{1}{(2\pi)^n} \int \frac{r_\star(\lambda)^{n-2}}{2} \int_{S^{n-1}} s(x, y, r_\star(\lambda)\omega; \hbar)\varkappa_\mu(\lambda)\mathrm{d}\omega\mathrm{d}\lambda + \mathcal{O}(\hbar^\infty),$$

where, according to (3.7), $r \mapsto s(x, y, r\omega; \hbar)$ is a classical symbol with compact support in $\mathbb{R}^*_+$ and principal part

$$s_0(x, y, \xi) = e^{\mathrm{i}(x-y)\cdot\xi}\vartheta(|\xi|^2 + V(x_0)).$$

To conclude, we go back to the original variable $\xi = r_\star(\lambda)\omega$. We have $\lambda = |\xi|^2 + V(x_0)$ and the Jacobian is

$$\mathrm{d}\xi = r_\star^{n-1}(\lambda)\frac{\mathrm{d}r_\star(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda\mathrm{d}\omega = \frac{r_\star(\lambda)^{n-2}}{2}\mathrm{d}\lambda\mathrm{d}\omega$$

so that

$$K^{f_{\hbar,\mu}}_{x_0,\hbar}(x, y) = \frac{1}{(2\pi)^n} \int (e^{\mathrm{i}(x-y)\cdot\xi}\vartheta(|\xi|^2 + V(x_0)) + \mathcal{O}(\hbar))\varkappa_\mu(|\xi|^2 + V(x_0))\mathrm{d}\xi + \mathcal{O}(\hbar^\infty),$$

where both errors are controlled uniformly for $(x, y) \in \mathcal{K}$ and $(x_0, \mu) \in \mathcal{A}$. Since $\varkappa = 1$ on $[1, M]$, $V(x_0) \geq 1$ and $\mu \leq M$, we identify the leading term as the kernel of the operator

$$K^{\vartheta_\mu}_{x_0,0} = \vartheta_\mu(-\Delta + V(x_0)),$$

this concludes the proof. ∎

To recover the asymptotics of $\vartheta_\mu(H_\hbar)$ from that of $f_{\hbar,\mu}(H_\hbar)$, we treat μ as a parameter, and we will rely on the following estimate on the derivative $\partial_\mu f_{\hbar,\mu}(H_\hbar)$.

Lemma 3.3. *Let ρ, ϑ be as in Notations 3.1 and set $\zeta_{\hbar,\mu}(\lambda) = \vartheta(\lambda)\rho_\hbar(\lambda - \mu)$. There exists a constant $C > 0$ such that for all $x_0 \in \{V \leq \mu_0\}$, $\mu \in \mathbb{R}_+$ and $\hbar \in (0, 1]$,*

$$\hbar^n \zeta_{\hbar,\mu}(H_\hbar)(x_0, x_0) \leq C.$$

Proof. By writing $\rho_\hbar$ in terms of its Fourier transform and applying Proposition 2.12, we obtain

$$\hbar^n \zeta_{\hbar,\mu}(H_\hbar)(x_0, x_0) = \frac{1}{(2\pi)^{n+1/2}\hbar} \int e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x_0,\xi) - x_0\cdot\xi - \mu t)} a(x_0, x_0, \xi, t; \hbar)\widehat{\rho}(t)\mathrm{d}t\mathrm{d}\xi + \mathcal{O}(\hbar^\infty),$$

where $a \in S^0$, $(x_0, \xi) \mapsto a(x_0, x_0, \xi, t; \hbar)$ has a fixed compact support for $t \in \mathrm{supp}(\widehat{\rho})$, $\hbar \in (0, 1]$ and the error (as well as all its derivatives) is controlled uniformly for $x_0 \in \{V \leq \mu\}$ and $\mu \in \mathbb{R}_+$.

We proceed like in the proof of Proposition 3.2, by decomposing $\eta(\xi) = r\omega$ for $(r,\omega) \in \mathbb{R}_+ \times S^{n-1}$ and applying the stationary phase method in the variables (r,t). According to (3.6)–(3.7),

$$\hbar^n \zeta_{\hbar,\mu}(H_\hbar)(x_0,x_0) = \frac{1}{(2\pi)^{n+1}\hbar} \int_{S^{n-1}} \int_{[-\tau,\tau]\times\mathbb{R}_+} e^{\frac{i}{\hbar}\psi(t,r,\omega,\mu)} b(x_0, r\omega, t; \hbar) r^{n-1} \mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\omega + \mathcal{O}(\hbar^\infty),$$

where $\psi(t,r,\omega,\mu) = t(r^2 + g(x,t) - \mu)$ and $b \in S^0$ is another classical symbol.

Like in the proof of Proposition 3.2, the equations for the critical point(s) are given by (3.8) with $\lambda = \mu$ and the parameter r is bounded away from 0. In particular, uniformly for (x_0, μ) in a (small) neighbourhood of $\{\mu \le V(x_0)\}$, there is no critical point and by applying Proposition A.16,

$$\zeta_{\hbar,\mu}(H_\hbar)(x_0,x_0) = \mathcal{O}(\hbar^\infty).$$

On the other hand, if μ is bounded away from $V(x_0)$, then there is a unique critical point $(r_\star, t) = (\sqrt{\mu - V(x_0)}, 0)$, and applying Proposition A.15, it holds

$$\hbar^n \zeta_{\hbar,\mu}(H_\hbar)(x_0,x_0) = \frac{r_\star^{n-2}/2}{(2\pi)^n} \int_{S^{n-1}} (b_0(x_0, r_\star\omega, 0) + \mathcal{O}(\hbar))\mathrm{d}\omega + \mathcal{O}(\hbar^\infty).$$

In both cases, this proves that there is a constant C such that for $x_0 \in \{V \le \mu_0\}$, $\mu \in \mathbb{R}_+$ and $\hbar \in (0,1]$,

$$\hbar^n \zeta_{\hbar,\mu}(H_\hbar)(x_0,x_0) \le C(1 + r_\star^{n-2}(\mu)). \qquad \blacksquare$$

3.3. From the regularised kernel to the projection kernel

To conclude the proof of Theorem II.1, it remains to replace $K_{x_0,\hbar}^{f_{\hbar,\mu}}$ by the rescaled kernel of the projection $\Pi_{\mu,\hbar} = \mathbb{1}_{[0,\mu]}(H_\hbar)$. To this end, we treat the energy level μ as a parameter, and we rely on the following Tauberian theorem.

Proposition 3.4 (Cf. [74, Theorem B.2.1]). *Let $N\colon [0,\infty) \to [0,\infty)$ be a function with $N(0) = 0$, $N' \ge 0$ and at most polynomial growth. Let ρ be a mollifier as in Notations 3.1. If $N' * \rho_\hbar(\lambda) \le 1 + \lambda^\alpha$ for some $\alpha \in \mathbb{R}_+$, then $|N(\lambda) - N * \rho_\hbar(\lambda)| \le C_\rho \hbar(1 + \lambda^\alpha)$ for all $\lambda \in \mathbb{R}_+$.*

This statement follows directly from [74, Theorem B.2.1] by rescaling the mollifier ρ at scale $\hbar$ as in the Notations 3.1. Moreover, the *counting function* N is allowed to depend on the parameter $\hbar$ as long as the condition $N' * \rho_\hbar(\lambda) \le C(1 + \lambda^\alpha)$ holds.

Let us sketch how this result comes into play. Recall that $\vartheta_\mu = \vartheta \mathbb{1}_{[0,\mu]}$, where $\vartheta \in C_c^\infty((\mu_0, M), \mathbb{R}_+)$. Treating μ as a parameter, we are interested in the kernel of the operator $K_{x_0,\hbar}^{\vartheta_\mu}$. To ease notation, let

$$K\colon \mu \mapsto K_{x_0,\hbar}^{\vartheta_\mu}(x,y) = \hbar^n \vartheta_\mu(H_\hbar)(x_0 + \hbar x, x_0 + \hbar y).$$

Then

$$K * \rho_\hbar(\mu) = \hbar^n \int \vartheta_\sigma(H_\hbar)(x_0 + \hbar x, x_0 + \hbar y)\rho_\hbar(\mu - \sigma)\mathrm{d}\sigma$$
$$= \hbar^n g_{\hbar,\mu}(H_\hbar)(x_0 + \hbar x, x_0 + \hbar y),$$

where using that ρ is even, we define

$$g_{\hbar,\mu}(\lambda) = \vartheta(\lambda) \int \mathbb{1}_{\{\sigma \le \mu\}} \rho_\hbar(\lambda - \sigma)\mathrm{d}\sigma. \tag{3.9}$$

This function is essentially the same as $f_{\hbar,\mu}$ appearing in Notations 3.1.

Proposition 3.5. *Let $\mathcal{K}_0 \Subset \{V < M\}^2$. It holds uniformly for $(\mu, x, y) \in \mathbb{R} \times \mathcal{K}_0 \times \mathcal{K}_0$,*

$$g_{\hbar,\mu}(H_\hbar)(x, y) = f_{\hbar,\mu}(H_\hbar)(x, y) + \mathcal{O}(\hbar^\infty).$$

Proof. One has

$$f_{\hbar,\mu}(\lambda) - g_{\hbar,\mu}(\lambda) = \vartheta(\lambda) \int (1 - \varkappa(\sigma))\mathbb{1}_{\{\sigma \le \mu\}}\rho_\hbar(\lambda - \sigma)\mathrm{d}\sigma.$$

Since the supports of ϑ and $1 - \varkappa$ are disjoint and $\rho \in \mathcal{S}$, we obtain

$$\| f_{\hbar,\mu} - g_{\hbar,\mu}\|_{L^\infty} = \mathcal{O}(\hbar^\infty),$$

and moreover $(f_{\hbar,\mu} - g_{\hbar,\mu})$ is supported inside (μ_0, μ_1) with $\mu_1 < M$. Using the spectral resolution of $H_\hbar$ for energy $< \mu_1$, by Lemma 2.2, this implies that

$$\|x \mapsto (f_{\hbar,\mu} - g_{\hbar,\mu})(H_\hbar)(x, x)\|_{C^0(\mathcal{K}_0)} \le C\hbar^{-n} \| f_{\hbar,\mu} - g_{\hbar,\mu}\|_{L^\infty} \max_{\lambda \le \mu_1} \|v_\lambda^2\|_{C^0(\mathcal{K}_0)},$$

where v_λ denotes the eigenfunction(s) of $H_\hbar$ with energy λ. Moreover, for any cutoff $\chi \in C_c^\infty(\{V < M\}, \mathbb{R}_+)$ such that $\chi \ge \mathbb{1}_{\mathcal{K}_0}$, by Lemma A.5,

$$\|v_\lambda^2\|_{C^0(\mathcal{K}_0)} \le \|v_\lambda \chi\|^2_{C^0(\mathbb{R}^n)} \le C\|(1 - \Delta)^{\lfloor n/4 \rfloor + 1}(v_\lambda \chi)\|_{L^2(\mathbb{R}^n)}.$$

Using the eigenvalue equation, $-H_\hbar v_\lambda = \lambda v_\lambda$, one verifies that for any $\ell \in \mathbb{N}$, there exists a constant $C_\ell > 0$ such that for any $\lambda < M$ and $x \in \operatorname{supp}(\chi)$,

$$|(1 - \Delta)^\ell (v_\lambda \chi)(x)| \le C_\ell \hbar^{-2\ell}|v_\lambda(x)|.$$

This implies that for any $\lambda < M$ and $\ell \in \mathbb{N}$,

$$\|(1 - \Delta)^\ell(v_\lambda \chi)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\hbar^{-2\ell}) \quad \text{and} \quad \|v_\lambda^2\|_{C^0(\mathcal{K}_0)} = \mathcal{O}(\hbar^{-(\lfloor n/2 \rfloor + 2)}).$$

We conclude that

$$\|x \mapsto (f_{\hbar,\mu} - g_{\hbar,\mu})(H_\hbar)(x, x)\|_{C^0(\mathcal{K}_0)} = \mathcal{O}(\hbar^\infty).$$

Since $f_{\hbar,\mu}, g_{\hbar,\mu} \ge 0$ by the Cauchy–Schwarz inequality, this completes the proof. ■

In addition,

$$\partial_\mu (K * \rho_\hbar)(\mu) = \hbar^n \vartheta(H_\hbar)\rho_\hbar(\mu - H_\hbar)(x_0 + \hbar x, x_0 + \hbar y),$$

which we studied in Proposition 3.3.

If $\mu \mapsto K^{\vartheta_\mu}_{x_0,\hbar}(x, y)$ would be non-increasing, one could directly combine Propositions 3.5, 3.3 and 3.2 with the Tauberian theorem to deduce the asymptotics of this kernel. Nevertheless, using the positivity of $K^{\vartheta_\mu}_{x_0,\hbar}$ on the diagonal as an additional trick, we obtain the desired result.

Proposition 3.6. *Let ϑ_μ be as in Notations 3.1. For any compact sets $\mathcal{A} \Subset \{(x_0, \mu) \in \mathbb{R}^n \times \mathbb{R}_+ : V(x_0) \le \mu_0 < \mu \le M\}$ and $\mathcal{K} \Subset \mathbb{R}^n$, there exists a constant C such that*

$$\max_{(x_0,\mu)\in\mathcal{A}} \max_{(x,y)\in\mathcal{K}\times\mathcal{K}} |K^{\vartheta_\mu}_{x_0,\hbar}(x, y) - K^{\vartheta_\mu}_{x_0,0}(x, y)| \le C\hbar.$$

Proof. Recall that we set $K_\lambda = K^{\vartheta_\lambda}_{x_0,\hbar}$ and define for $x_0, x, y \in \mathbb{R}^n$ and $\mu \in \mathbb{R}_+$,

$$N^1_{\hbar,\mu}(x) = \hbar^n \sum_{\lambda\le\mu} \vartheta(\lambda)|v_\lambda(x_0 + \hbar x)|^2,$$
$$N^2_{\hbar,\mu}(x, y) = \hbar^n \sum_{\lambda\le\mu} \vartheta(\lambda)|v_\lambda(x_0 + \hbar x) - v_\lambda(x_0 + \hbar y)|^2,$$

where (λ, ϕ_λ) are normalised eigenpairs of $H_\hbar$. Using the spectral resolution of operator $H_\hbar$, it holds

$$\begin{aligned} N^1_{\hbar,\lambda}(x_0 + \hbar x) &= K_\lambda(x, x), \\ N^2_{\hbar,\lambda}(x, y) &= K_\lambda(x, x) + K_\lambda(y, y) - 2K_\lambda(x, y). \end{aligned} \tag{3.10}$$

The *counting functions* $\lambda \mapsto N^j_{\hbar,\lambda}(x, y)$ for $j \in \{1, 2\}$ satisfy all the assumptions of Proposition 3.4. Indeed, they are non-decreasing, and we verify that if $\mu_0 < \mu_1 < M$ and $\hbar$ is sufficiently small, it holds uniformly for all $\mu \in \mathbb{R}_+$, $x_0 \in \{V \le \mu_0\}$ and $x, y \in \mathcal{K}$,

$$N^2_{\hbar,\mu}(x, y) \le 4 \max_{x_0\in\{V<\mu_1\}} N^1_{\hbar,\mu}(x_0) \le 4\hbar^n \max_{x_0\in\{V<\mu_1\}} \vartheta(H_\hbar)(x_0, x_0) < \infty.$$

The last bound follows from Proposition 2.13. From Proposition 3.3, the convolution $\partial_\lambda(K * \rho_\hbar)$ is uniformly bounded. Hence, the same holds for the derivatives of the counting functions $N^j_{\hbar,\lambda}(x, y)$ for $j \in \{1, 2\}$.

Thus, one can apply the Tauberian theorem to N^1 and N^2, and by linearity, we obtain

$$\max_{(x_0,\mu)\in\mathcal{A}} \max_{(x,y)\in\mathcal{K}} \left| K_\mu(x, y) - \int K_\lambda(x, y)\rho_\hbar(\mu - \lambda)\mathrm{d}\lambda \right| \le C\hbar.$$

On the other hand, according to Proposition 3.5, it also holds for $\mu \in \mathbb{R}_+$,

$$\begin{aligned} \int K_\lambda(x, y)\rho_\hbar(\mu - \lambda)\mathrm{d}\lambda &= \hbar^n g_{\hbar,\mu}(H_\hbar)(x_0 + \hbar x, x_0 + \hbar y) \\ &= \hbar^n f_{\hbar,\mu}(H_\hbar)(x_0 + \hbar x, x_0 + \hbar y) + \mathcal{O}(\hbar^\infty) \end{aligned}$$

with the required uniformity. By Proposition 3.2, we conclude that

$$K^{\vartheta_\mu}_{x_0,\hbar}(x,y) = K^{\vartheta_\mu}_{x_0,0}(x,y) + \mathcal{O}(\hbar). \qquad \blacksquare$$

We are almost done with the proof of Theorem II.1: it remains to add the kernel of a pseudodifferential operator.

Proposition 3.7. *Let $f \in C^\infty_c(\mathbb{R},\mathbb{R}_+)$ and let $f_\mu = f\mathbb{1}_{[0,\mu]}$ for $\mu \in \mathbb{R}_+$. For any compact sets $\mathcal{A} \Subset \{(x_0,\mu) \in \mathbb{R}^n \times \mathbb{R}_+ : V(x_0) < \mu < M\}$ and $\mathcal{K} \Subset \mathbb{R}^{2n}$, there exists a constant C such that for $\hbar \in (0,1]$,*

$$\max_{(x_0,\mu)\in\mathcal{A}} \max_{(x,y)\in\mathcal{K}} |K^{f_\mu}_{x_0,\hbar}(x,y) - K^{f_\mu}_{x_0,0}(x,y)| \le C\hbar.$$

Proof. Let us choose μ_j for $j \in \{0,1\}$ so that $V(x_0) < \mu_0 < \mu_1 \le \mu$ for all $(x_0,\mu) \in \mathcal{A}$ and decompose

$$f\mathbb{1}_{[0,\mu]} = \chi + \vartheta\mathbb{1}_{[0,\mu]},$$

where $\chi \in C^\infty_c(\mathbb{R},\mathbb{R}_+)$ is equal to 0 on $[\mu_1,\infty)$ and $\vartheta \in C^\infty_c((\mu_0,M),\mathbb{R}_+)$. By linearity, it holds for $\hbar \in [0,1]$,

$$K^{f_\mu}_{x_0,\hbar} = K^{\chi}_{x_0,\hbar} + K^{\vartheta_\mu}_{x_0,\hbar}.$$

Moreover, according to Propositions 3.6 and 2.15, there exists a constant C depending only on $(\mathcal{A},\mathcal{K})$ such that for all $(x,y) \in \mathcal{K}$,

$$|K^{\vartheta_\mu}_{x_0,\hbar}(x,y) - K^{\vartheta_\mu}_{x_0,0}(x,y)| \le C\hbar$$

and

$$|K^{\chi}_{x_0,\hbar}(x,y) - K^{\chi}_{x_0,0}(x,y)| \le C\hbar.$$

By combining these estimates, the proof is completed. $\blacksquare$

To conclude with the proof of Theorem II.1, one can choose any $f \in C^\infty_c(\mathbb{R},\mathbb{R}^+)$ equal to 1 on $[0,\mu]$.

Remark 3.8. Let us comment on the convergence of derivatives for the rescaled kernel at local scales. Choosing f equal to $\lambda \mapsto \lambda^{2k}$ on $[0,\mu]$ allows us to prove that the kernel of $H^k_\hbar \Pi_\hbar H^k_\hbar$ also admits a scaling limit in the bulk which is expressed in terms of the free Laplacian; for any $k \in \mathbb{N}_0$, as $\hbar \to 0$,

$$\begin{aligned}(-\Delta_x + V(x_0+\hbar x))^k(-\Delta_y + V(x_0+\hbar y))^k K^{\mathbb{1}_{[0,\mu]}}_{x_0,\hbar}(x,y)\\ \to (-\Delta_x + V(x_0))^k(-\Delta_y + V(x_0))^k K^{\mathbb{1}_{[0,\mu]}}_{x_0,0}(x,y)\end{aligned}$$

locally uniformly on $\mathbb{R}^{2n}$.

By Proposition A.6, this implies the convergence of the kernel $(x,y) \mapsto K^{\mathbb{1}_{[0,\mu]}}_{x_0,\hbar}(x,y)$ in the local C^{2k}-topology (as well as a rate of convergence $\mathcal{O}(\hbar)$).

3.4. Concentration inequalities for linear statistics

To conclude this section, we prove the law of large numbers of Theorem I.1 and the central limit Theorem I.3.

Recall that the free fermion point process, denoted by X, is the determinantal point process associated with the operator $\Pi_{\hbar,\mu} = \mathbb{1}_{(-\infty,\mu]}(H_\hbar)$, and $N = \operatorname{tr} \Pi_{\hbar,\mu}$. Its (normalised) intensity is denoted by ρ_N, (2.9). In particular, for any test function $f : \mathbb{R}^n \to \mathbb{R}_+$, $\mathbb{E}\mathrm{X}(f) = \operatorname{tr}[f \Pi_{\hbar,\mu}] = N \int f \mathrm{d}\rho_N$, where we view f as a (unbounded) positive multiplication operator. Let us denote

$$\mathbf{F} = \{f : \mathbb{R}^n \to \mathbb{R} : f \in \mathrm{Lip}_1, f(0) = 0\},$$

and recall that the Kantorovich distance is $\mathrm{d}_\mathrm{W}(\nu,\rho) = \sup\{\int f \mathrm{d}(\nu-\rho) : f \in \mathbf{F}\}$ for any probability measures ν, ρ on $\mathbb{R}^n$.

Lemma 3.9. *Let (μ, V) satisfy the hypothesis (H) and let $\varrho = Z^{-1}(\mu - V)_+^{n/2}$ be the corresponding density of states. Then, as $\hbar \to 0$ (or equivalently $N \to \infty$),*

$$\mathrm{d}_\mathrm{W}(\rho_N, \varrho) \to 0.$$

Proof. We begin with the elementary identity

$$\forall f \in \mathbf{F},\ \forall x \in \mathbb{R}^n, \quad |f(x)| \le |x|.$$

Let $\delta > 0$ be a small parameter and let

$$\epsilon(x_0) = 2\pi\hbar \frac{\omega_n^{-1/n}}{\sqrt{\mu - V(x_0)}} \quad \text{for } x_0 \in \{V < \mu\}$$

as in Theorem II.1. Let f_δ be as in Proposition 2.3 and let $f \in \mathbf{F}$. Then, using the spectral resolution of $\Pi_{\hbar,\mu}$,

$$\left|\frac{1}{N}\int_{V(x)\ge\mu+2\delta} f(x_0)\Pi_{\hbar,\mu}(x,x)\mathrm{d}x\right| \le 2\left(1 + \frac{mu}{\delta}\right) \sup_{V(x)\ge\mu+2\delta} |f(x)e^{-2\mathrm{f}_\delta(x)/\hbar}|,$$

and

$$\sup_{f\in\mathbf{F}} \sup_{V(x)\ge\mu+2\delta} |f(x)e^{-2\mathrm{f}_\delta(x)/\hbar}| \le \sup_{V(x)\ge\mu+2\delta} |x|e^{-2\mathrm{f}_\delta(x)/\hbar}.$$

The right-hand side is finite for every $\hbar \in (0,1]$ and tends to zero exponentially quickly as $\hbar \to 0$.

On the other hand, by Theorem II.1 and Proposition 2.10, as $\hbar \to 0$,

$$\begin{aligned}
&\left|\frac{1}{N}\int_{\{V(x)\le\mu-\delta\}} f(x)\Pi_\hbar(x,x)\mathrm{d}x - \int_{\{V(x)\le\mu-\delta\}} f(x)\varrho(x)\mathrm{d}x\right| \\
&\quad \le \int_{\{V(x)\le\mu-\delta\}} |x|\left|\frac{K_{x,\epsilon(x)}(0,0)}{N\epsilon(x)^n} - \frac{(2\pi\hbar)^n}{Z\omega_n\epsilon(x)^n}\right|\mathrm{d}x \to 0,
\end{aligned}$$

where we used that $K_{x,\epsilon(x)}(0,0) \to 1$ uniformly for all x in the compact set $\{V \le \mu - \delta\}$.

The function ϱ is uniformly bounded, so that

$$\limsup_{\delta\to 0} \sup_{f\in\mathrm{F}} \int_{\{\mu-\delta\le V(x)\le\mu+2\delta\}} f(x)\varrho(x)\mathrm{d}x = 0.$$

Similarly, by Proposition 2.13, ρ_N is uniformly bounded on compact subsets of $\{V < M\}$, so that

$$\limsup_{\delta\to 0} \sup_{f\in\mathrm{F}} \sup_{\hbar\in(0,1]} \int_{\{\mu-\delta\le V(x)\le\mu+2\delta\}} f(x)\rho_N(x)\mathrm{d}x = 0.$$

Combining these estimates concludes the proof. ∎

As consequence of Lemma 3.9, the (normalised) intensity ρ_N converges weakly to the density of states ϱ. To complete the proof of the law of large numbers, Theorem I.1, we rely on Lemma 3.9 and basic concentration bounds for determinantal processes.

Proposition 3.10. *Let (μ, V) satisfy the hypothesis (H). There exists a small constant $c > 0$ such that for any $\epsilon > 0$ with $\epsilon \le \hbar^{-1}$, there exists a (non-increasing) constant $C_\epsilon \ge 1$ and*

$$\mathbb{P}[\mathrm{d_W}(N^{-1}\mathrm{X}, \rho_N) \ge \epsilon] \le C_\epsilon \exp(-cN\epsilon^2).$$

Moreover, we can choose $C_\epsilon = e^{C\epsilon^{-n}}$ for some universal constant $C > 0$.

Proof. Let us denote $\widetilde{\mathrm{X}} := \mathrm{X} - \mathbb{E}\mathrm{X}$ for the recentred Fermion point process. By (A.4), for any $f \in C(\mathbb{R}^n, \mathbb{R})$,

$$\mathbb{E}[e^{\widetilde{\mathrm{X}}(f)}] = \det(\mathrm{I} + (e^f - 1)\Pi_{\hbar,\mu})e^{-\mathrm{tr}[f\Pi_{\hbar,\mu}]} \le \exp(\mathrm{tr}[(e^f - 1 - f)\Pi_{\hbar,\mu}]),$$

where we used the elementary bound

$$\det(\mathrm{I} + A) \le \exp(\mathrm{tr}(A))$$

valid for all finite-rank operators $A \ge -\mathrm{I}$ (here $A = \Pi_{\hbar,\mu}(e^f - 1)\Pi_{\hbar,\mu}$, is bounded from below by $-\mathrm{I}$).

Moreover, by Proposition 2.3 and using the spectral resolution of $\Pi_{\hbar,\mu}$, it holds for all $g \in C(\mathbb{R}^n, \mathbb{R}_+)$,

$$\mathrm{tr}(g\Pi_{\hbar,\mu}) \le N\Big(1 + 2\frac{\mu}{\delta}\Big) \sup_{x\in\mathbb{R}^n} (g(x)e^{-2\mathrm{f}_\delta(x)/\hbar}).$$

Using that $0 \le e^f - 1 - f \le f^2 e^f$, we obtain

$$\mathbb{E}[e^{\widetilde{\mathrm{X}}(f)}] \le \exp\Big(N\Big(1 + 2\frac{\mu}{\delta}\Big) \sup_{x\in\mathbb{R}^n} (f(x)^2 e^{f(x)-2\mathrm{f}_\delta(x)/\hbar})\Big).$$

By rescaling, this shows that for any $\lambda > 0$ with $\lambda \ll N\hbar^{-1}$, there exists a constant $C > 0$ such that if $\hbar$ is sufficiently small, then

$$\sup_{f\in\mathrm{F}} \mathbb{E}[e^{\lambda N^{-1}\widetilde{\mathrm{X}}(f)}] \le \exp\Big(\frac{C}{2N}\lambda^2\Big).$$

Here, we crucially use the fact that elements of $\mathbf{F}$ are globally 1-Lipschitz, so that, for c small enough,

$$\sup_{\lambda<cN\hbar^{-1}} \sup_{f\in\mathbf{F}} \sup_{x\in\mathbb{R}^d} (e^{\lambda N^{-1} f(x) - 2\mathrm{f}_\delta(x)/\hbar}) < +\infty.$$

By Markov's inequality, this yields a Gaussian tail bound for the random variable $\widetilde{\mathrm{X}}(f)$; for any $\epsilon > 0$ and $\lambda > 0$,

$$\sup_{f\in\mathbf{F}} \mathbb{P}[\widetilde{\mathrm{X}}(f) \geq \epsilon N] \leq e^{-\epsilon\lambda + C\lambda^2/2N},$$

so that, choosing $\lambda = cN\epsilon$ for $c > 0$ small enough,

$$\sup_{f\in\mathbf{F}} \mathbb{P}[\widetilde{\mathrm{X}}(f) \geq \epsilon N] \leq e^{-\epsilon\lambda + c\lambda^2/2N} = \exp\Big(-\frac{c}{2} N\epsilon^2\Big).$$

Upon replacing f with $-f$, we obtain a symmetric inequality. Hence, it holds for any $\epsilon \leq \hbar^{-1}$,

$$\sup_{f\in\mathbf{F}} \mathbb{P}[|\widetilde{\mathrm{X}}(f)| \geq \epsilon N] \leq 2\exp\Big(-\frac{c}{2} N\epsilon^2\Big). \tag{3.11}$$

To conclude the proof, it remains to use a compactness argument, but for this we need to localise the problem in space. Let $\chi \in C_c(\mathbb{R}^n, [0,1]) \cap \mathbf{F}$ with $\chi = 1$ on the compact $\{V \leq M\}$ so that for all $f \in \mathbf{F}$,

$$|\widetilde{\mathrm{X}}(f)| \leq |\widetilde{\mathrm{X}}(\chi f)| + \widetilde{\mathrm{X}}(g) + 2\mathbb{E}\mathrm{X}(g), \quad g(x) = (1-\chi(x))|x|.$$

Note that we used that by linearity,

$$|\widetilde{\mathrm{X}}((1-\chi)f)| \leq \mathrm{X}((1-\chi)|f|) + \mathbb{E}\mathrm{X}((1-\chi)|f|) \leq \widetilde{\mathrm{X}}(g) + 2\mathbb{E}\mathrm{X}(g)$$

as $|f(x)| \leq |x|$. Since $g \in \mathbf{F}$ and $\mathrm{supp}(\varrho) \subset \{V > M\}$, as in the proof of Lemma 3.9, $\mathbb{E}\mathrm{X}(g) \to 0$ exponentially quickly as $\hbar \to 0$. This implies that if $\hbar$ is sufficiently small, then

$$\mathbb{P}[\sup_{f\in\mathbf{F}} |\widetilde{\mathrm{X}}(f)| \geq 4\epsilon N] \leq \mathbb{P}[\sup_{f\in\mathbf{F}} |\widetilde{\mathrm{X}}(\chi f)| \geq 2\epsilon N] + 2\exp\Big(-\frac{c}{2} N\epsilon^2\Big).$$

We now use that the set $\{\chi f : f \in \mathbf{F}\}$ is compact for the uniform topology (by the Arzelà–Ascoli theorem) so that for any $\epsilon > 0$, there is a finite set $\mathbf{S}_\epsilon \subset \{\chi f : f \in \mathbf{F}\}$ such that for any $f \in \mathbf{F}$,

$$\text{there exists } g \in \mathbf{S}_\epsilon \text{ such that } \quad |\widetilde{\mathrm{X}}(\chi f) - \widetilde{\mathrm{X}}(g)| \leq N\epsilon,$$

where we used that the point process X has N particles. Since estimate (3.11) is uniform over all Lipschitz functions in $\mathbf{F}$, by a union bound, this implies that for any $\epsilon > 0$ with $\epsilon \ll \hbar^{-1}$,

$$\begin{aligned}\mathbb{P}[\sup_{f\in\mathbf{F}} |\widetilde{\mathrm{X}}(f)| \geq 4\epsilon N] &\leq \mathbb{P}[\sup_{g\in\mathbf{S}_\epsilon} |\widetilde{\mathrm{X}}(g)| \geq \epsilon N] + 2\exp(-cN\epsilon^2) \\ &\leq 2(|\mathbf{S}_\epsilon| + 1)\exp(-cN\epsilon^2).\end{aligned}$$

Since $\sup_{f\in\mathbf{F}} |\widetilde{\mathrm{X}}(f)| = N\mathrm{d_W}(N^{-1}\mathrm{X}, \rho_N)$, this completes the proof. The form of the constant C_ϵ follows from standard continuity arguments. ∎

By combining Lemma 3.9 and Proposition 3.10, the proof of Theorem I.1 is complete.

Let us now turn to the proof of Theorem I.3. This CLT follows from Corollary A.12 and showing that $\operatorname{var} \mathrm{X}(f) \to \infty$ as $\hbar \to 0$, which holds true in dimension $n \geq 2$.

Proof of Theorem I.3. Let us first remark that, for fixed $f \in C(\mathbb{R}^n, \mathbb{R})$ with at most exponential growth, for $\hbar > 0$ small enough, $\mathrm{X}(f) \in L^2$ (as a real random variable). Indeed, denoting by f_+ and f_- the positive and negative parts of f, respectively, then $\mathrm{X}(f) = \mathrm{X}(f_+) - \mathrm{X}(f_-)$, and X preserves positivity. Moreover, using the determinantal structure,

$$\mathbb{E}[\mathrm{X}(f_\pm)^2] \leq \operatorname{tr}[\Pi_\hbar f_\pm]^2 + \operatorname{tr}(\Pi_\hbar f_\pm^2),$$

and $\operatorname{tr}(\Pi_\hbar g) = \int \Pi_\hbar(x,x) g(x) < \infty$ as soon as $g \geq 0$ has at most exponential growth, by Proposition 2.3 (indeed, $\operatorname{rank}(\Pi_\hbar)$ is finite and the range consists of eigenfunctions with spacial decay at a rate $\hbar^{-1}|x|$ near infinity).

By Proposition 2.3, if f is supported on $\mathbb{R}^n \setminus \{V \leq \mu\}$, then we even have $\mathbb{E}\mathrm{X}(f) = \mathcal{O}(\hbar^\infty)$. In this case, by Lemma A.9,

$$\operatorname{var} \mathrm{X}(f) \leq \operatorname{tr}(f^2 \Pi_{\hbar,\mu}) = \mathbb{E}\mathrm{X}(f^2) = \mathcal{O}(\hbar^\infty).$$

Let $\chi \in C_c^\infty(\mathbb{R}^n, [0,1])$ be a cutoff such that $\chi \geq \mathbb{1}_{\{V \leq \mu\}}$. The previous estimate implies that if $f \in C(\mathbb{R}^n, \mathbb{R})$,

$$\operatorname{var} \mathrm{X}(f\chi) = \operatorname{var} \mathrm{X}(f) + \mathcal{O}(\hbar^\infty).$$

Note that the function $f\chi$ is uniformly bounded on $\mathbb{R}^n$. Hence, if we can show that $\operatorname{var} \mathrm{X}(f\chi) \to \infty$ as $\hbar \to \infty$, by Corollary A.12, we obtain as $\hbar \to 0$,

$$\frac{\widetilde{\mathrm{X}}(f\chi)}{\sqrt{\operatorname{var} \mathrm{X}(f)}} \Rightarrow \mathcal{N}_{0,1},$$

where $\widetilde{\mathrm{X}} := \mathrm{X} - \mathbb{E}\mathrm{X}$. Moreover, $\widetilde{\mathrm{X}}(f(1-\chi)) \to 0$ in L^2 so that by Slutsky's lemma, this implies the claim of Theorem I.3. So we can assume that $f \in C_c(\mathbb{R}^n, \mathbb{R}_+)$ and then

$$\operatorname{var} \mathrm{X}(f) = -\frac{1}{2} \operatorname{tr}([f, \Pi_{\hbar,\mu}]^2) = \frac{1}{2} \int (f(x) - f(y))^2 |\Pi_{\hbar,\mu}(x,y)|^2 \mathrm{d}x dy.$$

For any open set $\Omega \subset \{V < \mu\}$ and any continuous function $\epsilon \colon \Omega \to (0,1]$, we have a lower bound,

$$\operatorname{var} \mathrm{X}(f) \geq \frac{1}{2} \int \mathbb{1}_{x \in \Omega} (f(x) - f(x + \epsilon(x) z))^2 \epsilon(x)^{-n} |K_{\epsilon(x),x}(0,z)|^2 \mathrm{d}x dz,$$

where we used notation (1.4). By Lusin's theorem, since $\nabla f \in L^2(\Omega)$, by choosing a smaller open subset $\Omega \subset \{V \leq \mu\}$, we can assume that $f \in C^1(\Omega)$. Then, choosing

$$\epsilon(x) = 2\pi\hbar \frac{\omega_n^{-1/n}}{\sqrt{\mu - V(x)}}$$

and applying Theorem II.1, it holds as $\hbar \to 0$,

$$(f(x) - f(x + \epsilon(x)z))^2 \epsilon(x)^{-2} |K_{\epsilon,x}(0,z)|^2 \to (z \cdot \nabla f(x))^2 \frac{\mathrm{J}_{n/2}^2(c_n|z|)}{\omega_n |z|^n}$$

for every $x \in \Omega$, where we used expression (1.6) for the kernel K_{bulk} and $c_n = 2\pi \omega_n^{-1/n}$. By Fatou's lemma, this implies that

$$\begin{aligned}&\liminf_{\hbar \to 0} (\hbar^{n-2} \operatorname{var} \mathrm{X}(f)) \\ &\quad \geq \frac{c_n^2}{2(2\pi)^n} \int \mathbb{1}_{x \in \Omega} (\mu - V(x))^{(n-2)/2} (z \cdot \nabla f(x))^2 \mathrm{J}_{n/2}^2(c_n|z|) \frac{\mathrm{d}z}{|z|^n} \mathrm{d}x.\end{aligned}$$

By going to spherical coordinates, this integral factorises and we obtain

$$\begin{aligned}&\liminf_{\hbar \to 0} (\hbar^{n-2} \operatorname{var} \mathrm{X}(f)) \\ &\quad \geq \frac{n\omega_n}{2(2\pi)^n} \int_\Omega (\mu - V(x))^{(n-2)/2} |\nabla f(x)|^2 \mathrm{d}x \int_{\mathbb{R}_+} \mathrm{J}_{n/2}^2(r) r \mathrm{d}r.\end{aligned}$$

Hence, if $\int_\Omega |\nabla f|^2 > 0$, the first integral is positive and using the asymptotics of the Bessel functions as $r \to \infty$,

$$\mathrm{J}_{n/2}(r) = \sqrt{\frac{2}{\pi r}} \cos\Big(r - \frac{(n+1)\pi}{4}\Big) + \mathcal{O}(r^{-3/2}), \tag{3.12}$$

we obtain that

$$\liminf_{\hbar \to 0} (\hbar^{n-2} \operatorname{var} \mathrm{X}(f)) = +\infty. \tag{3.13}$$

This concludes the proof. ∎

4. The spectral projector at the edge at microscopic scale

This section is devoted to the proof of Theorem II.2, that is, the asymptotics of the rescaled kernel K_{ϵ,x_0} defined in (1.4), around a point $x_0 \in \mathbb{R}^n$ at the boundary of the droplet, $V(x_0) = \mu$. We also assume that the point is non-degenerate, $\nabla V(x_0) \neq 0$. In this case, upon an appropriate scaling $(\epsilon, \mathcal{U})$, the limiting kernel is

$$K_{\mathrm{edge}} \colon (x, y) \mapsto \mathbb{1}_{(-\infty,0]}(-\Delta + x_1),$$

where $x_1 = x \cdot \mathrm{e}_1$, and e_1 denotes the first vector of the canonical basis of $\mathbb{R}^n$. This kernel is given explicitly in terms of Airy and Bessel functions by (1.7).

The method of proof is the same as in Section 3; we first prove convergence of a regularised projection kernel and then we apply the Tauberian theorem (Proposition 3.4) to recover the asymptotics of the rescaled kernel K_{ϵ,x_0}. The main difference with the proof of Proposition 3.2 is that, when dealing with an oscillatory integral of the form

$\int e^{\mathrm{i}\phi/\hbar}a$, the stationary point $\nabla\phi = 0$ is degenerate (the Hessian is not invertible). This explains why K_{edge} involves the Airy function, which is the simplest degenerate oscillatory integral.

We again consider a regularised projection of the form $f_{\hbar,\mu}(H_\hbar)$ as in Notations 3.1 and denote

$$K_{x_0,\epsilon}^{f_{\hbar,\mu}} \colon (x, y) \mapsto \epsilon^n f_{\hbar,\mu}(H_\hbar)(x_0 + \epsilon\mathcal{U}^* x, x_0 + \epsilon\mathcal{U}^* y),$$

where $\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}$ and $\mathcal{U} \in \mathrm{SO}_n$ is chosen so that

$$\mathcal{U}(\nabla V(x_0)) = |\nabla V(x_0)|\mathrm{e}_1$$

as in Theorem II.2.

Proposition 4.1. *Let $\mathcal{K} \Subset \mathbb{R}^{2n}$, $\mathcal{A} \Subset \{(x_0, \mu) \in \mathbb{R}^n \times (0, M), \mu = V(x_0), \nabla V(x_0) \neq 0\}$. In the above setup, it holds*

$$\sup_{x_0\in\mathcal{A}} \sup_{(x,y)\in\mathcal{K}} |K_{\epsilon,x_0}^{f_{\hbar,\mu}}(x, y) - \vartheta(\mu)K_{\mathrm{edge}}(x, y)| \leq C\hbar^{1/3}.$$

Proof. Let us denote by $\vec{\gamma} = \nabla V(x_0)$, $\gamma = |\vec{\gamma}|$ and $\delta = \frac{\hbar}{\epsilon} = \hbar^{1/3}\gamma^{1/3}$. By (3.5), the rescaled kernel $K_{\epsilon,x_0}^{f_{\hbar,\mu}}$ at a point $(x, y) \in \mathcal{K}$ is given by

$$\frac{\epsilon^n}{(2\pi\hbar)^{n+1}} \int e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x_0+\epsilon\mathcal{U}^*x,\xi)-(x_0+\epsilon\mathcal{U}^*y)\cdot\xi-\lambda t)} \\ \times a(x_0 + \epsilon\mathcal{U}^* x, x_0 + \epsilon\mathcal{U}^* y, \xi, t; \hbar)\hat{\rho}(t)\varkappa_\mu(\lambda)\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}t + \mathcal{O}(\hbar^\infty),$$

where the error term is uniform for $(x_0, \mu) \in \mathcal{A}$ and $(x, y) \in \mathcal{K}$. As in the proof of Proposition 3.2, we can rewrite

$$e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x_0+\epsilon\mathcal{U}^*x,\xi)-(x_0+\epsilon\mathcal{U}^*y)\cdot\xi)}a(x_0 + \epsilon\,\mathcal{U}^* x, x_0 + \epsilon\,\mathcal{U}^* y, \xi, t; \hbar) \\ = e^{\frac{\mathrm{i}}{\hbar}\Psi(t,x_0,\xi)}e^{\frac{\mathrm{i}}{\delta}(\partial_x\phi(t,x_0,\xi)\cdot\mathcal{U}^*x-\xi\cdot\mathcal{U}^*y)}b(x, y, \xi, t; \delta),$$

where $\Psi(t, x_0, \xi)$ is as in Proposition 3.1 and b is a classical symbol with principal part at $t = 0$,

$$b_0(x, y, \xi, 0) = \vartheta(\mu + |\xi|^2).$$

The critical point for the phase $\Psi(t, x_0, \xi) - \lambda t$ is again given by $t = 0$, $\xi = 0$, $\lambda = \mu$, but it is degenerate. However, by Proposition 3.1, one can rewrite the phase as

$$\Psi(t, x_0, \xi) = t(|\eta(t, x_0, \xi)|^2 + g(t, x_0) + \mu),$$

where

$$g(t, x_0) = t^2\frac{\gamma^2}{12} + \mathcal{O}(t^3) \quad \text{and} \quad \eta(t, x_0, \xi) = \xi + t\frac{\vec{\gamma}}{2} + O(t^2).$$

We make a change of variable $\xi \leftarrow \eta$ and use that the Jacobian $|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}| = 1 + \mathcal{O}(t)$ is a smooth non-vanishing function for $t \in \operatorname{supp}(\widehat{\rho})$. Then

$$\sqrt{2\pi}\widehat{\rho}(t) b(x, y, \xi, t; \delta) \left|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right| = c(x, y, \eta, t; \delta)$$

is again a classical symbol, compactly supported in (t, η). Making a change variable $\lambda \leftarrow (\mu - \lambda)$ and letting $\chi_1 = \varkappa(\mu - \cdot)$ (according to Notations 3.1, the cutoff $\chi_1 \in C_c^\infty(\mathbb{R}, \mathbb{R}_+)$ can be chosen independent of μ and is equal to 1 on a neighbourhood of 0), this implies that for $(x, y) \in \mathcal{K}$,

$$\widetilde{K}^{\mu}_{\epsilon,x_0}(x, y) = \frac{\epsilon^n}{(2\pi\hbar)^{n+1}} \int \mathbb{1}_{\lambda \geq 0} e^{\mathrm{i}t\psi(\eta,t,\lambda)/\hbar} e^{\mathrm{i}\omega(x,y,t,\eta)/\delta} \times c(x, y, \eta, t; \delta)\chi_1(\lambda) \mathrm{d}\eta \mathrm{d}\lambda \mathrm{d}t + \mathcal{O}(\hbar^\infty),$$

where the error term is C^∞ with the required uniformity and the phases are given by

$$\begin{aligned} \omega(x, y, t, \eta) &= \partial_x \Psi(t, x_0, \xi(t, \eta)) \cdot \mathcal{U}^* x + \xi(t, \eta) \cdot \mathcal{U}^*(x - y), \\ \psi(\eta, t, \lambda) &= |\eta|^2 + g(t, x_0) + \lambda. \end{aligned}$$

Moreover, the classical symbol c has principal part at $t = 0$, for $x, y \in \mathcal{K}$,

$$c_0(x, y, \eta, 0) = \vartheta(\mu + |\eta|^2).$$

We claim that this integral can be localised to the set $\{|t| \leq \delta^{1-\alpha}, |\eta| \leq \delta^{1-\alpha}\}$ for any small $\alpha \in (0, 1)$ up to an arbitrary small error. Namely, if $\chi_2 \in C_c^\infty(\mathbb{R}^{n+1}, \mathbb{R}_+)$ is a cutoff which equals 1 on the unit ball B_{n+1}, we will show that the integral

$$J_\hbar = \int e^{\mathrm{i}t\psi(\eta,t,\lambda)/\hbar} e^{\mathrm{i}\omega(x,y,t,\eta)/\delta} c(x, y, \eta, t; \delta)(1 - \chi_2(\delta^{\alpha-1}(t, \eta))) \mathrm{d}\eta \mathrm{d}t$$

is $\mathcal{O}(\hbar^\infty)$. This relies on the fact that for $t \in \operatorname{supp}(\widehat{\rho})$ and $\lambda \in [0, \mu]$,

$$\partial_t [t\psi(\eta, t, \lambda)] \geq c(|t|^2 + |\eta|^2) \quad \text{and} \quad \partial_t^2 [t\psi(\eta, t, \lambda)] = O(t).$$

Let $\mathcal{D}_t = \partial_t (\frac{\cdot}{\partial_t [t\psi(\eta,t,\lambda)]})$. The previous bounds imply that if $(t, \eta) \notin \delta^{1-\alpha} B_{n+1}$, then for any $k \in \mathbb{N}$,

$$\left|\mathcal{D}_t^k \left(e^{\mathrm{i}\omega(x,y,t,\eta)/\delta} c(x, y, \eta, t; \delta)(1 - \chi(\delta^{\alpha-1}(t, \eta)))\right)\right| \leq C_k \delta^{-k(3-2\alpha)} = \mathcal{O}_k(\hbar^{-k(1-2\alpha/3)}).$$

Hence, repeated integration by parts shows that for any $k \in \mathbb{N}$,

$$\begin{aligned} J_\hbar &= (\mathrm{i}\hbar)^k \int e^{\mathrm{i}t\psi(\eta,t,\lambda)/\hbar} \mathcal{D}_t^k \left(e^{\mathrm{i}\omega(x,y,t,\eta)/\delta} c(x, y, \eta, t; \delta)(1 - \chi_2(\delta^{\alpha-1}(t, \eta)))\right) \mathrm{d}\xi \mathrm{d}\lambda \mathrm{d}t \\ &= \mathcal{O}_k(\hbar^{2k\alpha/3}). \end{aligned}$$

This proves that uniformly for all $(x_0, \mu) \in \mathcal{A}$ and $(x, y) \in \mathcal{K}$,

$$\widetilde{K}^{\mu}_{\epsilon,x_0}(x, y) = \frac{\epsilon^n}{(2\pi\hbar)^{n+1}} \int \mathbb{1}_{\lambda \geq 0} e^{\mathrm{i}t\psi(\eta,t,\lambda)/\hbar} e^{\mathrm{i}\omega(x,y,t,\eta)/\delta} \times c(x, y, \eta, t; \delta)\chi_1(\lambda)\chi_2(\delta^{\alpha-1}(t, \eta)) \mathrm{d}\eta \mathrm{d}\lambda \mathrm{d}t + \mathcal{O}(\hbar^\infty).$$

We can now perform a Taylor expansion of the two phases. By Proposition 3.1,

$$\psi(\gamma^{1/3}\mathcal{U}^*\eta, t\gamma^{-2/3}, \lambda) = \gamma^{2/3}\Big(|\eta|^2 + \frac{t^2}{12}\Big) + \lambda + \mathcal{O}(t^3),$$

$$t\gamma^{-2/3}\psi(\gamma^{1/3}\mathcal{U}^*\eta, t\gamma^{-2/3}, \gamma^{2/3}\lambda) = \frac{1}{3}\Big(\frac{t}{2}+\eta_1\Big)^3+\frac{1}{3}\Big(\frac{t}{2}-\eta_1\Big)^3+t(|\eta^\perp|^2+\lambda)+\mathcal{O}(t^4),$$

where we decompose $\eta = (\eta_1, \eta^\perp)$. In addition, by Proposition 3.1,

$$\partial_x\Psi(t, x_0, \xi) = t\vec{\gamma} + \mathcal{O}(t^2), \quad \xi(t,\eta) = \eta - \frac{t}{2}\vec{\gamma} + \mathcal{O}(t^2),$$

so that

$$\begin{aligned}\omega(x, y, t, \eta) &= \xi(t,\eta)\cdot\mathcal{U}^*(x-y) + \partial_x\Psi(t, x_0, \xi(t,\eta))\cdot\mathcal{U}^*x \\ &= \mathcal{U}\eta\cdot(x-y) + \frac{t}{2}\mathcal{U}\vec{\gamma}\cdot(x+y) + \mathcal{O}(t^2).\end{aligned}$$

Since $\mathcal{U}\vec{\gamma} = \gamma \mathrm{e}_1$, this implies that with the same scaling,

$$\omega(x, y, \gamma^{-2/3}t, \gamma^{1/3}\mathcal{U}^*\eta) = \gamma^{1/3}\Big(\eta^\perp\cdot(x-y) + x_1\Big(\frac{t}{2}+\eta_1\Big) + y_1\Big(\frac{t}{2}-\eta_1\Big)\Big) + \mathcal{O}(t^2).$$

Let us also decompose $x = (x_1, x^\perp)$ and $y = (y_1, y^\perp)$. These expansions imply that if we make a change of variables

$$u = \delta^{-1}\Big(\gamma\frac{t}{2}+\eta_1\Big), \quad v = \delta^{-1}\Big(\gamma\frac{t}{2}-\eta_1\Big), \quad z = \delta^{-1}\eta^\perp \quad \text{and} \quad s = \delta^{-2}(\lambda + |\eta^\perp|^2),$$

then we can rewrite

$$\begin{aligned}&e^{\mathbf{i}t\psi(\eta,t,\lambda)/\hbar}e^{\mathbf{i}\omega(x,y,t,\eta)/\delta}c(x, y, \eta, t; \delta) \\ &\quad= \exp\Big(\mathbf{i}\Big(\frac{u^3}{3} + (x_1+s)u + \frac{v^3}{3} + (y_1+s)v\Big) + \mathbf{i}z\cdot(x^\perp - y^\perp)\Big) \\ &\quad\times f(x, y, u, v, z, s; \delta).\end{aligned} \tag{4.1}$$

Note that we used in a crucial way that the errors coming from the expansions of the phases are given $\mathcal{O}(\hbar^{-1}t^4) = \mathcal{O}((\frac{u+v}{2})^4\delta)$ and $\mathcal{O}(\delta^{-1}t^2) = \mathcal{O}((\frac{u+v}{2})^2\delta)$, so that f is again a classical symbol with constant principal part,

$$f_0(x, y, u, v, z, s) = c_0(x, y, 0, 0) = \vartheta(\mu).$$

Hence, making the change of variables $(t, \eta, \lambda) \mapsto (u, v, z, s)$ as above (whose Jacobian is given by $\hbar\delta^n$) and using (4.1), we obtain

$$\begin{aligned}\widetilde{K}^\mu_{\epsilon,x_0}(x, y) = \frac{1}{(2\pi)^{n+1}}\int &\mathbb{1}_{s\geq|z|^2}e^{\mathbf{i}(u^3/3+(x_1+s)u+v^3/3+(y_1+s)v+z\cdot(x^\perp-y^\perp))} \\ &\times f(x, y, u, v, z, s; \delta)\chi_{3,\delta}(s, u, v, z)\mathrm{d}u\mathrm{d}v\mathrm{d}z\mathrm{d}s \\ &+ \mathcal{O}(\hbar^\infty),\end{aligned} \tag{4.2}$$

where the cutoff

$$\chi_{3,\delta}(s,u,v,z) = \chi_1(\delta^2(s-|z|^2))\chi_2\Big(\delta^\alpha\Big(\frac{u+v}{\gamma},\frac{u-v}{2},z\Big)\Big).$$

Since the cutoffs are arbitrary, we can assume that

$$\chi_{3,\delta}(s,u,v,z) = \chi_1(\delta^2(s-|z|^2))\chi_1(\delta^\alpha u)\chi_1(\delta^\alpha v)\chi_1(\delta^\alpha|z|).$$

Let us denote

$$\mathcal{D} = -\partial_{uv}\Big(\frac{\cdot}{(u^2+s+1)(v^2+s+1)}\Big)$$

and observe that performing repeated integrations by parts with respect to (u,v), for any smooth function $g\colon\mathbb{R}^2\to\mathbb{R}$ with $\|\partial_u^j\partial_v^\ell g\|_{L^\infty}<\infty$ for all $j,\ell\in\mathbb{N}_0$, it holds for any $k\in\mathbb{N}$,

$$\begin{aligned}
&\int e^{\mathbf{i}(u^3/3+(x_1+s)u+v^3/3+(y_1+s)v)}g(u,v)\chi_1(\delta^\alpha u)\chi_1(\delta^\alpha v)\mathrm{d}u\mathrm{d}v\\
&\quad=\int e^{\mathbf{i}(u^3/3+(s+1)u+v^3/3+(s+1)v)}\\
&\qquad\times\mathcal{D}^k(e^{\mathbf{i}(u(x_1-1)+v(y_1-1))}g(u,v)\chi_1(\delta^\alpha u)\chi_1(\delta^\alpha v))\mathrm{d}u\mathrm{d}v\\
&\quad=\mathcal{O}_k\Big(\int\frac{\mathrm{d}u\mathrm{d}v}{(u^2+s+1)^k(v^2+s+1)^k}\Big)=\mathcal{O}_k\Big(\frac{1}{(1+s)^{2k}}\Big).
\end{aligned}$$

These estimates imply that we can localise integral (4.2) on $\{s\le\delta^{-\alpha}\}$ and expand the symbol f with respect to δ; using that $f_0=\vartheta(\mu)$, we obtain for any $k\in\mathbb{N}$,

$$\begin{aligned}
\widetilde{K}^\mu_{\epsilon,x_0}(x,y)=\frac{\vartheta(\mu)}{(2\pi)^{n+1}}&\int_{s\ge|z|^2}e^{\mathbf{i}(u^3/3+(x_1+s)u+v^3/3+(y_1+s)v+\mathbf{i}z\cdot(x^\perp-y^\perp))}\\
&\qquad\times\chi_{3,\delta}(s,u,v,z)\chi_1(\delta^\alpha s)\mathrm{d}u\mathrm{d}v\mathrm{d}z\mathrm{d}s\\
&+\mathcal{O}_k\Big(\delta\int\mathbb{1}_{s\ge|z|^2}\frac{\mathrm{d}z\mathrm{d}s}{(1+s)^{2k}}\Big)+\mathcal{O}(\hbar^\infty).
\end{aligned}$$

Moreover,

$$\chi_1(\delta^2(s-|z|^2))\chi_1(\delta^\alpha|z|)=1$$

if $|z|^2\le s\le C\delta^{-\alpha}$ and δ is small enough, so that the leading term on the right-hand side of (4.2) factorises as

$$\begin{aligned}
\widetilde{K}^\mu_{\epsilon,x_0}(x,y)&=\vartheta(\mu)\int_0^\infty\chi_1(\delta^\alpha s)I_u(s)I_v(s)I_z(s)+\mathcal{O}(\delta),\\
I_u(s)&=\frac{1}{2\pi}\int e^{\mathbf{i}(u^3/3+(x_1+s)u)}\chi_1(\delta^\alpha u)\mathrm{d}u,\\
I_v(s)&=\frac{1}{2\pi}\int e^{\mathbf{i}(v^3/3+(y_1+s)v)}\chi_1(\delta^\alpha v)\mathrm{d}v,\\
I_z(s)&=\frac{1}{(2\pi)^{n-1}}\int_{\sqrt{s}\ge|z|}e^{\mathbf{i}z\cdot(x^\perp-y^\perp)}\mathrm{d}z.
\end{aligned}$$

Finally, by (A.2) and Lemma A.20, this implies that for any $k \in \mathbb{N}$,

$$\widetilde{K}^{\mu}_{\epsilon,x_0}(x,y) = \vartheta(\mu)\int_0^\infty \chi_1(\delta^\alpha s)\Big(\operatorname{Ai}(x_1+s) + \mathcal{O}_k\Big(\frac{\delta^{2\alpha k}}{(s+1)^k}\Big)\Big) \times \Big(\operatorname{Ai}(y_1+s) + \mathcal{O}_k\Big(\frac{\delta^{2\alpha k}}{(s+1)^k}\Big)\Big) \times \frac{\mathrm{J}_{(n-1)/2}(\sqrt{s}|x^\perp - y^\perp|)}{(2\pi|x^\perp - y^\perp|)^{(n-1)/2}} s^{(n-1)/4}\mathrm{d}s + \mathcal{O}(\delta)$$

uniformly for all $(x,y) \in \mathcal{K}$. Using the uniform bounds for $\nu \geq 0$, $\max_{r\geq 0} \frac{|\mathrm{J}_\nu(\sqrt{s}r)|}{r^\nu} \leq C_\nu s^{\nu/2}$, $\|\operatorname{Ai}\|_{L^\infty(\mathbb{R})} < \infty$ and that for any $r \in \mathbb{R}$,

$$\int_0^\infty |\operatorname{Ai}(r+s)|s^\nu \mathrm{d}s < \infty \quad \text{is of order } \mathcal{O}_\nu(r^{-\infty}) \text{ as } r \to \infty, \tag{4.3}$$

(cf. (A.19)), we conclude that

$$\widetilde{K}^{\mu}_{\epsilon,x_0}(x,y) = \vartheta(\mu)\int_0^\infty \operatorname{Ai}(x_1+s)\operatorname{Ai}(y_1+s)\frac{\mathrm{J}_{(n-1)/2}(\sqrt{s}|x^\perp - y^\perp|)}{(2\pi|x^\perp - y^\perp|)^{(n-1)/2}} s^{(n-1)/4}\mathrm{d}s + \mathcal{O}(\delta),$$

where the error term is uniform for all $(x_0,\mu) \in \mathcal{A}$ and $(x,y) \in \mathcal{K}$. Up to the factor $\vartheta(\mu)$, we identify that this kernel is exactly $K_{\text{edge}}(x,y)$ and as $\delta = \mathcal{O}(\hbar^{1/3})$, this completes the proof. ∎

Like in Section 3.3, we may use the asymptotics of Proposition 4.1 and the Tauberian theorem (Proposition 3.4) in order to obtain the edge asymptotics of the rescaled projection kernel $\Pi_{\hbar,\mu}$. This application of the Tauberian method is more subtle because the counting function changes regimes precisely at $\mu = V(x_0)$.

Recall that $\vartheta_\lambda = \vartheta\mathbb{1}_{[0,\lambda]}$ and let us denote

$$N^1_{\epsilon,\lambda}(x) := \epsilon^n \vartheta_\lambda(H_\hbar)(x,x), \quad \lambda \in \mathbb{R}_+,\ x \in \mathbb{R}^n.$$

Pay attention that the edge-scaling is different from that of Section 3.3 and $\epsilon = \mathcal{O}(\hbar^{2/3})$. The derivative of this function (with respect to λ) satisfies for $\lambda \in \mathbb{R}_+$,

$$\int N^{1\prime}_{\epsilon,\sigma}(x)\rho_\hbar(\lambda-\sigma)\mathrm{d}\sigma = \int N^1_{\epsilon,\sigma}(x)\rho_\hbar'(\lambda-\sigma)\mathrm{d}\sigma = \epsilon^n \zeta_{\hbar,\lambda}(H_\hbar)(x,x), \tag{4.4}$$

where

$$\zeta_{\hbar,\lambda} = \vartheta(\cdot)\rho_\hbar(\cdot - \lambda).$$

We expect that $\zeta_{\hbar,\lambda}(H_\hbar)(x,x) = \mathcal{O}(\hbar^\infty)$ for $\lambda < V(x)$ and this quantity becomes relevant when $\lambda = V(x)$. In this case, by adapting the proof of Proposition 4.1, the kernel of $\zeta_{\hbar,\lambda}(H_\hbar)$ can be controlled appropriately, locally uniformly (at scale ϵ). We obtain the following bounds.

Proposition 4.2. *Let $\vartheta \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$ be a cutoff and for $c \geq 1$, let*

$$\mathcal{A}_\hbar \Subset \{x \in \mathbb{R}^n,\ \lambda \in [0, M) : V(x) \geq \lambda - c\hbar^{2/3},\ \nabla V(x) \neq 0\} \tag{4.5}$$

be any compact set. There exists a constant $C > 0$ such that for all $(x, \lambda) \in \mathcal{A}_\hbar$,

$$\hbar^{2(n+1)/3} \zeta_{\hbar,\lambda}(H_\hbar)(x, x) \leq C.$$

Proof. By Proposition 2.12, it holds for $x \in \mathbb{R}^n$,

$$\zeta_{\hbar,\lambda}(H_\hbar)(x,x) = \frac{1}{(2\pi)^{n+1/2}\hbar^{n+1}} \int e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x,\xi) - x\cdot\xi - \lambda t)} a(x,\xi,t;\hbar)\hat{\rho}(t)\mathrm{d}t\mathrm{d}\xi + \mathcal{O}(\hbar^\infty),$$

where a is a classical symbol with principal part $a_0(x,\xi,0) = \vartheta(|\xi|^2 + V(x))$ at $t = 0$, the error term is independent of $\lambda \in \mathbb{R}_+$ and locally uniform. According to Proposition 3.1, the phase satisfies

$$\phi(t,x,\xi) - x\cdot\xi - \lambda t = t(|\eta|^2 + g(t,x) + V(x) - \lambda),$$

where

$$g(t,x) = t^2 \frac{|\nabla V(x)|^2}{12} + \mathcal{O}(t^4).$$

Hence, by a change of variable $\xi \leftarrow \eta$, we can rewrite

$$\zeta_{\hbar,\lambda}(H_\hbar)(x,x) = \frac{1}{(2\pi\hbar)^{n+1}} \int e^{\frac{\mathrm{i}t}{\hbar}(|\eta|^2 + g(t,x) + V(x) - \lambda)} b(x,\eta,t;\hbar)\mathrm{d}t\mathrm{d}\eta + \mathcal{O}(\hbar^\infty),$$

where $b(x,\eta,t;\hbar) = \sqrt{2\pi}\hat{\rho}(t)a(x,\xi,t;\delta)|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}|$ is again a classical symbol. By assumptions $V(x) \geq \lambda - c\hbar^{2/3}$, so that exactly as in the proof of Proposition 4.1, we can localise this integral in (t,η) at scale $\hbar^{1/3-\alpha}$ for any small $\alpha > 0$ up to an error which is $\mathcal{O}(\hbar^\infty)$. This means that for any cutoff $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R}_+)$ which equals 1 on $[-1, 1]$, it holds

$$\begin{aligned}\zeta_{\hbar,\lambda}(H_\hbar)(x,x) = \frac{1}{(2\pi\hbar)^{n+1}} \int & e^{\frac{\mathrm{i}t}{\hbar}(|\eta|^2 + g(t,x) + V(x) - \lambda)} b(x,\eta,t;\hbar) \\ & \times \chi(\hbar^{\alpha-1/3}t)\chi(\hbar^{\alpha-1/3}|\eta|)\mathrm{d}t\mathrm{d}\eta + \mathcal{O}(\hbar^\infty)\end{aligned}$$

uniformly for $(x,\lambda) \in \mathcal{A}_\hbar$. Let $\gamma(x) = \frac{|\nabla V(x)|}{2}$. By assumption, this function is bounded uniformly from above and below on $\mathcal{A}_\hbar$, and by making a change of variables

$$\hbar^{-1/3}\gamma^{2/3}t \leftarrow t \quad \text{and} \quad \hbar^{-1/3}\gamma^{-1/3}\eta \leftarrow \eta,$$

we obtain

$$\begin{aligned}&\hbar^{(2n+2)/3}\zeta_{\hbar,\lambda}(H_\hbar)(x,x) \\ &\quad = \frac{\gamma(x)^{(n-2)/3}}{(2\pi)^{n+1}} \int e^{\mathrm{i}(t^3/3 + t|\eta|^2 + \frac{t(V(x)-\lambda)}{(\gamma(x)\hbar)^{2/3}})} f(x,t,\eta;\hbar^{1/3}) \\ &\qquad\qquad \times \chi(\hbar^\alpha\gamma^{-2/3}t)\chi(\hbar^\alpha\gamma^{1/3}|\eta|)\mathrm{d}t\mathrm{d}\eta + \mathcal{O}(\hbar^\infty),\end{aligned}$$

where

$$f(x,t,\eta;\hbar^{1/3}) = e^{\mathrm{i}\mathcal{O}(t^4\hbar^{1/3})} b(x,\gamma^{1/3}\hbar^{1/3}\eta,\gamma^{-2/3}\hbar^{1/3}t;\hbar) = \vartheta(V(x)) + \mathcal{O}(\hbar^{1/3-4\alpha})$$

uniformly for $x \in \mathcal{A}_\hbar$ and $t, |\eta| \le C\hbar^{-\alpha}$.

If we further let $\eta = r\omega$, where $(r,\omega) \in \mathbb{R}_+ \times S^{n-1}$, this implies that for some $\theta(n) \in \mathbb{N}$,

$$\begin{aligned}
&\hbar^{(2n+2)/3}\zeta_{\hbar,\lambda}(H_\hbar)(x,x) \\
&\quad = \frac{\gamma(x)^{(n-2)/3}\vartheta(V(x))}{(2\pi)^{n+1}} \int_{S^{n-1}\times\mathbb{R}_+} \chi(\hbar^\alpha\gamma^{1/3}r)r^{n-1} \\
&\qquad \times \int e^{\mathrm{i}(t^3/3+tr^2+\frac{t(V(x)-\lambda)}{(\gamma(x)\hbar)^{2/3}})}\chi(\hbar^\alpha\gamma^{-2/3}t)\mathrm{d}t\mathrm{d}\omega\mathrm{d}r + \mathcal{O}(\hbar^{1/3-\theta\alpha}),
\end{aligned}$$

where the error term is uniform for $(x,\lambda) \in \mathcal{A}_\hbar$. Since the cutoff $\chi \in C_c^\infty(\mathbb{R},\mathbb{R}_+)$ is equal to 1 on a neighbourhood of 0, by Lemma A.20, it holds uniformly for $u \in \mathbb{R}_+$ and $(x,\lambda) \in \mathcal{A}$,

$$\frac{1}{2\pi}\int e^{\mathrm{i}(t^3/3+tu)}\chi(\hbar^\alpha\gamma^{-2/3}t)\mathrm{d}t = \mathrm{Ai}(u) + \mathcal{O}(\hbar^\infty).$$

By (4.3), we conclude that there exists a constant $C > 0$ such that

$$\sup_{\hbar\in(0,1]}\max_{(x,\lambda)\in\mathcal{A}} \hbar^{(2n+2)/3}\zeta_{\hbar,\lambda}(H_\hbar)(x,x) \le C.$$

This completes the proof. ∎

Let us now introduce again the counting functions

$$\begin{aligned}
N^1_{\epsilon,\lambda}(x) &:= \epsilon^n \sum_{\sigma\le\lambda}\vartheta(\sigma)|v_\sigma(x)|^2, \\
N^2_{\epsilon,\lambda}(x) &:= \epsilon^n \sum_{\sigma\le\lambda}\vartheta(\sigma)|v_\sigma(x)-v_\sigma(y)|^2,
\end{aligned}$$

where $\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}$.

Proposition 4.2 has the following consequence for the diagonal counting function.

Corollary 4.3. *Let $\mathcal{K} \Subset \mathbb{R}^n$ and $\mathcal{A} \Subset \{(x_0,\mu) \in \mathbb{R}^n \times (0,M), V(x_0) = \mu, \nabla V(x_0) \neq 0\}$. Then*

$$\max_{(x_0,\mu)\in\mathcal{A}}\max_{x\in\mathcal{K}} |N^1_{\epsilon,\mu}(x_0+\epsilon x) - N^1_{\epsilon,\cdot} * \rho_\hbar|_\mu(x_0+\epsilon x)| = \mathcal{O}(\hbar^{1/3}).$$

Proof. We work under general assumptions. Let N be as in Proposition 3.4 and define for $\lambda \in \mathbb{R}_+$, $\breve{N}(\lambda) := \min(N(\lambda), N(\mu+\hbar^{1-\alpha}))$, where $\alpha > 0$ is a small parameter. Then $(\breve{N})' * \rho_\hbar \le N' * \rho_\hbar$ pointwise and for $\lambda \le \mu$

$$(N-\breve{N}) * \rho_\hbar(\lambda) \le \hbar^{-1}\int_{\mathbb{R}_+}\rho\Big(\frac{\sigma-\lambda}{\hbar}\Big)(N(\sigma)-N(\mu+\hbar^{1-\alpha}))_+\mathrm{d}\sigma = \mathcal{O}(\hbar^\infty),$$

using that $\rho \in \mathcal{S}$ has superpolynomial decay and N has polynomial growth. Hence, if we assume that $\max_{[0,\mu+\hbar^{1-\alpha}]} N' * \rho_\hbar \le C\hbar^{-\beta}$, by applying Proposition 3.4 to $C^{-1}\hbar^{\beta}\check{N}$, we obtain the uniform bound

$$\max_{\lambda\le\mu} |N - N * \rho_\hbar| \le \max_{\lambda\le\mu} |\check{N} - \check{N} * \rho_\hbar| + \mathcal{O}(\hbar^\infty) \le \mathcal{O}(\hbar^{1-\beta}). \tag{4.6}$$

By (4.4), $\partial_\mu(N^1_{\hbar,\mu} * \rho_\hbar) = \epsilon^n \zeta_{\hbar,\mu}(H_\hbar)$, so that by Proposition 4.2, there exists a constant C such that

$$\max_{(x_0,\mu)\in\mathcal{A}} \max_{\lambda\le\mu+\hbar^{2/3}} \max_{x\in\mathcal{K}} \epsilon^n \zeta_{\hbar,\lambda}(H_\hbar)(x_0+\epsilon x) \le C\hbar^{-2/3}.$$

Here we used that since V is smooth on $\{V \le M\}$, $\mathcal{A}_\hbar = \{(x_0+\epsilon x, \lambda) : (x_0,\mu) \in \mathcal{A}, x \in \mathcal{K}, \lambda \le \mu + \hbar^{2/3}\}$ is a compact set of (4.5). Thus, if we apply bound (4.6) to $\lambda \mapsto N^1_{\hbar,\lambda}(x_0+\epsilon x)$ with $(\alpha,\beta) = (\frac{1}{3}, \frac{2}{3})$, we obtain the claim. ∎

Recall that $N^1_{\epsilon,\mu}(x) = \epsilon^n \vartheta_\mu(H_\hbar)(x,x)$ with $\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}$. Using Notations 3.1, by (3.9), we obtain

$$\int N^1_{\epsilon,\sigma}(x_0 + \epsilon \mathcal{U}^* x)\rho_\hbar(\mu-\sigma)\mathrm{d}\sigma = K^{f_{\hbar,\mu}}_{x_0,\epsilon}(x,x) + \mathcal{O}(\hbar^\infty),$$

where the error is controlled locally uniformly for $\{(x_0,\mu), V(x_0) = \mu, \nabla V(x_0) \ne 0\}$ and $x \in \mathbb{R}^n$. Choosing $\mathcal{U} \in \mathrm{SO}_n$ as in Theorem II.2, we conclude by Proposition 4.1 and Corollary 4.3 that

$$|N^1_{\epsilon,\mu}(x_0 + \epsilon\, \mathcal{U}^* x) - \vartheta(\mu) K_{\mathrm{edge}}(x,x)| = \mathcal{O}(\hbar^{1/3})$$

with the same uniformity.

Hence, choosing any cutoff ϑ equal to 1 on $[1,\mu]$, this implies that the rescaled projection kernel (1.4) obeys the relevant asymptotics (on the diagonal) in the edge case; for any compact sets $\mathcal{A} \subset \{V = \mu; \nabla V \ne 0\}$ and $\mathcal{K} \subset \mathbb{R}^n$,

$$\sup_{x_0\in\mathcal{A}} \sup_{x\in\mathcal{K}} |K_{x_0,\epsilon}(x,x) - K_{\mathrm{edge}}(x,x)| \le C\hbar^{1/3}.$$

To complete the proof of Theorem II.2, we can just adapt the proof of Proposition 3.6 using the previous estimates to obtain the relevant off-diagonal asymptotics. Note that in this case, the scalings are such that we cannot argue that any derivative of $K_{x_0,\epsilon}$ converges to that of K_{edge}.

Proof of Theorem II.2. Let us choose a cutoff $\vartheta \in C_c^\infty(\mathbb{R}_+, [0,1])$ such that $\vartheta \ge \mathbb{1}_{[0,\mu]}$ and let $K_\lambda = K^{\vartheta_\lambda}_{x_0,\epsilon} = K^{\mathbb{1}_{[0,\lambda]}}_{x_0,\epsilon}$, where $x_0 \in \mathcal{A}$ and $\epsilon = \hbar^{2/3}|\nabla V(x_0)|^{-1/3}$. This holds for any $\lambda \in [0,\mu]$ since we assume that the potential $V \ge 0$. We consider the counting function for $\sigma \ge 0$,

$$N^2_{\epsilon,\sigma}(x,y) = \epsilon^n \sum_{\lambda\le\sigma} |\phi_\lambda(x_0+\epsilon x) - \phi_\lambda(x_0+\epsilon y)|^2,$$

where (λ, ϕ_λ) are normalised eigenpairs of $H_\hbar$. Like in the proof of Proposition 3.6, the linear relationships (3.10) hold with $\hbar \leftarrow \epsilon$, so that if we proceed like in the proof of Corollary 4.3 for $N^2_{\epsilon,\sigma}(x, y)$, we conclude that for any compact sets $\mathcal{A} \subset \{V = \mu; \nabla V \neq 0\}$ and $\mathcal{K} \subset \mathbb{R}^n$,

$$\sup_{x_0 \in \mathcal{A}} \sup_{x,y \in \mathcal{K}} |K_{x_0,\epsilon}(x, y) - K_{\text{edge}}(x, y)| \le C\hbar^{1/3}.$$

This completes the proof. ∎

5. Mesoscopic fluctuations

The goal of this section is to prove Theorems III.1 and the Gaussian tail bounds of Theorem I.2. The arguments consist in controlling the variance of a mesoscopic, or macrocscopic linear statistics.

The first step is to study the model case of the free Laplacian in Section 5.1, which is helpful to understand the general picture. In this case, we obtain an (optimal) functional CLT as the intensity of the point process $\mu \to \infty$, see Theorem III.2 below.

5.1. Central limit theorem for the free Laplacian point process

In this section, we study linear statistics of the determinantal point process X_∞ associated with the free Laplacian, that is, with the operator

$$K_{\mathrm{b},\mu} = \mathbb{1}_{(-\infty,\mu^2]}(-\Delta)\colon (x, y) \mapsto \mu^{n/2} \frac{\mathrm{J}_{n/2}(\mu|x - y|)}{(2\pi|x - y|)^{n/2}},$$

in the regime where $\mu \to \infty$, or equivalently as the intensity $\frac{\mu^n \omega_n}{(2\pi)^n}$ of the particle tends to infinity, cf. (A.2). We obtain the following central limit theorem.

Theorem III.2. *Suppose that $n \ge 2$ and let $\sigma_n^2 = \frac{\omega_{n-1}}{(2\pi)^n}$. For any $g \in H^{1/2} \cap L^1(\mathbb{R}^n)$, it holds in distribution as $\mu \to \infty$,*

$$\mathrm{X}_{\infty,\mu}(g) := \frac{\mathrm{X}_\infty(g) - \mu^n\omega_n/(2\pi)^n \int g}{\sigma_n \mu^{(n-1)/2}} \to \mathcal{N}_{0,\Sigma^2(f)}, \ \textit{where}\ \Sigma^2(g) = \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2|\xi|\mathrm{d}\xi.$$

As in Theorem III.1, the interpretation is that the random process $\mathrm{X}_{\infty,\mu}$ converges in the sense of finite-dimensional distributions as $\mu \to \infty$ to a (centred) Gaussian field G on $\mathbb{R}^n$ with correlation kernel

$$\mathbb{E}\mathrm{G}(f)\mathrm{G}(g) = \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}|\xi|\mathrm{d}\xi, \quad f, g \in H^{1/2}(\mathbb{R}^n).$$

Note that the process $\mathrm{X}_{\infty,\mu}$ is exactly centred and the assumptions of Theorem III.2 are optimal in the sense that $\mathrm{X}_{\infty,\mu}$ is a priori defined on $L^1(\mathbb{R}^n)$ and the asymptotic variance $\Sigma^2(g)$ is finite if and only if $g \in H^{1/2}$. In dimension $n = 1$, $K_{\mathrm{b},\mu}\colon (x, y) \mapsto \frac{\sin(\mu|x-y|)}{\pi|x-y|}$ is the sine-kernel from random matrix theory and the counterpart of Theorem III.2 is a classical result. In this case, the CLT holds without (re)normalisation; we refer to [59,84] for different proofs.

The proof of Theorem III.2 relies on Corollary A.12 and the following lemma which controls the asymptotics of the variance of $\mathrm{X}_{\infty,\mu}(g)$ for general test functions.

Lemma 5.1. *For $n \geq 1$ and for every $g \in H^{1/2}(\mathbb{R}^n)$, it holds as $\mu \to \infty$,*

$$\operatorname{var} \mathrm{X}_{\infty}(g) = -\frac{1}{2}\operatorname{tr}([g, K_{\mathrm{b},\mu}]^2) \sim \sigma_n^2 \mu^{n-1} \Sigma^2(g).$$

Proof. By definition,

$$\operatorname{tr}([g, K_{\mathrm{b},\mu}]^2) = -\int |g(x) - g(y)|^2 |K_{\mathrm{b},\mu}(x, y)|^2 \mathrm{d}x\mathrm{d}y. \tag{5.1}$$

By Plancherel's formula, it holds for any $z \in \mathbb{R}^n$,

$$\int |g(x) - g(x + z)|^2 \mathrm{d}x = 4\int |\widehat{g}(\xi)|^2 \sin^2\Big(\frac{\xi \cdot z}{2}\Big)\mathrm{d}\xi.$$

Note that this identity makes sense for any $g \in H^{1/2}$ and by a change of variable, this implies that

$$\operatorname{tr}([g, K_{\mathrm{b},\mu}]^2) = -4\int |\widehat{g}(\xi)|^2 \sin^2\Big(\frac{\xi \cdot z}{2}\Big)|K_{\mathrm{b},\mu}(z, 0)|^2 \mathrm{d}\xi\mathrm{d}z.$$

We can use Plancherel's formula in the same way again using that $z \mapsto K_{\mathrm{bulk}}(z, 0)$ is the Fourier transform of $(2\pi)^{-n/2}\mathbb{1}_{|\cdot|\leq\mu}$ (see formula (A.1)). This yields

$$\operatorname{tr}([g, K_{\mathrm{b},\mu}]^2) = \frac{-1}{(2\pi)^n}\int |\widehat{g}(\xi)|^2 |\mathbb{1}_{|\zeta|\leq\mu} - \mathbb{1}_{|\zeta+\xi|\leq\mu}|^2 \mathrm{d}\xi\mathrm{d}\zeta.$$

Note that one has for any $\xi \in \mathbb{R}^n$,

$$\frac{1}{2}\int |\mathbb{1}_{|\zeta|\leq\mu} - \mathbb{1}_{|\zeta+\xi|\leq\mu}|^2 \mathrm{d}\zeta = \mu^n \int (\mathbb{1}_{|\zeta|\leq 1} - \mathbb{1}_{|\zeta+\xi/\mu|\leq 1, |\zeta|\leq 1})\mathrm{d}\zeta = \mu^n |B^n_{0,1} \setminus B^n_{|\xi|/\mu,1}|.$$

This shows that

$$\operatorname{var} \mathrm{X}_{\infty}(g) = \frac{\mu^n}{(2\pi)^n}\int |\widehat{g}(\xi)|^2 |B^n_{0,1} \setminus B^n_{|\xi|/\mu,1}| d\xi.$$

Moreover, the function

$$r \in [0, \infty) \mapsto \frac{|B^n_{0,1} \setminus B^n_{r,1}|}{r}$$

is clearly continuous and bounded and its value at 0 is

$$|B_{\mathbb{R}^{n-1}}(0, 1)| = \omega_{n-1}.$$

We conclude using the dominated convergence theorem: if $g \in H^{\frac{1}{2}}(\mathbb{R}^n)$, one has precisely $\int |\widehat{g}(\xi)|^2 |\xi| \mathrm{d}\xi < +\infty$, and the proof is complete. ∎

Remark 5.2. For the last step of the proof, we use the dominated convergence theorem; in fact, one can show that the convergence is monotone, using the convexity of Euclidean balls in $\mathbb{R}^n$. It follows that, if $g \in L^2$ but $g \notin H^{1/2}$, then

$$\operatorname{var} \mathrm{X}_{\infty,\mu}(g) < \infty \;\forall \mu \quad \text{but } \mu^{1-n} \operatorname{var} \mathrm{X}_{\infty,\mu}(g) \to +\infty \quad \text{as } \mu \to \infty.$$

Remark 5.3. From the Bessel functions asymptotics (3.12) and the explicit formula for the kernel $K_{\mathrm{b},\mu}$, one can derive an alternative proof of Lemma 5.1 on the more restrictive class $g \in C_c^2(\mathbb{R}^n)$.

Observe that, as $\mu|x-y| \to \infty$, by (A.2),

$$|K_{\mathrm{b},\mu}(x,y)|^2 = \mu^{n-1} \frac{4\cos^2(\mu|x-y| - (n+1)\pi/4)}{(2\pi|x-y|)^{n+1}} + \mathcal{O}\Big(\frac{\mu^{n-2}}{|x-y|^{n+2}}\Big).$$

Then, by formula (5.1),

$$\operatorname{var} \mathrm{X}_\infty(g) = \frac{1}{2}\int |g(x)-g(y)|^2 |K_{\mathrm{b},\mu}(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

and these asymptotics (and the Riemann–Lebesgue lemma) allow us to argue that for any $g \in C_c^2(\mathbb{R}^n)$, it holds as $\mu \to \infty$,

$$\operatorname{var} \mathrm{X}_\infty(g) \sim \mu^{n-1} \int \frac{|g(x)-g(y)|^2}{(2\pi|x-y|)^{n+1}} \mathrm{d}x\mathrm{d}y.$$

We recover the classical expression of the Sobolev–Slobodeckij $W^{1/2,2}(\mathbb{R}^n)$-seminorm via a singular integral kernel, which is equivalent to the Fourier space definition of the $H^{1/2}$-seminorm (see, for instance, [87, p. 155, Theorem 5]). In fact, together with Lemma 5.1, this provides a comparison of the constants in the definitions

$$\sigma_n^2 \Sigma^2(g) = \frac{1}{(2\pi)^{n+1}} \int \frac{|g(x)-g(y)|^2}{|x-y|^{n+1}} \mathrm{d}x\mathrm{d}y, \tag{5.2}$$

where $\sigma_n^2 = \frac{\omega_{n-1}}{(2\pi)^n}$ as in Theorem III.2.

Proof of Theorem III.2. Let us first assume that $g \in H^{1/2} \cap L^1 \cap L^\infty(\mathbb{R}^n)$. We make the extra assumption that $g \in L^\infty$ and $n \geq 2$ in order to apply Corollary A.12. Since X_∞ has constant intensity, $\mathbb{E}\mathrm{X}_\infty(g) = \frac{\mu^n \omega_n}{(2\pi)^n} \int g(x)\mathrm{d}x$ and Lemma 5.1 gives the asymptotics of $\operatorname{var} \mathrm{X}_\infty(g)$ which diverge if $g \neq 0$. Hence, by Corollary A.12, for any $t \in \mathbb{R}$, as $N \to \infty$

$$\mathbb{E}[e^{t\mathrm{X}_{\infty,\mu}(g)}] = \exp\Big(t^2\Big(\frac{\Sigma^2(g)}{2} + \mathcal{O}(\mu^{-\frac{n-1}{2}})\Big)\Big). \tag{5.3}$$

In this case, the random variable $\mathrm{X}_{\infty,\mu}(g)$ does not only converge in distribution, but in the sense of its Laplace transform, so that all its moments converge. Moreover, we can easily remove the technical condition $g \in L^\infty$ by using the following inequality for any $t \in \mathbb{R}$:

$$|\mathbb{E}e^{\mathrm{i}tX} - \mathbb{E}e^{\mathrm{i}tY}| \leq |t|\sqrt{\operatorname{var}(X-Y)}$$

for any two random variables X, Y defined on the same probability space. In particular, if $g \in H^{1/2} \cap L^1(\mathbb{R}^n)$ and $\chi \in C_c^\infty(\mathbb{R}^n)$, then by Lemma 5.1, it holds for any $t \in \mathbb{R}$,

$$\limsup_{\mu\to\infty} |\mathbb{E}[e^{\mathrm{i}t\mathrm{X}_{\infty,\mu}(g)}] - \mathbb{E}[e^{\mathrm{i}t\mathrm{X}_{\infty,\mu}(\chi)}]| = |t|\Sigma^2(g-\chi).$$

In addition, $\mathbb{E}[e^{\mathrm{i}t\mathrm{X}_{\infty,\mu}(\chi)}] \to e^{-t^2\Sigma^2(\chi)/2}$ as $\mu \to \infty$ by (5.3), so that for any small $\epsilon > 0$, choosing χ in such a way that

$$\Sigma^2(g-\chi) = \|\chi - g\|_{H^{1/2}} \le \epsilon$$

(by density), this implies that for any $t \in \mathbb{R}$,

$$\limsup_{\mu\to\infty} |\mathbb{E}[e^{\mathrm{i}t\mathrm{X}_{\infty,\mu}(g)}] - e^{-t^2\Sigma^2(g)/2}| = (1+t^2)\epsilon.$$

This establishes that the characteristic function

$$\mathbb{E}[e^{\mathrm{i}t\mathrm{X}_{\infty,\mu}(g)}] \to \mathbb{E}[e^{\mathrm{i}t\mathcal{N}_{0,\Sigma^2(g)}}] = e^{-t^2\Sigma^2(g)/2}$$

as $\mu \to \infty$ for any fixed $t \in \mathbb{R}$ and this implies Theorem III.2. ∎

5.2. *Mesoscopic commutator estimates*

The goal of this section is to prove Theorem III.1. The core of the argument is to obtain the asymptotics for the variance of smooth (mesoscopic) linear statistics of the free fermions point processes. Recall that for a test function $g \in C_c^\infty(\mathbb{R}^n)$,

$$\operatorname{var} T^*_{x_0,\epsilon}\mathrm{X}(g) = -\frac{1}{2}\operatorname{tr}([T^*_{x_0,\epsilon}g, \Pi_{\hbar,\mu}]^2), \tag{5.4}$$

where $T^*_{x_0,\epsilon}g = g(\frac{\cdot - x_0}{\epsilon})$ and $\Pi_{\hbar,\mu} = \mathbb{1}_{-\hbar^2\Delta+V\le\mu}$. Hence, the proof basically amounts to proving the following expansion for the Hilbert–Schmidt norm of the commutator,

$$\|[T^*_{x_0,\epsilon}g, \Pi_{\hbar,\mu}]\|^2_{\mathrm{J}^2} = -\operatorname{tr}([T^*_{x_0,\epsilon}g, \Pi_{\hbar,\mu}]^2).$$

Proposition 5.4. *Let* (μ, V) *satisfy* (H) *and let* $\epsilon\colon [0,1] \to [0,1]$ *be a non-increasing function such that* $\hbar^{1-\beta} \le \epsilon(\hbar) \le \hbar^\beta$ *for some* $\beta > 0$ *and let* $\delta(\hbar) = \frac{\hbar}{\epsilon(\hbar)}$. *There exists* $\alpha > 0$ *such that for any* $g \in C_c^\infty(\mathbb{R}^n)$*, as* $\hbar \to 0$,

$$-\operatorname{tr}([T^*_{x_0,\epsilon}g, \Pi_{\hbar,\mu}]^2) = 2\sigma_n(\mu - V(x_0))^{(n-1)/2}\delta(\hbar)^{1-n}\Sigma^2(g) + \mathcal{O}(\hbar^\alpha\delta(\hbar)^{1-n}),$$

where σ_n *and* Σ^2 *are as in Theorem* III.2 *and the error term is locally uniform for* $(x_0, \mu) \in \{(x,\lambda) \in \mathbb{R}^{n+1} : V(x) < \lambda < M\}$.

The proof of this result is divided into three steps and it is similar to the analysis carried out in the previous sections. The first step (Proposition 5.5) is to study the commutator between $T^*_{x_0,\epsilon}g$ and a smooth compactly supported function of $H_\hbar$, using the fact that such a spectral function is a pseudodifferential operator. The second step is to estimate the commutator with a projector over a spectral window of size $\hbar$ (Proposition 5.6). This allows us, in the last step, to replace the spectral projector $\Pi_{\hbar,\mu}$ by a Fourier integral operator using Proposition 2.11 and a frequency cutoff at scale $\hbar$. In this case, however, recovering the true projector from its mollification does not require to use a Tauberian theorem but is a direct consequence of Proposition 5.6.

Proposition 5.5. *Let (μ, V) satisfy (H) and $\epsilon = \delta(\hbar)^{-1}\hbar$, where $\delta\colon (0,1] \to (0,1]$ satisfies $\hbar \leq \delta(\hbar) \leq 1$. Then, for any $\chi \in C_c^\infty((-\infty, M), \mathbb{R})$, $g \in C_c^\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, it holds as $\hbar \to 0$,*

$$\operatorname{tr}([T^*_{x_0,\epsilon} g, \chi(H_\hbar)]^2) = \mathcal{O}(\delta(\hbar)^{2-n}).$$

Proof. We can apply Proposition 2.12 with $t = 0$. In particular, since $\|R_{\hbar,t}\|_{\mathrm{J}^1} = \mathcal{O}(\hbar^\infty)$ and $\|\chi(H_\hbar)\|_{\mathrm{J}^1} = \mathcal{O}(\hbar^{-n})$, we have

$$\begin{aligned}
\|[T^*_{x_0,\epsilon} g, R_{\hbar,0}]^2\|_{\mathrm{J}^1} &\leq 4\|g\|^2_{L^\infty}\|R_{\hbar,0}\|^2_{\mathrm{J}^1} = \mathcal{O}(\hbar^\infty),\\
|\operatorname{tr}([T^*_{x_0,\epsilon} g, \chi(H_\hbar)][T^*_{x_0,\epsilon} g, R_{\hbar,0}])| &\leq \|[T^*_{x_0,\epsilon} g, \chi(H_\hbar)]\|_{\mathrm{J}^1}\|[T^*_{x_0,\epsilon} g, R_{\hbar,0}]\|_{\mathrm{J}^1}\\
&\leq 4\|f\|^2_{L^\infty}\|\chi(H_\hbar)\|_{\mathrm{J}^1}\|R_{\hbar,0}\|_{\mathrm{J}^1}\\
&= \mathcal{O}(\hbar^\infty).
\end{aligned}$$

This implies that

$$\operatorname{tr}([T^*_{x_0,\epsilon} g, \chi(H_\hbar)]^2) = \operatorname{tr}([T^*_{x_0,\epsilon} g, I^{\phi,a}_{\hbar,0}]^2) + \mathcal{O}(\hbar^\infty),$$

where $a \in S^0$ is a classical symbol and $I^{\phi,a}_{\hbar,0}$ denotes the pseudodifferential operator

$$I^{\phi,a}_{\hbar,0}\colon (x,y) \mapsto \frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}((x-y)\cdot\xi)} a(x,y,\xi;\hbar)\mathrm{d}\xi.$$

Then, we have

$$\begin{aligned}
\operatorname{tr}([T^*_{x_0,\epsilon} g, I^{\phi,a}_{\hbar,0}]^2) &= \frac{-1}{(2\pi\hbar)^{2n}}\int (T^*_{x_0,\epsilon} g(x) - T^*_{x_0,\epsilon} g(y))^2 I^{\phi,a}_{\hbar,0}(x,y) I^{\phi,a}_{\hbar,0}(y,x)\mathrm{d}x\mathrm{d}y\\
&= \frac{-1}{(2\pi\hbar)^{2n}}\int (T^*_{x_0,\epsilon} g(x) - T^*_{x_0,\epsilon} g(y))^2 e^{\frac{\mathrm{i}}{\hbar}((x-y)\cdot(\xi-\zeta))}\\
&\qquad \times a(x,y,\xi;\hbar)a(y,x,\zeta;\hbar)\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

We are in position to apply Proposition A.18, the special case of the stationary phase lemma, in the variables (y,ζ) keeping (x,ξ) fixed. We obtain, for every $\ell \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{tr}([T^*_{x_0,\epsilon} g, I^{\phi,a}_{\hbar,0}]^2) = \frac{-1}{(2\pi\hbar)^n}\int \Big(\sum_{|\alpha|<\ell} \frac{(\mathrm{i}\hbar)^{|\alpha|}\partial_y^\alpha\partial_\zeta^\alpha}{\alpha!}&((T^*_{x_0,\epsilon} g(x) - T^*_{x_0,\epsilon} g(y))^2\\
&\times a(x,y,\xi;\hbar)a(y,x,\zeta;\hbar))|_{y=x,\zeta=\xi} + R_\ell(x,\xi)\Big)\mathrm{d}x\mathrm{d}\xi,
\end{aligned}$$

where the error $R_\ell(x,\xi) = \mathcal{O}(\hbar^\ell\epsilon^{-\ell}) = \mathcal{O}(\delta^\ell)$ is compactly supported in $\{|x - x_0| \leq C\epsilon$, $|\xi| \leq C\}$ for some $C > 0$ depending on the support of g.

To conclude the proof, it remains to take $\ell = 2$. Indeed, observe that the terms $k = 0$ and $k = 1$ in the sum vanish, and it remains

$$\frac{-1}{(2\pi\hbar)^n}\int R_2(x,\xi)\mathrm{d}x\mathrm{d}\xi = \mathcal{O}(\delta^{-n+2}). \qquad \blacksquare$$

Proposition 5.6. *Let* (μ, V) *satisfy* (H) *and* $g \in C_c^\infty(\mathbb{R}^n,\mathbb{R})$. *Let* $\epsilon\colon (0,1] \to (0,\epsilon_0]$ *be a small ($\hbar$-dependent) parameter. It holds as* $\hbar \to 0$,

$$\|T^*_{x_0,\epsilon}g\mathbb{1}_{H_\hbar\in[\mu-\hbar,\mu+\hbar]}T^*_{x_0,\epsilon}g\|_{\mathrm{J}^1} = \mathcal{O}(\epsilon^n\hbar^{1-n})$$

uniformly for $(x_0,\mu) \in \{(x,\lambda)\in\mathbb{R}^{n+1} : \lambda < M,\ \mathrm{supp}(T^*_{x,\epsilon_0}g)\subset\{V<\lambda\}\}$.

Proof. Let us fix (x_0,μ) such that $V(x_0) < \mu$. By functional calculus, we can bound

$$\mathbb{1}_{H_\hbar\in[\mu-\hbar,\mu+\hbar]} \leq \rho(\hbar^{-1}(H_\hbar-\mu))\chi(H_\hbar)$$

by choosing appropriate cutoff $\chi \in C_c^\infty(\mathbb{R},\mathbb{R}_+)$ with supp $\chi \subset (V(x_0), M)$ and a smooth mollifier ρ such that supp $\widehat{\rho} \subset [-\tau,\tau]$ (for a small $\tau > 0$ so that one can apply Proposition 2.12).

For any function $\vartheta\colon\mathbb{R}^n\to\mathbb{R}_+$ measurable with compact support, $T^*_{x_0,\epsilon}g\vartheta(H_\hbar)T^*_{x_0,\epsilon}g$ is a positive trace-class operator. Hence, by Proposition A.4,

$$\begin{aligned}\|T^*_{x_0,\epsilon}g\mathbb{1}_{H_\hbar\in[\mu-\hbar,\mu+\hbar]}T^*_{x_0,\epsilon}g\|_{\mathrm{J}^1} &\leq \mathrm{tr}[T^*_{x_0,\epsilon}g\rho(\hbar^{-1}(H_\hbar-\mu))\chi(H_\hbar)T^*_{x_0,\epsilon}g]\\ &= \mathrm{tr}[T^*_{x_0,\epsilon}g^2\rho(\hbar^{-1}(H_\hbar-\mu))\chi(H_\hbar)].\end{aligned}$$

Since

$$\rho(x) = \frac{1}{\sqrt{2\pi}}\int e^{\mathrm{i}x\cdot t}\widehat{\rho}(t)\mathrm{d}t,$$

by Proposition 2.12, we have

$$\begin{aligned}\rho(\hbar^{-1}(H_\hbar-\mu))\chi(H_\hbar)(x,y) = \frac{1/\sqrt{2\pi}}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(\phi(t,x,\xi)-y\cdot\xi-t\mu)}a(t,x,y,\xi;\hbar)\widehat{\rho}(t)\mathrm{d}\xi\mathrm{d}t\\ +\mathcal{O}_{\mathrm{J}^1}(\hbar^\infty),\end{aligned}$$

where the error is uniform for all $\mu\in\mathbb{R}$ and $a \in S^0$ is a classical symbol. This implies that

$$\begin{aligned}&\|T^*_{x_0,\epsilon}g\mathbb{1}_{H_\hbar\in[\mu-\hbar,\mu+\hbar]}T^*_{x_0,\epsilon}g\|_{\mathrm{J}^1}\\ &\quad\leq \frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(\Psi(t,x,\xi)-t\mu)}(T^*_{x_0,\epsilon}g(x))^2 b(t,x,\xi;\hbar)\mathrm{d}\xi\mathrm{d}t\mathrm{d}x + \mathcal{O}(\hbar^\infty),\end{aligned} \tag{5.5}$$

where Ψ is as in Proposition 3.1 and

$$b(t,x,\xi,\hbar) = a(t,x,x,\xi;\hbar)\frac{\widehat{\rho}(t)}{\sqrt{2\pi}}$$

has compact support.

We may assume that the parameter ϵ_0 is sufficiently small so that

- $\mathrm{supp}(T^*_{x_0,\epsilon}g) \subset \{V < \mu\}$ for any $\epsilon < \epsilon_0$,
- since $\mathrm{supp}\,\chi \subset (V(x_0), M)$, $(r,t) \mapsto b(t,x,r\omega;\hbar)$ has compact support in $\mathbb{R}^*_+ \times [-\tau,\tau]$ for any $x \in \mathrm{supp}(T^*_{x_0,\epsilon}g)$, $\omega \in S^{n-1}$ and $\hbar \in (0,1]$.

This is exactly the setting of the proof of Proposition 3.2. Thus, we can make a change of variables $\xi = r\omega$, where $(r,\omega) \in \mathbb{R}_+ \times S^{n-1}$ and apply the stationary phase method to integral (5.5) in the variables $(r,t) \in \mathbb{R}^*_+ \times [-\tau,\tau]$ for a fixed $(x,\omega) \in \mathbb{R}^n \times S^{n-1}$. By (3.8), the only critical point is $(r_\star, 0)$, where $r_\star(x) := \sqrt{\mu - V(x)}$ and the Hessian of the phase is non-degenerate with $\det \mathrm{Hess}\,\psi|_{r_\star} = 4r_\star^2$. Then, by Proposition A.15, there exists a classical symbol $s \in S^0$ such that

$$\frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}(\Psi(t,x,\xi)-t\mu)} b(t,x,\xi;\hbar)\mathrm{d}\xi\mathrm{d}t = r_\star(x)^{n-2}s(x;\hbar) + \mathcal{O}_x(\hbar^\infty),$$

where the error is controlled uniformly for all $x \in \mathrm{supp}(T^*_{x_0,\epsilon}g)$. By (5.5), this yields the bound

$$\|T^*_{x_0,\epsilon}g\mathbb{1}_{H_\hbar\in[\mu-\hbar,\mu+\hbar]}T^*_{x_0,\epsilon}g\|_{J^1} \le \frac{\mu^{n/2-1}}{(2\pi\hbar)^{n-1}}\int (T^*_{x_0,\epsilon}g(x))^2 s(x;\hbar)\mathrm{d}x + \mathcal{O}(\hbar^\infty).$$

Since $\|T^*_{x_0,\epsilon}g\|^2_{L^2} = \mathcal{O}(\epsilon^n)$, and $\max_{x\in\{V\le\mu\}}|s(x;\hbar)| \le C$ for a constant (independent of $\hbar$), this completes the proof. ∎

Let $f_{\hbar,\mu}$ be as in Notations 3.1 with $\vartheta = 1$ on a neighbourhood of μ. If we assume for now the asymptotics of the quantity $\|[T^*_{x_0,\epsilon}g, f_{\hbar,\mu}(H_\hbar)]\|^2_{J^2}$ as $\hbar \to 0$, we can prove Proposition 5.4 and then Theorem III.1. The remaining parts of the proof are given in the next section.

Proof of Proposition 5.4. Let us denote $G = T^*_{x_0,\epsilon}g$ viewed as a (bounded) self-adjoint multiplication operator. By the Cauchy–Schwarz inequality, we can compare

$$\begin{aligned}&|\mathrm{tr}([G,\Pi_{\hbar,\mu}]^2) - \mathrm{tr}([G, f_{\hbar,\mu}(H_\hbar)]^2)|\\ &\quad= |\mathrm{tr}([G,\Pi_{\hbar,\mu} - f_{\hbar,\mu}(H_\hbar)][G,\Pi_{\hbar,\mu} + f_{\hbar,\mu}(H_\hbar)])|\\ &\quad\le \|[G,\Pi_{\hbar,\mu} - f_{\hbar,\mu}(H_\hbar)]\|^2_{J^2} + 2\|[G,\Pi_{\hbar,\mu} - f_{\hbar,\mu}(H_\hbar)]\|_{J^2}\|[G, f_{\hbar,\mu}(H_\hbar)]\|_{J^2}.\end{aligned}$$

Hence, Proposition 5.4 follows if we can show there exists a small $\alpha > 0$ such that as $\hbar \to 0$,

$$\|[G, f_{\hbar,\mu}(H_\hbar)]\|_{J^2} = 2\sigma_n^2(\mu - V(x_0))^{(n-1)/2}(2\pi)^{-n+1}\delta^{1-n}\Sigma^2(f) + \mathcal{O}(\delta^{1-n}\hbar^\alpha) \quad (5.6)$$

and

$$\|[G,\Pi_{\hbar,\mu} - f_{\hbar,\mu}(H_\hbar)]\|^2_{J^2} = \mathcal{O}(\hbar^{2\alpha}\delta^{1-n}). \quad (5.7)$$

Asymptotics (5.6) follows by combining Propositions 5.7 and 5.8 below by using that

$$\|[G, f_{\hbar,\mu}(H_\hbar)]\|_{J^2} = 2(\mathrm{tr}(f^2_{\hbar,\mu}(H_\hbar)G^2) - \mathrm{tr}((f_{\hbar,\mu}(H_\hbar)G)^2)).$$

Note that the leading term (of order δ^{-n}) cancels in the expansions of both terms exactly.

To prove estimate (5.7), observe that since $\operatorname{supp}(\vartheta) \subset [0, M]$ with $\vartheta = 1$ on a neighbourhood of μ,

$$\mathbb{1}_{(-\infty,\mu]} - f_{\hbar,\mu} = g_{\hbar,\mu} + \chi,$$

where

$$g_{\hbar,\mu} = \vartheta(\mathbb{1}_{(-\infty,\mu]} - \varkappa_\mu * \rho_\hbar) \quad \text{and} \quad \chi \in C_c^\infty((-\infty, M), \mathbb{R}_+).$$

In particular, by Proposition 5.5, this implies that

$$\begin{aligned} \|[G, \Pi_{\hbar,\mu} - f_{\hbar,\mu}(H_\hbar)]\|_{\mathrm{J}^2}^2 &\le 2\|[G, g_{\hbar,\mu}(H_\hbar)]\|_{\mathrm{J}^2}^2 + 2\|[G, \chi(H_\hbar)]\|_{\mathrm{J}^2}^2 \\ &\le 2\|G g_{\hbar,\mu}^2(H_\hbar) G\|_{\mathrm{J}^1} + \mathcal{O}(\delta^{-n+2}). \end{aligned} \tag{5.8}$$

Since $\varkappa = 1$ on $[0, M]$ and $\rho \in S$, we can bound for any $\gamma > 0$,

$$g_{\hbar,\mu}^2 \le \mathbb{1}_{[\mu-\hbar^{1-\gamma},\mu+\hbar^{1-\gamma}]} + \mathcal{O}(\hbar^\infty).$$

Thus, one has

$$G g_{\hbar,\mu}^2(H_\hbar) G \le G(\mathbb{1}_{[\mu-\hbar^{1-\gamma},\mu+\hbar^{1-\gamma}]}(H_\hbar) + \mathcal{O}_{\mathrm{J}^1}(\hbar^\infty))G.$$

Then, since the operator norm of G is bounded, by the triangle inequality,

$$\|G g_{\hbar,\mu}^2(H_\hbar) G\|_{\mathrm{J}^1} \le \sum_k \|G \mathbb{1}_{[\mu_k-\hbar,\mu_k+\hbar]}(H_\hbar) G\|_{\mathrm{J}^1} + \mathcal{O}(\hbar^\infty),$$

where μ_k is a uniform mesh of the interval $[\mu - \hbar^{1-\gamma}, \mu + \hbar^{1-\gamma}]$ with spacing $2\hbar$.

Hence, by Proposition 5.6, we conclude that

$$\|G g_{\hbar,\mu}^2(H_\hbar) G\|_{\mathrm{J}^1} = \mathcal{O}(\epsilon^n \hbar^{1-\gamma-n}) = \mathcal{O}(\hbar^{\beta-\gamma}\delta^{1-n}).$$

Thus, if $\gamma > 0$ is small enough (compared to β), by (5.8), this completes the proof of estimate (5.7) for some $\alpha > 0$. ∎

Proof of Theorem III.1. According to the convention of Theorem III.1, using (5.4) together with Proposition 5.4, we obtain

$$\operatorname{var} \mathrm{X}_{\hbar,\epsilon}(g) = \sigma_n^{-2}\delta(\hbar)^{n-1} \operatorname{var} T_{x_0,g}^* \mathrm{X}(g) = \Sigma^2(g) + \mathcal{O}(\hbar^\alpha).$$

In dimension $n \ge 2$, we are therefore in a position to apply Corollary A.12. Indeed, up to normalisation, $\mathrm{X}_{\hbar,\epsilon}(g)$ corresponds to the linear statistic $\mathrm{X}(T_{x_0,\epsilon}^* g)$, where

$$\|T_{x_0,\epsilon}^* g\|_{L^\infty} \le \|g\|_{L^\infty}.$$

We conclude that as $\hbar \to 0$,

$$\mathrm{X}_{\hbar,\delta}(g) \Rightarrow \Sigma^2(g)^{1/2} \mathcal{N}_{0,1},$$

and the convergence holds in the sense of the Laplace transform. ∎

5.3. Mesoscopic commutator estimates with a regularised kernel

The goal of this section is to obtain asymptotics (5.6). Note that we can split

$$\|[T^*_{x_0,\epsilon}g, f_{\hbar,\mu}(H_\hbar)]\|_{J^2} = \operatorname{tr}(f^2_{\hbar,\mu}(H_\hbar)(T^*_{x_0,\epsilon}g)^2) - \operatorname{tr}((f_{\hbar,\mu}(H_\hbar)T^*_{x_0,\epsilon}g)^2).$$

We obtain separately the asymptotics of both terms in expansion (5.4) and show that the leading terms cancel while computing the correction term. Let us introduce the useful notation

$$g_\epsilon := \epsilon^{-n}T^*_{x_0,\epsilon}g \quad \text{for } \epsilon \in (0,1].$$

Proposition 5.7. *Recall Notations 3.1 and choose* $\vartheta = 1$ *on a neighbourhood of* μ. *Let* $\epsilon\colon[0,1]\to[0,1]$ *be a non-increasing function such that* $\delta(\hbar) = \frac{\hbar}{\epsilon(\hbar)}$ *satisfies* $\delta(\hbar) = \mathcal{O}(\hbar^\beta)$ *for some* $\beta \in (0,1]$. *For any* $g \in C^\infty_c(\mathbb{R}^n)$ *such that* $\operatorname{supp}(T^*_{x_0,\epsilon}g) \subset \{V \le \mu_0\}$ *for any* $\epsilon \le \epsilon(1)$, *it holds for any* $0<\alpha<\beta$ *as* $\hbar \to 0$,

$$\operatorname{tr}(f^2_{\hbar,\mu}(H_\hbar)T^*_{x_0,\epsilon}g) = \frac{|S^{n-1}|}{2(2\pi\delta)^n}\int g_\epsilon(x)\vartheta(\lambda)^2(\lambda - V(x))^{(n-2)/2}\mathbb{1}_{\mu\ge\lambda}\mathrm{d}x\mathrm{d}\lambda + \mathcal{O}(\hbar^\alpha\delta^{1-n}),$$

where the error term is locally uniform for $(x_0,\mu) \in \{(x,\lambda)\in\mathbb{R}^{n+1} : V(x)<\lambda<M\}$.

Proof. As in formula (3.4), it holds

$$f^2_{\hbar,\mu}(H_\hbar) = \frac{1}{2\pi\hbar^2}\int \varkappa_\mu(\lambda_1)\varkappa_\mu(\lambda_2)(I^{\phi,a}_{\hbar,t_1+t_2} + \mathcal{O}_{J^1}(\hbar^\infty))e^{-\frac{i}{\hbar}(t_1\lambda_1+t_2\lambda_2)} \times \hat\rho(t_1)\hat\rho(t_2)\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}t_1\mathrm{d}t_2,$$

where this estimate is uniform for $\lambda_1,\lambda_2 \in \operatorname{supp}(\varkappa)$ and $t_1,t_2 \in [-,\tau,\tau]$. Making a change of variables

$$t = t_1+t_2,\quad s = \frac{t_1-t_2}{2},\quad \lambda = \frac{\lambda_1+\lambda_2}{2},\quad \sigma = \lambda_1-\lambda_2,$$

we obtain that

$$\operatorname{tr}(f^2_{\hbar,\mu}(H_\hbar)T^*_{x_0,\epsilon}g) = \frac{1}{2\pi\hbar^2(2\pi\delta)^n}\int e^{\frac{i}{\hbar}(\phi(t,x,\xi)-\xi\cdot x - t\lambda - s\sigma)}g_\epsilon(x)a(x,x,\xi,t;\hbar) \times \Gamma_\mu(t,\Lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi\mathrm{d}\Lambda + \mathcal{O}(\hbar^\infty),$$

where, correspondingly, we let

$$\Gamma_\mu(t,\Lambda) = \hat\rho\Big(\frac{t}{2}+s\Big)\hat\rho\Big(\frac{t}{2}-s\Big)\varkappa\Big(\lambda+\frac{\sigma}{2}\Big)\varkappa\Big(\lambda-\frac{\sigma}{2}\Big)\mathbb{1}_{|\sigma/2|\le\mu-\lambda},\quad \Lambda = (s,\lambda,\sigma).$$

Let us make a change of variable $\xi \leftarrow \eta$ as in Proposition 3.1 and decompose η in polar coordinates: $\eta = r\omega$, where $(r,\omega)\in\mathbb{R}_+\times S^{n-1}$. We obtain

$$\operatorname{tr}(f^2_{\hbar,\mu}(H_\hbar)T^*_{x_0,\epsilon}g) = \frac{1}{2\pi\hbar^2(2\pi\delta)^n}\int e^{\frac{i}{\hbar}(t(r^2+g(t,x)+V(x)-\lambda)-s\sigma)}g_\epsilon(x)b(t,x,r,\omega;\hbar) \times \Gamma_\mu(t,\Lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}r\mathrm{d}\omega\mathrm{d}\Lambda + \mathcal{O}(\hbar^\infty),$$

where $b \in S^0$ is a classical symbol given by

$$b(t,x,r,\omega;\hbar) = a(x,x,\xi(t,x,r\omega),t;\hbar)|\mathrm{Jac}[\xi \leftarrow r\omega](t,x,r,\omega)|.$$

We now apply the stationary phase method as in the proof of Proposition 3.2 in the variable $(r,t) \in [c,\infty) \times [-2\tau, 2\tau]$ for a fixed $(x,\omega,\Lambda) \in \mathrm{supp}(g_\epsilon) \times S^{n-1} \times \mathbb{R}^3$. The equations for the critical point(s) are given by (3.8) (upon replacing x_0 by x). In particular, we can assume that $\mathrm{supp}(g_\epsilon) \subset \{V \le \mu_0\}$ for any $\epsilon \le \epsilon(1)$ and introduce a cutoff $\theta \in C^\infty(\mathbb{R},[0,1])$ such that $\theta(\lambda) = 0$ for $\lambda \le \mu_0$ and $\theta(\lambda) = 1$ for $\lambda \in \mathrm{supp}(\vartheta)$. Then, for $\lambda \in \mathrm{supp}(\theta)$, the (unique) critical point is given by $(r,t) = (r_\star(x,\lambda), 0)$, where

$$r_\star(x,\lambda) := \sqrt{\lambda - V(x)}. \tag{5.9}$$

By Proposition A.15, we obtain

$$\begin{aligned}\frac{1}{2\pi\hbar}\int e^{\frac{\mathrm{i}t}{\hbar}(r^2+g(t,x)+V(x)-\lambda)} b(t,x,r,\omega;\hbar)\Gamma_\mu(t,\Lambda)\mathrm{d}t\mathrm{d}r \\ = \theta(\lambda)d(x,\lambda,\omega;\hbar)\Gamma_\mu(0,\Lambda) + \mathcal{O}(\hbar^\infty),\end{aligned} \tag{5.10}$$

where $d \in S^0$ is again a classical symbol whose principal part is given by

$$d_0(x,\lambda,\omega) = \frac{1}{2r_\star(x,\lambda)} b_0(0,x,r_\star(x,\lambda),\omega) = \frac{1}{2} a_0(x,x,r_\star(x,\lambda)\omega,0) r_\star(x,\lambda)^{n-2}.$$

Here we used that according to (3.3),

$$\xi(0,x,\eta) = \eta \quad \text{and} \quad \mathrm{Jac}[\xi \leftarrow \eta](0,x,\eta) = 1.$$

Given the explicit formula for a_0, d_0 is independent of ω and by (5.9), it is given by

$$d_0(x,\lambda) = \frac{1}{2}\vartheta(V(x) + r_\star(x,\lambda)^2)^2 r_\star(x,\lambda)^{n-2} = \frac{1}{2}\vartheta(\lambda) r_\star(x,\lambda)^{n-2}. \tag{5.11}$$

Moreover, the error in (5.10) is uniform for all $x \in \mathrm{supp}(g_\epsilon)$, $\omega \in S^{n-1}$ and locally uniform for $\Lambda \in \mathbb{R}^3$. Hence, we conclude that

$$\begin{aligned}\mathrm{tr}(f^2_{\hbar,\mu}(H_\hbar) T^*_{x_0,\epsilon} g) = \frac{1}{\hbar(2\pi\delta)^n}\int e^{-\mathrm{i}s\sigma/\hbar} g_\epsilon(x) d(x,\lambda,\omega;\hbar)\Gamma_\mu(0,\Lambda)\theta(\lambda)\mathrm{d}x\mathrm{d}\omega\mathrm{d}\Lambda \\ + \mathcal{O}(\hbar^\infty).\end{aligned}$$

Next, let us observe that because of the cutoff θ ($\theta(\lambda) = 0$ for $\lambda \le \mu_0$ with $\mu_0 > 0$ and $\varkappa = 1$ on $[0,M]$)

$$\Gamma_\mu(0,\Lambda)\theta(\lambda) = \widehat{\rho}(s)^2 \mathbb{1}_{|\sigma/2| \le \mu - \lambda}\theta(\lambda).$$

This allows us to compute explicitly the integral with respect to σ. We obtain

$$\begin{aligned}\mathrm{tr}(f^2_{\hbar,\mu}(H_\hbar) T^*_{x_0,\epsilon} g) = \frac{1}{\hbar(2\pi\delta)^n}\int \frac{\sin(2(\mu-\lambda)s/\hbar)}{s/2\hbar} g_\epsilon(x) d(x,\lambda,\omega;\hbar) \\ \widehat{\rho}(s)^2\theta(\lambda)\mathbb{1}_{\mu \ge \lambda}\mathrm{d}s\mathrm{d}x\mathrm{d}\omega\mathrm{d}\lambda + \mathcal{O}(\hbar^\infty).\end{aligned}$$

We can now compute the integral with respect to (λ, s) using Proposition A.19. Since $\lambda \mapsto d(x,\lambda,\omega;\hbar)\theta(\lambda)\mathbb{1}_{\mu\geq\lambda}$ is L^∞_c (uniformly for $\omega \in S^{n-1}$ and $x \in \mathrm{supp}(g_\epsilon)$) and $\widehat{\rho}^2 \in C_c^\infty(\mathbb{R})$ with $\widehat{\rho}(0) = \frac{1}{\sqrt{2\pi}}$, this implies that for any $\gamma \in (0,1]$,

$$\mathrm{tr}(f^2_{\hbar,\mu}(H_\hbar)T^*_{x_0,\epsilon}g) = \frac{1}{(2\pi\delta)^n}\int g_\epsilon(x)d(x,\lambda,\omega;\hbar)\theta(\lambda)\mathbb{1}_{\mu\geq\lambda}\mathrm{d}x\mathrm{d}\omega\mathrm{d}\lambda + \mathcal{O}_\gamma(\delta^{1-\gamma}).$$

Here we also used that $\|g_\epsilon\|_{L^1(\mathbb{R}^n)} < \infty$ is independent of $\epsilon > 0$. Since $\delta \leq \hbar^\beta$, the error term is of order $\hbar^\alpha$ for any $0<\alpha<\beta$ and we can also replace the symbol d by its principal part (5.11) up to a negligible error. Since $\theta(\lambda) = 1$ for $\lambda \in \mathrm{supp}(\vartheta)$ and d_0 is independent of $\omega \in S^{n-1}$, this completes the proof. ∎

Proposition 5.8. *Let $\epsilon\colon [0,1] \to [0,1]$ be a non-increasing function such that $\hbar^{1-\beta} \leq \epsilon(\hbar) \leq \hbar^\beta$ for some $\beta > 0$ and let $\delta(\hbar) = \frac{\hbar}{\epsilon(\hbar)}$. There exists $\alpha > 0$ such that for any $g \in C_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned}\mathrm{tr}((f_{\hbar,\mu}(H_\hbar)T^*_{x_0,\epsilon}g)^2) = {} & \frac{|S^{n-1}|}{2(2\pi\delta)^n}\int g_\epsilon(x)^2\vartheta(\lambda)(\lambda - V(x))^{(n-2)/2}\mathbb{1}_{\lambda\leq\mu}\mathrm{d}x\mathrm{d}\lambda \\ & - \sigma_n^2\vartheta(\mu)\delta^{1-n}(\mu - V(x_0))^{(n-1)/2}\Sigma^2(f) + \mathcal{O}(\delta^{1-n}\hbar^\alpha),\end{aligned}$$

where σ_n and $\Sigma^2(f)$ are as in Theorem III.2 and the error term is locally uniform for $(x_0,\mu) \in \{(x,\lambda) \in \mathbb{R}^{n+1} : V(x) < \lambda < M\}$.

Recall the notations from (2.5):

$$I^{\phi,a}_{\hbar,t}(x,y) = \frac{1}{(2\pi\hbar)^n}\int e^{\frac{i}{\hbar}\psi(t,x,y,\xi)}a(t,x,y,\xi;\hbar)\mathrm{d}\xi, \quad \psi(t,x,y,\xi) := \phi(t,x,\xi) - y\cdot\xi.$$

The proof requires the following basic estimates on the decay of this kernel.

Lemma 5.9. *Suppose that $\epsilon \geq \hbar^{1-\beta}$ for $\beta \in (0,1]$. Suppose that the symbol $(t,x,y,\xi) \mapsto a(t,x,y,\xi;\hbar)$ is supported on $\{t \in [-\tau,\tau], x,y \in B^n_{0,\epsilon}, \xi \in \mathcal{K} : |t| \geq C\epsilon\}$, where $\mathcal{K} \Subset \mathbb{R}^n \setminus \{0\}$ and C is a large enough constant. Then, $I^{\phi,a}_{\hbar,t}(x,y) = \mathcal{O}(\hbar^\infty)$ for $(t,x,y) \in \mathrm{supp}(a)$.*

Proof. By (3.2), we have

$$\partial_\xi\psi(t,x,y,\xi) = (x-y) + 2t\xi + \mathcal{O}(t^2).$$

Hence, since $|\xi| > c$, $|t| \geq C\epsilon$ and $|x-y| \leq 2\epsilon$ on $\mathrm{supp}(a)$, one has $|\partial_\xi\psi(t,x,y,\xi)| \geq \epsilon$ and for any multi-index α with $|\alpha| \geq 1$, $|\partial_\xi^\alpha\psi(t,x,y,\xi)| \leq C_\alpha\epsilon$. In particular, for a multi-index α with $|\alpha| = 1$, the differential operator

$$\mathcal{L}_\xi\colon u \mapsto \partial_\xi^\alpha\Big(\frac{u}{\partial_\xi^\alpha\psi}\Big)$$

satisfies for $u \in C_c^\infty(\mathbb{R}^n)$ and every $k \in \mathbb{N}$,

$$\|\mathcal{L}_\xi^k u\|_{C^0} \leq \frac{C_k}{\epsilon^k}\|u\|_{C^k}.$$

By repeated integration by parts, we obtain for every $k \in \mathbb{N}$,

$$\int e^{\frac{\mathrm{i}}{\hbar}\psi(t,x,y,\xi)} a(t,x,y,\xi;\hbar)\mathrm{d}\xi = (\mathbf{i}\hbar)^k \int e^{\frac{\mathrm{i}}{\hbar}\psi(t,x,y,\xi)} \mathcal{L}_\xi^k a(t,x,y,\xi;\hbar)\mathrm{d}\xi.$$

Since $a \in S^0$ is supported in $\mathcal{K}$, the right-hand side is of order $\mathcal{O}_k((\hbar/\epsilon)^k)$. Because we assume that $\epsilon \geq \hbar^{1-\beta}$ for $\beta > 0$, this proves that

$$I_{\hbar,t}^{\phi,a}(x,y) = \mathcal{O}(\hbar^\infty). \qquad \blacksquare$$

Proof of Proposition 5.8. The argument splits into several parts: a reduction step, a first application of the stationary phase, a rescaling, and then the actual computation of the leading and subleading terms.

5.3.1. Reduction steps. We can assume that $\epsilon \leq \epsilon(1)$ is small enough so that $\mathrm{supp}(g_\epsilon) \subset B_{0,c}^n$ for all $\epsilon \in (0,\epsilon(1)]$ and $V(x) < \mu_0$ for all $x \in B_{0,c}^n$.

Using representation (3.4) and proceeding as in the proof of Proposition 5.5 for $t \in [-\tau,\tau]$, we obtain

$$\operatorname{tr}((f_{\hbar,\mu}(H_\hbar)g_\epsilon)^2) = \frac{\epsilon^{2n}}{2\pi\hbar^2}\int \operatorname{tr}(g_\epsilon I_{\hbar,t_1}^{\phi,a} g_\epsilon I_{\hbar,t_2}^{\phi,a\dagger}) e^{-\mathrm{i}t_1\lambda_1/\hbar - \mathrm{i}t_2\lambda_2/\hbar}\varkappa_\mu(\lambda_1)\varkappa_\mu(\lambda_2)$$
$$\times \widehat{\rho}(t_1)\widehat{\rho}(t_2)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}\lambda_1\mathrm{d}\lambda_2 + \mathcal{O}(\hbar^\infty),$$

where $I_{\hbar,t}^{\phi,a\dagger}$ denotes the adjoint of $I_{\hbar,t}^{\phi,a}$ and $a \in S^0$ is a classical symbol whose principal part satisfies

$$a_0(t,x,y,\xi) = \vartheta\Big(V\Big(\frac{x+y}{2} + |\xi|^2\Big)\Big) + \mathcal{O}_{x,y,\xi}(t), \quad x,y \in B_{0,c}^n, \quad \xi \in \mathbb{R}^n.$$

Note that we used that the operator $f_{\hbar,\mu}(H_\hbar)$ is self-adjoint, though $I_{\hbar,t}^{\phi,a}$ is not. Hence, the only relevant contribution to the trace in question is given by the oscillatory integral

$$\mathfrak{I}_\hbar = \frac{1}{2\pi\hbar^2(2\pi\delta)^{2n}}\int e^{\frac{\mathrm{i}}{\hbar}\Phi(x,y,\Lambda_1,\Lambda_2)} A(x,y,\Lambda_1,\Lambda_2)\mathrm{d}x\mathrm{d}y\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2,$$

where

$$\Lambda_1 = (t_1,x_1,\lambda_1), \quad \Lambda_2 = (t_2,x_2,\lambda_2)$$

and

$$\Phi(x,y,\Lambda_1,\Lambda_2) = \psi(t_1,x,\xi_1) - y\cdot\xi_1 - t_1\cdot\lambda_1 - \psi(t_2,x,\xi_2) + y\cdot\xi_2 + t_2\cdot\lambda_2,$$
$$A(x,y,\Lambda_1,\Lambda_2) = g_\epsilon(x)g_\epsilon(y)a(t_1,x,y,\xi_1;\hbar)\overline{a(t_2,y,x,\xi_2;\hbar)}$$
$$\times \varkappa_\mu(\lambda_1)\varkappa_\mu(\lambda_2)\widehat{\rho}(t_1)\widehat{\rho}(t_2).$$

Observe that there is a constant c such that the amplitude A is supported in

$$\{t \in [-\tau,\tau],\ x,y \in B_{0,c\epsilon}^n,\ \xi \in \mathcal{K}\}$$

for a $\mathcal{K} \Subset \mathbb{R}^n \setminus \{0\}$ (since $\vartheta(V(x)) = 0$ for all x in a neighbourhood of $\operatorname{supp}(g_\epsilon)$). Hence, by Lemma 5.9, we may add a cutoff $\chi(\frac{t_1+t_2}{2\epsilon^{1-\kappa}})$, where $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ is an even function which equals to 1 on a neighbourhood of 0 and $\kappa > 0$ is a small parameter, inside the integral $J_\hbar$ up to an error which is $\mathcal{O}(\hbar^\infty)$. Indeed, if $|\frac{t_1+t_2}{2\epsilon}| \geq C$, then either $|t_1| \geq C\epsilon$ or $|t_2| \geq C\epsilon$, and in either case Lemma 5.9 applies. The choice of cutoff $\chi(\frac{t_1+t_2}{2\epsilon^{1-\kappa}})$ is tuned for the sequel of the argument. In summary, we have

$$\mathfrak{I}_\hbar = \frac{1}{2\pi\hbar^2(2\pi\delta)^{2n}} \int e^{\frac{\mathrm{i}}{\hbar}\Phi(x,y,\Lambda_1,\Lambda_2)} B(x,y,\Lambda_1,\Lambda_2)\mathrm{d}x\mathrm{d}y\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2 + \mathcal{O}(\hbar^\infty), \tag{5.12}$$

where

$$\begin{aligned} B(x,y,\Lambda_1,\Lambda_2) &= g_\epsilon(x)g_\epsilon(y)a(t_1,x,y,\xi_1;\hbar)\overline{a(t_2,y,x,\xi_2;\hbar)} \\ &\quad \times \Gamma_\mu\Big(\frac{\lambda_1+\lambda_2}{2}, \lambda_1-\lambda_2\Big)\Theta_\epsilon\Big(\frac{t_1+t_2}{2}, t_1-t_2\Big), \\ \Gamma_\mu(\lambda,\sigma) &= \varkappa\Big(\lambda+\frac{\sigma}{2}\Big)\varkappa\Big(\lambda-\frac{\sigma}{2}\Big)\mathbb{1}_{|\sigma/2|\leq\mu-\lambda}, \\ \Theta_\epsilon(t,s) &= \hat{\rho}\Big(t+\frac{s}{2}\Big)\hat{\rho}\Big(t-\frac{s}{2}\Big)\chi(t\epsilon^{\kappa-1}). \end{aligned}$$

In preparation for the stationary phase, we introduce the new variables

$$r\omega = \frac{\xi_1+\xi_2}{2}, \quad \zeta = \xi_1-\xi_2, \quad t = \frac{t_1+t_2}{2}, \quad s = t_1-t_2, \quad \lambda = \frac{\lambda_1+\lambda_2}{2}, \quad \sigma = \lambda_1-\lambda_2,$$

where $(r,\omega) \in \mathbb{R}_+ \times S^{n-1}$, and we let

$$\Psi(t,s,r,\zeta,x,\omega,\lambda) = \phi\Big(t+\frac{s}{2}, x, r\omega+\frac{\zeta}{2}\Big) - \phi\Big(t-\frac{s}{2}, x, r\omega-\frac{\zeta}{2}\Big) - s\lambda.$$

By making this change of variables in (5.12), one has

$$\mathfrak{I}_\hbar = \frac{1}{\hbar(2\pi\delta)^{2n}} \int e^{-\frac{\mathrm{i}}{\hbar}(\zeta\cdot y+t\sigma)} g_\epsilon(x)g_\epsilon(y)\mathfrak{L}_\hbar(\Lambda)\Gamma_\mu(\lambda,\sigma)\mathrm{d}\Lambda\mathrm{d}\sigma + \mathcal{O}(\hbar^\infty), \tag{5.13}$$

where

$$\mathfrak{L}_\hbar(\Lambda) := \frac{1}{2\pi\hbar}\int e^{\frac{\mathrm{i}}{\hbar}\Psi(s,r,\Lambda)} d(s,r,\Lambda;\hbar)\Theta_\epsilon(t,s)\mathrm{d}s\mathrm{d}r, \qquad \Lambda = (t,\zeta,x,y,\omega,\lambda)$$

and we set

$$d(s,r,\Lambda;\hbar) = r^{n-1}a\Big(t+\frac{s}{2}, x, y, r\omega+\frac{\zeta}{2};\hbar\Big)\overline{a\Big(t-\frac{s}{2}, y, x, r\omega-\frac{\zeta}{2};\hbar\Big)}.$$

Remark 5.10. We can also write integral (5.13) as

$$\begin{aligned} \mathfrak{I}_\hbar = \frac{1}{2\pi\hbar^2(2\pi\delta)^{2n}} &\int e^{\frac{\mathrm{i}}{\hbar}(\Psi(t,s,r,\zeta,x,\omega,\lambda)-t\sigma)} g_\epsilon(x)\mathfrak{M}_\hbar(t,s,r,\zeta,x,\omega) \\ &\times \Gamma_\mu(\lambda,\sigma)\Theta_\epsilon(t,s)\mathrm{d}t\mathrm{d}s\mathrm{d}r\mathrm{d}\zeta\mathrm{d}x\mathrm{d}\omega\mathrm{d}\lambda\mathrm{d}\sigma + \mathcal{O}(\hbar^\infty), \end{aligned}$$

where

$$\mathfrak{M}_\hbar(t,s,r,\zeta,x,\omega) := \int e^{-\mathrm{i}\zeta\cdot y/\hbar} g_\epsilon(y) d(s,r,\Lambda;\hbar)\mathrm{d}y.$$

The point is that since $d \in S^0$ and $g \in C^\infty$, by making repeated integration by parts, we obtain if $|\zeta| \geq z$ for some $z > 0$, then for every $k \in \mathbb{N}_0$,

$$\mathfrak{M}_\hbar(t,s,r,\zeta,x,\omega) = \mathcal{O}_k(z^{-k}\hbar^k\epsilon^{-n-k})$$

uniformly for $(t,s) \in [-2\tau,2\tau]$, $x \in B^n_{0,c}$ and $r\omega \in \mathcal{K}$. Recall that $\delta = \hbar/\epsilon$ and we assume that $\delta \leq \hbar^\beta$ where $\beta > 0$. Hence, we conclude that $\mathfrak{M}_\hbar = \mathcal{O}(\hbar^\infty)$ if $|\zeta| \geq \delta^{1-\kappa}$ for any $\kappa > 0$. This argument allows us to include at will a cutoff $\chi(|\zeta|\delta^{\kappa-1})$ inside integral (5.13), where again $\chi \in C_c^\infty(\mathbb{R},[0,1])$ is even and equal to 1 on a neighbourhood of 0. Hence,

$$\mathrm{tr}((f_{\hbar,\mu}(H_\hbar), g_\epsilon)^2) = \mathfrak{I}'_\hbar + \mathcal{O}(\hbar^\infty),$$

where

$$\mathfrak{I}'_\hbar = \frac{1}{\hbar(2\pi\delta)^{2n}} \int e^{-\frac{\mathrm{i}}{\hbar}(\zeta\cdot y+t\sigma)} g_\epsilon(x)g_\epsilon(y)\mathfrak{L}_\hbar(\Lambda)\Gamma_\mu(\lambda,\sigma)\chi(|\zeta|\delta^{\kappa-1})\mathrm{d}\Lambda\mathrm{d}\sigma. \quad (5.14)$$

5.3.2. Stationary phase. We now apply the stationary phase method to the oscillatory integral $\mathfrak{L}_\hbar$ with respect to the variables (r,s), keeping $\Lambda = (t,\zeta,x,y,\omega,\lambda)$ fixed. The equations for the critical point(s) are

$$\begin{cases} 0 = \partial_r\Psi(s,r,\Lambda) = \omega\cdot\Big(\partial_\xi\phi\Big(t+\frac{s}{2},x,r\omega+\frac{\zeta}{2}\Big) - \partial_\xi\phi\Big(t-\frac{s}{2},x,r\omega-\frac{\zeta}{2}\Big)\Big), \\ 0 = \partial_s\Psi(s,r,\Lambda) = \frac{1}{2}\Big(\partial_t\phi\Big(t+\frac{s}{2},x,r\omega+\frac{\zeta}{2}\Big) + \partial_t\phi\Big(t-\frac{s}{2},x,r\omega-\frac{\zeta}{2}\Big)\Big) - \lambda. \end{cases}$$

By (3.2), using that

$$\Big(t+\frac{s}{2}\Big)\Big|r\omega+\frac{\zeta}{2}\Big|^2 - \Big(t-\frac{s}{2}\Big)\Big|r\omega-\frac{\zeta}{2}\Big|^2 = \Big(r^2+\frac{|\zeta|^2}{4}\Big)s + 2rt\,\zeta\cdot\omega$$

and considering even/odd terms in ζ, the phase is the following expansion for small times:

$$\Psi(s,r,\Lambda) = x\cdot\zeta + s\Big(V(x)+r^2+\frac{|\zeta|^2}{4}-\lambda\Big) + 2rt\zeta\cdot\omega + s\mathcal{O}(\varsigma) + \zeta\cdot\mathcal{O}(\varsigma^2), \quad (5.15)$$

where $\varsigma = \max\{|t|,|s|\} \leq 2\tau$. Similarly, by expanding the previous equations for small (t,s), we obtain

$$\begin{cases} 0 = \partial_r\Psi(s,r,\Lambda) = 2rs + 2\omega\cdot(t\zeta + s\,\mathcal{O}(\varsigma) + \zeta\mathcal{O}(\varsigma^2)), \\ 0 = \partial_s\Psi(s,r,\Lambda) = V(x) + r^2 + \frac{|\zeta|^2}{4} - \lambda + \mathcal{O}(\varsigma). \end{cases} \quad (5.16)$$

Since $r \geq c_0 \gg \tau$ and we may assume that $|\zeta| \leq c$ for any constant c (cf. Remark 5.10), these equations have the following consequences:

- If the parameter $\lambda \le \mu_0$, then the second equation has no solution $r \ge c$. In this case, there is no critical point and by Proposition A.16, we conclude that $I_\hbar = \mathcal{O}(\hbar^\infty)$. We can therefore include a cutoff $\theta(\lambda)$ in the integrand of (5.29), where $\theta \in C^\infty(\mathbb{R}, [0,1])$ satisfies $\theta(\lambda) = 0$ for $\lambda \le \mu_0$ and $\theta(\lambda) = 1$ for $\lambda \in \operatorname{supp}(\vartheta)$.
- For $\lambda \ge \mu_0$, (5.16) has a unique solution; $r_c > 0$ and s_c have the following expansion:

$$\begin{cases} r_c(\Lambda) = \sqrt{\lambda - V(x) - \dfrac{|\zeta|^2}{4}} + \mathcal{O}(\varsigma), \\ s_c(\Lambda) = -r_c^{-1} t\zeta \cdot \omega(1 + \mathcal{O}(\varsigma)). \end{cases} \tag{5.17}$$

In particular, the critical point at $t = 0$ is explicit,

$$(r_{c,0}, s_{c,0}) = \Big(\sqrt{\lambda - V(x) - \frac{|\zeta|^2}{4}}, 0\Big),$$

as well as the corresponding Hessian matrix. Indeed, using the first equation in (5.16), we compute

$$\begin{cases} \partial_{rr}\Psi(t, s_c, r_c, \zeta, x, \omega, \lambda)|_{t=0} = [s_c + \mathcal{O}(\varsigma)]_{t=0} = 0, \\ \partial_{sr}\Psi(t, s_c, r_c, \zeta, x, \omega, \lambda)|_{t=0} = [2r_c + \mathcal{O}(\varsigma)]_{t=0} = 2r_{c,0} > 0. \end{cases}$$

This shows that the Hessian is non-degenerate at $t = 0$ and its determinant is given by $\Sigma_{c,0}^2 = 4r_{c,0}^2$. This property is preserved in a small neighbourhood of $t = 0$ (which clearly contains $\operatorname{supp}(\Theta_\epsilon)$), so that we can apply Proposition A.15 to the integral $\mathfrak{L}_\hbar$, and we obtain

$$\mathfrak{L}_\hbar(\Lambda) = e^{\frac{i}{\hbar}\Phi(\Lambda)} e(\Lambda; \hbar)\Theta_\epsilon(t, s_c),$$
$$\Phi(\Lambda) := \Psi(t, s_c, r_c, \zeta, x, \omega, \lambda),$$

where $e \in S^0$ is again a classical symbol with principal part at time $t = 0$ along the diagonal $(x = y)$,

$$\begin{aligned} e_0(\Lambda_0) &= \Sigma_{c,0}^{-2} d_0(0, r_{c,0}, \Lambda_0) \\ &= \frac{1}{2} r_{c,0}^{n-2}(\zeta, x, \lambda)\vartheta\Big(V(x) + \Big|r_{c,0}\omega + \frac{\zeta}{2}\Big|^2\Big)\vartheta\Big(V(x) + \Big|r_{c,0}\omega - \frac{\zeta}{2}\Big|^2\Big), \end{aligned} \tag{5.18}$$

where $\Lambda_0 = (0, \zeta, x, x, \omega, \lambda)$.

Let us observe that $s_c = \zeta \cdot \mathcal{O}(t)$ and therefore the control parameter $\varsigma = t$ in the regime that we consider. By (5.15) and (5.17), this implies that the new phase has the following expansion for small t,

$$\begin{aligned} \Phi(\Lambda) &= x \cdot \zeta + 2r_c t\zeta \cdot \omega + s_c\mathcal{O}(t) + \zeta \cdot \mathcal{O}(t^2) \\ &= x \cdot \zeta + 2\sqrt{\lambda - V(x)}\, t\zeta \cdot (\omega + \mathcal{O}(t) + \mathcal{O}(|\zeta|^2)). \end{aligned} \tag{5.19}$$

Going back to formula (5.14), we conclude that

$$\begin{aligned} \mathfrak{I}'_\hbar = \frac{1}{\hbar(2\pi\delta)^{2n}} \int e^{-\frac{i}{\hbar}(\zeta \cdot y + t\sigma - \Phi(\Lambda))} g_\epsilon(x) g_\epsilon(y) e(\Lambda; \hbar)\Theta_\epsilon(t, s_c) \\ \times \Gamma_\mu(\lambda, \sigma)\theta(\lambda)\chi(|\zeta|\delta^{\kappa-1}) \mathrm{d}\Lambda \mathrm{d}\sigma + \mathcal{O}(\hbar^\infty). \end{aligned} \tag{5.20}$$

5.3.3. (Re)scaling. Let us first observe that because of the cutoff $\theta \le \mathbb{1}_{[\mu_0,\infty)}$ with $\mu_0 > \mu$ and $\varkappa = 1$ on $[0, M]$, we have

$$\Gamma_\mu(\lambda,\sigma)\theta(\lambda) = \mathbb{1}_{|\sigma/2|\le\mu-\lambda}\theta(\lambda).$$

This allows us to compute the integral with respect to σ in (5.20), since σ does not appear in Λ,

$$\frac{1}{2\hbar}\int e^{-\mathrm{i}t\sigma/\hbar}\Gamma_\mu(\lambda,\sigma)\mathrm{d}\sigma\theta(\lambda) = \mathbb{1}_{\mu\ge\lambda}\theta(\lambda)\frac{\sin(2(\mu-\lambda)t/\hbar)}{t}.$$

This implies that

$$\begin{aligned}\mathfrak{I}'_\hbar = \frac{2}{(2\pi\delta)^{2n}}\int e^{\frac{\mathrm{i}}{\hbar}(\Phi(\Lambda)-\zeta\cdot y)}g_\epsilon(y)g_\epsilon(x)e(\Lambda;\hbar)\Theta_\epsilon(t,s_\mathrm{c})\frac{\sin(2(\mu-\lambda)t/\hbar)}{t}\\ \times\theta(\lambda)\chi(|\zeta|\delta^{\kappa-1})\mathbb{1}_{\mu\ge\lambda}\mathrm{d}\Lambda + \mathcal{O}(\hbar^\infty).\end{aligned} \tag{5.21}$$

We can now make the following change of variables in integral (5.21):

$$y \leftarrow x_0 + \epsilon y, \quad x \leftarrow x_0 + \epsilon x, \quad t \leftarrow \epsilon t, \quad \zeta \leftarrow \delta\zeta,$$

where we recall that $\delta = \frac{\hbar}{\epsilon}$. In particular, by expansion (5.19), the phase satisfies for $|t| \le \epsilon^{-\kappa}$ and $|\zeta| \le \delta^{-\kappa}$,

$$\begin{aligned}&\frac{\Phi(\epsilon t,\delta\zeta,x_0+\epsilon x,x_0+\epsilon y,\omega,\lambda)-\delta\zeta\cdot(x_0+\epsilon y)}{\hbar}\\ &\quad= (x-y)\cdot\zeta + 2r_\star(x_0,\lambda)t\zeta\cdot(\omega+\mathcal{O}(\delta^{2-2\kappa})) + t\mathcal{O}(\epsilon\delta^{-\kappa}),\end{aligned}$$

where $r_\star$ is given by (5.9).

Moreover, by (5.18) Taylor expansions (in a neighbourhood of $t = 0$, $\zeta = 0$ and the diagonal $x = y$), one obtains

$$\begin{aligned}&e_0(t,\delta\zeta,x_0+\epsilon x,x_0+\epsilon y,\omega,\lambda)\\ &\quad= \frac{1}{2}r_{\mathrm{c},0}^{n-2}(\delta\zeta,x_\epsilon,\lambda)\vartheta\Big(V(x_\epsilon)+\Big|r_{\mathrm{c},0}\omega+\delta\frac{\zeta}{2}\Big|^2\Big)\vartheta\Big(V(x_\epsilon)+\Big|r_{\mathrm{c},0}\omega-\delta\frac{\zeta}{2}\Big|^2\Big)\\ &\qquad+\mathcal{O}_\Lambda(|t|)+\mathcal{O}_\Lambda(\epsilon|x-y|)\\ &\quad= \frac{1}{2}r_\star(x_\epsilon,\lambda)^{n-2}\vartheta(V(x_\epsilon)+r_\star(x_\epsilon,\lambda)^2)^2+\mathcal{O}_\Lambda(\delta^{2-2\kappa})+\mathcal{O}_\Lambda(|t|)\\ &\qquad+\mathcal{O}_\Lambda(\epsilon|x-y|),\end{aligned}$$

where the linear terms in ζ exactly cancel and we use the shorthand notation

$$x_\epsilon = x_0 + \epsilon x.$$

Finally, since $\widehat{\rho}$ is even with $\widehat{\rho}(0) = \frac{1}{\sqrt{2\pi}}$, it holds by (5.17),

$$\Theta_\epsilon(\epsilon t,s_\mathrm{c}) = \frac{1}{2\pi}(1+\mathcal{O}(t^2\epsilon^2))\chi(t\epsilon^\kappa).$$

We also emphasise that these expansions are all uniform for all $x, y \in \mathrm{supp}(g)$, $\omega \in S^{n-1}$, $\lambda \in \mathrm{supp}(\theta)$ and $(x_0, \mu) \in \mathcal{A}$. Altogether, this implies that as $\hbar \to 0$,

$$\begin{aligned} &e^{-\frac{\mathrm{i}}{\hbar}(\Phi(\epsilon t,\delta\zeta,x_0+\epsilon x,x_0+\epsilon y,\omega,\lambda)-\delta\zeta\cdot(x_0+\epsilon y))}\Theta_\epsilon(\epsilon t, s_{\mathrm{c}})e(\epsilon t,\delta\zeta,x_0+\epsilon x,x_0+\epsilon y,\omega,\lambda;\hbar)\\ &\quad= \frac{1}{4\pi}e^{\mathrm{i}(x-y+2r_\star(x_0,\lambda)t\omega)\cdot\zeta}\big(F(x_\epsilon,\lambda)+\Upsilon_{1;\epsilon,\delta}(\Lambda)+t\Upsilon_{2;\epsilon,\delta}(\Lambda)\\ &\qquad\qquad + (x-y+2r_\star(x_0,\lambda)t\omega)\cdot\Upsilon_{3;\epsilon,\delta}(\Lambda)\big)\chi(t\epsilon^\kappa), \end{aligned}$$

where the main term is given by

$$F(x,\lambda) = r_\star(x,\lambda)^{n-2}\vartheta\Big(V\Big(\frac{x+y}{2}\Big)+r_\star(x,\lambda)^2\Big)^2 = r_\star(x,\lambda)^{n-2}\vartheta(\lambda)^2$$

according to (5.9), and the error terms are smooth functions such that

$$\begin{aligned} \Upsilon_{1;\epsilon,\delta}(t,\zeta,x,y,\omega,\lambda) &= \mathcal{O}_\Lambda(\delta^{2-2\kappa})+\mathcal{O}_\Lambda(\hbar),\\ \Upsilon_{2;\epsilon,\delta}(t,\zeta,x,y,\omega,\lambda) &= \mathcal{O}_\Lambda(\epsilon^{1-\kappa}),\\ \Upsilon_{3;\epsilon,\delta}(t,\zeta,x,y,\omega,\lambda) &= \mathcal{O}_\Lambda(\epsilon). \end{aligned}$$

In particular, these errors (as well as their derivatives) are controlled uniformly for $\lambda \in \mathrm{supp}(\theta)$, $\omega \in S^{n-1}$, $x, y \in \mathrm{supp}(g)$, $(x_0, \mu) \in \mathcal{A}$, $|t| \le \epsilon^{-\kappa}$ and $|\zeta| \le \delta^{-\kappa}$.

According to (5.21), this allows us to rewrite the main contribution to $\mathfrak{I}'_\hbar$ as

$$\begin{aligned} \mathfrak{I}''_\hbar = \frac{1/2\pi}{((2\pi)^2\delta)^n}\int & e^{\mathrm{i}(x-y+2r_\star(x_0,\lambda)t\omega)\cdot\zeta}g(x)g(y)\\ &\times[F(x_\epsilon,\lambda)+\Upsilon_{1;\epsilon,\delta}(\Lambda)+t\Upsilon_{2;\epsilon,\delta}(\Lambda)+(x-y+2r_\star(x_0,\lambda)t\omega)\cdot\Upsilon_{3;\epsilon,\delta}(\Lambda)]\\ &\times\frac{\sin(2(\mu-\lambda)t/\delta)}{t}\chi(t\epsilon^\kappa)\chi(|\zeta|\delta^\kappa)\theta(\lambda)\mathbb{1}_{\mu\ge\lambda}\mathrm{d}\Lambda. \end{aligned} \tag{5.22}$$

By Proposition A.19, we argue that the error term $\Upsilon_{1;\epsilon,\delta}$ does not contribute significantly to the integral $\mathfrak{I}''_\hbar$. Indeed, the function

$$(\lambda, t) \mapsto \Upsilon_{1;\epsilon,\delta}(\Lambda)\chi(t\epsilon^\kappa)\theta(\lambda)$$

is smooth with respect to t, uniformly bounded with respect to λ with

$$\begin{aligned} &\|(\lambda,t)\mapsto\Upsilon_{1;\epsilon,\delta}(\Lambda)\chi(t\epsilon^\kappa)\theta(\lambda)\|_{L^\infty C^2}\\ &\quad+\|(\lambda,t)\mapsto\theta(\lambda)t^{\gamma-1}\partial_t(\Upsilon_{1;\epsilon,\delta}(\Lambda)\chi(t\epsilon^\kappa))\|_{L^\infty L^1}\le C\epsilon^{-\gamma\kappa}(\delta^{2-2\kappa}+\hbar), \end{aligned}$$

uniformly for $\omega \in S^{n-1}$, $x, y \in \mathrm{supp}(g)$, $(x_0, \mu) \in \mathcal{A}$ and $|\zeta| \le \delta^{-\gamma}$. Since one has $\hbar^{1-\beta} \le \epsilon, \delta \le \hbar^\beta$ and

$$\int \chi(|\zeta|\delta^\kappa)\mathrm{d}\zeta = \mathcal{O}(\delta^{-n\kappa}),$$

by choosing κ small enough (depending only on the dimension n and on $0 < \alpha < \beta$), this shows that the contribution of $\Upsilon_{1;\epsilon,\delta}$ in (5.22) is $\mathcal{O}(\delta^{1-n}\hbar^\alpha)$.

Similarly,

$$\|(\lambda,t)\mapsto\theta(\lambda)\Upsilon_{2;\epsilon,\delta}(\Lambda)\chi(t\epsilon^\kappa)\|_{L^\infty H^1}\le C\epsilon^{1-2\kappa},$$

so that by Proposition A.17, the contribution of $\Upsilon_{2;\epsilon,\delta}$ in (5.22) is $\mathcal{O}(\epsilon^{1-2\kappa}\delta^{-n+1-n\kappa})$, which is $O(\delta^{-n+1}\hbar^{\alpha})$ for small $\alpha > 0$.

Finally, note that we have tuned the pre-factor of $\Upsilon_{3;\epsilon,\delta}$ so that we can perform integration by parts with respect to ζ. Hence, the contribution of $\Upsilon_{3;\epsilon,\delta}$ in (5.22) is given by

$$\frac{\delta/2\pi}{((2\pi)^2\delta)^n}\int e^{\mathrm{i}(x-y+2R(x_0,\lambda)t\omega)\cdot\zeta} g(x)g(y)\,\mathrm{div}_\zeta\,\Upsilon_{3;\epsilon,\delta}(\Lambda)\frac{\sin(2(\mu-\lambda)t/\delta)}{t} \times\chi(t\epsilon^{\kappa})\chi(|\zeta|\delta^{\kappa})\theta(\lambda)\mathbb{1}_{\mu\geq\lambda}\mathrm{d}\Lambda.$$

Because of the change of variable $\zeta\leftarrow\delta\zeta$ that we performed,

$$|\mathrm{div}_\zeta\Upsilon_{3;\epsilon,\delta}(\Lambda)| = \mathcal{O}(\epsilon\delta)$$

and

$$\|(\lambda,t)\mapsto \mathrm{div}_\zeta\,\Upsilon_{3;\epsilon,\delta}(\Lambda)\chi(t\epsilon^{\kappa})\theta(\lambda)\|_{L^\infty C^2} + \|(\lambda,t)\mapsto \theta(\lambda)t^{\gamma-1}\partial_t(\mathrm{div}_\zeta\,\Upsilon_{3;\epsilon,\delta}(\Lambda)\chi(t\epsilon^{\kappa}))\|_{L^\infty L^1}\leq C\delta\epsilon^{1-\gamma\kappa}.$$

By Proposition A.19, we conclude that the contribution of $\Upsilon_{3;\epsilon,\delta}$ in (5.22) is also of order $O(\delta^{-n+1}\hbar^{\alpha})$ for small $\alpha > 0$ with the required uniformity.

In summary, as $\hbar\to 0$,

$$\mathfrak{I}''_\hbar = \frac{1/2\pi}{((2\pi)^2\delta)^n}\int e^{\mathrm{i}(x-y+2r_\star(x_0,\lambda)t\omega)\cdot\zeta} g(y)g(x)h(x_\epsilon,y_\epsilon,\lambda)\frac{\sin(2(\mu-\lambda)t/\delta)}{t} \times\chi(t\epsilon^{\kappa})\chi(|\zeta|\delta^{\kappa})\theta(\lambda)\mathbb{1}_{\mu\geq\lambda}\mathrm{d}\Lambda + \mathcal{O}(\delta^{1-n}\hbar^{\alpha}). \tag{5.23}$$

5.3.4. Leading term. Let us denote

$$\begin{aligned}F(\zeta,x,\lambda;\epsilon) &= \frac{g(x)\theta(\lambda)}{(2\pi)^n}\int e^{-\mathrm{i}y\cdot\zeta}h(x_\epsilon,y_\epsilon,\lambda)g(y)\mathrm{d}y\\ &= \frac{g(x)\theta(\lambda)}{(2\pi)^n}r_\star(x_\epsilon,\lambda)^{n-2}\int \widehat{g}(\zeta+\epsilon\xi)\widehat{v_{x_\epsilon}}(\xi)\mathrm{d}\xi,\end{aligned}$$

where

$$v_x\colon y\mapsto \vartheta\Big(V\Big(\frac{x+y}{2}\Big)+r_\star(x_0,\lambda)^2\Big)^2.$$

Since $g\in C_c^\infty(\mathbb{R}^n)$ and $v_x\in C_c^\infty(\mathbb{R}^n)$ for all $x\in B^n_{0,c}$ and $\lambda\in\mathrm{supp}(\theta)$, we have $F\in S^0(\mathbb{R}^n\times\mathbb{R}^{n+1})$ with compact support with respect to (λ,x) and

$$F_k(\zeta,x,\lambda) = \mathcal{O}_k(|\zeta|^{-\infty})$$

as $\zeta\to\infty$ for every $k\in\mathbb{N}_0$. In particular, the leading term on the right-hand side of (5.23) satisfies

$$\mathfrak{I}''_\hbar = \frac{1/2\pi}{(2\pi\delta)^n}\int e^{\mathrm{i}(x+2r_\star(x_0,\lambda)t\omega)\cdot\zeta}F(\zeta,x,\lambda;\epsilon)\frac{\sin(2(\mu-\lambda)t/\delta)}{t}\chi(t\epsilon^{\kappa}) \times\chi(|\zeta|\delta^{\kappa})\mathbb{1}_{\mu\geq\lambda}\mathrm{d}\zeta\mathrm{d}t\mathrm{d}x\mathrm{d}\lambda\mathrm{d}\omega + \mathcal{O}(\hbar^\infty).$$

Note that we used the decay of F to remove the cutoff $\chi(|\zeta|\delta^\kappa)$ up to another $\mathcal{O}(\hbar^\infty)$ error. We can now compute the integral with respect to ζ by Fourier's inversion formula, and we obtain for any $|t| \le \epsilon^{-\kappa}$ and $x \in \operatorname{supp}(g)$,

$$\begin{aligned}
&\int e^{\mathrm{i}(x+2r_\star(x_0,\lambda)t\omega)\cdot\zeta} F(\zeta, x, \lambda; \epsilon)\mathrm{d}\zeta \\
&\quad = g(x)\theta(\lambda)h(x_\epsilon, (x + 2r_\star(x_0,\lambda)t\omega)_\epsilon, \lambda)g(x + 2r_\star(x_0,\lambda)t\omega) \\
&\quad = g(x)g(x+2r_\star(x_0,\lambda)t\omega)r_\star(x_\epsilon,\lambda)^{n-2}\vartheta(V(x_\epsilon + \epsilon t r_\star(x_0,\lambda)\omega)+r_\star(x_\epsilon,\lambda)^2)^2\theta(\lambda) \\
&\quad = g(x)g(x + r_\star(x_0,\lambda)t\omega)r_\star(x_\epsilon,\lambda)^{n-2}\vartheta(\lambda) + \mathcal{O}_{x,\lambda,\omega}(\epsilon t)\theta(\lambda),
\end{aligned}$$

where we used that $V(x_\epsilon) + r_\star(x_\epsilon,\lambda)^2 = \lambda$ and that $\theta(\lambda) = 1$ for $\lambda \in \operatorname{supp}(\vartheta)$ to rewrite the leading term. Using again Proposition A.17, integrating the error term $\mathcal{O}_{x,\lambda,\omega}(\epsilon t)\theta(\lambda)$ over (λ, t), it contributes as $\mathcal{O}_\gamma(\epsilon\delta^{1-\gamma-n\kappa}) = \mathcal{O}(\delta\hbar^\alpha)$ to the previous integral, upon choosing the parameters $\gamma, \kappa > 0$ small enough. Hence, from (5.23), one has $\mathfrak{I}''_\hbar = \mathfrak{N}_\hbar + \mathcal{O}(\delta^{1-n}\hbar^\alpha)$, where

$$\begin{aligned}
\mathfrak{N}_\hbar = \frac{1/2\pi}{(2\pi\delta)^n}\int & g(x)g(x + 2r_\star(x_0,\lambda)t\omega)r_\star(x_\epsilon,\lambda)^{n-2}\frac{\sin(2(\mu-\lambda)t/\delta)}{t} \\
&\times \chi(t\epsilon^\kappa)\vartheta(\lambda)\mathbb{1}_{\mu\ge\lambda}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\lambda\,\mathrm{d}\omega.
\end{aligned} \tag{5.24}$$

To finish the proof, we split integral (5.24) in two parts, $\mathfrak{N}_\hbar = \mathfrak{R}_\hbar + \operatorname{Im}(\mathfrak{S}_\hbar)$, where

$$\mathfrak{R}_\hbar = \frac{1/2\pi}{(2\pi\delta)^n}\int g(x)^2 r_\star(x_\epsilon,\lambda)^{n-2}\frac{\sin(2(\mu-\lambda)t/\delta)}{t}\chi(t\epsilon^\kappa)\vartheta(\lambda)\mathbb{1}_{\mu\ge\lambda}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\lambda\,\mathrm{d}\omega$$

and

$$\begin{aligned}
\mathfrak{S}_\hbar = \frac{1/2\pi}{(2\pi\delta)^n}\int & g(x)\frac{g(x + 2r_\star(x_0,\lambda)t\omega) - g(x)}{t}e^{2\mathrm{i}(\mu-\lambda)t/\delta}r_\star(x_\epsilon,\lambda)^{n-2} \\
&\times \chi(t\epsilon^\kappa)\vartheta(\lambda)^2\mathbb{1}_{\mu\ge\lambda}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\lambda\,\mathrm{d}\omega.
\end{aligned} \tag{5.25}$$

Note that the integrand in $\mathfrak{R}_\hbar$ is independent of ω.

According to Proposition A.19, it holds for any $0 < \gamma \ll \kappa$,

$$\begin{aligned}
\mathfrak{R}_\hbar &= \frac{|S^{n-1}|/2\pi}{(2\pi\delta)^n}\int g_\epsilon(x)^2 r_\star(x,\lambda)^{n-2}\frac{\sin(2(\mu-\lambda)t/(\delta\epsilon^\kappa))}{t}\chi(t)\vartheta(\lambda)\mathbb{1}_{\mu\ge\lambda}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\lambda \\
&= \frac{|S^{n-1}|}{2(2\pi\delta)^n}\int g_\epsilon(x)^2 r_\star(x,\lambda)^{n-2}\vartheta(\lambda)\mathbb{1}_{\lambda\le\mu}\mathrm{d}x\,\mathrm{d}\lambda + \mathcal{O}(\epsilon^{\kappa(1-\gamma)}\delta^{1-\gamma}).
\end{aligned}$$

Since $\hbar^{1-\beta} \le \epsilon, \delta \le \hbar^\beta$, by choosing γ small enough (depending only on $0 < \alpha < \beta$ and κ), the error term is also $\mathcal{O}(\hbar^\alpha)$. By (5.23) and (5.24), this shows that

$$\mathfrak{N}_\hbar = \frac{|S^{n-1}|}{2(2\pi\delta)^n}\int g_\epsilon(x)^2 r_\star(x,\lambda)^{n-2}\vartheta(\lambda)\mathbb{1}_{\lambda\le\mu}\mathrm{d}x\,\mathrm{d}\lambda + \mathcal{O}(\delta^{1-n}\hbar^\alpha). \tag{5.26}$$

This coincides with the asymptotics in Proposition 5.7. Hence, to finish the proof, it remains to compute the leading asymptotics of $\operatorname{Im}(\mathfrak{S}_\hbar)$.

5.3.5. Subleading term. First of all, by Proposition A.19, when computing $\mathrm{Im}(\mathfrak{S}_\hbar)$ from (5.25), we may replace $r_\star(x_\epsilon,\lambda)$ by $r_\star(x_0,\lambda)$, up to an error of size $\mathcal{O}(\delta^{-n+1-\gamma}\epsilon^{1-\kappa}) = \mathcal{O}(\delta^{-n+1}\hbar^\alpha)$. It remains to compute the imaginary part of

$$\mathfrak{S}'_\hbar = \frac{1/2\pi}{(2\pi\delta)^n}\int g(x)\frac{g(x+2r_\star(x_0,\lambda)t\omega)-g(x)}{t}e^{2\mathrm{i}(\mu-\lambda)t/\delta}r_\star(x_0,\lambda)^{n-2} \times \chi(t\epsilon^\kappa)\vartheta(\lambda)^2\mathbb{1}_{\mu\geq\lambda}\mathrm{d}t\mathrm{d}x\mathrm{d}\lambda\mathrm{d}\omega.$$

Let us observe that using the invariance by rotation with respect to $\omega\in S^{n-1}$ of $r_\star(x_0,\lambda)$, one has

$$\int g(x)\omega\cdot\partial_x g(x)r_\star(x_0,\lambda)^{n-1}\frac{1}{t}\chi(t\epsilon^\kappa)\vartheta(\lambda)^2\mathbb{1}_{\mu\geq\lambda}\mathrm{d}t\mathrm{d}x\mathrm{d}\lambda\mathrm{d}\omega = 0.$$

In particular, letting

$$G\colon (x,\xi)\mapsto \frac{g(x+2\xi)-g(x)-2\xi\cdot\partial_x g(x)}{|\xi|^2},$$

one obtain that $(t,\lambda)\mapsto G(x,r_\star(x_0,\lambda)t\omega)$ is smooth with uniform controls in ω, and

$$\delta^n\mathfrak{S}'_\hbar = \frac{1}{(2\pi)^{n+1}}\int g(x)G(x,r_\star(x_0,\lambda)t\omega)r_\star(x_0,\lambda)^n t e^{2\mathrm{i}(\mu-\lambda)t/\delta} \times \chi(t\epsilon^\kappa)\vartheta(\lambda)^2\mathbb{1}_{\mu\geq\lambda}\mathrm{d}t\mathrm{d}x\mathrm{d}\lambda\mathrm{d}\omega.$$

We rewrite this as

$$\delta^{n-1}\mathfrak{S}'_\hbar = \frac{\mathbf{i}}{2(2\pi)^{n+1}}\int g(x)G(x,r_\star(x_0,\lambda)t\omega)r_\star(x_0,\lambda)^n\partial_\lambda(e^{2\mathrm{i}(\mu-\lambda)t/\delta}) \times \chi(t\epsilon^\kappa)\vartheta(\lambda)^2\mathbb{1}_{\mu\geq\lambda}\mathrm{d}t\mathrm{d}x\mathrm{d}\lambda\mathrm{d}\omega.$$

We can then perform an integration by parts with respect to λ to obtain

$$\begin{aligned}\delta^{n-1}\mathfrak{S}'_\hbar &= \frac{\mathbf{i}\vartheta(\mu)}{2(2\pi)^{n+1}}\int g(x)G(x,r_\star(x_0,\mu)t\omega)r_\star(x_0,\mu)^n\chi(t\epsilon^\kappa)\mathrm{d}t\mathrm{d}x\mathrm{d}\omega\\ &\quad-\frac{\mathbf{i}\pi}{(2\pi)^{n+2}}\int_{\lambda\leq\mu} g(x)e^{2\mathrm{i}(\mu-\lambda)t/\delta}\partial_\lambda[G(x,r_\star(x_0,\lambda)t\omega)\vartheta(\lambda)^2r_\star(x_0,\lambda)^n]\\ &\qquad\times\chi(t\epsilon^\kappa)\mathrm{d}t\mathrm{d}x\mathrm{d}\lambda\mathrm{d}\omega. \qquad (5.27)\end{aligned}$$

The function $\lambda\mapsto\partial_\lambda(G(x,r_\star(x_0,\lambda)t,\omega)\vartheta(\lambda)^2r_\star(x_0,\lambda)^n)$ is uniformly bounded with a fixed compact support and $t\mapsto\chi(t\epsilon^\kappa)$ is a smooth function with

$$\|\chi(\cdot\epsilon^\kappa)\|_{H^s}\leq C_s\|\chi(\cdot\epsilon^\kappa)\|_{L^2}\leq C_s\epsilon^{-\kappa/2}$$

for any $s\geq 0$ so that, by Proposition A.17, the second term on the right-hand side of (5.27) is $\mathcal{O}(\hbar^\alpha)$ for some $\alpha>0$. Hence, with $r_\star = r_\star(x_0,\mu)$, we obtain

$$\begin{aligned}\delta^{n-1}\mathfrak{S}_\hbar &= \frac{\mathbf{i}r_\star^n\vartheta(\mu)}{2(2\pi)^{n+1}}\int g(x)G(x,r_\star t\omega)\chi(t\epsilon^\kappa)\mathrm{d}t\mathrm{d}x\mathrm{d}\omega + \mathcal{O}(\hbar^\alpha)\\ &= \frac{\mathbf{i}r_\star^{n-1}\vartheta(\mu)}{(2\pi)^{n+1}}\int g(x)G(x,\xi)\frac{\chi(|\xi|\epsilon^\kappa/r_\star)}{|\xi|^{n-1}}\mathrm{d}\xi\mathrm{d}x + \mathcal{O}(\hbar^\alpha),\end{aligned}$$

where we made a change of variable $r_\star(x_0,\mu)t\omega \leftarrow \xi$. An extra factor 2 comes from the fact that this map is two-to-one ($t \in \mathbb{R}$). By symmetry, we can rewrite

$$\begin{aligned}\delta^{n-1}\mathfrak{S}_\hbar &= \frac{\mathbf{i}r_\star^{n-1}\vartheta(\mu)}{(2\pi)^{n+1}}\int g(x-\xi)G(x-\xi,\xi)\frac{\chi(|\xi|\epsilon^\kappa/r_\star)}{|\xi|^{n-1}}\mathrm{d}\xi\mathrm{d}x + \mathcal{O}(\hbar^\alpha)\\ &= \frac{-\mathbf{i}r_\star^{n-1}\vartheta(\mu)}{2(2\pi)^{n+1}}\int\Big(\Big(\frac{g(x-\xi)-g(x+\xi)}{|\xi|}\Big)^2\\ &\qquad + \frac{\xi}{4|\xi|^2}(\partial_x g(x-\xi)^2 - \partial_x g(x+\xi)^2)\Big)\\ &\qquad \times \frac{\chi(|\xi|\epsilon^\kappa/r_\star)}{|\xi|^{n-1}}\mathrm{d}\xi\mathrm{d}x + \mathcal{O}(\hbar^\alpha).\end{aligned}$$

In particular, the second term vanishes since it is an exact derivative with respect to x of an integrable function of (x,ξ) (here we use that χ and g have compact support). Hence, using the Hölder bound

$$|\chi(t\epsilon^\kappa)-1| \le c_\kappa t^\kappa \epsilon^{\gamma\kappa}$$

for any $\gamma \in (0,1)$ and that

$$\epsilon^{\gamma\kappa} = \mathcal{O}(\hbar^\alpha)$$

if $\alpha > 0$ is small enough, we obtain

$$\delta^{n-1}\mathfrak{S}_\hbar = \frac{-\mathbf{i}r_\star^{n-1}\vartheta(\mu)}{(2\pi)^{n+1}}\int\frac{(g(x-\xi/2)-g(x+\xi/2))^2}{|\xi|^{n+1}}\mathrm{d}\xi\mathrm{d}x + \mathcal{O}(\hbar^\alpha).$$

By formulas (5.26) and (5.2), we conclude that

$$\begin{aligned}\mathfrak{F}_\hbar = \frac{n|S^{n-1}|}{2(2\pi\delta)^n}\int g_\epsilon(x)^2 r_\star(x,\lambda)^{n-2}\vartheta(\lambda)\mathrm{d}x\mathrm{d}\lambda - \sigma_n^2 r_\star^{n-1}\vartheta(\mu)\delta^{1-n}\Sigma^2(f)\\ + \mathcal{O}(\delta^{1-n}\hbar^\alpha).\end{aligned}$$

Since $\mathrm{tr}((f_{\hbar,\mu}(H_\hbar)g_\epsilon)^2) = \mathfrak{F}_\hbar + \mathcal{O}(\hbar^\infty)$, this completes the proof. ∎

5.4. *Macroscopic commutators*

In this section, we turn to the proof of Theorem I.2. Let $g \in C_c^\infty(\{V < \mu\},\mathbb{R})$ be any smooth function compactly supported in the bulk of the droplet. Given the bounds for the Laplace functional of a determinantal point process from Proposition A.11, the main ingredient of the proof is a (sharp) bound for the variance

$$\mathrm{var}\,\mathrm{X}(g) = -\frac{1}{2}\mathrm{tr}([g,\Pi_{\hbar,\mu}]^2).$$

Using estimate (5.7) with $M = g$ and the next proposition, we obtain

$$\mathrm{var}\,\mathrm{X}(g) = \mathcal{O}(\hbar^{-n+1}). \tag{5.28}$$

Proposition 5.11. *Let (μ, V) satisfy (H) and let $f_{\hbar,\mu}$ be as in Notations 3.1 with $\mu_0 < \mu$. Then, for any $g \in C_c^\infty(\{V < \mu_0\})$,*

$$\mathrm{tr}([f_{\hbar,\mu}(H_\hbar), g]^2) = \mathcal{O}(\hbar^{-n+1}).$$

Proof. Using representation (3.4) and proceeding as in the proof of Proposition 5.5 for $t \in [-\tau, \tau]$, by bilinearity, we obtain

$$\mathrm{tr}([g, f_{\hbar,\mu}(H_\hbar)]^2) = \frac{1}{2\pi\hbar^2}\int \mathrm{tr}([g, I^{\phi,a}_{\hbar,t_1}][g, I^{\phi,a}_{\hbar,t_2}])e^{-\mathrm{i}t_1\lambda_1/\hbar-\mathrm{i}t_2\lambda_2/\hbar}\varkappa_\mu(\lambda_1)\varkappa_\mu(\lambda_2) \times \widehat{\rho}(t_1)\widehat{\rho}(t_2)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}\lambda_1\mathrm{d}\lambda_2 + \mathcal{O}(\hbar^\infty),$$

where we recall that $\varkappa_\mu = \varkappa\mathbb{1}_{(-\infty,\mu]}$, and $a \in S^0$ is a classical symbol.

Hence, the only relevant contribution to this trace is given by the oscillatory integral

$$J_\hbar = \frac{4\pi}{(2\pi\hbar)^{2n+2}}\int e^{\frac{\mathrm{i}}{\hbar}\Psi_1(t_1,t_2,x,y,\xi_1,\xi_2,\lambda_1,\lambda_2)}g(x)(g(y)-g(x))b(t_1,t_2,x,y,\xi_1,\xi_2;\hbar) \times \varkappa_\mu(\lambda_1)\varkappa_\mu(\lambda_2)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}x\mathrm{d}y,$$

where $b \in S^0$ is given by

$$b(t_1,t_2,x,y,\xi_1,\xi_2;\hbar) := a(t_1,x,y,\xi_1;\hbar)\overline{a}(t_2,y,x,\xi_2;\hbar)\widehat{\rho}(t_1)\widehat{\rho}(t_2).$$

In particular, b has compact support on $\mathbb{R}^{4n+2}$ and, provided τ is small enough, we can assume that for a fixed $\delta > 0$, for all $t \in [-\tau, \tau]$, $x \in \mathrm{supp}(g)$, $y \in \mathbb{R}^n$ and $\hbar \in (0, 1]$, the function $\xi \mapsto a(t, x, y, \xi; \hbar)\widehat{\rho}(t)$ is supported on $\{|\xi| \geq \delta\}$ (cf. Proposition 2.11).

5.4.1. Stationary phase method. The phase in $J_\hbar$ is given by

$$\Psi_1(t_1,t_2,x,y,\xi_1,\xi_2,\lambda_1,\lambda_2) := \phi(t_1,x,\xi_1) - y\cdot(\xi_1-\xi_2) - \phi(t_2,x_1,\xi_2) - t_1\lambda_1 + t_2\lambda_2.$$

To apply the stationary phase method, we make the change of variables

$$s = t_1 - t_2,\ \ t = \frac{t_1+t_2}{2},\ \ \zeta = \xi_1 - \xi_2,\ \ r\omega = \frac{\xi_1+\xi_2}{2},\ \ \varsigma = \lambda_1 - \lambda_2,\ \ \sigma = \frac{\lambda_1+\lambda_2}{2},$$

where $r \in \mathbb{R}_+$, $\omega \in S^{n-1}$, $t, s \in [-2\tau, 2\tau]$, $\sigma, \varsigma \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$. The Jacobian of this map is r^{n-1} and, introducing

$$c(t,s,x,y,\zeta,r,\omega;\hbar) = b\Big(t+\frac{s}{2}, t-\frac{s}{2}, x, y, r\omega+\frac{\zeta}{2}, r\omega-\frac{\zeta}{2};\hbar\Big)r^{n-1},$$

$$\Gamma_\mu(\varsigma,\sigma) = \varkappa_\mu\Big(\sigma+\frac{\varsigma}{2}\Big)\varkappa_\mu\Big(\sigma-\frac{\varsigma}{2}\Big),$$

$$\Psi_2(t,s,x,y,\zeta,r,\omega,\varsigma,\sigma) = \Psi_1(t_1,t_2,x,y,\xi_1,\xi_2,\lambda_1,\lambda_2),$$

we obtain

$$J_\hbar = \frac{4\pi}{(2\pi\hbar)^{2n+2}}\int e^{\frac{\mathrm{i}}{\hbar}\Psi_2(t,s,x,y,\zeta,r,\omega,\varsigma,\sigma)}g(x)(g(y)-g(x))c(t,s,x,y,\zeta,r,\omega;\hbar) \times \Gamma_\mu(\varsigma,\sigma)\mathrm{d}r\mathrm{d}t\mathrm{d}s\mathrm{d}\zeta\mathrm{d}y\mathrm{d}x\mathrm{d}\sigma\mathrm{d}\varsigma\mathrm{d}\omega. \tag{5.29}$$

We now apply the stationary phase method to this integral in the variables (y, r, s, ζ) while keeping $(t, x, \sigma, \varsigma, \omega)$ fixed. The equations for the critical point(s) are given by

$$0 = \partial_y \Psi_2(t, s, x, y, \zeta, r, \omega, \varsigma, \sigma) = -\zeta, \tag{5.30}$$

$$0 = \partial_r \Psi_2(t, s, x, y, \zeta, r, \omega, \varsigma, \sigma) = \omega \cdot (\partial_\xi \phi(t_1, x, \xi_1) - \partial_\xi \phi(t_2, x, \xi_2)), \tag{5.31}$$

$$0 = \partial_s \Psi_2(t, s, x, y, \zeta, r, \omega, \varsigma, \sigma) = \frac{1}{2}(\partial_t \phi(t_1, x, \xi_1) + \partial_t \phi(t_2, x, \xi_2)) - \sigma, \tag{5.32}$$

$$0 = \partial_\zeta \Psi_2(t, x, y, \zeta, r, \omega, \varsigma, \sigma) = \partial_\xi \phi(t_1, x, \xi_1) + \partial_\xi \phi(t_2, x, \xi_2) - y, \tag{5.33}$$

where we recall that

$$\xi_1 = r\omega + \frac{\zeta}{2}, \quad \xi_2 = r\omega + \frac{\zeta}{2}, \quad t_1 = t + \frac{s}{2}, \quad t_2 = t - \frac{s}{2}.$$

From these equations, we can now argue that there is at most one critical point within the support of c.

- Formula (5.30) implies that $\zeta = 0$ (equivalently $\xi_1 = \xi_2 = r\omega$).
- Recall that according to (3.2), $\partial_\xi \phi(t, x, \xi) = x + t\xi + \mathcal{O}(t^2)$. Then, (5.31) and the condition $\zeta = 0$ imply that

  $$0 = \omega \cdot (\partial_\xi \phi(t_1, x, r\omega) - \partial_\xi \phi(t_2, x, r\omega)) = s(r + \mathcal{O}(t)).$$

  Since $\xi \mapsto a(t, x, y, \xi; \hbar)$ is supported on $\{|\xi| > \delta\}$, one has $r > \delta$ at least when $\zeta = 0$, so that this equation is satisfied only if $s = 0$, provided τ is small enough (equivalently $t_1 = t_2 = t$).
- Using the conditions $\zeta = t = 0$, equations (5.32) and (3.2) then yield

  $$\sigma = \partial_t \phi(t, x, r\omega) = V(x) + r^2 + \mathcal{O}(t).$$

  Like in the proof of Proposition 3.2, we need to distinguish two cases:
  - If $\sigma \le V(x) + c_1$, for some constant $c_1 = c_1(\delta, \tau)$, then this equation has no solutions satisfying $r \ge \delta$. In this case, there is no critical point within the support of c and by Proposition A.16, we conclude that $J_\hbar = \mathcal{O}(\hbar^\infty)$. We can therefore include a cutoff $\theta(\sigma - V(x))$ in the integrand of (5.29), where $\theta \in C^\infty(\mathbb{R}, [0, 1])$ satisfies $\mathbb{1}_{[c_2,\infty)} \le \theta \le \mathbb{1}_{[c_1,\infty)}$ for a small enough $c_2 > c_1 > \tau$ such that $V(x) + c_2 < \mu_0$ for all x in the support of g.
  - For $\sigma \ge V(x) + c_1$, this equation has a unique (positive) solution which determines the critical value of r as a function of the other parameters,

    $$r_\star(t, x, \sigma, \omega) = \sqrt{\sigma - V(x)} + \mathcal{O}(t). \tag{5.34}$$

- Finally, using the conditions $(s, \zeta, r) = (0, 0, r_\star)$, condition (5.33) determines the critical value of y as a function of the other parameters,

  $$y_\star(t, x, \sigma, \omega) = \partial_\xi \phi(t, x, r_\star \omega) = x + \mathcal{O}(t),$$

  where we used again expansions (3.2) and (5.34).

This shows that the (unique) critical point is given by $(y, r, s, \zeta) = (y_\star, r_\star, 0, 0)$, and using (5.30)–(5.33), we verify that the Hessian of Ψ_2 in these variables evaluated at the critical point is given by the matrix

$$\mathrm{H} = -\begin{pmatrix} 0 & 0 & 0 & \mathrm{I} \\ 0 & 0 & \bullet & \bullet \\ 0 & \bullet & 0 & 0 \\ \mathrm{I} & \bullet & 0 & 0 \end{pmatrix}.$$

This matrix is non-degenerate, and by expanding over the second and third columns, we obtain

$$\begin{aligned} \det \mathrm{H} &= |\partial_r \partial_s \Psi_2|^2(t, s, x, y_\star, 0, r_\star, \omega, \varsigma, \sigma) \\ &= |\omega \cdot \partial_\xi |\partial_x \phi|^2(t, x, r_\star \omega)|^2 = 4(\sigma - V(x))^2 + \mathcal{O}(t^2), \end{aligned}$$

where we used expansion (3.2) to simplify the quantity for small t. Let us also observe that at the critical point, one has

$$\Psi_2(t, x, y_\star, 0, r_\star, \omega, \varsigma, \sigma) = -t\varsigma.$$

Hence, applying Proposition A.15 to integral (5.29) while keeping the variables $(t, x, \sigma, \varsigma, \omega)$ fixed, since $g \in C_c^\infty(\mathbb{R}^n)$, we obtain the expansion as $\hbar \to 0$, for any $k \in \mathbb{N}$,

$$J_\hbar = \frac{4\pi}{(2\pi\hbar)^{n+1}} \int e^{-\mathrm{i}t\varsigma/\hbar} e(t, x, \omega, \sigma; \hbar) \Gamma_\mu(\varsigma, \sigma) \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}\sigma\,\mathrm{d}\varsigma + \mathcal{O}(\hbar^\infty), \qquad (5.35)$$

where the (classical) symbol e is smooth and satisfies

$$\begin{aligned} e(t, x, \omega, \sigma; 0) &= g(x)(g(y_\star) - g(x))|\widehat{\rho}(t)|^2 a(t, x, y_\star, r_\star\omega; 0)\overline{a}(t, y_\star, x, r_\star\omega; 0) \\ &\quad \times \frac{r_\star^{n-1}}{\sqrt{\det \mathrm{H}}} \theta(\sigma - V(x)). \end{aligned}$$

In particular, the function $(t, x, \omega, \sigma, \varsigma) \mapsto e(t, x, \omega, \sigma; \hbar)\Gamma_\mu(\varsigma, \sigma)$ in (5.35) is compactly supported, L^∞ with respect to ς and C^∞ with respect to t. Hence, by Proposition A.17, subprincipal terms contribute as $\mathcal{O}(\hbar^{-n+1})$, so that

$$\begin{aligned} J_\hbar = \frac{4\pi}{(2\pi\hbar)^{n+1}} \int e^{-\mathrm{i}t\varsigma/\hbar} e(t, x, \omega, \sigma; 0) \varkappa\Big(\sigma + \frac{\varsigma}{2}\Big) \varkappa\Big(\sigma - \frac{\varsigma}{2}\Big) \mathbb{1}_{|\varsigma/2| \le \mu - \sigma} \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}\sigma\,\mathrm{d}\varsigma \\ + \mathcal{O}(\hbar^{-n+1}). \end{aligned}$$

Since $y_\star = x + \mathcal{O}(t)$ and g is smooth, the principal symbol of $e = \mathcal{O}(t)$, that is, there exists $F \in C_c^\infty$ such that

$$e(t, x, \omega, \sigma; 0) = tF(t, x, \omega, \sigma).$$

Integrating by parts with respect to ς, since $\varkappa = 1$ on $[0, M]$, we obtain

$$\begin{aligned} J_\hbar = \frac{4\pi}{(2\pi\hbar)^n} \int \sin\Big(\frac{2(\mu - \sigma)}{t}\Big) F(t, x, \omega, \sigma) \mathbb{1}_{\sigma \le \mu} \mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\omega \\ + \frac{4\pi}{(2\pi\hbar)^n} \int e^{\frac{2\mathrm{i}t}{\hbar}(\mu - \sigma)} F(t, x, \omega, \sigma) \Gamma'_\mu(\varsigma, \sigma) \mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}\zeta + \mathcal{O}(\hbar^{-n+1}), \end{aligned}$$

where

$$\Gamma'_\mu\colon (\varsigma,\sigma) \mapsto \frac{\mathrm{d}}{\mathrm{d}\varsigma}\Big[\varkappa\Big(\sigma+\frac{\varsigma}{2}\Big)\varkappa\Big(\sigma-\frac{\varsigma}{2}\Big)\Big]\mathbb{1}_{|\varsigma/2|\le\mu-\sigma}.$$

The function $\Gamma'_\mu \in L^\infty$ and $F \in C_c^\infty$, so by applying Proposition A.17 to both these integrals, we conclude that $J_\hbar = \mathcal{O}(\hbar^{-n+1})$. This completes the proof. ■

Remark 5.12. With extra work, it is possible to refine the proof to obtain the leading asymptotics of $\mathrm{tr}([f_{\hbar,\mu}(H_\hbar), g]^2)$. However, upon comparing these asymptotics with that of $\mathrm{tr}([\Pi_{\hbar,\mu} g]^2)$, there is another a priori uncontrolled $\mathcal{O}(\hbar^{-n+1})$ contribution coming from Lemma 5.6. In view of Conjecture 1, it is not clear how to bypass this difficulty.

Remark 5.13. The stationary phase in the proof of Proposition 5.11 still holds if the test function g depends on $\hbar$ at sufficiently large scales. In this fashion, one can give an alternative proof of the mesoscopic commutator estimate from Proposition 5.4 in the case $\epsilon \ge \hbar^\beta$ for $\beta > \frac{1}{2}$. Based on the counterpart of (5.35) at mesoscopic scales and Proposition A.19, one can recover the leading asymptotics of $\|[T^*_{x_0,\epsilon} g, f_{\hbar,\mu}(H_\hbar)]\|_{\mathrm{J}^2}$ when $\epsilon \to 0$ in the previous regime.

We are now ready to conclude the proof of Theorem I.2.

Proof of Theorem I.2. Let $f \in C_c^\infty(\{V < \mu\})$ and let $\widetilde{\mathrm{X}}(f) = \mathrm{X}(f) - \mathbb{E}[\mathrm{X}(f)]$. Suppose first that $f \le 0.69$. Then, by Proposition A.11 and estimate (5.28) applied to the test function $g = (e^f - 1) \in C_c^\infty(\{V < \mu\})$,

$$\log \mathbb{E}[e^{\widetilde{\mathrm{X}}(f)}] = \mathcal{O}(\operatorname{var} \mathrm{X}(g)) = \mathcal{O}(N\hbar)$$

as $\hbar \to 0$.

If now $f \in C_c^\infty(\{V < \mu\})$ is arbitrary and $\lambda \le \|f\|_{L^\infty}^{-1}\sqrt{N\hbar}$, by rescaling, one has

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda\widetilde{\mathrm{X}}(f)}{\sqrt{N\hbar}}\Big)\Big] \le \exp(C\lambda^2),$$

where C depends only on f. In particular, by Markov's inequality, for every $t > 0$,

$$\mathbb{P}[\widetilde{\mathrm{X}}(f) \ge \sqrt{N\hbar}t] = \mathbb{P}\Big[\frac{\lambda\widetilde{\mathrm{X}}(f)}{\sqrt{N\hbar}} \ge t\lambda\Big] \le \exp(C\lambda^2 - t\lambda).$$

The optimal value of λ for the right-hand side is

$$\lambda = \frac{t}{2C},$$

so that provided $t \le 2C\|f\|_{L^\infty}^{-1}\sqrt{N\hbar}$, we obtain

$$\mathbb{P}[\widetilde{\mathrm{X}}(f) \ge \sqrt{N\hbar}t] \le e^{-t^2/(2C)}.$$

Replacing f by $-f$ yields the symmetric inequality. ■

Appendix A.

A.1. Compact operators

We follow the conventions from [80, Chapters 1–3]. Throughout this article, we work on the Hilbert space $L^2(\mathbb{R}^n)$ equipped with the norm $\|\cdot\| = \|\cdot\|_{L^2}$. Recall that a (linear) operator B is bounded if

$$\|B\| = \sup_{\phi\in L^2, \|\phi\|\le 1} \|B\phi\| < +\infty.$$

If B is a bounded operator, it is called positive if $\langle B\phi, \phi\rangle = \langle \phi, B\phi\rangle \ge 0$ for all $\phi \in L^2(\mathbb{R}^n)$. Then, we write $A \ge B$ if A and B are symmetric and $(A - B)$ is positive. Moreover, we make use of the following conventions and basic properties.

Proposition A.1. (i) *Given any function $g \in L^\infty(\mathbb{R}^n)$, the operator $\phi \mapsto g\phi$ is bounded on $L^2(\mathbb{R}^n)$ and its norm is $\|g\|_{L^\infty}$. We will also denote by g this operator. In particular, this operator is positive if $g \ge 0$.*

(ii) *If A is a compact operator, we let $(\mu_k(A))_{k=1}^{+\infty}$ be its singular values as in* [80, Theorem 1.5]. *We define the Schatten norms for $p \ge 1$,*

$$\|A\|_{\mathrm{J}^p} = \left(\sum_{k=1}^{\infty} \mu_k^p(A)\right)^{1/p}.$$

Note that it holds for any bounded operator B and any $p \ge 1$, $\|BA\|_{\mathrm{J}^p}, \|AB\|_{\mathrm{J}^p} \le \|B\|\|A\|_{\mathrm{J}^p}$.

Proposition A.2 (Convergence of operators). *For a family of bounded operators $(B_\varepsilon)_{\varepsilon>0}$, we will use the following topology of convergence as $\varepsilon \to 0$,*

(i) *$B_\varepsilon \to 0$ in the strong operator topology if $\lim_{\varepsilon\to 0} \|B_\varepsilon\phi\| = 0$ for all $\phi \in L^2$. If $\|B_\varepsilon\| \le C$, it suffices to verify that $\lim_{\varepsilon\to 0} \|B_\varepsilon\phi\| = 0$ for all $\phi \in \mathcal{A}$, where $\mathcal{A}$ is dense in L^2.*

(ii) *$B_\varepsilon \to 0$ in the operator norm topology if $\lim_{\varepsilon\to 0} \|B_\varepsilon\| = 0$.*

(iii) *For $p \ge 1$, $B_\varepsilon \to 0$ in the J^p-norm topology if $\lim_{\varepsilon\to 0} \|B_\varepsilon\|_{\mathrm{J}^p} = 0$. Note that if $q > p \ge 1$, the J^p-norm topology is stronger than the J^q-norm topology.*

Then, we have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Proposition A.3 (Hilbert–Schmidt operators). (i) *We say that B is a Hilbert–Schmidt operator if $\|B\|_{\mathrm{J}^2} < +\infty$, in which case by* [80, Theorem 2.11], *B has an integral kernel and*

$$\|B\|_{\mathrm{J}^2} = \left(\int_{\mathbb{R}^{2n}} |B(x, y)|^2 \mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

(ii) *By* [80, Theorem 2.15], *it holds*

$$\|B\|_{\mathrm{J}^2}^2 = \sum_{k=1}^{\infty} |B\phi_k\|^2$$

for any orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^n)$.

Proposition A.4 (Trace-class operators). (i) *We say that B is a trace-class operator if $\|B\|_{\mathrm{J}^1} < +\infty$. Moreover, by* [80, Theorem 2.12]*, if $B \geq 0$ is trace-class and has a kernel which is continuous on $\mathbb{R}^{2n}$, and*

$$\|B\|_{\mathrm{J}^1} = \int_{\mathbb{R}^n} B(x,x)\mathrm{d}x.$$

(ii) *If B is trace-class, we define $\operatorname{tr} B = \sum_{k=1}^{+\infty} \langle \phi_k, B\phi_k\rangle$ for any orthonormal basis of $L^2(\mathbb{R}^n)$, see* [80, Theorem 3.1]*. Note that $\operatorname{tr}(\cdot)$ is a linear operator with $|\operatorname{tr} B| \leq \|B\|_{\mathrm{J}^1}$ and $\operatorname{tr} B = \|B\|_{\mathrm{J}^1}$ when $B \geq 0$.*

(iii) *If B is trace-class, then the Fredholm determinant $\det(1+B)$ is well defined and it is a continuous function on J^1. By* [80, Theorem 3.4],

$$|\det(1+B) - \det(1+A)| \leq \|A-B\|_{\mathrm{J}^1} e^{1+\|A\|_{\mathrm{J}^1}+\|B\|_{\mathrm{J}^1}}.$$

(iv) *By* [80, Corollary 3.8]*, if A, B are bounded operators such that AB and BA are trace-class, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\det(1+AB) = \det(1+BA)$.*

(v) *We say that B is locally trace-class if for any compact set $\mathcal{K} \Subset \mathbb{R}^n$, $\|B\mathbb{1}_{\mathcal{K}}\|_{\mathrm{J}^1} < +\infty$.*

Let $A, B, M \geq 0$ be bounded operators. If $B \geq A$ and BM is trace-class, then by Proposition A.4,

$$\operatorname{tr}[AM] = \operatorname{tr}[\sqrt{M}A\sqrt{M}] \leq \operatorname{tr}[\sqrt{M}B\sqrt{M}] = \operatorname{tr}[BM],$$

where $\sqrt{M}$ denotes the positive solution of $\sqrt{M}^2 = M$, see [80, Section 1.1]. Moreover, by Proposition A.1, if A is a finite-rank operator and B is bounded, then $\mu_k(AB) \leq \mathbb{1}_{k\leq \operatorname{Rank}(A)}\|AB\|$ and

$$\|AB\|_{\mathrm{J}^1} \leq \|AB\| \operatorname{Rank}(A).$$

A.2. Operators bounded from below

In this paper, the usual Laplacian Δ on $L^2(\mathbb{R}^n)$ plays a crucial role and we review below its basic properties.

An essentially self-adjoint operator, bounded from below, on a Hilbert space H is the data of a dense subspace $\mathcal{D}(A) \subset H$ and a linear map $A\colon \mathcal{D}(A) \to H$ such that $\{(x, Ax), x \in \mathcal{D}(A)\}$ is closed in $H \times H$ and there exists $C > 0$ such that, for every $x \in \mathcal{D}(A)$, one has $\langle x, Ax\rangle = \langle Ax, x\rangle \geq -C\|x\|_H^2$.

The point is that essentially self-adjoint operators, bounded from below, admit a spectral decomposition and a functional calculus; we refer to the textbook [49]. In particular, since $-\Delta \geq 0$ on $L^2(\mathbb{R}^n)$, under hypothesis (H), the Schrödinger operator

$$H_\hbar = -\hbar^2\Delta + V$$

is essentially self-adjoint, bounded from below, on $L^2(\mathbb{R}^n)$. Moreover, if $V \geq c$ for a constant $c \geq 0$, then $H_\hbar \geq c$.

A.2.1. Spectral properties of $-\Delta$ on $L^2(\mathbb{R}^n)$. Recall that $\mathcal{F}$ denotes the Fourier transform viewed as a unitary operator on $L^2(\mathbb{R}^n)$ and that $-\mathbf{i}\nabla = \mathcal{F}^* A \mathcal{F}$, where A is the multiplication operator, $A\phi : \xi \mapsto \xi\phi(\xi)$. This property of the Fourier transform renders explicit many properties of the operator $-\mathbf{i}\nabla$ or its functions. For instance, we define $-\Delta$ as a self-adjoint operator on $L^2(\mathbb{R}^n)$ with domain

$$H^2 = \{\phi \in L^2 : \xi \mapsto |\xi|^2 \mathcal{F}\phi(\xi) \in L^2(\mathbb{R}^n)\}.$$

Similarly, the classical L^2 Sobolev spaces can be defined using the Fourier transform. For $s \in \mathbb{R}$, let

$$H^s = \{\phi \in \mathcal{S}' : \xi \mapsto (1+|\xi|)^s \mathcal{F}\phi(\xi) \in L^2(\mathbb{R}^n)\},$$

where $\mathcal{S}'$ denotes the space of Schwartz distributions. There are alternative definitions of H^s, e.g., Remark 5.3.

For any $f \in L^\infty(\mathbb{R})$, we define $f(-\Delta) = \mathcal{F}^* f \mathcal{F}$, where f stands for a multiplication operator as in Proposition A.1 (i). In particular, $f(-\Delta)$ is a pseudodifferential operator in the sense of Definition 2.8, with semiclassical parameter $\hbar = 1$ and symbol $a = f(|\xi|^2)$. Hence, by Proposition A.3 (ii) and (v), if $f : \mathbb{R} \mapsto [0,\infty)$ satisfies $f(|\xi|^2) \in L^1(\mathbb{R}^n)$, the operator $f(-\Delta)$ is locally trace-class with integral kernel

$$(x,y) \in \mathbb{R}^{2n} \mapsto \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}(x-y)\cdot\xi} f(|\xi|^2) \mathrm{d}\xi.$$

In particular, this shows that for any $\mu > 0$ the kernel of the *bulk operator* is given by

$$K^{(n)}_{\mathrm{b},\mu} = \mathbb{1}_{(-\infty,\mu^2]}(-\Delta) : (x,y) \in \mathbb{R}^{2n} \mapsto \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbb{1}_{|\xi|\le\mu} e^{\mathbf{i}(x-y)\cdot\xi} \mathrm{d}\xi. \tag{A.1}$$

By going to spherical coordinates, one can rewrite

$$\begin{aligned}
K^{(n)}_{\mathrm{b},\mu}(x,y) &= \frac{1}{(2\pi)^n} \int_0^\mu \int_{S^{n-1}} e^{\mathbf{i}r|x-y|\mathrm{e}_1\cdot\theta} r^{n-1} \mathrm{d}\theta \mathrm{d}r \\
&= \frac{1}{(2\pi|x-y|)^n} \int_0^{\mu|x-y|} \int_{S^{n-1}} \cos(r\theta_1) r^{n-1} d\theta \mathrm{d}r \\
&= \frac{\Gamma((n-1)/2)^{-1}/\sqrt{\pi}}{(\sqrt{2\pi}|x-y|)^n} \int_0^{\mu|x-y|} r^{n-1} \left(\int_0^\pi \cos(r\cos\varphi)(\sin\varphi)^{n-2} \mathrm{d}\varphi \right) \mathrm{d}r \\
&= \frac{1}{(\sqrt{2\pi}|x-y|)^n} \int_0^{\mu|x-y|} r^{n/2} \mathrm{J}_{n/2-1}(r) \mathrm{d}r \\
&= \mu^{n/2} \frac{\mathrm{J}_{n/2}(\mu|x-y|)}{(2\pi|x-y|)^{n/2}},
\end{aligned} \tag{A.2}$$

where we used that the Bessel functions of the first kind are given by for any $\nu \in \mathbb{R}_+$ and $r \in \mathbb{R}_+$,

$$\mathrm{J}_\nu(r) = \frac{(r/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^\pi \cos(r\cos\varphi)(\sin\varphi)^{2\nu} \mathrm{d}\varphi$$

and $\frac{\mathrm{d}}{\mathrm{d}r}(r^\nu \mathrm{J}_\nu(r)) = r^\nu \mathrm{J}_{\nu-1}(r)$. In particular, the determinantal point process associated with the operator $K^{(n)}_{\mathrm{b},\mu}$ is both translation and rotation invariant on $\mathbb{R}^n$ with intensity $\frac{\mu^n \omega_n}{(2\pi)^n}$.

Let us also recall the following standard bounds.

Lemma A.5 (Sobolev embeddings). *Let $k \in \mathbb{N}_0$. There exists $C_k > 0$ such that, for every $f \in \mathcal{S}'(\mathbb{R}^n)$, for every $\hbar \in (0,1]$, one has*

$$\|f\|_{C^{2k}(\mathbb{R}^n)} \le C_k \hbar^{-2k-n/2} \|(1-\hbar^2\Delta)^{k+\lfloor n/4\rfloor+1} f\|_{L^2(\mathbb{R}^n)}.$$

The most direct argument relies on the properties of the Fourier transform.

Proof. The Fourier transform of $g \in L^1$ is continuous with

$$\|\widehat{g}\|_{C^0} \le (2\pi)^{n/2} \|g\|_{L^1}.$$

Hence, for every multi-index $\alpha \in \mathbb{N}_0^n$, one has

$$\|x \mapsto \partial_x^\alpha f(x)\|_{C^0} \le (2\pi)^{n/2} \|\xi \mapsto \xi^\alpha \widehat{f}(\xi)\|_{L^1},$$

and, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|\xi \mapsto \xi^\alpha \widehat{f}(\xi)\|_{L^1} \le \|\xi \mapsto \xi^\alpha (1+\hbar^2|\xi|^2)^{-k-\lfloor n/4\rfloor-1}\|_{L^2} \\ \times \|\xi \mapsto (1+\hbar^2|\xi|^2)^{k+\lfloor n/4\rfloor+1} \widehat{f}\|_{L^2}.\end{aligned}$$

By the above functional calculus for $-\mathbf{i}\nabla$ and since

$$\|\xi \mapsto \xi^\alpha (1+\hbar^2|\xi|^2)^{-k-\lfloor n/4\rfloor-1}\|_{L^2} \le C\hbar^{-|\alpha|-n/2}$$

if $|\alpha| \le 2k$, this completes the proof. ∎

By working on a (relatively) compact set, one can, in fact, replace 1 with any smooth positive potential.

Proposition A.6. *Let $V : \mathbb{R}^n \to [1,+\infty)$ and $\Omega \subset \mathbb{R}^n$ be a relatively compact open set such that V is C^∞ on a neighbourhood of $\overline{\Omega}$. Then, for every $k \in \mathbb{N}$, there exists $C_k(V)$ such that, for every $f \in \mathcal{S}'(\mathbb{R}^n)$ with compact support in Ω, for every $0 < \hbar \le 1$,*

$$\|f\|_{C^{2k}} \le C_k(V) \hbar^{-k-n/2} \|(V-\hbar^2\Delta)^{k+\lfloor n/4\rfloor+1} f\|_{L^2}.$$

Proof. According to Lemma A.5, it suffices to show that for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for any $f \in C_c^\infty(\Omega)$, uniformly for $0 < h \le 1$,

$$\|(1-\hbar^2\Delta)^k f\|_{L^2} \le C_k \|(V-\hbar^2\Delta)^k f\|_{L^2(\Omega)}. \tag{A.3}$$

Then the result follows by density.

Let us prove (A.3) by induction on k; the case $k = 0$ is trivial with $C_0 = 1$.

By assumptions, it holds for any $k \in \mathbb{N}_0$ and $f \in C_c^\infty(\Omega)$,

$$(V-\hbar^2\Delta)^{k+1} f = (1-\hbar^2\Delta)^{k+1} f + L_k f,$$

where the differential operator L_k takes the form

$$L_k = \sum_{|\alpha|\le 2k} g_{k;\alpha}(x,\hbar)\hbar^{|\alpha|}\partial^\alpha$$

and $g_{k;\alpha}$ is bounded, along with its derivatives, uniformly as $\hbar \to 0$. Indeed, letting $\widetilde{V} \in C_c^\infty(\mathbb{R}^n,\mathbb{R})$ be equal to V on $\overline{\Omega}$, the sequence $(L_k)_{k\ge 0}$ satisfy

$$\begin{aligned} L_0 &= \widetilde{V} - 1, \\ L_{k+1} &= (-\hbar^2\Delta + \widetilde{V})L_{k-1} + (\widetilde{V}-1)(-\hbar^2\Delta+1)^k. \end{aligned}$$

Hence, there exists C_k such that, for every $f \in C_c^\infty(\Omega)$,

$$\|(1-\hbar^2\Delta)^{k+1} f\|_{L^2} \le \|(V-\hbar^2\Delta)^{k+1} f\|_{L^2} + C_k \sum_{|\alpha|\le 2k} \|\hbar^{|\alpha|}\partial^\alpha f\|_{L^2}.$$

Moreover, one directly has

$$\sum_{|\alpha|\le 2k} \|\hbar^{|\alpha|}\partial^\alpha f\|_{L^2} \le C_k \|(1-\hbar^2\Delta)^k f\|_{L^2}$$

since all operators involved are Fourier multipliers.

In conclusion, by the induction hypothesis

$$\begin{aligned} \|(1-\hbar^2\Delta)^{k+1} f\|_{L^2} &\le \|(V-\hbar^2\Delta)^{k+1} f\|_{L^2} + C_k\|(V-\hbar^2\Delta)^k f\|_{L^2} \\ &\le C_k \|(V-\hbar^2\Delta)^{k+1} f\|_{L^2}, \end{aligned}$$

where we used the fact that $V - \hbar^2\Delta \ge 1$ as operators, and we upgraded the constant C_k from line to line. ∎

Remark A.7. The argument of Lemma A.5 also shows that if $f \in \mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$, then

$$\begin{aligned} \|f\|_{C^{2k}(\mathbb{R}^{2n})} &= \sum_{\alpha,\beta\in\mathbb{N}_0^n : |\alpha|+|\beta|\le 2k} \sup_{x,y\in\mathbb{R}^n} |\partial_x^\alpha \partial_y^\beta f(x,y)| \\ &\le C_k \hbar^{-2k-n} \\ &\quad \times \sum_{i,j\in\mathbb{N}_0 : i+j=k} \|(1-\hbar^2\Delta_x)^{i+\lfloor n/4\rfloor+1}(1-\hbar^2\Delta_y)^{j+\lfloor n/4\rfloor+1} f\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Then, by adjusting the proof of Proposition A.6, it is straightforward to show that under the same assumptions, for every $f \in \mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ with compact support in Ω,

$$\|f\|_{C^{2k}(\mathbb{R}^{2n})} \le C_k(V)\hbar^{-2k-n}\|(V-\hbar^2\Delta)_x^{k+\lfloor n/4\rfloor+1}(V-\hbar^2\Delta)_y^{k+\lfloor n/4\rfloor+1} f\|_{L^2}.$$

The only difference is that for a test function $f \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, for any $i,j \in \mathbb{N}_0$ with $i+j \le k$, it holds

$$(V-\hbar^2\Delta)_x^i (V-\hbar^2\Delta)_y^j f = (1-\hbar^2\Delta)_x^i(1-\hbar^2\Delta)_y^j f + L_{i,j} f,$$

where $L_{i,j}$ is an $\hbar$-differential operator, with bounded coefficients (by polynomials in $\|V\|_{C^{2k}(\bar{\Omega})}$ and $\hbar$), of orders $2i$ and $2j$ with respect to x and y, respectively, and of total order $2(i+j-1)$. Hence, since x, y are independent variables, we can bound

$$\|L_{i,j}\|_{L^2} \le C_k(V)(\|(1-\hbar^2\Delta)_x^i(1-\hbar^2\Delta)_y^{j-1}\|_{L^2} + \|(1-\hbar^2\Delta)_x^{i-1}(1-\hbar^2\Delta)_y^{j}\|_{L^2}).$$

The proof now follows by the same induction as in Proposition A.6.

A.3. Determinantal point processes

This class of point processes was introduced in [71] under the name *Fermion processes*. Macchi's fundamental contributions were physically motivated by electron-interference experiments and the goal was to provide a mathematical model for point processes which obey the Pauli exclusion principle; hence the name *Fermion processes*. In contrast to Poisson processes which describes independent particles, determinantal processes are well known to exhibit repulsion and are typically *hyperuniform* [90]. The mathematical theory of determinantal processes has been developed largely by Soshnikov [85] and Shirai–Takahashi [78,79]. We also refer to [54] for a survey with more probabilistic perspectives. There are many applications beyond the context of quantum mechanics discussed in the introduction which include random matrices and Coulomb gases, asymptotic representation theory, certain random tilings and two-dimensional growth processes, zeros of Gaussian analytic functions, etc. We refer to [13, 58, 85] for comprehensive reviews of these examples. In this section, we explain some basic concepts of the theory of determinantal point processes. The results are stated for the Hilbert space $L^2(\mathbb{R}^n)$ but they hold true for $L^2(\mathcal{E})$ if $\mathcal{E}$ is a locally compact, completely separable, Hausdorff space. We only focus on the Euclidean case for concreteness.

A.3.1. Point processes on $\mathbb{R}^n$. Let us recall that a (simple) point process is a random measure of the form $\Xi = \sum_{\lambda\in\Lambda} \boldsymbol{\delta}_\lambda$, where $\Lambda \subset \mathbb{R}^n$ is a countable set with no accumulation points. We refer to [58, Section 2] for the construction of such random measures. The law of a point process is usually characterised by its Laplace functional:

$$\psi_\Xi(f) = \mathbb{E}[e^{-\Xi(f)}] \quad \text{for all function } f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+).$$

We define the correlation functions $(R_k)_{k=1}^{+\infty}$ of the point process Ξ through its Laplace functional by

$$\psi_\Xi(f) = 1 + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathbb{R}^{n\times k}} \prod_{i=1}^{k} (e^{-f(z_i)} - 1) R_k(z_1,\dots,z_k) \mathrm{d}z_1 \cdots \mathrm{d}z_k,$$

when this expansion makes sense. For instance, if $\Xi = \sum_{j=1}^{N} \boldsymbol{\delta}_{x_j}$ and $\{x_1,\dots,x_N\}$ has a (symmetric) joint probability density function $\mathbb{P}_N$, then we verify that the correlation functions of Ξ are given by the marginals for $k \in \mathbb{N}$,

$$R_k(z_1,\dots,z_k) = \mathbb{1}_{k\le N} \frac{N!}{(N-k)!} \int_{\mathbb{R}^{n\times(N-k)}} \mathbb{P}_N[z_1,\dots,z_k,\mathrm{d}z_{k+1},\dots,\mathrm{d}z_N].$$

A.3.2. Determinantal processes. In this article, we rely on the following definition.

Definition A.8. A point process Ξ is called determinantal if its Laplace functional is of the form

$$\psi_{\Xi}(f) = \det[\mathrm{I} - (1 - e^{-f})K], \tag{A.4}$$

where K is a locally trace-class operator on $L^2(\mathbb{R}^n)$ and the right-hand side of (A.4) is a Fredholm determinant, see Proposition A.4 for the relevant definitions. Then, we say that the determinantal process Ξ is associated with the operator K and the correlation functions of Ξ are given by

$$R_k(z_1, \dots, z_k) = \det_{k \times k}[K(z_i, z_j)], \quad k \in \mathbb{N}.$$

Definition A.8 is equivalent to the usual definition of determinantal processes in terms of its correlation kernel K, see, e.g., [58, Section 2] or [85, Theorem 2]. Moreover, while a determinantal process has several correlation kernels, the associated operator is uniquely defined. In general, formula (A.4) makes sense for any (measurable) function $f : \mathbb{R}^n \to \mathbb{R}_+$. Moreover, if the operator K is trace-class, then by Proposition A.4 (iii), ψ_{Ξ} is continuous with respect to the L^∞-norm.

Definition A.8 has the following consequences for the distribution of a *linear statistic* $\Xi(f)$.

Lemma A.9. *Let Ξ be a determinantal process associated with the operator K. Then for any functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$,*

$$\mathbb{E}\, \Xi(f) = \mathrm{tr}(fK)$$

and, if defined,

$$\begin{aligned} \mathrm{cov}(\Xi(g), \Xi(f)) &= \mathrm{tr}(g(\mathrm{I} - K) fK) \\ &= \frac{1}{2} \mathrm{tr}([g, K][K, f]) + \mathrm{tr}(gfK(\mathrm{I} - K)). \end{aligned} \tag{A.5}$$

Proof. By definition of the correlation functions, see Definition A.8,

$$\mathbb{E}[\Xi(f)] = \int_{\mathbb{R}^n} f(z) R_1(z) \mathrm{d}z = \int_{\mathbb{R}^n} f(z) K(z, z) \mathrm{d}z = \mathrm{tr}[fK],$$

and for any $f, g \in L_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \mathbb{E}[\Xi(g)\Xi(f)] &= \iint_{\mathbb{R}^{2n}} g(z_1) f(z_2) R_2(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 + \int_{\mathbb{R}^n} g(z) f(z) R_1(z) \mathrm{d}z \\ &= \mathrm{tr}[gK]\, \mathrm{tr}[gK] - \mathrm{tr}[gKfK] + \mathrm{tr}[gfK]. \end{aligned}$$

This shows that the covariance between the random variables $\Xi(f)$ and $\Xi(g)$ is given by

$$\mathrm{cov}(\Xi(g), \Xi(f)) = \mathbb{E}[\Xi(g)\Xi(f)] - \mathbb{E}[\Xi(g)]\mathbb{E}[\Xi(f)] = \mathrm{tr}[g(\mathrm{I} - K) fK].$$

This formula is actually symmetric and can be written using commutators as follows:

$$\mathrm{tr}[g(\mathrm{I}-K)fK] = \frac{1}{2}\,\mathrm{tr}([g,\mathrm{I}-K][f,K]) + \mathrm{tr}(gfK(\mathrm{I}-K)),$$

where we used cyclicity of $\mathrm{tr}[\cdot]$.

To complete the proof, it remains to observe that $[g,\mathrm{I}-K] = [K,g]$. ■

Note that formula (A.5) implies that for any $g \in L^\infty_c(\mathbb{R}^n)$, the variance of the linear statistics $\Xi(g)$ is well defined and given in terms of Hilbert–Schmidt norm by

$$\mathrm{var}\,\Xi(g) = \|\sqrt{K}g\sqrt{\mathrm{I}-K}\|^2_{\mathrm{J}^2} = \frac{1}{2}\|[g,K]\|^2_{\mathrm{J}^2} + \|g\sqrt{K(\mathrm{I}-K)}\|^2_{\mathrm{J}^2}. \tag{A.6}$$

In the context of this article, it is relevant to recall the following [85, Theorem 3].

Proposition A.10. *If K is a self-adjoint locally trace-class operator on $L^2(\mathbb{R}^n)$, then it (uniquely) defines a determinantal process if and only if* $0 \le K \le \mathrm{I}$.

For instance, consider H an (unbounded) self-adjoint with a domain $\mathcal{D}(H)$ which is dense in $L^2(\mathbb{R}^n)$, and let $K_S = \mathbb{1}_S(H)$, where $S \subset \mathbb{R}$ is a Borel set. Then, by Proposition A.10, if K_S is locally trace-class on $\mathbb{R}^n$, there exists a determinantal process associated with K_S. In particular, if H is bounded from below and $\mathbb{1}_{(-\infty,\mu)}(H)$ is a finite-rank projection (as in our setting, thanks to Proposition 2.2), there is a determinantal process associated with this operator. Moreover, this process has almost surely $N = \mathrm{Rank}(\mathbb{1}_{(-\infty,\mu)}(H))$ particles, see [85, Theorem 4].

It is also of interest to record the following basic expansion for the Laplace functional (A.4) for small f. The proof of Proposition A.11 is inspired by that of [86, Theorem 1], except that we control directly the Laplace functional instead of the cumulants of the linear statistic $\Xi(f)$. This simplifies the argument and makes the result stronger. In particular, these asymptotics could certainly be of independent interest.

Proposition A.11. *Under the assumptions of Proposition A.10, it holds uniformly for all $f \in L^\infty_c(\mathbb{R}^n)$ with $f \le 0.69$,*

$$\psi_\Xi(-f) = \exp\Big(\mathbb{E}[\Xi(f)] + \mathrm{var}(\Xi(e^f-1))\Big(\frac{1}{2} + \mathcal{O}(\|f_+\|_{L^\infty})\Big)\Big).$$

Proof. By functional calculus, it holds for $g \in L^\infty_c(\mathbb{R}^n)$ with $\|g\|_{L^\infty} < 1$,

$$\log\det(1-gK) = \mathrm{tr}[\log(1-gK)] = -\sum_{\ell\in\mathbb{N}} \frac{1}{\ell}\,\mathrm{tr}[(gK)^\ell]. \tag{A.7}$$

Now, define $A_k = g(\mathrm{I}-K)g^kK$ for $k \in \mathbb{N}_0$ and observe that for every $k \in \mathbb{N}_0$ and $j \in \mathbb{N}$,

$$(gK)^j g^k K = (gK)^{j-1}g^{k+1}K + (gK)^{j-1}A_k;$$

repeated application of this formula yields, for every $\ell \ge 2$,

$$\mathrm{tr}[(gK)^\ell] = \mathrm{tr}[(gK)^{\ell-1}gK] = \mathrm{tr}[g^\ell K] + \sum_{j=2}^{\ell} \mathrm{tr}[(gK)^{\ell-j}A_{j-1}]. \tag{A.8}$$

Moreover, using that $0 \le K \le 1$ and cyclicity of $\mathrm{tr}[\cdot]$, we have for any $j, k \in \mathbb{N}$,

$$\mathrm{tr}[(gK)^k A_j] = \mathrm{tr}[\sqrt{K}(gK)^{k-1} g \sqrt{K}\sqrt{K} g(\mathrm{I}-K) g^j \sqrt{K}].$$

This implies that for $j, k \in \mathbb{N}$,

$$\begin{aligned} |\mathrm{tr}[(gK)^k A_j]| &\le \|(gK)^{k-1} g\| \|\sqrt{K} g(\mathrm{I}-K) g^j \sqrt{K}\|_{\mathrm{J}^1} \\ &\le \|g\|_{L^\infty}^k \sqrt{\|\sqrt{K} g \sqrt{\mathrm{I}-K}\|_{\mathrm{J}^2} \|\sqrt{\mathrm{I}-K} g^j \sqrt{K}\|_{\mathrm{J}^2}} \\ &= \|g\|_{L^\infty}^k \sqrt{\mathrm{var}\,\Xi(g)\,\mathrm{var}\,\Xi(g^j)} \end{aligned} \tag{A.9}$$

according to (A.6). Moreover, $\mathrm{var}\,\Xi(g^j) = \frac{1}{2}\|[g^j, K]\|_{\mathrm{J}^2}^2 + \mathrm{tr}[g^{2j} K(\mathrm{I}-K)]$ for $j \in \mathbb{N}$, and we can bound

$$\mathrm{tr}[g^{2j-2} gK(\mathrm{I}-K)g] \le \|g\|_{L^\infty}^{2(j-1)} \|gK(\mathrm{I}-K)g\|_{\mathrm{J}^1} = \|g\|_{L^\infty}^{2(j-1)} \mathrm{tr}[gK(\mathrm{I}-K)g].$$

Since $[K, g^j] = \sum_{k=1}^{j} g^{k-1}[K, g] g^{j-k}$, we also have $\|[K, g^j]\|_{\mathrm{J}^2} \le j\|g\|_{L^\infty}^{j-1} \|[K, g]\|_{\mathrm{J}^2}$ so that

$$\begin{aligned} \mathrm{var}\,\Xi(g^j) &\le j^2 \|g\|_{L^\infty}^{2(j-1)} \Big(\frac{1}{2}\|[K, g]\|_{\mathrm{J}^2} + \mathrm{tr}[g^2(\mathrm{I}-K)K]\Big) \\ &= j^2 \|g\|_{L^\infty}^{2(j-1)} \mathrm{var}\,\Xi(g). \end{aligned} \tag{A.10}$$

By (A.9), this shows that for any $j, k \in \mathbb{N}$,

$$|\mathrm{tr}[(gK)^k A_j]| \le j\|g\|_{L^\infty}^{k+j-1} \mathrm{var}\,\Xi(g).$$

Hence, by (A.8), we have shown that for any $\ell \ge 3$,

$$\begin{aligned} |\mathrm{tr}[(gK)^\ell] - \mathrm{tr}[g^\ell K]| &\le \|g\|_{L^\infty}^{\ell-2} \sum_{j=2}^{\ell} (j-1)\,\mathrm{var}\,\Xi(g) \\ &= \|g\|_{L^\infty}^{\ell-2} \frac{\ell(\ell-1)}{2} \mathrm{var}\,\Xi(g). \end{aligned} \tag{A.11}$$

Observe that if $g = 1 - e^{-f}$, then by linearity of $\mathrm{tr}[\cdot]$,

$$\sum_{\ell \in \mathbb{N}} \frac{1}{\ell} \mathrm{tr}[g^\ell K] = \mathrm{tr}\Big[\Big(\sum_{k \in \ell} \frac{1}{\ell} g^\ell\Big) K\Big] = -\mathrm{tr}[\log(1-g)K] = \mathrm{tr}[fK]. \tag{A.12}$$

By combining (A.7), (A.12) with bound (A.11) for $g = e^f - 1$ and $\ell \ge 3$, this shows that if $\|g\|_{L^\infty} < 1$,

$$\begin{aligned} \Big|\log\det(1 - gK) - \mathrm{tr}[fK] - \frac{1}{2}\mathrm{var}\,\Xi(g)\Big| &\le \frac{1}{2}\mathrm{var}\,\Xi(g) \sum_{\ell \ge 3} (\ell-1)\|g\|_{L^\infty}^{\ell-2} \\ &= \mathrm{var}\,\Xi(g) \frac{\|g\|_{L^\infty}}{(1 - \|g\|_{L^\infty})^2}, \end{aligned}$$

where we used that for $\ell = 2$, $\mathrm{tr}[(gK)^2] - \mathrm{tr}[g^2 K] = -\mathrm{tr}[A_1] = \mathrm{var}\,\Xi(g)$, according to formulas (A.6)–(A.8). By (A.4), we conclude that if $f \le 0.69 < \log 2$, then

$$\psi_\Xi(-f) = \exp\Big(\mathrm{tr}[fK] + \mathrm{var}\,\Xi(g)\Big(\frac{1}{2} + \mathcal{O}(\|f_+\|_{L^\infty})\Big)\Big),$$

where we used that under our assumptions,

$$\frac{\|g\|_{L^\infty}}{(1-\|g\|_{L^\infty})^2} \le C\|f_+\|_{L^\infty}$$

for a numerical constant $C > 0$. ∎

Like in [86], one can immediately deduce from the asymptotics of Proposition A.11 a CLT for linear statistics of general determinantal processes. We give the proof of this result since, compared to [86, Theorem 1], we can remove the (technical) conditions on the mean and variance of the linear statistic $\Xi_N(f_N)$.

Corollary A.12. *Let $(K_N)_{N\in\mathbb{N}}$ be a sequence of (self-adjoint) trace-class operators on $L^2(\mathbb{R}^n)$ with $0 \le K_N \le 1$ and let $(\Xi_N)_{N\in\mathbb{N}}$ be the associated determinantal processes on $\mathbb{R}^n$. Let $f_N \in L^\infty(\mathbb{R}^n)$ with $\|f_N\|_{L^\infty} \le C$ and assume that $\sigma_N^2 = \mathrm{var}[\Xi_N(f_N)] \to \infty$ as $N \to +\infty$. Then it holds uniformly for all $t \in \mathbb{R}$ with $|t| \le 0.69\sigma_N/C$,*

$$\psi_{\Xi_N}(t\sigma_N^{-1} f_N)\exp(t\sigma_N^{-1}\mathbb{E}[\Xi_N(f_N)]) = \exp\Big(t^2\Big(\frac{1}{2} + \mathcal{O}(t\sigma_N^{-1})\Big)\Big).$$

In particular, this implies that

$$\frac{\Xi_N(f_N) - \mathbb{E}[\Xi_N(f_N)]}{\sigma_N} \Rightarrow \mathcal{N}_{0,1},$$

where $\mathcal{N}_{0,1}$ denotes a standard Gaussian random variable.

Proof. By the Cauchy–Schwarz inequality, one has a crude bound

$$|\mathrm{var}[\Xi(1-e^{-f})] - \mathrm{var}[\Xi(f)] \le \sum_{\substack{k,j\in\mathbb{N}\\ k+j>2}} \frac{1}{k!j!}\sqrt{\mathrm{var}[\Xi(f^k)]\,\mathrm{var}[\Xi(f^j)]}.$$

Using bound (A.10), this implies that

$$\begin{aligned}|\mathrm{var}[\Xi(1-e^{-f})] - \mathrm{var}[\Xi(f)]| &\le \frac{\mathrm{var}[\Xi(f)]}{2}\sum_{\substack{k,j\in\mathbb{N}_0\\ k+j>0}} \frac{1}{k!j!}\|f\|_{L^\infty}^{k+j}\\ &= \frac{\mathrm{var}[\Xi(f)]}{2}(e^{2\|f\|_{L^\infty}} - 1)\\ &\le \|f\|_{L^\infty}\,\mathrm{var}[\Xi(f)]e^{2\|f\|_{L^\infty}}.\end{aligned}$$

Hence, by applying Proposition A.11, it shows that if $\|f\|_{L^\infty} \le 0.69$,

$$\psi_\Xi(f)e^{\mathbb{E}[\Xi(f)]} = \exp\Big(\mathrm{var}[\Xi(f)]\Big(\frac{1}{2} + \mathcal{O}(\|f\|_{L^\infty})\Big)\Big).$$

Making the change of variable $f \leftarrow t\sigma_N^{-1} f_N$ with $t \in \mathbb{R}$, we conclude that as $N \to +\infty$,

$$\psi_{\Xi_N}(t\sigma_N^{-1} f_N) e^{t\sigma_N^{-1}\mathbb{E}[\Xi_N(f_N)]} = \exp\Big(t^2\Big(\frac{1}{2} + \mathcal{O}(t\sigma_N^{-1})\Big)\Big).$$

As the convergence of the Laplace transform of the random variable $\frac{\Xi_N(f_N) - \mathbb{E}[\Xi_N(f_N)]}{\sigma_N}$ implies its convergence in distribution and

$$\mathbb{E}[e^{t\mathcal{N}(0,1)}] = e^{t^2/2}$$

for all $t \in \mathbb{R}$, this proves the claims. ∎

A.3.3. Weak convergence. In the context of Theorem II.3, we also recall the concept of convergence in distribution for point processes.

Definition A.13. We say that a sequence Ξ_N of point processes on $\mathbb{R}^n$ converges in distribution to a point process Ξ if the Laplace functionals $\psi_{\Xi_N}(f) \to \psi_{\Xi}(f)$ as $N \to +\infty$ for all $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$.

In the context of Proposition A.10, this has the following consequence.

Proposition A.14 (Weak convergence for determinantal processes). *Let $(K_N)_{N\in\mathbb{N}}$ be a sequence of self-adjoint operator on $L^2(\mathbb{R}^n)$ with $0 \le K_N \le \mathrm{I}$. Suppose that as $N \to \infty$, $K_N \to K$ locally uniformly, where K is locally trace-class, then there exists a determinantal process Ξ associated with the operator K and the determinantal processes Ξ_N associated with K_N converge weakly to Ξ as $N \to +\infty$.*

Proof. By assumptions, $K_N \to K$ in the weak operator topology as $n \to \infty$ (cf. Proposition A.1). Indeed, by density, it suffices to verify that $\langle \phi, K_N\phi\rangle \to \langle \phi, K\phi\rangle$ for any $\phi \in C_c(\mathbb{R}^n)$. This also implies that $0 \le K \le \mathrm{I}$ and by Proposition A.10, there exists a determinantal process associated with the operator K. Moreover, using that $K_N \ge 0$ and Proposition A.4 (i), it holds as $N \to \infty$,

$$\|\chi K_N\|_{\mathrm{J}^1} = \mathrm{tr}[\chi K_N] \to \mathrm{tr}[\chi K] = \|\chi K\|_{\mathrm{J}^1} < \infty$$

for any $\chi \in L_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$. As both K_N, K are positive operators, by [80, Theorem 2.19], this implies that $\|\chi(K_N - K)\|_{\mathrm{J}^1} \to 0$ for any such χ. Now, by formula (A.8) and Proposition A.4 (iii), the Laplace functionals of these point processes satisfy for every $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$.

$$|\psi_{\Xi_N}(f) - \psi_{\Xi}(f)| \le \|\chi(K_N - K)\|_{\mathrm{J}^1} e^{1 + \|\chi K_N\|_{\mathrm{J}^1} + \|\chi K\|_{\mathrm{J}^1}}$$

with $\chi = 1 - e^{-f}$. Since $\chi \ge 0$ with compact support,

$$\limsup_{N\to\infty} |\psi_{\Xi_N}(f) - \psi_{\Xi}(f)| \le e^{1+2\,\mathrm{tr}[\chi K]} \limsup_{N\to\infty} (\|\chi(K_N - K)\|_{\mathrm{J}^1} e^{\|\chi(K_N-K)\|_{\mathrm{J}^1}}) = 0.$$

According to Definition A.13, this completes the proof. ∎

A.4. Oscillatory integrals

The technical core of this article relies on standard techniques of semiclassical analysis, in particular, the stationary phase method to obtain the asymptotics of oscillatory integrals.

Proposition A.15 (Stationary phase lemma, [33, Proposition 5.2]). *Let $\Omega \subset \mathbb{R}^d$ be an open set, A be a compact and $x \mapsto \Phi(x,y) \in C^\infty(\Omega,\mathbb{R})$ for every $y \in A$. Suppose that for any $y \in A$, there is exactly one point $x_y \in \Omega$ such that $\partial_x \Phi(x_y, y) = 0$, and that the Hessian matrix,* Hess $\Phi(x_y; y)$*, is non-degenerate.*

Then, there exists a sequence of differential operators $(L_k)_{k\geq 0}$ on $\mathbb{R}^d$, depending on $y \in A$ such that L_k is of degree $2k$ and for every $\ell \geq 0$ and $\mathcal{K} \Subset \Omega$, there is a constant $C_{\ell,\Phi,\mathcal{K}}$ such that, for any $f \in C_c^\infty(\mathcal{K})$,

$$\sup_{y\in A}\left|\int e^{\frac{\mathrm{i}}{\hbar}\Phi(x,y)} f(x)\mathrm{d}x - e^{\frac{\mathrm{i}}{\hbar}\Phi(x_y,y)}(2\pi\hbar)^{d/2}\sum_{k=0}^{\ell}\hbar^k (L_k f)(x_y)\right| \leq C_{\ell,\Phi,\mathcal{K}}\,\hbar^{d/2+\ell+1}\|f\|_{C^{2\ell+d+1}}.$$

Moreover, one can also allow f to depend smoothly on the parameter $y \in A$, in which case the error can be controlled with respect to $C^k(A)$ for every $k \in \mathbb{N}_0$.

An explicit form for the operators L_k, depending on Φ, can be found in [53, Theorem 7.7.5]. In particular, it holds

$$L_0 = \frac{\mathbf{i}^{n_+ - n_-}}{\sqrt{\det(|\mathrm{Hess}(\Phi)(x_y, y)|)}},$$

where n_+ and n_- respectively denote the number of positive and negative eigenvalues of $\mathrm{Hess}(\Phi)(x_y, y)$.

In the special case where $d = 2n$ and $\Phi\colon \mathbb{R}^{2n} \ni (x,\xi) \mapsto x\cdot\xi \in \mathbb{R}$, one has

$$L_k = \frac{\mathbf{i}^k}{k!}\sum_{|\alpha|=k}\partial_x^\alpha \partial_\xi^\alpha. \tag{A.13}$$

In general, we apply Proposition A.15 in the following standard way. If $a \in S^0$ is a classical symbol (cf. Definition 2.7), then there exists another classical symbol $b \in S^0$ such that

$$\int_\Omega e^{\frac{\mathrm{i}}{\hbar}\Phi(x,y)} a(x,y;\hbar)\mathrm{d}x = b(y,\hbar) + \mathcal{O}(\hbar^\infty).$$

The error being controlled by $\|f\|_k$ for some $k \in \mathbb{N}$ allows us to apply Proposition A.15 to $\hbar$-dependent functions satisfying, uniformly as $\hbar \to 0$, a control of the form

$$\|f\|_{C^k} \leq C_k \hbar^{-k\delta} \quad \text{for some } \delta < \frac{1}{2}.$$

In this case, the terms in the expansion have decreasing magnitudes and the remainder is still $\mathcal{O}(\hbar^\infty)$ by choosing ℓ arbitrary large.

Proposition A.15 is complemented by the *non-stationary* phase lemma.

Proposition A.16 (Non-stationary phase lemma, [53, Theorem 7.7.1]). *Let $\Omega \subset \mathbb{R}^d$ be an open set, A be a compact and $x \mapsto \Phi(x, y) \in C^\infty(\Omega, \mathbb{R})$ for every $y \in A$. Suppose that $(x, y) \mapsto \partial_x \Phi(x, y)$ is bounded away from 0 on $\Omega \times A$. Then, for every $f \in C_c^\infty(\Omega)$, as $\hbar \to 0$,*

$$\int_\Omega e^{\frac{i}{\hbar}\Phi(x)} f(x)\mathrm{d}x = \mathcal{O}(\hbar^\infty).$$

An explicit estimate (involving derivatives of f and a lower bound on $|\partial_x \Phi|$) is [53, (7.7.1)′].

In the remainder of this section, we give more precise estimates in the case where $\phi\colon (x, y) \mapsto x \cdot y$.

Proposition A.17. *Let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Denote by $\hat{f}$ its Fourier transform with respect to the second variable. Define for $s \geq 0$,*

$$\|f\|^2_{L^\infty H^s} := \int \sup_{x\in\mathbb{R}^n} |\hat{f}(x, \xi)|^2 (1 + |\xi|)^{2s} \mathrm{d}\xi$$

and let $L^\infty H^s$ be the corresponding function space. If $f \in L^\infty H^s$ with $s > \frac{n}{2}$, then for any $\hbar \in (0, 1]$,

$$\left|\frac{1}{\hbar^n} \int e^{\frac{i}{\hbar}(x\cdot y)} f(x, y)\mathrm{d}x\mathrm{d}y\right| \leq C_s \|f\|_{L^\infty H^s}.$$

Proof. It holds

$$\frac{1}{\hbar^n} \int e^{-\frac{i}{\hbar}(x\cdot y)} f(x, y)\mathrm{d}x\mathrm{d}y = \frac{(2\pi)^{n/2}}{\hbar^n} \int \hat{f}\Big(x, \frac{x}{\hbar}\Big)\mathrm{d}x = (2\pi)^{n/2} \int \hat{f}(\hbar\xi, \xi)\mathrm{d}\xi.$$

Since $C_s = (2\pi)^{n/2} \int (1 + |\xi|)^{-2s}\mathrm{d}\xi < \infty$ if $s > n/2$, by the Cauchy–Schwarz inequality, we obtain the bound

$$\left|\frac{1}{\hbar^n} \int e^{\frac{i}{\hbar}(x\cdot y)} f(x, y)\mathrm{d}x\mathrm{d}y\right| \leq C_s \int \sup_{x\in\mathbb{R}^n} |\hat{f}(x, \xi)|^2 (1 + |\xi|)^{2s}\mathrm{d}\xi. \qquad \blacksquare$$

By assuming more regularity on f, we can bootstrap the previous proposition to obtain higher-order expansion of such oscillatory integral with a similar control of the remainder.

Proposition A.18. *Let $\ell, n \in \mathbb{N}$ and let $s > \ell + \frac{n}{2}$. Let $(L_k)_{k\in\mathbb{N}}$ be the sequence of differential operators given by (A.13). There exist constants $C_{\ell,s}$ such that for any function $f\colon \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{R}$ with $\partial_x^\alpha f \in L^\infty_x H^s_\xi$ for every $|\alpha| \leq \ell$, it holds for any $\hbar \in (0, 1]$,*

$$\left|\frac{1}{(2\pi\hbar)^n} \int e^{\frac{i}{\hbar}(x\cdot\xi)} f(x,\xi)\mathrm{d}x\mathrm{d}\xi - \sum_{k=0}^{\ell-1} \hbar^k L_k f(0,0)\right| \leq C_{\ell,s}\hbar^\ell \sum_{|\alpha|\leq\ell} \|\partial_x^\alpha f\|_{L^\infty_x H^s_\xi}. \quad \text{(A.14)}$$

This statement, more precise than Proposition A.15, is tailored to the case where f is smooth but oscillates rapidly with respect to the first variable: given $g \in C_c^\infty(\mathbb{R}^{2n}, \mathbb{R})$, we can apply Proposition A.18 to

$$f\colon (x, \xi) \mapsto g(\hbar^{-\delta}x, \xi)$$

for any $\delta < 1$ (since in this case, $\|\partial_x^\alpha f\|_{L_x^\infty H_\xi^s} = \mathcal{O}(\hbar^{-\delta\alpha\ell})$ for any $|\alpha| \le \ell$), whereas one needs $\delta < \frac{1}{2}$ to use the result of Proposition A.15.

Proof. We proceed by induction over $\ell \in \mathbb{N}_0$ utilizing that the case $\ell = 0$ follows from Proposition A.17. Hence, we assume that bound (A.14) holds for a given $\ell \in \mathbb{N}_0$. Define

$$F_1(x,\xi) = \frac{f(x,\xi) - f(x,\xi)|_{x_1=0}}{x_1}, \quad x,\xi \in \mathbb{R}^n.$$

Integrating by parts,

$$\frac{1}{\hbar^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)} x_1 F_1(x,\xi)\mathrm{d}x\mathrm{d}\xi = \frac{\mathbf{i}\hbar}{\hbar^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}\partial_{\xi_1}F_1(x,\xi)\mathrm{d}x\mathrm{d}\xi.$$

Note also that by (A.13) and Taylor's theorem, for every $k \in \mathbb{N}_0$,

$$\mathbf{i}L_k(\partial_{\xi_1}F_1)(0,0) = \frac{\mathbf{i}^{k+1}}{(k+1)!}\sum_{|\alpha|=k}\partial_x^\alpha\partial_\xi^\alpha\partial_{x_1}\partial_{\xi_1}f(0,0) = \frac{\mathbf{i}^{k+1}}{(k+1)!}\sum_{\substack{|\alpha|=k+1\\ \alpha_1>0}}\partial_x^\alpha\partial_\xi^\alpha f(0,0),$$

that is, we recover every terms in $L_{k+1}f(0,0)$ with $\alpha_1 > 0$.

Continuing in this fashion, one can write for $j \in \{1,\dots,n\}$,

$$F_j(x,\xi) = \frac{f(x,\xi)|_{x_1=\dots=x_{j-1}=0} - f(x,\xi)|_{x_1=\dots=x_j=0}}{x_j}, \quad x,\xi \in \mathbb{R}^n$$

and verify that for every $k \in \mathbb{N}_0$,

$$\mathbf{i}L_k(\partial_{\xi_j}F_j)(0,0) = \frac{\mathbf{i}^{k+1}}{(k+1)!}\sum_{\substack{|\alpha|=k+1,\alpha_j>0\\ \alpha_1=\dots=\alpha_j=0}}\partial_x^\alpha\partial_\xi^\alpha f(0,0). \tag{A.15}$$

Hence, in the end,

$$\frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}f(x,\xi)\mathrm{d}x = \frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}f(0,\xi)\mathrm{d}\xi\mathrm{d}x + \frac{\mathbf{i}\hbar}{(2\pi\hbar)^n}\sum_{j=1}^n\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}\partial_{\xi_j}F_j(x,\xi)\mathrm{d}x\mathrm{d}\xi.$$

For the first term, $\xi \mapsto f(0,\xi)$ belongs to H^s for $s > \frac{n}{2}+1$, which is a subset of $C^0 \cap L^2$ (by Proposition A.5). Hence, by a double Fourier transform,

$$\frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}f(0,\xi)\mathrm{d}\xi\mathrm{d}x = f(0,0).$$

Note also that by assumptions, the functions $\partial_{\xi_j}F_j$ are such that, for every $|\alpha| \le \ell$, one has $\partial_x^\alpha\partial_{\xi_j}F_j \in L_x^\infty H_\xi^{s-1}$, for instance, with

$$\|\partial_x^\alpha\partial_{\xi_1}F_1\|_{L_x^\infty H_\xi^{s-1}} \le \|\partial_x^{\alpha+(1,0,\dots,0)}f\|_{L_x^\infty H_\xi^s}. \tag{A.16}$$

Hence, using the induction hypothesis, this implies that if $s-1>\ell+\frac{n}{2}$,

$$\begin{aligned}&\Big|\frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}f(x,\xi)\mathrm{d}x-f(0,0)-\mathbf{i}\hbar\sum_{k=0}^{\ell-1}\sum_{j=1}^{n}\hbar^k L_k(\partial_{\xi_j}F_j)(0,0)\Big|\\&\quad\le C_{\ell,s-1}\hbar^{\ell+1}\sum_{j=1}^{n}\sum_{|\alpha|=\ell}\|\partial_x^\alpha\partial_{\xi_j}F_j\|_{L^\infty_x H^{s-1}_\xi}.\end{aligned}$$

Using (A.15) and bound (A.16), we conclude that $s>\ell+1+\frac{n}{2}$,

$$\begin{aligned}&\Big|\frac{1}{(2\pi\hbar)^n}\int e^{\frac{\mathrm{i}}{\hbar}(x\cdot\xi)}f(x,\xi)\mathrm{d}x-\sum_{k=0}^{\ell}\hbar^k L_k(\partial_{\xi_j}F_j)(0,0)\Big|\\&\quad\le C_{\ell,s-1}\,\hbar^{\ell+1}\sum_{|\alpha|=\ell+1}\|\partial_x^\alpha f\|_{L^\infty_x H^{s}_\xi},\end{aligned}$$

which completes the proof. ∎

To conclude this section, we provide another useful estimate on oscillatory integrals.

Proposition A.19. *Let $\mathcal{K}\Subset\mathbb{R}$ and let $f:\mathcal{K}\times\mathbb{R}\to\mathbb{R}$ be such that*

$$\|f\|_{L^\infty C^2}:=\sup_{(\lambda,t)\in\mathcal{K}\times\mathbb{R}}(|f(\lambda,t)|+|\partial_t f(\lambda,t)|+|\partial_t^2 f(\lambda,t)|)<+\infty.$$

Then for every $\gamma\in(0,1]$, there exists $C_{K,\gamma}$ such that, as $\delta\to0$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\frac{\sin(\lambda t/\delta)}{t}f(\lambda,t)\mathrm{d}t\mathrm{d}\lambda-2\pi\int f(\lambda,0)\mathrm{d}\lambda\Big|\\&\quad\le C_{K,\gamma}\delta^{1-\gamma}(\|f\|_{L^\infty C^2}+\|(t,\lambda)\mapsto t^{\gamma-1}\partial_t f(\lambda,t)\|_{L^\infty L^1}).\end{aligned}$$

Proof. Let us observe that by making an integration by parts, it holds for any $f\in C_c^{1,1}(\mathbb{R})$ and every $\lambda\neq0$,

$$\begin{aligned}\int_{\mathbb{R}}\frac{\sin(\lambda t)}{t}f(t)\mathrm{d}t&=-\frac{2}{\lambda}\int\sin\Big(\lambda\frac{t}{2}\Big)^2\partial_t\Big(\frac{f(t)}{t}\Big)\mathrm{d}t\\&=\frac{2}{\lambda}f(0)\int_{-\infty}^{+\infty}\frac{\sin(\lambda t/2)^2}{t^2}\mathrm{d}t\\&\quad+\int_{-\infty}^{+\infty}\frac{1-\cos(\lambda t)}{\lambda}\partial_t\Big(\frac{f(0)-f(t)}{t}\Big)\mathrm{d}t,\end{aligned}$$

where $\mathbb{R}\ni t\mapsto\partial_t(\frac{f(0)-f(t)}{t})$ is bounded. Indeed, by the Taylor integral theorem,

$$\Big|\partial_t\Big(\frac{f(0)-f(t)}{t}\Big)\Big|=\Big|\frac{f(t)-f(0)-tf'(t)}{t^2}\Big|=\Big|\frac{1}{2t^2}\int_0^t sf''(s)\mathrm{d}s\Big|\le\frac{1}{4}\|f''\|_{L^\infty}.$$

Moreover, $\int_{\mathbb{R}}(\frac{\sin(t)}{t})^2=\pi$ as shown by a Fourier transform.

Let now f be as in the claim and let $g\colon (\lambda,t)\mapsto \partial_t(\frac{f(\lambda,0)-f(\lambda,t)}{t})$. Then

$$\int_{\mathbb{R}^2}\frac{\sin(\lambda t/\delta)}{t}f(\lambda,t)\mathrm{d}t\mathrm{d}\lambda = 2\pi\int f(0,\lambda)\mathrm{d}\lambda + \int_{\mathbb{R}^2}\frac{1-\cos(\lambda\delta^{-1}t)}{\lambda\delta^{-1}}g(\lambda,t)\mathrm{d}t\mathrm{d}\lambda.$$

Using the Hölder bound $|1-\cos(u)|\le |u|^\gamma$, valid for all $u\in\mathbb{R}$, $\gamma\in(0,1)$, one has

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\frac{1-\cos(\lambda\delta^{-1}t)}{\lambda\delta^{-1}}g(\lambda,t)\mathrm{d}t\mathrm{d}\lambda\Big| &\le \delta^{1-\gamma}\int_{\mathcal{K}\times\mathbb{R}}|\lambda|^{\gamma-1}|t|^\gamma|g(\lambda,t)|\mathrm{d}t\mathrm{d}\lambda\\ &\le \delta^{1-\gamma}\Big(\int_{\mathcal{K}}|\lambda|^{\gamma-1}\mathrm{d}\lambda\Big)\Big(\int_{\mathbb{R}}|t|^\gamma\sup_{\lambda\in\mathcal{K}}|g(\lambda,t)|\mathrm{d}t\Big).\end{aligned}$$

It remains to decompose

$$\int_{\mathbb{R}}|t|^\gamma\sup_{\lambda\in\mathcal{K}}|g(\lambda,t)|\mathrm{d}t = \int_{[-1,1]}|t|^\gamma\sup_{\lambda\in\mathcal{K}}|g(\lambda,t)|\mathrm{d}t + \int_{\mathbb{R}\setminus[-1,1]}|t|^\gamma\sup_{\lambda\in\mathcal{K}}|g(\lambda,t)|\mathrm{d}t.$$

The integral on $[-1,1]$ is bounded by $\frac{1}{4}\|f\|_{L^\infty C^2}$. On $\mathbb{R}\setminus[-1,1]$, we can write

$$g(\lambda,t) = -\frac{f(\lambda,0)-f(\lambda,t)}{t^2} - \frac{\partial_t f(t)}{t},$$

and then

$$\int_{\mathbb{R}\setminus[-1,1]}|t|^\gamma\sup_{\lambda\in\mathcal{K}}|g(\lambda,t)|\mathrm{d}t \le C(\gamma)\|f\|_{L^\infty C^2} + \|(\lambda,t)\mapsto t^{\gamma-1}\partial_t f\|_{L^\infty L^1}.$$

This concludes the proof. ∎

A.5. The Airy function

The (classical) Airy function can be defined as the oscillatory integral; for $x\in\mathbb{R}$,

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\Big(\frac{t^3}{3}+xt\Big)\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathbf{i}(t^3/3+xt)}\mathrm{d}t. \tag{A.17}$$

These improper integrals converge and the definition can be extended to $x\in\mathbb{C}$ by considering a contour in the complex plane instead. The Airy function gives rise to the decaying solutions of the Schrödinger equation

$$(-\Delta+x)\,\mathrm{Ai}(x-\lambda) = \lambda\,\mathrm{Ai}(x-\lambda),\quad x\in\mathbb{R},\ \lambda\in\mathbb{C}.$$

For this reason, the Airy function arises in semiclassical approximations at (generic) boundary points as in Theorem II.2. The differential operator $-\Delta+x$ on $\mathbb{R}$ is essentially self-adjoint with absolutely continuous spectrum. Moreover, its *projection-valued measure* can be computed explicitly. Namely, for any $\lambda\in\mathbb{R}$, its kernel is given by

$$\frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}(-\Delta+x-(\lambda+\mathbf{i}\epsilon))^{-1}\colon (x,y)\in\mathbb{R}^2\mapsto \mathrm{Ai}(x-\lambda)\,\mathrm{Ai}(y-\lambda). \tag{A.18}$$

Before studying further the applications of the Airy functions to Schrödinger operators with linear potentials, let us prove a useful fact.

Lemma A.20. *Let* $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ *be any cutoff with* $\mathbb{1}_{[-1,1]} \leq \chi \leq \mathbb{1}_{[-2,2]}$. *For any* $k \in \mathbb{N}$ *and* $x \in \mathbb{R}$,

$$\left| \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}(t^3/3+xt)} \chi(\epsilon t) \mathrm{d}t - \mathrm{Ai}(x) \right| \leq \frac{C_k}{(\epsilon^{-2}+x)_+^k}.$$

Proof. By integrations by parts, we can rewrite for $x \in \mathbb{R}$,

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{\mathbf{i}}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}(t^3/3+t)} \partial_t \Big(\frac{e^{\mathrm{i}(x-1)t}}{t^2+1} \Big) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}(t^3/3+xt)} \chi(\epsilon t) \mathrm{d}t + \frac{\mathbf{i}}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}(t^3/3+t)} \partial_t \Big(\frac{e^{\mathrm{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \mathrm{d}t, \end{aligned}$$

where these integrals converge in the usual sense.

Let $\mathcal{D} = \partial_t(\frac{\cdot}{t^2+x})$. By making repeated integration by parts, it also holds for $x \geq -\epsilon^{-1}$ and for any $k \in \mathbb{N}$,

$$\begin{aligned} &\int_{\mathbb{R}} e^{\mathrm{i}(t^3/3+t)} \partial_t \Big(\frac{e^{\mathrm{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \mathrm{d}t \\ &\qquad = \int_{\mathbb{R}} e^{\mathrm{i}(t^3/3+t)} \mathcal{D}^k \Big(e^{\mathrm{i}(1-x)t} \partial_t \Big(\frac{e^{\mathrm{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \Big) \mathrm{d}t, \end{aligned}$$

where we verify by induction that

$$\mathcal{D}^k \Big(e^{\mathrm{i}(1-x)t} \partial_t \Big(\frac{e^{\mathrm{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \Big) = \begin{cases} 0 & \text{if } |t| \leq \epsilon^{-1}, \\ \mathcal{O}_k((t^2+1)^{-1}(\epsilon^{-2}+x)^{-k}) & \text{if } |t| \geq \epsilon^{-1}. \end{cases}$$

This proves the claim. ∎

Note that taking $\epsilon = 1$, Lemma A.20 and Proposition A.16 (applied with $\Phi(t) = t$ and $\hbar = \frac{1}{x}$) imply that as $x \to +\infty$,

$$\mathrm{Ai}(x) = \mathcal{O}(x^{-\infty}). \tag{A.19}$$

In fact, applying the steepest descent method to integral (A.17), one obtains the well-known asymptotics

$$\mathrm{Ai}(x) \sim \frac{e^{(-2/3)x^{3/2}}}{2\sqrt{\pi} x^{1/4}} \quad \text{as } x \to +\infty.$$

As a direct consequence of (A.19), the integral over $\lambda \in (-\infty, \mu]$ of the *resolvent kernel* (A.18) is convergent for any $\mu \in \mathbb{R}$. This implies in particular that the operator $\mathbb{1}_{-\Delta+x\leq\mu}$ is locally trace-class and its (integral) kernel is given by

$$K_{\mathrm{Airy},\mu} = \mathbb{1}_{-\Delta+x\leq\mu} \colon (x,y) \mapsto \int_{-\infty}^{\lambda} \mathrm{Ai}(x+\mu-\lambda)\,\mathrm{Ai}(y+\mu-\lambda) \mathrm{d}\lambda,$$

see, e.g., [12, Section A.2].

From this Airy kernel and the bulk kernel (A.1), one can formally construct an integral kernel for the edge operator $\mathbb{1}_{-\Delta+x_1\le 0}$ on $\mathbb{R}^n$ in arbitrary dimension $n \ge 2$ as in formula (1.7). Indeed, one can decompose

$$-\Delta + x_1 = H_1 + H_2,$$

where $H_1 = -\partial_{x_1}^2 + x_1$ is the one-dimensional Airy operator and $H_2 = -\Delta_{x^\perp}$ is the Laplacian on the orthogonal hyperplane (we use the coordinates $x = (x_1, x^\perp)$ with $x^\perp = (x_2, \dots, x_n)$ for $x \in \mathbb{R}^n$ with $n \ge 2$). These two operators commute, hence the *projection-valued measure* for $-\Delta + x_1$ is a convolution whose kernel can be written as for $\lambda \in \mathbb{R}$,

$$(x, y) \in \mathbb{R}^{2n} \mapsto \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\mu} K_{\mathrm{Airy},\mu}(x_1, y_1)\Big|_{\mu} \frac{\mathrm{d}}{\mathrm{d}\mu} K^{(n-1)}_{\mathrm{b},\sqrt{\mu}}(x^\perp, y^\perp)\Big|_{\lambda-\mu} \mathrm{d}\mu.$$

By (A.1), this integral can be restricted to $\{\mu \le \lambda\}$ since $K_{\mathrm{b},\mu} = 0$ for $\mu \le 0$. This shows that this integral converges since

$$\Big| \frac{\mathrm{d}}{\mathrm{d}\mu} K^{(n-1)}_{\mathrm{b},\sqrt{\mu}}(x^\perp, y^\perp)\big|_{\lambda-\mu} \Big| \le C_n \lambda^{(n-2)/2}$$

uniformly for $x, y \in \mathbb{R}^n$ and since $\mathrm{Ai}(\mu) = \mathcal{O}(\mu^{-\infty})$ as $\mu \to +\infty$ (cf. (A.19)). Then, by Fubini's theorem, we also obtain

$$\begin{aligned} K^{(n)}_{\mathrm{edge}} &= \mathbb{1}_{(-\infty,0]}(-\Delta + x_1): \\ &(x, y) \mapsto \int_{\mathbb{R}_-} \int_{-\infty}^{\lambda} \frac{\mathrm{d}}{\mathrm{d}\mu} K^{(n-1)}_{\mathrm{b},\sqrt{\mu}}(x^\perp, y^\perp)\Big|_{\lambda-\mu} \mathrm{Ai}(x-\mu)\,\mathrm{Ai}(y-\mu)\mathrm{d}\mu\mathrm{d}\lambda \\ &\qquad = \int_{\mathbb{R}_+} K^{(n-1)}_{\mathrm{b},\sqrt{\mu}}(x^\perp, y^\perp)\,\mathrm{Ai}(x+\mu)\,\mathrm{Ai}(y+\mu)\mathrm{d}\mu. \end{aligned}$$

This formula defines a locally trace-class operator and allows us to conjugate explicitly $-\Delta + x_1$ into a multiplication operator with a conjugation kernel that is well defined and L^2-unitary for functions in Schwartz space. In particular, $-\Delta + x_1$ is essentially self-adjoint, and using the explicit formula (A.2), this concludes the proof of (1.7).

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