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Closed geodesics and Frøyshov invariants of hyperbolic three-manifolds

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Abstract. Frøyshov invariants are numerical invariants of rational homology three-spheres derived from gradings in monopole Floer homology. In the past few years, they have been employed to solve a wide range of problems in 3- and 4-dimensional topology. In this paper, we look at connections with hyperbolic geometry for the class of minimal L-spaces. In particular, we study relations between Frøyshov invariants and closed geodesics using ideas from analytic number theory. We discuss two main applications of our approach. First, we derive explicit upper bounds for the Frøyshov invariants of minimal hyperbolic L-spaces purely in terms of volume and injectivity radius. Second, we describe an algorithm to compute Frøyshov invariants of minimal L-spaces in terms of data arising from hyperbolic geometry. As a concrete example of our method, we compute the Frøyshov invariants for all spin^c structures on the Seifert–Weber dodecahedral space. Along the way, we also prove several results about the eta invariants of the odd signature and Dirac operators on hyperbolic three-manifolds which might be of independent interest.

Keywords: Seiberg–Witten theory, hyperbolic three-manifold, spectral asymmetry, closed geodesics, Selberg trace formula, Floer homology.

Contents

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Introduction

Understanding the relationship between hyperbolic geometry and Floer-theoretic invariants of three-manifolds is one of the outstanding problems of low-dimensional topology. In our previous work [32], as a first step in this direction, we studied sufficient conditions for a hyperbolic rational homology sphere to be an L-space (i.e. to have simplest possible Floer homology [27]) in terms of its volume and complex length spectrum. Our approach was based on spectral geometry and its relation to hyperbolic geometry via the Selberg trace formula. It was implemented explicitly (taking as input computations from SnapPy [16]) to show that several manifolds of small volume in the Hodgson–Weeks census [23] are L-spaces.

In the present paper we focus our attention on the Frøyshov invariants of rational homology spheres. These are numerical invariants $h(Y, \mathfrak{s}) \in \mathbb{Q}$ indexed by spin^c structures which are extracted from the gradings in monopole Floer homology (see [26, Chapter 39] and [19]). The corresponding invariants in the context of Heegaard Floer homology are known as correction terms [46], and the identity $d(Y, \mathfrak{s}) = -2h(Y, \mathfrak{s})$ holds under the isomorphism between the theories (see [14, 28, 50] and subsequent papers). These invariants have been applied in recent years to a wide range of problems in three- and four-dimensional topology; see among the many [29, 45, 47]. Despite this, their computation in specific examples is still a very challenging problem, even under the assumption that Y is an L-space.

The first basic question about them in the spirit of the present paper is the following. Recall that given constants $V, \varepsilon > 0$, the set of hyperbolic rational homology spheres Y for which $\mathrm{vol}(Y) < V$ and $\mathrm{inj}(Y) > \varepsilon$ is finite [4, Chapter 5], and therefore so is the set of possible Frøyshov invariants $h(Y, \mathfrak{s})$.

Question 1. *Given $V, \varepsilon > 0$, can one provide an* explicit *upper bound on $|h(Y, \mathfrak{s})|$ for all hyperbolic rational homology spheres with* $\mathrm{vol}(Y) < V$ *and* $\mathrm{inj}(Y) > \varepsilon$*?*

Of course, this is a challenging question even when restricted to the smaller class of L-spaces with $\mathrm{vol}(Y) < V$ and $\mathrm{inj}(Y) > \varepsilon$. Another natural question in this spirit is the following.

Question 2. *Can one explicitly determine $h(Y, \mathfrak{s})$ in terms of data coming from hyperbolic geometry* (*e.g. volume, injectivity radius, etc.*)*?*

While we are not able to address these questions in the stated generality, we will answer them under the additional assumption that Y is a *minimal hyperbolic L-space*, i.e. a rational homology sphere (Y, g_{hyp}) equipped with a hyperbolic metric g_{hyp} for which sufficiently small perturbations of the Seiberg–Witten equations do not admit irreducible solutions. One of the main results of [32] is that a hyperbolic rational homology sphere Y for which $\lambda_1^* > 2$ (where λ_1^* is the first eigenvalue of the Hodge Laplacian acting on coexact 1-forms) is a minimal hyperbolic L-space; furthermore, the inequality $\lambda_1^* > 2$ can be verified algorithmically in concrete examples, taking as input the length spectrum up to a certain cutoff. The first result we present, which addresses Question 1, is the following.

Theorem 1. *Suppose Y is a minimal hyperbolic L-space with* $\mathrm{vol}(Y) < V$ *and* $\mathrm{inj}(Y) > \varepsilon$. *Then there exists an* explicit *constant $K_{V,\varepsilon}$ for which $|h(Y,\mathfrak{s})| \leq K_{V,\varepsilon}$ for all spin^c structures on Y. For example, if Y is a minimal hyperbolic L-space with* $\mathrm{vol}(Y) < 6.5$ *and* $\mathrm{inj}(Y) > 0.15$ (*e.g. any rational homology sphere in the Hodgson–Weeks census*[1] *with* $\lambda_1^* > 2$), *then for every spin^c structure $\mathfrak{s}$, the inequality $|h(Y,\mathfrak{s})| \leq 67658$ holds.*

In general, the dependence of $K_{V,\varepsilon}$ on V and ε is easily computable but does not admit a particularly pleasant closed form. In order to streamline our discussion, we take a simple approach to prove Theorem 1 leading to bounds which are not asymptotically optimal. More refined arguments might lead to significantly sharper estimates, especially in the case of small injectivity radius (cf. Remark 5.2).

Remark 1. While there are many examples of minimal L-spaces with volume < 6.5 and injectivity radius > 0.15, it is not known whether there are examples with arbitrarily large volume or arbitrarily small injectivity radius; cf. [32, Section 5]. In the opposite direction, we currently do not know any example of hyperbolic L-spaces which are not minimal hyperbolic L-spaces.

The key observation behind the proof of Theorem 1 is the following: for minimal hyperbolic L-spaces, the Frøyshov invariant $h(Y,\mathfrak{s})$ can be expressed in terms of the *eta invariants* η_{sign} and η_{Dir} of the odd signature operator $*d$ acting on coexact 1-forms and the Dirac operator D_{B_0} corresponding to the flat connection B_0^t on the determinant line bundle. Recall that both operators are first order, elliptic and self-adjoint, and are therefore diagonalizable in L^2 with real discrete spectrum unbounded in both directions. The eta invariant is a numerical invariant that intuitively measures the spectral asymmetry of an operator, i.e. the difference between the number of positive and negative eigenvalues [2]. Of course, in our cases of interest, both of these quantities are infinite, and the eta invariant is defined via suitable analytic continuation. While the latter was originally obtained using the heat kernel, in our setup it can also be understood in terms of closed geodesics via the Selberg trace formula for *odd test functions*. One should compare this with classical work of Millson [39] and Moscovici–Stanton [41] expressing the eta invariants in terms of values of suitable odd Selberg zeta functions. In particular, it is possible to provide explicit expressions for η_{sign} and η_{Dir} in terms of spectral and geometric data:

[1] The Hodgson–Weeks census [23] consists of approximatively 11 thousand closed oriented hyperbolic manifolds with $\mathrm{vol}(Y) < 6.5$ and $\mathrm{inj}(Y) > 0.15$, and most of these are rational homology spheres. It is currently not known what percentage of such manifolds it encompasses.

- In the case of η_{sign}, the geometric input is the complex length spectrum we have already exploited in [32]. The main difference is that in our previous paper we only needed the trace formula for even test functions. This is because we were interested in the Hodge Laplacian Δ acting on coexact 1-forms, and the trace formula involved spectral parameters t_j with $t_j^2 = \lambda_j^*$. When using even test functions, the choice of the sign t_j is irrelevant, but in fact there is a natural choice (for a fixed orientation of Y) because Δ is the square of $*d$.
- In the case of η_{Dir}, the relevant new geometric data is encoded in the *spin^c length spectrum* of $(Y, \mathfrak{s})$; this can be used to obtain information about the spectra of the corresponding Dirac operator via a specialization of the Selberg trace formula for the group
$$G = \{g \in \mathrm{GL}_2(\mathbb{C}) : |\det(g)| = 1\}.$$
This should be thought of as the spin^c analogue of the group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ which we studied in our previous paper.

Given this, Theorem 1 follows by applying ideas of analytic number theory and choosing suitable compactly supported test functions. The most notable inputs are *local Weyl laws* [42], which allow one to explicitly bound from above the number of eigenvalues of the odd signature and Dirac operators in any given interval. The trace formula relates the corresponding eta invariants quite explicitly to the hyperbolic geometry of the underlying manifold, which allows us to prove explicit upper bounds. For example, we will show that for any hyperbolic rational homology sphere Y with volume < 6.5 and injectivity radius > 0.15, the explicit inequalities

$$|\eta_{\text{sign}}| \leq 108267, \quad |\eta_{\text{Dir}}| \leq 108249$$

hold, where the second estimate is independent of the choice of spin^c structure. Let us remark again that these estimates are not optimal even within the range of our techniques.

The injectivity radius makes its appearance in the assumptions of these results because it equals half the length of the shortest closed geodesic [34, Section 4.3]. This implies that the relevant trace formula takes a particularly simple form when evaluated using test functions supported in the interval $[-2 \cdot \mathrm{inj}(Y), 2 \cdot \mathrm{inj}(Y)]$, as the sum over closed geodesics vanishes.

More generally, the explicit knowledge of the length spectrum up to a certain cutoff provides much more detailed information about Frøyshov invariants and can in fact be used to provide explicit computations for minimal hyperbolic L-spaces in the spirit of Question 2. In the second part of the paper, we first showcase the main ideas behind this approach in the simple case of the Weeks manifold. This is known [21] to be the hyperbolic three-manifold of smallest volume (≈ 0.94), and was shown in [32] to be a minimal hyperbolic L-space. The Weeks manifold admits very simple topological descriptions, and in particular its Frøyshov invariants can be computed in a purely topological fashion using the techniques of [33].

With this example in mind, we will focus as proof of concept on the significantly more challenging case of the Seifert–Weber dodecahedral space SW. It has volume ≈ 11.199, and while it was one of the first examples of closed hyperbolic manifolds to be discovered [51], it is a complicated space to study from the point of view of 3-dimensional topology. For example, it took 30 years to verify Thurston's conjecture that SW is not Haken [10], and it is a complicated space to study using standard techniques in Floer homology because it is not a branched double cover of S^3 and it does not admit a simple surgery description. In [31] we used our techniques to show that SW is a minimal hyperbolic L-space by taking into account its large symmetry group. The next result determines its Frøyshov invariants for all the spin^c structures on SW (recall that $H_1(\mathsf{SW};\mathbb{Z}) = (\mathbb{Z}/5\mathbb{Z})^3$).

Theorem 2. *Let $\mathfrak{s}_0$ be the unique spin structure on* SW*, and consider a spin^c structure $\mathfrak{s} = \mathfrak{s}_0 + x$ on* SW *for $x \in H^2(\mathsf{SW},\mathbb{Z})$. Then the Frøyshov invariant $h(\mathsf{SW},\mathfrak{s})$ is as in the following table, according to the value of the linking form $\mathsf{lk}(x,x) \in \mathbb{Q}/\mathbb{Z}$.*

$\mathsf{lk}(x,x)$	$h(\mathsf{SW},\mathfrak{s})$
0	$-1/2$
$1/5$	$-1/10$
$2/5$	$-7/10$
$3/5$	$-3/10$
$4/5$	$1/10$

Remark 2. As we will see in the proof, the case $\mathsf{lk}(x,x) = 0$ really encompasses two distinct cases: the unique spin structure and a family of 24 (non-spin) spin^c structures.

An important observation here is that the fractional parts of Frøyshov invariants of Y admit partial interpretations in terms of the linking form of Y and the topology of manifolds bounding Y; given the extra topological input, the problem boils down to computing eta invariants up to a certain small (but reasonable) error. Towards this end, the main limitation is that we can only access a limited amount of the length spectrum of SW. To prove Theorem 2, it is most convenient to use dilated Gaussians as test functions, because both the function and its Fourier transform (which is again a Gaussian) are rapidly decaying; as Gaussians are not compactly supported, we need to truncate certain infinite sums over closed geodesics that arise when estimating the η_{sign} and η_{Dir} via the odd trace formula for coexact 1-forms and spinors. The main step in the proof is then to estimate the error introduced in this procedure. This can be done by deriving explicit bounds on the number of closed geodesics of a given length, in the spirit of the prime geodesic theorem with error terms [11, Section 9.6]. Similarly, we will estimate the error introduced by truncation on the spectral side via (explicit and refined versions of) the local Weyl law.

Remark 3. The approach we implement for SW can be in principle carried out for any minimal hyperbolic L-space for which the linking form is known; the fundamental limitation comes from computing length spectra. In fact, the same ideas can also be exploited to obtain closed formulas for the Frøyshov invariants of minimal hyperbolic L-spaces with finitely many terms with an explicit (but impractical) upper bound on the number of terms. We will not pursue such a closed formula in the present paper.

Let us remark that explicit computations for the eta invariant of the odd signature operator η_{sign} for hyperbolic three-manifolds have been implemented in Snap [15], and are based on a Dehn filling approach [44]. The key insight is that for a fixed oriented compact four-manifold X bounding Y, the general Atiyah–Patodi–Singer index theorem [2] relates η_{sign} to the kernel of the odd signature operator on Y and the signature of X, both of which are topological invariants.

However, while the APS index theorem also holds for the spin^c Dirac operator, it is well known that the dimension of its kernel on Y is not a topological invariant [22]; this fact makes the computation of η_{Dir} much more subtle than its odd signature operator counterpart. In fact, the computations of Frøyshov invariants carried out in this paper for some explicit minimal hyperbolic L-spaces provide as a byproduct explicit examples of hyperbolic three-manifolds for which one can compute Dirac eta invariants to high accuracy (which are not zero for obvious geometric reasons, e.g. the existence of an orientation-reversing isometry), and to the best of our knowledge these are the first such examples. For example, our methods will show that for the unique spin structure on the Weeks manifold,

$$\eta_{\text{Dir}} = 0.989992\ldots.$$

This relies on the fact that the Weeks manifold is a minimal hyperbolic L-space, together with the computation of η_{sign} given in [15]; in particular, the value is as precise as the computations provided by Snap.

More generally, the odd trace formula allows one to obtain bounds on η_{Dir} provided one can access the spin^c length spectrum of $(Y, \mathfrak{s})$; in turn, we will describe an algorithm to compute spin^c length spectra taking as input information computed using SnapPy. In particular, our method could in principle be applied to compute the invariants η_{Dir} for any hyperbolic three-manifold, even though at a practical level it might be infeasible to obtain a decent approximation in a reasonable time.

Note for the reader. The paper is structured so that the various trace formulas (see Section 3 for the statements) can be treated as black boxes, and all subsequent sections are written in a way that is hopefully self-contained. In particular, in Section 4 we only use some complex analysis to provide an explicit formula for the eta invariants in terms of eigenvalues and complex (spin^c) lengths. Given this, the remainder of the paper only uses basic facts about Fourier transforms, and we will provide motivation and context for the tools from analytic number theory which we employ. Detailed proofs of the various trace formulas which we use can be found in the appendices; our discussion there assumes the reader to be familiar with the proof of the even trace formula for coexact 1-forms in our previous work [32, Appendix B].

Plan of the paper. Sections 1, 2 and 3 provide background about the main protagonists of the paper: Frøyshov invariants, spin^c length spectra, and trace formulas (both even and odd). In Section 4 we use the odd trace formulas to provide an explicit expression for the eta invariants of the odd signature and Dirac operators; this will be the main tool for the present paper. In particular, we prove Theorem 1 in Sections 5 and 6 by providing explicit bounds on the terms appearing in the sum. In Section 7, we show how our analytical expressions for eta invariants can be used to perform explicit computations on the Weeks manifold, the simplest minimal hyperbolic L-space. This example is propaedeutic for the significantly more challenging case of the Seifert–Weber space, which we discuss in detail in Sections 8–10.

1. Background on Frøyshov and eta invariants

In this section we review some background topics that will be central for the purposes of the paper.

1.1. Formal structure of monopole Floer homology and Frøyshov invariants

In [26] the authors associate to each three-manifold Y three $\mathbb{Z}[U]$-modules fitting in a long exact sequence

$$\cdots \xrightarrow{i_*} \widecheck{HM}_*(Y) \xrightarrow{j_*} \widehat{HM}_*(Y) \xrightarrow{p_*} \overline{HM}_*(Y) \xrightarrow{i_*} \cdots .$$

These are read respectively *HM-to*, *HM-from* and *HM-bar*, and we collectively refer to them as monopole Floer homology groups. Such invariants decompose along spin^c structures on Y; for example we have

$$\widecheck{HM}_*(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \widecheck{HM}_*(Y,\mathfrak{s}).$$

The reduced Floer homology group $HM_*(Y,\mathfrak{s})$ is defined to be the kernel of the map p_*. In this paper, we will be only interested in the case of rational homology spheres. In this situation the Floer homology groups for a fixed spin^c struture admit an absolute grading by a $\mathbb{Z}$-coset in $\mathbb{Q}$, and the action of U has degree -2. Furthermore,

$$\overline{HM}_*(Y,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}]$$

as graded modules (up to an overall shift), $\widehat{HM}_*(Y,\mathfrak{s})$ vanishes in degrees high enough, and p_* is an isomorphism in degrees low enough.

A rational homology sphere Y is called an *L-space* if $HM_*(Y,\mathfrak{s}) = 0$ for all spin^c structures. Given a spin^c rational homology sphere $(Y,\mathfrak{s})$, denote the minimum degree of a non-zero element in $i_*(\overline{HM}_*(Y,\mathfrak{s})) \subset \widecheck{HM}_*(Y,\mathfrak{s})$ by $-2h(Y,\mathfrak{s})$. The quantity $h(Y,\mathfrak{s})$ is then called the *Frøyshov invariant* of $(Y,\mathfrak{s})$.

1.2. Frøyshov invariants in terms of eta invariants

Recall [2, Theorem 4.14] that for an oriented three-manifold the odd signature operator B acts on even forms Ω^{even} as

$$(-1)^p(*d - d*) \quad \text{on } 2p\text{-forms}.$$

We identify 2-forms and 1-forms using $*$, so that the operator is

$$\begin{pmatrix} *d & d \\ d^* & 0 \end{pmatrix}$$

acting on $\Omega^1 \oplus \Omega^0$. This is a first order elliptic self-adjoint operator, and is diagonalizable in L^2 with real discrete spectrum unbounded in both directions. Its eta function is defined to be

$$\eta_{\mathrm{sign}}(s) = \sum_{\lambda \neq 0 \text{ eigenvalue}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s};$$

this sum defines a holomorphic function for $\mathrm{Re}(s)$ large. One of the key results of [2] is that it admits a meromorphic continuation to the entire complex plane, with a regular value at $s = 0$. In Section 4, we will quickly review the original proof of this (via the heat kernel) and then provide an alternative interpretation via the trace formula. Intuitively, $\eta_{\mathrm{sign}} = \eta_{\mathrm{sign}}(0)$ measures the spectral asymmetry of the operator. Similarly, the same procedure works for the Dirac operator D_{B_0} (and its perturbations), leading to η_{Dir}.

Remark 1.1. For our purposes it is convenient to notice (cf. [2, Proposition 4.20]) that the odd signature operator (assuming for simplicity $b_1(Y) = 0$) under the Hodge decomposition $\Omega^1 = d\Omega^0 \oplus d^*\Omega^2$ can be written as

$$\begin{pmatrix} *d & 0 & 0 \\ 0 & 0 & d \\ 0 & d^* & 0 \end{pmatrix}$$

acting on $d^*\Omega^2 \oplus d\Omega^0 \oplus \Omega^0$. Now, the block

$$\begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$$

has symmetric spectrum because d and d^* are adjoints; in particular, for $\mathrm{Re}(s)$ large enough we have

$$\eta_{\mathrm{sign}}(s) = \sum_{t_j} \frac{\mathrm{sgn}(t_j)}{|t_j|^s}, \tag{1}$$

where the sum runs only over the eigenvalues $\{t_j\}$ of $*d$ on *coexact* 1-forms (notice that all the t_j are non-zero). In particular, η_{sign} coincides with the spectral asymmetry of the action of $*d$ on coexact 1-forms.

Remark 1.2. We have $(*d)^2 = \Delta$ when acting on coexact 1-forms, and therefore the squares of the parameters t_j are exactly the eigenvalues λ_j^* of Δ we studied in [32]. The crucial extra information for the purposes of the present paper is the sign of these parameters.

The relation between eta invariants and the Frøyshov invariant is the following. Recall that a minimal L-space is a rational homology sphere admitting a metric for which small perturbations of the Seiberg–Witten equations do not have irreducible solutions.

Proposition 1.1. *Suppose (Y, g) is a minimal L-space. Then for each spin^c structure $\mathfrak{s}$,*

$$h(Y, \mathfrak{s}) = -\frac{\eta_{\text{sign}}}{8} - \frac{\eta_{\text{Dir}}}{2},$$

where η_{Dir} is the eta invariant of the Dirac operator D_{B_0} corresponding to the flat connection on the determinant line bundle B_0^t.

Classical examples of minimal L-spaces are rational homology spheres admitting metrics with positive scalar curvature. In [32], we showed that a hyperbolic rational homology sphere for which the first eigenvalue of the Hodge Laplacian coexact 1-forms λ_1^* satisfies $\lambda_1^* > 2$ is a minimal L-space (using the hyperbolic metric); we furthermore provided several examples of such spaces.

While Proposition 1.1 is well-known to experts, we will dedicate the rest of the section to its proof; our discussion will assume some familiarity with the content of [26], and will not be needed later except in Section 10 where we will briefly use the explicit form of the absolute grading (equation (3) below). The main idea behind the proof is the following. Under the assumption that there are no irreducible solutions, after a small perturbation the Floer chain complex has generator corresponding to the positive eigenspaces of (a small perturbation of) the Dirac operator D_{B_0}. From this description it readily follows that they are L-spaces [26, Chapter 22.7], and that $-2h(Y, \mathfrak{s})$ is the absolute grading of the critical point corresponding to the first positive eigenvalue; the goal is then to express this absolute grading in terms of eta invariants using the APS index theorem.

1.3. Proof of Proposition 1.1

We begin by recalling from [26, Chapter 28.3] how absolute gradings in monopole Floer homology are defined for torsion spin^c structures. As we only consider rational homology spheres, the spin^c structures in our context are automatically torsion. Consider a cobordism $(W, \mathfrak{s}_W)$ from S^3 to $(Y, \mathfrak{s})$; equip S^3 with a round metric and a small admissible perturbation, and denote by $[\mathfrak{a}_0]$ the first stable critical point of S^3, corresponding to the first positive eigenvalue of the Dirac operator. Consider on W a metric which is a product near the boundary, and denote by W^* the manifold obtained by attaching cylindrical ends. We then define for a critical point $[\mathfrak{a}]$ of Y the rational number

$$\mathrm{gr}^{\mathbb{Q}}([\mathfrak{a}]) = -\mathrm{gr}([\mathfrak{a}_0], W^*, \mathfrak{s}_W, [\mathfrak{a}]) + \frac{c_1(\mathfrak{s}_W)^2 - 2\chi(W) - 3\sigma(W)}{4}, \tag{2}$$

where:

- $\mathrm{gr}([\mathfrak{a}_0], W^*, \mathfrak{s}_W, [\mathfrak{a}])$ is the expected dimension of the moduli space of solutions of Seiberg–Witten equations in the spin^c structure $\mathfrak{s}_W$ that converge to $[\mathfrak{a}_0]$ and $[\mathfrak{a}]$. Concretely, this is the index of the linearized equations, after gauge fixing. Notice that because $b_1(Y) = 0$, there is only one homotopy class z of configurations, and we dropped it from the notation.

- $c_1(\mathfrak{s}_W)^2 \in \mathbb{Q}$ is the self-intersection of the class $c_1(\mathfrak{s}_W) \in H^2(W, \mathbb{Z})$. Recall that this is defined as
$$(\tilde{c} \cup \tilde{c})[W, \partial W],$$
where $\tilde{c} \in H^2(W, \partial W, \mathbb{Q})$ is any class whose image in $H^2(W, \mathbb{Q})$ is the same as the image $c_1(\mathfrak{s}_W)$ under the change of coefficients map $H^2(W, \mathbb{Z}) \to H^2(W, \mathbb{Q})$.

For our purposes, it will be convenient to work with a closed manifold X with boundary Y over which $\mathfrak{s}$ extends instead; this can be obtained from W by gluing in a ball D^4 to fill the S^3 boundary component. In this case the formula

$$\mathrm{gr}^{\mathbb{Q}}([\mathfrak{b}]) = -\mathrm{gr}(X^*, \mathfrak{s}_X, [\mathfrak{b}]) + \frac{c_1(\mathfrak{s}_X)^2 - 2\chi(X) - 3\sigma(X) - 2}{4} \in \mathbb{Q} \tag{3}$$

holds, where $\mathrm{gr}(X^*, \mathfrak{s}_X, [\mathfrak{b}])$ is again defined as the expected dimension of the relevant moduli space. This readily follows via the excision principle for the index from the definition in formula (2) and the computation for D^4 in [26, Chapter 27.4].

Let us point out the following observation.

Lemma 1.2. *Suppose (Y, g) is a minimal L-space. Then for each spin^c structure $\mathfrak{s}$, the Dirac operator D_{B_0} corresponding to the flat connection B_0^t has no kernel.*

Remark 1.3. As a consequence, the minimal hyperbolic L-spaces we exhibited in [32] (e.g. the Weeks manifold) do not admit harmonic spinors. These seem to be the first known examples of hyperbolic three-manifolds having no harmonic spinors; more examples can be found using the trace formula techniques we discuss later.

Proof of Lemma 1.2. Consider the small perturbation of the Chern–Simons–Dirac functional
$$\mathcal{L}_\delta = \mathcal{L} + \frac{\delta}{2}\|\Psi\|^2_{L^2},$$
for which the corresponding perturbed Dirac operator at $(B_0, 0)$ is $D_{B_0} + \delta$. For small values of $\pm\delta$, this operator has no kernel and the Seiberg–Witten equations still have no irreducible solutions (because Y is a minimal L-space). By adding additional small pertubations, we can assure that the spectra of these two operators are simple, and they still do not have kernel, so we obtain transversality in the sense of [26, Chapter 12]. In particular, they both determine chain complexes computing the Floer homology, hence both the two first stable critical points have the same absolute grading $-2h(Y, \mathfrak{s})$. This implies that the moduli space of solutions to the perturbed equations connecting them on a product cobordism $\mathbb{R} \times Y$ (for which the induced map is an isomorphism) is 0-dimensional, hence the spectral flow of the corresponding linearized operator at the reducible is also zero. After performing a small homotopy (cf. [26, Chapter 14]), this implies that the spectral flow for a family of operators of the form $D_{B_0} + f(t)$, where $f(t)$ is a monotone function with $f(t) = \pm\delta$ for $t > 1$ and $t < -1$ respectively, equals zero. In particular, D_{B_0} has no kernel. ∎

In this argument, one can avoid adding extra small perturbations to make the spectra simple when working with Morse–Bott singularities instead [32]. To streamline the argument below, we will assume that the first positive eigenspace of D_{B_0} is simple, and the general case can be dealt with by either adding a small perturbation or by working in a Morse–Bott setting. Consider first a reducible critical point $[\mathfrak{b}_0]$, which lies over the reducible solution $[B_0, 0]$. Under our assumptions, its absolute grading is exactly $-2h(Y, \mathfrak{s})$. Taking A to be a spin^c connection of $\mathfrak{s}_X$ restricting to B_0 in a neighborhood of the boundary, linearizing the equations at $(A, 0)$ we have

$$\mathrm{gr}(X^*, \mathfrak{s}_X, [\mathfrak{b}_0]) = \mathrm{ind}_{L^2(X)}(d^* + d^+) + 2\,\mathrm{ind}^{\mathbb{C}}_{L^2(X)}(D^+_A)$$

where

$$d^* + d^+ : i\Omega^1(X) \to i\Omega^0(X) \oplus i\Omega^+(X)$$

has index

$$b_1(X) - b_2^+(X) - 1 = -\frac{\chi(X) + \sigma(X) + 1}{2}.$$

The Atiyah–Patodi–Singer for the odd signature operator says in our case

$$\sigma(X) = \eta_{\mathrm{sign}} + \frac{1}{3}\int_X p_1,$$

where p_1 is the first Pontryagin form of the metric, and η_{sign} is the eta invariant of the odd signature operator. Let us point out that the different sign comes from our convention for boundary orientations from that of [2]: we think of X as a 'filling' of Y, rather than a 'cap', and this in turns implies that our boundary conditions involve the negative spectral projection [26, Chapter 17]. For the Dirac operator, we obtain

$$\mathrm{ind}^{\mathbb{C}}_{L^2(X)}(D^+_A) = \frac{1}{2}\eta_{D_{\mathrm{Dir}}} + \frac{1}{8}\int_X \left(-\frac{1}{3}p_1 + c_1(A)^2\right),$$

where $c_1(A)$ is the Chern form of A, and the term involving the dimension of the kernel of D_{B_0} is not present because of Lemma 1.2. Putting everything together, we see that

$$\mathrm{gr}^{\mathbb{Q}}([\mathfrak{b}_0]) = \tfrac{1}{4}\eta_{\mathrm{sign}} + \eta_{\mathrm{Dir}},$$

and the result follows.

2. Torsion spin^c structures and spin^c length spectra

Consider a closed geodesic γ (oriented and possibly a multiple of another one) in an oriented hyperbolic three-manifold. We denote its complex length by

$$\mathbb{C}\ell(\gamma) := \ell(\gamma) + i \cdot \mathrm{hol}(\gamma) \in \mathbb{R}^{>0} + i\mathbb{R}/2\pi\mathbb{Z},$$

where $\ell(\gamma)$ is the usual length of γ and $\mathrm{hol}(\gamma)$ is its holonomy, defined geometrically as follows (an algebraic interpretation is provided below). Consider a unit speed parametrization $\gamma : [0, \ell(\gamma)] \to Y$ of the geodesic, and fix an orthonormal basis $\{v, w\}$ of the normal plane N to γ at $\gamma(0)$ so that $\{\dot{\gamma}(0), v, w\}$ induces the orientation of Y. We use this convention to orient N. We obtain a new orthonormal basis $\{v', w'\}$ of N via parallel transport along γ and the identification $\gamma(0) = \gamma(\ell(\gamma))$, and $\mathrm{hol}(\gamma)$ is angle needed in order to obtain $\{v, w\}$ from $\{v', w'\}$ with a rotation in N.

In this section we discuss a refinement of this notion when the manifold is equipped with a torsion spin^c structure; this refinement of complex length appears on the geometric side of the trace formula for spinors. We also discuss how the refinement can be computed in explicit examples, taking as input SnapPy's Dirichlet domain and complex length spectrum computations.

2.1. Torsion spin^c structures and G

Let us begin by discussing the simpler case of a genuine spin structure $\mathfrak{s}_0$ on Y (see [24, Chapter IV] for additional background). The choice of a spin structure $\mathfrak{s}_0$ allows us to lift the holonomy of a closed geodesic $\gamma : S^1 \to Y$ from an element in $\mathbb{R}/2\pi\mathbb{Z}$ to an element in $\mathbb{R}/4\pi\mathbb{Z}$. From a topological perspective, this is because, using the Lie spin structure (the one corresponding to $1 \in \mathbb{Z}/2\mathbb{Z} = \Omega_1^{\mathrm{spin}}$, and the disconnected double cover of S^1) on the tangent bundle of γ, $\mathfrak{s}_0$ induces a spin structure on the normal bundle of γ; and a spin structure on a rank 2 real vector bundle is simply a trivialization well-defined up to adding an even number of twists. In particular, it makes sense to talk about $\mathrm{hol}(\gamma)/2$ as an element in $\mathbb{R}/2\pi\mathbb{Z}$.

Remark 2.1. For a covering space $S^1 \to S^1$, the pullback of the Lie spin structure is again the Lie spin structure. This implies that with our conventions, when considering multiples of a given closed geodesic γ, we have

$$\mathrm{hol}(\gamma^n)/2 = n \cdot (\mathrm{hol}(\gamma)/2) \quad \text{in } \mathbb{R}/2\pi\mathbb{Z}.$$

This would not hold if we chose instead the other spin structure on S^1 (corresponding to $0 \in \Omega_1^{\mathrm{spin}}$, and the connected double cover of S^1).

From the point of view of hyperbolic geometry, we think of $\pi_1(Y) \subset \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$. As $\mathbb{H}^3$ is simply connected, it admits a unique spin structure $\tilde{\mathfrak{s}}$, and its group of spin automorphisms is given by $\mathrm{SL}_2(\mathbb{C})$ (with ± 1 corresponding to the two spin automorphisms lifting $\mathrm{id} \in \mathrm{Isom}^+(\mathbb{H}^3)$). Hence a spin structure on Y is a lift

$$\begin{array}{ccc} & \nearrow & \mathrm{SL}_2(\mathbb{C}) \\ & & \downarrow \\ \pi_1(Y) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{C}) \end{array}$$

where the vertical map is the quotient by the subgroup $\{\pm 1\}$.

Notice that all orientable three-manifolds are spin because their second Stiefel–Whitney class w_2 vanishes [40, Chapter 11], so such a lift always exists. Any two such lifts differ by a homomorphism $\pi_1(Y) \to \{\pm 1\}$, i.e. an element in $H^1(Y, \mathbb{Z}/2)$. From this viewpoint, recall that complex length of a hyperbolic element γ is given by

$$\mathbb{C}\ell(\gamma) = 2\log\lambda_\gamma,$$

where λ_γ is the root of

$$x^2 - (\mathrm{tr}\gamma)x + 1 = 0$$

with $|\lambda_\gamma| > 1$. Of course, for an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ the trace is well-defined only up to sign, and we see that the two choices of the trace correspond to $\pm\lambda_\gamma$; this implies that the argument of λ_γ^2, which is $\mathrm{hol}(\gamma)$, is well-defined modulo 2π.

It is then clear that a spin structure, which provides a well-defined trace, also fixes a choice between $\pm\lambda_\gamma$; the argument of this choice, which is $\mathrm{hol}(\gamma)/2$, is well-defined modulo 2π. Very concretely, this is implies that the lift $\widetilde{\gamma} \in \mathrm{SL}_2(\mathbb{C})$ is conjugate to

$$\begin{pmatrix} e^{\ell(\gamma)/2+i\mathrm{hol}(\gamma)/2} & 0 \\ 0 & e^{-\ell(\gamma)/2-i\mathrm{hol}(\gamma)/2} \end{pmatrix}.$$

The case of general torsion spin^c structures is analogous, where we use instead the group

$$G = \{g \in \mathrm{GL}_2(\mathbb{C}) : |\det(g)| = 1\} \subset \mathrm{GL}_2(\mathbb{C}).$$

There is a natural map

$$\pi : G \to \mathrm{PSL}_2(\mathbb{C})$$

obtained by expressing

$$G = (\mathrm{SL}_2(\mathbb{C}) \times \mathrm{U}_1)/\{\pm 1\},$$

where $\mathrm{U}_1 \subset G$ is the subgroup of multiples of the identity, and projecting onto the first factor.

Now, $\mathbb{H}^3$ admits a unique spin^c structure, namely the torsion one induced by the spin structure. Because it is simply connected, it also admits a unique flat spin^c connection up to gauge. Arguing as in the case of spin structures, we see then that a torsion spin^c structure together with a flat spin^c connection on Y corresponds to a homomorphism

$$\varphi : \pi_1(Y) \to G$$

for which $\pi \circ \varphi : \pi_1(Y) \to \mathrm{PSL}_2(\mathbb{C})$ is the inclusion map. We see that two spin^c structures then differ by a homomorphism

$$\pi_1(Y) \to \mathrm{U}_1,$$

which can be thought of as an element in $H^1(Y, \mathbb{R}/\mathbb{Z})$. The Bockstein long exact sequence for the coefficients

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

reads in this case

$$H^1(Y,\mathbb{Z}) \to H^1(Y,\mathbb{R}) \to H^1(Y,\mathbb{R}/\mathbb{Z}) \to H^2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{R}).$$

The latter allows us to interpret topological classes of torsion spin^c structure as an affine space over the torsion of $H^2(Y,\mathbb{Z})$.

Throughout the paper, we will study rational homology spheres, for which all spin^c structures are torsion. In this case, we have identifications

$$H^1(Y,\mathbb{Q}/\mathbb{Z}) \cong H^1(Y,\mathbb{R}/\mathbb{Z}) \cong H^2(Y,\mathbb{Z})$$

We will further fix a base spin structure $\mathfrak{s}_0$ and accordingly identify spin^c structures with $H^1(Y,\mathbb{Q}/\mathbb{Z})$. For simplicity, φ will interchangeably refer to both the lift and the difference homomorphism. In this case, the element γ has lift conjugate to

$$\begin{pmatrix} e^{i\varphi(\gamma)}e^{\ell(\gamma)/2+i\mathrm{hol}(\gamma)/2} & 0 \\ 0 & e^{i\varphi(\gamma)}e^{-\ell(\gamma)/2-i\mathrm{hol}(\gamma)/2} \end{pmatrix}$$

and the spin^c length spectrum keeps track of the value $\varphi(\gamma)$. We will refer to the latter as the *twisting character*.

2.2. *Computations of spin^c lifts*

For eventual use of the trace formula for spinors on G, we need to explicitly compute the torsion spin^c data, namely $\mathrm{hol}(\gamma)/2$ and $\phi(\gamma)$ for representatives of all conjugacy classes in Γ up to some specified real length. We refer to this data as the *spin^c length spectrum* of $(Y,\mathfrak{s})$. We discuss a concrete algorithm to compute such information, taking as input data provided by SnapPy.

2.2.1. Input for spin^c length spectrum: SnapPy Dirichlet domain and complex length spectrum. In order to compute the spin^c length spectrum up to some real length threshold R, we take as input the following objects (both pre-computable in SnapPy):

- A Dirichlet domain D for $\Gamma = \pi_1(Y)$ centered at $o = (1,0,0,0)$ in the hyperboloid model of $\mathbb{H}^3$, i.e. the upper sheet of

  $$\{(t,x,y,z) : Q(t,x,y,z) = -1\},$$

  where $Q(t,x,y,z) = -t^2 + x^2 + y^2 + z^2$. In particular, this includes a list of all elements γ of the identity component $\mathrm{SO}(Q)^0$ of the orthogonal group for which some domain within the bisector of o and γo is a face $F \subset D$. For hyperbolic three-manifolds Y, such elements γ come in inverse pairs: for the above element γ, $\gamma^{-1}F$ is the face of D opposite F. The group Γ is generated by the face-pairing elements (see Section 2.2.4).
- For each conjugacy class C in Γ with $\ell(C) \le R$, a matrix g_C in $\mathrm{SO}(Q)^0$, which is the image in $\mathrm{SO}(Q)^0$ of some element of Γ representing C. One can extract the complex length of C from g_C, but of course it contains more information.

2.2.2. Converting $\Gamma \subset \mathrm{SO}(Q)^0$ to $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$. Minkowski space $\mathbb{R}^{1,3}$ can be identified with the space $i\mathfrak{u}_2$ of 2×2 hermitian matrices via the map

$$(t,x,y,z) \mapsto \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}.$$

Under this identification, the quadratic form Q equals the determinant of the above matrix. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $i\mathfrak{u}_2$ by

$$g \cdot X = gXg^*,$$

preserving the determinant. The resulting homomorphism

$$\mathrm{PSL}_2(\mathbb{C}) \xrightarrow{\sim} \mathrm{SO}(Q)^0$$

is in fact an isomorphism. To transform Γ to a corresponding subgroup of $\mathrm{PSL}_2(\mathbb{C})$ amounts to inverting the above isomorphism. In turn, this amounts to solving the following linear algebra problem:

Given $f \in \mathrm{End}(i\mathfrak{u}_2)$ preserving Q, find $g \in \mathrm{GL}_2(\mathbb{C})$ satisfying

$$Xg^* = (\det(g) \cdot g^{-1}) f(X) \quad \text{for all } X \in i\mathfrak{u}_2. \tag{4}$$

The entries of $\det(g) \cdot g^{-1}$ are linear in the entries of g, so (4) defines a linear algebra problem (over $\mathbb{R}$), which is readily solved. Rescaling the solution g to (4) as needed, we solve (4) with some matrix $g \in \mathrm{PSL}_2(\mathbb{C})$.

2.2.3. Reduction theory via Dirichlet domains for conjugacy classes $C \subset \Gamma$. We discuss how to express a given element $\gamma \in \Gamma$ in terms of the face-pairing elements γ_i of D.

Lemma 2.1. *Suppose $x \in \mathbb{H}^3 \setminus D$. There is some face-pairing element γ for which $d(\gamma^{-1} \cdot x, o) < d(x, o)$.*

Proof. Because $\gamma_1, \dots, \gamma_m$ support the faces of the Dirichlet domain D,

$$D = \{x \in \mathbb{H}^3 : d(x, o) \le d(x, \gamma_i \cdot o) \text{ for } i = 1, \dots, m\}.$$

In particular, if $x \notin D$, then

$$d(x, o) > d(x, \gamma_i \cdot o) = d(\gamma_i^{-1} \cdot x, o)$$

for some face-pairing element γ_i. ∎

This yields the following *Dirichlet domain reduction algorithm for* Γ:

- Initialize $x = \gamma \cdot o$. Let $L = []$ be an empty list. Note: if γ is not the identity element, then $x \notin D$.
- While x is not in D: find a face-pairing element γ for which $d(\gamma^{-1}x, o) < d(x, o)$; this is always possible by Lemma 2.1. Then:
  - Append γ to the right end of L.
  - Replace x by $\gamma^{-1}x$.
- Once $x \in D$, output $\delta = \prod_{g \in L} g$, product taken in left-to-right order.

Lemma 2.2. *The Dirichlet domain reduction algorithm for Γ terminates. Its output equals γ.*

Proof. Let $x = \gamma \cdot o$. If the while loop did not terminate, we would find an infinite sequence of face-pairing elements $\gamma_1, \gamma_2, \ldots$ for which $x, \gamma_1^{-1} \cdot x, \gamma_2^{-1} \cdot \gamma_1^{-1} \cdot x, \ldots$ all lie off D and for which

$$d(x,o) > d(\gamma_1^{-1} \cdot x, o) > d(\gamma_2^{-1} \cdot x, o) > \cdots .$$

But the latter is not possible since the action of Γ on $\mathbb{H}^3$ is properly discontinuous. Suppose now $L = [\gamma_1, \ldots, \gamma_n]$ at the point of the algorithm when $x \in D$. That means that

$$\gamma_n^{-1}\gamma_{n-1}^{-1} \cdots \gamma_1^{-1} \cdot (\gamma \cdot o) \in D.$$

The latter point is evidently Γ-equivalent to o and lies in D. Since o does not lie on the boundary of D, o is the unique point of D which is Γ-equivalent to o. So

$$(\gamma_n^{-1}\gamma_{n-1}^{-1} \cdots \gamma_1^{-1} \cdot \gamma) \cdot o = o.$$

Since the action of Γ on $\mathbb{H}^3$ is fixed-point free, it follows that $\gamma = \gamma_1 \cdots \gamma_n$. ∎

2.2.4. Computing one spin structure. As mentioned above, it is convenient to fix a base genuine spin structure $\mathfrak{s}_0$ (so that all the others spin^c structures are obtained by twisting it via a character). To compute the corresponding lift $\widetilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{C})$, we use the presentation of $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ afforded to us by the Dirichlet domain D. Let Γ_{abs} denote the (abstract) group with a generator $[\gamma]$ for each face-pairing element γ and satisfying the following relations:

- (Opposite face relations) $[\gamma] \cdot [\gamma^{-1}] = 1$ for all face-pairing elements γ.
- (Edge cycle relations) The edges of D are partitioned into *edge cycles*. This is an ordered sequence of edges $e_1, \ldots, e_n$ for which $\gamma_1 \cdot e_1 = e_2, \gamma_2 \cdot e_2 = e_3, \ldots, \gamma_n \cdot e_n = e_1$ for some $\gamma_1, \gamma_2, \ldots, \gamma_n$. Every edge cycle yields a corresponding relation: $[\gamma_n] \cdots [\gamma_1] = 1$.

The opposite face and edge cycle relations hold for the corresponding elements of Γ. The assignment

$$\iota : \Gamma_{\mathrm{abs}} \to \Gamma \subset \mathrm{PSL}_2(\mathbb{C}), \quad [\gamma] \mapsto \gamma, \tag{5}$$

thus extends to a well-defined homomorphism. According to the Poincaré polyhedron theorem [35], ι is an isomorphism.

For every face-pairing element $\gamma \in \Gamma \subset \mathrm{PSL}_2(\mathbb{C})$, choose arbitrarily a lift $\widetilde{\gamma} \in \mathrm{SL}_2(\mathbb{C})$. Because ι is well-defined, all of the defining relations have an associated sign. For example, if $[\gamma_n] \cdots [\gamma_1] = 1$ is an edge cycle relation, then

$$\widetilde{\gamma_n} \cdots \widetilde{\gamma_1} = \epsilon_{[\gamma_n]\cdots[\gamma_1]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\epsilon_{[\gamma_n]\cdots[\gamma_1]} = \pm 1$. Let $\epsilon_{[\gamma]} = \pm 1$ be unknowns we will solve for. Then

$$\gamma \mapsto \epsilon_{[\gamma]} \cdot \widetilde{\gamma}$$

extends to a well-defined lift of Γ to $\widetilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{C})$ iff the opposite face and edge cycle relations are satisfied. This immediately reduces to the following linear algebra problem (over $\mathbb{Z}/2\mathbb{Z}$):

$$\begin{cases} \epsilon_{[\gamma]}\epsilon_{[\gamma^{-1}]} = \epsilon_{[\gamma]\cdot[\gamma^{-1}]} & \text{opposite face relations,} \\ \epsilon_{[\gamma_n]}\cdots\epsilon_{[\gamma_1]} = \epsilon_{[\gamma_n]\cdots[\gamma_1]} & \text{edge cycle relations.} \end{cases}$$

We know abstractly that some solution must exist (because Y is spin), and we readily solve for one.

Remark 2.2. The collection of lifts $\widetilde{\Gamma}$ is a torsor for $H^1(\Gamma\backslash\mathbb{H}^3, \mathbb{Z}/2)$. No preferred lift exists in general. We note, however, that SW, the protagonist of the second part of our paper, satisfies $H^1(\mathsf{SW}, \mathbb{Z}/2) = 0$, so $\pi_1(\mathsf{SW})$ admits a unique lift to $\widetilde{\pi_1(\mathsf{SW})} \subset \mathrm{SL}_2(\mathbb{C})$.

2.2.5. Computing all homomorphisms $\Gamma \to \mathbb{Q}/\mathbb{Z}$. From the previous subsection, Γ_{abs} gives a presentation for Γ. Let $e_1, \dots, e_m$ denote the images of the face-pairing elements $\gamma_1, \dots, \gamma_m$ in the abelianization Γ^{ab}. The abelian group Γ^{ab} can be presented as

$$\mathbb{Z}\langle e_1, \dots, e_m\rangle / R,$$

where R are the abelian versions of the opposite face and edge cycle relations from Section 2.2.4. We assume that $b_1(Y) = 0$, so that $\Gamma^{\mathrm{ab}} = H_1(Y, \mathbb{Z})$ is a finite group. Using the Smith normal form, we compute a basis $b_1, \dots, b_m$ of $\mathbb{Z}\langle e_1, \dots, e_m\rangle$ for which a basis for the span of R is given $d_1b_1, \dots, d_mb_m$ for some non-zero integers $d_1, \dots, d_m$ with $d_i \mid d_{i+1}$. In particular,

$$\Gamma^{\mathrm{ab}} = H_1(Y, \mathbb{Z}) = \bigoplus \mathbb{Z}/d_i\mathbb{Z}.$$

Homomorphisms from Γ^{ab} to $\mathbb{Q}/\mathbb{Z}$ are uniquely determined by the images $\mathbf{y} := (y_1, \dots, y_m)$ of $b_1, \dots, b_m$; we define $\phi_{\mathbf{y}}$ by

$$b_i \mapsto y_i \in \left(\frac{1}{d_i}\mathbb{Z}\right)/\mathbb{Z}.$$

If $A = (b_1|\cdots|b_m)$ is the matrix with jth column b_j, then

$$\mathbf{y}A^{-1} = (\phi_{\mathbf{y}}(e_1), \dots, \phi_{\mathbf{y}}(e_m)).$$

2.2.6. Indexing all lifts of Γ *to* $\widetilde{\Gamma} \subset G$. We index the lifts of $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ to $\widetilde{\Gamma} \subset G$ as follows:

- Compute one spin lift $s_0 : \Gamma \to \Gamma_{\mathrm{spin}} \subset \mathrm{SL}_2(\mathbb{C})$, by the procedure described in Section 2.2.4.
- A general lift of Γ to G is of the form $\gamma \mapsto \phi(\gamma) \cdot s_0(\gamma)$ for some twisting character $\phi : \Gamma \to \mathrm{U}_1$.

Thus,

$$\left\{\gamma \mapsto e^{2\pi i \phi_{\mathbf{y}}(\gamma)} \cdot s(\gamma) : y_1 \in \left(\frac{1}{d_1}\mathbb{Z}\right)/\mathbb{Z}, \dots, y_m \in \left(\frac{1}{d_m}\mathbb{Z}\right)/\mathbb{Z}\right\}$$

parametrizes all lifts of Γ to G.

2.2.7. Computing the spin^c length spectrum. Suppose we have computed one lift $s_0 : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ specified by the images of the face-pairing generators for D; SnapPy provides the images in $\mathrm{SO}(Q)^0$ for all of the face-pairing generators, so we can apply the inverse isomorphism from Section 2.2.2 to each face-pairing generator followed by the procedure described in Section 2.2.4 to compute a lift s_0. Suppose also that we have computed a homomorphism $\phi : \Gamma \to \mathbb{Q}/\mathbb{Z}$, specified by its values on the face-pairing generators, e.g. the homomorphisms $\phi_{\mathbf{y}}$ from Section 2.2.5.

For every Γ-conjugacy class of translation length at most R, SnapPy specifies the image in $\mathrm{SO}(Q)^0$ of some representative element γ of its conjugacy class. Notice that γ lies in the group generated by the face-pairing elements of the Dirichlet domain D. Applying the Dirichlet domain reduction algorithm from Section 2.2.3, we can express $\gamma = \gamma_1 \cdots \gamma_k$ for some face-pairing elements $\gamma_1, \ldots, \gamma_k$ for D. Having computed s_0 on the face-pairing generators, $s_0(\gamma) = s_0(\gamma_1) \cdots s_0(\gamma_k)$ is some explicit element of $\mathrm{SL}_2(\mathbb{C})$. We readily compute its $\mathrm{SL}_2(\mathbb{C})$-conjugacy class:

$$s_0(\gamma) \sim_{\mathrm{SL}_2(\mathbb{C})} \begin{pmatrix} e^{u/2}e^{i\theta} & 0 \\ 0 & e^{-u/2}e^{-i\theta} \end{pmatrix},$$

and we see that $u = \ell(\gamma)$ and $\theta = \mathrm{hol}(\gamma)/2 \in \mathbb{R}/2\pi\mathbb{Z}$.

Finally, since the values of the twisting character ϕ are known on the face-pairing generators, we readily compute $\phi(\gamma) = \phi(\gamma_1) + \cdots + \phi(\gamma_k)$. In the lift $\widetilde{\Gamma} \subset G$ of Γ corresponding to s_0 and ϕ, we have

$$\widetilde{\gamma} \sim_G e^{2\pi i \phi} \cdot \begin{pmatrix} e^{u/2}e^{i\theta} & 0 \\ 0 & e^{-u/2}e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{u/2}e^{i\theta}e^{2\pi i \phi} & 0 \\ 0 & e^{-u/2}e^{-i\theta}e^{2\pi i \phi} \end{pmatrix}.$$

Following this procedure for every conjugacy class C of length at most R computes the spin^c length spectrum for the lift $\widetilde{\Gamma} \subset G$ of Γ corresponding to the pair $s_0 : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ and $\phi : \Gamma \to \mathbb{Q}/\mathbb{Z}$.

3. Trace formulas for functions, forms and spinors

While our previous work [32] only used the trace formula to sample coexact 1-form eigenvalues with even test functions, the present paper requires that we extend our toolkit to sample the eigenvalue spectrum for functions and spinors (the latter using both even and odd test functions). In this section, we collect statements of the trace formula specialized to the latter three contexts. For the purposes of our work, the various statements can be treated as a black box, and their detailed proofs can be found in the appendices to the paper. For simplicity, in the statements we will restrict to smooth compactly supported functions, but we will ultimately need to apply the trace formula using more general test functions; this is made precise in Section 3.5.

3.1. Notation and conventions

- Throughout the paper, we will use the convention for Fourier transforms

$$\widehat{H}(t) = \int_{\mathbb{R}} H(x)e^{-itx}\,dx.$$

- For a closed geodesic γ, we denote by γ_0 a prime geodesic which γ is a multiple of.
- When dealing with the odd trace formula, it will be important to have a clear orientation convention, as for example the spectrum of $*d$ on Y and $\bar{Y}$ are opposite to each other. We use the identification $\mathbb{H}^3 = \mathrm{PSL}_2/\mathrm{PSU}_2$ (obtained by thinking of the upper half-plane model), for which the tangent space at $(0,1)$ is $i\mathfrak{su}(2)$. We declare the (physicists') Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  to be a positively oriented basis.

3.2. Trace formula for functions

We begin with the classical trace formula for functions. Here, to each eigenvalue of the Laplacian operator,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

we associate the parameter $r_n \in \mathbb{R} \cup i[0,1]$ for which $\lambda_n = 1 + r_n^2$ (in particular, $\lambda_0 = i$). Non-zero eigenvalues less than 1 (i.e. such that $r_n \in i(0,1)$) are referred to as *small*; the value 1 is important because it is the bottom of the L^2-spectrum of the Laplacian on $\mathbb{H}^3$ (see [13]).

Theorem 3.1. *For $H \in C_c^\infty(\mathbb{R})$ an even test function, we have the identity*

$$\sum_{n=0}^{\infty} \widehat{H}(r_n) = \frac{\mathrm{vol}(Y)}{2\pi} \cdot (-H''(0)) + \sum_{[\gamma]\neq 1} \frac{\ell(\gamma_0)}{|1 - e^{\mathbb{C}\ell(\gamma)}|\,|1 - e^{-\mathbb{C}\ell(\gamma)}|} H(\ell(\gamma)).$$

This formula is well-known, and essentially follows from the computation in [32, Appendix B]. It can also be proved by direct integration via the Abel transform and integral kernels; see for example [13] (beware of an erroneous extra factor of 2).

3.3. Trace formula for coexact 1-forms

Each eigenvalue of the Hodge Laplacian $\Delta = (d + d^*)^2$ acting on coexact 1-forms,

$$0 < \lambda_1^* \leq \lambda_2^* \leq \cdots,$$

is of the form $\lambda_n^* = t_j^2$ for some j, where

$$\cdots \leq t_{-1} < 0 < t_0 \leq t_1 \leq t_2 \leq \cdots$$

are the eigenvalues of $*d$. In our previous paper, we were only concerned with the absolute values $|t_j|$ of the parameters and so even test functions sufficed for our purposes. In the present paper, however, the signs of the parameters t_j are crucial. Below, we state a variant of the trace formula which samples the coexact 1-form eigenvalue spectrum using odd test functions and is thus sensitive to these signs.

Theorem 3.2. *For $H \in C_c^\infty(\mathbb{R})$ an even test function, we have the identity*

$$\frac{1}{2}(b_1(Y)-1)\cdot \widehat{H}(0) + \frac{1}{2}\sum \widehat{H}(t_j) = \frac{\mathrm{vol}(Y)}{2\pi}\cdot (H(0)-H''(0)) + \sum \ell(\gamma_0)\frac{\cos(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}H(\ell(\gamma)).$$

For $K \in C_c^\infty(\mathbb{R})$ an odd test function, we have the identity

$$\frac{1}{2}\sum \widehat{K}(t_j) = i\sum \ell(\gamma_0)\frac{\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}K(\ell(\gamma)).$$

For the last identity, recall that if K is odd then its Fourier transform is purely imaginary.

3.4. Trace formula for spinors

As in Section 2, we fix a base spin structure $\mathfrak{s}_0$ and consider its twist by a character φ. The corresponding Dirac operator D_{B_0} has discrete spectrum

$$\cdots \le s_{-1} < 0 \le s_0 \le s_1 \le s_2 \le \cdots,$$

unbounded in both directions. Also, recall that we introduced the notation $\widetilde{\gamma}$ for the lift of γ corresponding to the base spin structure $\mathfrak{s}_0$.

Theorem 3.3. *For $H \in C_c^\infty(\mathbb{R})$ an even test function, we have the identity*

$$\frac{1}{2}\sum \widehat{H}(s_j) = \frac{\mathrm{vol}(Y)}{2\pi}\cdot\left(\frac{1}{4}H(0)-H''(0)\right) + \sum \ell(\gamma_0)\frac{\cos(\mathrm{hol}(\widetilde{\gamma}))\cos(\varphi(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}H(\ell(\gamma)).$$

For $K \in C_c^\infty(\mathbb{R})$ an odd test function, we have the identity

$$\frac{1}{2}\sum \widehat{K}(s_j) = i\sum \ell(\gamma_0)\frac{\sin(\mathrm{hol}(\widetilde{\gamma}))\cos(\varphi(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}K(\ell(\gamma)).$$

3.5. Allowing less regular functions

While for simplicity we stated the formulas only for smooth, compactly supported functions, the same conclusions hold with less regularity. For example, in our previous work

[32, Appendix C] we showed that the trace formula for coexact 1-forms holds for functions of the form $(\mathbf{1}_{[-a,a]})^{*4}$, the fourth convolution power of the indicator function of the interval $[-a,a]$. Rather than providing general statements, let us point out two specific instances that will be used in this work:

(1) The even trace formulas hold for $H(x) = (\mathbf{1}_{[-a,a]})^{*k}$ with $k \geq 6$ and the odd trace formulas hold for $K = H'$. This follows directly from the results in [32, Appendix C].

(2) The even trace formulas hold for a Gaussian $H(x) = e^{-x^2/2c}$, and the odd trace formulas hold for $K = H'$. This is proved in Appendix D.

In the proof of Theorem 1 we will only use functions of the first type, but our proof of Theorem 2 uses Gaussians.

4. Analytic continuation via the odd trace formula

The classical approach to the analytic continuation of the eta function (1) involves the Mellin transform and the asymptotic expansion of the trace of the heat kernel; we start by quickly reviewing the fundamental aspects of this for the reader's convenience. For our purposes, we will discuss a different interpretation using the odd trace formulas described in the previous section.

4.1. *Analytic continuation via the Mellin transform*

For the reader's convenience, we recall basic facts about the Mellin transform and its relevance for the definition of the η invariant. Our account loosely follows [13, Section 3.4], suitably adapted to the simpler situation we are dealing with. Given a continuous function $f : \mathbb{R}^{>0} \to \mathbb{R}$, we define its *Mellin transform* to be

$$M_f(s) = \int_0^\infty f(t) t^s \, \frac{dt}{t}.$$

This is essentially the Fourier transform for the multiplicative group $\mathbb{R}^{>0}$ equipped with the Haar measure dt/t. Assuming that $f(t) = O(t^{-a})$ for $t \to 0$, and $f(t) = O(t^{-n})$ for $t \to \infty$ for all n, we see that the integral absolutely converges on the half-plane $\mathrm{Re}(s) > a$, and that M_f is a holomorphic function there.

Under additional assumptions, M_f admits an explicit analytic continuation to the whole plane. For our purposes, suppose for example that f admits an asymptotic expansion

$$f(t) \sim \sum_{k=0}^{\infty} c_k t^{i_k} \qquad \text{for } t \to 0, \tag{6}$$

where $\{i_k\}$ is a strictly increasing sequence of real numbers with $\lim i_k = +\infty$. Then M_f extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $-i_k$ with residue c_k for all k, and holomorphic everywhere else. To see this, choose $L > 0$ and let us write

$M_f(s) = M_-(s) + M_+(s)$ where

$$M_-(s) = \int_0^L f(t)t^s \, \frac{dt}{t}, \quad M_+(s) = \int_L^\infty f(t)t^s \, \frac{dt}{t}.$$

Our assumption on the behavior of f at infinity implies that M_+ is an entire function. Regarding the first integral, the asymptotic expansion (6) implies that for a given N we can write

$$f(t) = \sum_{k=0}^{N-1} c_k t^{i_k} + r_N(t)$$

with $r_N(t) = O(t^{i_N})$ for $t \to 0$. Integrating, for $\mathrm{Re}(s) > -i_N$ we can therefore write

$$M_-(s) = \int_0^L f(t)t^s \, \frac{dt}{t} = \sum_{k=0}^{N-1} \frac{c_k \cdot L^{s+i_k}}{s+i_k} + \int_0^L r_N(t)t^s \, \frac{dt}{t}. \tag{7}$$

This expression provides a meromorphic continuation of M_- to the half-plane $\mathrm{Re}(s) > -i_N$. This continuation has simple poles at $-i_0, \dots, -i_{N-1}$ with residues $c_0, \dots, c_{N-1}$ and is holomorphic everywhere else in the half-plane. Because our choice of N was arbitrary, this proves the claim.

Remark 4.1. Though M_- and M_+ both depend on L, one readily checks that their sum does not.

The above discussion readily generalizes to the case in which $f(t) = O(t^{-n})$ for some fixed n when $t \to \infty$. In this case, the analytic continuation (obtained again by breaking the integral $M_f(s) = M_-(s) + M_+(s)$) defines a meromorphic continuation on the half-plane $\mathrm{Re}(s) < n$.

4.2. The heat kernel

We briefly recall the classical analytic continuation of the eta function of the odd signature operator

$$\eta_{\mathrm{sign}}(s) = \sum_{t_n \neq 0} \frac{\mathrm{sgn}(t_n)}{|t_n|^s}.$$

using the asymptotic expansion of the heat kernel; see [2] for details. The key observation to relate this to the previous discussion is the identity

$$|a|^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-|a|t} t^s \, \frac{dt}{t}, \tag{8}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{dt}{t}$ is the classical Gamma function.

In the simpler case of the Riemann zeta function, we have for $\mathrm{Re}(s) > 1$ (which ensures absolute convergence) the identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty e^{-nt} t^s \, \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t - 1} t^s \, \frac{dt}{t}.$$

Recall that $\Gamma(s)$ has poles at $s = 0, -1, -2, \ldots$ with residue $(-1)^k/k$ at $s = -k$. Given the asymptotic expansion for $t \to 0$ in terms of Bernoulli numbers

$$\frac{1}{e^t - 1} \sim \sum_{k=-1}^{\infty} \frac{B_{k+1}}{(k+1)!} t^k,$$

and $1/(e^t - 1) = O(t^{-n})$ for all n when $t \to \infty$, we recover the classical fact that $\zeta(s)$ admits a meromorphic extension with only one pole at $s = 1$.

Let us now focus on the case of the eta function of the odd signature operator (the case of the Dirac operator is analogous). In [2], the authors study the quantity[2]

$$K(t) = -\sum \frac{\operatorname{sgn}(t_n)}{2} \operatorname{erfc}(|t_n|\sqrt{t}),$$

where

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-\xi^2} d\xi$$

is the complementary error function. As 0 is not an eigenvalue by assumption, Weyl's law (Section 5) implies that $K(t) \to 0$ exponentially fast for $t \to \infty$. The derivative of $K(t)$ is computed to be

$$K'(t) = \frac{1}{\sqrt{4\pi t}} \sum t_n e^{-t_n^2 t}.$$

Using (8) and integrating by parts we get the formula

$$\eta_{\text{sign}}(2s) = \frac{2\sqrt{\pi}}{\Gamma(s + 1/2)} \int_0^{\infty} K'(t) t^{s+1} \frac{dt}{t} = -\frac{2s\sqrt{\pi}}{\Gamma(s + 1/2)} \int_0^{\infty} K(t) t^s \frac{dt}{t}. \tag{9}$$

The key point is that $K(t)$ admits an asymptotic expansion of the form

$$K(t) \sim \sum_{k \geq -n} a_k t^{k/2} \quad \text{for } t \to 0, \tag{10}$$

and therefore its Mellin transform has at worst a simple pole at 0. In (9), this pole is canceled by the term s in the numerator, and we see therefore that $\eta_{\text{sign}}(s)$ is holomorphic near 0. We conclude by mentioning that the asymptotic expansion (10) follows by interpreting K as the trace of the heat kernel on $\mathbb{R}^{\geq 0} \times Y$ with the (now-called) APS boundary conditions; see [2, Section 3] for details. As will define the analytic continuation of η in a completely different way, we will not need the K in what follows, and have discussed it only for motivational purposes.

4.3. *Analytic continuation via the trace formula*

While the analytic continuation via the heat kernel we have just discussed holds in general, for the specific case of hyperbolic three-manifolds we can take a different route using the

[2]To be precise, they study the odd signature operator on *coclosed* 1-forms, which leads to an additional term involving $b_1(Y)$ in their discussion.

odd trace formula. Let us discuss first in detail the case of the eta invariant for the odd signature operator (the case of the Dirac operator is essentially the same). Recall that in this case we only need to consider the spectral asymmetry of $*d$ acting on coexact 1-forms (1). Given an even test function H (in our case H will either be a Gaussian or compactly supported, as in Section 3.5), we consider the trace formula applied to the odd test function H'. Recalling that with our convention

$$\widehat{H'}(t) = it\widehat{H}(t),$$

we obtain the identity

$$\sum_n t_n \widehat{H}(t_n) = 2 \sum_{1\neq[\gamma]} \ell(\gamma_0) \cdot \frac{\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot H'(\ell(\gamma)). \tag{11}$$

For a fixed even test function G, and given $T > 0$, we apply this to the function $H(x) = G(x/T)$, for which $\widehat{H}(t) = T\widehat{G}(Tt)$, and obtain the family of identities

$$\sum_n Tt_n \widehat{G}(Tt_n) = 2 \sum_{1\neq[\gamma]} \ell(\gamma_0) \cdot \frac{\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot \frac{1}{T} G'\left(\frac{\ell(\gamma)}{T}\right). \tag{12}$$

We will denote either side of the identity by G_T (and refer to them as *spectral* and *geometric* respectively). Now, for any real number $a \neq 0$ we have the identity

$$\begin{aligned}\int_0^\infty aT\widehat{G}(aT)T^{s-1}dT &= a \cdot \int_0^\infty \widehat{G}(aT)T^s\,dT = a \cdot \int_0^\infty \widehat{G}(|a|T)T^s\,dT\\ &= \frac{\mathrm{sgn}(a)}{|a|^s} \cdot \int_0^\infty \widehat{G}(T)T^s\,dT,\end{aligned}$$

where we have used the fact that $\widehat{G}$ is even in the second equality and performed a change of variables in the last equality above. Therefore, for $\mathrm{Re}(s)$ large enough (so that the sums converge absolutely), we have

$$\begin{aligned}\eta_{\mathrm{sign}}(s) = \sum_{t_n} \frac{\mathrm{sgn}(t_n)}{|t_n|^s} &= \frac{\int_0^\infty \sum (t_n T\widehat{G}(t_nT))T^s\,\frac{dT}{T}}{\int_0^\infty \widehat{G}(T)T^s dT}\\ &= \frac{\int_0^\infty G_T T^s\,\frac{dT}{T}}{\int_0^\infty \widehat{G}(T)T^s dT}.\end{aligned} \tag{13}$$

We can recognize at the numerator and denominator the Mellin transforms of G_T and $T\widehat{G}(T)$ respectively. We have the following.

Lemma 4.1. *Suppose that G is either:*

- *a Gaussian function $x \mapsto e^{-x^2/(2c)}$;*
- *the k-th convolution power of an indicator function $\frac{1}{2a}\mathbf{1}_{[-a,a]}$ for some even $k \geq 6$; or*
- *one of the test functions introduced in Section 5.1 below.*

Then the identity

$$\eta_{\mathrm{sign}}(s) = \frac{\int_0^\infty G_T T^s \frac{dT}{T}}{\int_0^\infty T\widehat{G}(T) T^s \frac{dT}{T}}$$

holds in an open domain in $\mathbb{C}$ *containing* $s = 0$*, where we interpret the right hand side using the Mellin transform.*

Of course, the lemma is valid in much greater generality, but for simplicity we have restricted to the class of functions that will be used in the rest of the paper.

Proof of Lemma 4.1. Let us start with the case of a Gaussian. We claim that the quantity G_T is rapidly decaying for both $T \to 0$ and $T \to \infty$. The former follows by looking at the definition of G_T using the geometric side because by the prime geodesic theorem the number of prime closed geodesics of length less than x is $O(e^{2x}/(2x))$ (Section 9); for the latter, it follows by looking at its definition using the spectral side, because the Weyl law implies that the number of spectral parameters less than t in absolute value is $O(t^3)$ (Section 5), and $\widehat{G}$ is still a Gaussian. By the properties of the Mellin transform discussed in Section 4.1, this implies that the numerator $\int_0^\infty G_T T^s \frac{dT}{T}$ is an entire function of s. On the other hand, the integrand in the denominator $T\widehat{G}(T)$ is rapidly decaying for $T \to \infty$, and by looking at the Taylor series at $t = 0$ one obtains an asymptotic expansion with only odd exponents. Hence the denominator admits an analytic continuation to the whole plane which is regular and non-zero at 0 because $\int_0^\infty \widehat{G}(T)\, dT > 0$. The claim then follows by uniqueness of the analytic continuation.

The case of $(\frac{1}{2a}\mathbf{1}_{[-a,a]})^{*k}$ with $k \geq 6$ even is analogous. Assume for simplicity $a = 1$. First of all, the derivative of this function is regular enough for the odd trace formula to hold (see Section 3.5). Notice that in this case

$$\widehat{G}(t) = \left(\frac{\sin(t)}{t}\right)^k.$$

Looking at the spectral side, we see that

$$|G_T| \leq \frac{1}{T^{k-1}} \cdot \sum_n \frac{1}{|t_n|^{k-1}} = O(T^{-k+1})$$

for $T \to \infty$. Furthermore, G_T is still rapidly decaying for $T \to 0$, because G is compactly supported on the geometric side. The numerator is therefore a holomorphic function on the half-plane $\mathrm{Re}(s) < k - 1$. As in the case of Gaussians, $T\widehat{G}(T)$ has a Taylor expansion at 0 with only odd exponent terms, and $\int_0^\infty T\widehat{G}(T)\, \frac{dT}{T} > 0$; it is also $O(T^{-k+1})$ for $T \to \infty$. The denominator is therefore a meromorphic function on $\mathrm{Re}(s) < k - 1$, regular and non-vanishing at zero, and we conclude as in the case of the Gaussian.

Finally, the case of the test functions introduced in Section 5.1 follows in the same way, and is in fact much simpler because they are both compactly supported and smooth. ■

Let us unravel the statement of the above result in a way that is useful for concrete computations. If we choose a cutoff $L > 0$, we have the formula

$$\eta_{\mathrm{sign}} = \frac{\int_0^L G_T \, \frac{dT}{T} + \int_L^\infty G_T \, \frac{dT}{T}}{\int_0^\infty T\widehat{G}(T)\, \frac{dT}{T}}, \tag{14}$$

where at the numerator we evaluate the integral near zero using the geometric definition, and the integral near infinity using the spectral definition. Explicitly, for the eta invariant of the odd signature operator,

$$\begin{aligned}
\int_0^L G_T \frac{dT}{T} &= \sum \ell(\gamma_0)\cdot \frac{2\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \int_0^L \frac{1}{T} G'\left(\frac{\ell}{T}\right)\frac{dT}{T} \\
&= \sum \ell(\gamma_0)\cdot \frac{2\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\cdot\frac{1}{\ell(\gamma)}\cdot\int_{\ell/L}^\infty yG'(y)\,\frac{dy}{y} \\
&= \sum \ell(\gamma_0)\cdot \frac{2\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\cdot\frac{1}{\ell(\gamma)}\cdot\int_{\ell/L}^\infty G'(y)\,dy \\
&= \sum \ell(\gamma_0)\cdot \frac{2\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\cdot\frac{1}{\ell(\gamma)}\cdot\left(-G\left(\frac{\ell(\gamma)}{L}\right)\right).
\end{aligned}$$

and

$$\begin{aligned}
\int_L^\infty G_T \frac{dT}{T} &= \sum \int_L^\infty T t_n \widehat{G}(Tt_n)\,\frac{dT}{T} \\
&= \sum \mathrm{sgn}(t_n) \int_{L|t_n|}^\infty T\widehat{G}(T)\,\frac{dT}{T} \\
&= \sum \mathrm{sgn}(t_n) \int_{L|t_n|}^\infty \widehat{G}(T)\,dT,
\end{aligned}$$

where we have made a simple substitution in the integral.

Finally, the case of spinors follows in the same way by using the identity

$$\begin{aligned}
G_T &= \sum_n T s_n \widehat{G}(Ts_n) \\
&= 2\sum_{1\neq[\gamma]} \ell(\gamma_0)\cdot\frac{\sin(\mathrm{hol}(\widetilde{\gamma}))\cdot\cos(\varphi_{\widetilde{\gamma}})}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\cdot\frac{1}{T}G'\left(\frac{\ell(\gamma)}{T}\right)
\end{aligned}$$

instead. The only additional observation is that the computation in (13) still holds even when the Dirac operator has kernel. In fact, we have

$$\begin{aligned}
\eta_{\mathrm{Dir}}(s) = \sum_{s_n\neq 0}\frac{\mathrm{sgn}(s_n)}{|s_n|^s} &= \frac{\int_0^\infty \sum (s_n T\widehat{G}(s_nT))T^s\,\frac{dT}{T}}{\int_0^\infty \widehat{G}(T)T^s\,dT} \\
&= \frac{\int_0^\infty G_T T^s\,\frac{dT}{T}}{\int_0^\infty \widehat{G}(T)T^s\,dT}
\end{aligned}$$

as in the second row the parameters s_n equal to zero do not contribute.

5. Explicit local Weyl laws

In this section, we discuss upper bounds for the number of spectral parameters in a given interval for our operators of interest. These upper bounds are expressed purely in terms of injectivity radius and volume, and we refer to such bounds as *local Weyl laws*; see also [42].

To place this in context, recall the classical Weyl law: for every dimension n, there exists a constant C_n such that for every Riemannian n-manifold (X, g), the asymptotic

$$\#\{\text{eigenvalues } \lambda \text{ of } \Delta_g \text{ with } \sqrt{\lambda} \leq T\} \sim C_n \cdot \mathrm{vol}_g(X) \cdot T^n \tag{15}$$

holds, where Δ_g is the Hodge Laplacian acting on functions. Analogous versions hold more generally for squares of Dirac type operators, such as the Hodge Laplacian $(d + d^*)^2$ acting on k-forms or the Dirac Laplacian $D^2_{B_0}$ (see [5]). From (15), one expects the following local version of Weyl law to hold:

$$\#\{\text{eigenvalues } \lambda \text{ of } \Delta_g \text{ with } \sqrt{\lambda} \in [T, T+1]\} \sim n \cdot C_n \cdot \mathrm{vol}_g(X) \cdot T^{n-1}.$$

However, for our purposes it will be important not to just understand the asymptotic behavior of the number of the eigenvalues in a given interval, but also to provide concrete upper bounds on it; in particular, our goal is to prove upper bounds of the form

$$\#\{\text{eigenvalues } \lambda \text{ of } \Delta_g \text{ with } \sqrt{\lambda} \in [T, T+1]\} \leq A \cdot T^{n-1} + B \tag{16}$$

for explicit constants A, B, and similar bounds for the other operators we are interested in, namely the odd signature and Dirac operators. For our application, it is crucial to express the upper bound in (16) (and its analogues for all operators we study) *uniformly in the parameter T, where the constants A and B are expressed explicitly in terms of the geometry of X*. The leading constant A appearing in our upper bounds will generally be larger than the optimal one, $C'_n \cdot \mathrm{vol}_g(X)$.

To prove these local Weyl laws, we will evaluate the trace formulas using even test functions of support so small that the sum over the closed geodesics vanishes. The estimates we will prove are explicit in terms of volume and injectivity radius, but not of a nice form in terms of the input; for this reason, at the end of our computations we will specialize to the concrete case in which inj > 0.15 and vol < 6.5, e.g. the case of a manifold in the Hodgson–Weeks census. We will also assume throughout that $b_1 = 0$.

5.1. Preliminaries on test functions

Fix a value of $R > 0$ (which will later be a given lower bound on the injectivity radius). Choose a non-negative even function φ with support in $[-2, 2]$ and non-negative Fourier transform. Again, our convention is

$$\widehat{\varphi}(t) = \int \varphi(x) e^{-ixt}\, dt.$$

To simplify our discussion, we will make the following assumption.

Assumption. We will assume for simplicity throughout this section that both φ and $\widehat{\varphi}$ achieve their maximum at 0, and that the minimum m_R of $\widehat{\varphi}$ in $[-R, R]$ is achieved at $\pm R$.

Example 5.1. For the specialization $R = 0.15$, we will choose $\varphi_0 = \beta * \beta$, where

$$\beta(x) = \begin{cases} e^{-1/(1-x^2)} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

This function satisfies the assumptions, and $m_R \sim 0.19643$.

Remark 5.1. Of course, this implies that φ_0 satisfies the assumption also for $R < 0.15$. To obtain a function that satisfies the assumption for large values of R, we can look at functions of the form $\varphi_0(Cx)$.

Consider $\varphi_R(x) = \varphi(x/R)$, which is supported in $[-2R, 2R]$. We have

$$\varphi_R(0) = \varphi(0), \quad \varphi''_R(0) = \frac{1}{R^2}\varphi''(0),$$

and

$$\widehat{\varphi_R}(t) = \int \varphi\left(\frac{x}{R}\right)e^{-ixt}\,dx = \int \varphi(x')e^{iRx't}R\,dx' = R\widehat{\varphi}(Rt).$$

Consider also

$$\varphi_{R,\nu} = \varphi_R(x)(e^{i\nu x} + e^{-i\nu x}) = 2\varphi_R(x)\cos(\nu x).$$

Then

$$\varphi_{R,\nu}(0) = 2\varphi(0), \quad \varphi''_{R,\nu}(0) = 2\left(\frac{1}{R^2}\varphi''(0) - \nu^2\varphi(0)\right),$$

and

$$\widehat{\varphi_{R,\nu}}(t) = \widehat{\varphi_R}(t+\nu) + \widehat{\varphi_R}(t-\nu).$$

5.2. *Local Weyl law for coexact* 1*-forms*

We begin with the case of coexact 1-forms, which is the simplest to analyze. For $\nu \geq 0$, denote by $\delta^*(\nu)$ the number of spectral parameters t_j with $|t_j| \in [\nu, \nu+1]$. We evaluate the trace formula in Theorem 3.2 for the test function $\varphi_{R,\nu}(x)$ with R less than the injectivity radius of Y. As $\varphi_{R,\nu}$ is supported in $[-2R, 2R]$, and the injectivity radius is exactly half the length of the shortest geodesic of Y, the sum over the geodesics vanishes, and we have the identity

$$\frac{\mathrm{vol}(Y)}{2\pi}(\varphi_{R,\nu}(0) - \varphi''_{R,\nu}(0)) + \frac{1}{2}\widehat{\varphi_{R,\nu}}(0) = \sum_j \frac{1}{2}\widehat{\varphi_{R,\nu}}(|t_j|).$$

By the identities of the previous subsection we therefore obtain

$$\begin{aligned}\frac{\mathrm{vol}(Y)}{\pi}&\left(\varphi(0) - \frac{1}{R^2}\varphi''(0) + \nu^2\varphi(0)\right) + R\widehat{\varphi}(R\nu) \\ &= \frac{1}{2}\sum_j\left(\widehat{\varphi_R}(|t_j|+\nu) + \widehat{\varphi_R}(|t_j|-\nu)\right) \geq \sum_j \frac{1}{2}\widehat{\varphi_R}(|t_j|-\nu) \geq \frac{1}{2}R\cdot m_R\cdot\delta^*(\nu)\end{aligned}$$

using first the fact that $\widehat{\varphi_R}$ is non-negative, and then that

$$\widehat{\varphi_R}(t-\nu) = R \cdot \widehat{\varphi}(R \cdot (t-\nu)) \geq R \cdot m_R$$

for $t \in [\nu, \nu+1]$. For any choice of suitable test function φ, this provides the following upper bound on $\delta^*(\nu)$ in terms of $\mathrm{vol}(Y)$ and injectivity radius:

$$\delta^*(\nu) \leq \left(\frac{2 \cdot \mathrm{vol}(Y) \cdot \varphi(0)}{\pi R \cdot m_R}\right)\nu^2 + \frac{2}{R \cdot m_R}\left(R\widehat{\varphi}(0) + \frac{\mathrm{vol}(Y)}{\pi}\left(\varphi(0) - \frac{1}{R^2}\varphi''(0)\right)\right),$$

where we have used the fact that $\widehat{\varphi}$ achieves its maximum at 0. Working out the computations using our test function φ_0, we obtain the following.

Proposition 5.1. *Let Y be a hyperbolic rational homology sphere with* $\mathrm{vol}(Y) < 6.5$ *and* $\mathrm{inj}(Y) > 0.15$. *Then*

$$\delta^*(\nu) \leq 18.7\nu^2 + 2577.3.$$

Remark 5.2. The bound obtained with this approach gets worse as R goes to zero (and the same is true for the spectral density on functions and spinors we will study later). In fact, there cannot be an explicit bound independent of the injectivity radius: for example, for large Dehn filling on a cusped three-manifold, the spectrum of the Laplacian on coexact 1-forms becomes dense in $[0, \infty)$ (see [17]).

One way to obtain significantly better estimates in this case is to consider a Margulis number $\mu > 0$ for Y. Recall that for such a number, the set of points of Y with local injectivity radius $< \mu/2$ is the disjoint union of tubes around the finitely many closed geodesics with length $< \mu$. For example, in [38] it is shown $\mu = 0.1$ is a Margulis number for all closed oriented hyperbolic three-manifolds. The number of closed geodesics shorter than μ can be bounded above in terms of the volume [20], and one obtains improved estimates for the spectral density by applying the trace formula to test functions supported in $[-\mu, \mu]$. We will not pursue the exact output of this approach in the present work.

5.3. *Local Weyl law for eigenfunctions*

The case of functions is more involved because of the possible appearance of small eigenvalues (i.e. the ones corresponding to imaginary parameters r_j). Set $\delta(\nu)$ to be the number of parameters in $[\nu, \nu+1]$, and let δ_s be the number of small eigenvalues. We apply Theorem 3.1 choosing as before R to be less than the injectivity radius, again the sum over geodesics vanishes, and we get the identity

$$\sum \widehat{\varphi_{R,\nu}}(r_n) = -\frac{\mathrm{vol}(Y)}{2\pi}\varphi''_{R,\nu}(0).$$

Here we recall that $r_n \in \mathbb{R}^{\geq 0} \cup i[0,1]$ corresponds to the eigenvalue $\lambda_n = 1 + r_n^2$.

We start by bounding the number of small eigenvalues. For this purpose, we set $\nu = 0$. For real t, we have

$$\widehat{\varphi}(it) = \int_{\mathbb{R}} \varphi(x) e^{-tx}\, dx = \int_{\mathbb{R}} \varphi(x)\cosh(tx)\, dx,$$

which as a function of $t \in \mathbb{R}$ is non-negative, even, convex, and has minimum at 0. Therefore we have

$$\begin{aligned} -\frac{\mathrm{vol}(Y)\varphi''(0)}{2\pi R^2} &= \sum \widehat{\varphi_R}(r_n) \\ &\geq \sum_{r_n \text{ imaginary}} \widehat{\varphi_R}(r_n) = R \cdot \sum_{r_n \text{ imaginary}} \widehat{\varphi}(Rr_n) \\ &\geq R\widehat{\varphi}(0) \cdot \delta_s, \end{aligned}$$

so that

$$\delta_s \leq -\frac{\mathrm{vol}(Y)\varphi''(0)}{2\pi R^3 \widehat{\varphi}(0)}.$$

To bound large eigenvalues, we look at $\widehat{\varphi_{R,\nu}}$ for $\nu \neq 0$. Unfortunately, this is not necessarily positive on the imaginary axis. On the other hand, for $t \in [0, 1]$ we have

$$\begin{aligned} |\widehat{\varphi_{R,\nu}}(it)| &= \left| 2\int_{-2R}^{2R} \varphi(x/R)\cos(\nu x)e^{-xt}\,dx \right| \qquad \text{(because } \varphi \text{ is supported on } [-2,2]) \\ &= 2R \left| \int_{-2}^{2} \varphi(y)\cos(\nu R y)e^{-Ryt}\,dy \right| \\ &\leq 2R\varphi(0)\int_{-2}^{2} e^{-Rty}\,dy = 2\varphi(0)\cdot\frac{e^{2Rt}-e^{-2Rt}}{t} \leq 2\varphi(0)(e^{2R}-e^{-2R}), \end{aligned}$$

where the last inequality holds because $(e^{2Rt} - e^{-2Rt})/t$ is increasing for $t \in [0, 1]$. The trace formula then implies

$$\begin{aligned} &\frac{\mathrm{vol}(Y)}{\pi}\left(\nu^2\varphi(0) - \frac{1}{R^2}\varphi''(0)\right) + 2\varphi(0)\cdot(e^{2R}-e^{-2R})\delta_s \\ &\qquad \geq \sum_{\text{real}} \widehat{\varphi_{R,\nu}}(r_n) \geq \sum_{\text{real}} \widehat{\varphi_R}(r_n - \nu) \geq R \cdot m_R \cdot \delta(\nu) \end{aligned}$$

so that

$$\delta(\nu) \leq \left(\frac{\mathrm{vol}(Y)\cdot\varphi(0)}{\pi R \cdot m_R}\right)\nu^2 + \frac{1}{R\cdot m_R}\left(-\frac{\mathrm{vol}(Y)\varphi''(0)}{\pi R^2} + 2\varphi(0)\cdot(e^{2R}-e^{-2R})\delta_s\right).$$

Concretely, using again the test function φ_0, we obtain the following result.

Proposition 5.2. *Let Y be a hyperbolic rational homology sphere with* $\mathrm{vol}(Y) < 6.5$ *and* $\mathrm{inj}(Y) > 0.15$. *Then*

$$\delta_s \leq 637, \quad \delta(\nu) \leq 9.4\nu^2 + 4782.$$

5.4. Local Weyl law for spinors

This case is essentially the same as that of coexact 1-forms. For a given spin^c structure, denote by $\delta^D(\nu)$ the number of Dirac eigenvalues with absolute value in $[\nu, \nu + 1]$.

Choosing again $R = \mathrm{inj}(Y)$, we have

$$\frac{\mathrm{vol}(Y)}{2\pi}\left(\frac{1}{4}\varphi_{R,\nu}(0) - \varphi''_{R,\nu}(0)\right) = \sum_j \frac{1}{2}\widehat{\varphi_{R,\nu}}(s_j).$$

so that

$$\frac{\mathrm{vol}(Y)}{\pi}\left(\frac{1}{4}\varphi(0) - \frac{1}{R^2}\varphi''(0) + \nu^2\varphi(0)\right) \geq \frac{1}{2}R\cdot m_R \cdot \delta^D(\nu)$$

and

$$\delta^D(\nu) \leq \left(\frac{2\,\mathrm{vol}(Y)\cdot\varphi(0)}{\pi R\cdot m_R}\right)\nu^2 + \frac{2\,\mathrm{vol}(Y)}{\pi R\cdot m_R}\left(\frac{1}{4}\varphi(0) - \frac{1}{R^2}\varphi''(0)\right).$$

Specializing using the test function φ_0, we obtain the following result.

Proposition 5.3. *Let Y be a hyperbolic rational homology sphere with* $\mathrm{vol}(Y) < 6.5$ *and* $\mathrm{inj}(Y) > 0.15$. *Then for any spin^c structure,*

$$\delta^D(\nu) \leq 18.7\nu^2 + 2561.3.$$

6. Geometric bounds for the Frøyshov invariant

In this section we will use the local Weyl laws from the previous section to prove bounds on the eta invariants in terms of volume and injectivity radius. Combining this with Proposition 1.1, we will be able to prove Theorem 1.

Remark 6.1. Our goal here is to prove *explicit* bounds; we will not try to optimize the estimates within the set of choices we make, and simply make 'reasonable' choices for the parameters.

For simplicity of notation, assume that we have the inequalities

$$\begin{aligned}
\delta^*(\nu) &\leq A\nu^2 + B && \text{(coexact 1-forms)},\\
\delta_s &\leq C && \text{(0-form eigenvalues} < 1),\\
\delta(\nu) &\leq \frac{A}{2}\nu^2 + D && \text{(0-form eigenvalues} \geq 1),\\
\delta^D(\nu) &\leq A\nu^2 + E. && \text{(spinor eigenvalues)},
\end{aligned}$$

where the specific values of the constants for a manifold with vol < 6.5 and inj > 0.15 were determined in the previous section.

6.1. *Bounds on η_{sign}*

Recall from Section 4 (after setting $L = 1$) that for an arbitrary admissible test function G,

$$\eta_{\mathrm{sign}} = \frac{\int_0^1 G_T\,\frac{dT}{T} + \int_1^\infty G_T\,\frac{dT}{T}}{\int_0^\infty \widehat{G}(T)\,dT}$$

where

$$G_T = \sum_n Tt_n \widehat{G}(Tt_n) = 2 \sum_{1 \neq [\gamma]} \ell(\gamma_0) \cdot \frac{\sin(\mathrm{hol}(\gamma))}{|1 - e^{\mathbb{C}\ell(\gamma)}|\,|1 - e^{-\mathbb{C}\ell(\gamma)}|} \cdot \frac{1}{T} G'\Big(\frac{\ell(\gamma)}{T}\Big).$$

For our purposes, it is convenient to restrict our attention to $G(x) = (\frac{1}{2}\mathbf{1}_{[-1,1]})^{*k}$ for even $k \geq 6$. This function has support in $[-k, k]$, and its Fourier transform is $\widehat{G}(t) = \mathrm{sinc}(t)^k$ where $\mathrm{sinc}(t) := \sin(t)/t$. We will denote

$$c_k := \int_0^\infty \mathrm{sinc}(t)^k\, dt, \quad d_k := \int_1^\infty \mathrm{sinc}(t)^k\, dt.$$

Let us consider the numerator of the expression for η_{sign}; we will evaluate the first term using the geometric side of the trace formula and the second term using the spectral side. Starting with the second term, we have

$$\begin{aligned} \int_1^\infty G_T \frac{dT}{T} &= \int_1^\infty \sum_{t_n} Tt_n \cdot \mathrm{sinc}(Tt_n)^k \frac{dT}{T} \\ &= \int_1^\infty \sum_{t_n} \mathrm{sgn}(t_n) \cdot T|t_n| \cdot \mathrm{sinc}(T|t_n|)^k \frac{dT}{T} \\ &= \sum_{t_n} \mathrm{sgn}(t_n) \cdot \int_{|t_n|}^\infty \mathrm{sinc}(T)^k\, dT. \end{aligned}$$

We split the last sum into $\sum_{|t_n| \leq 2} + \sum_{|t_n| > 2}$. In absolute value, the first part can be bounded above as

$$\begin{aligned} \sum_{|t_n| \leq 2} \int_{|t_n|}^\infty \mathrm{sinc}(T)^k\, dT &\leq \delta^*(0) \cdot \int_0^\infty \mathrm{sinc}(T)^k\, dT + \delta^*(1) \cdot \int_1^\infty \mathrm{sinc}(T)^k\, dT \\ &\leq c_k \cdot B + d_k \cdot (A + B). \end{aligned}$$

For the second part, using $\mathrm{sinc}(T)^k \leq T^{-k}$, we have

$$\begin{aligned} \sum_{|t_n| \geq 2} \int_{|t_n|}^\infty \mathrm{sinc}(T)^k\, dT &\leq \sum_{|t_n| \geq 2} \int_{|t_n|}^\infty \frac{1}{T^k}\, dT \\ &= \frac{1}{k-1} \sum_{|t_n| \geq 2} \frac{1}{|t_n|^{k-1}} = \frac{1}{k-1} \sum_{m=2}^\infty \sum_{|t_n| \in [m, m+1]} \frac{1}{|t_n|^{k-1}} \\ &\leq \frac{1}{k-1} \sum_{m=2}^\infty \delta^*(m) \frac{1}{m^{k-1}} \leq \frac{1}{k-1} \sum_{m=2}^\infty (Am^2 + B) \frac{1}{m^{k-1}} \\ &= \frac{1}{k-1} \big(A(\zeta(k-3) - 1) + B(\zeta(k-1) - 1)\big) \end{aligned}$$

and therefore

$$\left| \int_1^\infty G_T \frac{dT}{T} \right| \leq B \cdot c_k + (A + B) \cdot d_k + \frac{1}{k-1} \big(A(\zeta(k-3) - 1) + B(\zeta(k-1) - 1)\big).$$

For the geometric side, using a simple substitution, we have

$$\int_0^1 G_T \frac{dT}{T} = 2\int_0^1 \sum \ell(\gamma_0)\frac{\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\frac{G'\left(\frac{\ell(\gamma)}{T}\right)}{T}\frac{dT}{T}$$
$$= -2\sum \ell(\gamma_0)\frac{\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\frac{G(\ell(\gamma))}{\ell(\gamma)}$$

As $\ell(\gamma) \geq 2\mathrm{inj}(Y)$, by taking absolute values we have

$$\left|\int_0^1 G_T \frac{dT}{T}\right| \leq \frac{1}{\mathrm{inj}(Y)}\sum \ell(\gamma_0)\frac{1}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}G(\ell(\gamma)).$$

To proceed, we notice that the right hand side is (up to a constant) the geometric side of the trace formula for functions, and we can provide bounds in terms of the spectral density of eigenfunctions. In particular, using Theorem 3.1, we have

$$\sum \ell(\gamma_0)\frac{1}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}G(\ell(\gamma)) = \frac{\mathrm{vol}(Y)}{2\pi}G''(0) + \sum_{r_n}\widehat{G}(r_n)$$
$$\leq \sum_{r_n}\widehat{G}(r_n),$$

because $G''(0)$ is negative. We deal with imaginary and real parameters separately. For imaginary parameters, we have

$$\widehat{G}(it) = \mathrm{sinc}(it)^k = \frac{\sinh(t)^k}{t^k},$$

which is increasing in t and therefore we obtain the inequality

$$\sum_{\text{imaginary } r_n}\widehat{G}(r_n) \leq \sinh(1)^k \cdot \delta_s \leq C\cdot\sinh(1)^k.$$

For the real parameters, using $\widehat{G} \leq 1$ and $\widehat{G} \leq 1/x^k$ respectively we have

$$\sum_{\text{real } r_n}\widehat{G}(r_n) = \sum_{r_n\in[0,1]}\widehat{G}(r_n) + \sum_{m=1}^{\infty}\sum_{r_n\in[m,m+1]}\widehat{G}(r_n)$$
$$\leq \delta(0)\cdot 1 + \sum_{m=1}^{\infty}\delta(m)\cdot\frac{1}{m^k} \leq D + \sum_{m=1}^{\infty}\frac{(A/2)m^2+D}{m^k}$$
$$= \frac{A}{2}\zeta(k-2) + D(\zeta(k)+1).$$

Hence, putting everything in Section 6.1 together, we have

$$\left|\int_0^1 G_T\frac{dT}{T}\right| \leq \frac{1}{\mathrm{inj}(Y)}\left(C\cdot\sinh(1)^k + \frac{A}{2}\zeta(k-2) + D(\zeta(k)+1)\right).$$

Putting everything together, we obtain the following.

Proposition 6.1. *For every even $k \geq 6$, we have*

$$|\eta_{\text{sign}}(Y)| \leq \frac{1}{c_k}\Bigg(B \cdot c_k + (A+B)\cdot d_k + \frac{1}{k-1}\big(A(\zeta(k-3)-1) + B(\zeta(k-1)-1)\big) + \frac{1}{\text{inj}(Y)}\Big(C \cdot \sinh(1)^k + \frac{A}{2}\zeta(k-2) + D(\zeta(k)+1)\Big)\Bigg),$$

where we have used the notation introduced above.

Choosing for example $k = 8$, and plugging in the constants we found in the previous section, we obtain the following explicit estimate.

Corollary 6.2. *If Y is a hyperbolic rational homology sphere with* $\text{vol}(Y) < 6.5$ *and* $\text{inj}(Y) > 0.15$, *then* $|\eta_{\text{sign}}(Y)| \leq 108267$.

6.2. *Bounds on η_{Dir}*

The discussion for the case of spinors is identical, with the final result obtained by substituting the constant D with E. This is because the quantity to take it is based on the identity

$$\begin{aligned} G_T &= \sum_n T s_n \widehat{G}(T s_n) \\ &= 2 \sum_{1 \neq [\gamma]} \ell(\gamma_0) \cdot \frac{\sin(\text{hol}(\widetilde{\gamma})) \cdot \cos(\varphi_{\widetilde{\gamma}})}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot \frac{1}{T} G'\left(\frac{\ell(\gamma)}{T}\right). \end{aligned}$$

Here we will again bound the integral $\int_1^\infty G_T \frac{dT}{T}$ using the bound for the spectral density $\delta^D(\nu)$, and the integral $\int_0^1 G_T \frac{dT}{T}$ using the trace formula for functions. The final result is the following:

Proposition 6.3. *For every even $k \geq 6$, we have*

$$|\eta_{\text{Dir}}(Y)| \leq \frac{1}{c_k}\Bigg(E \cdot c_k + (A+E)\cdot d_k + \frac{1}{k-1}\big(A(\zeta(k-3)-1) + E(\zeta(k-1)-1)\big) + \frac{1}{\text{inj}(Y)}\Big(C \cdot \sinh(1)^k + \frac{A}{2}\zeta(k-2) + D(\zeta(k)+1)\Big)\Bigg)$$

for all spin^c structures.

Setting again $k = 8$, we have the following explicit estimate.

Corollary 6.4. *If Y is a hyperbolic rational homology sphere with* $\text{vol}(Y) < 6.5$ *and* $\text{inj}(Y) > 0.15$, *then* $|\eta_{\text{Dir}}(Y)| \leq 108249$ *for every spin^c structure.*

6.3. *Proof of Theorem 1*

Given our discussion, Theorem 1 follows directly from the fact that for a minimal hyperbolic L-space,

$$h(Y, \mathfrak{s}) = -\eta_{\text{sign}}/8 - \eta_{\text{Dir}}/2$$

(see Proposition 1.1).

7. A concrete example: the Weeks manifold

In order to prove Theorem 1, we used test functions supported in $[-2 \cdot \text{inj}, 2 \cdot \text{inj}]$, as the only geometric input we had about the length spectrum was the injectivity radius. On the other hand, in specific examples one can access very concrete information about the length spectrum, and therefore one can apply the trace formula to a much larger class of test functions. In turn, one can use this to provide explicit computation of Frøyshov invariants. In this section, we show how this approach can be implemented in a simple example of minimal hyperbolic L-space, the Weeks manifold W; a similar approach works also for other small volume hyperbolic minimal L-spaces in the Hodgson–Weeks census we discussed in [32].

Recall that $H_1(\mathsf{W}, \mathbb{Z}) = (\mathbb{Z}/5\mathbb{Z})^2$. In our discussion, we will not discuss explicitly error bounds, to keep the section streamlined; we will deal with rigorous estimates of errors in our approximations when dealing with the Seifert–Weber dodecahedral space in the proof of Theorem 2. To this end, one can think of this section as both a warm-up exercise and a sanity check – the latter because the Frøyshov invariants of W can be computed directly with other purely topological methods. We have indeed the following.

Proposition 7.1. *The unique spin structure on* W *has Frøyshov invariant* $-1/2$*. For the remaining* 24 *spin*^{c} *structures the Frøyshov invariant is either* $\pm 1/10$*, or* $\pm 3/10$*, and each value is the invariant of exactly six spin*^{c} *structures.*

This is essentially showed in [33], using the fact that W is the branched double cover of the knot 9_{49} [36]. The key point is that the 9_{49} differs from an alternating knot only for an extra twist, and the authors showed that under favorable circumstances this allows computing the Frøyshov invariants in terms of a Goeritz matrix for the knot (generalizing the method of Ozsváth and Szabó [48], which applies to alternating knots).

Going back to our spectral approach, the signature eta invariant was computed in [15] to be

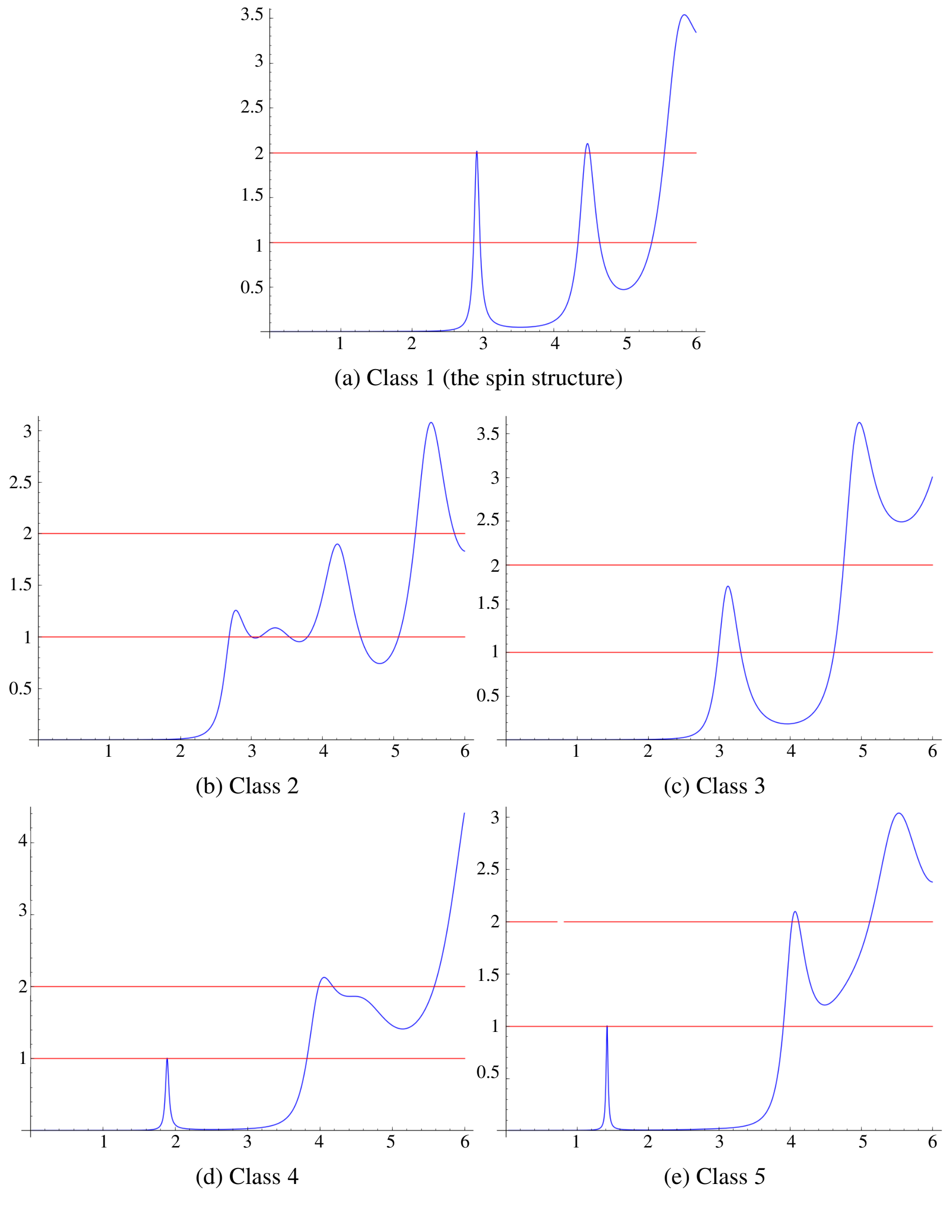
$$\eta_{\text{sign}} = 0.040028711\ldots$$ [3]

We compute the eta invariant for the Dirac operators using our explicit expression from Section 4. In order to do so, we first compute the spin^c length spectra up to cutoff $R = 6.5$ using the algorithm described in Section 2 (applied to data obtained from SnapPy). We then take the approach from [32] and use the even trace formula to obtain information about the spectrum. More specifically, using ideas of Booker and Strombergsson, for each spin^c structure $\mathfrak{s}$ we determine an explicit function

$$J_{\mathfrak{s}} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$$

with the property that if $\pm s$ are eigenvalues of the Dirac operator whose multiplicities add to m, then $J_{\mathfrak{s}}(|s|) \geq m$. The pictures in the range $[0, 6]$ can be found in Figure 1; Class 1

[3] A good approximation of this value can also be computed directly using our approach; we will see this in detail in the case of the Seifert–Weber dodecahedral space in the next section.



(a) Class 1 (the spin structure)

(b) Class 2

(c) Class 3

(d) Class 4

(e) Class 5

Fig. 1. Pictures for the Weeks manifold.

is the class of the spin structure, and the remaining 24 spin^c structures can be grouped in four groups (each consisting of six elements) with identical picture.

Remark 7.1. In fact, this is a consequence of the action of the isometry group D_{12} on the set of spin^c structures [36]. We will exploit symmetry under the isometry group in the more complicated case of SW in Section 10.

We compute an approximation to η_{Dir} via the formula (14) taking G to be a Gaussian (see Lemma 4.1). In this process, we only consider the sum over geodesics with length ≤ 6.5, and only sum over eigenvalues where a precise guess can be made. More specifically:

- In Classes 1, 4, 5, we consider the contribution of the smallest eigenvalue as suggested by the picture (and consider the rest of the spectrum as an error term). Let us point out that one can in principle use the method of [31] to actually prove that an eigenvalue is with the right multiplicity in the tiny window suggested by the picture; here we also use the fact that in the spin case eigenvalues always have even multiplicity, because the corresponding Dirac operator is quaternionic linear. The sign of the eigenvalue can also be determined by using the odd trace formula in Theorem 3.3.
- In Classes 2 and 3, where a precise value for the first eigenvalue is not available, we simply consider the whole spectrum to be an error term, and take a more dilated Gaussian (which has a narrower Fourier transform) to make such error smaller. This works well because there is a good spectral gap (but increases of course the error coming from truncating the sum over the geodesics).

Using formula (14) with $L = 1$, we have obtained the following approximate values for $h = -\eta_{\mathrm{sign}}/8 - \eta_{\mathrm{Dir}}/2$:

Spin^c class	Approximate value of h
1	$-0.49997\dots$
2	$0.09971\dots$
3	$-0.09994\dots$
4	$0.29994\dots$
5	$-0.29994\dots$

which are very close to the exact values in Proposition 7.1.

In order to make this approximate computation into an actual proof of Proposition 7.1, we need to provide estimates on the terms we did not take into account in the sum. Let us discuss the various steps:

(1) We know that the Frøyshov invariants are rational numbers, and in fact information about their fractional part can be obtained by purely topological means. For example, in the case of Weeks, one can show a priori that all Frøyshov invariants have the form $(2n + 1)/10$. Given this, one only needs to prove that the errors are less than $\frac{1}{20}$. In fact, by taking into account additional topological information coming from the linking form, one only needs to prove much weaker estimates for the error; this will be crucial in our proof of Theorem 2.

(2) To bound the error resulting from truncating the sum over geodesics, we need to have a good understanding of the geodesics with length in a certain interval; in a specific example, this can be done concretely using the trace formula for functions (in the same spirit as the prime geodesic theorem with errors [11]).

(3) To bound the error on the spectral side, the key observation is that large eigenvalues contribute very little to the error. For example (thinking about Classes 1, 4 and 5), picking $G(x) = e^{-x^2/2}$ (for which $\widehat{G}(t) = \sqrt{2\pi}e^{-x^2/2}$ and $\int_0^\infty \widehat{G}(t)\,dt = \pi$) and $L = 1$, an eigenvalue with $|t| > 3.5$ contributes to the eta invariant at most

$$\frac{1}{\pi}\int_{3.5}^{\infty} \sqrt{2\pi}e^{-x^2/2}\,dx = 0.00046\ldots.$$

The key point is then to get reasonable bounds on the spectral density (sharper than those in Section 5).

While we will not pursue the details of this approach here for the Weeks manifold, we will do so in subsequent sections for the more challenging case of the Seifert–Weber manifold in order to prove Theorem 2. In particular, each of the next three sections will address one of the three aspects discussed above. Notice that in the case of (2) and (3), the error estimate can be improved by computing a larger portion of the (spin^c) length spectrum.

Remark 7.2. Let us point out that combining Propositions 7.1 and 1.1 we obtain a formula for the Dirac eta invariants η_{Dir} of the Weeks manifold in terms of the signature eta invariant η_{sign}. In particular, this allows us to provide a computation of η_{Dir} which is as precise as the one provided by Snap; for example, as mentioned in the introduction we have

$$\eta_{\mathrm{Dir}} = 0.989992\ldots$$

for the unique spin structure on the Weeks manifold.

8. The linking form of the Seifert–Weber dodecahedral space SW

In this section we compute the linking form of the Seifert–Weber dodecahedral space SW. This will be the key (and only) topological input for our computation of its Frøyshov invariants, and will give us concrete information on their fractional parts. Recall that for oriented rational homology three-spheres, the linking form is the bilinear form

$$Q : H_1(Y,\mathbb{Z}) \times H_1(Y,\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

defined geometrically as follows: given elements x, y, choose n for which $ny = 0$ and pick a chain T with $\partial T = ny$. Then $Q(x, y)$ is the intersection number of x and T, divided by n. More abstractly, the corresponding map

$$Q^\# : H_1(Y,\mathbb{Z}) \to \mathrm{Hom}(H_1(Y,\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong H^1(Y,\mathbb{Q}/\mathbb{Z})$$

is the composition

$$H_1(Y,\mathbb{Z}) \cong H^2(Y,\mathbb{Z}) \cong H^1(Y,\mathbb{Q}/\mathbb{Z}),$$

where the first isomorphism is Poincaré duality and the second is the Bockstein homomorphism for the short exact sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

In particular, $Q^\#$ is an isomorphism and Q is non-degenerate.

Proposition 8.1. *For the explicit basis a, b, c of $H_1(\mathsf{SW}) \cong (\mathbb{Z}/5\mathbb{Z})^3$ identified in the proof below, the linking form Q_{SW} is represented by the matrix*

$$\begin{pmatrix} 0 & 2/5 & 3/5 \\ 2/5 & 0 & 2/5 \\ 3/5 & 2/5 & 0 \end{pmatrix}.$$

Remark 8.1. In fact, one can show that (up to scalar) every non-degenerate form has that shape with respect to some basis. But our point is that we also can identify the basis a, b, c geometrically in this specific case. Our computation also suggests that the linking form of a hyperbolic three-manifold is computable by taking a Dirichlet domain with face-pairing maps as input, so that the approach to the computation of the Frøyshov invariants of SW we will discuss in the upcoming sections works more generally.

Proof of Proposition 8.1. Recall that SW is obtained from a dodecahedron by identifying opposite faces by $3/5$ of a full counterclockwise rotation. The identification is described explicitly in [18]; see Figure 2. We find that:

- the 20 vertices are identified to a single point;
- the 30 edges are grouped into in 6 groups of 5 elements each, and we denote the corresponding generators of homology a, b, c, d, e, f.
- the 12 faces are identified in pairs as follows: $1 \leftrightarrow 12$, $2 \leftrightarrow 9$, $3 \leftrightarrow 10$, $4 \leftrightarrow 11$, $5 \leftrightarrow 7$, $6 \leftrightarrow 8$.

The six pairs of faces give us relations between the generators of the first homology. We will denote the corresponding chain by F_i, where i is the smallest of the two indices, and orient it with the opposite orientation as the one inherited as a subset of the plane. We get

$$\begin{array}{llllllll}
(\partial F_1) & a & +b & +c & +d & +e & & =0, \\
(\partial F_2) & -a & -b & +c & & +e & +f & =0, \\
(\partial F_3) & a & -b & -c & +d & & +f & =0, \\
(\partial F_4) & & b & -c & -d & +e & +f & =0, \\
(\partial F_5) & a & & +c & -d & -e & +f & =0, \\
(\partial F_6) & -a & +b & & +d & -e & +f & =0.
\end{array}$$

Simple elementary row operations reduce this system to

$$\begin{aligned} d &= a + 2b + 3c, \\ e &= 3a + 2b + c, \\ f &= 3a + 4b + 3c, \end{aligned}$$

and

$$5a = 5b = 5c = 0. \tag{17}$$

We therefore see that a, b, c generate the first homology $H_1(\mathsf{SW}) \cong (\mathbb{Z}/5)^3$. Furthermore, we can interpret equations (17) as the geometric identities

$$5a = \partial(F_1 - F_2 + F_3 + F_5 - F_6),$$

$$5b = \partial(F_1 - F_2 - F_3 + F_4 + F_6),$$
$$5c = \partial(F_1 + F_2 - F_3 - F_4 + F_5).$$

To compute the linking form it is convenient to introduce a second basis of the homology: let A, B, C be the generators corresponding to the oriented segments connecting the centers of the faces 12 to 1, 9 to 2 and 10 to 3 respectively. Using Figure 2, it is easy to compute that at the level of homology,

$$A = 4a + 2b + 4c,$$
$$B = 3a + b + 4c,$$
$$C = 4a + b + 3c.$$

For our choice of orientation, A has intersection $+1$ with F_1, and 0 with the other pairs; the analogous statement holds for B and F_2 and C and F_3 respectively. Using the geometric descriptions of chains bounding for $5a, 5b$ and $5c$ above, we therefore get the following values for the linking numbers between elements in A, B, C and a, b, c.

lk	a	b	c
A	$1/5$	$1/5$	$1/5$
B	$-1/5$	$-1/5$	$1/5$
C	$1/5$	$-1/5$	$-1/5$

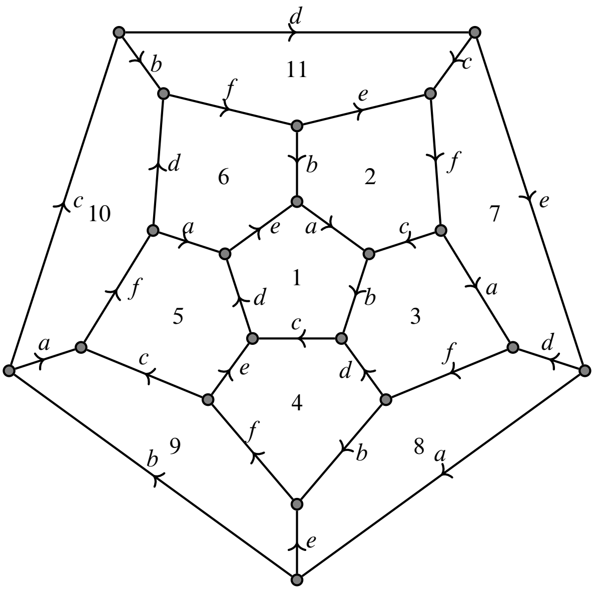


Fig. 2. A graphic depiction of the Seifert–Weber dodecahedral space. The twelfth face is the one formed by the five external edges, and we consider it to be on top of the others.

Finally, a simple change of basis to express everything in terms of the basis a, b, c concludes the proof. ∎

A useful observation for what follows is that $Q_{\mathsf{SW}}(x, x)$ for $x \neq 0$ attains possible values as in the following table.

$Q_{\mathsf{SW}}(x, x)$	Number of $x \neq 0$ that attain the value
0	24
1/5	30
2/5	20
3/5	20
4/5	30

Another useful observation can be made by looking at the isometry group of SW. Recall that the latter is isomorphic to the symmetric group S_5, and acts faithfully on the first homology group [37]. In fact, from the description in [37] we see that the natural map

$$\mathrm{Isom}(\mathsf{SW}) \hookrightarrow \mathrm{O}(Q_{\mathsf{SW}})$$

has image contained in $\mathrm{SO}(Q_{\mathsf{SW}})$. The latter has 120 elements, and we get therefore an isomorphism $\mathrm{Isom}(\mathsf{SW}) \cong \mathrm{SO}(Q_{\mathsf{SW}})$. We have therefore the following.

Corollary 8.2. *Consider two spin^c structures $\mathfrak{s}$, $\mathfrak{s}'$ on* SW *which are not spin. Then there is an isometry φ for which $\varphi^*\mathfrak{s} = \mathfrak{s}'$ if and only if*

$$Q_{\mathsf{SW}}(c_1(\mathfrak{s}), c_1(\mathfrak{s})) = Q_{\mathsf{SW}}(c_1(\mathfrak{s}'), c_1(\mathfrak{s}')).$$

Proof. This follows by the Witt extension theorem [49, Theorem 1.5.3]: because Q_{SW} is non-degenerate, given two elements $x, y \neq 0$ satisfying $Q(x) = Q(y)$, there is $A \in \mathrm{SO}(Q_{\mathsf{SW}})$ for which $Ax = y$. ∎

9. Explicit estimates for closed geodesics on SW

For practical purposes, the most effective way to evaluate the eta invariants using an expression such as (14) is to use a suitably dilated Gaussian (as demonstrated by the precise computations for the Weeks manifold in Section 7). The main complication is that the function is not compactly supported, and therefore we need to provide explicit bounds on the tail sums

$$\sum_{\ell(\gamma)\geq R} \ell(\gamma_0) \cdot \frac{2\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot \frac{1}{\ell} \cdot (-G(\ell/L)).$$

for the signature case and

$$\sum_{\ell(\gamma)\geq R} \ell(\gamma_0) \cdot \frac{2\sin(\mathrm{hol}(\tilde{\gamma}))\cdot\cos(\varphi_{\tilde{\gamma}})}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot (-G(\ell/L)),$$

in the Dirac case respectively. Here L is the parameter at which we split the integral defining the eta function, and R is the cutoff for geodesics (later on we will take them to be 1.7 and 7.5 respectively). For these purposes, we need a good understanding of the behavior of the lengths of geodesics, and more specifically of the quantity

$$\sum_{\ell(\gamma)\in[T-1/2,T+1/2]} \ell(\gamma_0).$$

To put this in context, the quantity

$$\sum_{\ell(\gamma)\leq T} \ell(\gamma_0) \tag{18}$$

naturally appears when studying the asymptotic number of prime geodesics, and in particular it plays a key role in the proof of the prime geodesic theorem

$$\#\{\text{prime geodesics } \gamma \text{ with } \ell(\gamma)\leq T\} \sim e^{2T}/(2T)$$

(see for example [13]). In fact, under the heuristic correspondence between closed geodesics on a hyperbolic manifold and prime numbers (under which $e^{\ell(\gamma)}$ is the analogue of p), the quantity (18) corresponds to the Chebyshev function

$$\psi(x) = \sum_{p^k\leq x} \log(p).$$

It is well-known [1, Chapter 4] that the prime number theorem

$$\#\{\text{prime numbers } p\leq x\} \sim x/\log(x)$$

is equivalent to the asymptotic

$$\psi(x) \sim x.$$

Precise asymptotics for (18) can be obtained by evaluating the trace formula for functions in Theorem 3.1 for a smoothed version of the function $\cosh(t)\mathbf{1}_{[-T,T]}$ (see also [11] for the case of hyperbolic surfaces). In our case, in order to obtain reasonable explicit constants, we will use instead combinations of Gaussian functions. Furthermore, while the general results of Section 5.3 apply to our case, we can obtain considerably sharper bounds for the spectrum using as input our knowledge of the length spectrum of SW up to cutoff $R = 8$ (rather than just the injectivity radius), as in [31]. Furthermore, we have computed the spin^c length spectrum of SW for all spin^c structures up to cutoff $R = 7.5$ using the method described in Section 2.

9.1. The spectral gap for the Laplacian on functions

The first step is to understand the spectral gap for the Laplacian on functions – as we saw in Section 5.3, small eigenvalues are the main cause of large error estimates. In our case, we have the following result.

Proposition 9.1. *The Laplacian on functions for* SW *has no small eigenvalues, and the smallest parameter satisfies* $r_1 > 2.8$*, corresponding to* $\lambda_1 > 8.84$.

Remark 9.1. The absence of small eigenvalues in the statement of Proposition 9.1 was to be expected, as it is a consequence of the generalized Ramanujan conjecture for GL_2. See for example [6].

The proof of the above proposition is based on the Booker–Strombergsson method applied to the trace formula for functions (which is indeed, in the case of surfaces, the original setup of their approach [7]). The main difference is that we now get two pictures, one for the imaginary parameters (Figure 3) and one for the real ones (Figure 4).[4] In both

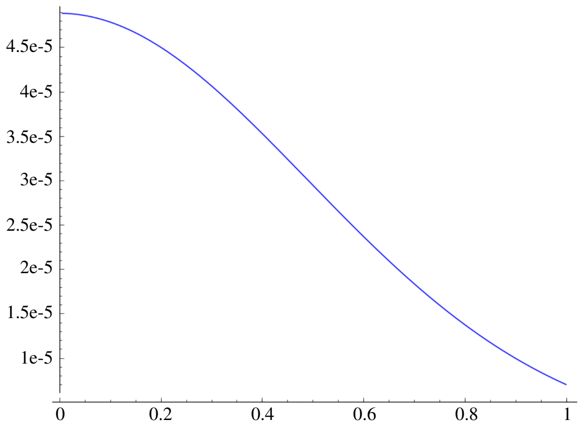


Fig. 3. The picture for the imaginary parameters for the Laplacian on functions on SW.

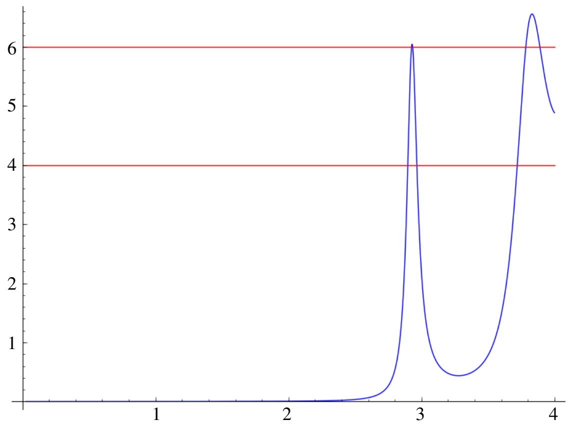


Fig. 4. The picture for the real parameters for the Laplacian on functions on SW.

[4] Note the very different scales of Figures 3 and 4.

cases, to exclude the parameter t, we minimized the quantity

$$\sum_{n=1}^{\infty} \widehat{H}(r_n) \tag{19}$$

over the same space of functions used in [32], subject to the constraint $\widehat{H}(t) = 1$, using the trace formula in Theorem 3.1. Here the cutoff is $R = 8$, as in [31]. Notice that $r_0 = i$ is always a spectral parameter (corresponding to $\lambda_0 = 0$), and is not included in the sum (19).

9.2. *Spectral density on functions*

We now use the spectral gap and our explicit knowledge of the length spectrum to provide refined bounds on the spectral density. These two extra ingredients will allow us to greatly improve the estimates from Section 5. Let us apply the trace formula to

$$H_\nu = \left(\tfrac{1}{2}\mathbf{1}_{[-1,1]}\right)^{*6} \cdot (e^{i\nu x} + e^{-i\nu x}) = 2 \cdot \left(\tfrac{1}{2}\mathbf{1}_{[-1,1]}\right)^{*6} \cdot \cos(\nu x),$$

which is the same kind of function we used in Section 5. In this case, we have

$$\begin{aligned} -H_\nu''(0) &= 2 * \left(\left(\left(\tfrac{1}{2}\mathbf{1}_{[-1,1]}\right)^{*6}\right)''(0) - \nu^2 \cdot \left(\tfrac{1}{2}\mathbf{1}_{[-1,1]}\right)^{*6}(0)\right) \\ &= \tfrac{11}{20}\nu^2 + \tfrac{1}{4}, \end{aligned}$$

and

$$\widehat{H_\nu}(t) = \left(\frac{\sin(t+\nu)}{t+\nu}\right)^6 + \left(\frac{\sin(t-\nu)}{t-\nu}\right)^6.$$

We will use the fact that

$$\left(\frac{\sin(x)}{x}\right)^6 \geq 0.777 \quad \text{for } x \in [-0.5, 0.5].$$

To prove upper bounds, we will look again at the trace formula. In our setup, we can compute explicitly an upper bound for the sum over geodesics

$$\begin{aligned} \left|\sum_\gamma \frac{\ell(\gamma_0)}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} H_\nu(\ell(\gamma))\right| &\leq \sum_\gamma \frac{\ell(\gamma_0)}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} H_0(\ell(\gamma)) \\ &\leq 4.827 \end{aligned}$$

because H_0 is supported in $[-6, 6]$ and we know the length spectrum up to cutoff $R = 8$. Furthermore, recalling that

$$|\sin(i+\nu)|^2 = \cos(\nu)^2 \sinh(1)^2 + \sin(\nu)^2 \cosh(1)^2 = \sinh(1)^2 + \sin(\nu)^2,$$

we see that the contribution of the zero eigenvalue is bounded above by

$$2 \cdot \left(\frac{\sinh(1)^2 + \sin(\nu)^2}{1+\nu^2}\right)^3 \tag{20}$$

We are interested in upper bounds for the number of spectral parameters in $[\nu - 1/2, \nu + 1/2]$ for $T \geq 3$. The quantity (20) is bounded above by 0.0055 for $\nu \geq 3$. Putting everything together, and recalling that SW has volume about 11.19, we get the following refined local Weyl law.

Proposition 9.2. *For the eigenvalues of the Laplacian on functions on* SW, *we have the upper bound*

$$\#\{\textit{parameters in } [\nu - 1/2, \nu + 1/2]\} \leq 1.262\nu^2 + 6.793 \quad \textit{for } \nu \geq 3.$$

9.3. Bounds for closed geodesics

We will apply the trace formula of Theorem 3.1 to the function

$$F_{c,T} = e^{-c(x-T)^2/2} + e^{-c(x+T)^2/2},$$

which has Fourier transform

$$\widehat{F}_{c,T} = \sqrt{2\pi/c} \cdot e^{-x^2/(2c)} \cdot (e^{iTx} + e^{-iTx}) = 2\sqrt{2\pi/c} \cdot e^{-x^2/(2c)} \cos(Tx).$$

Here $c > 0$ is a parameter to be determined later. Let us discuss its value at the various terms in the trace formula, starting from the spectral side. The contribution of the zero eigenvalue (corresponding to $r_0 = i$) is

$$2\sqrt{2\pi/c} \cdot e^{1/(2c)} \cosh(T).$$

Using Proposition 9.1, the contribution of the real parameters is bounded above (independenty of T) by

$$\begin{aligned} \sum_{r_n \text{ real}} \widehat{F}_{c,T}(r_n) &= \sum_{n=3}^{\infty} \sum_{r_n \in [n-1/2, n+1/2]} \widehat{F}_{c,T}(r_n) \\ &\leq \sum_{n=3}^{\infty} (1.262n^2 + 6.793) \cdot 2\sqrt{2\pi/c} \cdot e^{-\frac{(n-1/2)^2}{2c}}. \end{aligned} \tag{21}$$

The identity contribution is

$$-\frac{\operatorname{vol}(Y)}{2\pi} F''_{c,T}(0) = \frac{\operatorname{vol}(Y)}{\pi} \cdot e^{-cT^2/2} \cdot (c - c^2T^2) \tag{22}$$

(this is negative provided $T^2 > 1/c$).

All these terms are explicitly computable, and we will use this information to provide upper bounds on

$$\sum_{\ell(\gamma) \in [T-1/2, T+1/2]} \ell(\gamma_0)$$

as follows. Notice that

$$|1 - e^{\mathbb{C}\ell(\gamma)}|\,|1 - e^{-\mathbb{C}\ell(\gamma)}| \leq (e^{\ell} + 1)(1 + e^{-\ell}) = 2\cosh(\ell) + 2$$

and

$$F_{c,T}(\ell) \geq e^{-c/8} \quad \text{for } \ell \in [T-1/2, T+1/2].$$

We therefore have

$$\begin{aligned}\sum_{\gamma} \frac{\ell(\gamma_0)}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} F_{c,T}(\ell(\gamma)) \\ &\geq \sum_{\ell(\gamma)\in[T-1/2,T+1/2]} \frac{\ell(\gamma_0)}{2\cosh(T+1/2)+2} e^{-c/8} \\ &= \frac{e^{-c/8}}{2\cosh(T+1/2)+2} \cdot \sum_{\ell(\gamma)\in[T-1/2,T+1/2]} \ell(\gamma_0),\end{aligned}$$

and we can use the explicit upper bound for the first expression, obtained by combining the trace formula with the estimates (21) and (22), to get an upper bound for the sum $\sum_{\ell(\gamma)\in[T-1/2,T+1/2]} \ell(\gamma_0)$, depending on a parameter c. We see empirically that we obtain the best estimate for $c = 5$, and we have the following:

Proposition 9.3. *For $T \geq 7.5$, we have*

$$\sum_{\ell(\gamma)\in[T-1/2,T+1/2]} \ell(\gamma_0) \leq A(T),$$

where $A(T)$ is an explicit expression in T readily obtained from the discussion above.

We do not write the explicit expression of $A(T)$ as it is quite long and not particularly illuminating, but we remark that the leading term is of the form $C \cdot e^{2T}$.

9.4. The geometric error for η

Finally, we will conclude by bounding the error in the evaluation of the eta invariant using the standard Gaussian $G(x) = e^{-x^2/2}$ in the (14), and evaluating the geometric side up to length $R = 7.5$. We have $\widehat{G}(t) = \sqrt{2\pi} e^{-t^2/2}$, and

$$\int_0^\infty \widehat{G}(t)\, dt = \pi;$$

these errors correspond to

$$\frac{2}{\pi} \sum_{\ell(\gamma)\geq 7.5} \ell(\gamma_0) \cdot \frac{\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{\mathbb{C}\ell(\gamma)}|} \cdot \frac{1}{\ell(\gamma)} \cdot (-G(\ell/L))$$

for the signature case, and

$$\frac{2}{\pi} \sum_{\ell(\gamma)\geq 7.5} \ell(\gamma_0) \cdot \frac{\sin(\mathrm{hol}(\widetilde{\gamma})) \cdot \cos(\varphi_{\widetilde{\gamma}})}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot \frac{1}{\ell(\gamma)} \cdot (-G(\ell/L))$$

for the Dirac operator; again, L is the point at which we split the integral. We can therefore bound the error in the geometric side in all cases by

$$\frac{2}{\pi}\sum_{\ell(\gamma)\geq 7.5}\frac{\ell(\gamma_0)}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\frac{G(\ell(\gamma)/L)}{\ell(\gamma)}.$$

We can break the sum into two parts, one taking into account $\ell(\gamma)\in[7.5,8]$ and one for $\ell(\gamma)\geq 8$. The first part can be computed explicitly because we know the (standard) length spectrum of SW up to cutoff $R=8$. For the second part, noticing that

$$|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|\geq(e^{\ell}-1)(1-e^{-\ell})=2\sinh(\ell)-2,$$

we have

$$\begin{aligned}\frac{2}{\pi}\sum_{\ell(\gamma)\geq 8}\frac{\ell(\gamma_0)}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\frac{G(\ell(\gamma)/L)}{\ell(\gamma)}&\\ \leq\frac{1}{\pi}\sum_{n=8}^{\infty}\frac{e^{-n^2/(2L^2)}}{(\sinh(n)-1)\cdot n}&\cdot\sum_{\ell(\gamma)\in[n,n+1]}\ell(\gamma_0)\\ =\frac{1}{\pi}\sum_{n=8}^{\infty}\frac{e^{-n^2/(2L^2)}}{(\sinh(n)-1)\cdot n}&\cdot A(n+1/2),\end{aligned}$$

where A is the quantity from Proposition 9.3. Putting the pieces together, we obtain an estimate on the total error for the truncated sum depending on the choice of the parameter L. It is clear that the larger the value of L, the better our estimate for the error in the truncated sum will be; on the other hand, we will see in the next section that larger values of L lead to significantly worse errors coming from the spectral side of the formula for the eta invariant in (14). We empirically found that the value $L=1.7$ provides a very good bound for the sum of these two errors in our situation. In this case, for the geometric side we have the following.

Proposition 9.4. *When evaluating the η invariant for either the signature or the Dirac operator using a standard Gaussian $G(x)=e^{-x^2/2}$, splitting point $L=1.7$ and length cutoff $R=7.5$, the error coming from truncating the sum on the geometric side is bounded above by* 0.0376.[5]

10. The Frøyshov invariants of SW

In this section, we finally prove Theorem 2. Our proof is based on the fact that the Seifert–Weber is a minimal hyperbolic L-space [31]. More generally, the approach of this final section can in principle be adapted to any minimal hyperbolic L-space, provided the linking form and a good portion of the length spectrum have been computed.

[5]For the evaluation of this sum, we have used the fact that the general term decays extremely rapidly, as $A(n)=O(e^{2n})$; for example, the term $n=31$ is of the order of 10^{-57}.

10.1. The signature eta invariant

We have the following.

Proposition 10.1. *The signature eta invariant of* SW *is* $\eta_{\text{sign}} = 1.31111111\ldots.$

Our result will be obtained by combining the Chern–Simons computations obtained via SnapPy, together with computations involving the trace formula. In principle, this result can be obtained directly via Snap [15]; unfortunately, we were not able to run the software on our laptops. Also, the proof provides a good example of our technique, in a simpler setup than the Dirac case (where our understanding of the spectrum is less precise).

Proof of Proposition 10.1. SnapPy computes the Chern–Simons invariant of SW to be

$$\mathrm{cs} = -0.033333\ldots.$$

The relation

$$3\eta_{\text{sign}} = 2\mathrm{cs} + \tau \bmod 2\mathbb{Z}$$

holds [3], where τ, which is the number of 2-primary summands in $H_1(\mathsf{SW}, \mathbb{Z})$, is zero in our case. Thus

$$\eta_{\text{sign}} = -0.022222\ldots + \tfrac{2}{3}n \tag{23}$$

for some $n \in \mathbb{Z}$. Therefore, approximating the eta invariant to within error less than $\frac{1}{3}$ pins down its value to high accuracy. Approximating η_{sign} well is straightforward because we have a good understanding of the small coexact 1-form eigenvalues on SW (see Figure 5, which we used in [31] to show $\lambda_1^* > 2$):

- The smallest eigenspace corresponds to

$$\sqrt{\lambda_1^*} \in [1.42787720680237, 1.43033743858337] \tag{24}$$

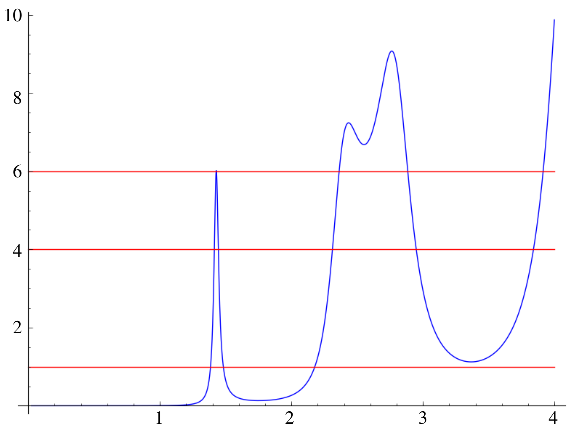


Fig. 5. The function $J_{\mathsf{SW}}(t)$ for coexact 1-forms, computed with cutoff $R = 8$ (cf. [31]).

and has multiplicity exactly 6 (see [31]). Furthermore, using the odd trace formula one readily shows that this eigenvalue is positive.

- The next spectral parameter is larger than 2.

We compute an approximation of the eta invariant using $G(x) = e^{-x^2/2}$ and $L = 1.7$ in (14), where

- we truncate the geometric sum at $R = 7.5$;
- in the spectral sum we approximate the first eigenvalue (with multiplicity 6) with the midpoint $\widetilde{\sqrt{\lambda_1^*}}$ of the interval in (24), and consider the remaining part of the spectral sum as an error term.

As $\int_0^\infty \widehat{G}(t)\,dt = \pi$, we have the approximation

$$\begin{aligned}\eta_{\text{sign}} &\approx \frac{1}{\pi}\biggl(6\cdot\int_{1.7\cdot\widetilde{\sqrt{\lambda_1^*}}}^{\infty}\sqrt{2\pi}e^{-t^2/2}\,dt\\ &\qquad -\sum_{\ell(\gamma)\le 7.5}\ell(\gamma_0)\cdot\frac{2\sin(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|}\cdot\frac{1}{\ell(\gamma)}\cdot e^{-\frac{\ell^2}{2}\cdot(1.7)^2}\biggr)\\ &= 1.31102382358\ldots,\end{aligned}$$

which is very close to the expression (23) for $n = 2$. All we need to do is provide an error bound for our computation.

The error arising from truncating the geometric sum at $R = 7.5$ is bounded above by 0.0376 by Proposition 9.4, while the error arising from approximating the value of the first eigenvalue is bounded above by

$$\frac{6}{\pi}\cdot\int_{1.7\times 1.427877}^{1.7\times 1.4303375}\sqrt{2\pi}e^{-t^2/2}\,dt \le 0.00105.$$

To estimate the error arising from truncating the spectral sum at the first eigenvalue, we can refine the estimates from Section 5 (which only involved volume and injectivity radius) because we have a direct knowledge of the length spectrum. In particular, we can use the test function

$$\cos(Tx)\cdot\left(\frac{1}{2\delta}\mathbf{1}_{[-\delta,\delta]}\right)^{*6}\quad\text{where } \delta = 8/6$$

and compute explicitly the value of the sum over geodesics (as we know it up to cutoff $R = 8$), and use it to provide upper bounds on the number of eigenvalues in specific intervals. In particular:

- there are at most 22 spectral parameters in $[2, 3]$, and each contributes at most
$$\frac{1}{\pi}\int_{1.7\times 2}^{\infty}\sqrt{2\pi}e^{-t^2/2}\,dt < 0.00068;$$
- there are at most 21 spectral parameters in $[3, 4]$, and each contributes at most
$$\frac{1}{\pi}\int_{1.7\times 3}^{\infty}\sqrt{2\pi}e^{-t^2/2}\,dt < 3.4\times 10^{-7};$$

- it is clear that the contribution of larger eigenvalues is negligible, as their number grows quadratically but the contribution decays superexponentially.

Taking all this into account, we obtain an error of at most 0.054, and the result follows. ∎

Remark 10.1. As the numerical result suggests, η_{sign} is a rational number. This follows because the invariant trace field $\mathbb{Q}(\sqrt{-1-2\sqrt{5}})$ of SW is CM-embedded (see [43]). In fact, one can show that

$$\eta_{\text{sign}} = 59/45$$

using the multiplicativity of the Chern–Simons invariants under covers, and the fact that SW admits a 60-fold cover with an orientation-reversing isometry (for which η_{sign} is therefore 0).

10.2. The spin structure

We now focus on the computation of $h(\mathsf{SW}, \mathfrak{s}_0)$ where $\mathfrak{s}_0$ is the unique spin structure on SW. This is the simplest spin^c structure to handle because

$$-2h(\mathsf{SW}, \mathfrak{s}_0) = \eta_{\text{sign}}/4 + \eta_{\text{Dir}} = n/4$$

for some $n \in \mathbb{Z}$. To see this, recall the classical theorem that every spin three-manifold bounds a compact spin four-manifold $(X, \mathfrak{s}_X)$; in fact, we can choose X to be simply connected [24, Chapter VII] (this will be useful later). Then, looking at the expression of the absolute grading (3) modulo integers, we have

$$\begin{aligned}\eta_{\text{sign}}/4 + \eta_{\text{Dir}} &= \tfrac{1}{4}(c_1(\mathfrak{s}_X)^2 - 2\chi(X) - 3\sigma(X) - 2) \bmod \mathbb{Z}\\ &= \tfrac{1}{4}(-2\chi(X) - 3\sigma(X) - 2) \bmod \mathbb{Z}.\end{aligned}$$

as $\text{gr}(X^*, \mathfrak{s}_X, [\mathfrak{b}])$ is an integer (it is the expected dimension of a moduli space) and $c_1(\mathfrak{s}_X)$ is a torsion class, and the claim follows.

Therefore, we only need to approximate η_{Dir} with a good error. The spectral picture can be found in Figure 6. Here, we computed the spin length spectrum up to cutoff $R = 7.5$ using the algorithm described in Section 2.

For a spin structure, the Dirac operator is quaternionic and therefore all eigenspaces have even multiplicity. We can therefore conclude that $|s_1| \geq 1.45$. We compute an approximation to the eta invariant using $G(x) = e^{-x^2/2}$ and $L = 1.7$ by truncating the geometric sum at $R = 7.5$ and considering the whole spectral side as an error term. We obtain

$$\begin{aligned}\eta_{\text{Dir}} &\approx -\frac{1}{\pi}\sum_{\ell(\gamma)\leq 7.5} \ell(\gamma_0)\cdot\frac{2\sin(\text{hol}(\widetilde{\gamma}))\cdot\cos(\varphi_{\widetilde{\gamma}})}{|1-e^{\ell+i\theta}|\,|1-e^{-(\ell+i\theta)}|}\cdot\frac{1}{\ell(\gamma)}\cdot e^{-\frac{\ell^2}{2}\cdot(1.7)^2}\\ &= 0.641083369621\ldots\end{aligned}$$

and therefore the approximate value

$$-2h(\mathsf{SW}, \mathfrak{s}_0) \approx 0.968861147398778.$$

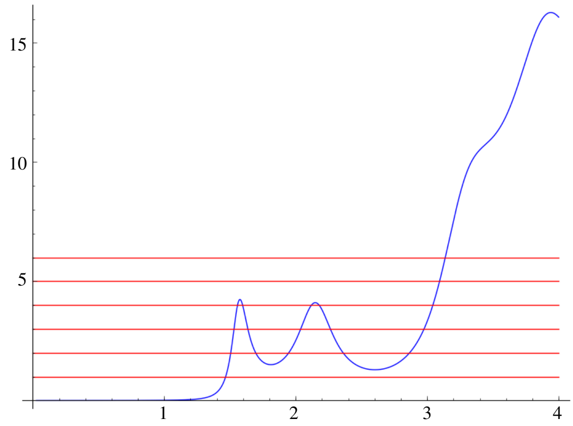


Fig. 6. The function $J_{\mathsf{SW},\mathfrak{s}_0}(t)$, computed with cutoff $R = 7.5$.

This shows $h(\mathsf{SW}, \mathfrak{s}_0) = -1/2$ provided we can bound the error term for η_{Dir} explicitly. Again, the error from truncating the sum on the geometric side is bounded above by 0.0376 by Proposition 9.4. For the spectral side, we bound the number of spectral parameters as in the proof of Proposition 10.1 using the even spinor trace formula for the function

$$\cos(Tx) \cdot \left(\frac{1}{2\delta}\mathbf{1}_{[-\delta,\delta]}\right)^{*6} \quad \text{where } \delta = 7.5/6 \tag{25}$$

(whose geometric side we can compute as it is supported in $[-7.5, 7.5]$). We find that there are at most 8 spectral parameters in $[1.45, 2]$, and each contributes at most

$$\frac{1}{\pi}\int_{1.7\cdot 1.45}^{\infty} \sqrt{2\pi} e^{-t^2/2}\, dt < 0.0108.$$

Furthermore, there are at most 12 parameters in $[2, 3]$ and 31 parameters in $[3, 4]$; larger parameters are again negligible. Putting everything together, we see that the error is bounded above by 0.1564. Therefore, we have

$$-2h(\mathsf{SW}, \mathfrak{s}_0) \in [0.812, 1.126]$$

and the result in Theorem 2 follows because 1 is the only number of the form $n/4$ in that interval.

10.3. Conclusion of the proof

We now complete the proof of Theorem 2. The arguments in this section will be quite ad hoc because the spin^c structures have a smaller spectral gap than the spin case, and in particular the error bounds we will be able to provide will not be as good as in the spin case. Of course, one could in principle obtain better bounds by computing larger portions of the length spectrum, but this is a quite challenging task from the computational viewpoint (compare with the discussion in [32]).

A corollary of the computation of the previous section is that the simply connected spin manifold $(X, \mathfrak{s}_X)$ bounding SW we fixed satisfies

$$\tfrac{1}{4}(-2\chi(X) - 3\sigma(X) - 2) \in \mathbb{Z}.$$

As $H^3(X, \partial X, \mathbb{Z}) = H_1(X, \mathbb{Z}) = 0$, the restriction map

$$H^2(X, \mathbb{Z}) \to H^2(\partial X, \mathbb{Z})$$

is surjective; therefore every spin^c structure $\mathfrak{s} = \mathfrak{s}_0 + x$ on SW extends to a spin^c structure $\tilde{\mathfrak{s}} = \mathfrak{s}_X + \tilde{x}$ on X. As $c_1(\tilde{\mathfrak{s}})$ equals $2\tilde{x}$ up to torsion elements, we have

$$\begin{aligned}\eta_{\text{sign}}/4 + \eta_{\text{Dir}} &= \tfrac{1}{4}(c_1(\tilde{\mathfrak{s}})^2 - 2\chi(X) - 3\sigma(X) - 2) \bmod \mathbb{Z}\\ &= \tilde{x}^2 \bmod \mathbb{Z}.\end{aligned}$$

As $H_1(X, \mathbb{Z}) = 0$, we have $\tilde{x}^2 = \operatorname{lk}(x, x) \bmod \mathbb{Z}$, hence

$$-2h(\mathsf{SW}, \mathfrak{s}) = \operatorname{lk}(x, x) \bmod \mathbb{Z}.$$

By the computations of Proposition 8.1 we know exactly the number of spin^c which are not spin and for which the self-linking number equals a given value in $(\frac{1}{5}\mathbb{Z})/\mathbb{Z}$. Also, we know from Corollary 8.2 that the isometry group acts transitively on the set of non-spin spin^c structures with fixed self-linking number, so we expect a priori to only have 5 possible spectral pictures for the remaining spin^c structures. This is indeed the case; see Figure 7.

Comparing the number of spin^c structures with a given linking form, we conclude the following.

Lemma 10.2. *The following statements hold:*

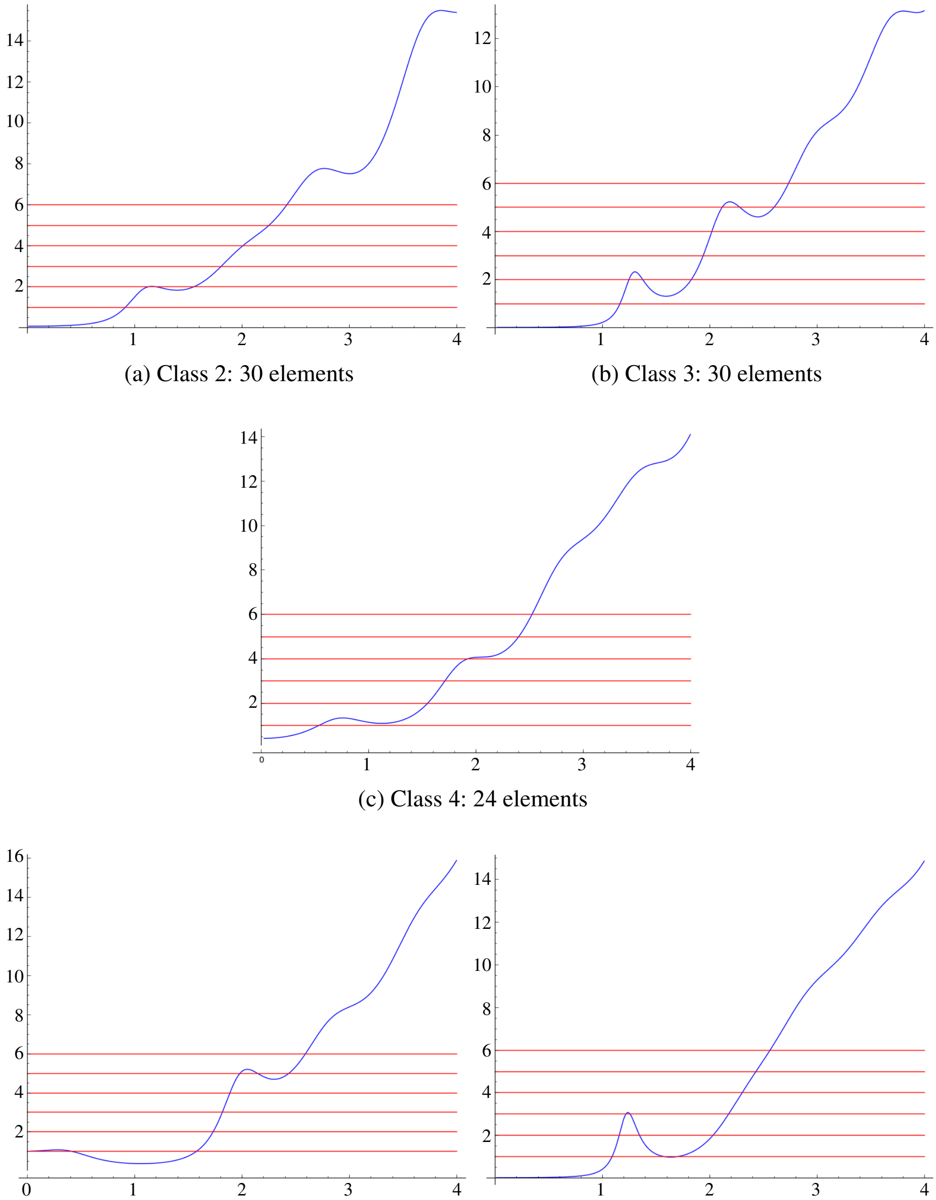
(1) *for the spin^c structures in Class* 2 *and* 3*,* $-2h$ *has fractional part* $1/5$ *and* $-1/5$*, or vice versa;*

(2) *for the spin^c structures in Class* 4*,* $-2h$ *has fractional part* 0*;*

(3) *for the spin^c structures in Class* 5 *and* 6*,* $-2h$ *has fractional part* $2/5$ *and* $-2/5$*, or vice versa.*

As in the case of the spin structure, we compute an approximation to the Dirac eta invariant using $G(x) = e^{-x^2/2}$ *and* $L = 1.7$ *by truncating the geometric sum at* $R = 7.5$ *and considering the whole spectral side as an error term. The results for the approximate value of* $-2h(Y, \mathfrak{s}) = \eta_{\text{sign}}/4 + \eta_{\text{Dir}}$ *are as in the following table.*

Spin^c class	Approximate value of $\eta_{\text{sign}}/4 + \eta_{\text{Dir}}$
2	0.292743626654778...
3	−0.133807427751222...
4	0.645708045190778...
5	0.456382720492778...
6	0.542040336591778...



(a) Class 2: 30 elements

(b) Class 3: 30 elements

(c) Class 4: 24 elements

(d) Class 5: 20 elements

(e) Class 6: 20 elements

Fig. 7. Pictures for SW.

We then again need to bound errors, as in the case of the spin structure. The error from truncating the sum on the geometric side is bounded above by 0.0376 (see Proposition 9.4). For the spectral side, using the even spinor trace formula for the function (25) we can provide decent bounds on spectral parameters on any given interval. Using this, we can conclude the proof; each case requires a slightly different argument, and we treat them separately.

Proof of Theorem 2 for Classes 2, 3 *and* 4. We begin with Class 4, which is the most involved because it has the smallest spectral gap. Using our refined estimates on the number of spectral parameters as in the previous sections using the test functions (25), we see that there are at most 1 eigenvalue in $[0.545, 1.05]$, 2 in $[1.05, 1.39]$, 4 in $[1.39, 1.85]$, 7 in $[1.85, 2.2]$, 15 in $[2.2, 3]$, 30 in $[3, 4]$, for a total error of 0.6279. We therefore conclude that

$$-2h \in [0.017, 1.274]$$

so that is $-2h = 1$ in this case, as it is an integer by Lemma 10.2.

For Classes 2 and 3 the gap is wider, so it is easier to get good bounds. For Class 2 we can obtain for example

$$-2h \in [-0.197, 0.784].$$

Because the fractional part is either $-1/5$ or $1/5$, we find that $-2h = 1/5$. This in turn implies that for Class 3, $-2h$ has fractional part $-1/5$; since we obtain with the same approach

$$-2h \in [-0.614, 0.347],$$

we conclude that $-2h = -1/5$ in this case. ■

The case of Classes 5 and 6, which have fractional part $2/5$ and $-2/5$ or vice versa, is more delicate because for Class 5 we can only prove a very small spectral gap. We first obtain more information on the fractional parts as follows.

Lemma 10.3. *For Class* 5*,* $-2h$ *has fractional part* $2/5$*, while for Class* 6 *it has fractional part* $3/5$*.*

Proof. The output of our computations described in Section 2 provides the description of a spin^c structure as a homomorphism

$$\varphi : \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},$$

where the natural basis is different from the one used in Section 8. It is easy to check that the linking form in this new basis is given by

$$\frac{1}{5}\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Indeed, this is the unique symmetric matrix that takes the correct values for the spin^c structures of Class 2, 3 and 4. The lemma is then proved by plugging in the explicit values for Classes 5 and 6. ■

Proof of Theorem 2 for Class 6. For Class 6, we compute using again the test functions (25) that there are at most 4 parameters in $[1, 1.5]$, 5 in $[1.5, 2]$, 16 in $[2, 3]$ and 30 in $[3, 4]$.

We therefore have a total error of at most 0.459, and therefore

$$-2h \in [0.083, 1.002],$$

hence by the previous lemma we have $-2h = 3/5$. ∎

Finally, to deal with the case of Class 5, we need some additional information about the small eigenvalues.

Lemma 10.4. *For spin^c Class 5, the smallest eigenvalue s_1 satisfies*

$$|s_1| \in [0.0408361, 0.4077692] \tag{26}$$

and has multiplicity 1. *Furthermore,* $|s_2| > 1.55$.

Proof. Using the same method as above involving the test functions (25) we conclude that there are at most two eigenvalues in this interval.

To prove that there is exactly one, we apply the trace formula to the Gaussian $e^{-(x/1.7)^2/2}$ by truncating the sum over geodesics at $R = 7.5$. The computations of the error bound involve the same quantities as in the proof of Proposition 9.4, and we obtain the following estimate for the sum over the spectrum:

$$\sum_{s_n} e^{-(1.7 \cdot s_n)^2/2} \in [0.9678, 1.0789].$$

Now, for $t \in [0.0408361, 0.4077692]$ we have $e^{-(1.7 \cdot t)^2/2} \in [0.786417, 0.9976]$ and the spectral picture shows that eigenvalues which are not in this interval are ≥ 1.55. The same trace formula argument used several times above to prove upper bounds on spectral density shows that there are at most 6 parameters in $[1.55, 2]$, 16 in $[2, 3]$, 30 in $[3, 4]$, from which we see that the contribution of the eigenvalues ≥ 1.55 has to be at most 0.2432. Using the fact that there are at most two eigenvalues in the interval, we conclude by direct inspection that there has to be exactly one. ∎

Proof of Theorem 2 *for Class* 5. For our purposes the main complication arises from the fact that it is hard to determine the sign of the small eigenvalue using the trace formula. This is because in order to study the sign we need to use odd test functions, and these will be small near zero. We will therefore study the two possibilities separately. We approximate the contribution of the small eigenvalue with the average 0.7157945 of

$$\frac{1}{\pi} \int_{1.7 \cdot 0.04}^{\infty} \sqrt{2\pi} e^{-t^2/2} \, dt \approx 0.945786,$$

$$\frac{1}{\pi} \int_{1.7 \cdot 0.41}^{\infty} \sqrt{2\pi} e^{-t^2/2} \, dt \approx 0.485803,$$

which introduces an error of at most 0.229992. The total error is then computed to be at most 0.3289 (luckily, $|s_2|$ is large). If the small eigenvalue s_1 were negative we would have

$$-2h \in [-0.588, 0.069]$$

which does not contain any number with fractional part $2/5$. Therefore $s_1 > 0$, and we have

$$-2h \in [0.8432, 1.5011],$$

from which we conclude $-2h = 7/5$. ∎

Appendix A. Generalities on the group G and the trace formula

All the trace formulas we need in the present paper are obtained by specializing the general trace formula for a cocompact torsion-free lattice $\widetilde{\Gamma}$ in the Lie group

$$G = \{g \in \mathrm{GL}_2(\mathbb{C}) : |\det(g)| = 1\}.$$

The strategy of the proof follows very closely that of [32, Appendix B], where we studied the case of $\mathrm{PGL}_2(\mathbb{C})$:

(1) express the very general trace formula in terms of irreducible representations and geometric data;

(2) understand the representation-theoretic incarnation of the differential operators under consideration;

(3) choose suitable test functions to isolate the relevant representations.

In [32, Appendix B], we tackled the first step for $\mathrm{PGL}_2(\mathbb{C})$: we derived a general trace formula for cocompact lattices in $\mathrm{PGL}_2(\mathbb{C})$ (which was called G there). So as not to repeat ourselves, we simply state the trace formula for G and highlight the differences between it and the very similar trace formula from [32].

Remark A.1. As will be clear from the discussion, this approach involving G is only strictly necessary when dealing with spinors; the trace formulas for functions and forms, both even and odd, can be derived from the general trace formula for $\mathrm{PGL}_2(\mathbb{C})$.

A.1. Notation for subgroups of G

We will work with the maximal torus $T \subset G$ of diagonal matrices; we will use the parametrizations

$$T = \left\{ \begin{pmatrix} e^{u/2}e^{i\alpha} & 0 \\ 0 & e^{-u/2}e^{i\beta} \end{pmatrix} : u \in \mathbb{R},\ \alpha, \beta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \tag{27}$$

and

$$T = \left\{ \begin{pmatrix} e^{i\phi}e^{v+i\theta} & 0 \\ 0 & e^{i\phi}e^{-v-i\theta} \end{pmatrix} : v \in \mathbb{R},\ \phi, \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}. \tag{28}$$

The center Z of G is the diagonal copy of U_1. We will equip T with the Haar measure

$$dt = du \frac{d\alpha}{2\pi} \frac{d\beta}{2\pi}. \tag{29}$$

The Weyl group W has two elements, and is generated by $(u, \alpha, \beta) \mapsto (-u, \beta, \alpha)$.

Remark A.2. Using this notation, the complex length of the image in $\mathrm{PGL}_2(\mathbb{C})$ of an element of T is $u + i2\theta$.

We will denote by B the subgroup of upper triangular matrices; the modular function is then

$$\delta : B \to \mathbb{R}^{>0}, \quad \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto |t_1/t_2|^2.$$

Finally, $K = \mathrm{U}_2$ is the maximal compact subgroup; we can identify

$$G/K = \mathrm{PGL}_2(\mathbb{C})/P\mathrm{U}_2 = \mathbb{H}^3$$

using the upper half-space model $\mathbb{H}^3 \equiv \mathbb{C} \times \mathbb{R}_{>0}$. The tangent space at $(0, 1)$ is then identified with $\mathfrak{p}_0 = i\mathfrak{su}(2)$. We will also denote by$\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$ its complexification.

A.2. Unitary representations of G

Every irreducible unitary representation of G is isomorphic to one of the following:

(a) the 1-dimensional representation with character $\det^k$ for some $k \in \mathbb{Z}$;

(b) the representation π_{s_1,s_2,n_1,n_2} with $s_1 = ir_1, s_2 = ir_2$ purely imaginary and $n_1, n_2 \in \mathbb{Z}$, defined as follows. Consider the character

$$\chi_{s_1,s_2,n_1,n_2} : B \to \mathbb{C}^*, \quad \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto |t_1|^{s_1} \cdot |t_2|^{s_2} \cdot \left(\frac{t_1}{|t_1|}\right)^{n_1} \cdot \left(\frac{t_2}{|t_2|}\right)^{n_2}.$$

We then define π_{s_1,s_2,n_1,n_2} to be the representation induced on G by $\delta^{1/2}\chi_{s_1,s_2,n_1,n_2}$. Because $|t_1|\,|t_2| = 1$, the representation π_{s_1,s_2,n_1,n_2} only depends on $s_1 - s_2, n_1, n_2$.

(c) complementary series representations, which are abstractly isomorphic to $\pi_{s_1,s_2,0,0}$ where the difference $s_1 - s_2$ is real and lies in the interval $[-1, 1]$.

Again, there are some coincidences among these representations.

Lemma A.1. *We have* $\pi_{s_1,s_2,n_1,n_2} \cong \pi_{t_1,t_2,m_1,m_2}$ *iff*

$$(t_1, t_2, m_1, m_2) = (s_1, s_2, n_1, n_2) \text{ or } (s_2, s_1, n_2, n_1) \in \frac{i\mathbb{R} \times i\mathbb{R}}{\langle i(1,1)\rangle} \times \mathbb{Z} \times \mathbb{Z}.$$

Proof. The proof is very similar to the case of $\mathrm{PGL}_2(\mathbb{C})$ covered in [32, Appendix B]. For every smooth compactly supported test function f on G,

$$\mathrm{trace}(\pi_{s_1,s_2,n_1,n_2}) = \widehat{Sf}(\chi^{-1}_{s_1,s_2,n_1,n_2}),$$

where we view Sf as a W-invariant function on the diagonal torus T. By the main theorem of Bouaziz [9], the Satake transform $f \mapsto Sf$ maps f onto a W-invariant compactly supported smooth function on T. Therefore, two representations π_{s_1,s_2,n_1,n_2} and π_{t_1,t_2,m_1,m_2} are isomorphic iff

$$\widehat{F}(\chi_{s_1,s_2,n_1,n_2}) = \widehat{F}(\chi_{t_1,t_2,m_1,m_2})$$

for every smooth, compactly supported function F on T which is W-invariant, i.e.

$$F(u,\alpha,\beta) = F(-u,\beta,\alpha).$$

The function F is W-invariant iff $\widehat{F}(s,n,m) = \widehat{F}(-s,m,n)$ (the forward implication is immediate from the definition, and the converse follows from Fourier inversion). Since

$$\begin{aligned}\widehat{F}(\chi^{-1}_{s_1,s_2,n_1,n_2}) &= \int F(t)\chi_{s_1,s_2,n_1,n_2}(t)\,dt \\ &= \int F(u,\alpha,\beta)\cdot e^{u/2\cdot(s_1-s_2)}\cdot e^{in_1\alpha}\cdot e^{in_2\beta}\,du\,\frac{d\alpha}{2\pi}\,\frac{d\beta}{2\pi} \\ &= \widehat{F}(-(s_1-s_2),-n_1,-n_2),\end{aligned}$$

the result follows. ∎

A.3. *The trace formula for G*

We have the following general result.

Proposition A.2. *Let F be any smooth, compactly supported, W-invariant function on the diagonal torus T. Let $\widetilde{\Gamma}\subset G$ be a cocompact torsion-free lattice. We have the identity*

$$\begin{aligned}&\sum_{s_1-s_2,n,m} m_{\widetilde{\Gamma}}(\pi_{s_1,s_2,n,m})\cdot\widehat{F}(\chi^{-1}_{s_1,s_2,n,m}) + \sum_k \frac{1}{|W|}\int_T |D(t^{-1})|^{1/2}\cdot\det(t)^k\cdot F(t)\,dt \\ &= -\frac{1}{8\pi}\cdot\mathrm{vol}(Y)\cdot\sum_{z\in\widetilde{\Gamma}\cap Z}\left(\frac{d^2}{dv^2}+\frac{d^2}{d\theta^2}\right)F\bigg|_{t=z} + \sum_{[\gamma]\neq 1}\ell(\gamma_0)\cdot|D(t_\gamma^{-1})|^{-1/2}\cdot F(t_\gamma).\end{aligned}$$

Here the second derivatives in v and θ are taken with respect to the parametrization (28) of T, and $|W|=2$.

This is in fact a generalization of the trace formula from [32], which can be recovered by using test functions on the diagonal torus T invariant by the center Z of G. The proof of this result is essentially identical to the one of the formula in [32], so we will focus on pointing out similarities and differences between Proposition A.2 and the trace formula from our first paper.

- The "identity contribution" to the trace formula has a contribution for each element of $\widetilde{\Gamma}\cap Z$. The group Z is compact so this sum is finite. In our case of interest, $\widetilde{\Gamma}$ is the lift of a lattice $\Gamma\subset \mathrm{PGL}_2(\mathbb{C})$, and $\widetilde{\Gamma}\cap Z=\{1\}$.
- Recall that the function $F(u,\alpha,\beta)$ is W-invariant iff

$$F(u,\alpha,\beta) = F(-u,\beta,\alpha).$$

  In particular, the formula $F(u,\alpha,\beta) = K_\pm(\alpha,\beta)\cdot H_\pm(u)$, where $K_\pm$ are symmetric and antisymmetric respectively and $H_\pm$ are even and odd respectively, defines W-invariant functions on T.

- The universal constant $-\frac{1}{8\pi}$ can be determined in several ways; for example, by specializing this formula to the case of the even trace formula for coexact 1-forms and comparing it with the one from [32].

Appendix B. The odd signature and Dirac operators

We need to specialize the general trace formula in Proposition A.2 to our cases of interest, namely to isolate the eigenvalue spectrum of the odd signature and Dirac operators. In [32], we showed how to interpret the Hodge Laplacian and its spectrum in terms of the representation theory of PGL_2; in particular, we identified coexact 1-eigenforms in terms of isotypic vectors in the irreducible representations, and the corresponding eigenvalue using the Casimir eigenvalue. We will do the same here, where we consider the maximal compact subgroup $K = \mathrm{U}_2$. In particular, we show the following two results.[6]

Proposition B.1. *Eigenforms of* $*d$ *correspond to* $\mathfrak{p}$*-isotypic vectors where* $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$. *An irreducible unitary representation of* G *contains a* $\mathfrak{p}$*-isotypic vector if and only if it is isomorphic to* $\pi_{s_1,s_2,1,-1}$ *with* s_1, s_2 *purely imaginary. Furthermore, each copy of* $\pi_{s_1,s_2,1,-1} \subset L^2(\widetilde{\Gamma}\backslash G)$ *contains exactly a* 1*-dimensional eigenspace of* $*d$, *spanned by an eigenform with eigenvalue* $i(s_1 - s_2)/2$.

Recall that a $\mathfrak{p}$-isotypic vector in a G-representation π is a non-zero element of $\mathrm{Hom}_K(\mathfrak{p}, \pi)$. Notice that as the Hodge Laplacian on coexact 1-forms is the square of $*d$, this result implies (after changing the parametrization) those in [32]; in that case we computed the corresponding eigenvalue via Kuga's lemma. Notice that while this result can be proved directly using the group PGL_2, the following one relies essentially on the larger group G.

Proposition B.2. *Eigenspinors correspond to* $S^\vee$*-isotypic vectors where* $S^\vee$ *is the dual of the standard representation of* $K = \mathrm{U}_2$ *on* $\mathbb{C}^2$. *An irreducible unitary representation of* G *contains an* $S^\vee$*-isotypic vector if and only if it is isomorphic to* $\pi_{s_1,s_2,-1,0}$ *with* s_1, s_2 *purely imaginary. Furthermore, each copy of* $\pi_{s_1,s_2,-1,0} \subset L^2(\widetilde{\Gamma}\backslash G)$ *contains exactly a* 1*-dimensional eigenspace of the Dirac operator, spanned by an eigenspinor with eigenvalue* $i(s_1 - s_2)/2$.

The present Appendix is dedicated to the proof of these results. The new main ingredient is to describe the Dirac and $*d$ operators in terms of representation theory, which we do in Sections B.1–B.3. We then identify the relevant representations in Section B.4, and the corresponding eigenvalues in Section B.5.

[6] Recall that our convention is that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a positively oriented basis of $\mathfrak{p}_0 = i\mathfrak{su}(2)$.

B.1. Generalities on associated bundles and connections for the principal K-bundle $G \to G/K$

B.1.1. The invariant connection on $G \to G/K$. We follow the notation of Section A.1. Note that $G \xrightarrow{\pi} G/K$ (with K acting on the right by multiplication) is a principal K-bundle. The assignment

$$H_g := (L_g)_* \mathfrak{p}_0 \subset T_g(G)$$

defines a subbundle $H \subset TG$. Observe that:

(1) The projection $H_g \to T_g(G) \xrightarrow{\pi_*} T_{gK}(G/K)$ is an isomorphism.

(2) The subbundle H is right K-invariant:

$$\begin{aligned}
(R_k)_* H_g &= (R_k)_*(L_g)_* \mathfrak{p}_0 \\
&= (L_g)_*(R_k)_* \mathfrak{p}_0 && \text{(left and right multiplications commute)} \\
&= [(L_g)_*(L_k)_*][(L_k)_*^{-1}(R_k)_*]\mathfrak{p}_0 \\
&= (L_g L_k)_*(\text{conjugation by } k^{-1})_* \mathfrak{p}_0 \\
&= (L_{gk})_* \mathrm{Ad}(k^{-1})\mathfrak{p}_0 \\
&= (L_{gk})_* \mathfrak{p}_0 && \text{(because } \mathfrak{p}_0 \text{ is invariant under } \mathrm{Ad}(K)).
\end{aligned}$$

(3) The subbundle H is left G-invariant: $(L_h)_* H_g = (L_h)_*(L_g)_* \mathfrak{p}_0 = H_{hg}$.

Items (1) and (2) imply that H defines a connection on $G \to G/K$. Item (3) implies that H is left G-invariant.

B.1.2. Associated bundles and covariant derivatives. Let $\rho : K \to \mathrm{GL}(V)$ be a representation of K. The associated vector bundle $\mathcal{F}_V := G \times_\rho V \to G/K$ has sections

$$\Gamma(\mathcal{F}_V) = \{s : G \to V : s(gk) = \rho(k)^{-1} s(g)\}.$$

The covariant derivative ∇^V on $\mathcal{F}_V$ associated to the connection $H \subset TG$ is given by

$$\nabla^V_X : \Gamma(\mathcal{F}_V) \to \Gamma(\mathcal{F}_V), \quad s \mapsto ds(\widetilde{X}),$$

where $\widetilde{X}$ denotes the horizontal lift of X to TG via the invariant connection H from Section B.1.1. Via the trivialization

$$\mathfrak{p}_0 \times G \to H, \quad (X, g) \mapsto (L_g)_* X,$$

from Section B.1.1, we may regard $\nabla^V s \in \Gamma(\mathcal{F}_V \otimes \Omega^1(G/K))$, which is the bundle associated to the representation $V \otimes \mathfrak{p}_0^\vee \simeq \mathrm{Hom}(\mathfrak{p}_0, V)$ of K, as the function

$$\nabla^V s : G \to \mathrm{Hom}(\mathfrak{p}_0, V), \quad g \mapsto \big(X \mapsto ds((L_g)_* X) = X(s \circ L_g)\big). \tag{30}$$

Let r denote the representation

$$r : K \to \mathrm{End}(\mathrm{Hom}(\mathfrak{p}_0, V)), \quad k \mapsto (T \mapsto \rho(k) \circ T \circ \mathrm{Ad}(k)^{-1}).$$

Note that $\nabla^V s$, as defined by (30), has the K-equivariance property

$$(\nabla^V s)(gk) = r(k)^{-1} \cdot (\nabla^V s)(g).$$

B.1.3. Dual picture for sections of $\mathcal{F}_V$ and the covariant derivative ∇^V. For our purposes it will be convenient to rephrase everything in terms of a dual picture, which is more manifestly representation-theoretic. There is a canonical isomorphism

$$\Phi^V : C^\infty(G,V) \xrightarrow{\sim} \operatorname{Hom}(V^\vee, C^\infty(G)), \quad s \mapsto (\ell \mapsto (g \mapsto \ell(s(g)))). \tag{31}$$

The inverse mapping $(\Phi^V)^{-1}$ is most easily expressed using a basis $\{v_i\}$ for V and its corresponding dual basis $\{v_i^\vee\}$ for $V^\vee$:

$$\begin{aligned}(\Phi^V)^{-1} : \operatorname{Hom}(V^\vee, C^\infty(G)) &\xrightarrow{\sim} C^\infty(G,V),\\ T &\mapsto \Big(g \mapsto \sum_i T(v_i^\vee)(g)\cdot v_i\Big).\end{aligned} \tag{32}$$

The map $\Phi := \Phi^V$ has the following equivariance properties:

(1) $\Phi(L_g s) = L_g\Phi(s)$ and $\Phi(R_g s) = R_g\Phi(s)$. In $C^\infty(G,V)$,

$$L_g s(x) = s(gx), \quad R_g s(x) = s(xg),$$

and on $\operatorname{Hom}(V^\vee, C^\infty(G))$, the corresponding actions are

$$V^\vee \to C^\infty(G) \xrightarrow{\text{left or right translation by } g} C^\infty(G).$$

(2) Φ carries the subspace $\Gamma(\mathcal{F}_V) = \{s : G \to S : s(gk) = \rho(k)^{-1}s(g)\}$ to the subspace $\operatorname{Hom}_K(V^\vee, C^\infty(G))$, where K acts by right translation on $C^\infty(G)$ and by $\rho^\vee$ on $V^\vee$.

(3) Upon combining (30) with the explicit formula for Φ^{-1} from (32), we see that Φ intertwines the covariant derivative ∇^V with the operator

$$\begin{aligned}&\Theta^V : \operatorname{Hom}(V^\vee, C^\infty(G)) \to \operatorname{Hom}((V\otimes\mathfrak{p}_0^\vee)^\vee, C^\infty(G)),\\ &T \mapsto \Big(\ell\otimes X \mapsto \Big(g \mapsto \sum_i \frac{d}{dt}\Big|_{t=0} [T(v_i^\vee)(ge^{tX})]\cdot \ell(v_i)\Big)\Big),\end{aligned}$$

where $\ell\otimes X \in V^\vee\otimes\mathfrak{p}_0 \cong (V\otimes\mathfrak{p}_0^\vee)^\vee$, i.e. there is a commutative diagram

$$\begin{array}{ccc} C^\infty(G,V) & \xrightarrow{\quad\nabla^V\quad} & C^\infty(G, V\otimes\mathfrak{p}_0^\vee) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \operatorname{Hom}(V^\vee, C^\infty(G)) & \xrightarrow{\Theta^V} & \operatorname{Hom}(V^\vee\otimes\mathfrak{p}_0, C^\infty(G)) \end{array}$$

The above restricts to a commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{F}_V) & \xrightarrow{\quad\nabla^V\quad} & \Gamma(\mathcal{F}_{V\otimes\mathfrak{p}_0^\vee}) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \operatorname{Hom}_K(V^\vee, C^\infty(G)) & \xrightarrow{\Theta^V} & \operatorname{Hom}_K(V^\vee\otimes\mathfrak{p}_0, C^\infty(G)) \end{array}$$

(4) K-equivariant maps $\alpha : V \to W$ induce maps $\Gamma(\mathcal{F}_V) \to \Gamma(\mathcal{F}_W)$ (by post-composition with α) and on $\operatorname{Hom}_K(V^\vee, C^\infty(G)) \to \operatorname{Hom}_K(W^\vee, C^\infty(G))$ (by pre-composition with $\alpha^\vee$). These induced maps commute with the isomorphism Φ.

B.1.4. Going automorphic. Everything in Sections B.1.1–B.1.3 is left invariant. So the connections, vector bundles, covariant derivatives and bundle maps defined thus far all descend to $\widetilde{\Gamma}\backslash G/K$. We denote descended objects with an $\bar{\bullet}$, e.g. for the descended vector bundle $\bar{\mathcal{F}}_V$ over $\widetilde{\Gamma}\backslash G/K$,

$$\Gamma(\bar{\mathcal{F}}_V) = \{s : \widetilde{\Gamma}\backslash G \to S : s(\widetilde{\Gamma}gk) = \rho(k)^{-1}s(\widetilde{\Gamma}g)\},$$

which is identified, via the isomorphism Φ^V, with $\mathrm{Hom}_K(V^\vee, C^\infty(\widetilde{\Gamma}\backslash G))$.

The descended covariant derivative $\bar{\nabla}^V$ and its incarnation $\bar{\Theta}^V$ in the dual picture admit the "same formulas mod $\widetilde{\Gamma}$". In particular,

$$\begin{aligned}\bar{\Theta}^V &: \mathrm{Hom}_K(V^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) \to \mathrm{Hom}_K((V \otimes \mathfrak{p}_0^\vee)^\vee, C^\infty(\widetilde{\Gamma}\backslash G)),\\ T &\mapsto \left(\ell \otimes X \mapsto \left(g \mapsto \sum_i \frac{d}{dt}\bigg|_{t=0} [T(v_i^\vee)(ge^{tX})] \cdot \ell(v_i)\right)\right),\end{aligned}$$

and there is a commutative diagram

$$\begin{array}{ccc} \Gamma(\bar{\mathcal{F}}_V) & \xrightarrow{\quad\bar{\nabla}^V\quad} & \Gamma(\bar{\mathcal{F}}_{V\otimes\mathfrak{p}_0^\vee}) \\ {\scriptstyle\bar{\Phi}}\downarrow & & \downarrow{\scriptstyle\bar{\Phi}} \\ \mathrm{Hom}_K(V^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) & \xrightarrow{\bar{\Theta}^V} & \mathrm{Hom}_K(V^\vee \otimes \mathfrak{p}_0, C^\infty(\widetilde{\Gamma}\backslash G)) \end{array}$$

Given a K-equivariant map $\alpha : V \to W$, we denote the associated bundle map (obtained by post-composition with α on $\Gamma(\bar{\mathcal{F}}_V)$ or with pre-composition by $\alpha^\vee$ in the dual picture) by $\bar{\alpha}$. We also have commutative diagrams

$$\begin{array}{ccc} \Gamma(\bar{\mathcal{F}}_V) & \xrightarrow{\quad\bar{\alpha}\quad} & \Gamma(\bar{\mathcal{F}}_{V\otimes\mathfrak{p}_0^\vee}) \\ {\scriptstyle\bar{\Phi}}\downarrow & & \downarrow{\scriptstyle\bar{\Phi}} \\ \mathrm{Hom}_K(V^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) & \xrightarrow{\bar{\alpha}} & \mathrm{Hom}_K(V^\vee \otimes \mathfrak{p}_0, C^\infty(\widetilde{\Gamma}\backslash G)) \end{array}$$

Because $\bar{\nabla}^V$ is built from the derivative of the right translation action of G on $C^\infty(\widetilde{\Gamma}\backslash G)$, the operators $\bar{\Theta}^V$ are amenable to representation-by-representation analysis. Likewise, the bundle maps $\bar{\alpha}$ are amenable to representation-by-representation analysis. For irreducible (unitary) representations π of G, denote by $\pi^\infty \subset \pi$ the dense subspace of smooth elements. We define the following π-isotypic versions of the operators $\bar{\Theta}^V$ and $\bar{\alpha}$:

$$\begin{aligned}\bar{\Theta}^V_\pi &: \mathrm{Hom}_K(V^\vee, \pi^\infty) \to \mathrm{Hom}_K((V \otimes \mathfrak{p}_0^\vee)^\vee, \pi^\infty),\\ &\qquad T \mapsto \left(\ell \otimes X \mapsto \sum_i (X \cdot T(v_i^\vee)\right) \cdot \ell(v_i)),\end{aligned} \tag{33}$$

and

$$\bar{\alpha}_\pi : \mathrm{Hom}_K(V^\vee, \pi^\infty) \to \mathrm{Hom}_k(W^\vee, \pi^\infty), \quad T \mapsto T \circ \alpha^\vee.$$

Given a subrepresentation $\pi \subset L^2(\widetilde{\Gamma}\backslash G)$, the above π-isotypic operators fit into commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}_K(V^\vee, \pi^\infty) & \xrightarrow{\bar{\Theta}^V_\pi} & \mathrm{Hom}_K((V \otimes \mathfrak{p}_0^\vee)^\vee, \pi^\infty) \\ \iota\downarrow & & \downarrow\iota \\ \mathrm{Hom}_K(V^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) & \xrightarrow{\bar{\Theta}^V} & \mathrm{Hom}_K((V \otimes \mathfrak{p}_0^\vee)^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) \end{array} \tag{34}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_K(V^\vee, \pi^\infty) & \xrightarrow{\bar{\alpha}_\pi} & \mathrm{Hom}_K(W^\vee, \pi^\infty) \\ \iota\downarrow & & \downarrow\iota \\ \mathrm{Hom}_K(V^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) & \xrightarrow{\bar{\alpha}} & \mathrm{Hom}_K(W^\vee, C^\infty(\widetilde{\Gamma}\backslash G)) \end{array} \tag{35}$$

where the vertical maps ι in both diagrams are induced by the inclusion $\pi \subset L^2(\widetilde{\Gamma}\backslash G)$.

B.2. Automorphic incarnation of the spinor bundle and Dirac operator

Denote as before by S the standard representation of U_2 on $\mathbb{C}^2$.

Definition B.1. Let $\widetilde{\Gamma} \subset G$ be a lattice lifting the closed hyperbolic three-manifold lattice $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$. The spinor bundle over $\Gamma\backslash\mathbb{H}^3 = \widetilde{\Gamma}\backslash\mathbb{H}^3 = \widetilde{\Gamma}\backslash G/K$ associated to the lift $\widetilde{\Gamma} \subset G$ of Γ is the vector bundle $\bar{\mathcal{F}}_S$. It comes equipped with the covariant derivative $\bar{\nabla}^S$.

B.2.1. Clifford multiplication on $\bar{\mathcal{F}}_S$. The linear map $C : S \otimes \mathfrak{p}_0 \to S$ given by $2i$ times matrix multiplication is U_2-equivariant:

$$\begin{aligned} C((\mathrm{std} \otimes \mathrm{Ad})(k)(s \otimes X)) &= C(\mathrm{std}(k)s \otimes \mathrm{Ad}(k)X) = 2i\cdot(kXk^{-1})(ks) \\ &= k(2i\cdot Xs) = \mathrm{std}(k)C(s \otimes X). \end{aligned}$$

In what follows, we will make use of the (physicists') Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{36}$$

Remark B.1. In the hyperbolic metric, the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are orthogonal to each other, but have norm 2. To see why the latter holds, notice that

$$e^{t\sigma_3} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

In the upper half-space model, this maps $(0, 0, 1)$ to $(0, 0, e^{2t})$, and the geodesic segment connecting them has length $2t$. On the other hand, it has length $\|\sigma_3\| \cdot t$, so $\|\sigma_3\| = 2$. So the multiplier 2 in the above definition of C guarantees that Clifford multiplication is isometric.

Remark B.2. The above Clifford multiplication is compatible with the orientation $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ in the sense that

$$C(\sigma_1)C(\sigma_2)C(\sigma_3) = 8$$

is positive.

The hyperbolic metric on $\mathfrak{p}_0$ defines a K-equivariant isomorphism $\iota : \mathfrak{p}_0^\vee \to \mathfrak{p}_0$. We define C' to be the composition of C with ι:

$$C' = S \otimes \mathfrak{p}_0^\vee \xrightarrow{1\otimes\iota} S \otimes \mathfrak{p}_0 \xrightarrow{C} S.$$

We will refer to the associated map on sections,

$$\Gamma(\bar{\mathcal{F}}_{S\otimes\mathfrak{p}_0^\vee}) \xrightarrow{\bar{C}'} \Gamma(\bar{\mathcal{F}}_S),$$

as *Clifford multiplication*.

B.2.2. The Dirac operator on $\Gamma(\bar{\mathcal{F}}_S)$.

Definition B.2. The *Dirac operator* $\bar{D}$ on $\Gamma(\bar{\mathcal{F}}_S)$ is defined to be the composition

$$\Gamma(\bar{\mathcal{F}}_S) \xrightarrow{\bar{\nabla}^S} \Gamma(\bar{\mathcal{F}}_{S\otimes\mathfrak{p}_0^\vee}) \xrightarrow{C'} \Gamma(\bar{\mathcal{F}}_S).$$

Let X_1, X_2, X_3 be an oriented orthonormal basis for $\mathfrak{p}_0$. The canonical isomorphism $\mathrm{Hom}(\mathfrak{p}_0, S) \simeq \mathfrak{p}_0^\vee \otimes S$ can be expressed as

$$T \mapsto \sum_{i=1}^{3} \langle \bullet, X_i\rangle \otimes T(X_i),$$

where X_i is any orthonormal basis for $\mathfrak{p}$. By the discussion from Sections B.1.2 and B.2.1, the Dirac operator is given by the formula

$$\begin{aligned} &\bar{D} : \Gamma(\bar{\mathcal{F}}_S) \to \Gamma(\bar{\mathcal{F}}_S),\\ &(g \mapsto s(g)) \mapsto \left(g \mapsto \sum_{i=1}^{3} 2i\cdot m_{X_i} X_i(s \circ L_g)\right)\\ &\qquad = \left(g \mapsto \sum_{i=1}^{3} 2i\cdot m_{X_i} \frac{d}{dt}\bigg|_{t=0} s(ge^{tX_i})\right), \end{aligned}$$

where m_{X_i} denotes matrix multiplication by $X_i \in \mathfrak{p}_0 = i\mathfrak{su}(2)$.

In the dual picture, $\bar{D} = \bar{C}' \circ \bar{\Theta}^S$. Unraveling the formula for $\bar{\Theta}^S$ from (33), we find that $\bar{D}$ equals

$$\begin{aligned} &\bar{D} : \mathrm{Hom}_K(S^\vee, C^\infty(\tilde{\Gamma}\backslash G)) \to \mathrm{Hom}_K(S^\vee, C^\infty(\tilde{\Gamma}\backslash G)),\\ &T \mapsto \sum_{i=1}^{3}\left(S^\vee \xrightarrow{2i\cdot m_{X_i}^\vee} S^\vee \xrightarrow{T} C^\infty(\tilde{\Gamma}\backslash G) \xrightarrow{f\mapsto(g\mapsto \frac{d}{dt}|_{t=0} f(ge^{tX_i}))} C^\infty(\tilde{\Gamma}\backslash G)\right). \end{aligned}$$

B.2.3. Representation-by-representation analysis of the Dirac operator. For every irreducible representation π of G, we define the π-isotypic Dirac operators

$$\bar{D}_\pi := \bar{C}'_\pi \circ \bar{\Theta}^S_\pi.$$

Suppose $\pi \subset L^2(\tilde{\Gamma}\backslash G)$ is a subrepresentation. By the commutativity of the diagrams (34) and (35), we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_K(S^\vee, \pi^\infty) & \xrightarrow{\bar{D}_\pi} & \mathrm{Hom}_K(S^\vee, \pi^\infty) \\ \iota\downarrow & & \downarrow\iota \\ \mathrm{Hom}_K(S^\vee, C^\infty(\tilde{\Gamma}\backslash G)) & \xrightarrow{\bar{D}} & \mathrm{Hom}_K(S^\vee, C^\infty(\tilde{\Gamma}\backslash G)) \end{array} \tag{37}$$

where the vertical maps ι in both diagrams are induced by the inclusion $\pi \subset L^2(\tilde{\Gamma}\backslash G)$.

We will show in Proposition B.5 that if π is an irreducible (unitary) representation of G and $\mathrm{Hom}_K(S^\vee, \pi^\infty)$ is non-zero, then it is 1-dimensional. For all such representations π, $\bar{D}_\pi$ thus necessarily acts on $\mathrm{Hom}_K(S^\vee, \pi^\infty)$ by some scalar $\lambda_{\mathrm{Dirac},\pi}$ (which will be later computed in Proposition B.7). Thus, $\iota(\mathrm{Hom}_K(S^\vee, \pi^\infty))$ corresponds to an eigenline for the Dirac operator $\bar{D}$ with eigenvalue $\lambda_{\mathrm{Dirac},\pi}$. In particular, the natural decomposition

$$L^2(\bar{\mathcal{F}}_S) = \bigoplus_{\pi \in \hat{G}} m_{\tilde{\Gamma}}(\pi) \cdot \mathrm{Hom}_K(S^\vee, \pi^\infty),$$

which is the analogue of the Matsushima decomposition we used in [32], corresponds to the eigenspace decomposition of the Dirac operator.

B.3. Automorphic incarnation of q-form bundles and the $*d$ operator

Retain all notation from Section B.2. As in [32], in this setup it is more convenient to consider the operators on complex-valued forms; to this end, we will consider the K-representation $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$. In particular, the vector bundle $\bar{\mathcal{F}}_{\bigwedge^q \mathfrak{p}^\vee}$ is naturally identified with the vector bundle of (complex-valued) q-forms over $\tilde{\Gamma}\backslash\mathbb{H}^3$.

B.3.1. Hodge star on $\bar{\mathcal{F}}_{\bigwedge^q \mathfrak{p}^\vee}$. Consider the Hodge star operator

$$* : \bigwedge^q \mathfrak{p}^\vee \to \bigwedge^{3-q} \mathfrak{p}^\vee.$$

The action of U_2 on $\mathfrak{p}^\vee$ factors through PU_2, the group of orientation preserving isometries of $\mathbb{H}^3 = G/K$ stabilizing eK. Thus, the actions of K on $\bigwedge^q \mathfrak{p}^\vee$ and $\bigwedge^{3-q} \mathfrak{p}^\vee$ commute with $*$. We get the following result immediately.

Lemma B.3. *The associated map on sections,*

$$\Gamma(\bar{\mathcal{F}}_{\bigwedge^q \mathfrak{p}^\vee}) \xrightarrow{\bar{*}} \Gamma(\bar{\mathcal{F}}_{\bigwedge^{3-q} \mathfrak{p}^\vee}),$$

is identified with Hodge star for the hyperbolic metric on $\tilde{\Gamma}\backslash\mathbb{H}^3$.

B.3.2. Exterior derivative. Let $\overline{\nabla} := \overline{\nabla}^{\bigwedge^q \mathfrak{p}^\vee}$ be the covariant derivative on $\overline{\overline{\mathscr{F}}}_{\bigwedge^q \mathfrak{p}^\vee}$ induced by the invariant connection on $G \to G/K$ described in Section B.1.2. Let $\overline{\Theta} := \overline{\Theta}^{\bigwedge^q \mathfrak{p}^\vee}$ be the corresponding operator in the dual picture, described in Section B.1.3. Composing with the wedge product map, $\wedge : \mathfrak{p}^\vee \otimes \bigwedge^q \mathfrak{p}^\vee \to \bigwedge^{q+1} \mathfrak{p}^\vee$, which is K-equivariant, recovers the exterior derivative:

$$d = \Gamma(\overline{\overline{\mathscr{F}}}_{\bigwedge^q \mathfrak{p}^\vee}) \xrightarrow{\overline{\nabla}} \Gamma(\overline{\overline{\mathscr{F}}}_{\mathfrak{p}^\vee \otimes \bigwedge^q \mathfrak{p}^\vee}) \xrightarrow{\overline{\wedge}} \Gamma(\overline{\overline{\mathscr{F}}}_{\bigwedge^{q+1} \mathfrak{p}^\vee}).$$

In the dual picture, applying formula (33) for $V = \bigwedge^q \mathfrak{p}^\vee$ yields the following formula for $d = \overline{\wedge} \circ \overline{\Theta}^{\bigwedge^q \mathfrak{p}^\vee}$:

$$\begin{aligned} d &: \mathrm{Hom}_K(\textstyle\bigwedge^q \mathfrak{p}, C^\infty(\widetilde{\Gamma}\backslash G)) \to \mathrm{Hom}_K(\textstyle\bigwedge^{q+1} \mathfrak{p}, C^\infty(\widetilde{\Gamma}\backslash G)), \\ \omega &\mapsto \Big(X_0 \wedge \cdots \wedge X_q \mapsto \sum_i X_i \cdot \omega(X_0 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_q)\Big). \end{aligned} \tag{38}$$

See [8, Chapter 1, Section 1]. Implicitly in (38), we have used the canonical duality between $\bigwedge^q \mathfrak{p}^\vee$ and $\bigwedge^q \mathfrak{p}$.

B.3.3. Representation-by-representation analysis of $*d$. For every irreducible representation π of G, we define the π-isotypic $*d$ operators

$$(*d)_\pi := \overline{*}_\pi \circ *\overline{\wedge}_\pi \circ \overline{\Theta}_\pi^{\bigwedge^1 \mathfrak{p}^\vee}.$$

Suppose $\pi \subset L^2(\widetilde{\Gamma}\backslash G)$ is a subrepresentation. By the commutativity of the diagrams (34) and (35), it follows that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty) & \xrightarrow{(*d)_\pi} & \mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty) \\ \iota\downarrow & & \downarrow\iota \\ \mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, C^\infty(\widetilde{\Gamma}\backslash G)) & \xrightarrow{*d} & \mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, C^\infty(\widetilde{\Gamma}\backslash G)) \end{array} \tag{39}$$

is commutative, where the vertical maps ι in both diagrams are induced by the inclusion $\pi \subset L^2(\widetilde{\Gamma}\backslash G)$.

Similarly to the case of the Dirac operator, we will show in Proposition B.6 that if π is an irreducible (unitary) representation of G and $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty)$ is non-zero, then it is 1-dimensional. For all such representations π, $(*d)_\pi$ thus necessarily acts on $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty)$ by some scalar $\lambda_{*d,\pi}$ (which will be computed in Proposition B.8). Thus, $\iota(\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty))$ is an eigenline for the odd signature operator with eigenvalue $\lambda_{*d,\pi}$. In particular, the natural decomposition

$$L^2(\Omega^1) = \bigoplus_{\pi \in \widehat{G}} m_{\widetilde{\Gamma}}(\pi) \cdot \mathrm{Hom}_K(\textstyle\bigwedge^1 \mathfrak{p}, \pi^\infty),$$

which is the essentially the Matsushima decomposition we used in [32], corresponds to the eigenspace decomposition of the odd signature operator. Notice that of course closed forms are in the kernel of $*d$; as in [32], we will identify exactly which representations correspond to coexact forms.

B.4. Frobenius reciprocity and K-isotypic vectors for induced representations

Let $M = K \cap B$ be the subgroup of diagonal unitary matrices. Let χ_{n_1,n_2} denote the character of M given by

$$\chi_{n_1,n_2} : M \to \mathbb{C}^\times, \quad \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto t_1^{n_1} \cdot t_2^{n_2},$$

which is obtained by restricting the family of characters χ_{s_1,s_2,n_1,n_2} of B.

Suppose $\pi \subset L^2(\widetilde{\Gamma}\backslash G)$ is a subrepresentation isomorphic to π_{s_1,s_2,n_1,n_2}. The following lemma will allow us to characterize whether π contributes to the spinor (resp. $*d$) eigenvalue spectrum. And if π contributes, it will allow us to write down explicit vectors in $\mathrm{Hom}_K(S^\vee, \pi^\infty)$ (resp. $\mathrm{Hom}_K(S^\vee, \pi^\infty)$) whose images in $\mathrm{Hom}_K(S^\vee, C^\infty(\widetilde{\Gamma}\backslash G))$ (resp. in $\mathrm{Hom}_K(S^\vee, C^\infty(\widetilde{\Gamma}\backslash G))$) are Dirac eigenspinors (resp. $*d$ eigen-1-forms), per the discussion from Section B.2 (resp. Section B.3).

Lemma B.4. *Let (V, ρ) be a finite-dimensional representation of K. Then the map*

$$\begin{aligned}\mathrm{Hom}_M(V^\vee, \chi_{n_1,n_2}) &\to \mathrm{Hom}_K(V^\vee, \pi_{s_1,s_2,n_1,n_2}),\\ T &\mapsto \big(\ell \mapsto \chi_{s_1,s_2,n_1,n_2}(b)\delta(b)^{1/2} \cdot T(\rho^\vee(k)\ell)\big),\end{aligned}$$

where δ is the modular function, is an isomorphism.

In particular, this reduces the problem of determining $V^\vee$-isotypic vectors to the (essentially trivial) problem of understanding $V^\vee$ as a representation of M.

Proof of Lemma B.4. This is proved in [32, Appendix B], to which we refer for a more detailed discussion (and unraveling of the formulas). The isomorphism is the composite of the Frobenius reciprocity isomorphism

$$\begin{aligned}\mathrm{Hom}_M(V^\vee, \chi_{n_1,n_2}) &\to \mathrm{Hom}_K(V^\vee, \mathrm{Ind}_M^K \chi_{n_1,n_2}),\\ T &\mapsto \big(\ell \mapsto f_\ell(k) := T(\rho^\vee(k)\ell)\big),\end{aligned}$$

and the map

$$\begin{aligned}\mathrm{Hom}_K(V^\vee, \mathrm{Ind}_M^K \chi_{n_1,n_2}) &\to \mathrm{Hom}_K(V^\vee, \pi_{s_1,s_2,n_1.n_2}),\\ S &\mapsto \big(\ell \mapsto g_\ell(bk) := \chi_{s_1,s_2,n_1,n_2}(b)\delta(b)^{1/2} \cdot [S(\ell)](k)\big).\end{aligned}$$

The latter is an isomorphism because $G = BK$, $M = B \cap K$, and $\chi_{s_1,s_2,n_1,n_2} \cdot \delta^{1/2}$ restricted to M equals χ_{n_1,n_2}. ∎

B.4.1. Representations contributing to the spinor spectrum. We need to apply Lemma B.4 to the representation $S^\vee$, where again $S = \mathbb{C}^2$ is the standard representation of U_2.

Proposition B.5. *Let π be an irreducible unitary representation of G. Then $\mathrm{Hom}_K(S^\vee, \pi^\infty)$ is either* 0- *or* 1-*dimensional. It is non-zero* (*and hence* 1-*dimensional*) *if and only if* π *is isomorphic to* $\pi_{s_1,s_2,-1,0}$ (*or equivalently* $\pi_{s_2,s_1,0,-1}$), *in which case a basis vector for* $\mathrm{Hom}_K(S^\vee, \pi^\infty)$ *is given by*

$$\ell \mapsto f_\ell(bk) := \chi_{s_1,s_2,-1,0}(b)\delta(b)^{1/2} \cdot [\mathrm{std}^\vee(k)\ell](e_1).$$

Proof. We easily check that $\mathrm{Hom}_K(S^\vee, \pi^\infty) = 0$ if π is isomorphic to some power of the determinant. So we focus on the case $\pi \cong \pi_{s_1,s_2,n_1,n_2}$.

By Lemma B.4, $\mathrm{Hom}_K(S^\vee, \pi^\infty)$ is naturally isomorphic to $\mathrm{Hom}_M(S^\vee, \chi_{n_1,n_2})$. Since $S^\vee|_M$ is isomorphic to $\chi_{-1,0} \oplus \chi_{0,-1}$, it follows that:

- $\mathrm{Hom}_K(S^\vee, \pi^\infty_{s_1,s_2,n_1,n_2})$ is non-zero iff $(n_1, n_2) = (-1, 0)$ or $(0, -1)$. If non-zero, it is exactly 1-dimensional.
- The intertwining map $\ell \mapsto \ell(e_1)$ is a basis vector for the 1-dimensional space $\mathrm{Hom}_M(S^\vee, \chi_{-1,0})$. Its image under the isomorphism from Lemma B.4, namely

$$\ell \mapsto f_\ell(bk) := \chi_{s_1,s_2,-1,0}(b)\delta(b)^{1/2} \cdot [\mathrm{std}^\vee(k)\ell](e_1),$$

  thus defines a basis vector for the 1-dimensional space $\mathrm{Hom}_K(S^\vee, \pi_{s_1,s_2,-1,0})$.

This proves the proposition. ■

B.4.2. Representations contributing to the coclosed 1-form spectrum. We need to apply Lemma B.4 to the representation $\bigwedge^1 \mathfrak{p} = \mathfrak{p}$, where again $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$ and $\mathfrak{p}_0 = i\mathfrak{su}(2)$ is acted on by K via the adjoint representation. Denoting again the Pauli matrices in (36) by σ_i, we find that

$$w_1 = \sigma_1 + i\sigma_2, \quad w_0 = \sigma_3, \quad w_{-1} = \sigma_1 - i\sigma_2 \tag{40}$$

are the M-weight vectors for $\bigwedge^1 \mathfrak{p}$ of respective weights $(1, -1)$, $(0, 0)$ and $(-1, 1)$. We denote the corresponding dual basis by $w_1^\vee, w_0^\vee, w_{-1}^\vee$.

Proposition B.6. *Let* π *be an irreducible unitary representation of* G. *Then the space* $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty)$ *is either* 0- *or* 1-*dimensional. It is non-zero* (*and hence* 1-*dimensional*) *if and only if* π *is isomorphic to* $\pi_{s_1,s_2,1,-1}$ (*or equivalently* $\pi_{s_2,s_1,-1,1}$) *or* $\pi_{s_1,s_2,0,0}$; *only the representations of the form* $\pi_{s_1,s_2,1,-1}$ *contribute to the coclosed* 1-*form spectrum, in which case a basis vector for* $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty)$ *is given by*

$$v \mapsto f_v(bk) := \chi_{s_1,s_2,1,-1}(b)\delta(b)^{1/2} \cdot w_1^\vee(\mathrm{Ad}(k)v).$$

Proof. We easily check that $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty) = 0$ if π is isomorphic to some power of the determinant. So we focus on the case $\pi \cong \pi_{s_1,s_2,n_1,n_2}$.

By Lemma B.4, $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty)$ is naturally isomorphic to $\mathrm{Hom}_M(\bigwedge^1 \mathfrak{p}, \chi_{n_1,n_2})$. Since $\bigwedge^1 \mathfrak{p}|_M$ is isomorphic to $\chi_{1,-1} \oplus \chi_{0,0} \oplus \chi_{-1,1}$, it follows that

- $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi^\infty_{s_1,s_2,n_1,n_2})$ is non-zero iff $(n_1, n_2) = (1, -1)$, $(0, 0)$, or $(-1, 1)$. If non-zero, it is exactly 1-dimensional. Furthermore, as argued in [32, Lemma B.11], the representations $\pi_{s_1,s_2,1,-1}$ and $\pi_{s_1,s_2,-1,1}$, isomorphic respectively to $\pi_{s_1-s_2,\pm 1}$ in the notation from [32, Appendix B], only contribute to the coclosed 1-form spectrum, while the representations $\pi_{s_1,s_2,0,0}$, isomorphic to $\pi_{s_1-s_2,0}$ in the notation from [32, Appendix B], only contribute to the closed 1-form spectrum.

- The intertwining map $v \mapsto w_1^\vee(v)$ is a basis vector for the 1-dimensional space $\mathrm{Hom}_M(\bigwedge^1 \mathfrak{p}, \chi_{1,-1})$. Its image under the isomorphism from Lemma B.4, namely

$$\ell \mapsto f_v(bk) := \chi_{s_1,s_2,1,-1}(b)\delta(b)^{1/2} \cdot w_1^\vee(\mathrm{Ad}(k)v),$$

thus defines a basis vector for the 1-dimensional space $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi_{s_1,s_2,1,-1})$.

This proves the proposition. ■

B.5. Representation-by-representation calculation of Dirac and $*d$ eigenvalues

We conclude the proofs of the main results of this Appendix by computing the eigenvalues $\lambda_{*d,\pi}$ and $\lambda_{\mathrm{Dirac},\pi}$ of the eigenforms/eigenspinors corresponding to isotypic vectors.

Remark B.3. While in [32, Appendix B] we performed the computations in the natural Killing metric, and then normalized the result to obtain the answer for the hyperbolic metric, we will now work directly with the hyperbolic metric.

Proposition B.7 (Calculation of $\lambda_{\mathrm{Dirac},\pi}$). *Let $\pi = \pi_{s_1,s_2,-1,0}$. Then*

$$\lambda_{\mathrm{Dirac},\pi} = i \cdot \frac{s_1 - s_2}{2}.$$

Proof. The space $\mathrm{Hom}_{T\cap K}(S^\vee, \chi_{-1,0})$ has basis $\ell \mapsto \ell(e_1)$. Thus according to Lemma B.4, the unique (up to scaling) element T of $\mathrm{Hom}_K(S^\vee, \pi_{s_1,s_2,-1,0})$ equals

$$T(\ell) = f_\ell : G \to \mathbb{C}, \quad f_\ell(bk) = \delta^{1/2}(b)\chi_{s_1,s_2,0,-1}(b) \cdot [\mathrm{std}^\vee(k)\ell](e_1).$$

Suppose $\bar{D}_{\pi_{s_1,s_2,-1,0}} T = cT$. Then, as $T(e_1^\vee)(1) = 1$, we have

$$\lambda_{\mathrm{Dirac},\pi} = c = cT(e_1^\vee)(1) = (\bar{D}_{\pi_{s_1,s_2,-1,0}} T)(e_1^\vee)(1),$$

so we will evaluate the latter next. We calculate:

$$\begin{aligned} m_{\sigma_1}^\vee &: e_1^\vee \mapsto e_2^\vee, && e_2^\vee \mapsto e_1^\vee, \\ m_{\sigma_2}^\vee &: e_1^\vee \mapsto -ie_2^\vee, && e_2^\vee \mapsto ie_1^\vee, \\ m_{\sigma_3}^\vee &: e_1^\vee \mapsto e_1^\vee, && e_2^\vee \mapsto -e_2^\vee, \end{aligned}$$

Therefore we have the explicit expression

$$\begin{aligned} \bar{D}_{\pi_{s_1,s_2,-1,0}} T(e_1^\vee) &= \frac{2i}{2^2} \cdot \big(\sigma_1 \cdot Tm_{\sigma_1}^\vee(e_1^\vee) + \sigma_2 \cdot Tm_{\sigma_2}^\vee(e_1^\vee) + \sigma_3 \cdot Tm_{\sigma_3}^\vee(e_1^\vee)\big) \\ &= \frac{i}{2} \cdot (\sigma_1 \cdot f_{e_2^\vee} + \sigma_2 \cdot f_{-ie_2^\vee} + \sigma_3 \cdot f_{e_1^\vee}), \end{aligned}$$

where the factor 2^2 comes from the fact that the Pauli matrices have norm 2 in the hyperbolic metric (see Remark B.1). To evaluate the above expression at 1, we express each Pauli matrix in the form $K_i + B_i$, where $K_i \in \mathfrak{k}$ and $B_i \in \mathfrak{b}$ (here $\mathfrak{k}$ and $\mathfrak{b}$ are the Lie algebras of K and B respectively):

$$\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: K_1 + B_1,$$
$$\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - 2\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} =: K_2 + B_2,$$
$$\sigma_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: K_3 + B_3$$

Notice that $B_1, B_2 \in \mathfrak{n}$, the Lie algebra of the group N of unipotent matrices. Because the functions f_ℓ are invariant under left translation by elements of N,

$$\begin{aligned}\bar{D}_{\pi_{s_1,s_2,-1,0}} T(e_1^\vee) &= \frac{i}{2} \cdot (\sigma_1 \cdot f_{e_2^\vee} + \sigma_2 \cdot f_{-ie_2^\vee} + \sigma_3 \cdot f_{e_1^\vee}) \\ &= \frac{i}{2} \cdot (K_1 \cdot f_{e_2^\vee} + K_2 \cdot f_{-ie_2^\vee} + B_3 \cdot f_{e_1^\vee}).\end{aligned}$$

We calculate each summand, evaluated at 1, individually:

$$\begin{aligned}(K_1 \cdot f_{e_2^\vee})(1) &= 1 \cdot \frac{d}{dt}\bigg|_{t=0} [e_2^\vee](e^{-tK_1} e_1) = [e_2^\vee](-K_1 e_1) = -1, \\ (K_2 \cdot f_{-ie_2^\vee})(1) &= 1 \cdot \frac{d}{dt}\bigg|_{t=0} [-ie_2^\vee](e^{-tK_2} e_1) = [-ie_2^\vee](-K_2 e_1) \\ &= [-ie_2^\vee](-ie_2) = -1, \\ (B_3 \cdot f_{e_1^\vee})(1) &= \frac{d}{dt}\bigg|_{t=0} \delta^{1/2} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \chi_{s_1,s_2,-1,0} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot [e_1^\vee](e_1) \\ &= \frac{d}{dt}\bigg|_{t=0} e^{2t} \cdot e^{t \cdot (s_1 - s_2)} \cdot 1 = s_1 - s_2 + 2.\end{aligned}$$

Summing these contributions, $\bar{D}_{\pi_{s_1,s_2,-1,0}}$ acts on $\mathrm{Hom}_K(S^\vee, \pi_{s_1,s_2,-1,0})$ by the scalar

$$\begin{aligned} c = \bar{D}_{\pi_{s_1,s_2,-1,0}} T(e_1^\vee)(1) &= \frac{i}{2} \cdot ((-1) + (-1) + (s_1 - s_2 + 2)) \\ &= i \cdot \left(\frac{s_1 - s_2}{2}\right), \end{aligned} \tag{41}$$

and the proof is complete. ∎

We conclude by computing $\lambda_{*d,\pi}$, obtaining the desired refinement of the result of [32].

Proposition B.8 (Calculation of $\lambda_{*d,\pi}$). *Let* $\pi = \pi_{s_1,s_2,1,-1}$. *Then*

$$\lambda_{*d,\pi} = i \cdot \frac{s_1 - s_2}{2}.$$

Proof. We will use the notation from (40). According to Proposition B.6, the unique (up to scaling) element T of $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi_{s_1,s_2,1,-1})$ equals

$$T(v) = f_v : G \to \mathbb{C}, \quad f_v(bk) = \delta^{1/2}(b)\chi_{s_1,s_2,1,-1}(b) \cdot w_1^\vee(\mathrm{Ad}(k)v).$$

Recall the π-isotypic version $(*d)_\pi$ of $*d$ from Section B.3.3. Because the space $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi_{s_1,s_2,1,-1})$ is 1-dimensional, $(*d)_\pi$ acts on it by a scalar. We next calculate the eigenvalue of $(*d)_\pi$ acting on $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi)$ for $\pi = \pi_{s_1,s_2,1,-1}$. To lighten notation, we refer to d_π, $*_\pi$ and $(*d)_\pi$ simply by d, $*$, and $*d$ respectively. Suppose $(*d)T = cT$. Then

$$\lambda_{*d,\pi} = c = cT(w_1)(1) = ((*d)T)(w_1)(1),$$

so we will evaluate the latter next. Observe that

$$\begin{aligned} dT(\sigma_1 \wedge \sigma_2) &= \sigma_1 \cdot T(\sigma_2) - \sigma_2 \cdot T(\sigma_1), \\ dT(\sigma_2 \wedge \sigma_3) &= \sigma_2 \cdot T(\sigma_3) - \sigma_3 \cdot T(\sigma_2), \\ dT(\sigma_3 \wedge \sigma_1) &= \sigma_3 \cdot T(\sigma_1) - \sigma_1 \cdot T(\sigma_3), \end{aligned}$$

implying that

$$\begin{aligned} 2 \cdot *dT(\sigma_3) &= dT(\sigma_1 \wedge \sigma_2) = \sigma_1 \cdot T(\sigma_2) - \sigma_2 \cdot T(\sigma_1), \\ 2 \cdot *dT(\sigma_1) &= dT(\sigma_2 \wedge \sigma_3) = \sigma_2 \cdot T(\sigma_3) - \sigma_3 \cdot T(\sigma_2), \\ 2 \cdot *dT(\sigma_2) &= dT(\sigma_3 \wedge \sigma_1) = \sigma_3 \cdot T(\sigma_1) - \sigma_1 \cdot T(\sigma_3), \end{aligned}$$

where the factor of 2 appears because the norm of the Pauli matrices in the hyperbolic metric is 2 (see Remark B.1). It thus follows that

$$\begin{aligned} 2 \cdot *dT(w_1) &= 2 \cdot (*dT(\sigma_1) + i * dT(\sigma_2)) \\ &= \sigma_2 \cdot T(\sigma_3) - \sigma_3 \cdot T(\sigma_2) + i(\sigma_3 \cdot T(\sigma_1) - \sigma_1 \cdot T(\sigma_3)). \end{aligned}$$

To evaluate the above expression at 1, recall the decomposition of the Pauli matrices $\sigma_i = K_i + B_i$ introduced in the proof of Proposition B.7. Again, the functions f_v are invariant under left translation by elements of N, and therefore we compute

$$\begin{aligned} 2 \cdot *dT(w_1) &= \sigma_2 \cdot T(\sigma_3) - \sigma_3 \cdot T(\sigma_2) + i(\sigma_3 \cdot T(\sigma_1) - \sigma_1 \cdot T(\sigma_3)) \\ &= K_2 \cdot f_{\sigma_3} - B_3 \cdot f_{\sigma_2} + i(B_3 \cdot f_{\sigma_1} - K_1 \cdot f_{\sigma_3}). \end{aligned}$$

We calculate each summand, evaluated at 1, individually. The terms involving B_3 are

$$\begin{aligned} (B_3 \cdot f_{\sigma_2})(1) &= \frac{d}{dt}\bigg|_{t=0} \delta^{1/2}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \chi_{s_1,s_2,1,-1}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot w_1^\vee(\sigma_2) \\ &= \frac{d}{dt}\bigg|_{t=0} e^{2t} \cdot e^{t\cdot(s_1-s_2)} \cdot \frac{1}{2}(-i) \\ &= (2 + s_1 - s_2) \cdot \frac{1}{2}(-i), \end{aligned}$$

$$
\begin{aligned}
(B_3 \cdot f_{\sigma_1})(1) &= \frac{d}{dt}\bigg|_{t=0} \delta^{1/2}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \chi_{s_1,s_2,1,-1}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot w_1^\vee(\sigma_1) \\
&= \frac{d}{dt}\bigg|_{t=0} e^{2t} \cdot e^{t\cdot(s_1-s_2)} \cdot \frac{1}{2} \\
&= (2+s_1-s_2)\cdot\frac{1}{2},
\end{aligned}
$$

while those involving the K_i are

$$
\begin{aligned}
(K_2 \cdot f_{\sigma_3})(1) &= 1 \cdot \frac{d}{dt}\bigg|_{t=0} w_1^\vee(\mathrm{Ad}(e^{tK_2})\sigma_3) \\
&= w_1^\vee([K_2,\sigma_3]) = w_1^\vee(2\sigma_2) = -i, \\
(K_1 \cdot f_{\sigma_3})(1) &= 1 \cdot \frac{d}{dt}\bigg|_{t=0} w_1^\vee(\mathrm{Ad}(e^{tK_1})\sigma_3) \\
&= w_1^\vee([K_1,\sigma_3]) = w_1^\vee(2\sigma_1) = 1.
\end{aligned}
$$

Summing these contributions, $*d$ acts on $\mathrm{Hom}_K(\bigwedge^1 \mathfrak{p}, \pi_{s_1,s_2,1,-1})$ by the scalar

$$
\begin{aligned}
c &= *dT(w_1)(1) \\
&= \frac{1}{2}\cdot\left(-i-(2+s_1-s_2)\cdot\frac{1}{2}(-i)+i\left((2+s_1-s_2)\cdot\frac{1}{2}-1\right)\right) \\
&= i\cdot\frac{s_1-s_2}{2},
\end{aligned}
$$

completing the proof. ■

Appendix C. Proof of the explicit trace formulas

We will now prove the results in Section 3, starting with the more involved case of spinors in Theorem 3.3.

C.1. Isolating the spinor spectrum

We discuss suitable choices of test function to specialize the general trace formula for G in Proposition A.2. According to Proposition B.5, the irreducible unitary representations of G contributing to the spinor spectrum are precisely those isomorphic to $\pi_{s_1,s_2,-1,0}$ or $\pi_{s_1,s_2,0,-1}$ for purely imaginary s_1, s_2. To isolate the latter representations, we use test functions of the form

$$
F\begin{pmatrix} e^{u/2}e^{i\alpha} & 0 \\ 0 & e^{-u/2}e^{i\beta} \end{pmatrix} = \begin{cases} H_+(u)(e^{i\alpha}+e^{i\beta}), & H_+ \text{ even}, \\ H_-(u)(e^{i\alpha}-e^{i\beta}), & H_- \text{ odd}. \end{cases}
$$

We now discuss how each term in the trace formula simplifies when choosing test functions of the latter shapes.

C.1.1. Regular geometric terms for spinors. For our normalization of Haar measure (29), the covolume of the centralizer of γ equals $\ell(\gamma_0)$. If its conjugacy class in G is represented by

$$\begin{pmatrix} e^{\ell/2}e^{i\theta}e^{i\phi} & 0 \\ 0 & e^{-\ell/2}e^{-i\theta}e^{i\phi} \end{pmatrix},$$

then its complex length equals $\ell + i2\theta$ and its corresponding summand on the geometric side of the trace formula from Proposition A.2 equals

$$\ell_0 \cdot \frac{1}{|1-e^{\ell+i2\theta}|\,|1-e^{-\ell-i2\theta}|} \cdot H_\pm(\ell)\cdot(e^{i(\theta+\phi)} \pm e^{i(-\theta+\phi)}).$$

Here we use the fact that the Weyl discriminant is the same as in the trace formula from [32]:

$$|D(t_\gamma)|^{1/2} = \frac{1}{|1-e^{\mathbb{C}\ell(\overline{\gamma})}|\,|1-e^{-\mathbb{C}\ell(\overline{\gamma})}|},$$

where $\overline{\gamma}$ denotes the image of γ in $\mathrm{PGL}_2(\mathbb{C})$. In particular, it is independent of ϕ.

C.1.2. Contribution of central elements for spinors. Let $z \in Z \subset G$ be central, so

$$z = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

for some $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Recalling that for the two different parametrizations of T we have $u = 2v$, the contribution of the conjugacy class of z in the trace formula of Proposition A.2 is then

$$\begin{aligned}
-\frac{1}{8\pi}\cdot \mathrm{vol}(Y)\cdot\left(\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial \theta^2}\right)\bigg|_{t=z} F \\
&= -\frac{1}{8\pi}\cdot \mathrm{vol}(Y)\cdot\left(\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial \theta^2}\right)\bigg|_{(v,\theta)=(0,0)} H_+(2v)(e^{i(\theta+\phi)}+e^{i(-\theta+\phi)}) \\
&= \frac{1}{8\pi}\cdot \mathrm{vol}(Y)\cdot 2e^{i\phi}\cdot (H_+(0)-4H_+''(0)) \\
&= \frac{1}{4\pi}\cdot \mathrm{vol}(Y)\cdot e^{i\phi}\cdot (H_+(0)-4H_+''(0)).
\end{aligned}$$

In our case, we will be interested in lifts of torsion-free lattices $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$, so that we will only need to consider the case $z = 1$, corresponding to $\phi = 0$.

C.1.3. Regular spectral terms for spinors. Let $s_1 = ir_1, s_2 = ir_2 \in i\mathbb{R}$. The summand on the spectral side of the trace formula from Proposition A.2 corresponding to π_{s_1,s_2,n_1,n_2} then equals

$$\begin{aligned}
\hat{F}(\chi^{-1}_{s_1,s_2,n_1,n_2}) &= \int_T F(t)\chi_{s_1,s_2,n_1,n_2}(t)\,dt \\
&= \iiint H_\pm(u)\cdot(e^{i\alpha}\pm e^{i\beta})\cdot e^{u/2\cdot s_1}e^{-u/2\cdot s_2}\cdot e^{in_1\alpha}\cdot e^{in_2\beta}\,du\,\frac{d\alpha}{2\pi}\,\frac{d\beta}{2\pi}.
\end{aligned}$$

In the case of $H_+ \cdot (e^{i\alpha} + e^{i\beta})$, the latter equals

$$\widehat{H_+}\left(\frac{r_2 - r_1}{2}\right) \cdot \begin{cases} +1 & \text{if } (n_1, n_2) = (-1, 0) \text{ or } (0, -1), \\ 0 & \text{otherwise,} \end{cases}$$

while in the case of $H_- \cdot (e^{i\alpha} - e^{i\beta})$, it equals

$$\widehat{H_-}\left(\frac{r_2 - r_1}{2}\right) \cdot \begin{cases} +1 & \text{if } (n_1, n_2) = (-1, 0), \\ -1 & \text{if } (n_1, n_2) = (0, -1), \\ 0 & \text{otherwise.} \end{cases}$$

C.1.4. Contribution of $\det^k$ *to the spectral side.* For every $k \in \mathbb{Z}$, the contribution of $\det^k$ to the trace formula for the test function $H_\pm(u) \cdot (e^{i\alpha} \pm e^{i\beta})$ equals

$$\begin{aligned} &\frac{1}{|W|} \int_T |D(t^{-1})|^{1/2} \cdot \det(t)^k \cdot F(t)\, dt \\ &= \frac{1}{|W|} \iiint \left(e^u + e^{-u} - (e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)})\right) \cdot e^{k \cdot i(\alpha+\beta)} \\ &\qquad\qquad \cdot H_\pm(u) \cdot (e^{i\alpha} \pm e^{i\beta}) \frac{d\alpha}{2\pi} \frac{d\beta}{2\pi}\, du \\ &= 0, \end{aligned}$$

since the integral of every non-trivial character over the torus $\mathrm{U}_1 \times \mathrm{U}_1$ equals 0.

C.2. Trace formula for spinors

We are finally ready to prove the trace formulas that appear in Theorem 3.3.

C.2.1. Even spinor trace formula. Let H_+ be a smooth, compactly supported, even test function on $\mathbb{R}$. Let $\widetilde{\Gamma}$ be the lift to G of the closed hyperbolic three-manifold group $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$ (so that $\widetilde{\Gamma} \cap Z = \{1\}$). For $\gamma \in \widetilde{\Gamma}$, suppose that γ is conjugate to

$$\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} e^{u/2} e^{i\alpha} & 0 \\ 0 & e^{-u} e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{u/2} e^{i\theta} e^{i\phi} & 0 \\ 0 & e^{-u/2} e^{-i\theta} e^{i\phi} \end{pmatrix}.$$

Let $z = z_1/z_2$. Let ℓ_0 denote the length of the primitive closed geodesic some multiple of which corresponds to γ. Unwinding all terms in the trace formula from Proposition A.2 as in Section C.1 yields the following trace formula:

$$\begin{aligned} &\sum m_{\widetilde{\Gamma}}(\pi_{s_1, s_2, -1, 0}) \cdot \widehat{H_+}\left(\frac{r_2 - r_1}{2}\right) \\ &\qquad = \frac{1}{4\pi} \cdot \mathrm{vol}(Y) \cdot (H_+(0) - 4H_+''(0)) \\ &\qquad\quad + \sum_{[\gamma]} \ell_0 \cdot \frac{1}{|1 - z|\,|1 - z^{-1}|} \cdot (e^{i\alpha} + e^{i\beta}) \cdot H_+(u), \end{aligned}$$

where the LHS is summed over all isomorphism classes of irreducible unitary representations of G isomorphic to some $\pi_{s_1,s_2,-1,0}$. Since H_+ is even, the left side is real. So taking real parts yields

$$\begin{aligned}\sum m_{\tilde\Gamma}(\pi_{s_1,s_2,-1,0})\cdot \widehat{H_+}\left(\frac{r_2-r_1}{2}\right)&\\ &= \frac{1}{4\pi}\cdot \mathrm{vol}(Y)\cdot (H_+(0)-4H_+''(0))\\ &\quad+\sum_{[\gamma]}\ell_0\cdot\frac{1}{|1-z|\,|1-z^{-1}|}\cdot(\cos\alpha+\cos\beta)\cdot H_+(u)\\ &= \frac{1}{4\pi}\cdot \mathrm{vol}(Y)\cdot (H_+(0)-4H_+''(0))\\ &\quad+\sum_{[\gamma]}\ell_0\cdot\frac{1}{|1-z|\,|1-z^{-1}|}\cdot(2\cos\theta\cos\phi)\cdot H_+(u). \end{aligned} \tag{42}$$

Finally, by Proposition B.7 the Dirac eigenvalue on $\mathrm{Hom}_K(S^\vee,\pi_{s_1,s_2,0,-1})$, with orientation $\sigma_1\wedge\sigma_2\wedge\sigma_3$, equals $i\,\frac{s_1-s_2}{2}=\frac{r_2-r_1}{2}$. Thus, by Proposition B.5 we can reexpress the trace formula from (42) geometrically as

$$\begin{aligned}\frac{1}{2}\cdot\sum_{\text{Dirac eigenvalues for }\tilde\Gamma} m_{\tilde\Gamma}(\lambda)\cdot\widehat{H_+}(\lambda)&\\ &=\frac{1}{2\pi}\cdot\mathrm{vol}(Y)\cdot\left(\frac{1}{4}\cdot H_+(0)-H_+''(0)\right)\\ &\quad+\sum_{[\gamma]}\ell_0\cdot\frac{1}{|1-z|\,|1-z^{-1}|}\cdot(\cos\theta\cos\phi)\cdot H_+(u).\end{aligned}$$

This is the even trace formula in Theorem 3.3.

C.2.2. Odd spinor trace formula: representation-theoretic form. Let H_- be a smooth, compactly supported, odd test function on $\mathbb{R}$. Using the notation of the previous subsection, unwinding all terms in the trace formula from Proposition A.2 as in Section C.1 yields the following trace formula:

$$\begin{aligned}\sum m_{\tilde\Gamma}(\pi_{s_1,s_2,-1,0})\cdot\widehat{H_-}\left(\frac{r_2-r_1}{2}\right)&\\ &=-\sum_{[\gamma]}\ell_0\cdot\frac{1}{|1-z|\,|1-z^{-1}|}\cdot(e^{i\alpha}-e^{i\beta})\cdot H_-(u),\end{aligned}$$

where the LHS is summed over all isomorphism classes of irreducible unitary representations of G isomorphic to some $\pi_{s_1,s_2,-1,0}$. Since H_- is odd, its Fourier transform is purely imaginary. So looking at imaginary parts gives

$$\begin{aligned}\sum m_{\widetilde{\Gamma}}(\pi_{s_1,s_2,-1,0}) \cdot \widehat{H_-}\left(\frac{r_2-r_1}{2}\right) \\ &= i\sum_{[\gamma]} \ell_0 \cdot \frac{1}{|1-z|\,|1-z^{-1}|} \cdot (\sin\alpha - \sin\beta) \cdot H_-(u) \\ &= i\sum_{[\gamma]} \ell_0 \cdot \frac{1}{|1-z|\,|1-z^{-1}|} \cdot (2\sin\theta\cos\phi) \cdot H_-(u). \qquad (43)\end{aligned}$$

By (B.7), the Dirac eigenvalue on $\mathrm{Hom}_K(S^\vee, \pi_{s_1,s_2,-1,0})$, with orientation $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$, equals $i\,\frac{s_1-s_2}{2} = \frac{r_2-r_1}{2}$. Thus by Proposition B.5, we can reexpress the above trace formula from (43) geometrically:

$$\begin{aligned}\frac{1}{2} \sum_{\text{Dirac eigenvalues for } \widetilde{\Gamma}} m_{\widetilde{\Gamma}}(\lambda) \cdot \widehat{H_-}(\lambda) \\ &= i\sum_{[\gamma]} \ell_0 \cdot \frac{1}{|1-z|\,|1-z^{-1}|} \cdot (\sin\theta\cos\phi) \cdot H_-(u). \qquad (44)\end{aligned}$$

This is the odd trace formula in Theorem 3.3; let us point out again that in (44), the Dirac operator is taken relative to the orientation $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ on $\mathbb{H}^3$.

C.3. The trace formula for coexact 1-forms

Specializing the trace formula to isolate coexact 1-forms is more straightforward than specializing the trace formula to isolate spinors as in Section C.2. We content ourselves with highlighting the main differences between specializing to 1-forms versus specializing to spinors.

- Irreducible unitary subrepresentations of $L^2(\widetilde{\Gamma}\backslash G)$ contributing to the coclosed 1-form spectrum are precisely those isomorphic to $\pi_{s_1,s_2,1,-1}$ and $\pi_{s_1,s_2,-1,1}$ for $s_1, s_2 \in i\mathbb{R}$. We isolate those representations using the test functions $H_\pm(u) \cdot (e^{i(\alpha-\beta)} \pm e^{i(\beta-\alpha)})$, where H_+ is even and H_- is odd.
- For the latter test functions $H_\pm(u) \cdot (e^{i(\alpha-\beta)} \pm e^{i(\beta-\alpha)})$, the summand corresponding to the representation $\pi_{s_1,s_2,1,-1}$ equals $\widehat{H_\pm}(\frac{r_2-r_1}{2})$. By Proposition B.8, $\frac{r_2-r_1}{2}$ is the eigenvalue of $*d$ acting on the $\bigwedge^1 \mathfrak{p}$-isotypic vector of π (for the orientation $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ on $\mathbb{H}^3$). For a subrepresentation $\pi \subset L^2(\widetilde{\Gamma}\backslash G)$ isomorphic to $\pi_{s_1,s_2,1,-1}$, the latter equals the eigenvalue of $*d$ acting on the $\bigwedge^1 \mathfrak{p}$-isotypic vectors of $\pi_{s_1,s_2,1,-1}$.
- The contribution of $\det^k$ to the trace formula for 1-forms is non-trivial only if $k = 0$. If $k = 0$, the contribution equals 0 for test functions $H_-(u) \cdot (e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)})$ for odd H_- and equals $-\widehat{H_+}(0)$ for the test function

  $$H_+(u) \cdot (e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)}) = 2H_+(2v)\cos(2\theta)$$

  with H_+ even.

- In the even case for the test function $2H_+(2v)\cos(2\theta)$ the identity contribution is $\frac{1}{2\pi} \cdot \mathrm{vol}(Y) \cdot 2(H_+(0) - H_+''(0))$.

From this, one readily obtains the formulas for coexact 1-forms in Theorem 3.2.

Appendix D. Admissibility of Gaussian functions

Gaussian functions were convenient to use at several points in our arguments. The next result proves that functions of sufficiently fast decay, e.g. Gaussians, are admissible for use in the trace formula.

Proposition D.1. *Let f be an even or odd smooth function on $\mathbb{R}$. Suppose that*

$$\sum_{n=0}^{\infty} e^n \cdot \sup_{x\in[n,n+1]} |f(x)| < \infty.$$

Then both the geometric and spectral sides of the spinor trace formula for closed hyperbolic three-manifolds converge absolutely for the test function f and they are equal. The same statement holds for the 1-form trace formula applied to even or odd test functions and the 0-form trace formula applied to even test functions f.

Proof. We focus on the spinor case (arguments in the other cases being identical). Convergence of the spectral side of the trace formula for f follows because $\widehat{f}$ is Schwartz and the number of spectral parameters in $[\nu, \nu+1]$ is quadratic in ν (see Section 5)

Convergence of the geometric side of the trace formula follows by our hypothesis. Indeed,

$$\begin{aligned}\sum_{\gamma} \left| \ell(\gamma_0) \cdot \frac{(\cos \text{ or } \sin)\theta \cos\phi}{|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|} \cdot f(\ell(\gamma)) \right| \\ &\leq \sum_{n=0}^{\infty} \Big(\sum_{\ell(\gamma)\in[n,n+1]} \ell(\gamma_0) \Big) \cdot \frac{C}{e^n} \cdot \sup_{x\in[n,n+1]} |f(x)| \\ &\leq \sum_{n=0}^{\infty} D \cdot e^{2n} \cdot \frac{C}{e^n} \cdot \sup_{x\in[n,n+1]} |f(x)| \\ &= CD \cdot \sum_{n=0}^{\infty} e^n \cdot \sup_{x\in[n,n+1]} |f(x)| < \infty. \end{aligned} \tag{45}$$

Above, we have used the fact that $\sum_{\ell(\gamma)\in[L,L+1]} \ell(\gamma_0)$ has order of magnitude e^{2L} (cf. the discussion in Section 9) and that $|1-e^{\mathbb{C}\ell(\gamma)}|\,|1-e^{-\mathbb{C}\ell(\gamma)}|$ has order of magnitude $e^{\ell(\gamma)}$ for $\ell(\gamma)$ large.

The trace formula for f will be a consequence of the limit of the trace formulas applied to the test functions $f(x) \cdot b(x/R)$, where b is a smooth even, compactly supported bump function with $\widehat{b}$ everywhere positive and $b(0) = 1$. As $R \to \infty$, the geometric and spectral sides of the trace formula for $f(x) \cdot b(x/R)$ respectively converge pointwise to the geometric and spectral sides of the trace formula for f; this is immediate for the geometric side, and it follows on the spectral side because $f(x) \cdot b(x/R)$ has Fourier transform $(\widehat{f} * g_R)(t)$, where

$$g_R(t) := \frac{1}{2\pi} \cdot R \cdot \widehat{b}(Rt)$$

is an approximate identity (i.e. it is everywhere positive of total mass 1 and concentrated increasingly near the origin as $R \to \infty$). Here we use the fact that for our definition of Fourier transform, $\widehat{f \cdot g} = \frac{1}{2\pi} \widehat{f} * \widehat{g}$. It therefore suffices to prove that the tails of the geometric and spectral sides of the trace formula, applied to the test function $f(x) \cdot b(x/R)$, approach 0 uniformly as $R \to \infty$.

Uniform smallness of the tail on the geometric side follows by our hypothesis. Indeed,

$$\begin{aligned}
\sum_{\ell(\gamma) \geq k} \left| \ell(\gamma_0) \cdot \frac{(\cos \text{ or } \sin)\theta \cos\phi}{|1 - e^{\mathbb{C}\ell(\gamma)}|\,|1 - e^{-\mathbb{C}\ell(\gamma)}|} \cdot f(\ell(\gamma)) \right| & \\
\leq CD \cdot \sum_{n=k}^{\infty} e^n \cdot \sup_{x \in [n, n+1]} |f(x) b(x/R)| & \qquad \text{(exactly as in (45))} \\
\leq CD \cdot \|b\|_\infty \cdot \sum_{n=k}^{\infty} e^n \cdot \sup_{x \in [n, n+1]} |f(x)|, &
\end{aligned}$$

which converges uniformly to 0 as $k \to \infty$, being the tail of one fixed convergent sum (independent of R).

Uniform smallness of the tail on the spectral side follows by the following estimate:

$$\begin{aligned}
\widehat{f} * g_R(t) &= \int_{\mathbb{R}} \widehat{f}(t - y) g_R(y)\, dy \\
&= \int_{y \in [-|t|/2, +|t|/2]} \widehat{f}(t - y) g_R(y)\, dy \\
&\quad + \int_{y \notin [-|t|/2, +|t|/2]} \widehat{f}(t - y) g_R(y)\, dy \\
&= O\Big(\sup_{y \in [|t|/2, 3|t|/2]} |\widehat{f}(y)| \Big) + O\bigg(\|\widehat{f}\|_\infty \cdot \int_{y \notin [-R|t|/2, +R|t|/2]} \widehat{b}(z)\, dz \bigg).
\end{aligned}$$

Since $\widehat{b}$ and $\widehat{f}$ are Schwartz, for every $p > 0$ there is some constant C_p for which the latter is bounded above by $C_p \langle t \rangle^{-p}$ uniformly in R. In particular, since the number of spectral parameters in $[\nu, \nu + 1]$ grows quadratically with ν, taking any $p > 3$, the tail of the spectral side of the trace formula converges to 0 uniformly in R. ∎

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