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Almost commuting matrices and stability for product groups

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Abstract. We prove that any product of two non-abelian free groups, $\Gamma = \mathbb{F}_m \times \mathbb{F}_k$, for $m, k \geq 2$, is not Hilbert–Schmidt stable. This means that there exist asymptotic representations $\pi_n : \Gamma \to \mathrm{U}(d_n)$ with respect to the normalized Hilbert–Schmidt norm which are not close to actual representations. As a consequence, we prove the existence of contraction matrices A, B such that A almost commutes with B and B^*, with respect to the normalized Hilbert–Schmidt norm, but A, B are not close to any matrices A', B' such that A' commutes with B' and B'^*. This settles in the negative a natural version of a question concerning almost commuting matrices posed by Rosenthal in 1969.

Keywords: almost commuting matrices, group stability, von Neumann algebras, lifting properties, spectral gap, almost representations.

1. Introduction and statement of main results

A famous question, which can be traced back to the foundations of quantum mechanics [60], is whether two matrices A, B, which almost commute with respect to a given norm, must be close to two commuting matrices A', B'. It was first explicitly posed by Rosenthal [50] for the normalized Hilbert–Schmidt norm and by Halmos [25] for the operator norm. Almost commuting matrices have since been studied extensively and found applications in several areas of mathematics, including operator algebras and group theory, quantum physics and computer science (see, e.g., the introductions of [16, 36]). The most interesting case of this question is when the matrices are contractions, and "almost" and "close" are taken independent of their sizes. The answer depends both on the types of matrices considered and the norms chosen. Historically, research has focused on the operator norm. In this situation, the answer is positive for self-adjoint matrices by a remarkable result of Lin [35] (see also [20, 27, 33]), but negative for unitary and general matrices by results of Voiculescu [58] and Choi [10], respectively (see [13, 17] for related results).

More recently, several works [18, 19, 21–24, 51] studied the question for the normalized Hilbert–Schmidt norm and obtained affirmative answers for pairs of self-adjoint,

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unitary and normal matrices. In fact, the answer is positive if at least one of the matrices is normal, see Remark 1.1 (1). Moreover, this question has a positive answer within permutation matrices [2]. However, these results only apply when at least one of the matrices is normal, leaving wide open the general situation when neither matrix is normal.

We make progress on this problem by proving that, in contrast to the case of normal matrices, a version of Rosenthal's question [50] has a negative answer for non-normal matrices. The version that we consider is natural from the perspective of (self-adjoint) operator algebras. Specifically, it requires that A almost commutes not only with B but also with its adjoint B^*.

Theorem A. *There exist sequences of contractions $A_n, B_n \in \mathbb{M}_{d_n}(\mathbb{C})$, for some $d_n \in \mathbb{N}$, such that*

(a) $\lim_{n\to\infty} \|A_n B_n - B_n A_n\|_2 = \lim_{n\to\infty} \|A_n B_n^* - B_n^* A_n\|_2 = 0$, *and*

(b) $\inf_{n\in\mathbb{N}}(\|A_n - A_n'\|_2 + \|B_n - B_n'\|_2) > 0$, *for any sequences of matrices $A_n', B_n' \in \mathbb{M}_{d_n}(\mathbb{C})$ such that $A_n'B_n' = B_n'A_n'$ and $A_n'B_n'^* = B_n'^*A_n'$, for every $n \in \mathbb{N}$.*

For $A = (a_{i,j})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{C})$, we denote by $\|A\|$, $\|A\|_2 = (\frac{1}{n}\sum_{i,j=1}^n |a_{i,j}|^2)^{\frac{1}{2}}$ and $\tau(A) = \frac{1}{n}\sum_{i=1}^n a_{i,i}$ the operator norm, normalized Hilbert–Schmidt norm and normalized trace of A, respectively.

Remark 1.1. We continue with two remarks on the statement of Theorem A.

(1) The conclusion of Theorem A fails if one of the matrices is normal. Moreover, the following holds: let $A_n, B_n \in \mathbb{M}_{d_n}(\mathbb{C})$ be contractions such that $\|A_nB_n - B_nA_n\|_2 \to 0$ and B_n is normal, for every $n \in \mathbb{N}$. Then there are $A_n', B_n' \in \mathbb{M}_{d_n}(\mathbb{C})$ such that $A_n'B_n' = B_n'A_n'$ and $A_n'B_n'^* = B_n'^*A_n'$, for every $n \in \mathbb{N}$, and $\|A_n - A_n'\|_2 + \|B_n - B_n'\|_2 \to 0$ (see Lemma 7.1).

(2) Theorem A complements a result of von Neumann [59, Theorem 9.7] which implies the existence of contractions $A_n \in \mathbb{M}_{k_n}(\mathbb{C})$, for some $k_n \to \infty$, such that any contractions $B_n \in \mathbb{M}_{k_n}(\mathbb{C})$, which verify condition (a), must satisfy $\|B_n - \tau(B_n)1\|_2 \to 0$. In particular, A_n, B_n are close to the commuting matrices A_n, $\tau(B_n)1$. Thus, the pair A_n, B_n does not satisfy the conclusion of Theorem A, for any choice of contractions $B_n \in \mathbb{M}_{k_n}(\mathbb{C})$.

Theorem A is a consequence of a non-stability result for the product group $\mathbb{F}_2 \times \mathbb{F}_2$, see Theorem B. To motivate the latter result, we note that whether almost commuting matrices are near commuting ones is a prototypical stability problem. In general, following [28, 56], stability refers to a situation when elements which "almost" satisfy an equation must be "close" to elements satisfying the equation exactly. In recent years, there has been a considerable amount of interest in the study of group stability (see, e.g., [30,54]). For a countable group Γ, one can define stability with respect to any class $\mathcal{C}$ of metric groups endowed with bi-invariant metrics. This requires that any asymptotic homomorphism from Γ to a group in $\mathcal{C}$ is close to an actual homomorphism [2, 3, 14, 54]. Specializing to the class $\mathcal{C}$ of unitary groups endowed with the normalized Hilbert–Schmidt norms leads to the following notion of stability introduced in [6, 23].

Definition 1.2. A sequence of maps $\varphi_n\colon \Gamma \to \mathrm{U}(d_n)$, for some $d_n \in \mathbb{N}$, is called an *asymptotic homomorphism* if it satisfies $\lim_{n\to\infty} \|\varphi_n(gh) - \varphi_n(g)\varphi_n(h)\|_2 = 0$, for every $g, h \in \Gamma$. The group Γ is called *Hilbert–Schmidt stable* (or HS-*stable*) if for any asymptotic homomorphism $\varphi_n\colon \Gamma \to \mathrm{U}(d_n)$, we can find homomorphisms $\rho_n\colon \Gamma \to \mathrm{U}(d_n)$ such that $\lim_{n\to\infty} \|\varphi_n(g) - \rho_n(g)\|_2 = 0$, for every $g \in \Gamma$.

The class of HS-stable groups includes the free groups $\mathbb{F}_m$, virtually abelian groups and one-relator groups with non-trivial center [23], certain graph product groups [5], and is closed under free products. Moreover, the product of two HS-stable groups is HS-stable, provided that one of the groups is abelian [23, Theorem 1] or, more generally, amenable [31, Corollary D].

However, it remained a basic open problem whether HS-stability is closed under general direct products and, specifically, if $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable (see [30, Remark 1.4]). We settle this problem in the negative by proving that the product of two non-abelian free groups is not HS-stable. Moreover, we show the following.

Theorem B. *The group $\mathbb{F}_k \times \mathbb{F}_m$ is not flexibly* HS-*stable, for any integers $k, m \geq 2$.*

Before discussing results related to Theorem B, let us recall the notion of flexible HS-stability. It was shown in [6] that all infinite residually finite groups Γ with Kazhdan's property (T) (e.g., $\mathrm{SL}_m(\mathbb{Z})$, for $m \geq 3$) are not HS-stable. The proof builds on the observation that any sequence of homomorphisms $\rho_n\colon \Gamma \to \mathrm{U}(d_n)$ with $d_n \to \infty$ can be perturbed slightly to obtain an asymptotic homomorphism $\varphi_n\colon \Gamma \to \mathrm{U}(d_n - 1)$. To account for this method of constructing asymptotic homomorphisms, the following weakening of the notion of HS-stability was suggested in [6].

Definition 1.3. A countable group Γ is called *flexibly* HS-*stable* if for any asymptotic homomorphism $\varphi_n\colon \Gamma \to \mathrm{U}(d_n)$, we can find homomorphisms $\rho_n\colon \Gamma \to \mathrm{U}(D_n)$, for some integers $D_n \geq d_n$, such that

$$\lim_{n\to\infty} \frac{D_n}{d_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|\varphi_n(g) - p_n\rho_n(g)p_n\|_2 = 0,$$

for every $g \in \Gamma$, where $p_n\colon \mathbb{C}^{D_n} \to \mathbb{C}^{d_n}$ denotes the orthogonal projection for every $n \in \mathbb{N}$.

If a Connes-embeddable countable group Γ is flexibly HS-stable, then it must be residually finite. On the other hand, deciding if a residually finite group is flexibly HS-stable or not is a challenging problem. For instance, while the arithmetic groups $\mathrm{SL}_m(\mathbb{Z})$, $m \geq 3$, are not HS-stable by [6], it is open whether they are flexibly HS-stable. The first examples of residually finite groups which are not flexibly HS-stable were found only recently in [32], where certain groups with the relative property (T), including $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, were shown to have this property.

Theorem B provides the only other known examples of non-flexibly HS-stable residually finite groups, and the first examples that do not have infinite subgroups with the relative property (T). Moreover, these are the first examples of residually finite non-HS-stable groups that neither satisfy property (T;FD) (see [6, Section 4.2]) nor have infinite subgroups with the relative property (T).

Remark 1.4. We now compare Theorem B with two related results concerning other notions of stability. A countable group Γ is called P-*stable* if it is stable with respect to the class of finite permutation groups endowed with the normalized Hamming distance (see [30] for a survey on P-stability). As shown in [30, Corollary B], P-stability is not closed under direct products. Given the similarity between the notions of HS-stability and P-stability [2], it should not be surprising that HS-stability is not closed under direct products. We note however that the methods of [30] cannot be adapted to prove Theorem B. The approach of [30], which exploits the discrete aspects of P-stability, allows us to prove that the group $\mathbb{F}_2 \times \mathbb{Z}$ is not P-stable, despite being HS-stable by [23, Theorem 1]. As we explain at the end of the introduction, to prove Theorem B we introduce an entirely new approach based on ideas from the theory of von Neumann algebras.

A countable group Γ is called W*-*tracially stable* if it is stable with respect to the class of unitary groups of tracial von Neumann algebras endowed with their 2-norms [23]. Theorem B strengthens [31, Theorem E] which showed that $\mathbb{F}_k \times \mathbb{F}_m$ is not W*-tracially stable, for any integers $k, m \geq 2$. Indeed, being W*-tracial stable is stronger than being HS-stable, which corresponds to restricting to the unitary groups of finite-dimensional von Neumann algebras.

Next, we mention two operator algebraic consequences of Theorem B and explain how the first of these implies Theorem A. Let (M_n, τ_n), $n \in \mathbb{N}$, be tracial von Neumann algebras and ω be a free ultrafilter on $\mathbb{N}$. The *tracial ultraproduct von Neumann algebra* $\prod_\omega M_n$ is defined as the quotient $\ell^\infty(\mathbb{N}, M_n)/\mathcal{I}_\omega(\mathbb{N}, M_n)$ of the C*-algebra $\ell^\infty(\mathbb{N}, M_n)$ of sequences $(x_n) \in \prod_{\mathbb{N}} M_n$ with $\sup \|x_n\| < \infty$ by its ideal $\mathcal{I}_\omega(\mathbb{N}, M_n)$ of sequences (x_n) such that $\lim_{n\to\omega} \|x_n\|_2 = 0$.

First, by [31, Proposition C], if P, Q are commuting separable subalgebras of a tracial ultraproduct $\prod_\omega M_n$, and P is amenable, then there are commuting von Neumann subalgebras P_n, Q_n of M_n, for all $n \in \mathbb{N}$, such that $P \subset \prod_\omega P_n$ and $Q \subset \prod_\omega Q_n$. In contrast, Theorem B implies that, without the amenability assumption, this lifting property fails in certain matricial ultraproducts.

Corollary C. *There exist a sequence $(d_n) \subset \mathbb{N}$ and commuting separable von Neumann subalgebras P, Q of $\prod_\omega \mathbb{M}_{d_n}(\mathbb{C})$ such that the following holds: there are no commuting von Neumann subalgebras P_n, Q_n of $\mathbb{M}_{d_n}(\mathbb{C})$, for all $n \in \mathbb{N}$, such that $P \subset \prod_\omega P_n$ and $Q \subset \prod_\omega Q_n$.*

By [31, Theorem B], the conclusion of Corollary C holds for the ultrapower M^ω of certain, fairly complicated, examples of II$_1$ factors M. Corollary C provides the first natural examples of tracial ultraproducts that satisfy its conclusion. We conjecture that this phenomenon holds for any ultraproduct II$_1$ factor $\prod_\omega M_n$.

Since $\mathbb{F}_2 \times \mathbb{F}_2$ is not HS-stable by Theorem B, we can take P and Q in Corollary C to be generated by pairs of unitaries $\{U_1, U_2\}$ and $\{V_1, V_2\}$, respectively. Then P and Q can be generated by contractions A and B. Represent $A = (A_n)$, $B = (B_n)$, where

$A_n, B_n \in \mathbb{M}_{d_n}(\mathbb{C})$ are contractions for every $n \in \mathbb{N}$. By Corollary C, there are no commuting von Neumann subalgebras P_n, Q_n of $\mathbb{M}_{d_n}(\mathbb{C})$, for all $n \in \mathbb{N}$, such that $A \in \prod_\omega P_n$ and $B \in \prod_\omega Q_n$. This implies that, after passing to a subsequence, the contractions A_n, B_n satisfy the conclusion of Theorem A.

Second, Theorem B can be reformulated as a property of the full group C*-algebra $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$. This has been an important object of study since the work of Kirchberg [34] showed that certain properties of $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ (being residually finite or having a faithful trace) are equivalent to Connes' embedding problem (see [41, 44]).

Theorem B implies the existence of a $*$-homomorphism

$$\varphi\colon \mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2) \to \prod_\omega \mathbb{M}_{d_n}(\mathbb{C}),$$

for a sequence $(d_n) \subset \mathbb{N}$, which does not "lift" to a $*$-homomorphism

$$\widetilde{\varphi}\colon \mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2) \to \ell^\infty(\mathbb{N}, \mathbb{M}_{d_n}(\mathbb{C})).$$

Specifically, there is no $*$-homomorphism $\widetilde{\varphi}$ such that $\pi \circ \widetilde{\varphi} = \varphi$, where $\pi\colon \ell^\infty(\mathbb{N}, \mathbb{M}_{d_n}) \to \prod_\omega \mathbb{M}_{d_n}$ is the quotient homomorphism. We do not know if φ admits a unital completely positive (ucp) lift $\widetilde{\varphi}$. If no ucp lift exists, then it would follow that $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ does not have Kirchberg's local lifting property (LLP) (see [41, Corollary 3.12]). Whether $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ has the LLP is an open problem which goes back to [41] (see also [42, 44]).

Comments on the proof of Theorem B

We end the introduction with a detailed outline of the proof of Theorem B. Let us first reduce it to a simpler statement. As we prove in Lemma 2.6, if Γ_1, Γ_2 are HS-stable, then $\Gamma_1 \times \Gamma_2$ is flexibly HS-stable if and only if it is HS-stable. Also, if $(\Gamma_1 * \Lambda_1) \times (\Gamma_2 * \Lambda_2)$ is HS-stable, for groups Γ_1, Γ_2, Λ_1, Λ_2, then $\Gamma_1 \times \Gamma_2$ must be HS-stable. These facts imply that proving Theorem B is equivalent to showing that $\mathbb{F}_2 \times \mathbb{F}_2$ is not HS-stable. To prove the latter statement, we will reason by contradiction assuming that $\mathbb{F}_2 \times \mathbb{F}_2$ *is* HS-stable.

The proof of Theorem B is divided into two parts, which we discuss separately below. A main novelty of our approach is the use of ideas and techniques from the theory of (infinite-dimensional) von Neumann algebras to prove a statement concerning finite unitary matrices. We combine small perturbation results for von Neumann algebras with finite-dimensional analogues of two key ideas (the use of deformations and spectral gap arguments) from Popa's deformation/rigidity theory.

The first part of the proof, which occupies Sections 3–5, is devoted to proving the following result. This asserts, roughly speaking, that if $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable, then given any (arbitrarily large) finite sets $\mathcal{U} = \{U_1, \ldots, U_k\}$ and $\mathcal{V} = \{V_1, \ldots, V_m\}$ of unitaries of arbitrary dimension which almost commute (in the sense that $\|[U, V]\|_2 \approx 0$, for all $U \in \mathcal{U}$, $V \in \mathcal{V}$), we can find sets of unitaries $\widetilde{\mathcal{U}} = \{\widetilde{U}_1, \ldots, \widetilde{U}_k\}$ and $\widetilde{\mathcal{V}} = \{\widetilde{V}_1, \ldots, \widetilde{V}_m\}$ which commute and are close to $\mathcal{U}$ and $\mathcal{V}$.

Proposition 1.5. *If $\mathbb{F}_2 \times \mathbb{F}_2$ is* HS*-stable, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds: for every $k, m, n \in \mathbb{N}$ and every $U_1, \dots, U_k, V_1, \dots, V_m \in \mathrm{U}(n)$ satisfying $\frac{1}{km} \sum_{i=1}^{k} \sum_{j=1}^{m} \|[U_i, V_j]\|_2^2 \leq \delta$, we can find $\widetilde{U}_1, \dots, \widetilde{U}_k, \widetilde{V}_1, \dots, \widetilde{V}_m \in \mathrm{U}(n)$ such that*

(1) $[\widetilde{U}_i, \widetilde{V}_j] = 0$, *for every* $1 \leq i \leq k$ *and* $1 \leq j \leq m$,

(2) $\frac{1}{k} \sum_{i=1}^{k} \|U_i - \widetilde{U}_i\|_2^2 \leq \varepsilon$ *and* $\frac{1}{m} \sum_{j=1}^{m} \|V_j - \widetilde{V}_j\|_2^2 \leq \varepsilon$.

To illustrate the strength of the conclusion of Proposition 1.5, we make the following remark.

Remark 1.6. Suppose that $k, m \in \mathbb{N}$. Then $\mathbb{F}_k \times \mathbb{F}_m$ is HS-stable if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds: for every $n \in \mathbb{N}$ and every $U_1, \dots, U_k, V_1, \dots, V_m \in \mathrm{U}(n)$ satisfying $\|[U_i, V_j]\|_2 \leq \delta$, for all $1 \leq i \leq k$ and $1 \leq j \leq m$, we can find $\widetilde{U}_1, \dots, \widetilde{U}_k, \widetilde{V}_1, \dots, \widetilde{V}_m \in \mathrm{U}(n)$ such that

$$[\widetilde{U}_i, \widetilde{V}_j] = 0, \quad \|U_i - \widetilde{U}_i\|_2 \leq \varepsilon \quad \text{and} \quad \|V_j - \widetilde{V}_j\|_2 \leq \varepsilon,$$

for all $1 \leq i \leq k$ and $1 \leq j \leq m$. In view of this, Proposition 1.5 can be interpreted as follows: if $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable, then $\mathbb{F}_k \times \mathbb{F}_m$ is HS-stable and, moreover, it satisfies an "averaged" version of HS-stability, *uniformly* over all $k, m \in \mathbb{N}$.

We continue with some comments on the proof of Proposition 1.5 under the stronger assumption that $\mathbb{F}_3 \times \mathbb{F}_3$ is HS-stable. The proof of Proposition 1.5 has three main ingredients. All subalgebras of matrix algebras considered below are taken to be von Neumann (i.e., self-adjoint) subalgebras.

The first ingredient is a small perturbation result for subalgebras of a tensor product of three matrix algebras

$$M = \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C})$$

(see Lemma 3.5). Informally, we prove that any subalgebra $P \subset M$ which almost contains $\mathbb{M}_k(\mathbb{C}) \otimes 1 \otimes 1$ and is almost contained in $\mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \otimes 1$ must be close to a subalgebra of the form $\mathbb{M}_k(\mathbb{C}) \otimes S \otimes 1$, for some subalgebra $S \subset \mathbb{M}_n(\mathbb{C})$. Here, for subalgebras $P, Q \subset M$ and $\varepsilon > 0$, we say that P is ε-contained in Q if for all $x \in P$ with $\|x\| \leq 1$ there is $y \in Q$ with $\|x - y\|_2 \leq \varepsilon$, and that P is ε-close to Q if $P \subset_\varepsilon Q$ and $Q \subset_\varepsilon P$ [40]. A crucial aspect of Lemma 3.5 is that the constants involved are independent of $k, n, m \in \mathbb{N}$. Its proof is based on ideas from [11, 45, 46] and in particular uses the basic construction as in [11].

The second ingredient in the proof of the Proposition 1.5 is the existence of pairs of unitaries satisfying the following "spectral gap" condition: for a universal constant $\kappa > 0$ and every $n \in \mathbb{N}$, we can find $X_1, X_2 \in \mathrm{U}(n)$ such that

$$\|x - \tau(x)1\|_2 \leq \kappa(\|[X_1, x]\|_2 + \|[X_2, x]\|_2),$$

for every $x \in \mathbb{M}_n(\mathbb{C})$ (see Lemma 4.3). This is a consequence of a result of Hastings [26, 43] on quantum expanders.

To finish the proof of Proposition 1.5, we combine the first two ingredients with a "matrix trick". Let $U_1, \dots, U_k, V_1, \dots, V_m \in \mathrm{U}(n)$ such that $\|[U_i, V_j]\|_2 \approx 0$ for every i, j. Let $X_1, X_2 \in \mathrm{U}(k)$ and $Y_1, Y_2 \in \mathrm{U}(m)$ be pairs of unitaries with spectral gap. We define $M = \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C})$ and unitaries $Z_1, Z_2, Z_3, T_1, T_2, T_3 \in M$ by letting

$$Z_1 = X_1 \otimes 1 \otimes 1, \quad Z_2 = X_2 \otimes 1 \otimes 1, \quad Z_3 = \sum_{i=1}^{k} e_{i,i} \otimes U_i \otimes 1,$$

$$T_1 = 1 \otimes 1 \otimes Y_1, \quad T_2 = 1 \otimes 1 \otimes Y_2, \quad T_3 = \sum_{j=1}^{m} 1 \otimes V_j \otimes e_{j,j}.$$

Then $\|[Z_i, T_j]\|_2 \approx 0$, for every $1 \leq i, j \leq 3$. Since $\mathbb{F}_3 \times \mathbb{F}_3$ is assumed to be HS-stable, there are unitaries $Z_i', T_j' \in M$ such that $[Z_i', T_j'] = 0$, $\|Z_i - Z_i'\|_2 \approx 0$ and $\|T_j - T_j'\|_2 \approx 0$, for every $1 \leq i, j \leq 3$.

Let P be the subalgebra of M generated by Z_1', Z_2', Z_3', and let Q be its commutant. Since P almost commutes with T_1, T_2, using the spectral gap property of Y_1, Y_2 via an argument inspired by [47,48], we deduce that P is almost contained in $\mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \otimes 1$. Similarly, it follows that Q is almost contained in $1 \otimes \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C})$. Since P is the commutant of Q, the bicommutant theorem implies that $\mathbb{M}_k(\mathbb{C}) \otimes 1 \otimes 1$ is almost contained in P. The first ingredient of the proof now provides commuting subalgebras $R, S \subset \mathbb{M}_n(\mathbb{C})$ such that P is close to $\mathbb{M}_k(\mathbb{C}) \otimes R \otimes 1$ and Q is close to $1 \otimes S \otimes \mathbb{M}_m(\mathbb{C})$. At this point, the conclusion of Proposition 1.5 follows easily.

In *the second part of the proof* of Theorem B, presented in Section 6, we construct a counterexample to the conclusion of Proposition 1.5 and derive that $\mathbb{F}_2 \times \mathbb{F}_2$ is not HS-stable. Our construction, which we describe in detail below, is inspired by Popa's malleable deformation for noncommutative Bernoulli actions, see [46,57], and its variant introduced in [29].

Construction 1.7. Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

(1) We denote $M_n = \bigotimes_{k=1}^n \mathbb{M}_2(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$ and $A_n = \bigotimes_{k=1}^n \mathbb{C}^2 \cong \mathbb{C}^{2^n}$. We view A_n as a subalgebra of M_n, where we embed $\mathbb{C}^2 \subset \mathbb{M}_2(\mathbb{C})$ as the diagonal matrices.

(2) For $1 \leq i \leq n$, let $X_{n,i} = 1 \otimes \cdots \otimes 1 \otimes \sigma \otimes 1 \otimes \cdots \otimes 1 \in A_n$, where $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathbb{C}^2$ is placed on the i-th tensor position.

(3) Let G_n be a finite group of unitaries which generates $A_n \otimes M_n$.

(4) We define $U_t \in \mathrm{U}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ by $U_t = P + e^{it}(1 - P)$, where $P\colon \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ is the orthogonal projection onto the one-dimensional space spanned by $e_1 \otimes e_2 - e_2 \otimes e_1$.

(5) We identify $M_n \otimes M_n = \bigotimes_{k=1}^n (\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}))$ and let $\theta_{t,n}$ be the automorphism of $M_n \otimes M_n$ given by $\theta_{t,n}(\bigotimes_{k=1}^n x_k) = \bigotimes_{k=1}^n U_t x_k U_t^*$.

(6) Finally, consider the following two sets of unitaries in $M_n \otimes M_n$: $\mathcal{U}_n = \{X_{n,i} \otimes 1 \mid 1 \leq i \leq n\}$ and $\mathcal{V}_{t,n} = G_n \cup \theta_{t,n}(G_n)$.

Then $\mathcal{U}_n$ and $\mathcal{V}_{t,n}$ almost commute: $\|[U,V]\|_2 \leq 4t$, for $U \in \mathcal{U}_n$, $V \in \mathcal{V}_{t,n}$. This is because $\mathcal{U}_n$ commutes with G_n and $\|\theta_{t,n}(U) - U\|_2 \leq 2t$, for every $U \in \mathcal{U}_n$. Using this, we show that if $t > 0$ is small enough, then the sets $\mathcal{U}_n$, $\mathcal{V}_{t,n}$ contradict the conclusion of Proposition 1.5 for large $n \in \mathbb{N}$.

To informally outline our argument, suppose that $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable. Then Proposition 1.5 provides commuting subalgebras P_n, Q_n of $M_n \otimes M_n$ such that P_n almost contains $\mathcal{U}_n$ and Q_n almost contains $\mathcal{V}_{t,n}$. Thus, P_n almost commutes with $\mathcal{V}_{t,n}$ and hence with the generating groups G_n and $\theta_{t,n}(G_n)$ of $A_n \otimes M_n$ and $\theta_{t,n}(A_n \otimes M_n)$, respectively. By passing to commutants, we derive that P_n is almost contained in both $A_n \otimes 1$ and $\theta_{t,n}(A_n \otimes 1)$. By perturbing P_n slightly, we can assume that P_n is a subalgebra of $A_n \otimes 1$ which is almost contained in $\theta_{t,n}(A_n \otimes 1)$ (see Corollary 3.3).

Assume for a moment that $n = \infty$ in the above construction. Then $(\theta_{t,\infty})_{t\in\mathbb{R}}$ recovers the malleable deformation of the Bernoulli action on the hyperfinite II_1 factor $M_\infty = \bar{\bigotimes}_{k=1}^{\infty} \mathbb{M}_2(\mathbb{C})$, see [46,57]. In this case, if a subalgebra P of $M_\infty \otimes 1$ is almost contained in $\theta_{t,\infty}(M_\infty \otimes 1)$, then it must have a finite-dimensional direct summand [29]. While this result cannot be used in our finite-dimensional setting, we use the intuition behind its proof and a dimension argument to derive a contradiction.

Since $P_n \subset A_n \otimes 1$ is almost contained in $\theta_{t,n}(A_n \otimes 1)$, it is almost contained in the subspace spanned by tensors from $A_n = \bigotimes_{k=1}^{n} \mathbb{C}^2$ of length at most l, for some l independent on n. This forces the dimension of P_n to be at most polynomial in n. But P_n also almost contains $\mathcal{U}_n$ and so all tensors of length 1. This forces the dimension of P_n to grow exponentially in n, giving a contradiction as $n \to \infty$.

Remark 1.8. The last step of the proof is of independent interest, so let us explain it in more detail. Let C_n be a subalgebra of $A_n = \bigotimes_{k=1}^{n} \mathbb{C}^2$ which satisfies

$$\frac{1}{n}\sum_{i=1}^{n} \|X_{n,i} - \mathrm{E}_{C_n}(X_{n,i})\|_2^2 \leq \varepsilon, \qquad (\star)$$

where $\varepsilon \in (0, \frac{1}{8})$ and E_{C_n} is the conditional expectation onto C_n. Lemma 6.6 implies that

$$\liminf \frac{\log_2(\dim(C_n))}{n} \geq 1 - \mathrm{H}(4\varepsilon),$$

where

$$\mathrm{H}(\delta) = -\delta\log_2(\delta) - (1-\delta)\log_2(1-\delta)$$

is the binary entropy function of $\delta \in (0,1)$.

In fact, this estimate is essentially sharp: Lemma 6.8 implies that for every $n \in \mathbb{N}$, there is a subalgebra C_n of A_n which satisfies $(\star)$ and that

$$\limsup \frac{\log_2(\dim(C_n))}{n} \leq 1 - \mathrm{H}\Big(\frac{\varepsilon}{8}\Big).$$

Remark 1.9. Let $\Gamma = \mathbb{F}_k \times \mathbb{F}_m$, for integers $k, m \geq 2$. Note that $\mathcal{U}_n$ and $\mathcal{V}_{t,n}$ almost commute in the operator norm: $\|[U,V]\| \leq 4t$, for $U \in \mathcal{U}_n$, $V \in \mathcal{V}_{t,n}$. Using this fact, a close inspection of the proof of Theorem B shows that we proved the following stronger statement: the asymptotic homomorphism $\varphi_n\colon \Gamma \to \mathrm{U}(d_n)$, which witnesses that Γ is not flexibly HS-stable, is an asymptotic homomorphism in the operator norm, i.e.,

$$\lim_{n\to\infty} \|\varphi_n(gh) - \varphi_n(g)\varphi_n(h)\| = 0,$$

for all $g, h \in \Gamma$. Therefore, Γ fails a hybrid notion of stability which weakens both the notion of matricial stability studied in [12, 15] and (flexible) HS-stability.

2. Preliminaries

While the main results of this paper concern matrix algebras, the proofs are based on ideas and techniques from the theory of von Neumann algebras. Moreover, our proofs often extend with no additional effort from matrix algebras to general tracial von Neumann algebras. As such, it will be convenient to work in the latter framework. In this section, we recall several basic notions and constructions concerning von Neumann algebras (see [1, 52] for more information).

2.1. Von Neumann algebras

For a complex Hilbert space $\mathcal{H}$, we denote by $\mathbb{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and by $\mathrm{U}(\mathcal{H}) = \{u \in \mathbb{B}(\mathcal{H}) \mid u^*u = uu^* = 1\}$ the group of unitary operators on $\mathcal{H}$. For $x \in \mathbb{B}(\mathcal{H})$, we denote by $\|x\|$ its operator norm. A set of operators $S \subset \mathbb{B}(\mathcal{H})$ is called *self-adjoint* if $x^* \in S$, for all $x \in S$. We denote by S' the *commutant* of S, i.e., the set of operators $y \in \mathbb{B}(\mathcal{H})$ such that $xy = yx$, for all $x \in S$.

A self-adjoint subalgebra $M \subset \mathbb{B}(\mathcal{H})$ is a *von Neumann algebra* if it is closed in the weak operator topology. By von Neumann bicommutant's theorem, a unital self-adjoint subalgebra $M \subset \mathbb{B}(\mathcal{H})$ is a von Neumann algebra if and only if it is equal to its bicommutant, $M = (M')'$. From now on, we assume that all von Neumann algebras M are unital. We denote by $\mathcal{Z}(M) = M' \cap M$ the *center* of M, by $(M)_1 = \{x \in M \mid \|x\| \leq 1\}$ the *unit ball* of M, by $M_+ = \{x \in M \mid x \geq 0\}$ the set of *positive* elements of M, and by $\mathcal{U}(M)$ the group of unitary operators in M. We call M a *factor* if $\mathcal{Z}(M) = \mathbb{C}1$. Two projections $p, q \in M$ are Murray–von Neumann *equivalent* if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. A linear functional $\varphi\colon M \to \mathbb{C}$ is called

(a) a *state* if $\varphi(1) = 1$ and $\varphi(x) \geq 0$, for every $x \in M_+$,

(b) *faithful* if having $\varphi(x) = 0$, for some $x \in M_+$, implies that $x = 0$, and

(c) *normal* if $\sup \varphi(x_i) = \varphi(\sup x_i)$, for any increasing net $(x_i) \subset M_+$.

A *tracial von Neumann algebra* is a pair (M, τ) consisting of a von Neumann algebra M and a *trace* τ, i.e., a faithful normal state $\tau\colon M \to \mathbb{C}$ which satisfies $\tau(xy) = \tau(yx)$, for all $x, y \in M$. We endow M with the 1- and 2-norms given by $\|x\|_1 = \tau((x^*x)^{\frac{1}{2}})$ and $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$, for all $x \in M$. Then $\|xy\|_2 \leq \|x\|\|y\|_2$ and $\|xy\|_2 \leq \|x\|_2\|y\|$, for all $x, y \in M$. We denote by $\mathrm{L}^2(M)$ the Hilbert space obtained as the closure of M with respect to $\|\cdot\|_2$, and consider the standard representation $M \subset \mathbb{B}(\mathrm{L}^2(M))$ given by the left multiplication action of M on $\mathrm{L}^2(M)$. For further reference, we recall the Powers–Størmer inequality (see [9, Proposition 6.2.4] and [1, Theorem 7.3.7])

$$\|h-k\|_2^2 \leq \|h^2-k^2\|_1 \leq \|h-k\|_2\|h+k\|_2, \quad \text{for every } h,k \in M_+, \tag{2.1}$$

and the following inequality

$$|\tau(p)-\tau(q)| \leq \|p-q\|_2^2, \quad \text{for every projections } p,q \in M. \tag{2.2}$$

The latter inequality holds because $\|p-q\|_2^2 = \tau(p)+\tau(q)-2\tau(pq)$ and $\tau(pq) \leq \min\{\tau(p),\tau(q)\}$. We also note that (2.1) and (2.2) more generally hold when $\tau: M \to \mathbb{C}$ is a semifinite trace.

The matrix algebra $\mathbb{M}_n(\mathbb{C}) = \mathbb{B}(\mathbb{C}^n)$ with its *normalized trace* $\tau: \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ given by

$$\tau(x) = \frac{1}{n}\sum_{i=1}^n x_{i,i}, \quad \text{for } x = (x_{i,j})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{C}),$$

is a tracial von Neumann algebra. The associated 2-norm is the *normalized Hilbert–Schmidt norm*

$$\|x\|_2 = \left(\frac{1}{n}\sum_{i,j=1}^n |x_{i,j}|^2\right)^{\frac{1}{2}}, \quad \text{for } x = (x_{i,j})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{C}).$$

Since $\mathbb{M}_n(\mathbb{C})$ has a trivial center, it is a tracial factor. Any tracial factor is either finite-dimensional and isomorphic to $\mathbb{M}_n(\mathbb{C})$, for some $n \in \mathbb{N}$, or infinite-dimensional and called a II_1 *factor*.

Moreover, any finite-dimensional von Neumann algebra M is isomorphic to a direct sum of matrix algebras and therefore is tracial. Indeed, if $z_1,\dots,z_k \in \mathcal{Z}(M)$ are the minimal projections, then $M = \bigoplus_{i=1}^k Mz_i$, where Mz_i is a finite-dimensional factor and thus a matrix algebra, for $1 \leq i \leq k$. We claim that there is a finite subgroup $G \subset \mathcal{U}(M)$ which generates M. If $M = \mathbb{M}_n(\mathbb{C})$, we can take G to be a group of unitaries of the form $(\varepsilon_i\delta_{\sigma(i),j})_{i,j=1}^n$, where $\varepsilon_1,\dots,\varepsilon_n \in \{\pm 1\}$ and σ is a permutation of $\{1,\dots,n\}$. In general, if $G_i \subset \mathcal{U}(Mz_i)$ is a generating group, for every $1 \leq i \leq k$, then the finite group $G = \{u_1 \oplus \cdots \oplus u_k \mid u_1 \in G_1,\dots,u_k \in G_k\}$ generates M.

A subalgebra of a matrix algebra $\mathbb{M}_n(\mathbb{C})$ is a von Neumann subalgebra if and only if it is self-adjoint. Nevertheless, for consistency, we will call self-adjoint subalgebras of $\mathbb{M}_n(\mathbb{C})$ von Neumann algebras.

2.2. The basic construction

Let (M,τ) be a tracial von Neumann algebra together with a von Neumann subalgebra $Q \subset M$. Then we have an embedding $\mathrm{L}^2(Q) \subset \mathrm{L}^2(M)$. We denote by $e_Q: \mathrm{L}^2(M) \to \mathrm{L}^2(Q)$ the orthogonal projection onto $\mathrm{L}^2(Q)$. We denote by $\mathrm{E}_Q: M \to Q$ the *conditional expectation* onto Q, i.e., the unique map satisfying $\tau(\mathrm{E}_Q(x)y) = \tau(xy)$, for all $x \in M$

and $y \in Q$. If we consider the natural embedding $M \subset \mathrm{L}^2(M)$, then $e_Q(M) \subset Q$ and $\mathrm{E}_Q = e_{Q|M}$.

Jones' basic construction $\langle M, e_Q\rangle$ of the inclusion $Q \subset M$ is defined as the von Neumann subalgebra of $\mathbb{B}(\mathrm{L}^2(M))$ generated by M and e_Q. Let $J\colon \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ be the involution given by $J(x) = x^*$, for all $x \in M$. Then $\langle M, e_Q\rangle$ is equal to both $JQ'J$, the commutant of the right multiplication action of Q on $\mathrm{L}^2(M)$, and the weak operator closure of the span of $\{xe_Q y \mid x, y \in M\}$.

The basic construction admits a normal semifinite trace $\mathrm{Tr}\colon \langle M, e_Q\rangle_+ \to [0, +\infty]$ which satisfies

$$\mathrm{Tr}(xe_Q y) = \tau(xy), \quad \text{for every } x, y \in M.$$

It also admits a normal semifinite center-valued tracial weight $\Phi\colon \langle M, e_Q\rangle_+ \to \widehat{\mathcal{Z}(Q)}_+$ which satisfies $\mathrm{Tr} = \tau \circ \Phi$, $\Phi(X^*X) = \Phi(XX^*)$, for every $X \in \langle M, e_Q\rangle$, and $\Phi(ST^*) = \mathrm{E}_{\mathcal{Z}(Q)}(T^*S)$, for all bounded right Q-linear operators $S, T\colon \mathrm{L}^2(Q) \to \mathrm{L}^2(M)$ (see [1, Section 9.4]). Here, $\widehat{\mathcal{Z}(Q)}_+$ denotes the set of positive operators affiliated with $\mathcal{Z}(Q)$. If S and T are the left multiplication operators by x and y^*, for $x, y \in M$, then $ST^* = xe_Q y$ and $T^*S = \mathrm{E}_Q(yx)$. Thus, we conclude that

$$\Phi(xe_Q y) = \mathrm{E}_{\mathcal{Z}(Q)}(yx), \quad \text{for every } x, y \in M.$$

If $z \in \mathcal{Z}(Q)$, then $e_Q JzJ = e_Q z^*$ and thus

$$\Phi(xe_Q yJzJ) = \Phi(xe_Q z^* y) = \mathrm{E}_{\mathcal{Z}(Q)}(z^* yx) = \Phi(xe_Q y)z^*, \quad \text{for every } x, y \in M.$$

Therefore, it follows that

$$\Phi(TJzJ) = \Phi(T)z^*, \quad \text{for every } T \in \langle M, e_Q\rangle_+ \text{ and } z \in \mathcal{Z}(Q). \tag{2.3}$$

We note that if M is finite-dimensional, then $\langle M, e_Q\rangle$ is finite-dimensional and Tr and Φ are finite, i.e., $\mathrm{Tr}(1) < \infty$ and $\Phi(1) \in \mathcal{Z}(Q)$. We next record two well-known properties of Φ.

Lemma 2.1. *Given two projections $p, q \in \langle M, e_Q\rangle$, the following hold:*

(1) *p is equivalent to a subprojection of q if and only if $\Phi(p) \leq \Phi(q)$.*

(2) *There is a projection $r \in \langle M, e_Q\rangle$ such that $r \leq p$ and $\Phi(r) = \min\{\Phi(p), \Phi(q)\}$.*

Proof. For (1), see [1, Proposition 9.1.8.]. For (2), it is easy to see that any maximal projection $r \leq p$, which is equivalent to a subprojection of q, has the desired property. ■

2.3. *Almost containment*

Let us recall the notion of ε-containment studied in [11, 37, 40]. Let $P \subset pMp$, $Q \subset qMq$ be von Neumann subalgebras of a tracial von Neumann algebra (M, τ), for projections $p, q \in M$. For $\varepsilon \geq 0$, we write $P \subset_\varepsilon Q$ and say that P is *ε-contained* in Q if

$\|x - \mathrm{E}_Q(x)\|_2 \le \varepsilon$, for all $x \in (P)_1$. We also define the distance between P and Q by letting

$$\mathbf{d}(P, Q) := \min\{\varepsilon \ge 0 \mid P \subset_\varepsilon Q \text{ and } Q \subset_\varepsilon P\}.$$

Convention. To specify the trace τ, we sometimes write $\|x\|_{2,\tau}$, $\subset_{\varepsilon,\tau}$, $\mathbf{d}_\tau$ instead of $\|x\|_2$, $\subset_\varepsilon$, $\mathbf{d}$.

In the rest of this subsection, we prove several useful lemmas. We start with two well-known results.

Lemma 2.2. *The following hold:*

(1) *Let M be a von Neumann algebra with a faithful normal semifinite trace τ. If $p, q \in M$ are equivalent finite projections, then there is $u \in \mathcal{U}(M)$ satisfying $upu^* = q$ and $\|u - 1\|_2 \le 3\|p - q\|_2$.*

(2) *Let (M, τ) be a tracial von Neumann algebra, $P \subset M$ a von Neumann subalgebra and $u \in \mathcal{U}(M)$. Then there is $v \in \mathcal{U}(P)$ satisfying $\|u - v\|_2 \le 3\|u - \mathrm{E}_P(u)\|_2$.*

Proof. For (1), see [11, Lemma 2.2]. For (2), using the polar decomposition of $\mathrm{E}_P(u)$ we find $v \in \mathcal{U}(P)$ such that $\mathrm{E}_P(u) = v|\mathrm{E}_P(u)|$. Then $\|u - v\|_2 \le \|u - \mathrm{E}_P(u)\|_2 + \|1 - |\mathrm{E}_P(u)|\|_2$. Since $\|1 - |\mathrm{E}_P(u)|\|_2 \le \|1 - |\mathrm{E}_P(u)|^2\|_2 = \|u^*u - \mathrm{E}_P(u)^*\mathrm{E}_P(u)\|_2 \le 2\|u - \mathrm{E}_P(u)\|_2$, (2) follows. ∎

Lemma 2.3. *Let (M, τ) be a tracial von Neumann algebra and $P \subset pMp$, $Q \subset qMq$ be von Neumann subalgebras, for some projections $p, q \in M$. Assume that $P \subset_\varepsilon Q$ and $\|p - q\|_2 \le \varepsilon$, for some $\varepsilon > 0$. Then $Q' \cap qMq \subset_{4\varepsilon} P' \cap pMp$.*

Proof. Let $y \in (Q' \cap qMq)_1$. Then for every $u \in \mathcal{U}(P)$, we have that $[u, pyp] = p[u, y]p$ and thus $\|[u, pyp]\|_2 \le \|[u, y]\|_2 = \|[u - \mathrm{E}_Q(u), y]\|_2 \le 2\|u - \mathrm{E}_Q(u)\|_2 \le 2\varepsilon$. Since this holds for every $u \in \mathcal{U}(P)$, it follows that $\|pyp - \mathrm{E}_{P' \cap pMp}(pyp)\|_2 \le 2\varepsilon$. Since $y = qyq$, we also have that $\|y - pyp\|_2 = \|qyq - pyp\|_2 \le 2\|p - q\|_2 \le 2\varepsilon$. By combining the last two inequalities, we derive that $\|y - \mathrm{E}_{P' \cap pMp}(y)\|_2 \le \|y - \mathrm{E}_{P' \cap pMp}(pyp)\|_2 \le 4\varepsilon$, which finishes the proof. ∎

The following lemma is a simple application of the basic construction.

Lemma 2.4. *Let (M, τ) be a tracial von Neumann algebra and let $P, Q \subset M$ be von Neumann subalgebras. Assume that P is finite-dimensional and $G \subset \mathcal{U}(P)$ is a finite subgroup which generates P and satisfies $\frac{1}{|G|}\sum_{U \in G} \|U - \mathrm{E}_Q(U)\|_2^2 \le \varepsilon$, for some $\varepsilon > 0$. Then $P \subset_{\sqrt{2\varepsilon}} Q$.*

Proof. Consider the basic construction $\langle M, e_Q \rangle$ with its canonical semifinite trace Tr: $\langle M, e_Q \rangle \to \mathbb{C}$. Denote $f = \frac{1}{|G|}\sum_{U \in G} U e_Q U^* \in \langle M, e_Q \rangle$. Then an easy calculation shows that

$$\|f - e_Q\|_{2,\mathrm{Tr}}^2 = \frac{1}{|G|}\sum_{U \in G} \|U - \mathrm{E}_Q(U)\|_2^2 \le \varepsilon. \tag{2.4}$$

Since G is a group, f commutes with G and thus with P. Therefore, if $x \in (P)_1$, then using that x commutes with f and (2.4), we get that

$$\begin{aligned}\|x - \mathrm{E}_Q(x)\|_2^2 &= \frac{1}{2}\|xe_Q - e_Q x\|_{2,\mathrm{Tr}}^2 = \frac{1}{2}\|x(e_Q - f) - (e_Q - f)x\|_{2,\mathrm{Tr}}^2 \\ &\leq 2\|e_Q - f\|_{2,\mathrm{Tr}}^2 \leq 2\varepsilon.\end{aligned}$$

This proves the conclusion. ■

Our next goal is to establish the following useful elementary lemma.

Lemma 2.5. *Let (M,τ) be a tracial factor and $P \subset M$ a von Neumann subalgebra. Let $\varepsilon \in (0, \frac{1}{8}]$ and assume that $q \in M$ is a projection such that $\tau(1-q) = \varepsilon$. Then there is a von Neumann subalgebra $Q \subset qMq$ such that $\mathbf{d}(P,Q) \leq 14\varepsilon^{\frac{1}{4}}$.*

Proof. We claim that P or $P' \cap M$ contains a projection p such that $\tau(p) \in [\varepsilon, \varepsilon^{\frac{1}{2}}]$. Assume that P does not contain a projection p with $\tau(p) \in [\varepsilon, \varepsilon^{\frac{1}{2}}]$. Since $2\varepsilon < \varepsilon^{\frac{1}{2}}$, there is a minimal projection $r \in P$ such that $\tau(r) > \varepsilon^{\frac{1}{2}}$. Let $z \in P$ be the smallest central projection such that $r \leq z$. Then we can find $d \in \mathbb{N}$ such that $\tau(z) = d\tau(r)$, $Pz \cong \mathbb{M}_d(\mathbb{C})$ and there is a $*$-isomorphism $\theta: rMr \to z(P' \cap M)z$ such that $\tau(\theta(x)) = d\tau(x)$, for every $x \in rMr$. Since M is a factor and $\tau(1-q) = \varepsilon$, rMr contains a projection of trace ε. Thus, $P' \cap M$ contains a projection of trace $d\varepsilon$. Since $d\varepsilon \geq \varepsilon$ and $d\varepsilon = (d\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}} \leq (d\tau(r))\varepsilon^{\frac{1}{2}} = \tau(z)\varepsilon^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}}$, the claim follows.

Let p be a projection in P or $P' \cap M$ with $\tau(p) \in [\varepsilon, \varepsilon^{\frac{1}{2}}]$. Since $\tau(1-p) \leq 1 - \varepsilon = \tau(q)$ and M is a factor, there is projection $q_0 \in qMq$ such that $\tau(q_0) = \tau(1-p) \geq 1 - \varepsilon^{\frac{1}{2}}$. Then $1-p$ and q_0 are equivalent projections in M such that $\|(1-p) - q_0\|_2 \leq \|p\|_2 + \|1 - q_0\|_2 = 2\tau(p)^{\frac{1}{2}} \leq 2\varepsilon^{\frac{1}{4}}$. By Lemma 2.2 (1), we can find a unitary $u \in M$ such that $u(1-p)u^* = q_0$ and $\|u - 1\|_2 \leq 6\varepsilon^{\frac{1}{4}}$.

Finally, since $1-p$ belongs to P or $P' \cap M$, $(1-p)P(1-p)$ is a von Neumann algebra. Define $Q := u(1-p)P(1-p)u^* \oplus \mathbb{C}(q - q_0)$. Then $Q \subset qMq$ is a von Neumann subalgebra and for $x \in (P)_1$,

$$\|x - u(1-p)x(1-p)u^*\|_2 \leq 2\|1 - u(1-p)\|_2 \leq 2(\|1-u\|_2 + \|p\|_2) \leq 14\varepsilon^{\frac{1}{4}}.$$

Since $u(1-p)x(1-p)u^* \in Q$, we get that $P \subset_{14\varepsilon^{\frac{1}{4}}} Q$. Conversely, let $y \in (Q)_1$ and write $y = uxu^* + \alpha(q - q_0)$, for some $x \in (P)_1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Then

$$\|y - x\|_2 \leq \|uxu^* - x\|_2 + \|q - q_0\|_2 \leq 2\|u-1\|_2 + \|1 - q_0\|_2 \leq 13\varepsilon^{\frac{1}{4}}.$$

Hence, $Q \subset_{13\varepsilon^{\frac{1}{4}}} P$, and the conclusion follows. ■

We end this section by illustrating the usefulness of Lemma 2.5 in proving the following assertion.

Lemma 2.6. *Let Γ_1 and Γ_2 be* HS*-stable countable groups. Then $\Gamma_1 \times \Gamma_2$ is* HS*-stable if and only if it is flexibly* HS*-stable.*

Proof. Let $\Gamma = \Gamma_1 \times \Gamma_2$. To prove the lemma, we only have to argue that if Γ is flexibly HS-stable, then it is HS-stable. To this end, assume that Γ is flexibly HS-stable.

For $n \in \mathbb{N}$, denote by τ_n the normalized trace of $\mathbb{M}_n(\mathbb{C})$. Let $\pi_n\colon \Gamma \to \mathrm{U}(k_n)$ be an asymptotic homomorphism. Since Γ is flexibly HS-stable, we can find homomorphisms $\rho_n\colon \Gamma \to \mathrm{U}(K_n)$, for $K_n \geq k_n$, such that $\lim_{n\to\infty} \frac{K_n}{k_n} = 1$ and, denoting by $e_n\colon \mathbb{C}^{K_n} \to \mathbb{C}^{k_n}$ the orthogonal projection,

$$\lim_{n\to\infty} \|\pi_n(g) - e_n\rho_n(g)e_n\|_{2,\tau_{k_n}} = 0, \quad \text{for every } g \in \Gamma. \tag{2.5}$$

Since $\lim_{n\to\infty} \frac{K_n}{k_n} = 1$ and $\|e_n - 1\|_{2,\tau_{K_n}} = \sqrt{\frac{K_n - k_n}{K_n}}$, for every $n \in \mathbb{N}$, we get that

$$\lim_{n\to\infty} \|e_n - 1\|_{2,\tau_{K_n}} = 0. \tag{2.6}$$

Let $P_n \subset \mathbb{M}_{K_n}(\mathbb{C})$ be the von Neumann algebra generated by $\rho_n(\Gamma_1)$. By using (2.6) and applying Lemma 2.5, we can find a von Neumann subalgebra $Q_n \subset e_n\mathbb{M}_{K_n}(\mathbb{C})e_n \equiv \mathbb{M}_{k_n}(\mathbb{C})$ such that

$$\lim_{n\to\infty} \mathbf{d}_{\tau_{K_n}}(P_n, Q_n) = 0. \tag{2.7}$$

Next, by combining (2.6), (2.7) and Lemma 2.3 we derive that

$$\lim_{n\to\infty} \mathbf{d}_{\tau_{K_n}}(P_n' \cap \mathbb{M}_{K_n}(\mathbb{C}), Q_n' \cap \mathbb{M}_{k_n}(\mathbb{C})) = 0. \tag{2.8}$$

Since $\rho_n(\Gamma_1) \subset P_n$ and $\rho_n(\Gamma_2) \subset P_n' \cap \mathbb{M}_{K_n}(\mathbb{C})$, combining (2.5), (2.6), (2.7) and (2.8) implies that

$$\lim_{n\to\infty} \|\pi_n(g_1) - \mathrm{E}_{Q_n}(\pi_n(g_1))\|_{2,\tau_{k_n}} = 0$$

and

$$\lim_{n\to\infty} \|\pi_n(g_2) - \mathrm{E}_{Q_n' \cap \mathbb{M}_{k_n}(\mathbb{C})}(\pi_n(g_2))\|_{2,\tau_{k_n}} = 0,$$

for every $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$. By putting together these facts and Lemma 2.2 (2), we can find maps $\sigma_n^1\colon \Gamma_1 \to \mathcal{U}(Q_n)$ and $\sigma_n^2\colon \Gamma_2 \to \mathcal{U}(Q_n' \cap \mathbb{M}_{k_n}(\mathbb{C}))$ such that

$$\lim_{n\to\infty} \|\pi_n(g_1) - \sigma_n^1(g_1)\|_{2,\tau_{k_n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\pi_n(g_2) - \sigma_n^2(g_2)\|_{2,\tau_{k_n}} = 0,$$

for every $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$.

Then (σ_n^1) and (σ_n^2) are asymptotic homomorphisms of Γ_1 and Γ_2, respectively. Since Γ_1 and Γ_2 are HS-stable, we can find homomorphisms $\delta_n^1\colon \Gamma_1 \to \mathcal{U}(Q_n)$ and $\delta_n^2\colon \Gamma_2 \to \mathcal{U}(Q_n' \cap \mathbb{M}_{k_n}(\mathbb{C}))$ such that

$$\lim_{n\to\infty} \|\sigma_n^1(g_1) - \delta_n^1(g_1)\|_{2,\tau_{k_n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\sigma_n^2(g_2) - \delta_n^2(g_2)\|_{2,\tau_{k_n}} = 0,$$

for every $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$. Then we have $\lim_{n\to\infty} \|\pi_n(g_1) - \delta_n^1(g_1)\|_{2,\tau_{k_n}} = 0$ and $\lim_{n\to\infty} \|\pi_n(g_2) - \delta_n^2(g_2)\|_{2,\tau_{k_n}} = 0$, for every $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$.

Finally, since δ_n^1 and δ_n^2 have commuting images for every $n \in \mathbb{N}$, we can define a homomorphism $\delta_n\colon \Gamma \to \mathrm{U}(k_n)$ by letting $\delta_n(g_1, g_2) = \delta_n^1(g_1)\delta_n^2(g_2)$, for every $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$. It is then clear that $\lim_{n\to\infty} \|\pi_n(g) - \delta_n(g)\|_{2,\tau_{k_n}} = 0$, for every $g \in \Gamma$. This proves that Γ is HS-stable. ∎

3. Perturbation results

In this section, we study the almost containment relation for tracial von Neumann algebras. A crucial feature of the results is that they do not depend on the dimensions of the algebras involved.

3.1. A "small perturbation" lemma

Our main result is the following small perturbation lemma. If P and Q are subalgebras of a tracial von Neumann algebra such that P is almost contained in Q, we show that P must be close to a subalgebra of $\mathbb{M}_2(\mathbb{C}) \otimes Q$. For a tracial von Neumann algebra (M, τ), we equip $\mathbb{M}_2(\mathbb{C}) \otimes M$ with the trace $\tilde{\tau}$ given by the formula $\tilde{\tau}(\sum_{i,j=1}^{2} e_{i,j} \otimes x_{i,j}) = \frac{1}{2}(\tau(x_{1,1}) + \tau(x_{2,2}))$.

Lemma 3.1. *Let (M, τ) be a tracial von Neumann algebra and let $P, Q \subset M$ be von Neumann subalgebras. Assume that $P \subset_\varepsilon Q$, for some $\varepsilon \in (0, \frac{1}{200})$. Then there exists a $*$-homomorphism $\theta: P \to \mathbb{M}_2(\mathbb{C}) \otimes Q$ such that $\|\theta(x) - e_{1,1} \otimes x\|_{2,\tilde{\tau}} \leq 30\varepsilon^{\frac{1}{8}}$, for every $x \in (P)_1$.*

Lemma 3.1 implies that there is a non-trivial $*$-homomorphism from P to $\mathbb{M}_2(\mathbb{C}) \otimes Q$. This generalizes [11, Theorem 4.7] where the same conclusion was proved assuming that Q is a factor. As shown in [45, Theorem A.2] under certain conditions (e.g., if $P, Q \subset M$ are irreducible subfactors and $P \subset M$ is regular) $P \subset_\varepsilon Q$ implies the existence of $u \in \mathcal{U}(M)$ such that $uPu^* \subset Q$. However, such a strong conclusion does not hold in general even for irreducible subfactors $P, Q \subset M$ (see [49, Proposition 5.5]).

The proof of Lemma 3.1 relies on ideas of Christensen [11] and Popa [45, 46]. As in [11], we use the basic construction $\langle M, e_Q \rangle$ and find a projection $f \in P' \cap \langle M, e_Q \rangle$ which is close to e_Q. Then, inspired by an argument in [46], we show that after replacing f by $fJzJ$, for a projection $z \in \mathcal{Z}(Q)$ close to 1, one may assume that $e_{1,1} \otimes f$ is subequivalent to $1 \otimes e_Q$ in $\mathbb{M}_2(\mathbb{C}) \otimes \langle M, e_q \rangle$.

Proof of Lemma 3.1. Let $\mathrm{Tr}: \langle M, e_Q \rangle \to \mathbb{C}$ be the canonical tracial weight. Since

$$\|ue_Qu^* - e_Q\|_{2,\mathrm{Tr}}^2 = 2(1 - \mathrm{Tr}(ue_Qu^*e_Q)) = 2(1 - \tau(u\mathrm{E}_Q(u)^*)) = 2\|u - \mathrm{E}_Q(u)\|_2^2,$$

we get that

$$\|ue_Qu^* - e_Q\|_{2,\mathrm{Tr}} \leq \sqrt{2}\varepsilon, \quad \text{for every } u \in \mathcal{U}(P). \tag{3.1}$$

Let $\mathcal{C} \subset \langle M, e_Q \rangle$ be the weak operator closure of the convex hull of the set $\{ue_Qu^* \mid u \in \mathcal{U}(P)\}$. Then $\mathcal{C}$ is $\|\cdot\|_{2,\mathrm{Tr}}$-closed and admits an element h of minimal $\|\cdot\|_{2,\mathrm{Tr}}$-norm which satisfies $0 \leq h \leq 1$, $h \in P' \cap \langle M, e_Q \rangle$ and $\|h - e_Q\|_{2,\mathrm{Tr}} \leq \sqrt{2}\varepsilon$ by (3.1) (see [11, Section 3] or [3, Lemma 14.3.3]).

Define the spectral projection $f := \mathbf{1}_{[1-\delta^{\frac{1}{2}},1]}(h)$, where $\delta = \sqrt{2}\varepsilon$. Then $f \in P' \cap \langle M, e_Q \rangle$ and $\|f - e_Q\|_{2,\mathrm{Tr}} \leq \delta^{\frac{1}{2}}(1 - \delta^{\frac{1}{2}})^{-1}$ by [11, Lemma 2.1]. As $\varepsilon < \frac{1}{200}$, we have that $(1 - \delta^{\frac{1}{2}})^{-1} < \sqrt{2}$ and thus

$$\|f - e_Q\|_{2,\mathrm{Tr}} \leq 2\varepsilon^{\frac{1}{2}}. \tag{3.2}$$

Since $\|e_Q\|_{2,\mathrm{Tr}} = 1$ and $2\varepsilon^{\frac{1}{2}} < 1$, we get that $\|f + e_Q\|_{2,\mathrm{Tr}} \leq 3$. Combining the Powers–Størmer inequality (2.1) and (3.2), we further get that

$$\|f - e_Q\|_{1,\mathrm{Tr}} \leq \|f - e_Q\|_{2,\mathrm{Tr}} \cdot \|f + e_Q\|_{2,\mathrm{Tr}} \leq 6\varepsilon^{\frac{1}{2}}. \tag{3.3}$$

Let $\Phi: \langle M, e_Q\rangle_+ \to \widehat{\mathcal{Z}(Q)}_+$ be the center-valued tracial weight defined in Section 2.2. Define the spectral projection $z := \mathbf{1}_{[0,\varepsilon^{\frac{1}{4}}]}(\Phi(|f - e_Q|))$. Then $z \in \mathcal{Z}(Q)$ and $\varepsilon^{\frac{1}{4}}(1-z) \leq \Phi(|f - e_Q|)$. Since $\mathrm{Tr} = \tau \circ \Phi$, (3.3) implies that $\tau(\Phi(|f - e_Q|)) = \|f - e_Q\|_{1,\mathrm{Tr}} \leq 6\varepsilon^{\frac{1}{2}}$. Thus, $\tau(1-z) \leq 6\varepsilon^{\frac{1}{4}}$ and hence

$$\|1 - z\|_{2,\tau} \leq 3\varepsilon^{\frac{1}{8}}. \tag{3.4}$$

Let $J: \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ be the canonical involution given by $J(x) = x^*$, and put $z' = JzJ$. Since $\langle M, e_Q\rangle = JQ'J$, we have that $z' \in M' \cap \mathcal{Z}(\langle M, e_Q\rangle)$. Thus, $g = fz' \in P' \cap \langle M, e_Q\rangle$ is a projection. Moreover, we have that $e_Q z' = e_Q z$, while (2.3) gives that $\Phi(x)z = \Phi(xz')$, for every $x \in \langle M, e_Q\rangle$. Altogether, since $\Phi(|f - e_Q|)z \leq \varepsilon^{\frac{1}{4}} z$, we get that

$$\begin{aligned}\Phi(|g - e_Q z|) &= \Phi(|fz' - e_Q z'|) = \Phi(|f - e_Q|z') = \Phi(|f - e_Q|)z \\ &\leq \varepsilon^{\frac{1}{4}} z = \varepsilon^{\frac{1}{4}} \Phi(e_Q z).\end{aligned}$$

From this we deduce that

$$(1 - \varepsilon^{\frac{1}{4}})\Phi(e_Q z) \leq \Phi(g) \leq (1 + \varepsilon^{\frac{1}{4}})\Phi(e_Q z). \tag{3.5}$$

By Lemma 2.1 (2), we can find a projection $p_1 \in \langle M, e_Q\rangle$ such that $p_1 \leq g$ and $\Phi(p_1) = \min\{\Phi(g), \Phi(e_Q z)\}$. Put $p_2 = g - p_1$. Since $\Phi(p_1) \leq \Phi(e_Q z)$, by Lemma 2.1 (1), there is a projection $q_1 \in \langle M, e_Q\rangle$ such that $q_1 \leq e_Q z$ and q_1 is equivalent to p_1. Since $\Phi(g - p_1) = \max\{0, \Phi(g) - \Phi(e_Q z)\}$, (3.5) implies that $\Phi(g - p_1) \leq \varepsilon^{\frac{1}{4}}\Phi(e_Q z)$ and thus $\mathrm{Tr}(g - p_1) \leq \varepsilon^{\frac{1}{4}}$. Similarly, since p_1 and q_1 are equivalent, we have $\Phi(p_1) = \Phi(q_1)$, thus $\Phi(e_Q z - q_1) = \Phi(e_Q z - p_1) \leq \varepsilon^{\frac{1}{4}}\Phi(e_Q z)$ and hence $\mathrm{Tr}(e_Q z - q_1) \leq \varepsilon^{\frac{1}{4}}$.

The last paragraph gives that $\|g - p_1\|_{2,\mathrm{Tr}} \leq \varepsilon^{\frac{1}{8}}$ and $\|e_Q z - q_1\|_{2,\mathrm{Tr}} \leq \varepsilon^{\frac{1}{8}}$. Moreover, (3.2) gives that

$$\|g - e_Q z\|_{2,\mathrm{Tr}} = \|(f - e_Q)z'\|_{2,\mathrm{Tr}} \leq \|f - e_Q\|_{2,\mathrm{Tr}} \leq 6\varepsilon^{\frac{1}{2}}. \tag{3.6}$$

As $\varepsilon < \frac{1}{200}$, we have that $6\varepsilon^{\frac{1}{2}} < \varepsilon^{\frac{1}{8}}$ and the triangle inequality gives that $\|p_1 - q_1\|_{2,\mathrm{Tr}} \leq 3\varepsilon^{\frac{1}{8}}$. By [11, Lemma 2.2], there is a partial isometry $v_1 \in \langle M, e_Q\rangle$ such that $v_1 v_1^* = p_1$, $v_1^* v_1 = q_1$ and $\|v_1 - p_1\|_{2,\mathrm{Tr}} \leq 6\|p_1 - q_1\|_{2,\mathrm{Tr}} \leq 18\varepsilon^{\frac{1}{8}}$. Since $\mathrm{Tr}(p_2) = \mathrm{Tr}(g - p_1) \leq \varepsilon^{\frac{1}{4}}$, we get $\|p_2\|_{2,\mathrm{Tr}} \leq \varepsilon^{\frac{1}{8}}$ and so

$$\|v_1 - g\|_{2,\mathrm{Tr}} \leq \|v_1 - p_1\|_{2,\mathrm{Tr}} + \|p_2\|_{2,\mathrm{Tr}} \leq 19\varepsilon^{\frac{1}{8}}. \tag{3.7}$$

Next, since $\Phi(p_2) = \Phi(g - p_1) \leq \varepsilon^{\frac{1}{4}}\Phi(e_Q z) \leq \Phi(e_Q z)$, by Lemma 2.1 (1) we can find a partial isometry $v_2 \in \langle M, e_Q\rangle$ with $v_2 v_2^* = p_2$ and $v_2^* v_2 \leq e_Q$. Then

$$\|v_2\|_{2,\mathrm{Tr}} = \|p_2\|_{2,\mathrm{Tr}} \leq \varepsilon^{\frac{1}{8}}. \tag{3.8}$$

Let $v = (v_1 \quad v_2) \in \mathbb{M}_{1,2}(\mathbb{C}) \otimes \langle M, e_Q\rangle$. Then v is a partial isometry with $v^*v \in \mathbb{M}_2(\mathbb{C}) \otimes \langle M, e_Q\rangle$, $v^*v \leq 1 \otimes e_Q$ and $vv^* = g$. As $g \in P' \cap \langle M, e_Q\rangle$, the map $P \ni x \mapsto v^*xv \in \mathbb{M}_2(\mathbb{C}) \otimes \langle M, e_Q\rangle$ is a $*$-homomorphism. If $x \in P$, then

$$v^*xv \in (1 \otimes e_Q)(\mathbb{M}_2(\mathbb{C}) \otimes \langle M, e_Q\rangle)(1 \otimes e_Q) = \mathbb{M}_2(\mathbb{C}) \otimes Qe_Q.$$

Since $Q \ni y \mapsto ye \in Qe_Q$ is a $*$-isomorphism, there is a $*$-homomorphism $\theta\colon P \to \mathbb{M}_2(\mathbb{C}) \otimes Q$ such that

$$\theta(x)(1 \otimes e_Q) = v^*xv, \quad \text{for every } x \in P. \tag{3.9}$$

As $6\varepsilon^{\frac{1}{2}} < \varepsilon^{\frac{1}{8}}$, (3.4) and (3.6) give that $\|g - e_Q\|_{2,\mathrm{Tr}} \leq \|g - e_Q z\|_{2,\mathrm{Tr}} + \|1 - z\|_{2,\tau} \leq 4\varepsilon^{\frac{1}{8}}$. Let $x \in (P)_1$. Since g commutes with x, we have $gxg = xg$, and the above inequality and (3.7) imply

$$\begin{aligned}\|v_1^*xv_1 - xe_Q\|_{2,\mathrm{Tr}} &\leq \|v_1^*xv_1 - gxg\|_{2,\mathrm{Tr}} + \|xg - xe_Q\|_{2,\mathrm{Tr}}\\ &\leq 2\|v_1 - g\|_{2,\mathrm{Tr}} + \|g - e_Q\|_{2,\mathrm{Tr}} \leq 42\varepsilon^{\frac{1}{8}}.\end{aligned} \tag{3.10}$$

Finally, for $y = \sum_{i,j}^2 e_{i,j} \otimes y_{i,j} \in \mathbb{M}_2(\mathbb{C}) \otimes \langle M, e_Q\rangle$, we denote

$$\widetilde{\mathrm{Tr}}(y) = \frac{1}{2}(\mathrm{Tr}(y_{1,1}) + \mathrm{Tr}(y_{2,2})) \quad \text{and} \quad \|y\|_{2,\widetilde{\mathrm{Tr}}} = (\widetilde{\mathrm{Tr}}(y^*y))^{\frac{1}{2}}.$$

Then $\|z(1 \otimes e_Q)\|_{2,\widetilde{\mathrm{Tr}}} = \|z\|_{2,\tilde{\tau}}$, for every $z \in \mathbb{M}_2(\mathbb{C}) \otimes M$. This fact together with (3.8), (3.9) and (3.10) gives that for every $x \in (P)_1$, we have

$$\begin{aligned}&\|\theta(x) - e_{1,1} \otimes x\|_{2,\tilde{\tau}}\\ &\quad = \|\theta(x)(1 \otimes e_Q) - e_{1,1} \otimes xe_Q\|_{2,\widetilde{\mathrm{Tr}}} = \|v^*xv - e_{1,1} \otimes xe_Q\|_{2,\widetilde{\mathrm{Tr}}}\\ &\quad = \Big(\frac{\|v_1^*xv_1 - xe_Q\|_{2,\mathrm{Tr}}^2 + \|v_1^*xv_2\|_{2,\mathrm{Tr}}^2 + \|v_2^*xv_1\|_{2,\mathrm{Tr}}^2 + \|v_2^*xv_2\|_{2,\mathrm{Tr}}^2}{2}\Big)^{\frac{1}{2}}\\ &\quad \leq 30\varepsilon^{\frac{1}{8}},\end{aligned}$$

which finishes the proof. ∎

3.2. From almost containment to containment

Let P and Q be von Neumann subalgebras of a tracial von Neumann algebra. If P is close to a subalgebra of Q, then P is almost contained in Q. In this subsection, we use Lemma 3.1 to prove that the converse holds provided that Q is a factor (see Corollary 3.2) or a finite-dimensional abelian algebra (see Corollary 3.3).

Corollary 3.2. *For any $\varepsilon > 0$, there is $\delta = \delta_1(\varepsilon) > 0$ such that the following holds. Let (M, τ) be a tracial von Neumann algebra and $P, Q \subset M$ be von Neumann subalgebras such that $P \subset_\delta Q$. Assume that Q is a factor. Then there exists a von Neumann subalgebra $R \subset Q$ such that* $\mathbf{d}(P, R) \leq \varepsilon$.

Proof. Given $\varepsilon > 0$, we will prove that any $\delta > 0$ such that $\delta < 10^{-16}$ and $400\delta^{\frac{1}{16}} < \varepsilon$ works. Assume that $P \subset_\delta Q$. Then Theorem 3.1 gives a $*$-homomorphism $\theta\colon P \to \mathbb{M}_2(\mathbb{C}) \otimes Q$ with $\|\theta(x) - e_{1,1} \otimes x\|_{2,\tilde{\tau}} \leq 30\delta^{\frac{1}{8}}$, for all $x \in (P)_1$. Let $q = \theta(1)$ and $A = \theta(P)$. Then $\|e_{1,1} \otimes 1 - q\|_{2,\tilde{\tau}} \leq 30\delta^{\frac{1}{8}}$ and $\mathbf{d}_{\tilde{\tau}}(e_{1,1} \otimes P, A) \leq 30\delta^{\frac{1}{8}}$.

Since Q is a factor, we can find projections $r \in q(\mathbb{M}_2(\mathbb{C}) \otimes Q)q$ and $s \in e_{1,1} \otimes Q$ such that $\tilde{\tau}(r) = \tilde{\tau}(s) = \min\{\tilde{\tau}(q), \frac{1}{2}\}$. By (2.2), we get that $\tilde{\tau}(q - r) \leq |\tilde{\tau}(q) - \frac{1}{2}| \leq \|q - e_{1,1} \otimes 1\|_{2,\tilde{\tau}}^2 \leq 900\delta^{\frac{1}{4}}$, so $\|q - r\|_{2,\tilde{\tau}} \leq 30\delta^{\frac{1}{8}}$. In particular, $\tilde{\tau}(q) \geq \frac{1}{2} - 900\delta^{\frac{1}{4}} \geq \frac{1}{3}$. Similarly, $\|e_{1,1} \otimes 1 - s\|_{2,\tilde{\tau}} \leq 30\delta^{\frac{1}{8}}$, thus

$$\|s - r\|_{2,\tilde{\tau}} \leq \|s - e_{1,1} \otimes 1\|_{2,\tilde{\tau}} + \|e_{1,1} \otimes 1 - q\|_{2,\tilde{\tau}} + \|q - r\|_{2,\tilde{\tau}} \leq 90\delta^{\frac{1}{8}}. \tag{3.11}$$

By Lemma 2.5, we can find a von Neumann subalgebra $B \subset r(\mathbb{M}_2(\mathbb{C}) \otimes Q)r$ such that

$$\mathbf{d}_{\tilde{\tau}}(A, B) \leq 14\Big(\frac{\tilde{\tau}(q - r)}{\tilde{\tau}(q)}\Big)^{\frac{1}{4}} \leq 200\delta^{\frac{1}{16}}. \tag{3.12}$$

Since $\tilde{\tau}(r) = \tilde{\tau}(s)$ and Q is a factor, using Lemma 2.2 (1) and (3.11) we find a unitary $u \in \mathbb{M}_2(\mathbb{C}) \otimes Q$ such that $uru^* = s$ and $\|u - 1\|_{2,\tilde{\tau}} \leq 3\|s - r\|_{2,\tilde{\tau}} \leq 270\delta^{\frac{1}{8}}$. Let $C = uBu^* \oplus \mathbb{C}(e_{1,1} \otimes 1 - s)$. Then $C \subset e_{1,1} \otimes Q$ is a von Neumann subalgebra such that

$$\mathbf{d}_{\tilde{\tau}}(B, C) \leq 2\|u - 1\|_{2,\tilde{\tau}} + \|e_{1,1} \otimes 1 - s\|_{2,\tilde{\tau}} \leq 570\delta^{\frac{1}{8}}. \tag{3.13}$$

If $R \subset Q$ is a von Neumann subalgebra such that $C = e_{1,1} \otimes R$, then (3.12) and (3.13) imply that

$$\begin{aligned}\mathbf{d}_\tau(P, R) = \sqrt{2}\mathbf{d}_{\tilde{\tau}}(e_{1,1} \otimes P, C) &\leq \sqrt{2}(\mathbf{d}_{\tilde{\tau}}(e_{1,1} \otimes P, A) + \mathbf{d}_{\tilde{\tau}}(A, B) + \mathbf{d}_{\tilde{\tau}}(B, C))\\ &\leq \sqrt{2}(30\delta^{\frac{1}{8}} + 200\delta^{\frac{1}{16}} + 570\delta^{\frac{1}{8}}) \leq 400\delta^{\frac{1}{16}}.\end{aligned}$$

This finishes the proof of the lemma. ∎

Corollary 3.3. *For any $\varepsilon > 0$, there is $\delta = \delta_2(\varepsilon) > 0$ such that the following holds. Let (M, τ) be a tracial von Neumann algebra and $P, Q \subset M$ be finite-dimensional von Neumann subalgebras such that $P \subset_\delta Q$. Assume that Q is abelian. Then there exists a von Neumann subalgebra $R \subset Q$ such that* $\mathbf{d}(P, R) \leq \varepsilon$.

Proof. Given $\varepsilon > 0$, we will prove that any $\delta > 0$ such that $\delta < \frac{1}{200}$ and $200\delta^{\frac{1}{8}} < \varepsilon$ works. Assume that $P \subset_\delta Q$. We will first show that P has a large abelian direct summand.

Let $z \in \mathcal{Z}(P)$ be the largest projection such that Pz is abelian. Since $P(1 - z)$ has no abelian direct summand, we can find a projection $p \in P(1 - z)$ with $\tau(p) \geq \frac{\tau(1-z)}{3}$ and a unitary $u \in P(1 - z)$ such that p and upu^* are orthogonal. Thus,

$$\|[u, p]\|_{2,\tau} = \sqrt{2}\|p\|_{2,\tau} \geq \sqrt{\frac{2}{3}}\|1 - z\|_{2,\tau}.$$

On the other hand, since $P \subset_\delta Q$ and Q is abelian, we get that

$$\|[u, p]\|_{2,\tau} \leq 2(\|u - \mathrm{E}_Q(u)\|_{2,\tau} + \|p - \mathrm{E}_Q(p)\|_{2,\tau}) \leq 4\delta.$$

By combining the last two inequalities, we derive that

$$\|1-z\|_{2,\tau} \leq 4\sqrt{\frac{3}{2}}\delta \leq 5\delta. \tag{3.14}$$

Let $\{p_i\}_{i=1}^m$ and $\{q_j\}_{j=1}^n$ be the minimal projections of Pz and Q, so that $Pz = \bigoplus_{i=1}^m \mathbb{C}p_i$ and $Q = \bigoplus_{j=1}^n \mathbb{C}q_j$. Since $P \subset_\delta Q$, Lemma 3.1 provides a $*$-homomorphism $\theta: P \to \mathbb{M}_2(\mathbb{C}) \otimes Q$ such that $\|\theta(x) - e_{1,1} \otimes x\|_{2,\tilde{\tau}} \leq 30\delta^{\frac{1}{8}}$, for every $x \in (P)_1$. In particular, using (3.14) we get that

$$\begin{aligned}\|\theta(z) - e_{1,1} \otimes 1\|_{2,\tilde{\tau}} &\leq \|\theta(z) - e_{1,1} \otimes z\|_{2,\tilde{\tau}} + \|e_{1,1} \otimes (1-z)\|_{2,\tilde{\tau}} \\ &\leq 30\delta^{\frac{1}{8}} + 5\delta \leq 31\delta^{\frac{1}{8}}.\end{aligned} \tag{3.15}$$

Write $\theta(z) = \sum_{j=1}^n \alpha_j \otimes q_j$, where $\alpha_j \in \mathbb{M}_2(\mathbb{C})$ is a projection, for every $1 \leq j \leq n$. Let S be the set of all $j \in \{1,\dots,n\}$ such that α_j has rank one. Define $w = \sum_{j\in S} q_j \in Q$. If $j \notin S$, then α_j is equal to 0 or 1 and thus

$$\|\alpha_j \otimes q_j - e_{1,1} \otimes q_j\|_{2,\tilde{\tau}}^2 = \frac{\tau(q_j)}{2}.$$

This implies that

$$\|1-w\|_{2,\tau}^2 = \sum_{j\notin S} \tau(q_j) = 2\sum_{j\notin S} \|\alpha_j \otimes q_j - e_{1,1} \otimes q_j\|_{2,\tilde{\tau}}^2 \leq 2\|\theta(z) - e_{1,1} \otimes 1\|_{2,\tilde{\tau}}^2.$$

In combination with (3.15), we derive that

$$\|1-w\|_{2,\tau} \leq 31\sqrt{2}\delta^{\frac{1}{8}}. \tag{3.16}$$

Since $\theta(z)(1\otimes w) = \sum_{j\in S} \alpha_j \otimes q_j$ and α_j has rank one, for every $j \in S$, we get that there is a partition $S = S_1 \sqcup \cdots \sqcup S_m$ such that $\theta(p_i)(1 \otimes w) = \sum_{j\in S_i} \alpha_j \otimes q_j$, for every $1 \leq i \leq m$. Define a $*$-homomorphism $\rho: P \to Q$ by letting $\rho(1-z) = 0$ and $\rho(p_i) = \sum_{j\in S_i} q_j$, for every $1 \leq i \leq m$.

Claim 3.4. *We have that $\|\rho(x) - x\|_{2,\tau} \leq 150\delta^{\frac{1}{8}}$, for every $x \in (P)_1$.*

Proof. Let $x \in (P)_1$ and write $x = c_0 + \sum_{i=1}^m c_i p_i$, where $c_0 \in (P(1-z))_1$ and $c_i \in \mathbb{C}$ satisfies $|c_i| \leq 1$, for $0 \leq i \leq m$. Let $y = \sum_{i=1}^m c_i p_i$. Then

$$e_{1,1} \otimes \rho(x) = \sum_{i=1}^m \sum_{j\in S_i} c_i(e_{1,1} \otimes q_j),$$

$$\theta(y)(1\otimes w) = \sum_{i=1}^m c_i\theta(p_i)(1\otimes w) = \sum_{i=1}^m \sum_{j\in S_i} c_i(\alpha_j \otimes q_j).$$

Since the projections $\{q_j\}_{j\in S}$ are pairwise orthogonal, the sets $\{S_i\}_{i=1}^m$ partition S, and $|c_i| \leq 1$, for every $1 \leq i \leq m$, we get that

$$\|e_{1,1} \otimes \rho(x) - \theta(y)(1\otimes w)\|_{2,\tilde{\tau}}^2 \leq \sum_{j\in S} \|e_{1,1} \otimes q_j - \alpha_j \otimes q_j\|_{2,\tilde{\tau}}^2 \leq \|e_{1,1} \otimes 1 - \theta(z)\|_{2,\tilde{\tau}}^2.$$

In combination with (3.15), we derive that

$$\|e_{1,1} \otimes \rho(x) - \theta(y)(1 \otimes w)\|_{2,\tilde{\tau}} \leq 31\delta^{\frac{1}{8}}. \tag{3.17}$$

Since $\|\theta(y) - e_{1,1} \otimes y\|_{2,\tilde{\tau}} \leq 30\delta^{\frac{1}{8}}$, using (3.16) and (3.17), we further get that

$$\begin{aligned}\|\rho(x) - y\|_{2,\tau} &= \sqrt{2}\|e_{1,1} \otimes \rho(x) - e_{1,1} \otimes y\|_{2,\tilde{\tau}} \\ &\leq \sqrt{2}(\|e_{1,1} \otimes \rho(x) - \theta(y)(1 \otimes w)\|_{2,\tilde{\tau}} + \|\theta(y)(1 \otimes (1 - w))\|_{2,\tilde{\tau}} \\ &\qquad + \|\theta(y) - e_{1,1} \otimes y\|_{2,\tilde{\tau}}) \\ &\leq 140\delta^{\frac{1}{8}}.\end{aligned}$$

As $x - y \in (P(1 - z))_1$, (3.4) implies that $\|x - y\|_{2,\tau} \leq \|1 - z\|_{2,\tau} \leq 5\delta$. The last displayed inequality gives that $\|\rho(x) - x\|_{2,\tau} \leq \|\rho(x) - y\|_{2,\tau} + \|1 - z\|_{2,\tau} \leq 140\delta^{\frac{1}{8}} + 5\delta \leq 150\delta^{\frac{1}{8}}$, proving the claim. ∎

Finally, Claim 3.4 gives that $\mathbf{d}_\tau(P, \rho(P)) \leq 150\delta^{\frac{1}{8}}$. Let $R = \rho(P) \bigoplus \mathbb{C}(1 - w)$. Since $\rho(1) = w$, R is a von Neumann subalgebra of Q. By (3.16), we get that $\mathbf{d}_\tau(\rho(P), R) \leq \|1 - w\|_{2,\tau} \leq 31\sqrt{2}\delta^{\frac{1}{8}} \leq 50\delta^{\frac{1}{8}}$. Thus, we conclude that $\mathbf{d}_\tau(P, R) \leq \mathbf{d}_\tau(P, \rho(P)) + \mathbf{d}_\tau(\rho(P), R) \leq 200\delta^{\frac{1}{8}}$.

This completes the proof of Corollary 3.3. ∎

The following consequence of Corollary 3.2 is a key ingredient of the proof of Proposition 1.5.

Lemma 3.5. *For any $\varepsilon > 0$, there is $\delta = \delta_3(\varepsilon) > 0$ such that the following holds. For $i \in \{1, 2, 3\}$, let M_i be a finite-dimensional factor and denote by 1_{M_i} its unit. Let $M = M_1 \otimes M_2 \otimes M_3$ and $P \subset M$ be a von Neumann subalgebra. Assume that $M_1 \otimes 1_{M_2} \otimes 1_{M_3} \subset_\delta P$ and $P \subset_\delta M_1 \otimes M_2 \otimes 1_{M_3}$. Then there exists a von Neumann subalgebra $S \subset M_2$ such that $\mathbf{d}(P, M_1 \otimes S \otimes 1_{M_3}) \leq \varepsilon$.*

Proof. Let $\delta_1\colon (0, +\infty) \to (0, +\infty)$ be the function provided by Corollary 3.2. Let $\varepsilon > 0$. Let $\varepsilon' > 0$ such that $\varepsilon' \leq \frac{\varepsilon}{2}$ and $\delta(\varepsilon') + \varepsilon' \leq \frac{1}{4}\delta_1(\frac{\varepsilon}{8})$. We will show that $\delta := \delta(\varepsilon')$ works.

Since M_1, M_2 are factors, so is $M_1 \otimes M_2$. Since $P \subset_{\delta(\varepsilon')} M_1 \otimes M_2 \otimes 1_{M_3}$, Corollary 3.2 implies the existence of a von Neumann subalgebra $Q \subset M_1 \otimes M_2$ such that

$$\mathbf{d}(P, Q \otimes 1_{M_3}) \leq \varepsilon' \leq \frac{\varepsilon}{2}. \tag{3.18}$$

As $M_1 \otimes 1_{M_2} \otimes 1_{M_3} \subset_{\delta_1(\varepsilon')} P$, (3.18) implies that $M_1 \otimes 1_{M_2} \subset_{\delta_1(\varepsilon')+\varepsilon'} Q$. Let $N = Q' \cap (M_1 \otimes M_2)$. Since $(M_1 \otimes 1_{M_2})' \cap (M_1 \otimes M_2) = 1_{M_1} \otimes M_2$ and $4(\delta_1(\varepsilon') + \varepsilon') \leq \delta_1(\frac{\varepsilon}{8})$, Lemma 2.3 gives that

$$N \subset_{\delta_1(\frac{\varepsilon}{8})} 1_{M_1} \otimes M_2.$$

Since M_2 is a factor, applying Corollary 3.2 gives a von Neumann subalgebra $R \subset M_2$ such that

$$\mathbf{d}(N, 1_{M_1} \otimes R) \leq \frac{\varepsilon}{8}.$$

As $M_1 \otimes M_2$ is a finite-dimensional factor, the bicommutant theorem gives that $N' \cap (M_1 \otimes M_2) = Q$. By applying Lemma 2.3 again, we deduce that if $S = R' \cap M_2$, then

$$\mathbf{d}(Q, M_1 \otimes S) \leq \frac{\varepsilon}{2}. \tag{3.19}$$

Finally, by combining (3.18) and (3.19) we derive that $\mathbf{d}(P, M_1 \otimes S \otimes 1_{M_3}) \leq \varepsilon$. ∎

4. Pairs of unitary matrices with spectral gap

The goal of this section is to prove the following two results giving pairs of unitary matrices with spectral gap properties. These results provide the first step towards proving Theorem B. For $n \in \mathbb{N}$, we denote by τ the normalized trace on $\mathbb{M}_n(\mathbb{C})$, and by $\|\cdot\|_2$ and $\|\cdot\|_1$ the associated norms.

Proposition 4.1. *There is a constant $\eta > 0$ such that the following holds. Let $A = \mathbb{M}_k(\mathbb{C})$ and $B = \mathbb{M}_n(\mathbb{C})$, for $k, n \in \mathbb{N}$. Then there are $Z_1, Z_2 \in \mathcal{U}(A \otimes 1)$ such that*

$$\|x - \mathrm{E}_{1\otimes B}(x)\|_2 \leq \eta(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2), \quad \text{for every } x \in A \otimes B.$$

Proposition 4.1 suffices to prove that $\mathbb{F}_3 \times \mathbb{F}_3$, and thus $\mathbb{F}_m \times \mathbb{F}_n$, for all $m, n \geq 3$, is not HS-stable. However, to prove the failure of HS-stability for $\mathbb{F}_2 \times \mathbb{F}_2$, we will need the following result.

Proposition 4.2. *There is a constant $\eta > 0$ such that the following holds. Let $A = \mathbb{M}_k(\mathbb{C})$, $B = \mathbb{M}_n(\mathbb{C})$ and $w \in \mathcal{U}(A \otimes B)$, for $k, n \in \mathbb{N}$. Then there are $Z_1, Z_2 \in \mathcal{U}(\mathbb{M}_3(\mathbb{C}) \otimes A \otimes B)$ such that*

(1) $\|x - \mathrm{E}_{1\otimes 1\otimes B}(x)\|_2 \leq \eta(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2)$, *for every* $x \in \mathbb{M}_3(\mathbb{C}) \otimes A \otimes B$,

(2) $Z_1 \in \mathbb{M}_3(\mathbb{C}) \otimes A \otimes 1$, *and*

(3) $Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & w & 0 \end{pmatrix}$.

4.1. Pairs of unitary matrices with spectral gap

The proofs of Propositions 4.1 and 4.2 rely on the following result.

Lemma 4.3. *There exist a constant $\kappa > 0$, a sequence (k_n) of natural numbers with $k_n \to \infty$, and a pair of unitaries $(u_n, v_n) \in \mathrm{U}(k_n)^2$, for every $n \in \mathbb{N}$, such that*

$$\|x - \tau(x)1\|_2 \leq \kappa(\|[u_n, x]\|_2 + \|[v_n, x]\|_2), \quad \text{for every } x \in \mathbb{M}_{k_n}(\mathbb{C}).$$

Moreover, we can take $k_n = n$, for every $n \in \mathbb{N}$.

This result is likely known to experts, but, for completeness, we indicate how it follows from the literature. We give two proofs of the main assertion based on property (T) and quantum expanders, respectively. The second proof will allow us to also derive the moreover assertion.

4.2. First proof of the main assertion of Lemma 4.3

The first proof combines an argument from the proof of [8, Proposition 3.9(4)], which we recall below, with the fact that $\Gamma := \mathrm{SL}_3(\mathbb{Z})$ is 2-generated. Indeed, by [55], the following two matrices generate Γ:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ has Kazhdan's property (T) (see, e.g., [9, Theorem 12.1.14]), we can find $\kappa > 0$ such that if $\rho\colon \Gamma \to \mathrm{U}(\mathcal{H})$ is any unitary representation and $P\colon \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto the subspace of $\rho(\Gamma)$-invariant vectors, then

$$\|\xi - P(\xi)\| \leq \kappa(\|\rho(a)\xi - \xi\| + \|\rho(b)\xi - \xi\|), \quad \text{for every } \xi \in \mathcal{H}. \tag{4.1}$$

Since Γ is residually finite and has property (T), it has a sequence of finite-dimensional irreducible representations $\pi_n\colon \Gamma \to \mathrm{U}(k_n)$, $n \in \mathbb{N}$, with $k_n \to \infty$ (see the proof of [8, Proposition 3.9 (4)]). Alternatively, if p is a prime, then any non-trivial representation of $\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z})$ has dimension at least $\frac{p-1}{2}$ (see [53, Exercise 3.0.9]). Thus, we can take π_n to be any irreducible representation of Γ factoring through $\mathrm{SL}_3(\mathbb{Z}/p_n\mathbb{Z})$, for any $n \in \mathbb{N}$, where (p_n) is a sequence of primes with $p_n \to \infty$.

Since π_n is irreducible, the only matrices which are invariant under the unitary representation $\rho_n\colon \Gamma \to \mathrm{U}(\mathbb{M}_{k_n}(\mathbb{C}))$ given by $\rho_n(g)x = \pi_n(g)x\pi_n(g)^*$ are the scalar multiples of the identity. Thus, applying inequality (4.1) to ρ_n gives that $\|x - \tau(x)1\|_2 \leq \kappa(\|[\pi_n(a), x]\|_2 + \|[\pi_n(b), x]\|_2)$, for every $x \in \mathbb{M}_{k_n}(\mathbb{C})$. Hence, $u_n = \pi_n(a)$ and $v_n = \pi_n(b)$ satisfy the conclusion of Lemma 4.3.

We are grateful to one of the referees for pointing out that the following fact holds.

Lemma 4.4. *Let Γ be a non-virtually abelian, residually finite, countable group. Then Γ admits a sequence of finite-dimensional irreducible representations $\pi_n\colon \Gamma \to \mathrm{U}(k_n)$, $n \in \mathbb{N}$, with $k_n \to \infty$.*

This fact is used in the above proof for $\Gamma = \mathrm{SL}_3(\mathbb{Z})$. As pointed out by the referee, Lemma 4.4 shows that this fact is not specific to $\mathrm{SL}_3(\mathbb{Z})$ and that its proof does not need to use property (T).

Proof of Lemma 4.4. Assume by contradiction that there exists $M \in \mathbb{N}$ such that for every irreducible finite-dimensional representation $\pi\colon \Gamma \to \mathrm{U}(k)$, we have that $k \leq M$. Let $\{\Gamma_n\}_n$ be a decreasing sequence of finite index normal subgroups of Γ such that $\bigcap_n \Gamma_n = \{e\}$. Denote by $G = \varprojlim \Gamma/\Gamma_n$ the profinite completion of Γ with respect to $\{\Gamma_n\}$. Since $\bigcap_n \Gamma_n = \{e\}$, the natural homomorphism $\Gamma \to G$ gives a dense embedding $\Gamma \subset G$. Since G is compact, the Peter–Weyl theorem implies that every irreducible continuous unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is finite-dimensional. Since $\Gamma \subset G$ is dense, the restriction of π to Γ is still irreducible and thus $\dim(\mathcal{H}) \leq m$. Hence, every irreducible continuous unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ satisfies $\dim(\mathcal{H}) \leq m$. Applying [38, Proposition 3.1] gives that G has an open abelian subgroup G_0. Then $\Gamma_0 = \Gamma \cap G_0$ is a finite index abelian subgroup of Γ. This proves that Γ is virtually abelian, which is a contradiction. ■

We next give a second proof of the main assertion of Lemma 4.3 showing that one can take $k_n = n$. This relies on the notion of quantum expanders introduced in [7,26] (see also [43]). For a related application of quantum expanders, see the recent article [39]. For $k \geq 2$ and a k-tuple of unitaries $u = (u_1, u_2, \dots, u_k) \in \mathrm{U}(n)^k$, let $T_u \colon \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be the operator given by

$$T_u(x) = \sum_{i=1}^{k} u_i x u_i^*, \quad \text{for every } x \in \mathbb{M}_n(\mathbb{C}).$$

Endow $\mathbb{M}_n(\mathbb{C})$ with the normalized Hilbert–Schmidt norm, note that the space $\mathbb{M}_n(\mathbb{C}) \ominus \mathbb{C}1$ of matrices of trace zero is T_u-invariant, and denote by T_u^0 the restriction of T_u to $\mathbb{M}_n(\mathbb{C}) \ominus \mathbb{C}1$.

Remark 4.5. We clearly have that $\|T_u^0\| \leq k$. Moreover, equality holds if $k = 2$. To see this, let $u = (u_1, u_2)$. Then $T_u(u_2^* u_1) = 2u_1 u_2^*$ and $\|u_2^* u_1 - \alpha 1\|_2 = \|u_1 u_2^* - \alpha 1\|_2 = \sqrt{1 - |\alpha|^2}$, where $\alpha = \tau(u_2^* u_1) = \tau(u_1 u_2^*)$. If $|\alpha| < 1$, then since $T_u(u_2^* u_1 - \alpha 1) = 2(u_1 u_2^* - \alpha 1)$, we get that $\|T_u^0\| = 2$. If $|\alpha| = 1$, then $u_1 = \alpha u_2$ and so $T_u(x) = 2u_1 x u_1^*$, for every $x \in \mathbb{M}_n(\mathbb{C})$, which gives that $\|T_u^0\| = 2$.

A sequence of k-tuples $u^n = (u_1^n, \dots, u_k^n) \in \mathrm{U}(n)^k$ is called a *quantum expander* if $\sup_n \|T_{u^n}^0\| < k$. By Remark 4.5, this forces that $k \geq 3$. The following result due to Hastings [26] (formulated here following [43, Lemma 1.8]) shows that random unitaries provide quantum expanders for $k \geq 3$.

Lemma 4.6. *For $n \in \mathbb{N}$, let μ_n be the Haar measure of* $\mathrm{U}(n)$. *Then for every $\varepsilon > 0$, we have that*

$$\lim_{n \to \infty} \mu_n^k(\{u \in \mathrm{U}(n)^k \mid \|T_u^0\| \leq 2\sqrt{k-1} + \varepsilon k\}) = 1.$$

4.3. *Proof of Lemma* 4.3

We claim that there is $N \in \mathbb{N}$ such that the main assertion of Lemma 4.3 holds for any constant κ greater than $3 + 2\sqrt{2}$ and $k_n = n$, for every $n > N$. Assuming this claim, note that if $n \in \mathbb{N}$ is fixed, then we can find two unitaries $u_n, v_n \in \mathrm{U}(n)$ such that $\{u_n, v_n\}' \cap \mathbb{M}_n(\mathbb{C}) = \mathbb{C}1$. Using the compactness of the unit ball of $\mathbb{M}_n(\mathbb{C})$ with respect to the $\|\cdot\|_2$-norm, we can find a constant $\kappa_n > 0$ such that $\|x - \tau(x)1\|_2 \leq \kappa_n(\|[u_n, x]\|_2 + \|[v_n, x]\|_2)$, for every $x \in \mathbb{M}_n(\mathbb{C})$. It is now clear that the moreover assertion of Lemma 4.3 holds after replacing κ with $\max\{\kappa, \kappa_1, \dots, \kappa_N\}$.

To prove our claim, fix a constant $\kappa > 3 + 2\sqrt{2}$. Note that

$$\frac{1}{\kappa} < \frac{1}{3 + 2\sqrt{2}} = 3 - 2\sqrt{2}$$

and let $\varepsilon := \frac{3-2\sqrt{2}-\frac{1}{\kappa}}{2} > 0$. By applying Lemma 4.6 in the case $k = 3$, we deduce that

$$\lim_{n\to\infty} \mu_n^3(\{u \in \mathrm{U}(n)^3 \mid \|T_u^0\| \leq 2\sqrt{2} + 2\varepsilon\}) = 1. \tag{4.2}$$

Let S_n be set of pairs of unitaries $(u_1, u_2) \in \mathrm{U}(n)^2$ such that $\|T^0_{(u_1,u_2,I)}\| \leq 2\sqrt{2} + 2\varepsilon$. Since

$$\|\mathbb{T}^0_{(u_1,u_2,u_3)}\| = \|T^0_{(u_3^*u_1,u_3^*u_1,I)}\|, \quad \text{for every } (u_1,u_2,u_3) \in \mathrm{U}(n)^3,$$

(4.2) implies that $\lim_{n\to\infty} \mu_n^2(S_n) = 1$.

Now, let $(u_1, u_2) \in S_n$. Then for every $x \in \mathbb{M}_n(\mathbb{C}) \ominus \mathbb{C}1$, we have

$$(2\sqrt{2} + 2\varepsilon)\|x\|_2 \geq \|u_1xu_1^* + u_2xu_2^* + x\|_2 \geq 3\|x\|_2 - \|[u_1, x]\|_2 - \|[u_2, x]\|_2,$$

and hence $\|x\|_2 \leq \kappa(\|[u_1, x]\|_2 + \|[u_2, x]\|_2)$. If $x \in \mathbb{M}_n(\mathbb{C})$, then applying this last inequality to $x - \tau(x)1 \in \mathbb{M}_n(\mathbb{C}) \ominus \mathbb{C}1$ gives that

$$\|x - \tau(x)1\|_2 \leq \kappa(\|[u_1, x]\|_2 + \|[u_2, x]\|_2).$$

Since $\lim_{n\to\infty} \mu_n^2(S_n) = 1$, we have that $S_n \neq \emptyset$, for n large enough. Then any pair $(u_n, v_n) \in S_n$, for n large enough, will satisfy the conclusion of Lemma 4.3.

We end this subsection by proving the following property for pairs of unitaries with spectral gap.

Lemma 4.7. *Let $\kappa > 0$ and $(u_1, u_2) \in \mathrm{U}(n)^2$ such that $\|x - \tau(x)1\|_2 \leq \kappa(\|[u_1, x]\|_2 + \|[u_2, x]\|_2)$, for all $x \in \mathbb{M}_n(\mathbb{C})$. Then $\|x\|_2 \leq 10^5\kappa^6(\|u_1xv - x\|_2 + \|u_2xv - x\|_2)$, for all $v \in \mathrm{U}(n)$ and $x \in \mathbb{M}_n(\mathbb{C})$.*

Proof. We claim that $\delta := \|u_1 - u_2\|_2 \geq \frac{1}{8\kappa^2}$. Indeed, $\|u_1 - \tau(u_1)1\|_2 \leq \kappa\|[u_1, u_2]\|_2 \leq 2\kappa\delta$ and similarly $\|u_2 - \tau(u_2)1\|_2 \leq 2\kappa\delta$. If $x \in \mathrm{U}(n)$ and $\tau(x) = 0$, then $\|[u_1, x]\|_2 \leq 2\|u_1 - \tau(u_1)1\|_2 \leq 4\kappa\delta$ and similarly $\|[u_2, x]\|_4 \leq 2\kappa\delta$. Thus, $1 = \|x\|_2 \leq \kappa(\|[u_1, x]\|_2 + \|[u_2, x]\|_2) \leq 8\kappa^2\delta$, proving our claim. Note also that since $1 = \|x\|_2 \leq \kappa(\|[u_1, x]\|_2 + \|[u_2, x]\|_2) \leq 4\kappa$, we have that $\kappa \geq \frac{1}{4}$.

To prove the conclusion of the lemma, let $x \neq 0$ and put $\varepsilon = \|x\|_2^{-1}(\|u_1xv - x\|_2 + \|u_2xv - x\|_2)$. Let $w \in \{u_1, u_2\}$. Then $wxx^*w^* = (wxv)(wxv)^*$ and the Cauchy–Schwarz inequality implies that

$$\|wxx^*w^* - xx^*\|_1 \leq \|(wxv - x)(wxv)^*\|_1 + \|x(wxv - x)^*\|_1 \leq 2\varepsilon\|x\|_2^2.$$

Let $y = (xx^*)^{\frac{1}{2}}$ and $\alpha = \tau(y) \geq 0$. Then the Powers–Størmer inequality (2.1) implies that

$$\|wyw^* - y\|_2 \leq \|wy^2w^* - y^2\|_1^{\frac{1}{2}} = \|wxx^*w^* - xx^*\|_1^{\frac{1}{2}} \leq \sqrt{2\varepsilon}\|x\|_2,$$

for every $w \in \{u_1, u_2\}$.

Thus, we get that $\|y - \alpha 1\|_2 \leq 2\kappa\sqrt{2\varepsilon}\|x\|_2$. Since $\|y\|_2 = \|x\|_2$, it follows that

$$\alpha \geq (1 - 2\kappa\sqrt{2\varepsilon})\|x\|_2.$$

Next, by the polar decomposition we can find a unitary $z \in \mathrm{U}(n)$ such that $x = yz$. Then we have $\|x - \alpha z\|_2 = \|y - \alpha 1\|_2 \le 2\kappa\sqrt{2\varepsilon}\|x\|_2$. Since

$$\|(u_1 - u_2)xv\|_2 \le \|u_1xv - x\|_2 + \|u_2xv - x\|_2 = \varepsilon\|x\|_2,$$

we further get that

$$\alpha\delta = \|(u_1 - u_2)(\alpha z)v\|_2 \le \|(u_1 - u_2)xv\|_2 + \|x - \alpha z\|_2 \le (\varepsilon + 2\kappa\sqrt{2\varepsilon})\|x\|_2.$$

Since $x \neq 0$, by combining the fact that $\delta \ge \frac{1}{8\kappa^2}$ with the last inequality, we conclude that

$$\frac{1 - 2\kappa\sqrt{2\varepsilon}}{8\kappa^2} \le \varepsilon + 2\kappa\sqrt{2\varepsilon}.$$

Since $\kappa \ge \frac{1}{4}$ and $\varepsilon \le 4$, it follows that $\varepsilon \ge \frac{1}{10^5\kappa^6}$, which finishes the proof. ∎

4.4. *Proof of Proposition* 4.1

Let $\kappa > 0$ be as given by Lemma 4.3. The moreover assertion of Lemma 4.3 provides $u, v \in \mathcal{U}(A)$ such that $\|x - \tau(x)\|_2 \le \kappa(\|[u, x]\|_2 + \|[v, x]\|_2)$, for every $x \in A$.

It is a standard fact that $Z_1 = u \otimes 1$, $Z_2 = v \otimes 1 \in \mathcal{U}(A \otimes 1)$ satisfy the conclusion for $\eta = \sqrt{2}\kappa$. For completeness, let us recall the argument. Let $\{\xi_i\}_{i \in I}$ be an orthonormal basis of B with respect to the scalar product given by its trace. Let $x \in A \otimes B$ and write $x = \sum_{i \in I} x_i \otimes \xi_i$, with $x_i \in A$. Then $\mathrm{E}_{1\otimes B}(x) = \sum_{i \in I} \tau(x_i)1 \otimes \xi_i$, $[Z_1, x] = \sum_{i \in I}[u, x_i] \otimes \xi_i$, $[Z_2, x] = \sum_{i \in I}[v, x_i] \otimes \xi_i$ and therefore

$$\begin{aligned}\|x - \mathrm{E}_{1\otimes B}(x)\|_2^2 &= \sum_{i \in I} \|x_i - \tau(x_i)1\|_2^2 \le \sum_{i \in I} \kappa^2(\|[u, x_i]\|_2 + \|[v, x_i]\|_2)^2 \\ &\le \eta^2 \sum_{i \in I}(\|[u, x_i]\|_2^2 + \|[v, x_i]\|_2^2) = \eta^2(\|[Z_1, x]\|_2^2 + \|[Z_2, x]\|_2^2) \\ &\le (\eta(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2))^2.\end{aligned}$$

This finishes the proof.

4.5. *Proof of Proposition* 4.2

Let $\kappa > 0$ be as given by Lemma 4.3. The moreover assertion of Lemma 4.3 gives $u, v \in \mathcal{U}(A)$ such that $\|x - \tau(x)1\|_2 \le \kappa(\|[u, x]\|_2 + \|[v, x]\|_2)$, for every $x \in A$. The proof of Proposition 4.1 shows that

$$\|x - \mathrm{E}_{1\otimes B}(x)\|_2 \le \sqrt{2}\kappa(\|[u \otimes 1, x]\|_2 + \|[v \otimes 1, x]\|_2), \quad \text{for every } x \in A \otimes B. \tag{4.3}$$

By Lemma 4.7, $\|x\|_2 \le 10^5\kappa^6(\|uxt - x\|_2 + \|vxt - x\|_2)$, for every $t \in \mathcal{U}(A)$ and $x \in A$. Then the argument from the proof of Proposition 4.1 implies that for every $t \in \mathcal{U}(A)$ and $x \in A \otimes B$, we have

$$\|x\|_2 \le \sqrt{2} \cdot 10^5\kappa^6(\|(u \otimes 1)x(t \otimes 1) - x\|_2 + \|(v \otimes 1)x(t \otimes 1) - x\|_2). \tag{4.4}$$

Define $Z_1, Z_2 \in \mathcal{U}(\mathbb{M}_3(\mathbb{C}) \otimes A \otimes B)$ by letting

$$Z_1 = \begin{pmatrix} u \otimes 1 & 0 & 0 \\ 0 & u \otimes 1 & 0 \\ 0 & 0 & v \otimes 1 \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & w & 0 \end{pmatrix}.$$

Then Z_1, Z_2 satisfy conditions (2) and (3) from the conclusion and

$$Z_2 Z_1 Z_2^* = \begin{pmatrix} v \otimes 1 & 0 & 0 \\ 0 & u \otimes 1 & 0 \\ 0 & 0 & w(u \otimes 1)w^* \end{pmatrix}.$$

We will show that condition (1) is satisfied for $\eta = 10^7(1+\kappa^6)$. To this end, fix $x \in \mathbb{M}_3(\mathbb{C}) \otimes A \otimes B$ with $x = x^*$. Write $x = [x_{ij}]$, where $x_{i,j} \in A \otimes B$ are such that $x_{i,j}^* = x_{j,i}$, for every $1 \leq i, j \leq 3$. Our goal is to show that

$$\|x - \mathrm{E}_{1\otimes 1\otimes B}(x)\|_2 \leq \frac{\eta}{2}(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2). \tag{4.5}$$

Towards this goal, we denote $\varepsilon = \|[Z_1, x]\|_2 + \|[Z_2, x]\|_2$ and record the following elementary fact.

Fact 4.8. Let $d_1, d_2, d_3 \in \mathcal{U}(A \otimes B)$ and put $d = [d_i \delta_{i,j}] \in \mathcal{U}(\mathbb{M}_3(\mathbb{C}) \otimes A \otimes B)$. Then we have $\|[d, x]\|_2^2 = \sum_{i,j=1}^3 \|d_i x_{i,j} d_j^* - x_{i,j}\|_2^2$, hence $\|d_i x_{i,j} d_j^* - x_{i,j}\|_2 \leq \|[d, x]\|_2$, for every $1 \leq i, j \leq 3$.

Fact 4.8 implies that $\|[(u \otimes 1), x_{1,1}]\|_2 \leq \|[Z_1, x]\|_2 \leq \varepsilon$ and $\|[(v \otimes 1), x_{1,1}]\|_2 \leq \|[Z_2 Z_1 Z_2^*, x]\|_2 \leq 3\varepsilon$. Together with (4.3) this gives that

$$\|x_{1,1} - y\|_2 \leq (\sqrt{2}\kappa) \cdot (4\varepsilon) \leq \frac{\eta}{6}\varepsilon, \tag{4.6}$$

where $y := \mathrm{E}_{1\otimes B}(x_{1,1}) \in 1 \otimes B$.

By using Fact 4.8, we also derive that $\|(u \otimes 1)x_{1,2}(u \otimes 1)^* - x_{1,2}\|_2 \leq \|[Z_1, x]\|_2 \leq \varepsilon$ and that $\|(v \otimes 1)x_{1,2}(u \otimes 1)^* - x_{1,2}\|_2 \leq \|[Z_2 Z_1 Z_2^*, x]\|_2 \leq 3\varepsilon$. Applying (4.4) to $x = x_{1,2}$ and $t = u^*$, we get

$$\|x_{1,2}\|_2 \leq (\sqrt{2} \cdot 10^5 \kappa^6) \cdot (4\varepsilon) \leq \frac{\eta}{6}\varepsilon. \tag{4.7}$$

Next, note that

$$[Z_2, x] = \begin{pmatrix} x_{3,1} - x_{1,2} & x_{3,2} - x_{1,3}w & x_{3,3} - x_{1,1} \\ x_{1,1} - x_{2,2} & x_{1,2} - x_{2,3}w & x_{1,3} - x_{2,1} \\ wx_{2,1} - x_{3,2} & wx_{2,2} - x_{3,3}w & wx_{2,3} - x_{3,1} \end{pmatrix}.$$

Since $\|[Z_2, x]\|_2 \leq \varepsilon$, we get that $\|x_{2,2} - x_{1,1}\|_2 \leq \varepsilon$ and $\|x_{3,3} - x_{1,1}\|_2 \leq \varepsilon$. Together with (4.6), this gives that

$$\|x_{2,2} - y\|_2 \leq (\sqrt{2} \cdot 8\kappa + 1)\varepsilon \leq \frac{\eta}{6}\varepsilon, \quad \|x_{3,3} - y\|_2 \leq (\sqrt{2} \cdot 8\kappa + 1)\varepsilon \leq \frac{\eta}{6}\varepsilon. \tag{4.8}$$

Since $\|[Z_2, x]\|_2 \le \varepsilon$, we also get that $\|x_{2,3}w - x_{1,2}\|_2 \le \varepsilon$ and $\|x_{3,1} - x_{1,2}\|_2 \le \varepsilon$. Together with (4.7), this gives that

$$\|x_{2,3}\|_2 \le (8 \cdot \sqrt{2} \cdot 10^4 \kappa^6 + 1)\varepsilon \le \frac{\eta}{6}\varepsilon, \quad |x_{3,1}\|_2 \le (8 \cdot \sqrt{2} \cdot 10^4 \kappa^6 + 1)\varepsilon \le \frac{\eta}{6}\varepsilon. \tag{4.9}$$

Since $x = x^*$, by using (4.6)–(4.9), we get that

$$\begin{aligned}\|x - 1 \otimes y\|_2^2 &= \|x_{1,1} - y\|_2^2 + \|x_{2,2} - y\|_2^2 + \|x_{3,3} - y\|_2^2 \\ &\quad + 2\|x_{1,2}\|_2^2 + 2\|x_{1,3}\|_2^2 + 2\|x_{2,3}\|_2^2 \le 9\Big(\frac{\eta}{6}\varepsilon\Big)^2.\end{aligned}$$

Since $1 \otimes y \in 1 \otimes 1 \otimes B$, we get that $\|x - \mathrm{E}_{1\otimes 1\otimes B}(x)\|_2 \le \|x - 1 \otimes y\|_2 \le \frac{\eta}{2}\varepsilon$, hence (4.5) holds.

Finally, given $x \in \mathbb{M}_3(\mathbb{C}) \otimes A \otimes B$, write $x = x_1 + ix_2$, where $x_1 = x_1^*$ and $x_2 = x_2^*$. Then $\|[u, x]\|_2^2 = \|[u, x_1]\|_2^2 + \|[u, x_2]\|_2^2$, for every unitary u and by using (4.5) for x_1 and x_2, we get that

$$\begin{aligned}\|x - \mathrm{E}_{1\otimes 1\otimes B}(x)\|_2 &\le \|x_1 - \mathrm{E}_{1\otimes 1\otimes B}(x_1)\|_2 + \|x_2 - \mathrm{E}_{1\otimes 1\otimes B}(x_2)\|_2 \\ &\le \frac{\eta}{2}(\|[Z_1, x_1]\|_2 + \|[Z_2, x_1]\|_2 + \|[Z_1, x_2]\|_2 + \|[Z_2, x_2]\|_2) \\ &\le \eta(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2).\end{aligned}$$

This finishes the proof.

5. Proof of Proposition 1.5

This section is devoted to the proof of Proposition 1.5. We first prove Proposition 1.5 under the stronger assumption that $\mathbb{F}_3 \times \mathbb{F}_3$ is HS-stable, since the proof is more transparent in this case and relies on the simpler Proposition 4.1 instead of Proposition 4.2.

5.1. *Proof of Proposition 1.5 assuming that $\mathbb{F}_3 \times \mathbb{F}_3$ is HS-stable*

Let $\varepsilon > 0$. Let $\eta > 0$ be the constant provided by Proposition 4.1. Let $\delta_3 \colon (0, +\infty) \to (0, +\infty)$ be the function provided by Lemma 3.5. Let $\varepsilon_0 > 0$ such that $\varepsilon_0 < \frac{\varepsilon}{24}$ and $16\eta\varepsilon_0 < \delta_3(\frac{\varepsilon}{16})$.

Since $\mathbb{F}_3 \times \mathbb{F}_3$ is HS-stable, we can find $\delta > 0$ such that for any finite-dimensional factor M and $Z_\alpha, T_\beta \in \mathcal{U}(M)$ such that $\|[Z_\alpha, T_\beta]\|_2^2 \le \delta$, for every $\alpha, \beta \in \{1, 2, 3\}$, we can find $\widetilde{Z}_\alpha, \widetilde{T}_\beta \in \mathcal{U}(M)$ such that $\|\widetilde{Z}_\alpha - Z_\alpha\|_2 \le \varepsilon_0$, $\|\widetilde{T}_\beta - T_\beta\|_2 \le \varepsilon_0$ and $[\widetilde{Z}_\alpha, \widetilde{T}_\beta] = 0$, for every $\alpha, \beta \in \{1, 2, 3\}$.

Let $k, m, n \in \mathbb{N}$ and $U_1, \ldots, U_k, V_1, \ldots, V_m \in U(n)$ such that

$$\frac{1}{km}\sum_{i=1}^{k}\sum_{j=1}^{m} \|[U_i, V_j]\|_2^2 \le \delta.$$

Denote $A = \mathbb{M}_k(\mathbb{C})$, $B = \mathbb{M}_n(\mathbb{C})$, $C = \mathbb{M}_m(\mathbb{C})$ and $M = A \otimes B \otimes C$. By applying Lemma 4.1 twice, we can find $Z_1, Z_2 \in \mathcal{U}(A \otimes 1 \otimes 1)$ and $T_1, T_2 \in \mathcal{U}(1 \otimes 1 \otimes C)$ such that

$$\|x - \mathrm{E}_{1\otimes B\otimes C}(x)\|_2 \leq \eta(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2), \quad \text{for every } x \in M, \tag{5.1}$$

and

$$\|x - \mathrm{E}_{A\otimes B\otimes 1}(x)\|_2 \leq \eta(\|[T_1, x]\|_2 + \|[T_2, x]\|_2), \quad \text{for every } x \in M. \tag{5.2}$$

Let $Z_3 \in \mathcal{U}(A \otimes B \otimes 1)$ and $T_3 \in \mathcal{U}(1 \otimes B \otimes C)$ be given by

$$Z_3 = \sum_{i=1}^{k} e_{i,i} \otimes U_i \otimes 1 \quad \text{and} \quad T_3 = \sum_{j=1}^{m} 1 \otimes V_j \otimes e_{j,j}.$$

Then $[Z_3, T_3] = \sum_{i=1}^{k} \sum_{j=1}^{m} e_{i,i} \otimes [U_i, V_j] \otimes e_{j,j}$ and thus

$$\|[Z_3, T_3]\|_2^2 = \frac{1}{km} \sum_{i=1}^{k} \sum_{j=1}^{m} \|[U_i, V_j]\|_2^2 \leq \delta.$$

On the other hand, $[Z_\alpha, T_\beta] = 0$ if $\alpha, \beta \in \{1, 2, 3\}$ are not both equal to 3. Altogether, we get that

$$\|[Z_\alpha, T_\beta]\|_2^2 \leq \delta, \quad \text{for every } \alpha, \beta \in \{1, 2, 3\}.$$

The second paragraph of the proof implies that there are $\widetilde{Z}_\alpha, \widetilde{T}_\beta \in \mathcal{U}(M)$ such that $\|\widetilde{Z}_\alpha - Z_\alpha\|_2 \leq \varepsilon_0$, $\|\widetilde{T}_\beta - T_\beta\|_2 \leq \varepsilon_0$ and $[\widetilde{Z}_\alpha, \widetilde{T}_\beta] = 0$, for all $\alpha, \beta \in \{1, 2, 3\}$.

Denote by $P \subset M$ the von Neumann subalgebra generated by $\{\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3\}$. Let $x \in (P)_1$. If $\beta \in \{1, 2\}$, then since x commutes with $\widetilde{T}_\beta$, we get that $\|[T_\beta, x]\|_2 \leq 2\|\widetilde{T}_\beta - T_\beta\|_2 \leq 2\varepsilon_0$ and (5.2) gives that $\|x - \mathrm{E}_{A\otimes B\otimes 1}(x)\|_2 \leq 4\eta\varepsilon_0$. As $x \in (P)_1$ is arbitrary, we get $P \subset_{4\eta\varepsilon_0} A \otimes B \otimes 1$.

Similarly, using that $Q = P' \cap M$ commutes with $\widetilde{Z}_1$, $\widetilde{Z}_2$ and (5.2), we get that $Q \subset_{4\eta\varepsilon_0} 1 \otimes B \otimes C$. Since M is a finite-dimensional factor, the bicommutant theorem gives that $Q' \cap M = P$. By applying Lemma 2.3, we derive that $A \otimes 1 \otimes 1 \subset_{16\eta\varepsilon_0} P$.

Since $16\eta\varepsilon_0 < \delta_3(\frac{\varepsilon}{16})$, by combining the last two paragraphs and Lemma 3.5 we find a von Neumann subalgebra $S \subset B$ such that

$$\mathbf{d}(P, A \otimes S \otimes 1) \leq \frac{\varepsilon}{16}. \tag{5.3}$$

Denote $T = S' \cap B$. By Lemma 2.3, we get that

$$\mathbf{d}(Q, 1 \otimes T \otimes C) \leq \frac{\varepsilon}{4}. \tag{5.4}$$

Since $\widetilde{Z}_3 \in P$, (5.3) gives that $\|\widetilde{Z}_3 - \mathrm{E}_{A\otimes S\otimes 1}(\widetilde{Z}_3)\|_2 \leq \frac{\varepsilon}{16}$. As $\|Z_3 - \widetilde{Z}_3\|_2 \leq \varepsilon_0$, we get that

$$\|Z_3 - \mathrm{E}_{A\otimes S\otimes 1}(Z_3)\|_2 \leq \frac{\varepsilon}{16} + 2\varepsilon_0 < \frac{\varepsilon}{3}.$$

Similarly, by (5.4) we get

$$\|T_3 - \mathrm{E}_{1\otimes T\otimes C}(T_3)\|_2 \leq \frac{\varepsilon}{4} + 2\varepsilon_0 < \frac{\varepsilon}{3}.$$

The last two inequalities imply that

$$\frac{1}{k}\sum_{i=1}^{k}\|U_i - \mathrm{E}_S(U_i)\|_2^2 \leq \frac{\varepsilon}{9} \quad \text{and} \quad \frac{1}{m}\sum_{j=1}^{m}\|V_j - \mathrm{E}_T(V_j)\|_2^2 \leq \frac{\varepsilon}{9}.$$

Finally, by Lemma 2.2 we can find $\widetilde{U}_i \in \mathcal{U}(S)$, $\widetilde{V}_j \in \mathcal{U}(T)$ such that $\|U_i - \widetilde{U}_i\|_2 \leq 3\|U_i - \mathrm{E}_S(U_i)\|_2$ and $\|V_j - \widetilde{V}_j\|_2 \leq 3\|V_j - \mathrm{E}_T(V_j)\|_2$, for every $1 \leq i \leq k$ and $1 \leq j \leq m$. Since S and T commute, the conclusion follows.

5.2. *Proof of Proposition* 1.5

Assume that $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable. Let $\varepsilon > 0$. Let $\eta > 0$ be the constant provided by Proposition 4.2. Let $\delta_3\colon (0,+\infty) \to (0,+\infty)$ be the function provided by Lemma 3.5. Let $\varepsilon_0 > 0$ be such that $\varepsilon_0 < \frac{\varepsilon}{24}$ and $16\eta\varepsilon_0 < \delta_3(\frac{\varepsilon}{32})$.

Since $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable, we can find $\delta > 0$ such that for any finite-dimensional factor M and $Z_\alpha, T_\beta \in \mathcal{U}(M)$ such that $\|[Z_\alpha, T_\beta]\|_2^2 \leq \delta$, for every $\alpha, \beta \in \{1,2\}$, we can find $\widetilde{Z}_\alpha, \widetilde{T}_\beta \in \mathcal{U}(M)$ such that $\|\widetilde{Z}_\alpha - Z_\alpha\|_2 \leq \varepsilon$, $\|\widetilde{T}_\beta - T_\beta\|_2 \leq \varepsilon$ and $[\widetilde{Z}_\alpha, \widetilde{T}_\beta] = 0$, for every $\alpha, \beta \in \{1,2\}$.

Let $k, m, n \in \mathbb{N}$ and $U_1, \ldots, U_k, V_1, \ldots, V_m \in U(n)$ such that

$$\frac{1}{km}\sum_{i=1}^{k}\sum_{j=1}^{m}\|[U_i, V_j]\|_2^2 \leq \delta.$$

Denote $A = \mathbb{M}_k(\mathbb{C})$, $B = \mathbb{M}_n(\mathbb{C})$, $C = \mathbb{M}_m(\mathbb{C})$ and $M = \mathbb{M}_3(\mathbb{C}) \otimes A \otimes B \otimes C \otimes \mathbb{M}_3(\mathbb{C})$. Let $W \in \mathcal{U}(A \otimes B)$ and $W' \in \mathcal{U}(B \otimes C)$ be given by $W = \sum_{i=1}^{k} e_{i,i} \otimes U_i$ and $W' = \sum_{j=1}^{m} V_j \otimes e_{j,j}$. By applying Lemma 4.2 twice, we can find $Z_1, Z_2, T_1, T_2 \in \mathcal{U}(M)$ such that

(1) $\|x - \mathrm{E}_{1\otimes 1\otimes B\otimes C\otimes \mathbb{M}_3(\mathbb{C})}(x)\|_2 \leq \eta(\|[Z_1, x]\|_2 + \|[Z_2, x]\|_2)$, for every $x \in M$,

(2) $Z_1 \in \mathbb{M}_3(\mathbb{C}) \otimes A \otimes 1 \otimes 1 \otimes 1$,

(3) $Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & W & 0 \end{pmatrix} \otimes 1 \otimes 1 \in \mathbb{M}_3(\mathbb{C}) \otimes A \otimes B \otimes 1 \otimes 1$,

(4) $\|x - \mathrm{E}_{\mathbb{M}_3(\mathbb{C})\otimes A\otimes B\otimes 1\otimes 1}(x)\|_2 \leq \eta(\|[T_1, x]\|_2 + \|[T_2, x]\|_2)$, for every $x \in M$,

(5) $T_1 \in 1 \otimes 1 \otimes 1 \otimes C \otimes \mathbb{M}_3(\mathbb{C})$, and

(6) $T_2 = 1 \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & W' & 0 \end{pmatrix} \in 1 \otimes 1 \otimes B \otimes C \otimes \mathbb{M}_3(\mathbb{C})$.

Next, we have $[Z_2, T_2] = e_{3,2} \otimes [W \otimes 1, 1 \otimes W'] \otimes e_{3,2} = \frac{1}{km}\sum_{i=1}^{k}\sum_{j=1}^{m} e_{3,2} \otimes e_{i,i} \otimes [U_i, V_j] \otimes e_{j,j} \otimes e_{3,2}$ and thus $\|[Z_2, T_2]\|_2^2 = \frac{1}{9km}\sum_{i=1}^{k}\sum_{j=1}^{m}\|[U_i, V_j]\|_2^2 \leq \delta$. On the other hand, $[Z_\alpha, T_\beta] = 0$ if $\alpha, \beta \in \{1,2\}$ are not both equal to 2. Altogether, we have that $\|[Z_\alpha, T_\beta]\|_2^2 \leq \delta$, for every $\alpha, \beta \in \{1,2\}$.

The second paragraph of the proof implies that we can find $\widetilde{Z}_\alpha, \widetilde{T}_\beta \in \mathcal{U}(M)$ such that $\|\widetilde{Z}_\alpha - Z_\alpha\|_2 \leq \varepsilon_0$, $\|\widetilde{T}_\beta - T_\beta\|_2 \leq \varepsilon_0$ and $[\widetilde{Z}_\alpha, \widetilde{T}_\beta] = 0$, for every $\alpha, \beta \in \{1, 2\}$.

Let $P \subset M$ be the von Neumann subalgebra generated by $\{Z_1, Z_2\}$. Let $x \in (P)_1$. If $\beta \in \{1, 2\}$, then since x commutes with $\widetilde{T}_\beta$, we get that $\|[T_\beta, x]\|_2 \leq 2\|\widetilde{T}_\beta - T_\beta\|_2 \leq 2\varepsilon_0$ and (4) gives that $\|x - \mathrm{E}_{\mathbb{M}_3(\mathbb{C})\otimes A\otimes B\otimes 1\otimes 1}(x)\|_2 \leq 4\eta\varepsilon_0$. As $x \in (P)_1$ is arbitrary, we get $P \subset_{4\eta\varepsilon_0} \mathbb{M}_3(\mathbb{C}) \otimes A \otimes B \otimes 1 \otimes 1$.

Similarly, using that $Q = P' \cap M$ commutes with $\widetilde{Z}_1$, $\widetilde{Z}_2$ and (1), we deduce that $Q \subset_{4\eta\varepsilon_0} 1 \otimes 1 \otimes B \otimes C \otimes \mathbb{M}_3(\mathbb{C})$. Since M is a finite-dimensional factor, the bicommutant theorem gives that $Q' \cap M = P$. By applying Lemma 2.3, we get that $\mathbb{M}_3(\mathbb{C}) \otimes A \otimes 1 \otimes 1 \otimes 1 \subset_{16\eta\varepsilon_0} P$.

Since $16\eta\varepsilon_0 < \delta_3(\frac{\varepsilon}{32})$, using the last two paragraphs and Lemma 3.5, we find a von Neumann subalgebra $S \subset B$ such that

$$\mathbf{d}(P, \mathbb{M}_3(\mathbb{C}) \otimes A \otimes S \otimes 1 \otimes 1) \leq \frac{\varepsilon}{32}. \tag{5.5}$$

Denote $T = S' \cap B$. By Lemma 2.3, we get that

$$\mathbf{d}(Q, 1 \otimes 1 \otimes T \otimes C \otimes \mathbb{M}_3(\mathbb{C})) \leq \frac{\varepsilon}{8}.$$

Since $\widetilde{Z}_2 \in P$, formula (5.5) gives that $\|\widetilde{Z}_2 - \mathrm{E}_{\mathbb{M}_3(\mathbb{C})\otimes A\otimes S\otimes 1\otimes 1}(\widetilde{Z}_2)\|_2 \leq \frac{\varepsilon}{32}$. Since $\|Z_2 - \widetilde{Z}_2\|_2 \leq \varepsilon_0$, we get that $\|Z_2 - \mathrm{E}_{\mathbb{M}_3(\mathbb{C})\otimes A\otimes S\otimes 1\otimes 1}(Z_2)\|_2 \leq \frac{\varepsilon}{32} + 2\varepsilon_0 < \frac{\varepsilon}{6}$. Similarly, using (5.4) we get that $\|T_2 - \mathrm{E}_{1\otimes 1\otimes T\otimes C\otimes \mathbb{M}_3(\mathbb{C})}(T_2)\|_2 \leq \frac{\varepsilon}{8} + 2\varepsilon_0 < \frac{\varepsilon}{6}$. By the definition of Z_2, T_2, the last two inequalities imply that

$$\frac{1}{3k}\sum_{i=1}^{k} \|U_i - \mathrm{E}_S(U_i)\|_2^2 \leq \frac{\varepsilon}{36} \quad \text{and} \quad \frac{1}{3m}\sum_{j=1}^{m} \|V_j - \mathrm{E}_T(V_j)\|_2^2 \leq \frac{\varepsilon}{36}.$$

Finally, by Lemma 2.2 we can find $\widetilde{U}_i \in \mathcal{U}(S)$, $\widetilde{V}_j \in \mathcal{U}(T)$ such that $\|U_i - \widetilde{U}_i\|_2 \leq 3\|U_i - \mathrm{E}_S(U_i)\|_2$ and $\|V_j - \widetilde{V}_j\|_2 \leq 3\|V_j - \mathrm{E}_T(V_j)\|_2$, for every $1 \leq i \leq k$ and $1 \leq j \leq m$. Since S and T commute, the conclusion follows.

6. Proof of Theorem B

6.1. *Construction*

In this section, we prove Theorem B by constructing a counterexample to the conclusion of Proposition 1.5. We start by recalling our construction presented in the introduction.

Notation 6.1. Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

(1) We denote $M_n = \bigotimes_{k=1}^n \mathbb{M}_2(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$ and $A_n = \bigotimes_{k=1}^n \mathbb{C}^2 \cong \mathbb{C}^{2^n}$. We view A_n as a subalgebra of M_n, where we embed $\mathbb{C}^2 \subset \mathbb{M}_2(\mathbb{C})$ as the diagonal matrices.

(2) For $1 \leq i \leq n$, let $X_{n,i} = 1 \otimes \cdots \otimes 1 \otimes \sigma \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{U}(A_n)$, where $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathbb{C}^2$ is placed on the i-th tensor position.

(3) Let $G_n \subset \mathcal{U}(A_n \otimes M_n)$ be a finite subgroup which generates $A_n \otimes M_n$.

(4) We define $U_t \in \mathrm{U}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ by $U_t = P + e^{it}(1-P)$, where $P: \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ is the orthogonal projection onto the one-dimensional space spanned by $e_1 \otimes e_2 - e_2 \otimes e_1$.

(5) We identify $M_n \otimes M_n = \bigotimes_{k=1}^n (\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}))$, and let $\theta_{t,n}$ be the automorphism of $M_n \otimes M_n$ given by $\theta_{t,n}(\bigotimes_{k=1}^n x_k) = \bigotimes_{k=1}^n U_t x_k U_t^*$.

(6) Finally, consider the following two sets of unitaries in $M_n \otimes M_n$: $\mathcal{U}_n = \{X_{n,i} \otimes 1 \mid 1 \leq i \leq n\}$ and $\mathcal{V}_{t,n} = G_n \cup \theta_{t,n}(G_n)$.

We begin with the following elementary lemma. For $t \in \mathbb{R}$, we let $\rho_t = \frac{1+\cos(t)}{2} \in [0,1]$. We endow M_n with its unique trace τ and the scalar product given by $\langle x, y \rangle = \tau(y^* x)$, for every $x, y \in M_n$. For $1 \leq l \leq n$, we denote by $e_l: M_n \to M_n$ the orthogonal projection onto the subspace of tensors of length at most l, i.e., the span of $\bigotimes_{k=1}^n x_k$, with $x_k \in \mathbb{M}_2(\mathbb{C})$ and $|\{k \mid x_k \neq 1\}| \leq l$.

Lemma 6.2. *The following hold:*

(1) $P(x \otimes 1)P = \tau(x)P$, *for every* $x \in \mathbb{M}_2(\mathbb{C})$.

(2) $\tau((x \otimes 1)U_t(y \otimes 1)U_t^*) = \rho_t \tau(xy) + (1-\rho_t)\tau(x)\tau(y)$, *for every* $x, y \in \mathbb{M}_2(\mathbb{C})$.

(3) $\mathrm{E}_{U_t(\mathbb{M}_2(\mathbb{C})\otimes 1)U_t^*}(x \otimes 1) = \rho_t U_t(x \otimes 1)U_t^*$, *for every* $x \in \mathbb{M}_2(\mathbb{C})$ *with* $\tau(x) = 0$.

(4) $\|\mathrm{E}_{\theta_{t,n}(M_n \otimes 1)}(x \otimes 1)\|_2^2 \leq (1-\rho_t^{2l})\|e_l(x)\|_2^2 + \rho_t^{2l}\|x\|_2^2$, *for every* $x \in M_n$ *and* $1 \leq l \leq n$.

Proof. It is immediate that $P(e_{i,j} \otimes 1)P$ is equal to $\frac{1}{2}P$, if $i = j$, and 0, if $i \neq j$, which implies (1), where $e_{i,j}$ is the matrix whose (i,j) entry is equal to 1 and all other entries are equal to 0. Part (2) follows via a straightforward calculation by using (1) and that $\tau(P) = \frac{1}{4}$. If $x \in \mathbb{M}_2(\mathbb{C})$ and $\tau(x) = 0$, then (2) gives $\tau((x \otimes 1)U_t(y \otimes 1)U_t^*) = \rho_t \tau(xy) = \rho_t \tau(U_t(x \otimes 1)U_t^* U_t(y \otimes 1)U_t^*)$, for every $y \in \mathbb{M}_2(\mathbb{C})$. This clearly implies (3).

To prove (4), for $0 \leq i \leq n$, we denote by $V_i \subset M_n$ the span of tensors of the form $\bigotimes_{k=1}^n x_k$, such that $x_k = 1$ or $\tau(x_k) = 0$, for every $1 \leq k \leq n$, and $|\{k \mid x_k \neq 1\}| = i$. Let $f_i: M_n \to M_n$ be the orthogonal projection onto V_i. If $x = \bigotimes_{k=1}^n x_k \in V_i$, then using part (3) we get that

$$\mathrm{E}_{\theta_{t,n}(M_n \otimes 1)}(x \otimes 1) = \bigotimes_{k=1}^n \mathrm{E}_{U_t(\mathbb{M}_2(\mathbb{C})\otimes 1)U_t^*}(x_k \otimes 1) = \rho_t^i \theta_{t,n}(x \otimes 1).$$

Thus, for every $x \in M_n$ we have $\mathrm{E}_{\theta_{t,n}(M_n \otimes 1)}(x \otimes 1) = \sum_{i=0}^n \rho_t^i \theta_{t,n}(f_i(x) \otimes 1)$ and therefore

$$\|\mathrm{E}_{\theta_{t,n}(M_n \otimes 1)}(x \otimes 1)\|_2^2 = \sum_{i=0}^n \rho_t^{2i} \|f_i(x)\|_2^2. \tag{6.1}$$

Since $\sum_{i=0}^l \|f_i(x)\|_2^2 = \|e_l(x)\|_2^2$ and $\sum_{i=l+1}^n \|f_i(x)\|_2^2 = \|x\|_2^2 - \|e_l(x)\|_2^2$, (6.1) implies part (4). ∎

Next, we show that the sets of unitaries $\mathcal{U}_n$ and $\mathcal{V}_{t,n}$ almost commute.

Lemma 6.3. *We have that* $\|[U,V]\|_2 \le \|[U,V]\| \le 4|t|$, *for all* $U \in \mathcal{U}_n$, $V \in \mathcal{V}_{t,n}$.

Proof. Let $U \in \mathcal{U}_n$ and $V \in \mathcal{V}_{t,n}$. If $V \in G_n \subset A_n \otimes M_n$, then as $U \in A_n \otimes 1$ and A_n is abelian, we get $[U,V]=0$. Thus, we may assume that $V=\theta_{t,n}(Y)$, for $Y \in G_n$. Then since $[\theta_{t,n}(U),V]=0$, we get

$$\begin{aligned}\|[U,V]\| &\le 2\|U-\theta_{t,n}(U)\| = 2\|(\sigma\otimes 1)-U_t(\sigma\otimes 1)U_t^*\| \\ &\le 4\|U_t-1\| = 4|e^{it}-1| \le 4|t|. \end{aligned}$$ ■

6.2. A consequence of HS-*stability of* $\mathbb{F}_2\times\mathbb{F}_2$

To prove Theorem B, we show that if $t>0$ is small enough, the almost commuting sets of unitaries $\mathcal{U}_n$ and $\mathcal{V}_{t,n}$ contradict the conclusion of Proposition 1.5 for large $n \in \mathbb{N}$. To this end, we first use Proposition 1.5 to deduce the following.

Corollary 6.4. *Assume that* $\mathbb{F}_2\times\mathbb{F}_2$ *is* HS-*stable. Then for every* $\varepsilon>0$, *there exists* $t>0$ *such that the following holds: for every* $n\in\mathbb{N}$, *we can find a von Neumann subalgebra* $C\subset A_n$ *such that*

(1) $C\otimes 1 \subset_\varepsilon \theta_{t,n}(A_n\otimes 1)$, *and*

(2) $\frac{1}{n}\sum_{i=1}^n \|X_{n,i}-\mathrm{E}_C(X_{n,i})\|_2^2 \le \varepsilon$.

Proof. Let $\varepsilon\in(0,1)$. Let $\eta>0$ such that $\eta<\frac{\varepsilon^2}{256}$ and $\eta<\frac{\delta_2(\frac{\varepsilon}{2})^2}{64}$, where $\delta_2\colon(0,+\infty)\to(0,+\infty)$ is the function provided by Corollary 3.3.

By Lemma 6.3, we have that $\frac{1}{|\mathcal{U}_n|\cdot|\mathcal{V}_{t,n}|}\sum_{U\in\mathcal{U}_n,V\in\mathcal{V}_{t,n}}\|[U,V]\|_2^2 \le 16t^2$, for every $n\in\mathbb{N}$ and $t\in\mathbb{R}$. Since $\mathbb{F}_2\times\mathbb{F}_2$ is HS-stable, Proposition 1.5 implies that if $t>0$ is small enough, then the following holds: given any $n\in\mathbb{N}$, we can find a von Neumann subalgebra $P\subset M_n\otimes M_n$ such that

$$\frac{1}{|\mathcal{U}_n|}\sum_{U\in\mathcal{U}_n}\|U-\mathrm{E}_P(U)\|_2^2 \le \eta \tag{6.2}$$

and

$$\frac{1}{|\mathcal{V}_{t,n}|}\sum_{V\in\mathcal{V}_{t,n}}\|V-\mathrm{E}_{P'}(V)\|_2^2 \le \eta. \tag{6.3}$$

Then (6.3) gives that

$$\frac{1}{|G_n|}\sum_{V\in G_n}\|V-\mathrm{E}_{P'}(V)\|_2^2 \le 2\eta \quad\text{and}\quad \frac{1}{|G_n|}\sum_{V\in\theta_{t,n}(G_n)}\|V-\mathrm{E}_{P'}(V)\|_2^2 \le 2\eta.$$

Since G_n generates $A_n\otimes M_n$, by Lemma 2.4 we conclude that

$$A_n\otimes M_n \subset_{2\sqrt{\eta}} P' \quad\text{and}\quad \theta_{t,n}(A_n\otimes M_n)\subset_{2\sqrt{\eta}} P'. \tag{6.4}$$

Since $M_n\otimes M_n$ is a finite-dimensional factor, then the bicommutant theorem gives that $(P')'=P$. Since $A_n\subset M_n$ is a maximal abelian subalgebra, we have that $(A_n\otimes M_n)'=$

$A_n \otimes 1$. By combining these facts with (6.4) and Lemma 2.3, we derive that $P \subset_{8\sqrt{\eta}} A_n \otimes 1$ and $P \subset_{8\sqrt{\eta}} \theta_{t,n}(A_n \otimes 1)$.

Since $A_n \otimes 1$ is abelian and we have chosen $\eta > 0$ so that $8\sqrt{\eta} \leq \delta_2(\frac{\varepsilon}{2})$, Corollary 3.3 implies that we can find a von Neumann subalgebra $Q \subset A_n \otimes 1$ such that $\mathbf{d}(P, Q) \leq \frac{\varepsilon}{2}$. Since $8\sqrt{\eta} \leq \frac{\varepsilon}{2}$, we also have that $P \subset_{\frac{\varepsilon}{2}} \theta_{t,n}(A_n \otimes 1)$. By combining the last two facts, we derive that $Q \subset_{\varepsilon} \theta_{t,n}(A_n \otimes 1)$.

Thus, if $C \subset A_n$ is a von Neumann subalgebra such that $Q = C \otimes 1$, then condition (1) is satisfied. To verify condition (2), let $U \in \mathcal{U}(M_n \otimes M_n)$. Since $\mathbf{d}(P, Q) \leq \frac{\varepsilon}{2}$, we have that

$$\begin{aligned}\|U - \mathrm{E}_Q(U)\|_2 &\leq \|U - \mathrm{E}_P(U)\|_2 + \|\mathrm{E}_P(U) - \mathrm{E}_Q(\mathrm{E}_P(U))\|_2 + \|\mathrm{E}_Q(\mathrm{E}_P(U) - U)\|_2 \\ &\leq 2\|U - \mathrm{E}_P(U)\|_2 + \frac{\varepsilon}{2}.\end{aligned}$$

Hence, $\|U - \mathrm{E}_Q(U)\|_2^2 \leq 2(4\|U - \mathrm{E}_P(U)\|_2^2 + \frac{\varepsilon^2}{4}) = 8\|U - \mathrm{E}_P(U)\|_2^2 + \frac{\varepsilon^2}{2}$. In combination with (6.2), we derive that

$$\frac{1}{n}\sum_{i=1}^{n} \|X_{n,i} - E_C(X_{n,i})\|_2^2 = \frac{1}{|\mathcal{U}_n|}\sum_{U \in \mathcal{U}_n} \|U - \mathrm{E}_Q(U)\|_2^2 \leq 8\eta + \frac{\varepsilon^2}{2}.$$

Since $\eta < \frac{\varepsilon^2}{256}$ and $\varepsilon \in (0, 1)$, we have that $8\eta + \frac{\varepsilon^2}{2} < \varepsilon$ and condition (2) follows. ■

Let $\varepsilon \in (0, \frac{1}{16})$. Assuming that $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable, Corollary 6.4 implies that there is $t > 0$ such that for every $n \in \mathbb{N}$, we can find a subalgebra $C_n \subset A_n$ such that

(a) $C_n \otimes 1 \subset_{\varepsilon} \theta_{t,n}(A_n \otimes 1)$, and

(b) $\frac{1}{n}\sum_{i=1} \|X_{n,i} - \mathrm{E}_{C_n}(X_{n,i})\|_2^2 < \varepsilon$.

We will derive a contradiction as $n \to \infty$ by showing that (a) and (b) imply the following incompatible facts:

- $\dim(C_n z_n) \leq P(n)$, where P is a polynomial (see Lemma 6.5), and
- $\dim(C_n z_n) \geq 2^{\kappa n}$, where $\kappa \in (0, 1)$ (see Lemma 6.6), for a projection $z_n \in C_n$.

6.3. A polynomial upper bound on dimension

Lemma 6.5. *Let $C \subset M_n$ be a von Neumann subalgebra such that $C \otimes 1 \subset_{\varepsilon} \theta_{t,n}(M_n \otimes 1)$, for some $n \in \mathbb{N}$, $t \in (0, \frac{\pi}{4}]$ and $\varepsilon \in [0, \frac{1}{16}]$. Then there exists a projection $z \in \mathcal{Z}(C)$ such that*

$$\dim(Cz) \leq 2(6n)^{64\frac{\varepsilon}{t^2}+1} \quad \textit{and} \quad \tau(z) \geq \frac{1}{2}.$$

Proof. For simplicity, we denote $D = \theta_{t,n}(M_n \otimes 1)$. Since $C \otimes 1 \subset_{\varepsilon} D$, we get that

$$\|\mathrm{E}_D(u \otimes 1)\|_2 \geq 1 - \varepsilon, \quad \text{for every } u \in \mathcal{U}(C). \tag{6.5}$$

Let $\{z_j\}_{j=1}^m$ be an enumeration of the minimal projections of $\mathcal{Z}(C)$. Then $C = \bigoplus_{j=1}^m Cz_j$, where Cz_j is a factor and thus isomorphic to a matrix algebra $\mathbb{M}_{n_j}(\mathbb{C})$, for some $n_j \in \mathbb{N}$.

Assume that S is the set of $1 \leq j \leq m$ such that $\|E_D(u \otimes 1)\|_2^2 \geq (1-4\varepsilon)\|u\|_2^2 = (1-4\varepsilon)\tau(z_j)$, for every $u \in \mathcal{U}(Cz_j)$. Let $T = \{1,\dots,m\} \setminus S$. Then for every $j \in T$, there exists $u_j \in \mathcal{U}(Cz_j)$ such that

$$\|\mathrm{E}_D(u_j \otimes 1)\|_2^2 \leq (1-4\varepsilon)\tau(z_j). \tag{6.6}$$

We will prove that $z = \sum_{j \in S} z_j \in \mathcal{Z}(C)$ satisfies the conclusion. To estimate $\tau(z)$, for every $j \in S$, let $u_j = 1$. Denote by λ^m the Haar measure of $\mathbb{T}^m$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. By applying (6.5) to $\sum_{j=1}^m \mu_j u_j \in \mathcal{U}(C)$, for $\mu_1,\dots,\mu_m \in \mathbb{T}$, we get that

$$\begin{aligned}\sum_{j=1}^m \|\mathrm{E}_D(u_j \otimes 1)\|_2^2 &= \int_{\mathbb{T}^m} \left\|\mathrm{E}_D\left(\sum_{j=1}^m \mu_j u_j \otimes 1\right)\right\|_2^2 d\lambda^m(\mu_1,\dots,\mu_m)\\ &\geq (1-\varepsilon)^2 \geq 1-2\varepsilon. \end{aligned}\tag{6.7}$$

On the other hand, (6.6) implies that $\sum_{j=1}^m \|\mathrm{E}_D(u_j \otimes 1)\|_2^2 \leq \tau(z) + (1-4\varepsilon)\tau(1-z)$. In combination with (6.7), we deduce that $\tau(z) \geq \frac{1}{2}$.

To estimate $\dim(Cz)$, let l be the smallest positive integer such that $\rho_t^{2l} \leq 1-8\varepsilon$. We claim that

$$\|e_l(u)\|_2^2 \geq \frac{\|u\|_2^2}{2}, \quad \text{for every } j \in S \text{ and } u \in \mathcal{U}(Cz_j). \tag{6.8}$$

If $u \in \mathcal{U}(Cz_j)$, for some $j \in S$, then Lemma 6.2 (4) gives that

$$\begin{aligned}(1-4\varepsilon)\|u\|_2^2 &\leq \|\mathrm{E}_D(u \otimes 1)\|_2^2\\ &\leq \rho_t^{2l}(\|u\|_2^2 - \|e_l(u)\|_2^2) + \|e_l(u)\|_2^2\\ &\leq (1-8\varepsilon)(\|u\|_2^2 - \|e_l(u)\|_2^2) + \|e_l(u)\|_2^2,\end{aligned}$$

which implies (6.8).

If $j \in S$, then since Cz_j is isomorphic to the matrix algebra $\mathbb{M}_{n_j}(\mathbb{C})$, it admits an orthonormal basis $\mathcal{B}_j$ whose every element is of the form $\frac{u}{\|u\|_2}$, for some $u \in \mathcal{U}(Cz_j)$. Then $\mathcal{B} = \bigcup_{j \in S} \mathcal{B}_j$ is an orthonormal basis for $Cz = \bigoplus_{j \in S} Cz_j$ and (6.8) implies that $\|e_l(\xi)\|_2^2 \geq \frac{1}{2}$, for every $\xi \in \mathcal{B}$. Recall that e_l is the orthogonal projection onto the subspace $W_l \subset M_n$ of tensors of length at most l and let $\mathcal{O}$ be an orthonormal basis for W_l. Then we have that

$$\dim(Cz) = |\mathcal{B}| \leq 2\sum_{\xi \in \mathcal{B}} \|e_l(\xi)\|_2^2 = 2 \sum_{\xi \in \mathcal{B}, \eta \in \mathcal{O}} |\langle \xi, \eta\rangle|^2 \leq 2|\mathcal{O}| = 2\dim(W_l). \tag{6.9}$$

On the other hand, we have the following crude estimate:

$$\dim(W_l) = \sum_{i=0}^l 3^i \binom{n}{i} \leq (l+1)3^l n^l \leq (6n)^l. \tag{6.10}$$

Next, note that $x \leq |\log(1-x)| \leq 2x$, for every $x \in [0, \frac{1}{2}]$. Since $\varepsilon \in [0, \frac{1}{16}]$, we get that $|\log(1-8\varepsilon)| \leq 16\varepsilon$. Since $t \in (0, \frac{\pi}{4}]$, we also have that $1 - \rho_t = \frac{1-\cos(t)}{2} \in [0, \frac{1}{2}]$ and

$1-\rho_t \geq \frac{t^2}{8}$. Thus, $|\log(\rho_t)| \geq 1-\rho_t \geq \frac{t^2}{8}$. By using these facts and the definition of l, we derive that

$$l \leq \frac{|\log(1-8\varepsilon)|}{2|\log(\rho_t)|} + 1 \leq 64\frac{\varepsilon}{t^2} + 1. \tag{6.11}$$

Combining (6.9), (6.10) and (6.11) implies that $\dim(Cz) \leq 2(6n)^{64\frac{\varepsilon}{t^2}+1}$, as desired. ∎

6.4. An exponential lower bound on dimension

Let $\mathrm{H}\colon (0,1) \to (0,1]$ be the binary entropy function given by $\mathrm{H}(\delta) = -\delta\log_2(\delta) - (1-\delta)\log_2(1-\delta)$.

Lemma 6.6. *Let $C \subset A_n$ be a von Neumann subalgebra such that*

$$\frac{1}{n}\sum_{i=1}^{n} \|X_{n,i} - \mathrm{E}_C(X_{n,i})\|_2^2 \leq \varepsilon, \quad \text{for some } \varepsilon \in \left[0, \frac{1}{8}\right] \text{ and } n \in \mathbb{N}.$$

Then $\dim(Cz) \geq 2^{n-\mathrm{H}(4\varepsilon)n-3}$*, for any projection $z \in C$ with $\tau(z) \geq \frac{1}{2}$.*

Proof. Let $p = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. For $1 \leq i \leq n$, let $p_i = 1 \otimes \cdots \otimes 1 \otimes p \otimes 1 \otimes \cdots \otimes 1$, where p is placed on the i-th tensor position. Then $X_{n,i} = 2p_i - 1$ and so $X_{n,i} - \mathrm{E}_C(X_{n,i}) = 2(p_i - \mathrm{E}_C(p_i))$, for every $1 \leq i \leq n$. Thus, the hypothesis rewrites as

$$\sum_{i=1}^{n} \|p_i - \mathrm{E}_C(p_i)\|_2^2 \leq \frac{\varepsilon}{4}. \tag{6.12}$$

Let $\{q_j\}_{j=1}^m$ be the minimal projections of C such that $C = \bigoplus_{j=1}^m \mathbb{C}q_j$. We claim that

$$\|p - \mathrm{E}_C(p)\|_2^2 = \sum_{j=1}^{m} \frac{\tau(pq_j)\tau((1-p)q_j)}{\tau(q_j)}, \quad \text{for every projection } p \in A_n. \tag{6.13}$$

Since $\mathrm{E}_C(p) = \sum_{j=1}^m \frac{\tau(pq_j)}{\tau(q_j)} q_j$, we get that $\|\mathrm{E}_C(p)\|_2^2 = \sum_{j=1}^m \frac{\tau(pq_j)^2}{\tau(q_j)}$. By combining the last fact with the identity

$$\|p - E_C(p)\|_2^2 = \|p\|_2^2 - \|E_C(p)\|_2^2 = \tau(p) - \|E_C(p)\|_2^2,$$

(6.13) follows.

By combining (6.12) and (6.13), we deduce that

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\tau(p_i q_j)\tau((1-p_i)q_j)}{\tau(q_j)} \leq \frac{\varepsilon}{4}. \tag{6.14}$$

Next, let S be the set of $1 \leq j \leq m$ such that

$$\frac{1}{n}\sum_{i=1}^{n} \tau(p_i q_j)\tau((1-p_i)q_j) < \varepsilon\tau(q_j)^2.$$

Define $T = \{1, \dots, m\} \setminus S$. Let $r = \sum_{j \in T} q_j$. Since $\varepsilon\tau(q_j) \leq \frac{1}{n}\sum_{i=1}^n \frac{\tau(p_i q_j)\tau((1-p_i)q_j)}{\tau(q_j)}$, for every $j \in T$, by using (6.14) we derive that $\varepsilon\tau(r) = \varepsilon\sum_{j\in T}\tau(q_j) \leq \frac{\varepsilon}{4}$ and thus $\tau(r) \leq \frac{1}{4}$.

Claim 6.7. *We have that* $\tau(q_j) \leq 2^{H(4\varepsilon)n-n+1}$ *for every* $j \in S$.

Proof. We identify A_n with $\mathrm{L}^\infty(\{0,1\}^n, \mu)$, where μ is the uniform probability measure on $\{0,1\}^n$. Then p_i is identified with the characteristic function of the set $\{x \in \{0,1\}^n \mid x_i = 0\}$, for every $1 \leq i \leq n$, and $\tau(\mathbf{1}_Y) = \mu(Y) = \frac{|Y|}{2^n}$, for every $Y \subset \{0,1\}^n$.

For $x, y \in \{0,1\}$, we denote the normalized Hamming distance:

$$\mathrm{d_H}(x,y) = \frac{|\{i \in \{1,\dots,n\} \mid x_i \neq y_i\}|}{n}.$$

Let $j \in S$ and $Z_j \subset \{0,1\}^n$ be such that $q_j = \mathbf{1}_{Z_j}$. As $\tau(p_i q_j) = \mu(\{x \in Z_j \mid x_i = 0\})$ and $\tau((1-p_i)q_j) = \mu(\{x \in Z_j \mid x_i = 1\})$, the inequality $\varepsilon\tau(q_j)^2 > \frac{1}{n}\sum_{i=1}^n \tau(p_i q_j) \times \tau((1-p_i)q_j)$ rewrites as

$$\begin{aligned}
\varepsilon\mu(Z_j)^2 &> \frac{1}{n}\sum_{i=1}(\mu(\{x \in Z_j \mid x_i = 0\}) \cdot \mu(\{x \in Z_j \mid x_i = 1\})) \\
&= \frac{1}{2n}\sum_{i=1}^n (\mu \times \mu)(\{(x,y) \in Z_j \times Z_j \mid x_i \neq y_i\}) \\
&= \frac{1}{2}\int_{Z_j \times Z_j} \mathrm{d_H}(x,y)\,\mathrm{d}(\mu\times\mu)(x,y).
\end{aligned}$$

By Fubini's theorem, we can find $x \in Z_j$ such that $\int_{Z_j} \mathrm{d_H}(x,y)\,\mathrm{d}\mu(y) < 2\varepsilon\mu(Z_j)$. This implies that $\mu(\{y \in Z_j \mid \mathrm{d_H}(x,y) \geq 4\varepsilon\}) < \frac{\mu(Z_j)}{2}$ and hence $\mu(\{y \in Z_j \mid \mathrm{d_H}(x,y) < 4\varepsilon\}) > \frac{\mu(Z_j)}{2}$. Thus,

$$\begin{aligned}
\mu(Z_j) &< 2\mu(\{y \in \{0,1\}^n \mid \mathrm{d_H}(x,y) < 4\varepsilon\}) \\
&= 2\mu\Big(\Big\{y \in \{0,1\}^n \mid \frac{1}{n}\sum_{i=1}^n y_i < 4\varepsilon\Big\}\Big) \\
&\leq \frac{1}{2^{n-1}}\sum_{i=0}^{\lfloor 4\varepsilon n\rfloor}\binom{n}{i}.
\end{aligned}$$

Since $\sum_{i=0}^{\lfloor \delta n\rfloor}\binom{n}{i} \leq 2^{\mathrm{H}(\delta)n}$, for all $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2}]$ (see [4, (4.7.4)]) in combination with the last displayed inequality, we conclude that $\tau(q_j) = \mu(Z_j) \leq 2^{H(4\varepsilon)n-n+1}$. ∎

To finish the proof of Lemma 6.6, let $z \in C$ be a projection with $\tau(z) \geq \frac{1}{2}$. Since $\tau(1-r) = 1 - \tau(r) \geq \frac{3}{4}$, we have that $\tau(z(1-r)) \geq \frac{1}{4}$. Since $z(1-r) \in C(1-r) = \bigoplus_{j\in S}\mathbb{C}q_j$, there is a subset $S_0 \subset S$ such that $z(1-r) = \sum_{j\in S_0} q_j$. By Claim (6.7), we get that $\frac{1}{4} \leq \tau(z(1-r)) = \sum_{j\in S_0}\tau(q_j) \leq |S_0| 2^{H(4\varepsilon)n-n+1}$ and thus $|S_0| \geq 2^{n-H(4\varepsilon)n-3}$. Since $Cz \supset Cz(1-r) = \bigoplus_{j\in S_0}\mathbb{C}q_j$, we have that $\dim(Cz) \geq |S_0|$ and the conclusion follows. ∎

Although this will not be used later, we show that the estimate provided by Lemma 6.6 is optimal.

Lemma 6.8. *Let* $\varepsilon \in (0,1)$. *Then there is* $c > 0$ *such that for any* $n \in \mathbb{N}$, *we can find a von Neumann subalgebra* $C \subset A_n$ *satisfying* $\frac{1}{n}\sum_{i=1}^{n} \|X_{n,i} - \mathrm{E}_C(X_{n,i})\|_2^2 \leq \varepsilon$ *and* $\dim(C) \leq c\sqrt{n}2^{n-\mathrm{H}(\frac{\varepsilon}{8})n}$.

Proof. We use the notation and the calculations established in the proof of Lemma 6.6. For $x \in \{0,1\}^n$ and $\delta > 0$, we denote by $\mathrm{B}_\delta(x) = \{y \in \{0,1\}^n \mid \mathrm{d_H}(x,y) \leq \delta\}$ the ball of radius δ centered at x. Let $x_1,\dots,x_m \in \{0,1\}^n$ be a maximal set such that $\mathrm{d_H}(x_j,x_k) > \frac{\varepsilon}{4}$, for every $j \neq k$. Then $\{0,1\}^n = \bigcup_{j=1}^{m} \mathrm{B}_{\frac{\varepsilon}{4}}(x_j)$. Let $Z_1 = \mathrm{B}_{\frac{\varepsilon}{4}}(x_1)$ and $Z_j = \mathrm{B}_{\frac{\varepsilon}{4}}(x_j) \setminus (\bigcup_{k=1}^{j-1} \mathrm{B}_{\frac{\varepsilon}{4}}(x_k))$, for $2 \leq j \leq m$. Then the sets $\{Z_j\}_{j=1}^m$ form a partition of $\{0,1\}^n$. Let $q_j = \mathbf{1}_{Z_j}$, for $1 \leq j \leq m$.

Define $C = \bigoplus_{j=1}^{m} \mathbb{C}q_j$. We will show that C satisfies the conclusion of the lemma. Let $1 \leq j \leq m$. Since $Z_j \subset \mathrm{B}_{\frac{\varepsilon}{4}}(x_j)$ we have that $\mathrm{d_H}(x,y) \leq \frac{\varepsilon}{2}$, for every $x, y \in Z_j$. Thus, we get that

$$\frac{1}{n}\sum_{i=1}^{n} \tau(p_i q_j)\tau((1-p_i)q_j) = \frac{1}{2}\int_{Z_j \times Z_j} \mathrm{d_H}(x,y)\,\mathrm{d}(\mu\times\mu)(x,y) \leq \frac{\varepsilon}{4}\mu(Z_j)^2$$
$$= \frac{\varepsilon}{4}\tau(q_j)^2.$$

By combining this inequality with (6.13) and the fact that $\sum_{j=1}^{m} \tau(q_j) = 1$, we get that

$$\frac{1}{n}\sum_{i=1}^{n} \|X_{n,i} - \mathrm{E}_C(X_{n,i})\|_2^2 = \frac{4}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\tau(p_i q_j)\tau((1-p_i)q_j)}{\tau(q_j)} \leq \varepsilon.$$

On the other hand, we have that $\dim(C) = m$. To prove the desired lower bound for m, note that the balls $\{\mathrm{B}_{\frac{\varepsilon}{8}}(x_j)\}_{j=1}^m$ are pairwise disjoint. Thus, we deduce that

$$2^n \geq \sum_{j=1}^{m} |\mathrm{B}_{\frac{\varepsilon}{8}}(x_j)| = m\sum_{i=0}^{\lfloor \frac{n\varepsilon}{8} \rfloor} \binom{n}{i}. \tag{6.15}$$

For $\delta \in (0, \frac{1}{2})$ such that δn is an integer, we have $\sum_{i=0}^{\delta n}\binom{n}{i} \geq \frac{2^{\mathrm{H}(\delta)n}}{\sqrt{8n\delta(1-\delta)}}$ (see [4, (4.7.4)]). This implies that

$$\sum_{i=0}^{\lfloor \frac{n\varepsilon}{8} \rfloor} \binom{n}{i} \geq \frac{2^{\mathrm{H}(\frac{\lfloor n\varepsilon/8 \rfloor}{n})n}}{\sqrt{n\varepsilon(1-\frac{\varepsilon}{8})}}.$$

Since the sequence $\{n\mathrm{H}(\frac{\varepsilon}{8}) - n\mathrm{H}(\frac{\lfloor \frac{n\varepsilon}{8} \rfloor}{n})\}$ is bounded, it follows that we can find a constant $c > 0$ depending only on ε such that

$$\sum_{i=0}^{\lfloor \frac{n\varepsilon}{8} \rfloor} \binom{n}{i} \geq \frac{2^{\mathrm{H}(\frac{\varepsilon}{8})n}}{c\sqrt{n}}.$$

In combination with (6.15), we derive that $\dim(C) = m \leq c\sqrt{n}2^{n-\mathrm{H}(\frac{\varepsilon}{8})n}$, which finishes the proof. ∎

6.5. *Proof of Theorem* B

Assume by contradiction that $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable. Let $\varepsilon \in (0, \frac{1}{16})$. Then by Corollary 6.4, there exists $t > 0$ such that for every $n \in \mathbb{N}$, we can find a von Neumann subalgebra $C_n \subset A_n$ such that

(a) $C_n \otimes 1 \subset_\varepsilon \theta_{t,n}(A_n \otimes 1)$, and

(b) $\frac{1}{n} \sum_{i=1}^{n} \|X_{n,i} - \mathrm{E}_{C_n}(X_{n,i})\|_2^2 \leq \varepsilon$.

Using (a), Lemma 6.5 gives a projection $z_n \in C_n$ such that $\dim(C_n z_n) \leq 2(6n)^{64\frac{\varepsilon}{t^2}+1}$ and $\tau(z_n) \geq \frac{1}{2}$. On the other hand, using (b), Lemma 6.6 implies that $\dim(C_n z_n) \geq 2^{n-\mathrm{H}(4\varepsilon)n-3}$. Thus, $2(6n)^{64\frac{\varepsilon}{t^2}+1} \geq 2^{n-\mathrm{H}(4\varepsilon)n-3}$, for all $n \in \mathbb{N}$. Since $\mathrm{H}(4\varepsilon) < 1$, letting $n \to \infty$ gives a contradiction.

7. Proofs of Theorem A and Corollary C

In this section, we give the proofs of Theorem A and Corollary C, and justify item (1) of Remark 1.1.

In preparation for the proofs of Theorem A and Corollary C, we note that, as $\mathbb{F}_2 \times \mathbb{F}_2$ is not HS-stable by Theorem B, there are sequences $U_{n,1}, U_{n,2}, V_{n,1}, V_{n,2} \in \mathrm{U}(d_n)$, for some $d_n \in \mathbb{N}$, such that

(1) $\|[U_{n,p}, V_{n,q}]\|_2 \to 0$, as $n \to \infty$, for every $1 \leq p, q \leq 2$, and

(2) $\inf_{n\in\mathbb{N}}(\|U_{n,1} - \widetilde{U}_{n,1}\|_2 + \|U_{n,2} - \widetilde{U}_{n,2}\|_2 + \|V_{n,1} - \widetilde{V}_{n,1}\|_2 + \|V_{n,2} - \widetilde{V}_{n,2}\|_2) > 0$, for any sequences $\widetilde{U}_{n,1}, \widetilde{U}_{n,2}, \widetilde{V}_{n,1}, \widetilde{V}_{n,2} \in \mathrm{U}(d_n)$ such that $[\widetilde{U}_{n,p}, \widetilde{V}_{n,q}] = 0$, for every $1 \leq p, q \leq 2$.

Consider the matricial ultraproduct $M = \prod_\omega \mathbb{M}_{d_n}(\mathbb{C})$. Letting $U_p = (U_{n,p})$, $V_q = (V_{n,q}) \in \mathcal{U}(M)$, condition (1) implies that $[U_p, V_q] = 0$, for every $1 \leq p, q \leq 2$. Let P and Q be the von Neumann subalgebras of M generated by $\{U_1, U_2\}$ and $\{V_1, V_2\}$, respectively. Then P and Q commute.

7.1. *Proof of Corollary* C

Assume by contradiction that the conclusion of Theorem A is false. Then we can find commuting von Neumann subalgebras P_n, Q_n of $\mathbb{M}_{d_n}(\mathbb{C})$, for all $n \in \mathbb{N}$, such that $P \subset \prod_\omega P_n$ and $Q \subset \prod_\omega Q_n$.

Thus, $U_p \in \mathcal{U}(\prod_\omega P_n)$ and $V_p \in \mathcal{U}(\prod_\omega Q_n)$, so we can find $\widetilde{U}_{n,p} \in \mathcal{U}(P_n)$ and $\widetilde{V}_{n,p} \in \mathcal{U}(Q_n)$, for every $n \in \mathbb{N}$, such that $U_p = (\widetilde{U}_{n,p})$ and $V_p = (\widetilde{V}_{n,p})$, for every $1 \leq p \leq 2$. But then we have that

$$\lim_{n\to\omega} \|U_{n,p} - \widetilde{U}_{n,p}\|_2 = \lim_{n\to\omega} \|V_{n,p} - \widetilde{V}_{n,p}\|_2 = 0, \quad \text{for every } 1 \leq p \leq 2.$$

Since $[\widetilde{U}_{n,p}, \widetilde{V}_{n,q}] = 0$, for every $1 \leq p, q \leq 2$, this contradicts (2), which finishes the proof.

7.2. Proof of Theorem A

Assume by contradiction that the conclusion of Theorem A is false.

Let $1 \leq p \leq 2$. Let $f:\mathbb{T} \to [-\frac{1}{2}, \frac{1}{2}]$ be a Borel function satisfying $\exp(2\pi i f(z)) = z$, for all $z \in \mathbb{T}$, and define $h_p = f(U_p)$. Then $h_p \in M$ is self-adjoint, generates the same von Neumann algebra as U_p and satisfies $\|h_p\| \leq \frac{1}{2}$ and $U_p = \exp(2\pi i h_p)$. Similarly, let $k_p = f(V_p)$.

Let $A = h_1 + ih_2$ and $B = k_1 + ik_2$. Then $\|A\| \leq 1$ and $\|B\| \leq 1$. As $[h_p, k_q] = 0$, for any $1 \leq p, q \leq 2$, we have $[A, B] = [A, B^*] = 0$. Represent $A = (A_n)$ and $B = (B_n)$, where $A_n, B_n \in \mathbb{M}_{d_n}(\mathbb{C})$ satisfy $\|A_n\|, \|B_n\| \leq 1$, for every $n \in \mathbb{N}$. Then

$$\lim_{n\to\omega} \|[A_n, B_n]\|_2 = \lim_{n\to\omega} \|[A_n, B_n^*]\|_2 = 0.$$

Since the conclusion of Theorem A is assumed false, we can find $A_n', B_n' \in \mathbb{M}_{d_n}(\mathbb{C})$ such that $\lim_{n\to\omega} \|A_n - A_n'\|_2 = \lim_{n\to\omega} \|B_n - B_n'\|_2 = 0$ and $[A_n', B_n'] = [A_n, B_n'^*] = 0$, for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$, denote by P_n and Q_n the von Neumann subalgebras of $\mathbb{M}_{d_n}(\mathbb{C})$ generated by A_n' and B_n'. Then P_n and Q_n commute and $\lim_{n\to\omega} \|A_n - \mathrm{E}_{P_n}(A_n)\|_2 = \lim_{n\to\omega} \|B_n - \mathrm{E}_{Q_n}(B_n)\|_2 = 0$.

Then $A \in \prod_\omega P_n$ and $B \in \prod_\omega Q_n$, hence $h_1, h_2 \in \prod_\omega P_n$ and $k_1, k_2 \in \prod_\omega Q_n$. Thus, $U_p \in \mathcal{U}(\prod_\omega P_n)$ and $V_p \in \mathcal{U}(\prod_\omega Q_n)$, for every $1 \leq p \leq 2$, and the proof of Corollary C gives a contradiction.

7.3. Almost versus near commuting when one matrix is normal

The following result generalizes Remark 1.1 (1).

Lemma 7.1. *Let (M_n, τ_n), $n \in \mathbb{N}$, be a sequence of tracial von Neumann algebras. Let $x_n, y_n \in (M_n)_1$ such that y_n is normal, for every $n \in \mathbb{N}$, and $\|[x_n, y_n]\|_2 \to 0$. Then there are $x_n', y_n' \in M_n$ such that $x_n'y_n' = y_n'x_n'$ and $x_n'y_n'^* = y_n'^*x_n'$, for every $n \in \mathbb{N}$, and $\|x_n - x_n'\|_2 + \|y_n - y_n'\|_2 \to 0$.*

This result can be proved quantitatively by adapting [18, 21]. Instead, as in [22], we give a short proof using tracial ultraproducts.

Proof of Lemma 7.1. Let us consider the ultraproduct von Neumann algebra $M = \prod_\omega M_n$, where ω is a free ultrafilter on $\mathbb{N}$. Let P and Q be the von Neumann subalgebras of M generated by $x = (x_n)$ and $y = (y_n)$. Then $[x, y] = 0$. Since y is normal, we get that $[x, y^*] = 0$, so P and Q commute. Since y is normal, we also get that Q is abelian. By applying [24, Theorem 2.7] or [31, Proposition C], we can represent $x = (x_n')$ and $y = (y_n')$ so that the von Neumann subalgebras of M_n generated by x_n' and y_n' commute, for all $n \in \mathbb{N}$. Since $\|x_n - x_n'\|_2 + \|y_n - y_n'\|_2 \to 0$, as $n \to \omega$, the conclusion follows. ∎

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References

[1] Anantharaman, C., Popa, S.: An introduction to II_1 factors. https://www.math.ucla.edu/~popa/Books/IIun.pdf, visited on 16 June 2024 (2010)

[2] Arzhantseva, G., Păunescu, L.: Almost commuting permutations are near commuting permutations. J. Funct. Anal. **269**, 745–757 (2015) Zbl 1368.20025 MR 3350728

[3] Arzhantseva, G., Păunescu, L.: Constraint metric approximations and equations in groups. J. Algebra **516**, 329–351 (2018) Zbl 1427.20054 MR 3863481

[4] Ash, R.: Information theory. Interscience Tracts Pure Appl. Math. 19, Interscience Publishers John Wiley & Sons, New York (1965) Zbl 0141.34904 MR 229475

[5] Atkinson, S.: Some results on tracial stability and graph products. Indiana Univ. Math. J. **70**, 1167–1187 (2021) Zbl 1484.46050 MR 4284110

[6] Becker, O., Lubotzky, A.: Group stability and Property (T). J. Funct. Anal. **278**, article no. 108298 (2020) Zbl 1516.20100 MR 4027744

[7] Ben-Aroya, A., Ta-Shma, A.: Quantum expanders and the quantum entropy difference problem. arXiv:quant-ph/0702129 (2007)

[8] Boutonnet, R., Chifan, I., Ioana, A.: II_1 factors with nonisomorphic ultrapowers. Duke Math. J. **166**, 2023–2051 (2017) Zbl 1384.46040 MR 3694564

[9] Brown, N. P., Ozawa, N.: C^*-algebras and finite-dimensional approximations. Grad. Stud. Math. 88, American Mathematical Society, Providence, RI (2008) Zbl 1160.46001 MR 2391387

[10] Choi, M. D.: Almost commuting matrices need not be nearly commuting. Proc. Amer. Math. Soc. **102**, 529–533 (1988) Zbl 0649.15005 MR 928973

[11] Christensen, E.: Subalgebras of a finite algebra. Math. Ann. **243**, 17–29 (1979) Zbl 0393.46049 MR 543091

[12] Dadarlat, M.: Obstructions to matricial stability of discrete groups and almost flat K-theory. Adv. Math. **384**, article no. 107722 (2021) Zbl 1477.19005 MR 4238917

[13] Davidson, K. R.: Almost commuting Hermitian matrices. Math. Scand. **56**, 222–240 (1985) Zbl 0563.15010 MR 813638

[14] De Chiffre, M., Glebsky, L., Lubotzky, A., Thom, A.: Stability, cohomology vanishing, and nonapproximable groups. Forum Math. Sigma **8**, article no. e18 (2020) Zbl 1456.22002 MR 4080477

[15] Eilers, S., Shulman, T., Sørensen, A. P. W.: C^*-stability of discrete groups. Adv. Math. **373**, article no. 107324 (2020) Zbl 1466.46044 MR 4131398

[16] Enders, D., Shulman, T.: Almost commuting matrices, cohomology, and dimension. Ann. Sci. Éc. Norm. Supér. (4) **56**, 1653–1683 (2023) Zbl 07827697 MR 4706570

[17] Exel, R., Loring, T.: Almost commuting unitary matrices. Proc. Amer. Math. Soc. **106**, 913–915 (1989) Zbl 0677.15003 MR 975641

[18] Filonov, N., Kachkovskiy, I.: A Hilbert–Schmidt analog of Huaxin Lin's theorem. arXiv:1008.4002 (2010)

[19] Filonov, N., Safarov, Y.: On the relation between an operator and its self-commutator. J. Funct. Anal. **260**, 2902–2932 (2011) Zbl 1239.47029 MR 2774059

[20] Friis, P., Rørdam, M.: Almost commuting self-adjoint matrices – a short proof of Huaxin Lin's theorem. J. Reine Angew. Math. **479**, 121–131 (1996) Zbl 0859.47018 MR 1414391

[21] Glebsky, L.: Almost commuting matrices with respect to normalized Hilbert–Schmidt norm. arXiv:1002.3082 (2010)

[22] Hadwin, D., Li, W.: A note on approximate liftings. Oper. Matrices **3**, 125–143 (2009) Zbl 1181.46042 MR 2500597

[23] Hadwin, D., Shulman, T.: Stability of group relations under small Hilbert–Schmidt perturbations. J. Funct. Anal. **275**, 761–792 (2018) Zbl 1411.46046 MR 3807776

[24] Hadwin, D., Shulman, T.: Tracial stability for C^*-algebras. Integral Equations Operator Theory **90**, article no. 1 (2018) Zbl 1396.46045 MR 3767651

[25] Halmos, P. R.: Some unsolved problems of unknown depth about operators on Hilbert space. Proc. Roy. Soc. Edinburgh Sect. A **76**, 67–76 (1976) Zbl 0346.47001 MR 451002

[26] Hastings, M. B.: Random unitaries give quantum expanders. Phys. Rev. A (3) **76**, article no. 032315 (2007) MR 2486279

[27] Hastings, M. B.: Making almost commuting matrices commute. Comm. Math. Phys. **291**, 321–345 (2009) Zbl 1189.15018 MR 2530163

[28] Hyers, D. H.: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. USA **27**, 222–224 (1941) Zbl 0061.26403 MR 4076

[29] Ioana, A.: Rigidity results for wreath product II_1 factors. J. Funct. Anal. **252**, 763–791 (2007) Zbl 1134.46041 MR 2360936

[30] Ioana, A.: Stability for product groups and property (τ). J. Funct. Anal. **279**, article no. 108729 (2020) Zbl 1485.20004 MR 4134896

[31] Ioana, A., Spaas, P.: A class of II_1 factors with a unique McDuff decomposition. Math. Ann. **375**, 177–212 (2019) Zbl 1436.46049 MR 4000239

[32] Ioana, A., Spaas, P., Wiersma, M.: Cohomological obstructions to lifting properties for full C*-algebras of property (T) groups. Geom. Funct. Anal. **30**, 1402–1438 (2020) Zbl 1461.46049 MR 4181828

[33] Kachkovskiy, I., Safarov, Y.: Distance to normal elements in C^*-algebras of real rank zero. J. Amer. Math. Soc. **29**, 61–80 (2016) Zbl 1339.47054 MR 3402694

[34] Kirchberg, E.: On non-semisplit extensions, tensor products and exactness of group C^*-algebras. Invent. Math. **112**, 449–489 (1993) Zbl 0803.46071 MR 1218321

[35] Lin, H.: Almost commuting selfadjoint matrices and applications. In: Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun. 13, American Mathematical Society, Providence, RI, 193–233 (1997) Zbl 0881.46042 MR 1424963

[36] Loring, T. A., Sørensen, A. P. W.: Almost commuting self-adjoint matrices: The real and self-dual cases. Rev. Math. Phys. **28**, article no. 1650017 (2016) Zbl 1362.46054 MR 3544231

[37] McDuff, D.: Central sequences and the hyperfinite factor. Proc. London Math. Soc. (3) **21**, 443–461 (1970) Zbl 0204.14902 MR 281018

[38] Moore, C. C.: Groups with finite dimensional irreducible representations. Trans. Amer. Math. Soc. **166**, 401–410 (1972) Zbl 0236.22010 MR 302817

[39] Mulcahy, G., Sinclair, T.: Malnormal matrices. Proc. Amer. Math. Soc. **150**, 2969–2982 (2022) Zbl 1506.15023 MR 4428882

[40] Murray, F. J., von Neumann, J.: On rings of operators. IV. Ann. of Math. (2) **44**, 716–808 (1943) Zbl 0060.26903 MR 9096

[41] Ozawa, N.: About the QWEP conjecture. Internat. J. Math. **15**, 501–530 (2004) Zbl 1056.46051 MR 2072092

[42] Ozawa, N.: Tsirelson's problem and asymptotically commuting unitary matrices. J. Math. Phys. **54**, article no. 032202 (2013) Zbl 1281.81023 MR 3059438

[43] Pisier, G.: Quantum expanders and geometry of operator spaces. J. Eur. Math. Soc. (JEMS) **16**, 1183–1219 (2014) Zbl 1307.46045 MR 3226740

[44] Pisier, G.: Tensor products of C^*-algebras and operator spaces. The Connes–Kirchberg problem. London Math. Soc. Stud. Texts 96, Cambridge University Press, Cambridge (2020) Zbl 1453.46002 MR 4283471

[45] Popa, S.: On a class of type II_1 factors with Betti numbers invariants. Ann. of Math. (2) **163**, 809–899 (2006) Zbl 1120.46045 MR 2215135

[46] Popa, S.: Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I. Invent. Math. **165**, 369–408 (2006) Zbl 1120.46043 MR 2231961

[47] Popa, S.: On Ozawa's property for free group factors. Int. Math. Res. Not. IMRN **2007**, article no. rnm036 (2007) Zbl 1134.46039 MR 2344271

[48] Popa, S.: On the superrigidity of malleable actions with spectral gap. J. Amer. Math. Soc. **21**, 981–1000 (2008) Zbl 1222.46048 MR 2425177

[49] Popa, S., Sinclair, A. M., Smith, R. R.: Perturbations of subalgebras of type II_1 factors. J. Funct. Anal. **213**, 346–379 (2004) Zbl 1063.46047 MR 2078630

[50] Rosenthal, P.: Are almost commuting matrices near commuting matrices? Amer. Math. Monthly **76**, 925–926 (1969) MR 1535586

[51] Said, M.: Almost commuting elements in non-commutative symmetric operator spaces. Ph.D. thesis, University of California, https://www.math.uci.edu/~mathphysics/preprints/msthesis.pdf, visited 16 June 2024 (2014)

[52] Takesaki, M.: Theory of operator algebras. I. Springer, New York (1979) Zbl 0436.46043 MR 548728

[53] Tao, T.: Expansion in finite simple groups of Lie type. Grad. Stud. Math. 164, American Mathematical Society, Providence, RI (2015) Zbl 1336.20015 MR 3309986

[54] Thom, A.: Finitary approximations of groups and their applications. In: Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures, World Scientific, Hackensack, NJ, 1779–1799 (2018) Zbl 1445.20039 MR 3966829

[55] Trott, S. M.: A pair of generators for the unimodular group. Canad. Math. Bull. **5**, 245–252 (1962) Zbl 0107.02503 MR 141716

[56] Ulam, S. M.: A collection of mathematical problems. Interscience Tracts Pure Appl. Math., Interscience Publishers, New York (1960) Zbl 0086.24101 MR 120127

[57] Vaes, S.: Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa). Astérisque **311**, 237–294 (2007) Zbl 1194.46085 MR 2359046

[58] Voiculescu, D.: Asymptotically commuting finite rank unitary operators without commuting approximants. Acta Sci. Math. (Szeged) **45**, 429–431 (1983) Zbl 0538.47003 MR 708811

[59] von Neumann, J.: Approximative properties of matrices of high finite order. Port. Math. **3**, 1–62 (1942) Zbl 0026.23302 MR 6137

[60] von Neumann, J.: Proof of the ergodic theorem and the H-theorem in quantum mechanics. Eur. Phys. J. H **35**, 201–237 (2010)