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Rigidity of the ball for an isoperimetric problem with strong capacitary repulsion

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Abstract. We consider a variational problem involving competition between surface tension and charge repulsion. We show that, as opposed to the case of weak (short-range) interactions where we proved ill-posedness of the problem in a previous paper, when the repulsion is stronger the perimeter dominates the capacitary term at small scales. In particular, we prove existence of minimizers for small charges as well as their regularity. Combining this with the stability of the ball under small $C^{1,\gamma}$ perturbations, this ultimately leads to the minimality of the ball for small charges. We cover in particular the borderline case of the 1-capacity where both terms in the energy are of the same order.

Keywords: De Giorgi perimeter, capacity, charged drops, Reifenberg regularity, potential theory.

1. Introduction

In this paper, we consider a geometric variational problem motivated by models for charged liquid drops recently studied in a series of papers [7, 18, 19, 30–32]. One of the main features of these problems is the strong competition between surface tension and charge repulsion. In particular, as opposed to the much studied Gamow liquid drop model (see [5, 22]), the non-local effects often dominate the cohesive forces leading to singular behaviors. The aim of the paper is to consider the case of very strong short-range repulsion between the charges, thus completing the program started in [18].

We now introduce the model. Given $\alpha \in (0, N)$ and a measurable set $E \subset \mathbb{R}^N$, we define the Riesz interaction energy of E by

$$\mathcal{I}_\alpha(E) = \inf_{\mu(E)=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}}. \tag{1.1}$$

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This energy coincides with the inverse of the α-capacity; see (2.9). Letting $P(E)$ denote the perimeter of E (see [27]), we consider for every charge $Q > 0$ the functional

$$\mathcal{F}_{\alpha,Q}(E) = P(E) + Q^2 \mathcal{I}_\alpha(E).$$

While postponing the discussion about the precise class in which we are minimizing, the aim of the paper is to study for $m > 0$ the problem

$$\min_{|E|=m} \mathcal{F}_{\alpha,Q}(E).$$

By a scaling argument, up to renaming the constant Q, it is enough to consider the case $m = \omega_N$, where ω_N is the volume of the unit ball B_1.

This question is motivated by the model for an electrically charged liquid drop in absence of gravity, introduced by Lord Rayleigh [34] in the physically relevant case $N = 3$ and $\alpha = 2$, and later investigated by many authors (see for instance [12, 18, 21, 30, 31, 38, 40]). We proved in [18] that, quite surprisingly, for every $N \geq 2$ and $\alpha \in (1, N)$ (in particular in the Coulomb case $\alpha = 2$), the problem is ill-posed. Indeed, in that case, we can show that

$$\inf_{|E|=\omega_N,\, E \text{ smooth}} \mathcal{F}_{\alpha,Q}(E) = P(B_1).$$

In words, starting from smooth sets the lower semicontinuous envelope of the energy $\mathcal{F}_{\alpha,Q}$ in $L^1(\mathbb{R}^N)$ reduces to the perimeter. To restore well-posedness of the problem one needs to impose some extra regularity conditions such as bounds on the curvature [18], entropic terms [30] or the convexity of competitors [19].

Later on, it was shown in [31] that at least if $N = 2$ and $\alpha = 1$, the problem admits the ball as unique minimizer as long as Q lies below an explicit threshold, and that non-existence occurs otherwise.

The aim of this paper is to complement the picture in the case $\alpha \in (0, 1]$ for $N \geq 2$.

1.1. Main results

As already mentioned above, the first difficulty with this model is to properly define the class of competitors. Indeed, while both the perimeter and $\mathcal{I}_\alpha$ are well defined in the class of smooth compact sets, this class does not enjoy good compactness properties. For variational problems involving the perimeter, the usual setup is the one of sets of finite perimeter (see [27]) where we identify two sets E and F if they are equal up to a Lebesgue-negligible set. However, it is not hard to see that $\mathcal{I}_\alpha$ is not well-behaved under such identification (we have $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(F)$ if $E = F$ outside a set of zero α-capacity; see [18]). As advocated in [31, 32] for $N = 2$ and $\alpha = 1$, we will consider here the class

$$\mathcal{S} = \{E \subset \mathbb{R}^N : E \text{ is compact and } P(E) = \mathcal{H}^{N-1}(\partial E) < +\infty\}. \tag{1.2}$$

We will always identify sets in $\mathcal{S}$ which differ only on a set of Lebesgue measure zero (and thus actually agree $\mathcal{H}^{N-1}$ a.e.); see Remark 2.1. The variational problem we consider is thus

$$\min_{|E|=\omega_N,\, E\in\mathcal{S}} \mathcal{F}_{\alpha,Q}(E). \tag{1.3}$$

The main result of this paper is the following.

Theorem 1.1. *For every $N \geq 2$ and $\alpha \in (0,1]$, there exists $Q_0 = Q_0(N,\alpha) > 0$ such that for every $Q \leq Q_0$, balls are the only minimizers of (1.3).*

We recall that by [18], if $\alpha \in (1, N)$, problem (1.3) does not admit minimizers for any value of $Q > 0$. The proof of Theorem 1.1 follows the same general scheme as in [22] (see also [1,3,4,11,29] where similar strategies have been used). Inspired by the proof [6] of the quantitative isoperimetric inequality, the idea is to prove first existence of (generalized) minimizers for the problem. Then the challenge is to prove regularity estimates for minimizers which are uniform in Q. This allows one, by compactness, to reduce the problem to a second order Taylor expansion of the energy close to the ball (this is the so-called Fuglede type argument). As we will now see, in our case all three steps present serious difficulties. Let us point out that when $N = 2$ and $\alpha = 1$, the proof in [31] is of totally different nature. Indeed, it uses a combination of convexification and Brunn–Minkowski inequalities.

While it could be interesting to see if this argument could be extended to the case $\alpha \in (0, 1)$, it is intrinsically limited to $N = 2$. We start with existence of minimizers:

Theorem 1.2. *For every $N \geq 2$ and $\alpha \in (0,1]$, there exists $Q_1 = Q_1(N,\alpha) > 0$ such that for every $Q \leq Q_1$ minimizers of (1.3) exist.*

This result is proven in Theorem 3.11 where we prove actually a bit more. Indeed, we show that if $\alpha < 1$, minimizers exist for every $Q > 0$, at least in a generalized sense (see Definition 2.2). As in [3, 10, 17, 19], a classical first step is to transform the volume constraint into a penalization; see Lemma 3.1. Following for example [3, 14, 17, 23, 33] we would then like to prove Theorem 1.2 through a concentration-compactness argument. However, since the class $\mathcal{S}$ is not closed under L^1 convergence and because of the issues related to the lower semicontinuity of $\mathcal{I}_\alpha$ raised above, it is not clear how to argue directly for $\mathcal{F}_{\alpha,Q}$. The idea is thus to first regularize the functional by penalizing concentration of the charge. While we believe that the precise choice of regularization is not essential, in line with [30], for $\varepsilon > 0$, we replace $\mathcal{I}_\alpha$ by

$$\mathcal{I}_{\alpha,\varepsilon}(E) = \min_{\mu(E)=1} \left\{ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} + \varepsilon \int_{\mathbb{R}^N} \mu^2 \right\}.$$

In particular, this functional is well defined in L^1, i.e. $\mathcal{I}_{\alpha,\varepsilon}(E) = \mathcal{I}_{\alpha,\varepsilon}(F)$ if $E = F$ a.e., and we can prove in Proposition 3.7 the existence of (generalized) minimizers.

In order to conclude the proof of Theorem 1.2 and send ε to zero, we show in Proposition 3.10 that minimizers enjoy density estimates which are uniform in ε. This is a consequence of a first almost-minimality property of minimizers proven in Proposition 3.8. Indeed, using a relatively simple lower bound from [31] on the Riesz interaction energy of the union of two disjoint sets, we show that there exists a constant $C > 0$ such that if E is a minimizer for the regularized functional and F is such that $E \triangle F \subset B_r$, then

$$P(E) \le P(F) + C(Q^2 + r^\alpha)r^{N-\alpha}. \tag{1.4}$$

The desired density bounds then follow from [20]; see also [27, 37] when $\alpha < 1$.

We then turn to regularity:

Theorem 1.3. *For every $N \ge 2$ and $\alpha \in (0, 1]$, there exist $Q_2 = Q_2(N, \alpha) > 0$ and $\gamma = \gamma(N, \alpha) \in (0, 1/2)$ such that for $Q \le Q_2$ minimizers of (1.3) are uniformly (in Q) $C^{1,\gamma}$.*

This result is contained in Proposition 3.12 for $\alpha < 1$ (see also Remark 3.15) and Proposition 3.21 for $\alpha = 1$. On the one hand, we see from (1.4) that when $\alpha < 1$, we may directly appeal to the classical regularity theory for almost-minimizers of the perimeter (see [27, 37]), and there is nothing to prove.

On the other hand, when $\alpha = 1$ the situation is much more delicate. In fact, Proposition 3.21 may be seen as one of the main achievements of this paper. When $\alpha = 1$, while (1.4) is in general too weak to obtain $C^{1,\gamma}$ regularity, it is still strong enough to yield Reifenberg flatness of E (see Definition 3.13) when $Q \ll 1$ as recently shown in[1] [20].

In order to improve it to the full $C^{1,\gamma}$ regularity we rely on a second almost-minimality property. We show in Proposition 3.16 that if μ_E is the optimal charge distribution for E, i.e. μ_E is a minimizer in (1.1), and if $E \triangle F \subset B_r$, then

$$P(E) \le P(F) + C\left(Q^2\left(\int_{B_r} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} + r^N\right). \tag{1.5}$$

The proof is inspired by [7, Proposition 4.5]. Notice, however, that we have to deal with difficulties which are quite different from the ones in [7]. Indeed, on the one hand our operator is smooth (here it is just the half Laplacian, see (2.5)) as opposed to [7] where the heart of the problem is the presence of irregular coefficients. On the other hand, in our case the charge distribution μ_E is, a priori, just a measure while in [7] it is known to be a function in L^∞. In fact, in light of (1.5), the main point here is to prove good integrability properties of μ_E. This is done in Lemmas 3.17 and 3.18 where we prove that for any $\gamma \in (0, 1/2)$, if E is a sufficiently Reifenberg flat set, then

$$\left(\int_{B_r} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \le C r^{N-1+2\gamma}. \tag{1.6}$$

By comparing this result with the case of the ball, we can see that this estimate is optimal. It may be seen as an extension to irregular domains of the boundary regularity for the fractional Laplacian developed in [35]. As in the case of the Laplacian considered in [25], the main ingredient for the proof is the monotonicity formula of Alt–Caffarelli–Friedman.

[1] As a matter of fact, the present paper served as motivation for [20].

Combining (1.5) and (1.6) we find that also for $\alpha = 1$, minimizers of (1.3) are actually classical almost-minimizers of the perimeter and thus Theorem 1.3 follows.

Let us point out that our proof actually applies to a more general class of functionals. Indeed, for $\Lambda \geq 0$ we say that E is a Λ-*minimizer* of $\mathcal{F}_{\alpha,Q}$ if for every set F,

$$\mathcal{F}_{\alpha,Q}(E) \leq \mathcal{F}_{\alpha,Q}(F) + \Lambda|E \triangle F|.$$

Arguing as in Theorem 1.3 we can prove the following result (we state it only for $\alpha = 1$ since $\alpha \in (0, 1)$ is simpler):

Theorem 1.4. *For every $\gamma \in (0, 1/2)$, there is $\bar{Q} = \bar{Q}(N, \gamma) > 0$ such that for every $Q \leq \bar{Q}$ and $\Lambda \geq 0$, every Λ-minimizer E of $\mathcal{F}_{1,Q}$ is $C^{1,\gamma}$ regular outside a singular set Σ with $\Sigma = \emptyset$ if $N \leq 7$, Σ is locally finite if $N = 8$ and satisfies $\mathcal{H}^s(\Sigma) = 0$ if $s > N - 8$ and $N \geq 9$.*

In particular, this answers a question left open in [32].

Thanks to Theorem 1.3 and the quantitative isoperimetric inequality, if Q is small enough then up to translation any minimizer of (1.3) is nearly spherical. By this we mean that fixing $\gamma \in (0, 1)$ (which remains implicit), $|E| = \omega_N$, the barycenter of E is at zero and there is $\phi : \partial B_1 \to \mathbb{R}$ with $\|\phi\|_{C^{1,\gamma}(\partial B_1)} \leq 1$ such that

$$\partial E = \{(1 + \phi(x))x : x \in \partial B_1\}.$$

The proof of Theorem 1.1 is thus concluded once we show the minimality of the ball among nearly spherical sets:

Theorem 1.5. *Let $\alpha \in (0, 2)$. There exist $Q_3 = Q_3(N, \alpha, \gamma) > 0$ and $\varepsilon = \varepsilon(N, \alpha, \gamma) > 0$ such that for every nearly spherical set E with $\|\phi\|_{W^{1,\infty}(\partial B_1)} \leq \varepsilon$, and every $Q \leq Q_3$,*

$$\mathcal{F}_{\alpha,Q}(B_1) \leq \mathcal{F}_{\alpha,Q}(E).$$

Moreover, equality is attained only if $E = B_1$.

This shows in particular the stability of the ball under small $C^{1,\gamma}$ perturbations if Q is small enough. The counterpart of Theorem 1.5 for the Coulomb case $\alpha = 2$ has been obtained in [18, Theorem 1.7]. The main point of the proof is to show in Proposition 4.5 that

$$\mathcal{I}_\alpha(B_1) - \mathcal{I}_\alpha(E) \leq C([\phi]^2_{H^{\alpha/2}(\partial B_1)} + [\phi]^2_{H^{(2-\alpha)/2}(\partial B_1)}).$$

The proof of this quantitative estimate for $\mathcal{I}_\alpha$ follows the same general strategy as in [18]. As there, the difficulty comes from the fact that the optimal measure μ_E is not explicitly given in terms of E. There are however some important differences between the Coulomb case $\alpha = 2$ and the non-local case $\alpha \in (0, 2)$. In the Coulomb case, the charges are concentrated on the boundary. This makes it easier to express the Riesz interaction energy in terms of ϕ with respect to the case $\alpha \in (0, 2)$ where the optimal measure μ_E has support equal to E. Another difficulty here comes from the blow-up of μ_E near ∂E.

The last result of this paper is a non-existence result in dimension 2.

Theorem 1.6. *Let $N = 2$ and $\alpha \in (0, 1]$. Then there exists $Q_4 = Q_4(N, \alpha) > 0$ such that for $Q \geq Q_4$, there are no minimizers of* (1.3).

The paper is divided into four parts. In Section 2, we collect the precise notation and definitions used in the paper. In Section 3, we prove Theorems 1.2 and 1.3 about existence and regularity of minimizers for (1.3). We then prove Theorem 1.5 in Section 4. The last short section is dedicated to the proof of the non-existence result, Theorem 1.6.

2. Notation

We will use the notation $A \lesssim B$ to indicate that there exists a constant $C > 0$, typically depending on the dimension N and on α, such that $A \leq CB$ (we will specify when C depends on other quantities). We write $A \sim B$ if $A \lesssim B \lesssim A$. We use in hypotheses the notation $A \ll B$ to indicate that there exists a (typically small) universal constant $\varepsilon > 0$ depending only on N and α such that if $A \leq \varepsilon B$ then the conclusion of the statement holds.

For a measurable set $E \subset \mathbb{R}^N$ and an open set $\Omega \subset \mathbb{R}^N$, we denote by $|E|$ the Lebesgue measure of E and by $P(E, \Omega)$ its relative perimeter in Ω. When $\Omega = \mathbb{R}^N$ we simply write $P(E)$ (see [27]). We use ∂E for the topological boundary of E, $\partial^M E$ for the measure-theoretic boundary and $\partial^* E$ for the reduced boundary.

2.1. Fractional Sobolev spaces, Laplacians and capacities

We collect here some standard notation and basic properties of fractional Sobolev spaces, Laplacians and capacities. We refer for instance to [9, 24, 26] for more information. For the Fourier transform we use the convention

$$\hat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2i\pi\xi\cdot x} u(x)\, dx.$$

For $s \in \mathbb{R}$ we then define the (homogeneous) H^s seminorm as

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi.$$

When there is no risk of confusion we simply write $[u]_{H^s}$ for $[u]_{H^s(\mathbb{R}^N)}$. We define the s-fractional Laplacian by its Fourier transform:

$$\widehat{(-\Delta)^s u} = |\xi|^{2s} \hat{u},$$

so that by the Parseval identity,

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u(-\Delta)^s u.$$

For $s \in (0,1)$, there exists $C(N,s) > 0$ such that

$$(-\Delta)^s u(x) = C(N,s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \tag{2.1}$$

where the integral is understood in the principal value sense. We then have an alternative formula for the H^s seminorm,

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{C(n,s)}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dx\, dy. \tag{2.2}$$

We will also use fractional Sobolev spaces defined on the unit sphere ∂B. For these we take (2.2) as starting point and define, for $\phi : \partial B \to \mathbb{R}$ and $s \in (0,1)$,

$$[\phi]^2_{H^s(\partial B)} = \int_{\partial B \times \partial B} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N-1+2s}}\, dx\, dy. \tag{2.3}$$

Let us point out a slight abuse of notation here: we do not distinguish between the volume measure on $\mathbb{R}^N$ and the one on the sphere. We recall that for $0 < s < s' < 1$, if we denote $\bar{\phi} = \frac{1}{P(B)} \int_{\partial B} \phi$ we have

$$\int_{\partial B} (\phi - \bar{\phi})^2 \lesssim [\phi]^2_{H^s(\partial B)} \le 2^{2(s'-s)} [\phi]^2_{H^{s'}(\partial B)} \lesssim \int_{\partial B} |\nabla \phi|^2, \tag{2.4}$$

where we write $\nabla\phi$ for the tangential gradient and where the implicit constants depend on N, s and s'. Indeed, the first inequality follows from Cauchy–Schwarz as

$$\begin{aligned}
\int_{\partial B} (\phi - \bar{\phi})^2 &\le \frac{1}{P(B)^2} \int_{\partial B} \left(\int_{\partial B} |\phi(x) - \phi(y)|\, dy \right)^2 dx \\
&\le \frac{1}{P(B)^2} \int_{\partial B} \left(\int_{\partial B} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N-1+2s}}\, dy \right) \left(\int_{\partial B} |x-y|^{N-1+2s}\, dy \right) dx \\
&\lesssim \int_{\partial B \times \partial B} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N-1+2s}}\, dx\, dy.
\end{aligned}$$

The second inequality in (2.4) is immediate while the third can be deduced from [8, Proposition 2.4 and Remark 2.8].

For $\alpha \in (0,N)$ and μ a Radon measure, we define the Riesz interaction energy of μ as

$$I_\alpha(\mu) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}}.$$

With this notation, definition (1.1) becomes

$$\mathcal{I}_\alpha(E) = \min_{\mu(E)=1} I_\alpha(\mu).$$

Remark 2.1. Recalling the definition (1.2) of $\mathcal{S}$, we see that for $\alpha \in (0,1]$, $\mathcal{I}_\alpha$ is well defined in $\mathcal{S}$ in the sense that if $E, F \in \mathcal{S}$ with $|E \triangle F| = 0$ then actually $\mathcal{H}^{N-1}(E \triangle F) = 0$ and thus $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(F)$ (since $\mathcal{I}_\alpha(E \triangle F) = \infty$, see e.g. [28, Theorem 8.7]).

For every $\alpha \in (0, N)$, there exists a constant $C'(N,\alpha) > 0$ such that for any Radon measure μ, the associated potential

$$u(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\alpha}}$$

satisfies (see [26])

$$(-\Delta)^{\alpha/2} u = C'(N,\alpha)\mu \quad \text{in } \mathbb{R}^N. \tag{2.5}$$

In particular, since

$$I_\alpha(\mu) = \int_{\mathbb{R}^N} u\, d\mu = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} u (-\Delta)^{-\alpha/2}\mu = C'(N,\alpha) \int_{\mathbb{R}^N} \mu(-\Delta)^{-\alpha/2}\mu,$$

we have

$$I_\alpha(\mu) = C'(N,\alpha)[\mu]^2_{H^{-\alpha/2}(\mathbb{R}^N)}. \tag{2.6}$$

Similarly,

$$I_\alpha(\mu) = \frac{1}{C'(N,\alpha)}[u]^2_{H^{\alpha/2}(\mathbb{R}^N)}. \tag{2.7}$$

Moreover, if E is compact and μ_E is the equilibrium measure of E, i.e. $\mathcal{I}_\alpha(E) = I_\alpha(\mu_E)$, then the corresponding potential u_E satisfies

$$u_E \equiv \mathcal{I}_\alpha(E) \quad \text{on } E; \tag{2.8}$$

see [18, 24] for a precise justification. We finally point out that if we define the fractional capacity as

$$C_\alpha(E) = \frac{1}{\mathcal{I}_\alpha(E)},$$

then it is not hard to check that at least for smooth enough sets E, $u_E/\mathcal{I}_\alpha(E)$ is a minimizer of

$$\min_{v \geq \chi_E,\, v\to 0 \text{ at } \infty} [v]^2_{H^{\alpha/2}(\mathbb{R}^N)},$$

so that by (2.7) and (2.2),

$$C_\alpha(E) = \frac{C(N,\alpha/2)}{C'(N,\alpha)} \inf_{u \in C_c^\infty(\mathbb{R}^N),\, u \geq \chi_E} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dx\, dy. \tag{2.9}$$

We refer to [24, 28] for further information on fractional capacities.

2.2. *Generalized sets and minimizers*

For (possibly finite) sequences $\widetilde{E} = (E^i)_{i\geq 1}$ of sets $\widetilde{\mu} = (\mu^i)_{i \geq 1}$ of measures, we define

$$I_\alpha(\widetilde{\mu}) = \sum_i I_\alpha(\mu^i), \tag{2.10}$$

$$\mathcal{I}_\alpha(\widetilde{E}) = \inf_{\widetilde{\mu}} \Big\{ I_\alpha(\widetilde{\mu}) : \sum_i \mu^i(E^i) = 1 \Big\}, \quad P(\widetilde{E}) = \sum_i P(E^i). \tag{2.11}$$

Notice that since $I_\alpha(\widetilde{\mu} + \widetilde{\mu}') \geq I_\alpha(\widetilde{\mu})$, when minimizing over $\widetilde{\mu}$, we may assume without loss of generality that μ^i is concentrated on E^i.

Definition 2.2. Define a *generalized set* to be a collection $\widetilde{E} = (E^i)_{i\geq 1}$ of sets as above, and set

$$|\widetilde{E}| = \sum_i |E^i|. \tag{2.12}$$

For $Q > 0$, we define the *energy* of the generalized set as

$$\mathcal{F}_{\alpha,Q}(\widetilde{E}) = P(\widetilde{E}) + Q^2 \mathcal{I}_\alpha(\widetilde{E}).$$

We say that $\widetilde{E} \in \mathcal{S}^{\mathbb{N}}$ is a (*volume-constrained*) *generalized minimizer* for $\mathcal{F}_{\alpha,Q}$ if, for any collection $\widetilde{F} \in \mathcal{S}^{\mathbb{N}}$ of sets with $|\widetilde{F}| = |\widetilde{E}|$, we have

$$\mathcal{F}_{\alpha,Q}(\widetilde{E}) \leq \mathcal{F}_{\alpha,Q}(\widetilde{F}).$$

3. Existence and regularity of minimizers

3.1. Relaxation of the volume constraint

For $\Lambda > 0$, we relax the volume constraint by considering

$$\mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{E}) = \mathcal{F}_{\alpha,Q}(\widetilde{E}) + \Lambda\big||\widetilde{E}| - \omega_N\big|. \tag{3.1}$$

Our first result is that for Λ large enough the relaxed problem coincides with the constrained one.

Lemma 3.1. *For every $\alpha \in (0, N)$, $Q > 0$ and every $\Lambda \gg 1 + Q^2$, we have*

$$\inf_{\widetilde{E}\in\mathcal{S}^{\mathbb{N}}} \{\mathcal{F}_{\alpha,Q}(\widetilde{E}) : |\widetilde{E}| = \omega_N\} = \inf_{\widetilde{E}\in\mathcal{S}^{\mathbb{N}}} \mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{E}). \tag{3.2}$$

Moreover, for such Λ, if $\widetilde{E}$ is a minimizer of the right-hand side of (3.2)*, then $|\widetilde{E}| = \omega_N$.*

Proof. Since the left-hand side of (3.2) is larger than the right-hand side, it is enough to prove the remaining inequality. Let $\Lambda \gg 1 + Q^2$ and assume that there exists $\widetilde{E}$ with $|\widetilde{E}| \neq \omega_N$ and

$$\mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{E}) \leq \inf_{\widetilde{E}\in\mathcal{S}^{\mathbb{N}}} \{\mathcal{F}_{\alpha,Q}(\widetilde{E}) : |\widetilde{E}| = \omega_N\}.$$

Using B_1 as competitor we find

$$P(\widetilde{E}) + Q^2 \mathcal{I}_\alpha(\widetilde{E}) + \Lambda\big||\widetilde{E}| - \omega_N\big| \lesssim 1 + Q^2. \tag{3.3}$$

In particular, if $t = \omega_N^{1/N}|\widetilde{E}|^{-1/N}$, then we can write $t = 1 + \delta$ with $|\delta| \lesssim \Lambda^{-1}(1 + Q^2) \ll 1$. We now use $t\widetilde{E} = (tE^i)_{i\geq 1}$, which satisfies $|t\widetilde{E}| = \omega_N$, as competitor and find, using Taylor expansion, (3.3) and $\Lambda \gg 1 + Q^2$, that

$$\Lambda|\delta| \lesssim \delta\big((N-1)P(\widetilde{E}) - (N-\alpha)Q^2\mathcal{I}_\alpha(\widetilde{E})\big). \tag{3.4}$$

Now if $\delta \geq 0$, this implies

$$\Lambda\delta \lesssim \delta P(\widetilde{E}) \overset{(3.3)}{\lesssim} \delta(1 + Q^2)$$

and thus $\Lambda \lesssim 1 + Q^2$, contrary to the hypothesis $\Lambda \gg 1 + Q^2$. In the case $\delta \le 0$ we reach the same contradiction since (3.4) yields this time

$$\Lambda|\delta| \lesssim |\delta| Q^2 \mathcal{I}_\alpha(\widetilde{E}) \overset{(3.3)}{\lesssim} |\delta|(1 + Q^2). \qquad \blacksquare$$

3.2. The regularized functional

As mentioned in the introduction, since the capacitary term $\mathcal{I}_\alpha$ is not well defined in L^1, which is the natural setting to minimize the perimeter, we will first show existence of (generalized) minimizers for a regularized energy. For $\varepsilon > 0$ and a positive measure μ we define

$$I_{\alpha,\varepsilon}(\mu) = I_\alpha(\mu) + \varepsilon \int_{\mathbb{R}^N} \mu^2 = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x - y|^{N-\alpha}} + \varepsilon \int_{\mathbb{R}^N} \mu^2$$

with the understanding that $\mathcal{I}_{\alpha,\varepsilon}(\mu) = \infty$ if $\mu \notin L^2(\mathbb{R}^N)$. We then define, for $\widetilde{\mu} = (\mu^i)_{i \ge 1}$ and $\widetilde{E} = (E^i)_{i \ge 1}$, in analogy with (2.10) and (2.11),

$$I_{\alpha,\varepsilon}(\widetilde{\mu}) = \sum_i I_{\alpha,\varepsilon}(\mu^i), \quad \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) = \inf_{\widetilde{\mu}} \Big\{ I_{\alpha,\varepsilon}(\widetilde{\mu}) : \sum_i \mu^i(E^i) = 1 \Big\}. \tag{3.5}$$

Remark 3.2. If E and F are two measurable sets with $|E \triangle F| = 0$ then $\mathcal{I}_{\alpha,\varepsilon}(E) = \mathcal{I}_{\alpha,\varepsilon}(F)$. Indeed, every measure μ with $I_{\alpha,\varepsilon}(\mu) < \infty$ is in L^2 and thus $\mu(E) = \mu(F)$. When considering $\mathcal{I}_{\alpha,\varepsilon}$ instead of $\mathcal{I}_\alpha$ we can therefore identify sets which agree Lebesgue a.e.

Lemma 3.3. *Let $\widetilde{E} = (E_i)_{i \ge 1}$ be a generalized set with $|\widetilde{E}| \in (0, \infty)$. Then (recall definition (2.12))*

$$\frac{\varepsilon}{|\widetilde{E}|} \le \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) \le \frac{c(N,\alpha)}{|\widetilde{E}|^{\frac{N-\alpha}{N}}} + \frac{\varepsilon}{|\widetilde{E}|}, \tag{3.6}$$

where

$$c(N,\alpha) = \int_{B_1 \times B_1} \frac{1}{|x - y|^{N-\alpha}}\, dx\, dy.$$

Proof. We start with the upper bound. Let $m = |\widetilde{E}|$ and for every $i \ge 1$ let B^i be a ball such that $|B^i| = |E^i|$. Choosing $\mu^i = \chi_{E^i}/m$ in the definition of $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})$ and recalling the Riesz rearrangement inequality, we get

$$\begin{aligned}
\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) &\le \frac{1}{m^2} \sum_i \int_{E^i \times E^i} \frac{dx\, dy}{|x - y|^{N-\alpha}} + \frac{\varepsilon}{m} \\
&\le \frac{1}{m^2} \sum_i \int_{B^i \times B^i} \frac{dx\, dy}{|x - y|^{N-\alpha}} + \frac{\varepsilon}{m} \\
&= \frac{c(N,\alpha)}{m^2} \sum_i |E^i|^{1+\alpha/N} + \frac{\varepsilon}{m} \\
&\le \frac{c(N,\alpha)}{m^{\frac{N-\alpha}{N}}} + \frac{\varepsilon}{m}.
\end{aligned}$$

To obtain the lower bound we simply observe that for every $\widetilde{\mu} = (\mu_i)_{i \geq 1}$ with $\sum_i \mu^i(E^i) = 1$ we have, by Cauchy–Schwarz,

$$\varepsilon^{1/2} = \varepsilon^{1/2} \sum_i \mu^i(E^i) \leq \Big(\sum_i |E^i|\Big)^{1/2} \Big(\varepsilon \sum_i \int_{E^i} (\mu^i)^2\Big)^{1/2} \leq m^{1/2} \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\mu})^{1/2}. \tag{3.7}$$

The desired bound follows by minimizing in $\widetilde{\mu}$. ■

As a consequence of Lemma 3.3, we can prove the existence of an optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})$.

Corollary 3.4. *For every $\varepsilon > 0$ and every generalized set $\widetilde{E} = (E^i)_{i \geq 1}$ with $|\widetilde{E}| < \infty$ and $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) < \infty$, there exists a unique optimal measure $\widetilde{\mu}$ for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})$.*

Proof. Uniqueness follows from strict convexity of the energy so we only focus on the existence part of the statement. We first notice that from the definition (3.5) of $\mathcal{I}_{\alpha,\varepsilon}$ we have

$$\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) = \inf\Big\{\sum_i q_i^2 \mathcal{I}_{\alpha,\varepsilon}(E^i) : \sum_i q_i = 1\Big\}. \tag{3.8}$$

Hence, the existence of an optimal $\widetilde{\mu}$ follows if we can prove that, on the one hand, for every fixed set E of finite volume, there exists an optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(E)$, and on the other hand, there exists an optimal distribution $(q_i)_{i \geq 1}$ of charges for (3.8).

We thus start by considering a fixed set E with $|E| + \mathcal{I}_{\alpha,\varepsilon}(E) < \infty$ and prove the existence of an optimal charge μ. If μ_n is a minimizing sequence, arguing as in (3.7) we find that for every $R > 0$,

$$\mu_n(B_R^c) \leq \varepsilon^{-1/2} |E \cap B_R^c|^{1/2} \mathcal{I}_{\alpha,\varepsilon}(\mu_n)^{1/2}.$$

Therefore μ_n is tight and we can extract a sequence converging weakly in $L^2(\mathbb{R}^N)$ to a measure μ with $\mu(E) = 1$. By lower semicontinuity of $\mathcal{I}_{\alpha,\varepsilon}$ (see [24, (1.4.5)]), μ is a minimizer for $\mathcal{I}_{\alpha,\varepsilon}(E)$.

We now turn to the existence of an optimal charge distribution $(q_i)_{i \geq 1}$. For this we first observe that from the first inequality in (3.6),

$$\sum_i \mathcal{I}_{\alpha,\varepsilon}(E^i)^{-1} \leq \frac{1}{\varepsilon} \sum_i |E^i| < \infty$$

and thus in particular $\lim_{I \to \infty} \sum_{i \geq I} \mathcal{I}_{\alpha,\varepsilon}(E^i)^{-1} = 0$. Now for every $(q_i)_{i \geq 1}$ and every $I \in \mathbb{N}$, by Cauchy–Schwarz,

$$\sum_{i \geq I} q_i \leq \Big(\sum_{i \geq I} q_i^2 \mathcal{I}_{\alpha,\varepsilon}(E^i)\Big)^{1/2} \Big(\sum_{i \geq I} \mathcal{I}_{\alpha,\varepsilon}(E^i)^{-1}\Big)^{1/2},$$

so that tightness of minimizing sequences follows, leading to the existence of an optimal distribution $(q_i)_{i \geq 1}$. ■

In order to prove that generalized minimizers are almost-minimizers of the perimeter, we will need the following lemma which is adapted from [31, Lemma 2] (see also [32, Lemma 13]).

Lemma 3.5. *For every generalized set* $\widetilde{E} = (E \cup F) \times (E^i)_{i\geq 2}$ *with* E *and* F *sets of positive measure such that* $|E \cap F| = 0$ *define* $\widetilde{F} = F \times (E^i)_{i\geq 2}$. *Then*

$$\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) \geq \mathcal{I}_{\alpha,\varepsilon}(\widetilde{F}) - \frac{\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F})^2}{\mathcal{I}_{\alpha,\varepsilon}(E)}. \tag{3.9}$$

Proof. We first show that

$$\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) \geq \min_{\theta\in[0,1]} [\theta^2 \mathcal{I}_{\alpha,\varepsilon}(E) + (1-\theta)^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{F})]. \tag{3.10}$$

Let $\widetilde{\mu} = (\mu^i)_{i\geq 1}$ be optimal for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})$. We may assume without loss of generality that $\mu^1(E) \neq 0$ and $\mu^1(F) + \sum_{i\geq 2} \mu^i(E^i) \neq 0$ since otherwise in the first case we would have $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) = \mathcal{I}_{\alpha,\varepsilon}(\widetilde{F})$, and in the second case $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) = \mathcal{I}_{\alpha,\varepsilon}(E)$, which both imply (3.9). We now define

$$\mu = \frac{\mu^1|_E}{\mu^1(E)}, \quad \nu^1 = \frac{\mu^1|_F}{1-\mu^1(E)}, \quad \nu^i = \frac{\mu^i}{1-\mu^1(E)}, \ \forall i \geq 2.$$

With this definition, μ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(E)$ and $\widetilde{\nu} = (\nu^i)_{i\geq 1}$ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F})$ and we have

$$\begin{aligned}
\mathcal{I}_{\alpha,\varepsilon}(\mu^1) &\geq (\mu^1(E))^2 \left(\int_{E\times E} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} + \varepsilon \int_E \mu^2 \right) \\
&\quad + (1-\mu^1(E))^2 \left(\int_{F\times F} \frac{d\nu^1(x)\, d\nu^1(y)}{|x-y|^{N-\alpha}} + \varepsilon \int_F (\nu^1)^2 \right) \\
&= (\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(\mu) + (1-\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(\nu^1),
\end{aligned}$$

so that by definition (3.5) of $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{\nu})$,

$$\begin{aligned}
\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) &\geq (\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(\mu) + (1-\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\nu}) \\
&\geq (\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(E) + (1-\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{F}).
\end{aligned}$$

This proves (3.10). Optimizing in θ together with the inequality $(1+t)^{-1} \geq 1-t$ for $t \geq 0$ yields (3.9). ∎

3.3. Existence of generalized minimizers for the regularized energy

In line with (3.1), we introduce the regularized energy

$$\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\widetilde{E}) = P(\widetilde{E}) + Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) + \Lambda \big| |\widetilde{E}| - \omega_N \big|.$$

The aim of this section is to prove the existence of minimizers for this functional. We start with the simple observation that for $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, minimizing among classical or generalized sets gives the same infimum energy.

Lemma 3.6. *For every* $\alpha \in (0, N)$, Q, Λ, $\varepsilon > 0$, *we have*

$$\inf \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) = \inf \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\widetilde{E}). \tag{3.11}$$

Proof. Since the left-hand side of (3.11) is larger than the right-hand side, it is enough to prove that for every $\delta > 0$ and every generalized set $\widetilde{E} = (E^i)_{i\geq 1}$, we can construct a set E with $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) \leq \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\widetilde{E}) + \delta$. For $I \in \mathbb{N}$ and $R > 0$, let $F^i = E^i \cap B_R$ if $i \leq I$ and $F^i = \emptyset$ otherwise, and set $\widetilde{F} = (F^i)_{i\geq 1}$. We first observe that for each fixed i, $\lim_{R\to\infty} |E^i \cap B_R| = |E^i|$. Combining this with the fact that $\sum_i |E^i| < \infty$ we see that we can choose I and R large enough so that

$$\Lambda \Big| \sum_{i=1}^{I} |F^i| - \omega_N \Big| \leq \Lambda \big| |\widetilde{E}| - \omega_N \big| + \delta. \tag{3.12}$$

Moreover, thanks to the co-area formula we may further assume that

$$\sum_{i=1}^{I} P(F^i) \leq P(\widetilde{E}) + \delta. \tag{3.13}$$

We now turn to the last term in the energy. Let $\widetilde{\mu} = (\mu^i)_{i\geq 1}$ be the optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})$ given by Corollary 3.4. We then set

$$\nu^i = \frac{\mu^i|_{F^i}}{\sum_{i=1}^{I} \mu^i(F^i)} \quad \text{for } i \leq I$$

and $\nu^i = 0$ otherwise so that $\widetilde{\nu} = (\nu^i)_{i\geq 1}$ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F})$. Since $\sum_{i=1}^{I} \mu^i(F^i)$ converges to 1 as $I \to \infty$ and $R \to \infty$, we can also assume that I and R are chosen such that in addition to (3.12) and (3.13) we have

$$\begin{aligned} Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\nu}) &= \frac{Q^2}{(\sum_{i=1}^{I} \mu^i(F^i))^2} \left(\sum_{i=1}^{I} \int_{F^i \times F^i} \frac{d\mu^i(x)\, d\mu^i(y)}{|x-y|^{N-\alpha}} + \varepsilon \int_{F^i} (\mu^i)^2 \right) \\ &\leq Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) + \delta. \end{aligned} \tag{3.14}$$

We finally choose for every $i \leq I$ a point $x^i \in \mathbb{R}^N$ such that $\min_{i\neq j} |x^i - x^j| \gg R$ and define

$$E = \bigcup_{i=1}^{I} (F^i + x^i) \quad \text{and} \quad \nu(x) = \sum_{i=1}^{I} \nu^i(x - x^i).$$

Since $F^i \subset B_R$ by construction, the sets $F^i + x^i$ are pairwise disjoint and from (3.12) and (3.13) we have

$$\begin{aligned} P(E) + \Lambda \big| |E| - \omega^N \big| &= \sum_{i=1}^{I} P(E^i) + \Lambda \Big| \sum_{i=1}^{I} |F^i| - \omega_N \Big| \\ &\leq P(\widetilde{E}) + \Lambda \big| |\widetilde{E}| - \omega_N \big| + 2\delta. \end{aligned}$$

Finally, we observe that ν is admissible for $\mathcal{I}_{\alpha,\varepsilon}(E)$ with

$$\begin{aligned} Q^2 \mathcal{I}_{\alpha,\varepsilon}(\nu) &= Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\nu}) + Q^2 \sum_{i\neq j} \int_{F^i \times F^j} \frac{d\nu^i(x)\, d\nu^j(y)}{|x-y|^{N-\alpha}} \\ &\leq Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\nu}) + \frac{Q^2}{\min_{i\neq j} |x^i - x^j|^{N-\alpha}} \overset{(3.14)}{\leq} Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\nu}) + 2\delta \end{aligned}$$

provided $\min_{i\neq j}|x^i - x^j|$ is large enough. Since $\mathcal{I}_{\alpha,\varepsilon}(E) \le \mathcal{I}_{\alpha,\varepsilon}(\nu)$, we find as anticipated that

$$\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) \le \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\widetilde{E}) + 4\delta. \qquad \blacksquare$$

We can now prove the existence of generalized minimizers for $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. This will be proven by a concentration-compactness argument which relies on isoperimetric effects to avoid the loss of mass, together with the semicontinuity of $\mathcal{I}_{\alpha,\varepsilon}$ with respect to L^1_{loc} convergence. This type of argument is relatively standard by now (see for instance [3] which we closely follow or [14,17,23,33]). However, we face here the additional difficulty that we need to avoid not only loss of volume but also loss of charge in the limit.

Proposition 3.7. *For every $\alpha \in (0,1]$, $Q > 0$, $\varepsilon > 0$ and $\Lambda \gg 1 + Q^2$, generalized minimizers of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$ exist.*

Proof. Let $(E_n)_{n\ge1}$ be a (classical) minimizing sequence for $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. By Lemma 3.6 it is also a minimizing sequence in the class of generalized sets. Using for instance the ball B_1 as competitor we have $\sup_n \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E_n) \lesssim 1 + Q^2$. In particular, if we let $m_n = |E_n|$, after possibly taking a subsequence we have $m_n \to m \in (0,\infty)$. Fix $L \gg m^{1/N}$ and consider a partition of $\mathbb{R}^N$ into cubes $(Q_{i,n})_{i\ge1}$ where $Q_{i,n} = [0,L]^N + z_i$ with $z_{i,n} \in (L\mathbb{Z})^N$. We let $m_{i,n} = |E_n \cap Q_{i,n}|$ and assume without loss of generality that for every n, $m_{i,n}$ is decreasing in i. Moreover, we tacitly consider from now on only the indices i such that $m_{i,n} > 0$. We let μ_n be the optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(E_n)$ and set $q_{i,n} = \mu_n(Q_{i,n})$.

We start by proving tightness of $(m_{i,n})_{i\ge1}$ and $(q_{i,n})_{i\ge1}$. For $m_{i,n}$, we argue as usual that thanks to the relative isoperimetric inequality (recall that with our choice of L, $|Q_{i,n} \cap E_n| \le |Q_{i,n}|/2$)

$$\sum_i m_{i,n}^{\frac{N-1}{N}} \lesssim \sum_i P(E_n, Q_{i,n}) = P(E_n) \lesssim 1 + Q^2.$$

Since $m_{i,n} \le m/i$ we conclude that for every $I \in \mathbb{N}$,

$$\sum_{i\ge I} m_{i,n} \le \left(\frac{m}{I}\right)^{1/N} \sum_{i\ge I} m_{i,n}^{\frac{N-1}{N}} \lesssim (1+Q^2)\left(\frac{m}{I}\right)^{1/N}. \tag{3.15}$$

For $q_{i,n}$ we argue as in (3.7) and obtain, invoking Cauchy–Schwarz twice,

$$\sum_{i\ge I} q_{i,n} \le \sum_{i\ge I} m_{i,n}^{1/2}\left(\int_{E_n\cap Q_{i,n}} \mu_n^2\right)^{1/2} \le \left(\sum_{i\ge I} m_{i,n}\right)^{1/2}\left(\int_{\mathbb{R}^N} \mu_n^2\right)^{1/2}$$
$$\overset{(3.15)}{\lesssim} \varepsilon^{-1/2}(1+Q^2)\left(\frac{m}{I}\right)^{\frac{1}{2N}}.$$

Therefore, up to taking a subsequence we have $\lim_{n\to\infty} m_{i,n} = m_i$ with $\sum_i m_i = m$ and $\lim_{n\to\infty} q_{i,n} = q_i$ with $\sum_i q_i = 1$.

We now construct a generalized set $\widetilde{E}$ which will be our generalized minimizer. By the perimeter bound, up to a subsequence we have, for every i, $E_n - z_{i,n} \to E^i$ in L^1_{loc} for some sets E^i. Moreover, $\mu_n^i = \mu_n(\cdot + z_{i,n})$ converges weakly in L^2 to some μ^i. We can further assume that $|z_{i,n} - z_{j,n}| \to a_{ij} \in [0,\infty]$ for all i, j. We now say that $i \sim j$ if $a_{ij} < \infty$ and denote by $[i]$ the equivalence class of i. Notice that if $i \sim j$ then E^i and E^j are translates of each other. For each equivalence class we denote

$$m_{[i]} = \sum_{j \sim i} m_j \quad \text{and} \quad q_{[i]} = \sum_{j \sim i} q_j$$

so that $\sum_{[i]} m_{[i]} = m$ and $\sum_{[i]} q_{[i]} = 1$. For every i, using the convergence of $E_n - z_{i,n}$ to E^i and of μ_n^i to μ^i, and the definition of the equivalence relation, we have

$$|E^i| = m_{[i]} \quad \text{and} \quad \mu^i(E^i) = q_{[i]}.$$

Up to relabeling, we may now assume that each equivalence class $[i]$ consists of a single element. If we set $\widetilde{E} = (E^i)_{i\ge1}$ and $\widetilde{\mu} = (\mu^i)_{i\ge1}$, we have just shown that $\widetilde{\mu}$ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})$. Let us finally prove that

$$P(\widetilde{E}) + Q^2 \mathcal{I}_{\alpha,\varepsilon}(\widetilde{\mu}) + \Lambda\big||\widetilde{E}| - \omega_N\big| \le \liminf_{n\to\infty} P(E_n) + Q^2 \mathcal{I}_{\alpha,\varepsilon}(\mu_n) + \Lambda\big||E_n| - \omega_N\big|.$$

We consider each term of the energy separately. Since $|\widetilde{E}| = m = \lim_{n\to\infty} |E_n|$, the volume term is not a problem. For the first term, we fix $I \in \mathbb{N}$ and $R > 0$. If n is large enough, we can assume that $|z_{i,n} - z_{j,n}| \gg R$ for $i, j \le I$ distinct. By the co-area formula we can find for every $i \le I$ a radius $R_n \in (R, 2R)$ such that

$$\sum_{i\le I} \mathcal{H}^{N-1}(\partial B_{R_n}(z_{i,n}) \cap E_n) \lesssim \frac{1}{R}.$$

If $E^{i,R_n} = (E_n - z_{i,n}) \cap B_{R_n}$, we thus have

$$\sum_{i\le I} P(E^{i,R_n}) \le P(E_n) + \frac{C}{R}.$$

From this bound we see that E^{i,R_n} converges in L^1_{loc} to a set $E^{i,R}$ which itself converges to E^i as $R \to \infty$. We thus have

$$\sum_{i\le I} P(E^i) \le \sum_{i\le I} \liminf_{R\to\infty} P(E^{i,R}) \le \sum_{i\le I} \liminf_{R\to\infty} \liminf_{n\to\infty} P(E^{i,R_n}) \le \liminf_{n\to\infty} P(E_n).$$

For the last term, we use similarly the fact that for every fixed $I \in \mathbb{N}$ and $R > 0$,

$$\begin{aligned}\sum_{i\le I} \mathcal{I}_{\alpha,\varepsilon}(\mu^i|_{B_R}) &\le \liminf_{n\to\infty} \sum_{i\le I} \mathcal{I}_{\alpha,\varepsilon}(\mu_n^i|_{B_R}) \le \liminf_{n\to\infty} \mathcal{I}_{\alpha,\varepsilon}\Big(\sum_{i\le I} \mu^n|_{B_R(z_{i,n})}\Big) \\ &\le \liminf_{n\to\infty} \mathcal{I}_{\alpha,\varepsilon}(\mu_n).\end{aligned}$$

■

3.4. First almost-minimality property and existence of minimizers for the original problem

In this section we use Lemma 3.5 to prove a first almost-minimality property for generalized minimizers of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. In order to pass to the limit $\varepsilon \to 0$ it is crucial that the estimates are uniform in ε.

Proposition 3.8. *There exists $C > 0$ depending only on N and $\alpha \in (0, N)$ with the following property. For every $Q > 0$, $\varepsilon \in (0, 1)$ and $\Lambda \sim 1 + Q^2$ for which Lemma 3.1 applies, every generalized minimizer $\widetilde{E} = (E^i)_{i\ge 1}$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$ is an almost minimizer of the perimeter in the sense that for every $i \ge 1$, $x \in \mathbb{R}^N$ and $r \ll 1$,*

$$P(E^i) \le P(F) + C(Q^2 + r^\alpha)r^{N-\alpha} \quad \forall F \,\triangle\, E^i \subset B_r(x). \tag{3.16}$$

Proof. Without loss of generality we may assume that $i = 1$ and $x = 0$. To simplify notation a bit we set $E = E^1$. Using $\widetilde{F} = F \times (E^i)_{i\ge 2}$ as competitor and the minimality of $\widetilde{E}$ we have after simplifications

$$P(E) \le P(F) + Q^2\big(\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E})\big) + \Lambda|E \,\triangle\, F|. \tag{3.17}$$

Since $P(E \cap F) + P(E \cup F) \le P(E) + P(F)$, it is enough to prove (3.16) under the additional condition $E \subset F$ or $F \subset E$. If $E \subset F$ then $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) \ge \mathcal{I}_{\alpha,\varepsilon}(\widetilde{F})$ and thus (3.16) follows from $|E \,\triangle\, F| \lesssim r^N$.

We are left with the case $F \subset E$. Writing $E = F \cup (E \setminus F)$ and appealing to (3.9) from Lemma 3.5, we have

$$\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) \le \frac{\mathcal{I}^2_{\alpha,\varepsilon}(\widetilde{F})}{\mathcal{I}_{\alpha,\varepsilon}(E \setminus F)}. \tag{3.18}$$

Now on the one hand, since by Lemma 3.1, $|E| + \sum_{i\ge 2} |E^i| = \omega_N$, it follows that $|F| + \sum_{i\ge 2} |E^i| = \omega_N - |E \setminus F| \gtrsim 1$ (recall that $r \ll 1$) and thus by (3.6) of Lemma 3.3,

$$\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F}) \lesssim 1.$$

On the other hand, since $E \setminus F \subset B_r$ we have

$$\mathcal{I}_{\alpha,\varepsilon}(E \setminus F) \ge \mathcal{I}_{\alpha,\varepsilon}(B_r) + \varepsilon \inf_{\mu(B_r)=1} \int_{B_r} \mu^2 \gtrsim r^{-(N-\alpha)} + \varepsilon r^{-N} \ge r^{-(N-\alpha)}. \tag{3.19}$$

Putting these two things together, (3.18) yields

$$\mathcal{I}_{\alpha,\varepsilon}(\widetilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}) \lesssim r^{N-\alpha}.$$

Plugging this together with $|E \,\triangle\, F| \lesssim r^N$ in (3.17) concludes the proof of (3.16). ∎

Remark 3.9. From (3.19) we see that we can improve (3.16) to

$$P(E^i) \le P(F) + C\big(Q^2 \min(r^{-\alpha}, \varepsilon^{-1}) + 1\big)r^N \quad \forall F \,\triangle\, E^i \subset B_r(x).$$

This means that for every $\alpha \in (0, N)$, if $r \le \varepsilon^{1/\alpha}$ then the classical regularity theory for perimeter almost-minimizers applies (see [27]). In particular, for $\alpha = 2$, this gives a very elementary proof of the regularity of minimizers for the functional considered in [7, 30] if the permittivity of the droplet is assumed to coincide with the permittivity of the vacuum (see however [7, Remark 4.6] where it is observed that this assumption would also simplify their proof).

At this point we see the difference between the cases $\alpha > 1$ and $\alpha \leq 1$. Indeed, in the latter case, thanks to (3.16), we may appeal to the regularity theory for almost-minimizers of the perimeter (since $N - \alpha \geq N - 1$). We start with the simpler part which consists of the density estimates. Since the cases $\alpha < 1$ and $\alpha = 1$ are treated differently, we introduce the notation $\mathbf{1}_{\alpha=1} = 1$ if $\alpha = 1$ and $\mathbf{1}_{\alpha=1} = \infty$ if $\alpha \in (0, 1)$.

Proposition 3.10. *For every $\alpha \in (0, 1]$ and $Q > 0$ let $\Lambda \sim 1 + Q^2$ be such that Proposition 3.8 applies. Then, for every $\varepsilon \in (0, 1]$, and every generalized minimizer $\widetilde{E} = (E^i)_{i\geq 1}$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, if $\max(Q^2 r^{1-\alpha}, r) \ll 1$ and $x \in \partial^M E^i$ (recall that ∂^M is the measure-theoretic boundary),*

$$\min(|E^i \cap B_r(x)|, |B_r(x) \setminus E^i|) \gtrsim r^N \tag{3.20}$$

and

$$P(E^i, B_r(x)) \sim r^{N-1}. \tag{3.21}$$

As a consequence, there exists $Q_1 > 0$ such that for $Q \leq \bar{Q} \leq Q_1 \mathbf{1}_{\alpha=1}$, up to the choice of a representative, every generalized minimizer is made up of finitely many E^i, each of which is connected, is in $\mathcal{S}$ and has $\partial E^i = \partial^M E^i$. Moreover, the number of such components as well as their diameter depends only on $\bar{Q}$.

Proof. Estimates (3.20) and (3.21) follow directly from [20, Proposition 3.1]. For $\alpha < 1$ they can also be obtained (under slightly stronger hypothesis on r) from the more classical theory; see for instance [27].

The regularity of the minimizers as well as the bound on the number and diameter of the connected components is classical (see e.g. [22]) once we observe that for every $Q \leq \bar{Q} \leq Q_1 \mathbf{1}_{\alpha=1}$ there is $\bar{r}$ depending only on $\bar{Q}$ such that $\max(Q^2 \bar{r}^{1-\alpha}, \bar{r}) \ll 1$ for every $Q \leq \bar{Q}$. The fact that we may assume that each component E^i of $\widetilde{E}$ is a single connected component follows from $\mathcal{I}_{\alpha,\varepsilon}(E \cup F) \geq \mathcal{I}_{\alpha,\varepsilon}(E \times F)$ for any disjoint E, F. ■

Before stating the full conclusions of the regularity theory for perimeter almost-minimizers, let us conclude the proof of the existence of generalized volume-constrained minimizers of $\mathcal{F}_{\alpha,Q}$.

Theorem 3.11. *Let Q_1 be given by Proposition 3.10. Then for every $0 < Q \leq \bar{Q} \leq Q_1 \mathbf{1}_{\alpha=1}$ there exist generalized minimizers $\widetilde{E} = (E^i)_{i=1}^I \in \mathcal{S}^{\mathbb{N}}$ of*

$$\min_{\widetilde{E} \in \mathcal{S}^{\mathbb{N}}} \{\mathcal{F}_{\alpha,Q}(\widetilde{E}) : |\widetilde{E}| = \omega_N\}.$$

Moreover, for each $i \leq I$, E^i is a perimeter almost-minimizer in the sense of (3.16) and both I and $\operatorname{diam}(E^i)$ are bounded by a constant depending only on $\bar{Q}$.

Proof. Let $\Lambda \sim 1 + \bar{Q}^2$ be such that both Lemma 3.1 and Proposition 3.7 apply. By the latter, for every $\varepsilon \in (0, 1]$ and $Q \le \bar{Q}$, there exists a generalized minimizer $\widetilde{E}_\varepsilon$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. Moreover, by Proposition 3.10, $\widetilde{E}_\varepsilon = (E^i_\varepsilon)^I_{i=1}$ for some connected sets $E^i \in \mathcal{S}$, where both I and the diameters of E^i depend only on $\bar{Q}$. Thanks to the uniform density bounds (3.20) and (3.21) we can extract a subsequence for which E^i_ε converges both in L^1 and in the Kuratowski sense to some $E^i \in \mathcal{S}$. By compactness of perimeter almost-minimizers (see [20]), E^i satisfies (3.16). We set $\widetilde{E} = (E^i)_{i \le I}$. Using $\mathcal{I}_{\alpha,\varepsilon}(\widetilde{E}_\varepsilon) \ge \mathcal{I}_\alpha(\widetilde{E}_\varepsilon)$ and the fact that $\mathcal{I}_\alpha$ is lower semicontinuous under this convergence (see e.g. [18, Theorem 4.2]), we obtain

$$\liminf_{\varepsilon \to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\widetilde{E}_\varepsilon) \ge \mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{E}).$$

We now prove that

$$\inf_{\widetilde{F} \in \mathcal{S}^{\mathbb{N}}} \mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{F}) \ge \limsup_{\varepsilon \to 0} \inf_{\widetilde{F} \in \mathcal{S}^{\mathbb{N}}} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\widetilde{F}),$$

which combined with the previous inequality will show that $\widetilde{E}$ is a generalized minimizer of $\mathcal{F}_{\alpha,Q,\Lambda}$ as

$$\mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{E}) \ge \inf_{\widetilde{F} \in \mathcal{S}^{\mathbb{N}}} \mathcal{F}_{\alpha,Q,\Lambda}(\widetilde{F}).$$

Arguing exactly as in Lemma 3.6, we see that it is enough to prove that for every $F \in \mathcal{S}$, there exists a sequence F_ε such that

$$\limsup_{\varepsilon \to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(F_\varepsilon) \le \mathcal{F}_{\alpha,Q,\Lambda}(F). \tag{3.22}$$

By [36] applied to F^c, we can find smooth compact sets F_δ with $F \subset F_\delta$, $P(F^\delta) \le P(F) + \delta$ and $\big||F| - |F^\delta|\big| \le \delta$. Since $\mathcal{I}_\alpha(F) \ge \mathcal{I}_\alpha(F^\delta)$ as $F \subset F^\delta$, we have (actually there is equality)

$$\limsup_{\delta \to 0} \mathcal{F}_{\alpha,Q,\Lambda}(F^\delta) \le \mathcal{F}_{\alpha,Q,\Lambda}(F),$$

and we can thus further assume that F is smooth in the proof of (3.22). For smooth sets, by [18, Proposition 2.16],[2] we can find for every $\delta > 0$ a function $f_\delta \in L^\infty(F)$ with $\int_F f_\delta = 1$ and such that

$$\mathcal{I}_\alpha(f_\delta) \le \mathcal{I}_\alpha(F) + \delta.$$

Since for every $\delta > 0$, $\lim_{\varepsilon \to 0} \mathcal{I}_{\alpha,\varepsilon}(f_\delta) = \mathcal{I}_\alpha(f_\delta)$, a diagonal argument shows that $\mathcal{I}_\alpha(F) = \lim_{\varepsilon \to 0} \mathcal{I}_{\alpha,\varepsilon}(F)$. Using $F_\varepsilon = F$ we conclude the proof of (3.22).

As $\widetilde{E}$ is a generalized minimizer of $\mathcal{F}_{\alpha,Q,\Lambda}$, Lemma 3.1 implies that $|\widetilde{E}| = \omega_N$ and thus $\widetilde{E}$ is also a volume-constrained generalized minimizer of $\mathcal{F}_{\alpha,Q}$. ■

We end this section by recalling the regularity properties of generalized minimizers and show in particular that for small charge Q they are actually classical minimizers. We start with the case $\alpha < 1$.

[2] The statement of [18, Proposition 2.16] requires F to be connected but the proof works for disconnected sets as well.

Proposition 3.12. *For $\alpha \in (0,1)$ and $Q > 0$ let $\widetilde{E} = (E^i)_{i=1}^I$ be a volume-constrained generalized minimizer of $\mathcal{F}_{\alpha,Q}$. Then the sets $\partial^* E^i$ (recall that ∂^* denotes the reduced boundary) are $C^{1,\frac{1}{2}(1-\alpha)}$ regular. Moreover, if we denote $\Sigma_i = \partial E^i \setminus \partial^* E^i$, then for every i, Σ_i is empty if $N \le 7$, is at most finite if $N = 8$, and satisfies $\mathcal{H}^s(\Sigma_i) = 0$ if $s > N - 8$ and $N \ge 9$.*

In addition, for $Q \ll 1$, $\widetilde{E} = E_Q$ is a classical volume-constrained minimizer of $\mathcal{F}_{\alpha,Q}$, $\Sigma(E_Q) = \emptyset$ and for every $\beta < \frac{1}{2}(1-\alpha)$, E_Q converges in $C^{1,\beta}$ to B_1 as $Q \to 0$.

Proof. The conclusion follows from the classical regularity theory for perimeter almost-minimizers; see [27, 37] and the fact that by the quantitative isoperimetric inequality, up to translation and relabeling,

$$\Big(|E^1 \triangle B_1| + \sum_{i\ge 2} |E^i|\Big)^2 \lesssim P(\widetilde{E}) - P(B_1) \le Q^2 \mathcal{I}_\alpha(B_1),$$

which implies in conjunction with (3.20) that for Q small enough, $E^i = \emptyset$ for $i \ge 2$ (so that $\widetilde{E} = E^1$ is a classical minimizer) together with the convergence to B_1. ∎

For $\alpha = 1$ it is well known that in general (3.16) does not even imply C^1 regularity. In order to state the counterpart of Proposition 3.12 in this case, let us first recall the definition of Reifenberg flat sets.

Definition 3.13. Let $\delta, r_0 > 0$ and $x \in \mathbb{R}^N$. We say that E is (δ, r_0)-*Reifenberg* flat in $B_{r_0}(x)$ if for every $B_r(y) \subset B_{r_0}(x)$, there exists a hyperplane $H_{y,r}$ containing y and such that

- we have
$$\frac{1}{r} d\big(\partial E \cap B_r(y), H_{y,r} \cap B_r(y)\big) \le \delta,$$
where d denotes the Hausdorff distance;
- one of the connected components of
$$\{d(\cdot, H_{y,r}) \ge 2\delta r\} \cap B_r(y)$$
is included in E and the other in E^c.

We say that E is *uniformly* (δ, r_0)-*Reifenberg flat* if the above conditions hold for every $x \in \partial E$.

Proposition 3.14. *Let $\alpha = 1$. There exists $Q_2 > 0$ such that for every $Q \le Q_2$, every volume-constrained generalized minimizer of $\mathcal{F}_{\alpha,Q}$ is a classical minimizer. Moreover, for every $\delta > 0$, there exist $Q_\delta, r_\delta > 0$ such that for every $Q \le Q_\delta$, every volume-constrained minimizer E_Q of $\mathcal{F}_{\alpha,Q}$ is uniformly (δ, r_δ)-Reifenberg flat and up to translation,*

$$|E_Q \triangle B_1|^2 \lesssim Q^2.$$

Proof. The proof goes exactly as that for Proposition 3.12, replacing the classical regularity theory by [20, Corollary 1.4]. ∎

3.5. Second almost-minimality property and regularity of minimizers

The aim of this section is to prove that in the case $\alpha = 1$, we can pass from the Reifenberg flatness of volume-constrained minimizers of $\mathcal{F}_{\alpha,Q}$ stated in Proposition 3.14 to almost $C^{1,1/2}$ regularity. This will be obtained by proving a second almost-minimality property for minimizers together with a higher integrability result for the optimal measure μ.

Remark 3.15. Let us point out that using a similar proof for $\alpha \in (0, 1)$, it would be possible to improve the $C^{1,\frac{1}{2}(1-\alpha)}$ regularity from Proposition 3.12 to almost $C^{1,1/2}$. However, in this case, the proof of the integrability of μ can be greatly simplified by appealing directly to [35] (see also (4.5) below). Moreover, we expect that any $C^{1,\beta}$ regularity may be improved to higher regularity through the Euler–Lagrange equation (see [32]).

We start with the quasi-minimality property.

Proposition 3.16. *There exists $C > 0$ depending only on N with the following property. If $Q \leq 1$ and E is a volume-constrained minimizer of $\mathcal{F}_{1,Q}$ with μ_E the corresponding $1/2$-harmonic measure, i.e. $\mathcal{I}_1(E) = I_1(\mu_E)$, then for every $x \in \mathbb{R}^N$ and $0 < r \ll 1$,*

$$P(E) \leq P(F) + C\left(Q^2\left(\int_{B_r(x)} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} + r^N\right) \quad \forall E \,\triangle\, F \subset B_r(x).$$

Proof. Without loss of generality we may assume that $x = 0$ and $\mu_E \in L^{\frac{2N}{N+1}}(B_r)$ since otherwise there is nothing to prove. By Lemma 3.1, there exists a universal constant $\Lambda > 0$ (recall that $Q \leq 1$) such that E is a minimizer of

$$\mathcal{F}_{1,Q}(E) + \Lambda\big||E| - \omega_N\big|.$$

Arguing as in the proof of Proposition 3.8, we see that it is enough to prove that for every $F \subset E$ with $E \setminus F \subset B_r$,

$$\mathcal{I}_1(F) \leq \mathcal{I}_1(E) + C\left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}. \tag{3.23}$$

In order to prove (3.23) we follow the general strategy of [7, Proposition 4.5] and use

$$\mu = \left(\mu_E + \frac{\mu_E(E \setminus F)}{|F|}\right)\chi_F$$

as a competitor for $\mathcal{I}_1(F)$. We define

$$u_E(x) = \int_E \frac{d\mu_E(y)}{|x - y|^{N-1}} \quad \text{and} \quad u(x) = \int_E \frac{d\mu(y)}{|x - y|^{N-1}},$$

the potentials associated to μ_E and μ. We recall from (2.5) that u solves on $\mathbb{R}^N$ the equation

$$(-\Delta)^{1/2} u = C'(N, 1)\mu,$$

and that by (2.7),

$$\frac{1}{C'(N,1)}[u]^2_{H^{1/2}} = \int_E u\, d\mu = I_1(\mu).$$

Let us notice that since $u_E = \mathcal{I}_1(E)$ on E (recall (2.8)), and since $\mu_E(E) = \mu(E) = 1$,

$$\int_E u_E\, d(\mu - \mu_E) = 0. \tag{3.24}$$

Since $\mathcal{I}_1(F) \leq I_1(\mu)$ and $\mathcal{I}_1(E) = I_1(\mu_E)$, we have

$$\begin{aligned}\mathcal{I}_1(F) - \mathcal{I}_1(E) &\leq \int_E u\, d\mu - \int_E u_E\, d\mu_E \\ &= \int_E (u - u_E)\, d(\mu - \mu_E) + \int_E u\, d\mu_E - \int_E u_E\, d\mu_E.\end{aligned}$$

Using Fubini we have

$$\int_E u\, d\mu_E = \int_E u_E\, d\mu \overset{(3.24)}{=} \int_E u_E\, d\mu_E$$

and we get

$$\mathcal{I}_1(F) - \mathcal{I}_1(E) \leq \int_E (u - u_E)\, d(\mu - \mu_E) = \frac{1}{C'(N,1)}[u - u_E]^2_{H^{1/2}}.$$

We now estimate $[u - u_E]^2_{H^{1/2}}$. For this, using the Hölder inequality and Sobolev embedding we write

$$\begin{aligned}[u - u_E]^2_{H^{1/2}} &= \int_E (u - u_E)\, d(\mu - \mu_E) \\ &\leq \|u - u_E\|_{L^{\frac{2N}{N-1}}} \|\mu - \mu_E\|_{L^{\frac{2N}{N+1}}} \\ &\lesssim [u - u_E]_{H^{1/2}} \|\mu - \mu_E\|_{L^{\frac{2N}{N+1}}}.\end{aligned}$$

Using the Young inequality leads to

$$\mathcal{I}_1(F) - \mathcal{I}_1(E) \lesssim [u - u_E]^2_{H^{1/2}} \lesssim \|\mu - \mu_E\|^2_{L^{\frac{2N}{N+1}}}.$$

We are left with estimating $\|\mu - \mu_E\|_{L^{\frac{2N}{N+1}}}$. By definition of μ, we have $\mu - \mu_E = \frac{\mu_E(E\setminus F)}{|F|}\chi_F - \mu_E \chi_{E\setminus F}$ and thus

$$\begin{aligned}\|\mu - \mu_E\|^2_{L^{\frac{2N}{N+1}}} &= \left(\int_E \left|\frac{\mu_E(E\setminus F)}{|F|}\chi_F - \mu_E\chi_{E\setminus F}\right|^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \\ &= \left(\int_F \left(\frac{\mu_E(E\setminus F)}{|F|}\right)^{\frac{2N}{N+1}} + \int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \\ &\lesssim \frac{\mu_E(E\setminus F)^2}{|F|^{\frac{N-1}{N}}} + \left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \\ &\overset{|F|\gtrsim 1}{\lesssim} \mu_E(E\setminus F)^2 + \left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}.\end{aligned}$$

Finally, by the Hölder inequality,

$$\mu_E(E\setminus F)^2 \le \left(\int_{E\setminus F}\mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}|E\setminus F|^{\frac{N-1}{N}} \lesssim \left(\int_{E\setminus F}\mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}.$$

This concludes the proof of (3.23). ■

From Proposition 3.16, we see that in order to prove that E is a perimeter almost-minimizer in the classical sense, it is enough to show decay estimates for $\|\mu_E\|_{L^{\frac{2N}{N+1}}(B_r(x))}$ for $x \in \partial E$. We start by proving the following Hölder estimate for the potentials.

Lemma 3.17. *For every $\delta > 0$, there exists $\gamma \in (0, 1/2)$ with $\gamma \to 1/2$ as $\delta \to 0$ such that if E is a bounded (δ, r_0)-Reifenberg flat domain then*

$$|1 - \mathcal{I}_1(E)^{-1}u_E| \lesssim \frac{d(\cdot, \partial E)^\gamma}{r_0^\gamma}, \tag{3.25}$$

where $u_E(x) = \int_E \frac{d\mu_E(y)}{|x-y|^{N-1}}$ and μ_E is such that $\mathcal{I}_1(E) = I_1(\mu_E)$.

Proof. By scaling we may assume that $r_0 = 1$. We follow the ideas from the proofs of [25, 39] and use the Alt–Caffarelli–Friedman monotonicity formula to show (3.25). Let $u = 1 - \mathcal{I}_1(E)^{-1}u_E$ and v be the harmonic extension of u to $\mathbb{R}^{N+1}_+$. Since $u \in [0, 1]$, also $v \in [0, 1]$. Notice that since $u \le 1$, it is enough to prove (3.25) in $\{d(\cdot, \partial E) \ll 1\}$. For every $x \in \mathbb{R}^{N+1}_+$ and every $r > 0$, we let $B^+_r(x) = B_r(x) \cap \mathbb{R}^{N+1}_+$ and $\partial^+ B_r(x) = \partial B_r(x) \cap \mathbb{R}^{N+1}_+$. We claim that

$$\frac{1}{r^{N-1}}\int_{B^+_r(x)}|\nabla v|^2 \lesssim r^{2\gamma}\int_{B^+_1(x)}\frac{|\nabla v|^2}{|x-y|^{N-1}} \quad \forall 0 < r \ll 1 \tag{3.26}$$

for some exponent $\gamma > 0$ with $\gamma \to 1/2$ as $\delta \to 0$ and

$$\sup_{\mathbb{R}^{N+1}_+}\int_{B^+_1(x)}\frac{|\nabla v|^2}{|x-y|^{N-1}} \lesssim 1. \tag{3.27}$$

Provided (3.26) and (3.27) hold, we can conclude the proof of (3.25) using the Poincaré inequality, Campanato's criterion and $v = 0$ in $E \times \{0\}$. Eventually we show that $\gamma \to 1/2$ as $\delta \to 0$. We devote a step to each of these three claims.

Step 1: Proof of (3.26). We first observe that it is enough to consider $x \in \partial E \times \{0\}$. Indeed, assume the statement is proven in that case. Then for $x \notin \partial E \times \{0\}$, using either odd reflection or even reflection with respect to $x_{N+1} = 0$ we may assume that v is harmonic in $B_r(x)$ for every $r \le \bar r = \min(1, d(x, \partial E \times \{0\}))$. It is then a classical fact that

$$r \mapsto \frac{1}{r^{N+1}}\int_{B_r(x)}|\nabla v|^2$$

is increasing (this follows for instance from subharmonicity of $|\nabla v|^2$, which is itself a consequence of the Bochner formula). Therefore for any $0 \le \gamma \le 1$,

$$\frac{1}{r^{N-1}}\int_{B_r^+(x)}|\nabla v|^2 \le \left(\frac{r}{\bar r}\right)^2\frac{1}{\bar r^{N-1}}\int_{B_{\bar r}^+(x)}|\nabla v|^2 \le \left(\frac{r}{\bar r}\right)^{2\gamma}\frac{1}{\bar r^{N-1}}\int_{B_{\bar r}^+(x)}|\nabla v|^2.$$

If $\bar r \ll 1$ then (3.26) follows from the case $x \in \partial E \times \{0\}$. If instead $\bar r \gtrsim 1$, then

$$\left(\frac{r}{\bar r}\right)^{2\gamma}\frac{1}{\bar r^{N-1}}\int_{B_{\bar r}^+(x)}|\nabla v|^2 \lesssim r^{2\gamma}\int_{B_{\bar r}^+(x)}\frac{|\nabla v|^2}{|x-y|^{N-1}} \le r^{2\gamma}\int_{B_1^+(x)}\frac{|\nabla v|^2}{|x-y|^{N-1}},$$

which proves (3.26) also in this case.

Let now $x \in \partial E \times \{0\}$. Without loss of generality we may assume that $x = 0$. For every $r > 0$, let[3]

$$\lambda(r) = \min\left\{\frac{\int_{\partial^+ B_r}|\nabla_\tau v|^2}{\int_{\partial^+ B_r} v^2} : v = 0 \text{ on } E \times \{0\} \cap \partial B_r^+\right\}$$

be the first eigenvalue of the Laplacian on the half-sphere with Dirichlet boundary conditions on E. Define then the function

$$\gamma(\lambda) = \sqrt{\left(\frac{N-1}{2}\right)^2 + \lambda} - \frac{N-1}{2}$$

and then

$$\overline{\gamma} = \min_{r\le 1}\gamma(r^2\lambda(r)).$$

We claim that for $r \in (0, 1]$, the function

$$\Phi(r) = \frac{1}{r^{2\overline{\gamma}}}\int_{B_r^+}\frac{|\nabla v|^2}{|x|^{N-1}}$$

is increasing. For this we follow almost verbatim the proof of [39, Theorem 2.6] (see also [39, Lemmas 2.10, 2.11]). In particular, a regularization argument is required to make rigorous all the computations below but we refer the reader to [39] for the details. Computing the logarithmic derivative of Φ, we have

$$\frac{\Phi'}{\Phi} = -2\frac{\overline{\gamma}}{r} + \left(\int_{\partial^+ B_r}\frac{|\nabla v|^2}{|x|^{N-1}}\right)\left(\int_{B_r^+}\frac{|\nabla v|^2}{|x|^{N-1}}\right)^{-1},$$

and it is therefore enough to prove that

$$\left(\int_{\partial^+ B_r}\frac{|\nabla v|^2}{|x|^{N-1}}\right)\left(\int_{B_r^+}\frac{|\nabla v|^2}{|x|^{N-1}}\right)^{-1} \ge 2\frac{\overline{\gamma}}{r}. \tag{3.28}$$

[3] We denote by ∇_τ the tangential gradient on the sphere and by ∂_ν the normal derivative.

We first claim that

$$\int_{B_r^+} \frac{|\nabla v|^2}{|x|^{N-1}} \le \frac{1}{r^{N-1}} \int_{\partial^+ B_r} v\partial_\nu v + \frac{N-1}{2r^N} \int_{\partial^+ B_r} v^2. \tag{3.29}$$

For this we notice that since $\Gamma = |x|^{1-N}$ is the Green function of the Laplacian on $\mathbb{R}^{N+1}$, we have $\Delta\Gamma \le 0$ and moreover, since it is radially symmetric, we have $\partial_{N+1}\Gamma = 0$ if $x_{N+1} = 0$. Using integration by parts we have (using the fact that $v\partial_\nu v = 0$ on $\partial B_r^+ \cap \{x_{N+1} = 0\}$)

$$\begin{aligned}
\int_{B_r^+} |\nabla v|^2 \Gamma &= \int_{\partial B_r^+} v\Gamma\partial_\nu v - \int_{B_r^+} \frac{1}{2}\nabla(v^2)\cdot\nabla\Gamma \\
&= \int_{\partial^+ B_r} \Gamma v\partial_\nu v - \int_{\partial B_r^+} \frac{1}{2} v^2 \partial_\nu \Gamma + \frac{1}{2}\int_{B_r^+} v^2 \Delta\Gamma \\
&\le \int_{\partial^+ B_r} \Gamma v\partial_\nu v - \int_{\partial^+ B_r} \frac{1}{2} v^2 \partial_\nu \Gamma \\
&= \frac{1}{r^{N-1}} \int_{\partial^+ B_r} v\partial_\nu v + \frac{N-1}{2r^N} \int_{\partial^+ B_r} v^2.
\end{aligned}$$

This proves (3.29). We thus have

$$\begin{aligned}
&\left(\int_{\partial^+ B_r} \frac{|\nabla v|^2}{|x|^{N-1}}\right)\left(\int_{B_r^+} \frac{|\nabla v|^2}{|x|^{N-1}}\right)^{-1} \\
&\quad = \left(r^{1-N}\int_{\partial^+ B_r} |\nabla v|^2\right)\left(\int_{B_r^+} |\nabla v|^2 \Gamma\right)^{-1} \\
&\quad \overset{(3.29)}{\ge} \left(r^{1-N}\int_{\partial^+ B_r} |\nabla v|^2\right)\left(r^{1-N}\left(\int_{\partial^+ B_r} v\partial_\nu v + \frac{N-1}{2r}\int_{\partial^+ B_r} v^2\right)\right)^{-1} \\
&\quad \ge \left(\int_{\partial^+ B_r} |\nabla_\tau v|^2 + \int_{\partial^+ B_r} |\partial_\nu v|^2\right) \\
&\qquad \times \left(\left(\int_{\partial^+ B_r} v^2\right)^{1/2}\left(\int_{\partial^+ B_r} (\partial_\nu v)^2\right)^{1/2} + \frac{N-1}{2r}\int_{\partial^+ B_r} v^2\right)^{-1} \\
&\quad = \left(\frac{\int_{\partial^+ B_r} |\nabla_\tau v|^2}{\int_{\partial^+ B_r} v^2} + \frac{\int_{\partial^+ B_r} |\partial_\nu v|^2}{\int_{\partial^+ B_r} v^2}\right)\left(\left(\frac{\int_{\partial^+ B_r} |\partial_\nu v|^2}{\int_{\partial^+ B_r} v^2}\right)^{1/2} + \frac{N-1}{2r}\right)^{-1} \\
&\quad \ge \min_{t>0} \frac{\lambda(r) + t^2}{t + \frac{N-1}{2r}}.
\end{aligned}$$

A direct computation shows that the above minimum is attained for $t_{\min} = \frac{1}{r}\gamma(r^2\lambda(r))$ and that $\min_{t>0} \frac{\lambda(r)+t^2}{t+\frac{N-1}{2r}} = 2t_{\min} = \frac{2}{r}\gamma(r^2\lambda(r))$ so that eventually

$$\left(\int_{\partial^+ B_r} \frac{|\nabla v|^2}{|x|^{N-1}}\right)\left(\int_{B_r^+} \frac{|\nabla v|^2}{|x|^{N-1}}\right) \ge \frac{2}{r}\gamma(r^2\lambda(r)) \ge \frac{2}{r}\overline{\gamma}.$$

This concludes the proof of (3.28). By monotonicity of Φ we have

$$\frac{1}{r^{2\overline{\gamma}+N-1}}\int_{B_r^+}|\nabla v|^2\le\Phi(r)\le\Phi(1)=\int_{B_1^+}\frac{|\nabla v|^2}{|x|^{N-1}}$$

and the proof of (3.26) with $\gamma=\overline{\gamma}$ is complete.

Step 2: Proof of (3.27). For $R\gg 1$ we have

$$\begin{aligned}\int_{B_1^+(x)}\frac{|\nabla v|^2}{|x-y|^{N-1}}&\le\int_{B_R^+(x)}\frac{|\nabla v|^2}{|x-y|^{N-1}}\\&\overset{(3.29)}{\lesssim}\frac{1}{R^{N-1}}\int_{\partial^+B_R(x)}|v|\,|\partial_\nu v|+\frac{1}{R^N}\int_{\partial^+B_R(x)}v^2\\&\overset{|v|\le1}{\lesssim}\frac{1}{R^{N-1}}\int_{\partial^+B_R(x)}|\partial_\nu v|+1.\end{aligned}$$

Since $v(z)=1-\mathcal{I}_1(E)^{-1}\int_E\frac{d\mu(y)}{|z-y|^{N-1}}$ and since E is bounded, if R is large enough and $z\in\partial^+B_R(x)$ then

$$|\nabla v(z)|\lesssim\frac{\mathcal{I}_1(E)^{-1}}{|z|^N}$$

and thus

$$\frac{1}{R^{N-1}}\int_{\partial^+B_R(x)}|\partial_\nu v|\lesssim\frac{\mathcal{I}_1(E)^{-1}}{R^{N-1}}.$$

Letting $R\to\infty$, we conclude the proof of (3.27).

Step 3: Asymptotic of $\bar{\gamma}$ *and conclusion.* We finally show that $\overline{\gamma}\to 1/2$ as $\delta\to 0$. Since $\gamma(\lambda)$ is an increasing function of λ and since for every $r>0$, $\lambda(r)$ is monotone under inclusion (i.e. if we make the dependence in E explicit, then $F\subset E$ implies $\lambda_F(r)\le\lambda_E(r)$), it is enough to prove that $\overline{\gamma}_{H_\delta}(r)\to 1/2$ where

$$H_\delta=\{x_1\le-\delta\}.$$

If $\delta=0$, then $\overline{\gamma}_{H_0}(r)=1/2$ [39, Proposition 2.12]. Since $\overline{\gamma}_{H_\delta}(r)$ does not depend on r, it is enough to consider $r=1$ and drop the dependence on r.

The proof is then concluded by observing that $\delta\mapsto\lambda_{H_\delta}$ is continuous as $\delta\to 0$. Indeed, this can be proven by an easy Γ-convergence argument. If u_δ is a minimizer for λ_{H_δ}, then up to normalization we may assume that $\int_{\partial^+B_1}u_\delta^2=1$ so that u_δ is bounded in $H^1(\partial^+B_1)$ and its trace on H_δ is bounded in $H^{1/2}$. Therefore, a subsequence converges weakly in $H^1(\partial^+B_1)$ to a function u_0 which vanishes on H_0 (by compact embedding of $H^{1/2}$ in L^2 for instance). Hence u_0 is admissible for λ_{H_0} and we have

$$\lambda_{H_0}\ge\liminf_{\delta\to0}\lambda_{H_\delta}\ge\liminf_{\delta\to0}\int_{\partial^+B_1}|\nabla_\tau u_\delta|^2\ge\int_{\partial^+B_1}|\nabla_\tau u_0|^2\ge\lambda_{H_0}.\qquad\blacksquare$$

We now convert estimate (3.25) on the potential into the desired statement on μ_E.

Lemma 3.18. *For every $\gamma \in (0, 1/2)$, there exists $\delta_0 > 0$ such that for every $r_0 > 0$ and every (δ, r_0)-Reifenberg flat domain E with $\delta \le \delta_0$, $\mu_E \in L^{\frac{2N}{N+1}}_{\mathrm{loc}}(\mathbb{R}^N)$ and for every $x \in \mathbb{R}^N$ and $r < r_0/2$ we have*

$$\left(\int_{B_r(x)} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \lesssim r^{N-1+2\gamma}, \tag{3.30}$$

where the implicit constant depends on N, γ, r_0 and $|E|$.

Proof. Let $\gamma = \gamma(\delta)$ be given by Lemma 3.17. We first derive from (3.25) the following estimate on μ_E:

$$\mu_E \lesssim d(\cdot, \partial E)^{-(1-\gamma)}. \tag{3.31}$$

If u_E denotes the associated potential, then

$$\begin{aligned}
C'(N,1)\mu_E(x) &\overset{(2.5)}{=} (-\Delta)^{1/2} u_E(x) \\
&\overset{(2.1)}{=} C(N,1/2)\int_{E^c} \frac{\mathcal{I}_1(E) - u_E(y)}{|x-y|^{N+1}}\,dy \\
&\overset{(3.25)}{\lesssim} \frac{\mathcal{I}_1(E)}{r_0^\gamma}\int_{E^c} \frac{d(y,\partial E)^\gamma}{|x-y|^{N+1}}\,dy \\
&\lesssim \frac{\mathcal{I}_1(E)}{r_0^\gamma}\int_{B^c_{d(x,\partial E)}(x)} \frac{dz}{|z|^{N+1-\gamma}} \\
&\lesssim \frac{\mathcal{I}_1(E)}{r_0^\gamma} d(x,\partial E)^{-(1-\gamma)} \lesssim d(x,\partial E)^{-(1-\gamma)},
\end{aligned}$$

where in the last line we have used the fact that if B is a ball of measure $|E|$ then $\mathcal{I}_1(E) \le \mathcal{I}_1(B)$. This follows for instance from the fractional Pólya–Szegő inequality [15] and the capacitary definition (2.9) of $\mathcal{I}_1$ (see also [2]).

We now prove (3.30). We may assume without loss of generality that $x = 0$ and $|\gamma - 1/2| \ll 1$. For $P > 0$, we set $\mu_P = \min\{\mu_E, P\}$. Clearly μ_P is an integrable function and $\mu_P \to \mu$ a.e. in E. Moreover, since $0 \le \mu_P \le \mu$, μ_P satisfies inequality (3.31). We first claim that there exist $C_0, C_1 > 0$ such that for every $x \in \partial E$ and every $r \le r_0/2$ there exists a set $A(x) \subset \partial E$ such that

$$\sharp A(x) \le C_1 \delta^{1-N} \tag{3.32}$$

and

$$\int_{B_r(x)} \mu_P^{\frac{2N}{N+1}} \le C_0 r^{N - \frac{2N}{N+1}(1-\gamma)} + \sum_{y \in A(x)} \int_{B_{5\delta r}(y)} \mu_P^{\frac{2N}{N+1}}. \tag{3.33}$$

Again, there is no loss of generality by restricting ourselves to $x = 0$. Recall that by Definition 3.13, since E is (δ, r_0)-Reifenberg flat, for every $r \le r_0/2$ there exists an hyperplane H_r such that

$$d(\partial E \cap B_r, H_r \cap B_r) \le \delta r.$$

In particular, if $N_r = \{y \in B_r : d(y, H_r) > 2\delta r\}$ we have $d(y, \partial E) \sim d(y, H_r)$ for $y \in N_r$, so that

$$\int_{B_r} \mu_P^{\frac{2N}{N+1}} \le \int_{N_r} \mu_P^{\frac{2N}{N+1}} + \int_{B_r \cap N_r^c} \mu_P^{\frac{2N}{N+1}}$$
$$\overset{(3.31)}{\le} C r^{N-1} \int_{2\delta r}^{r} \frac{dt}{t^{\frac{2N}{N+1}(1-\gamma)}} + \int_{B_r \cap N_r^c} \mu_P^{\frac{2N}{N+1}}$$
$$\le C_0 r^{N - \frac{2N}{N+1}(1-\gamma)} + \int_{B_r \cap N_r^c} \mu_P^{\frac{2N}{N+1}}.$$

We now estimate the last term on the right-hand side. By the triangle inequality, for every $x \in N_r^c \cap B_r$, $d(x, \partial E \cap B_r) \le 3\delta r$ and thus setting $r_1 = 5\delta r$ we find that $\{B_{r_1}(y)\}_{y \in \partial E \cap B_r}$ is a covering of $N_r^c \cap B_r$. By the Vitali covering lemma we can extract a finite subset of points $A \subset \partial E \cap B_r$ such that

- $\{B_{r_1/5}(y)\}_{y \in A}$ consists of pairwise disjoint balls,
- $\{B_{r_1}(y)\}_{y \in A}$ is still a covering of $N_r^c \cap B_r$.

Since $B_{r_1/5}(y) = B_{\delta r}(y) \subset N_r^c \cap B_{(1+\delta)r}$ for $y \in A$, the first condition gives

$$r_1^N \sharp A \lesssim |N_r^c \cap B_{(1+\delta)r}| \sim (\delta r) r^{N-1},$$

which, by definition of r_1, yields (3.32). The second condition gives

$$\int_{B_r \cap N_r^c} \mu_P^{\frac{2N}{N+1}} \le \sum_{y \in A} \int_{B_{r_1}(y)} \mu_P^{\frac{2N}{N+1}},$$

concluding the proof of (3.33).

For $k \ge 0$, we set $r_k = (5\delta)^k r$ and define recursively $A_0 = \{0\}$ and

$$A_k = \bigcup_{x \in A_{k-1}} A(x).$$

From (3.32) we have

$$\sharp A_k \le (C_1 \delta^{1-N})^k, \tag{3.34}$$

and thus applying recursively (3.33) we find, for $K \ge 0$,

$$\int_{B_r} \mu_P^{\frac{2N}{N+1}} \le C_0 \sum_{k=0}^{K} (\sharp A_k) r_k^{N - \frac{2N}{N+1}(1-\gamma)} + \sum_{y \in A_{K+1}} \int_{B_{r_{K+1}}(y)} \mu_P^{\frac{2N}{N+1}}.$$

By definition of μ_P we have

$$\sum_{y \in A_{K+1}} \int_{B_{r_{K+1}}(y)} \mu_P^{\frac{2N}{N+1}} \le (\sharp A_{K+1}) |B_{r_{K+1}}| P^{\frac{2N}{N+1}}$$
$$\lesssim (C_1 \delta^{1-N})^K (5\delta)^{KN} r^N P^{\frac{2N}{N+1}}$$
$$= (5^N C_1 \delta)^K r^N P^{\frac{2N}{N+1}}.$$

Thus, if $5^N C_1\delta < 1$ we can let $K \to \infty$ to obtain, from the definition of r_k and (3.34),

$$\int_{B_r} \mu_P^{\frac{2N}{N+1}} \le C_0\Big(\sum_{k\ge 0} (C_2\delta^{1-\frac{2N}{N+1}(1-\gamma)})^k\Big) r^{N-\frac{2N}{N+1}(1-\gamma)},$$

where $C_2 = C_1 5^{N-\frac{2N}{N+1}(1-\gamma)}$. Finally, if $|\gamma - 1/2| \ll 1$, then $\frac{2N}{N+1}(1-\gamma) < 1$ and thus provided δ is small enough, the sum converges and we have (notice that all the constants involved are independent of P)

$$\left(\int_{B_r} \mu_P^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \lesssim r^{N-1+2\gamma}.$$

Letting $P \to \infty$ concludes the proof of (3.30). ■

Remark 3.19. We point out that this estimate is essentially optimal, as can be seen from the case $E = B_1$; see [24, Chapter II.13] and Section 4 below.

Remark 3.20. A quick inspection of the proof shows that for every $q < 2$, $\mu_E \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ if E is δ-Reifenberg flat with δ small enough. This is again optimal in light of the boundary behavior of the 1/2-harmonic measure of the ball; see [24, Chapter II.13] and Section 4 below. This higher integrability (with respect to $L^{\frac{2N}{N+1}}$) would not, however, be sufficient by itself to obtain the regularity of volume-constrained minimizers of $\mathcal{F}_{\alpha,Q}$, so that we need the more precise estimate (3.30).

Combining Proposition 3.14 with Proposition 3.16 and Lemma 3.18 we find that for small charge Q every volume constrained minimizer of $\mathcal{F}_{1,Q}$ is also a perimeter almost-minimizer for which the classical regularity theory applies (see [27]) so that we have the counterpart of Proposition 3.12.

Proposition 3.21. *Let $\alpha = 1$. For every $\gamma \in (0, 1/2)$ there exists $Q(\gamma, N) > 0$ such that for every $Q \le Q(\gamma, N)$, every volume-constrained minimizer E_Q of $\mathcal{F}_{1,Q}$ is $C^{1,\gamma}$ with uniformly bounded $C^{1,\gamma}$ norm. As a consequence, for every $\beta < \gamma$, up to translation, E_Q converges in $C^{1,\beta}$ to B_1 as $Q \to 0$.*

4. Rigidity of the ball for small charges

In this section we prove Theorem 1.5, i.e. we show that for every $\alpha \in (0, 2)$ and small enough charge Q, the ball is the unique minimizer of $\mathcal{F}_{\alpha,Q}$ under volume constraints in the class of nearly spherical sets.

Before embarking on the proof let us set some notation and make a few preliminary remarks. First, recall that fixing an arbitrary $\gamma \in (0, 1)$, we say that E is *nearly spherical* if $|E| = |B|$ (where $B = B_1$ is the unit ball), E has barycenter 0 and there exists $\phi : \partial B \to \mathbb{R}$ with $\|\phi\|_{C^{1,\gamma}(\partial B)} \le 1$ such that

$$\partial E = \{(1 + \phi(x))x : x \in \partial B\}.$$

With a slight abuse of notation we still denote by ϕ its 0-homogeneous extension outside ∂B, that is, the function $\mathbb{R}^N \ni x \mapsto \phi(x/|x|)$. We recall from [16] that if E is nearly spherical, we have

$$\left|\int_{\partial B} \phi\right| \lesssim \int_{\partial B} \phi^2. \tag{4.1}$$

In particular, if $\|\phi\|_{W^{1,\infty}(\partial B)} \ll 1$, recalling the notation $\bar{\phi} = \frac{1}{P(B)} \int_{\partial B} \phi$ we have, for $s \in (0,1)$,

$$\int_{\partial B} \phi^2 \lesssim \int_{\partial B} (\phi - \bar{\phi})^2 \overset{(2.4)}{\lesssim} [\phi]^2_{H^s(\partial B)}. \tag{4.2}$$

For two Radon measures μ and ν on $\mathbb{R}^N$, we define the positive bilinear operator (see [24])

$$I_\alpha(\mu,\nu) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x)\, d\nu(y)}{|x-y|^{N-\alpha}}.$$

In particular, we have $I_\alpha(\mu) = I_\alpha(\mu,\mu)$. We let μ_E be the optimal measure of E and let $u_E(x) = \int_E \frac{d\mu_E}{|x-y|^{N-\alpha}}$ be the associated potential. When there is no risk of confusion we drop the index E from both. In the specific case of the unit ball B we have, by [24, Chapter II.13],

$$\mu_B(x) = \frac{C_\alpha}{(1-|x|^2)^{\alpha/2}} \sim \frac{1}{d(x,\partial B)^{\alpha/2}}.$$

We sometimes write $\phi_x = \phi(x)$. In particular, if E is nearly spherical and ϕ is the corresponding parametrization, we set $T(x) = (1+\phi_x)x$ for $x \in B$, so that $E = T(B)$. We then define $g = T_\#^{-1}\mu_E$ (which is a probability measure on B) so that

$$\mathcal{I}_\alpha(E) = \int_{B\times B} \frac{dg_x\, dg_y}{|T(x)-T(y)|^{N-\alpha}}. \tag{4.3}$$

We can now begin the proofs. We first prove that g has the same behavior as μ_B close to ∂B.

Lemma 4.1. *Let $\alpha \in (0,2)$ and let E be a nearly spherical set. Then its optimal measure satisfies, for $x \notin \partial B$,*

$$g(x) \lesssim \frac{1}{d(x,\partial B)^{\alpha/2}} \sim \mu_B(x). \tag{4.4}$$

Proof. The proof resembles the proof of Lemma 3.18, taking advantage of the regularity of E to obtain a sharp estimate. We first show that $\mu = \mu_E$ satisfies

$$\mu(x) \lesssim \frac{1}{d(x,\partial E)^{\alpha/2}}. \tag{4.5}$$

Recall that by (2.5) and (2.8),

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = 0, & x \in E^c, \\ u(x) - \mathcal{I}_\alpha(E) = 0, & x \in E. \end{cases}$$

Thus, by the boundary regularity theory for the fractional Laplacian developed in [35],

$$u(x) - \mathcal{I}_\alpha(E) \lesssim d(x, \partial E)^{\alpha/2}.$$

Hence, arguing as in the proof of Lemma 3.18 we compute, for $x \in E$,

$$\begin{aligned} C'(N,\alpha)\mu(x) &\overset{(2.5)}{=} (-\Delta)^{\alpha/2}u(x) \overset{(2.1)}{=} C(N,\alpha/2)\int_{E^c} \frac{\mathcal{I}_\alpha(E) - u(y)}{|x-y|^{N+\alpha/2}}\,dy \\ &\lesssim \int_{E^c} \frac{d(y,\partial E)^{\alpha/2}}{|x-y|^{N+\alpha}}\,dy \lesssim \int_{B^c_{d(x,\partial E)}(x)} \frac{dz}{|z|^{N+\alpha-\alpha/2}} \\ &\lesssim d(x,\partial E)^{-\alpha/2}. \end{aligned}$$

Since $g(x) = (1+\phi_x)^N \mu((1+\phi_x)x)$ with $|\phi| \le 1/4$, up to choosing δ small enough we obtain

$$g(x) \lesssim \mu((1+\phi_x)x) \lesssim \frac{1}{d((1+\phi_x)^{\alpha/2}x, \partial E)}.$$

Thus (4.4) follows provided

$$d((1+\phi_x)x, \partial E) \sim \big|1 - |x|\big|. \tag{4.6}$$

Let us prove (4.6). Since

$$d((1+\phi_x)x, \partial E) = \min_{y\in\partial B} |(1+\phi_x)x - (1+\phi_y)y|,$$

testing with $y = x/|x|$ we obtain the upper bound

$$d((1+\phi_x)x, \partial E) \le (1+\phi_x)\big|1-|x|\big| \lesssim |1-|x||.$$

To get the lower bound we may assume that $\big|1-|x|\big| \ll 1$. Squaring we get

$$\begin{aligned} &d((1+\phi_x)^2x, \partial E) \\ &= \min_{y\in\partial B} |(1+\phi_x)x - (1+\phi_y)y|^2 \\ &= \min_{y\in\partial B} \{(1+\phi_x)^2|x|^2 - 2(1+\phi_x)(1+\phi_y)x\cdot y + |1+\phi_y|^2\} \\ &= \min_{y\in\partial B} \Big\{(1+\phi_x)^2|x|^2 - 2(1+\phi_x)(1+\phi_y)|x| + |1+\phi_y|^2 \\ &\qquad\qquad + 2(1+\phi_x)(1+\phi_y)x\cdot\Big(\frac{x}{|x|} - y\Big)\Big\} \\ &= \min_{y\in\partial B} \Big\{(1+\phi_x)^2\Big||x| - \frac{1+\phi_y}{1+\phi_x}\Big|^2 + 2(1+\phi_x)(1+\phi_y)(|x| - y\cdot x)\Big\} \\ &\gtrsim \min_{y\in\partial B} \Big\{\Big||x| - 1 + \frac{\phi_x - \phi_y}{1+\phi_x}\Big|^2 + (|x| - y\cdot x)\Big\}. \end{aligned}$$

Now for every y, either $|x| - y\cdot x \gtrsim \big||x|-1\big|^2$ or $|x| - y\cdot x \ll \big||x|-1\big|^2$. The first case directly leads to the conclusion of the proof of (4.6). In the second case, writing

$x = r\sigma$ with $\sigma \in \partial B$, this means that $|\sigma - y|^2 = \frac{1}{2r}(|x| - x \cdot y) \ll \big||x| - 1\big|^2$ and thus $|\phi_x - \phi_y| \lesssim |\sigma - y| \ll \big||x| - 1\big|$ from which we find that for every $y \in \partial B$,

$$\left||x| - 1 + \frac{\phi_x - \phi_y}{1 + \phi_x}\right|^2 + (|x| - y \cdot x) \gtrsim \big||x| - 1\big|^2,$$

and the claim follows as well. ■

Next we state and prove two lemmas giving the Taylor expansion of the term $|T(x) - T(y)|^{-(N-\alpha)}$ appearing in (4.3).

Lemma 4.2. *For $x, y \in B$, we have*

$$|T(x) - T(y)|^2 = |x - y|^2\big(1 + \phi_x + \phi_y + \phi_x\phi_y + \psi(x, y)\big), \tag{4.7}$$

where

$$\psi(x, y) = \frac{1}{2}(|x|^2 + |y|^2)\left(\frac{\phi_x - \phi_y}{|x - y|}\right)^2 + (|x| + |y|)\left(1 - \frac{1}{2}(\phi_x + \phi_y)\right)\frac{\phi_x - \phi_y}{|x - y|}. \tag{4.8}$$

Proof. Expanding the squares we get

$$\begin{aligned}|T(x) - T(y)|^2 &= \big|(x - y) + \tfrac{1}{2}((x + y)(\phi_x - \phi_y) + (x - y)(\phi_x + \phi_y))\big|^2\\ &= |x - y|^2 + (|x|^2 - |y|^2)(\phi_x - \phi_y) + |x - y|^2(\phi_x + \phi_y)\\ &\quad + \tfrac{1}{4}|x + y|^2|\phi_x - \phi_y|^2 + \tfrac{1}{4}|x - y|^2|\phi_x + \phi_y|^2\\ &\quad + \tfrac{1}{2}(|x|^2 - |y|^2)(\phi_x^2 - \phi_y^2).\end{aligned}$$

Comparing with (4.7) we are left with the proof of

$$\tfrac{1}{4}|x + y|^2|\phi_x - \phi_y|^2 + \tfrac{1}{4}|x - y|^2|\phi_x + \phi_y|^2 = \tfrac{1}{2}(|x|^2 + |y|^2)(\phi_x - \phi_y)^2 + |x - y|^2\phi_x\phi_y.$$

For this we write $(\phi_x + \phi_y)^2 = (\phi_x - \phi_y)^2 + 4\phi_x\phi_y$ to get

$$\begin{aligned}\tfrac{1}{4}|x + y|^2|\phi_x - \phi_y|^2 + \tfrac{1}{4}|x - y|^2|\phi_x + \phi_y|^2 &= \tfrac{1}{4}|\phi_x - \phi_y|^2(|x + y|^2 + |x - y|^2) + |x - y|^2\phi_x\phi_y\\ &= \tfrac{1}{2}(|x|^2 + |y|^2)(\phi_x - \phi_y)^2 + |x - y|^2\phi_x\phi_y.\end{aligned}$$ ■

As a consequence we get the following Taylor expansion of $|T(x) - T(y)|^{-(N-\alpha)}$.

Lemma 4.3. *Set $\widehat{\alpha} = N - \alpha$. If $\|\phi\|_{W^{1,\infty}(\partial B)} \ll 1$ then for $x, y \in B$,*

$$|T(x) - T(y)|^{-(N-\alpha)} = |x - y|^{-(N-\alpha)}\left(\left(1 - \frac{\widehat{\alpha}}{2}\phi_x\right)\left(1 - \frac{\widehat{\alpha}}{2}\phi_y\right) - \frac{\widehat{\alpha}}{2}\psi(x, y) + \zeta(x, y)\right), \tag{4.9}$$

where

$$|\zeta(x,y)| \lesssim \phi_x^2 + \phi_y^2 + \psi(x,y)^2, \tag{4.10}$$

and where ψ *is the function defined in* (4.8).

Proof. Let us first point out that under our hypothesis we have $\|\psi\|_{L^\infty} \ll 1$. Indeed, this follows from $\|\phi\|_{W^{1,\infty}(\partial B)} \ll 1$ and

$$\frac{|x|+|y|}{|x-y|} \lesssim \frac{1}{\left|\frac{x}{|x|} - \frac{y}{|y|}\right|} + 1. \tag{4.11}$$

This inequality may be easily seen using for instance polar coordinates. That is, if $x = r\sigma$ and $y = sv$ then

$$\begin{aligned} \left(\frac{|x|+|y|}{|x-y|}\right)^2 &= \frac{r^2+s^2}{|r-s|^2 + rs|\sigma - v|^2} \\ &= \frac{|r-s|^2}{|r-s|^2 + rs|\sigma - v|^2} + \frac{2rs}{|r-s|^2 + rs|\sigma - v|^2} \\ &\le 1 + \frac{2}{|\sigma - v|^2}. \end{aligned}$$

We then obtain the result by (4.7) and Taylor expansion. ■

The next result contains one of the key linearization estimates we need to obtain our rigidity result.

Lemma 4.4. *Let* E *be a nearly spherical set with* $\|\phi\|_{W^{1,\infty}(\partial B)} \ll 1$. *Then for every* $\alpha \in (0,2)$ *and* $\varepsilon > 0$,

$$\left| \mathcal{I}_\alpha(E) - I_\alpha\left(\left(1 - \frac{\widehat{\alpha}}{2}\phi\right) g\right)\right| \lesssim_\varepsilon [\phi]^2_{H^{(2-\alpha)/2}(\partial B)} + [\phi]^2_{H^{1/2+\varepsilon}(\partial B)}, \tag{4.12}$$

where $(1 - \frac{\widehat{\alpha}}{2}\phi)g$ *is seen as a measure on* B *(recall that* ϕ *is extended by* 0*-homogeneity on* $\mathbb{R}^N$*) and* $\widehat{\alpha} = N - \alpha$.

Proof. We fix $\varepsilon > 0$. In view of (4.9) and (4.3), it is enough to prove that

$$\left|\int_{B\times B} \frac{\psi(x,y)}{|x-y|^{N-\alpha}}\, dg_x\, dg_y\right| + \left|\int_{B\times B} \frac{\zeta(x,y)}{|x-y|^{N-\alpha}}\, dg_x\, dg_y\right| \lesssim_\varepsilon [\phi]^2_{H^{(2-\alpha)/2}(\partial B)} + [\phi]^2_{H^{1/2+\varepsilon}(\partial B)}.$$

Recall that from the proof of Lemma 4.3, $\|\phi\|_{L^\infty(\partial B)} \ll 1$ implies $\|\psi\|_{L^\infty} \ll 1$. Moreover, by the radial symmetry of μ_B and the 0-homogeneity of ϕ we have

$$\begin{aligned} \int_{B\times B} \frac{\phi_x^2}{|x-y|^{N-\alpha}}\, dg_x\, dg_y &\overset{(4.4)}{\lesssim} \int_{B\times B} \phi_x^2 \frac{d\mu_B(x)\, d\mu_B(y)}{|x-y|^{N-\alpha}} = \frac{I_\alpha(B)}{\mathcal{H}^{N-1}(\partial B)} \int_{\partial B} \phi^2 \\ &\overset{(4.2)}{\lesssim} [\phi]^2_{H^{(2-\alpha)/2}(\partial B)}. \end{aligned} \tag{4.13}$$

Hence, by (4.10) we are left with the proof of

$$\left|\int_{B\times B}\frac{\psi(x,y)}{|x-y|^{N-\alpha}}\,dg_x\,dg_y\right|\lesssim_\varepsilon [\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)}.$$

By symmetry in x and y we have

$$\int_{B\times B}(\phi_x-\phi_y)\frac{|x|+|y|}{|x-y|^{N-\alpha}}\,dg_x\,dg_y=0.$$

Moreover, the Young inequality yields

$$(|x|+|y|)|\phi_x+\phi_y|\frac{|\phi_x-\phi_y|}{|x-y|}\lesssim(|x|^2+|y|^2)\left(\frac{\phi_x-\phi_y}{|x-y|}\right)^2+\phi_x^2+\phi_y^2,$$

so that by (4.13) and the definition (4.8) of ψ, we just need to prove

$$\int_{B\times B}(|x|^2+|y|^2)\left(\frac{\phi_x-\phi_y}{|x-y|}\right)^2\frac{1}{|x-y|^{N-\alpha}}\,dg_x\,dg_y \lesssim_\varepsilon [\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)}.$$

Using (4.11) and (4.4) we have

$$\begin{aligned}\int_{B\times B}(|x|^2+|y|^2)\left(\frac{\phi_x-\phi_y}{|x-y|}\right)^2\frac{1}{|x-y|^{N-\alpha}}\,dg_x\,dg_y &\lesssim\int_{B\times B}\left[\left(\frac{\phi_x-\phi_y}{\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}\right)^2+(\phi_x-\phi_y)^2\right]\frac{d\mu_B(x)\,d\mu_B(y)}{|x-y|^{N-\alpha}}\\ &\lesssim\int_{B\times B}\left(\frac{\phi_x-\phi_y}{\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}\right)^2\frac{d\mu_B(x)\,d\mu_B(y)}{|x-y|^{N-\alpha}}+\int_{\partial B}\phi^2,\end{aligned}$$

where we have used the Young inequality and formula (4.13) to estimate the second term in the last inequality. Thanks to (4.2), we may further reduce the proof of (4.12) to

$$\int_{B\times B}\left(\frac{\phi_x-\phi_y}{\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}\right)^2\frac{d\mu_B(x)\,d\mu_B(y)}{|x-y|^{N-\alpha}}\lesssim_\varepsilon [\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)}. \tag{4.14}$$

Recalling that $\mu_B(x)\lesssim(1-|x|)^{-\alpha/2}$ and writing x and y in polar coordinates $x=r\sigma$ and $y=sv$, with $r,s\in\mathbb{R}$ and $\sigma,v\in\partial B$, we get

$$\begin{aligned}&\int_{B\times B}\left(\frac{\phi_x-\phi_y}{\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}\right)^2\frac{d\mu_B(x)\,d\mu_B(y)}{|x-y|^{N-\alpha}}\\ &\lesssim\int_{\partial B\times\partial B}\left(\frac{\phi(\sigma)-\phi(v)}{|\sigma-v|}\right)^2\left[\int_0^1\int_0^1\frac{r^{N-1}s^{N-1}}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}}\frac{dr\,ds}{(|r-s|^2+rs|\sigma-v|^2)^{(N-\alpha)/2}}\right]\times d\sigma\,dv\\ &=\int_{\partial B\times\partial B}\left(\frac{\phi(\sigma)-\phi(v)}{|\sigma-v|}\right)^2F(|\sigma-v|)\,d\sigma\,dv,\end{aligned}$$

where

$$F(\theta) = \int_0^1 \int_0^1 \frac{r^{N-1} s^{N-1}}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}} \frac{dr\, ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}}.$$

We claim that for $\theta \in (0,2)$,

$$F(\theta) \lesssim_\varepsilon \frac{1}{\theta^{N-\alpha-1}} + \frac{1}{\theta^{N-2+2\varepsilon}}. \tag{4.15}$$

It is enough to prove this estimate for $\theta \ll 1$. To this end we first estimate

$$\begin{aligned}
&\int_0^{1/2} \int_0^1 \frac{r^{N-1} s^{N-1}}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}} \frac{dr\, ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}} \\
&\quad\lesssim \int_0^{1/2} r^{N-1}\left[\int_0^1 \frac{1}{|1-s|^{\alpha/2}} \frac{ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}}\right] dr \\
&\quad\lesssim \int_0^{1/2} r^{N-1}\left[\int_0^{3/4} \frac{ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}}\right] dr + \int_0^{1/2}\left[\int_{3/4}^1 \frac{ds}{|1-s|^{\alpha/2}}\right] dr \\
&\quad\lesssim \int_0^{1/2} r^{N-1}\left[\int_0^{3/4} \frac{ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}}\right] dr + 1.
\end{aligned}$$

Using the change of variables $s = rt$ we then compute

$$\begin{aligned}
\int_0^{3/4} \frac{ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}} &= r^{1-N+\alpha} \int_0^{3/(4r)} \frac{dt}{(|1-t|^2 + t\theta^2)^{(N-\alpha)/2}} \\
&\lesssim r^{1-N+\alpha}\left[1 + \int_{1/2}^2 \frac{dt}{(|1-t| + \theta)^{N-\alpha}} + \int_2^{3/(4r)} \frac{dt}{t^{N-\alpha}}\right] \\
&\lesssim r^{1-N+\alpha}\left[1 + \frac{1}{\theta^{N-\alpha-1}} + r^{N-\alpha-1}\right].
\end{aligned}$$

We thus conclude that

$$\begin{aligned}
&\int_0^{1/2} \int_0^1 \frac{r^{N-1} s^{N-1}}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}} \frac{dr\, ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}} \\
&\qquad\lesssim 1 + \int_0^{1/2} r^\alpha\left[1 + \frac{1}{\theta^{N-\alpha-1}} + r^{N-\alpha-1}\right] dr \lesssim 1 + \frac{1}{\theta^{N-\alpha-1}}. \qquad (4.16)
\end{aligned}$$

We now focus on the integral between $1/2$ and 1 which we split as

$$\begin{aligned}
&\int_{1/2}^1 \int_0^1 \frac{r^{N-1} s^{N-1}}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}} \frac{dr\, ds}{(|r-s|^2 + rs\theta^2)^{(N-\alpha)/2}} \\
&\lesssim \int_{1/2}^1 \int_0^{1/4} \frac{1}{|1-r|^{\alpha/2}}\, dr\, ds + \int_{1/2}^1 \int_{1/4}^1 \frac{1}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}} \frac{dr\, ds}{(|r-s|^2 + \theta^2)^{(N-\alpha)/2}} \\
&\lesssim 1 + \int_0^{1/2} \int_{-1}^1 \frac{1}{t^{\alpha/2}|t-w|^{\alpha/2}} \frac{dt\, dw}{(w^2 + \theta^2)^{(N-\alpha)/2}},
\end{aligned}$$

where in the last line we have made the change of variables $r = 1-t$ and $s = 1-t+w$.

We now prove that for every $w \in (-1,1)$,

$$\int_0^{1/2} \frac{dt}{t^{\alpha/2}|t-w|^{\alpha/2}} \lesssim |w|^{1-\alpha} + 1 + \chi_{\alpha=1}\big|\log|w|\big|. \tag{4.17}$$

Since the left-hand side of (4.17) increases when we replace w by $|w|$, it is enough to prove it for $w > 0$. We then obtain (4.17) from

$$\int_0^{1/2} \frac{dt}{t^{\alpha/2}|t-w|^{\alpha/2}} \le \int_0^{w/2} \frac{dt}{t^{\alpha/2}w^{\alpha/2}} + \int_{\frac{w}{2}}^{2w} \frac{dt}{w^{\alpha/2}|t-w|^{\alpha/2}} + \int_{2w}^{2} \frac{dt}{t^{\alpha}}$$
$$\lesssim w^{1-\alpha} + 1 + \chi_{\alpha=1}|\log w|.$$

Using (4.17) we then find

$$\int_0^{1/2}\int_{-1}^{1} \frac{1}{t^{\alpha/2}|t-w|^{\alpha/2}} \frac{dt\,dw}{(w^2+\theta^2)^{(N-\alpha)/2}} \lesssim_\varepsilon \frac{1}{\theta^{N-\alpha-1}} + \frac{1}{\theta^{N-2+2\varepsilon}}.$$

This proves

$$\int_{1/2}^{1}\int_0^{1} \frac{r^{N-1}s^{N-1}}{|1-r|^{\alpha/2}|1-s|^{\alpha/2}} \frac{dr\,ds}{(|r-s|^2+rs\theta^2)^{(N-\alpha)/2}} \lesssim_\varepsilon \frac{1}{\theta^{N-\alpha-1}} + \frac{1}{\theta^{N-2+2\varepsilon}},$$

which together with (4.16) concludes the proof of (4.15).

We thus find

$$\int_{B\times B} \left(\frac{\phi_x-\phi_y}{\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}\right)^2 \frac{d\mu_B(x)\,d\mu_B(y)}{|x-y|^{N-\alpha}}$$
$$\lesssim_\varepsilon \int_{\partial B\times\partial B} \frac{(\phi(\sigma)-\phi(v))^2}{|\sigma-v|^{N-\alpha+1}}\,d\sigma\,dv + \int_{\partial B\times\partial B} \frac{(\phi(\sigma)-\phi(v))^2}{|\sigma-v|^{N+2\varepsilon}}\,d\sigma\,dv$$
$$\overset{(2.3)}{=} [\phi]^2_{H^{(2-\alpha)/2}(\partial B)} + [\phi]^2_{H^{1/2+\varepsilon}(\partial B)},$$

which is (4.14). ■

We may now conclude the proof of the stability inequality for nearly spherical sets.

Proposition 4.5. *If E is a nearly spherical set with $\|\phi\|_{W^{1,\infty}(\partial B)} \ll 1$, then for $\alpha \in (0,2)$ and $\varepsilon > 0$,*

$$\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E) \lesssim_\varepsilon [\phi]^2_{H^{\alpha/2}(\partial B)} + [\phi]^2_{H^{(2-\alpha)/2}(\partial B)} + [\phi]^2_{H^{1/2+\varepsilon}(\partial B)}. \tag{4.18}$$

As a consequence,

$$\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E) \lesssim P(E) - P(B). \tag{4.19}$$

Proof. Using the same notation as above and using the equality $I_\alpha(g) = I_\alpha(g-\mu_B) + 2I_\alpha(g-\mu_B,\mu_B) + I_\alpha(\mu_B)$, we have

$$\begin{aligned}\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E) &= I_\alpha(\mu_B) - \mathcal{I}_\alpha(E)\\ &= I_\alpha(\mu_B) - I_\alpha(g) + I_\alpha(g) - \mathcal{I}_\alpha(E)\\ &= -I_\alpha(g-\mu_B) - 2I_\alpha(g-\mu_B,\mu_B) + I_\alpha(g) - \mathcal{I}_\alpha(E).\end{aligned}$$

We now notice that by optimality of μ_B we know that u_B is constant in B (recall (2.8)) and thus, since $\int_B \mu_B = \int_B g = 1$,

$$I_\alpha(g-\mu_B,\mu_B)=\int_B u_B(g-\mu_B)=u_B(0)\int_B(g-\mu_B)=0.$$

Using (4.12) we can compute

$$\begin{aligned}\mathcal{I}_\alpha(B)-\mathcal{I}_\alpha(E)+I_\alpha(g-\mu_B)&\\ &\le I_\alpha(g)-I_\alpha\left(\left(1-\frac{\widehat{\alpha}}{2}\phi\right)g\right)+C_\varepsilon([\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)})\\ &=-\frac{\widehat{\alpha}^2}{4}I_\alpha(\phi g)+\widehat{\alpha}I_\alpha(g,\phi g)+C_\varepsilon([\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)})\\ &\lesssim_\varepsilon I_\alpha(g,\phi g)+[\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)}.\end{aligned}$$

We further decompose the term $I_\alpha(g,\phi g)$ as follows:

$$\begin{aligned}I_\alpha(g,\phi g)&=I_\alpha(\mu_B,\phi g)+I_\alpha(g-\mu_B,\phi g)\\ &=I_\alpha(\mu_B,\phi\mu_B)+I_\alpha(\mu_B,\phi(g-\mu_B))+I_\alpha(g-\mu_B,\phi g).\end{aligned}$$

We now observe that since μ_B is radially symmetric and since ϕ is 0-homogeneous,

$$I_\alpha(\mu_B,\phi\mu_B)=C\int_{\partial B}\phi\overset{(4.1)}{\lesssim}\int_{\partial B}\phi^2.$$

By (4.2), we therefore have

$$\begin{aligned}\mathcal{I}_\alpha(B)-\mathcal{I}_\alpha(E)+I_\alpha(g-\mu_B)\lesssim_\varepsilon I_\alpha(\mu_B,\phi(g-\mu_B))+I_\alpha(g-\mu_B,\phi g)&\\ +[\phi]^2_{H^{(2-\alpha)/2}(\partial B)}+[\phi]^2_{H^{1/2+\varepsilon}(\partial B)}.&\end{aligned}\tag{4.20}$$

We first estimate $I_\alpha(g-\mu_B,\phi g)$. We notice that

$$\begin{aligned}I_\alpha(\phi g)&\le\left(\int_{B\times B}\frac{\phi_x^2\,dg_x\,dg_y}{|x-y|^{N-\alpha}}\right)^{1/2}\left(\int_{B\times B}\frac{\phi_y^2\,dg_x\,dg_y}{|x-y|^{N-\alpha}}\right)^{1/2}\\ &\overset{(4.13)}{\lesssim}\int_{\partial B}\phi^2\overset{(4.2)}{\lesssim}[\phi]^2_{H^{\alpha/2}(\partial B)}.\end{aligned}$$

Thus, the Cauchy–Schwarz inequality for I_α (recall that it is a positive bilinear operator) gives

$$I_\alpha(g-\mu_B,\phi g)\le I_\alpha(g-\mu_B)^{1/2}I_\alpha(\phi g)^{1/2}\lesssim I_\alpha(g-\mu_B)^{1/2}[\phi]_{H^{\alpha/2}(\partial B)}.\tag{4.21}$$

We now turn to $I_\alpha(\mu_B,\phi(g-\mu_B))$. For this we use the fact that u_B is constant on B to write

$$I_\alpha(\mu_B,\phi(g-\mu_B))=u_B(0)\int_B\phi(g-\mu_B).$$

Let ρ be a smooth, positive cut-off function with $\rho = 1$ on B and $\rho = 0$ on B_2^c. We then set $\Phi = \phi\rho$ so that

$$\begin{aligned}\int_B \phi(g-\mu_B) &= \int_{\mathbb{R}^N} \Phi(g-\mu_B)\\ &\le [\Phi]_{H^{\alpha/2}(\mathbb{R}^N)}[g-\mu_B]_{H^{-\alpha/2}(\mathbb{R}^N)}\\ &\overset{(2.6)}{\lesssim} [\Phi]_{H^{\alpha/2}(\mathbb{R}^N)} I_\alpha(g-\mu_B)^{1/2}.\end{aligned}$$

We finally show that

$$[\Phi]^2_{H^{\alpha/2}(\mathbb{R}^N)} \lesssim [\phi]^2_{H^{\alpha/2}(\partial B)} + \int_{\partial B}\phi^2. \tag{4.22}$$

For all x, y,

$$\begin{aligned}(\Phi_x-\Phi_y)^2 = (\phi_x\rho_x-\phi_y\rho_y)^2 &\lesssim (\phi_x-\phi_y)^2\rho_x^2 + \rho_y^2(\rho_x-\rho_y)^2\\ &\lesssim (\phi_x-\phi_y)^2 + \phi_y^2(x-y)^2,\end{aligned}$$

so that

$$\begin{aligned}[\Phi]^2_{H^{\alpha/2}(\mathbb{R}^N)} &\overset{(2.2)}{\lesssim} \int_{B_2\times B_2}\frac{(\Phi_x-\Phi_y)^2}{|x-y|^{N+\alpha}} \lesssim \int_{B_2\times B_2}\frac{(\phi_x-\phi_y)^2}{|x-y|^{N+\alpha}} + \int_{B_2\times B_2}\frac{\phi_y^2}{|x-y|^{N+\alpha-2}}\\ &\lesssim \int_{B_2\times B_2}\frac{(\phi_x-\phi_y)^2}{|x-y|^{N+\alpha}} + \int_{\partial B}\phi^2.\end{aligned}$$

Using polar coordinates we now write

$$\begin{aligned}&\int_{B_2\times B_2}\frac{(\phi_x-\phi_y)^2}{|x-y|^{N+\alpha}}\\ &= \int_{\partial B\times\partial B}(\phi(\sigma)-\phi(v))^2\left[\int_0^2\int_0^2 r^{N-1}s^{N-1}\frac{dr\,ds}{((r-s)^2+rs|\sigma-v|^2)^{(N+\alpha)/2}}\right]d\sigma\,dv.\end{aligned}$$

Arguing as for (4.15) we have

$$\int_0^2\int_0^2 r^{N-1}s^{N-1}\frac{dr\,ds}{((r-s)^2+rs|\sigma-v|^2)^{(N+\alpha)/2}} \lesssim \frac{1}{|\sigma-v|^{N-1+\alpha}},$$

which concludes the proof of (4.22). Recalling (4.2) we find

$$I_\alpha(\mu_B,\phi(g-\mu_B)) \lesssim_\varepsilon I_\alpha(g-\mu_B)^{1/2}[\phi]_{H^{\alpha/2}(\partial B)} + [\phi]^2_{H^{1/2+\varepsilon}(\partial B)}. \tag{4.23}$$

Plugging (4.21) and (4.23) into (4.20) we get

$$\begin{aligned}&\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E) + I_\alpha(g-\mu_B)\\ &\qquad\lesssim_\varepsilon I_\alpha(g-\mu_B)^{1/2}[\phi]_{H^{\alpha/2}(\partial B)} + [\phi]^2_{H^{(2-\alpha)/2}(\partial B)} + [\phi]^2_{H^{1/2+\varepsilon}(\partial B)}.\end{aligned}$$

Using the Young inequality we conclude the proof of (4.18).

Since E is nearly spherical we have[4] (see [16])

$$\int_{\partial B} |\nabla \phi|^2 \lesssim P(E) - P(B),$$

so that (4.19) follows using (2.4). ■

We can now conclude the proof of Theorem 1.5.

Proof of Theorem 1.5. Let E be a nearly spherical set with $\|\phi\|_{W^{1,\infty}(\partial B)} \ll 1$. If $\mathcal{F}_{\alpha,Q}(E) \le \mathcal{F}_{\alpha,Q}(B)$, then rearranging terms we find

$$P(E) - P(B) \le Q^2(\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E)) \overset{(4.19)}{\lesssim} Q^2(P(E) - P(B)).$$

This implies that either $P(E) = P(B)$ and thus $E = B$ by the isoperimetric inequality or $1 \lesssim Q^2$, which proves the claim. ■

5. Non-existence in dimension 2

We show here a non-existence result in dimension 2: in $N = 2$ minimizers in $\mathcal{S}$ (and hence classical minimizers) cannot exist for large Q.

Theorem 5.1. *Let $N = 2$ and $\alpha \in (0, 1]$. Then for $Q \gg 1$ there are no minimizers of*

$$\min\{\mathcal{F}_{\alpha,Q}(E) : |E| = \omega_N,\ E \in \mathcal{S}\}. \tag{5.1}$$

Proof. Let us point out that although the case $\alpha = 1$ is already covered by [31] (with an explicit threshold between existence and non-existence) we will still include it in the proof. We follow the ideas of [22, Theorem 3.3] in the streamlined version of [13]. For $\nu \in \partial B_1$ and $t \in \mathbb{R}$, we let

$$H^+_{\nu,t} = \{x \cdot \nu \ge t\}, \quad H^-_{\nu,t} = \{x \cdot \nu < t\}, \quad H_{\nu,t} = \{x \cdot \nu = t\}.$$

We then define, for any measure μ and set E,

$$\mu^\pm_{\nu,t} = \mu|_{H^\pm_{\nu,t}} \quad \text{and} \quad E^\pm_{\nu,t} = E \cap H^\pm_{\nu,t}.$$

Assume that E is a minimizer of (5.1). Comparing the energy of E with the one of two infinitely far apart copies of $E^\pm_{\nu,t}$ with measures $\mu^\pm_{\nu,t}$, we have

$$\mathcal{F}_{\alpha,Q}(E) \le P(E^+_{\nu,t}) + P(E^-_{\nu,t}) + Q^2 I_\alpha(\mu^+_{\nu,t}) + Q^2 I_\alpha(\mu^-_{\nu,t}).$$

Using $P(E) = P(E^+_{\nu,t}) + P(E^-_{\nu,t}) - 2\mathcal{H}^1(E \cap H_{\nu,t})$ and $I_\alpha(E) = I_\alpha(\mu^+_{\nu,t}) + I_\alpha(\mu^-_{\nu,t}) + 2I_\alpha(\mu^+_{\nu,t}, \mu^-_{\nu,t})$, this simplifies to

$$Q^2 I_\alpha(\mu^+_{\nu,t}, \mu^-_{\nu,t}) \le \mathcal{H}^1(E \cap H_{\nu,t}).$$

[4] This is the only place where we use the assumption that the barycenter of E is 0.

We now integrate this inequality in t and ν to get

$$\begin{aligned}|E| &\gtrsim \int_{\partial B_1}\int_{\mathbb{R}} \mathcal{H}^1(E\cap H_{\nu,t})\,d\nu\\ &\geq Q^2\int_{\partial B_1}\int_{\mathbb{R}} \mathcal{I}_\alpha(\mu^+_{\nu,t},\mu^-_{\nu,t})\,d\nu\\ &= Q^2\int_{\partial B_1}\int_{\mathbb{R}}\int_{H^+_{\nu,t}\times H^-_{\nu,t}} \frac{d\mu(x)\,d\mu(y)}{|x-y|^{2-\alpha}}\,d\nu\\ &\gtrsim Q^2\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{d\mu(x)\,d\mu(y)}{|x-y|^{1-\alpha}},\end{aligned}$$

where we have used the fact that for every (x,y),

$$\int_{\partial B_1}\int_{\mathbb{R}} \chi_{H^+_{\nu,t}\times H^-_{\nu,t}}(x,y)\,dt\,d\nu \sim |x-y|.$$

Since $|E| = \omega_N$, this yields the estimate

$$1 \gtrsim \frac{Q^2}{d^{1-\alpha}}$$

where $d = \operatorname{diam}(E)$. If $\alpha = 1$ this already gives the conclusion so that we are left with the case $\alpha < 1$. Since for $N = 2$, $P(E) \gtrsim d$, we get the lower bound

$$\mathcal{F}_{\alpha,Q}(E) \gtrsim Q^{\frac{2}{1-\alpha}}.$$

For a generalized set $\widetilde{E}_r$ made up of n copies of the ball of radius $r = n^{-1/2}$, we have

$$\mathcal{F}_{\alpha,Q}(\widetilde{E}_r) \lesssim nr + \frac{Q^2}{nr^{2-\alpha}} = r^{-1} + Q^2 r^\alpha.$$

Optimizing in r by choosing $r = Q^{-2/(1+\alpha)}$, we find, by minimality of E,

$$Q^{\frac{2}{1-\alpha}} \lesssim \mathcal{F}_{\alpha,Q}(E) \leq \mathcal{F}_{\alpha,Q}(\widetilde{E}_r) \lesssim Q^{\frac{2}{1+\alpha}},$$

which is absurd if $Q \gg 1$. ■

Remark 5.2. While we believe that the same result holds for $N \geq 3$, it is well known that this kind of argument gives useful information only when $\alpha > N-2$, which is compatible with $\alpha \leq 1$ only if $N = 2$.

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