

Filtrations on instanton homology

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Abstract. In earlier work of the authors, the Khovanov complex of a knot or link appeared as the first page in a spectral sequence abutting to the instanton homology. The quantum and (co)homological gradings on Khovanov homology do not survive as gradings, but we show that they survive as filtrations.

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1. Introduction

The Khovanov cohomology $\text{Kh}(K)$ of an oriented knot or link is defined in [3] as the cohomology of a cochain complex $(\mathbf{C} = \mathbf{C}(D), d_{\text{Kh}})$ associated to a plane diagram D for K :

$$\text{Kh}(K) = H(\mathbf{C}, d_{\text{Kh}}).$$

The free abelian group \mathbf{C} carries both a cohomological grading h and a quantum grading q . The differential d_{Kh} increases h by 1 and preserves q , so that the Khovanov cohomology is bigraded. We write $\mathcal{F}^{i,j} \mathbf{C}$ for the decreasing filtration defined by the bigrading, so $\mathcal{F}^{i,j} \mathbf{C}$ is generated by elements whose cohomological grading is not less than i and whose quantum grading is not less than j . In general, given abelian groups with a decreasing filtration indexed by $\mathbb{Z} \oplus \mathbb{Z}$, we will say that a group homomorphism ϕ has *order* $\geq (s, t)$ if $\phi(\mathcal{F}^{i,j}) \subset \mathcal{F}^{i+s, j+t}$. So d_{Kh} has order $\geq (1, 0)$.

In [5], a new invariant $I^\sharp(K)$ was defined using singular instantons, and it was shown that $I^\sharp(K)$ is related to $\text{Kh}(K^\dagger)$ through a spectral sequence. The notation K^\dagger here denotes the mirror image of K . Building on the results of [5], we establish the following theorem in this paper.

Theorem 1.1. *Given an oriented link K in \mathbb{R}^3 , and a diagram D^\dagger for K^\dagger , one can construct a differential d_\sharp on the Khovanov complex $\mathbf{C} = \mathbf{C}(D^\dagger)$ such that the homology of (\mathbf{C}, d_\sharp) is $I^\sharp(K)$. The differential d_\sharp is equal to $d_{\text{Kh}} = d_{\text{Kh}}(D^\dagger)$ to leading order in the q -filtration: that is both differentials have order $\geq (1, 0)$, and the difference $d_\sharp - d_{\text{Kh}}$ has order $\geq (1, 2)$.*

The differential d_\sharp depends (*a priori*) on more than just a choice of diagram for K . It depends also on choices of perturbations and metrics, required to make moduli spaces of instantons transverse. The fact that, with appropriate choices, the complex which computes I^\sharp can be made to coincide with \mathbf{C} as an abelian group is easily seen from the authors' earlier paper [5]. The new content in the above theorem is that $d_\sharp - d_{\text{Kh}}$ has order ≥ 2 with respect to the quantum filtration. (The quantum degree is of constant parity on the complex \mathbf{C} , so order ≥ 2 is inevitable once the leading-order parts agree.)

As a consequence of the above theorem, the filtrations by i and j lead to two spectral sequences abutting to $I^\sharp(K)$, whose E_2 and E_1 terms respectively are both isomorphic to $\text{Kh}(K^\dagger)$. Indeed, there is such a spectral sequence for every positive linear combination of i and j . The next theorem addresses the topological invariance of these spectral sequences. We write \mathcal{C} for the category whose objects are finitely-generated differential abelian groups filtered by $\mathbb{Z} \oplus \mathbb{Z}$ with differentials of order $\geq (1, 0)$, and whose morphisms are differential group homomorphisms of order $(0, 0)$ up to chain-homotopies of order $\geq (-1, 0)$.

Theorem 1.2. *In the category \mathfrak{C} , the isomorphism class of $(\mathbf{C}, d_{\#})$ depends only on the oriented link K .*

From the above theorem, it follows that the various pages of the resulting spectral sequences are invariants of K , as the next corollary states. (A similar result for a related spectral sequence [8] involving the Heegaard Floer homology of the branched double cover was established by Baldwin [1].)

Corollary 1.3. *For the homological and quantum filtrations by i and j , the isomorphism type of all the pages (E_r, d_r) , for $r \geq 2$ or $r \geq 1$ respectively, are invariants of K . More generally, for the filtration by $ai + bj$ with $a, b \geq 1$, the same is true when $r \geq a + 1$.*

Proof of the corollary. See for example [6], Chapter IX, Proposition 3.5, though the notation there uses increasing filtrations rather than the decreasing filtrations of this paper. \square

Corollary 1.4. *The homological and quantum filtrations i and j give rise to filtrations of the instanton homology $I^{\#}(K)$, as do the combinations $ai + bj$. These filtrations are invariants of K .* \square

Our results leave open a functoriality question for the pages (E_r, d_r) of the spectral sequences. For example, Theorem 1.2 does not imply that there is a functor to \mathfrak{C} from the category whose objects are oriented links and whose morphisms are isotopies. We do expect that a result of this sort is true however. The issue is similar to the ones that arise in proving that Khovanov cohomology is functorial [2].

We do know that the homology groups $I^{\#}(K)$ are functorial for cobordisms [5]. Thus, if K_1 and K_0 are oriented links and $S \subset [a, b] \times \mathbb{R}^3$ is a cobordism from K_1 to K_0 (which we allow to be non-orientable), then there is a map

$$I^{\#}(S) : I^{\#}(K_1) \longrightarrow I^{\#}(K_0)$$

which is well-defined up to an overall sign. The following proposition describes how the filtrations behave under such a map.

Proposition 1.5. *The map $I^{\#}(S) : I^{\#}(K_1) \rightarrow I^{\#}(K_0)$ resulting from a cobordism S is represented at the chain level by a map $\mathbf{C}_1 \rightarrow \mathbf{C}_0$ of order*

$$\geq \left(\frac{1}{2}(S \cdot S), \chi(S) + \frac{3}{2}(S \cdot S) \right).$$

Here the term $S \cdot S$ is the self-intersection number of the surface S defined with respect to a push-off which, at the two ends, has total linking number 0 with K_1 and K_0 respectively.

Note that the self-intersection number which appears in the proposition is zero if S is an oriented cobordism, and is always even. These results for the filtrations of I^\sharp should be compared to the corresponding statements for Khovanov homology, where it is known that an orientable cobordism S gives rise to a map that preserves the homological grading and maps elements of quantum degree j to elements of degree $j + \chi(S)$.

Remark. There is also a reduced version of the instanton homology, denoted by $I^\sharp(K)$ in [5], which is related to the reduced Khovanov homology $\text{Khr}(K^\dagger)$. Theorem 1.1 and Proposition 1.5 can be formulated for these reduced versions with no essential change to the wording.

The remainder of this paper is organized as follows. In Sections 2 through 8, we focus on the q -filtration. Section 2 introduces a quantum filtration on the instanton homology of an unlink. Section 3 reviews the “cube of resolutions” in the context of instanton homology, from [5], and this is used in the following section to extend the q -filtration to the case of a general link. Rather than working with traditional diagrams (plane projections) of links, we introduce the slightly more flexible notion of a *pseudo-diagram* (Definition 4.1). With one additional hypothesis on the pseudo-diagram, we prove that the differential on the cube complex preserves the q -filtration (Proposition 4.6). Sections 5 through 6 examine how the q -filtration behaves when a pseudo-diagram is altered by an isotopy or by adding or dropping crossings, and we can then treat Reidemeister moves by regarding a single Reidemeister move as a sequence of isotopies and add-drops of crossings. The h -filtration is somewhat simpler to deal with than the q -filtration, but follows the same outline: it is discussed in Section 9. The proofs of the main results are then given in Section 10. The final section contains some simple examples, to illustrate the use of pseudo-diagrams, as well as a more complicated example: the $(4, 5)$ -torus knot.

2. Unlinks

Let $K \subset \mathbb{R}^3$ be an oriented link that is isotopic to the standard p -component unlink U_p . According to [5], Proposition 8.11, we have an isomorphism

$$I^\sharp(K) \xrightarrow{\gamma} V^{\otimes p} \tag{1}$$

that depends only on the given orientation and a choice of ordering of the components of K . Here

$$V = \langle \mathbf{v}_+, \mathbf{v}_- \rangle$$

is a free abelian group on two generators. This isomorphism arises in [5] as a consequence of an excision property which is used to establish a Künneth product formula for I^\sharp of a split link $K_1 \amalg K_2$.

On the other hand, there is a more direct way to compute $I^\sharp(K)$ for an unlink K , working from the definition of instanton homology. We recall the definition in outline. From K , we form a new link $K^\sharp \subset S^3$, the union of K and a standard Hopf link near infinity, and we equip S^3 with an orbifold metric with cone-angle π along K^\sharp . We let ω denote an arc joining the two components of the Hopf link, and we form the configuration space $\mathcal{C}^\omega(S^3, K^\sharp)$ consisting of singular $\mathrm{SO}(3)$ connections on the complement of K^\sharp , with w_2 equal to the dual of ω and with holonomy asymptotically conjugate to the element

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

on small circles linking K^\sharp . We form $\mathcal{B}^\omega(S^3, K^\sharp)$ as the quotient of $\mathcal{C}^\omega(S^3, K^\sharp)$ by the determinant-1 gauge group. In $\mathcal{B}^\omega(S^3, K^\sharp)$ we consider the critical points of the perturbed Chern–Simons functional $\mathrm{CS} + f$, where f is a holonomy perturbation chosen to achieve a Morse–Smale transversality condition for the formal gradient flow. For the *unperturbed* Chern–Simons functional, the set of critical points is the space of flat connections in $\mathcal{B}^\omega(S^3, K^\sharp)$, and these can be identified with the representation variety

$$\mathcal{R}(K, \mathbf{i})$$

consisting of homomorphisms $\rho: \pi_1(\mathbb{R}^3 \setminus K) \rightarrow \mathrm{SU}(2)$ with $\rho(m) \sim \mathbf{i}$, for all meridians m of K .

In the case of an unlink, the fundamental group of $\mathbb{R}^3 \setminus K$ is free on p generators: we can specify generators by giving explicit choices of meridians, oriented consistently with the given orientation of K . After making these choices, we have an identification

$$\mathcal{R}(K, \mathbf{i}) = (S^2)^p,$$

(where the 2-sphere is the conjugacy class of \mathbf{i} in $\mathrm{SU}(2)$). This representation variety sits in $\mathcal{B}^\omega(S^3, K^\sharp)$ as a Morse–Bott critical set for CS . A product of 2-spheres carries an obvious Morse function with critical points only in even index. By choosing f above so that its restriction to $\mathcal{R}(K, \mathbf{i})$ is equal to such a Morse function, we can arrange that the critical points of $\mathrm{CS} + f$ consist of exactly 2^p points, all in the same index mod 2. The differential in the instanton homology is then zero, and $I^\sharp(K)$ is the free abelian group generated by these critical points. Thus we obtain an isomorphism,

$$I^\sharp(K) \xrightarrow{\beta} H_*(S^2)^{\otimes p} \tag{2}$$

as the composite

$$\begin{aligned} I^\sharp(K) &= H_*(\mathcal{R}(K, \mathbf{i})) \\ &= H_*((S^2)^p) \\ &= H_*(S^2)^{\otimes p}. \end{aligned}$$

We now have two different ways to identify $I^\sharp(K)$ with the tensor product of p copies of a free abelian group of rank 2, through the isomorphisms γ and β of

equations (1) and (2) respectively. Combining the first isomorphism with inverse of the second, we have a map

$$\epsilon = \gamma \circ \beta^{-1}: H_*(S^2)^{\otimes p} \longrightarrow V^{\otimes p}. \quad (3)$$

Using the $\mathbb{Z}/4$ grading on instanton homology (for example) it is easy to see that, in the case $p = 1$, this map is the isomorphism

$$H_*(S^2) \longrightarrow V$$

that sends the 2-dimensional generator to \mathbf{v}_+ and the 0-dimensional generator to \mathbf{v}_- (given appropriate conventions about orientations, to fix the signs). For larger p , it does not follow that this map is simply the p 'th tensor power of the isomorphism from the $p = 1$ case. (The map potentially involves instantons on the cobordisms that are used in the proof of the excision property.) Indeed, ϵ will in general depend on the choice of metric and perturbation.

What we *can* say about ϵ is this. Make V a graded abelian group by putting \mathbf{v}_+ and \mathbf{v}_- in degrees 1 and -1 respectively, and give $V^{\otimes p}$ the tensor-product grading. Similarly, grade $H_*(S^2)$ so that the 2-dimensional generator is in degree 1 and the 0-dimensional generator is in degree -1 and grade $H_*(S^2)^{\otimes p}$ accordingly. We refer to these gradings on both sides as the Q -grading. In the case of $H_*(S^2)^{\otimes p}$, this grading is the ordinary homological grading on the manifold $\mathcal{R}(K, \mathbf{i})$, shifted down by p . The isomorphism ϵ in (3) preserves the Q -grading modulo 4 (essentially because the instanton homology is $\mathbb{Z}/4$ graded). Furthermore, we can write it as

$$\epsilon = \epsilon_0 + \epsilon_1 + \cdots \quad (4)$$

where ϵ_0 preserves the Q -grading and ϵ_i increases the Q -grading by $4i$. The term ϵ_0 can be computed by looking only at flat connections on the excision cobordism, and it is not hard to see that ϵ_0 is indeed the p 'th tensor power of the standard map. The terms ϵ_i for i positive arise from instantons with non-zero energy.

Our conclusion is that the map ϵ respects the decreasing filtration defined by the Q -gradings on the two sides, and that the induced map on the associated graded objects of these two filtrations is standard.

Remark. In the authors' earlier paper [5], the group $V = \langle \mathbf{v}_+, \mathbf{v}_- \rangle$ appears with a mod-4 grading in which \mathbf{v}_+ and \mathbf{v}_- have degrees 0 and $-2 \pmod{4}$ respectively. The mod 4 grading in [5] is not the same as the grading that we are considering here.

Now let S be a cobordism (not necessarily orientable) from an unlink K_1 to another unlink K_0 . The cobordism S (when equipped with an I -orientation, to fix the overall sign) induces a map

$$I^\sharp(S): I^\sharp(K_1) \longrightarrow I^\sharp(K_0),$$

or equivalently

$$I^\sharp(S): V^{\otimes b_0(K_1)} \longrightarrow V^{\otimes b_0(K_0)}.$$

The Q -grading on $V^{\otimes b_0(K_1)}$ and $V^{\otimes b_0(K_0)}$ defines a decreasing filtration on each of them. We wish to see what the effect of $I^\sharp(S)$ is on this Q -filtration.

Lemma 2.1. *For a cobordism S as above, the induced map $I^\sharp(S)$ has order greater than or equal to*

$$\chi(S) + S \cdot S - 4 \left\lfloor \frac{S \cdot S}{8} \right\rfloor \quad (5)$$

with respect to the filtration defined by Q .

Proof. Choose small perturbations f_1 and f_0 for the Chern–Simons functional on the two ends to achieve the Morse–Smale condition; choose them so that all the critical points have even index, as in the discussion above. Let β_1 and β_0 be critical points for K_1^\sharp and K_0^\sharp and let $M(S; \beta_1, \beta_0)$ be the corresponding moduli space. The map I^\sharp is defined by counting points in zero-dimensional moduli spaces of this sort; but we consider first the dimension formula in general. For each $[A] \in M(S; \beta_1, \beta_0)$ we can find a nearby configuration $[A']$ which is asymptotic to points β'_1 and β'_0 in the critical set of the *unperturbed* functional. Let us write $\kappa = \kappa(A)$ for the *topological action* of the solution A , by which we mean the integral

$$\frac{1}{8\pi^2} \int \text{tr}(F_{A'} \wedge F_{A'}).$$

This quantity is a homotopy invariant of A , independent of the choice of nearby path A' . We write $M(S; \beta_1, \beta_0)$ as a union of parts $M_\kappa(S; \beta_1, \beta_0)$ of different actions κ .

We claim that the dimension of $M_\kappa(S; \beta_1, \beta_0)$ is given by the formula

$$\dim M_\kappa(S; \beta_1, \beta_0) = 8\kappa + \chi(S) + \frac{1}{2}(S \cdot S) + Q(\beta_1) - Q(\beta_0). \quad (6)$$

To verify this, note first that if we change β_1 to a different β'_1 while keeping κ and β_0 unchanged, then the change in $\dim M$ is equal to the change in the Morse index of the critical points in the representation variety, which is $Q(\beta_1) - Q(\beta'_1)$. A similar remark applies to β_0 , with an opposite sign. So it is enough to check the formula for one particular choice of β_1 and β_0 . So we take β_1 to be the critical point of top Morse index, corresponding to the generator $\mathbf{v}_+ \otimes \cdots \otimes \mathbf{v}_+$, and β_0 to be the critical point of lowest Morse index, corresponding to the generator $\mathbf{v}_- \otimes \cdots \otimes \mathbf{v}_-$. So $Q(\beta_1) = b_0(K_1)$ and $Q(\beta_0) = -b_0(K_0)$. In this particular case, the dimension of $M_\kappa(S; \beta_1, \beta_0)$ is equal to the dimension of

$$M_\kappa(\bar{S}; u_0, u_0)$$

where \bar{S} is the closed surface obtained from S by adding disks to all boundary components, regarded as a cobordism from the empty link U_0 to itself, and u_0 is the unique critical point in $\mathcal{B}^\omega(S^3, U_0^\#)$ (the generator of $I^\#(U_0) = \mathbb{Z}$). The dimension in this case can be read off from the dimension formula for the case of a closed manifold (see [5], Lemma 2.11) which gives

$$\dim M_\kappa(\bar{S}; u_0, u_0) = 8\kappa + \chi(\bar{S}) + \frac{1}{2}(\bar{S} \cdot \bar{S}). \quad (7)$$

Taking account of the added disks, we can write this as

$$8\kappa + \chi(S) + \frac{1}{2}(S \cdot S) + b_0(K_1) + b_0(K_0), \quad (8)$$

which coincides with the formula (6) in this case. This verifies the formula (6) for the general case.

To continue with the proof of the lemma, we make two observations about the action $\kappa(A)$:

- (1) $\kappa(A) = \frac{1}{16}(S \cdot S)$ modulo $\frac{1}{2}\mathbb{Z}$ and
- (2) $\kappa(A)$ is non-negative, as long as the perturbations are small.

The first of these can be read off from the formula in Proposition 2.7 of [5], applied to the closed surface \bar{S} in \mathbb{R}^4 , using the fact that $p_1(P_\Delta)$ is divisible by 4 when P_Δ is an $SU(2)$ bundle. The second of these observations follows from the non-negativity of the action for solutions of the unperturbed equations. Together, these observations tell us that

$$8\kappa \geq \frac{1}{2}(S \cdot S) - 4 \left\lfloor \frac{S \cdot S}{8} \right\rfloor.$$

The matrix entries of $I^\#(S)$ arise from moduli spaces of dimension zero; and for such moduli spaces we have

$$\begin{aligned} Q(\beta_0) - Q(\beta_1) &= 8\kappa + \chi(S) + \frac{1}{2}(S \cdot S) \\ &\geq \chi(S) + S \cdot S - 4 \left\lfloor \frac{S \cdot S}{8} \right\rfloor \end{aligned}$$

because of the above inequality for κ . This last quantity is the expression (5) in the lemma. \square

The lemma above is rather artificial (the maps involved are often zero in any case), but the method of proof adapts with essentially no change to yield a more applicable version. We take K_1 , K_0 , and S as above, and we consider a smooth, finite-dimensional family of metrics and perturbations on the cobordism, parametrized by manifold G . We then have parametrized moduli spaces $M(S, \beta_1, \beta_0)_G$ over the space G . Counting isolated points in these parametrized moduli spaces gives rise to maps

$$m_G(S): V^{\otimes b_0(K_1)} \longrightarrow V^{\otimes b_0(K_0)}.$$

Just as in the case above where G is a point, we obtain an inequality for Q -grading:

Lemma 2.2. *The map $m_G(S)$ has order greater than or equal to*

$$\chi(S) + S \cdot S - 4 \left\lfloor \frac{S \cdot S}{8} \right\rfloor + \dim G$$

with respect to the decreasing filtration defined by Q .

Proof. The proof is unchanged, except that the formula for the dimension of the moduli space has an extra term $\dim G$. \square

Corollary 2.3. *If $S \cdot S < 8$, then $m_G(S)$ has order greater than or equal to*

$$\chi(S) + S \cdot S + \dim G.$$

3. The cube

Let K be a link in \mathbb{R}^3 . Figure 1 shows three copies of the standard closed ball B^3 , each containing a pair of arcs: L_0 , L_1 and L_2 respectively. By a *crossing* of K we will mean an embedding of pairs

$$c : (B^3, L_2) \hookrightarrow (\mathbb{R}^3, K)$$

which is orientation-preserving on B^3 and satisfies $c(L_2) = c(B^3) \cap K$.

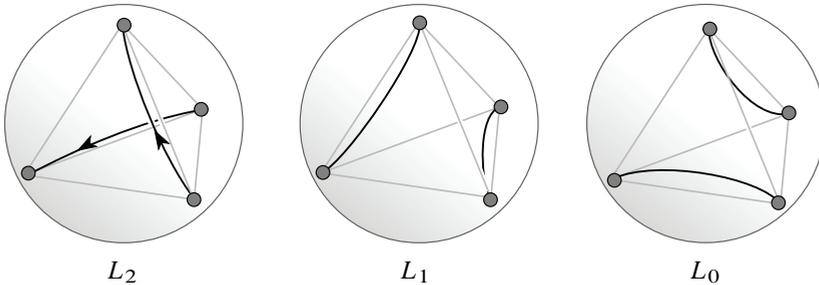


Figure 1. A closed 3-ball, containing a pair of arcs in three different ways.

The figure also shows a standard orientation for the pair of arcs $L_2 \subset B^3$. If the link K is also given an orientation, then we will say that c is a *positive crossing* if $c : L_2 \rightarrow K$ is either orientation-preserving or orientation-reversing on both components of L_2 . Otherwise, if c is orientation-reversing on exactly one component of L_2 , we say that c is a *negative crossing*.

Let N be a finite set of disjoint crossings for K . For each $v \in \mathbb{Z}^N$, let $K_v \subset \mathbb{R}^3$ be the link obtained from K by replacing $c(L_2) \subset K$ by either $c(L_0)$, $c(L_1)$ or $c(L_2)$,

according to the value of $v(c) \bmod 3$, for each crossing $c \in N$. Thus $K_v = K$ in the case that $v: N \rightarrow \mathbb{Z}$ is the constant 2.

Following the prescription of [5], we choose generic metrics and holonomy perturbations for each link $K_v^\sharp \subset S^3$ so as to achieve the Morse–Smale condition. In order to fix signs for the maps arising later from cobordisms, we also need to choose for each v a basepoint in $\mathcal{B}^\omega(S^3, K_v^\sharp)$. We refer to the choice of metric, perturbation and basepoint as the *auxiliary data* for K_v . We then have a complex

$$(C_v, d_v)$$

that computes the instanton homology $I^\sharp(K_v)$.

For each $v \geq u$ we have a standard cobordism S_{vu} from K_v to K_u , as in [5], Sections 6.1 and 7.2. This cobordism comes with a family of metrics G'_{vu} defined in [5], Section 7.2. The dimension of G'_{vu} is $|v - u|_1$ (the sum of the coefficients of $v - u$) and it is acted on by a 1-dimensional group of translations. The quotient family $\check{G}'_{vu} = G'_{vu}/\mathbb{R}$ has dimension $|v - u|_1 - 1$ if $v \neq u$. The norm $|v - u|_1$ is also equal to $-\chi(S_{vu})$.

Definition 3.1. We say that a cobordism S_{vu} (or sometimes, a pair (v, u)) with $v, u \in \mathbb{Z}^N$ is of *type* n for $n \geq 0$ if $v \geq u$ and

$$\max\{v(c) - u(c) \mid c \in N\} = n.$$

In particular, (v, u) has *type* 0 if and only if $v = u$.

In the case that S_{vu} has type 1, 2, or 3, the authors defined in [5] a larger family of metrics, \check{G}_{vu} containing \check{G}'_{vu} . In the case of type 1, the space \check{G}_{vu} coincided with \check{G}'_{vu} ; for type 2 and $|N| = 1$, the inclusion of \check{G}'_{vu} in \check{G}_{vu} was the inclusion of a half-line in \mathbb{R} , and for type 3 it was the inclusion of a “quadrant” in an open pentagon. In all cases, the dimensions of \check{G}_{vu} and \check{G}'_{vu} are equal.

If we choose an I -orientation for each cobordism S_{vu} and an orientation for the family of metrics \check{G}'_{vu} (or equivalently the family \check{G}_{vu} , when defined), then we have oriented, parametrized moduli spaces of instantons,

$$M_{vu}(\beta, \alpha) \longrightarrow \check{G}_{vu}$$

for each pair of critical points $\beta \in \mathfrak{C}_v$ and $\alpha \in \mathfrak{C}_u$, whenever the pair (v, u) has type 3 or less. Consistency conditions are imposed on the chosen orientations: see for example [5], Lemmas 6.1 and 6.2. In addition to the auxiliary data for each, secondary perturbations on the cobordisms must be chosen, to ensure that the moduli spaces are regular. By counting points in zero-dimensional parametrized moduli spaces, we obtain maps between the corresponding groups C_v and C_u . Following the notation of [5] we write these maps as

$$\check{m}_{vu}: C_v \longrightarrow C_u.$$

The orientation conventions which are specified in [5] lead to extra signs in the various gluing formulae, so it is convenient to introduce the following variant: we define

$$f_{vu}: C_v \longrightarrow C_u$$

by the formula

$$f_{vu} = (-1)^{s(v,u)} \check{m}_{vu}$$

where

$$s(v, u) = \frac{1}{2}|v - u|_1(|v - u|_1 - 1) + \sum_{c \in N} v(c). \quad (9)$$

(In [5] the notation f_{vu} was reserved for the case of type 0 or 1, and the notation j_{vu} or k_{vu} was used for type 2 or 3. For efficiency however, we here adopt f_{vu} for all these cases.) It is also convenient to define $\check{m}_{vv} = d_v$ for the case that $v = u$ – i.e. the case of type 0 – so that

$$f_{vv} = (-1)^{\sum v(c)} d_v.$$

Some chain-homotopy formulae involving these maps are proved in [5]. For (v, u) a pair of type 0, 1, or 2, the formulae all take the same basic form, given in the following proposition.

Proposition 3.2. *For (v, u) of type $n \leq 2$, we have*

$$\sum_{\substack{w \\ v \geq w \geq u}} f_{wu} f_{vw} = 0.$$

(There is also a formula in [5] for the case of type 3, but this involves additional terms; see (22) in Section 6.) In the case of type 0, so that $v = u$, the formula in the proposition says $d_v^2 = 0$, expressing the fact that d_v is a differential.

We write

$$\mathbf{C}(N) = \bigoplus_{v: N \rightarrow \{0,1\}} C_v.$$

This is a sum of the complexes indexed by the vertices of a cube of dimension $|N|$. We write $\mathbf{F} = \mathbf{F}(N)$ for the map

$$\mathbf{F}: \mathbf{C}(N) \longrightarrow \mathbf{C}(N)$$

given by

$$\mathbf{F} = \bigoplus_{\substack{v,u: N \rightarrow \{0,1\} \\ v \geq u}} f_{vu}.$$

Note that the summands f_{vu} in this definition all have type 0 or 1. Proposition 3.2 tells us that $\mathbf{F}^2 = 0$, so we have a complex.

We have had to choose auxiliary data for each K_v , secondary perturbations for the moduli spaces associated to the cobordisms S_{vu} , as well as consistent I -orientations for the cobordisms and orientations for the families of metrics \check{G}'_{vu} . We refer to this collection of choices as *auxiliary data for (K, N)* .

The following is proved in [5].

Theorem 3.3. *For any two collections of crossings, N and N' , and any corresponding choices of auxiliary data, the complexes $(\mathbf{C}(N), \mathbf{F}(N))$ and $(\mathbf{C}(N'), \mathbf{F}(N'))$ are quasi-isomorphic.*

Of course, it is sufficient to deal with the case that N' is obtained from N by forgetting just one crossing; and this is how the proof is given in [5] (see Proposition 6.11 of that paper). We will later refine this theorem, replacing “quasi-isomorphic” with “chain-homotopy equivalent.” As a special case we can take N' to be empty, and we obtain the following result.

Corollary 3.4 ([5], Theorem 6.8). *For any collection N of crossings of K , the homology of the complex $(\mathbf{C}(N), \mathbf{F}(N))$ is isomorphic to $I^\sharp(K)$.*

4. The q -filtration on cubes

We continue to consider $K \subset \mathbb{R}^3$ with a collection N of crossings, and the complex $(\mathbf{C}(N), \mathbf{F}(N))$ defined in the previous subsection. We will suppose that the collection N has the following property.

Definition 4.1. We will say that a link K with a collection N of crossings is a *pseudo-diagram* if, for all $v: N \rightarrow \{0, 1\}$, the link $K_v \subset \mathbb{R}^3$ is an unlink. In this case, we refer to the unlinks K_v as the *resolutions* of K .

As in Section 2, whenever K_v is an unlink, we can choose the auxiliary data so that the corresponding differential d_v is zero, in which case C_v can be identified with the homology of the representation variety, $\mathcal{R}(K_v, \mathbf{i}) = (S^2)^{p(v)}$ by the isomorphism β of (2). When this is done, we say that we have chosen *good auxiliary data* for (K, N) .

The terminology in Definition 4.1 is chosen because the condition holds when N is the set of crossings that arises from a plane diagram of K , but the case of a plane diagram is special in other ways: for example, the cobordisms S_{vu} , for $v, u: N \rightarrow \{0, 1\}$, are always orientable if N arises from a diagram, whereas Definition 4.1 certainly allows some S_{vu} to be non-orientable. In particular, the self-intersection numbers $S_{vu} \cdot S_{vu}$ may be non-zero. For $v \geq u$ we define

$$\sigma(v, u) = S_{vu} \cdot S_{vu}.$$

In the case that $w \geq v \geq u$, we have $\sigma(w, v) + \sigma(v, u) = \sigma(w, u)$, so we can consistently define $\sigma(v, u)$ even when we do *not* have $v \geq u$ by insisting on the additivity property. Thus, for example, $\sigma(v, u) = -\sigma(u, v)$. We can extend this notation beyond the cube $\{0, 1\}^N$ to all elements $v \in \mathbb{Z}^N$ with the property that K_v is an unlink. Thus, if v, u both have this property we may consistently define

$$\sigma(v, u) = S_{vw} \cdot S_{vw} - S_{uw} \cdot S_{uw}$$

where w is any chosen third point with $v \geq w$ and $u \geq w$.

Lemma 4.2. *Suppose $v \in \mathbb{Z}^N$ is such that K_v is an unlink, and suppose that $v - u$ is divisible by 3, so that $K_u \cong K_v$ is also an unlink. Then*

$$\sigma(v, u) = \frac{2}{3} \sum_c (v(c) - u(c)).$$

Proof. It is enough to consider only the case that v and u differ at only a single crossing c_* , with $v(c_*) - u(c_*) = 3$. In this case, the cobordism S_{vu} is a composite of three cobordisms, $S_{vv'}$, $S_{v'v''}$, and $S_{v''u}$, with $v'(c_*) = v(c) - 1$ and $v''(c_*) = v(c) - 2$. As explained in [5], the cobordism $S_{vv''}$ (the composite of the first two) can be described as a connected sum of the opposite of $S_{v''u}$ with standard pair (S^4, \mathbb{RP}^2) , where the \mathbb{RP}^2 has self-intersection 2 in S^4 . So S_{vu} is obtained from the composite of $S_{v''u}$ with its opposite, by summing with this \mathbb{RP}^2 . So $S_{vu} \cdot S_{vu} = 2$. Thus $\sigma(v, u) = (2/3)(v(c_*) - u(c_*))$ in this case. \square

Suppose now that (K, N) is a pseudo-diagram, and let us choose good auxiliary data. As in Section 2 we obtain an isomorphism

$$\beta: I^\sharp(K_v) \longrightarrow V^{\otimes b_0(K_v)}$$

via the identifications

$$\begin{aligned} I^\sharp(K_v) &= C_v \\ &\cong H_2(S^2)^{\otimes b_0(K_v)} \\ &= V^{\otimes b_0(K_v)} \end{aligned} \tag{10}$$

because the differential d_v is zero. As before, we give $V^{\otimes b_0(K_v)}$ a grading Q , by declaring that the generators \mathbf{v}_+ and \mathbf{v}_- in V have Q -grading 1 and -1 .

We define the q -grading on C_v by shifting the Q -grading by some correction terms. Choose first an orientation for K . At each crossing $c \in N$, one of the resolutions 0 or 1 is preferred as the *oriented* resolution. We write $o: N \rightarrow \{0, 1\}$ for the function that assigns to each crossing its oriented resolution. Thus K_o can be oriented in such a way that its orientation agrees with the original orientation of K outside the crossing-balls. For $v: N \rightarrow \{0, 1\}$ we then set

$$q = Q - \left(\sum_{c \in N} v(c) \right) + \frac{3}{2} \sigma(v, o) - n_+ + 2n_- \tag{11}$$

on C_v , where n_+ and n_- are the number of positive and negative crossings respectively, so that

$$n_+ + n_- = |N|.$$

With the exception of the self-intersection term $\sigma(v, o)$ (which is zero in the case arising from a plane diagram), these correction terms are essentially the same as those presented by Khovanov in [3]. The q -gradings on all the vertices of the cube gives us a grading q on $\mathbf{C}(N)$. Note that the correction terms in the formula above do not depend on a choice of orientation for K if K is a knot rather than a link.

We can also define q on C_v when v more generally is in \mathbb{Z}^N rather than $\{0, 1\}^N$ subject to the constraint that K_v is an unlink: we use the same formula. Then we have what follows.

Lemma 4.3. *Suppose v is such that K_v is an unlink, and let u be such that $v - u$ is divisible by 3 in \mathbb{Z}^N , so that $K_v = K_u$. Identify C_v with C_u as abelian groups, via the isomorphisms β ; see (10). Then the q -gradings on C_v and C_u coincide.*

Proof. In the definition of q , the terms Q , n_+ and n_- are unchanged on replacing v by u . The remaining terms are $-\sum v(c)$ and $(3/2)\sum\sigma(v, o)$, and the changes in these terms cancel, as an immediate consequence of Lemma 4.2. \square

We also note the following result.

Lemma 4.4. *The parity of q on $\mathbf{C}(N)$ is constant, and is equal to the number of components of K mod 2.*

Proof. At each $v \in \{0, 1\}^N$, the parity of Q on C_v is equal to the number of components of K_v . (This follows immediately from the definition.) So it is clear that the parity of q is constant on each C_v . We must check that its parity is independent of v . For this we consider two adjacent vertices v, v' in the cube $\{0, 1\}^N$. The term $\sum v(c)$ which appears in the definition of q then changes by 1 between v and v' . The change in the term $(3/2)\sigma(v, o)$ is equal to $(1/2)\sigma(v, v')$ mod 2, which is zero if the cobordism $S_{vv'}$ is orientable and is equal to 1 if it is non-orientable, because the self-intersection number of an $\mathbb{R}P^2$ in \mathbb{R}^4 is equal to 2 mod 4. In the orientable case, the number of components of K_v and $K_{v'}$ differ by 1, so the parity of Q changes by 1. In the non-orientable case, the parity of Q is unchanged. Altogether, exactly two of the first three terms in the definition of q change parity. So the parity of q is indeed constant.

Now let K_o denote the oriented resolution of our original knot K . To obtain the oriented resolution, we must set $o(c) = 0$ at every positive crossing and $o(c) = 1$ at every negative crossing. So when $v = o$, we have

$$\sum_{v \in N} v(c) = n_-.$$

At this vertex of the cube, we therefore have

$$\begin{aligned} q &= b_0(K_o) - n_- - n_+ \\ &= b_0(K_o) - |N| \end{aligned}$$

mod 2. The cobordism from K to K_o is orientable, and is obtained by attaching $|N|$ 1-handles. As above, each 1-handle addition changes the number of components by 1. So $b_0(K_o) - |N| = b_0(K) \pmod{2}$. This concludes the proof of the lemma. \square

Although we can define the q -grading on $\mathbf{C}(N)$ whenever (K, N) is a pseudo-diagram, it is not the case (*a priori*, at least) that the differential $\mathbf{F}(N): \mathbf{C}(N) \rightarrow \mathbf{C}(N)$ respects the decreasing filtration defined by q . For this, we need an additional condition.

Definition 4.5. We say that (K, N) has *small self-intersection numbers* if $S_{vu} \cdot S_{vu} \leq 6$ for all $v \geq u$ in $\{0, 1\}^N$.

Proposition 4.6. *If (K, N) is a pseudo-diagram with small self-intersection numbers, then the differential $\mathbf{F}(N): \mathbf{C}(N) \rightarrow \mathbf{C}(N)$ has order ≥ 0 with respect to the decreasing filtration defined by q .*

Proof. The map $\mathbf{F} = \mathbf{F}(N)$ is the sum of the maps f_{vu} , each of which is obtained by counting instantons on a cobordism S_{vu} over a family of metrics of dimension $-\chi(S_{vu}) - 1$. Because the self-intersection number of S_{vu} is at most 6, we can apply Corollary 2.3 to the map f_{vu} . That corollary tells us that, with respect to the decreasing filtration defined by Q on C_v and C_u , the map f_{vu} has order $\geq S_{vu} \cdot S_{vu} - 1$. If we instead consider the decreasing filtration \mathcal{F}^j defined by q instead of Q , then we obtain

$$\text{ord}_q f_{vu} \geq -\frac{1}{2}(S_{vu} \cdot S_{vu}) + \sum_c (v(c) - u(c)) - 1. \quad (12)$$

Now we need the following lemma.

Lemma 4.7. *If (K, N) is a pseudo-diagram, we have*

$$(S_{vu} \cdot S_{vu}) \leq 2 \sum_c (v(c) - u(c)),$$

for all pairs $v \geq u$ in $\{0, 1\}^N$.

Proof. This can immediately be reduced to the case that v and u differ at only one crossing c . In that case, S_{vu} is the union of some cylinders (corresponding to components of K_v that do not pass through the crossing) and a single non-trivial piece that is either a pair of pants or a twice-punctured $\mathbb{R}P^2$. The links K_v and K_u are trivial by hypothesis, and the self-intersection number $S_{vu} \cdot S_{vu}$ is equal to that self-intersection

of the closed surface obtained from the cobordism by adding disks. Thus the self-intersection number is equal to that of either a surface that is either a union of spheres only, or a union of spheres with a single $\mathbb{R}P^2$. In the former case, the self-intersection is zero. In the latter case, the self-intersection of an $\mathbb{R}P^2$ in \mathbb{R}^4 is equal to either 2 or -2 ; see [7]. In all cases, the self-intersection number is no larger than 2. \square

To continue with the proof of the Proposition, it follows from the lemma now that the right-hand side of (12) is ≥ -1 , and so $f_{vu}(\mathcal{F}^j) \subset \mathcal{F}^{j-1}$ for all j . However, since the q -grading takes values of only one parity, it is in fact the case that $f_{vu}(\mathcal{F}^j) \subset \mathcal{F}^j$ for all j . \square

5. Isotopy and ordering

Given a link K with a set of crossings N and choices of auxiliary data, we have constructed in the previous section a q -filtered complex $\mathbf{C}(K, N)$ with its differential $\mathbf{F}(K, N)$. We now begin the task of investigating to what extent this filtered complex is dependent on the choices made. For reference, let us introduce the category \mathfrak{C}_q whose objects are filtered, finitely-generated abelian groups with a differential of order ≥ 0 , and whose morphisms are chain maps of order ≥ 0 modulo chain-homotopies of order ≥ 0 . (As elsewhere in this paper we continue to refer to a differential group as a “chain complex” even when there is no \mathbb{Z} -grading. All our differential groups will be at least $\mathbb{Z}/4$ -graded.) Our complex $\mathbf{C}(K, N)$ is an element of \mathfrak{C}_q .

We consider what happens when we keep the crossings the same but change K by an isotopy and change the choice of auxiliary data. That is, we fix a set of crossings N for K , and we suppose that K' is isotopic to K by an isotopy that is constant inside the union

$$B(N) = \bigcup_{c \in N} c(B^3).$$

Thus the trace of this isotopy is a cobordism T from K to K' that is topologically a cylinder in $\mathbb{R} \times \mathbb{R}^3$ and is metrically a cylinder inside $\mathbb{R} \times B(N)$. We choose auxiliary data for both (K, N) and (K', N) , giving rise to chain complexes

$$\mathbf{C}(K, N) \quad \text{and} \quad \mathbf{C}(K', N)$$

constructed from the cubes of resolutions.

From the cobordism T we obtain cobordisms T_{vv} from K_v to K'_v for all $v: N \rightarrow \{0, 1\}$. For $v \geq u$, we also obtain a cobordism T_{vu} from K_v to K'_u equipped with a family of metrics H_{vu} of dimension $\sum(v(c) - u(c))$: these cobordisms and metrics are all the same outside $B(N)$, while inside $B(N)$ they coincide with the metrics G_{vu} with which the cobordisms S_{vu} were earlier equipped. By counting instantons over the cobordisms T_{vu} equipped with these metrics and generic secondary perturbations, we obtain a chain map of the cube complexes,

$$\mathbf{T}: \mathbf{C}(K, N) \longrightarrow \mathbf{C}(K', N).$$

By the usual sorts of arguments, two different choices of metrics and secondary perturbations on the interior of the cobordism give rise to chain maps \mathbf{T} that differ by chain homotopy, and it follows that $\mathbf{C}(K, N)$ and $\mathbf{C}(K', N)$ are chain-homotopy equivalent.

Now we introduce q -gradings. For this we suppose that (K, N) is a pseudo-diagram with small self-intersection numbers, as in Definition 4.5. Of course, (K', N) then shares these properties. For (K, N) and (K', N) we ensure that our chosen auxiliary data is good. The complexes $\mathbf{C}(K, N)$ and $\mathbf{C}(K', N)$ both then have q -filtrations preserved by the differential (Proposition 4.6). Arguing as in the proof of Proposition 4.6, we see that the chain map \mathbf{T} is filtration-preserving, as are the chain-homotopies between different chain maps \mathbf{T} and $\tilde{\mathbf{T}}$ arising from different choices of metrics and secondary perturbations on the cobordisms. (The dimensions of the families of metrics on T_{vu} are larger by 1 than those on S_{vu} , but this only helps us. The proofs are otherwise the same.) Thus we get the following result.

Proposition 5.1. *Let (K, N) be a pseudo-diagram with small self-intersection numbers, and suppose we have an isotopy from K to K' relative to $B(N)$. Let $\mathbf{C}(K, N)$, $\mathbf{C}(K', N)$ be the q -filtered complexes obtained from (K, N) and (K', N) via choices of good auxiliary data. Then the isotopy gives rise to a well-defined isomorphism $\mathbf{C}(K, N) \rightarrow \mathbf{C}(K', N)$ in the category \mathfrak{C}_q .*

Remark. As usual in Floer theory, a special case of the above proposition is the case that $K = K'$ and the isotopy is trivial, in which case the objects $\mathbf{C}(K, N)$ and $\mathbf{C}(K', N)$ differ only in that the auxiliary data may be chosen differently.

We now have a diagram of maps on homology groups:

$$\begin{array}{ccc} H(\mathbf{C}(K, N)) & \xrightarrow{\mathbf{T}_*} & H(\mathbf{C}(K', N)) \\ \downarrow & & \downarrow \\ I^\#(K) & \xrightarrow{I^\#(T)} & I^\#(K'). \end{array} \quad (13)$$

The maps $I^\#(T)$ is the isomorphism induced by a cylindrical cobordism in the usual way, and \mathbf{T}_* is induced by the chain map \mathbf{T} . The vertical maps are the isomorphisms of Corollary 3.4. It is a straightforward application of the usual ideas, to show that this diagram commutes. To do this, we remember that the isomorphisms of Corollary 3.4 are obtained as a composite of maps, forgetting the crossings of N one by one. Thus one should only check the commutativity of a diagram such as this one, where $N' = N \setminus \{c_*\}$:

$$\begin{array}{ccc} H(\mathbf{C}(K, N)) & \xrightarrow{\mathbf{T}^{(N)}_*} & H(\mathbf{C}(K', N)) \\ \downarrow & & \downarrow \\ H(\mathbf{C}(K, N')) & \xrightarrow{\mathbf{T}^{(N')}_*} & H(\mathbf{C}(K', N')). \end{array}$$

A chain homotopy between the composite chain-maps at the level of these cubes is constructed by counting instantons on cobordisms T_{vu} with v and u in the appropriate cubes, to obtain a map

$$\mathbf{C}(K, N) \longrightarrow \mathbf{C}(K', N').$$

There is an additional point concerning the commutativity of the square (13). Corollary 3.4 provides the isomorphisms which are the vertical arrows in the diagram, but the construction of these isomorphisms depends on a choice of ordering for the set of crossings N , because the map is defined by “removing one crossing at a time”. Despite this apparent dependence, the isomorphism $H_*(\mathbf{C}(N)) \rightarrow I^\sharp(K)$ is actually independent of the ordering, up to overall sign. To verify this, it is of course enough to consider the effect of changing the order of just two crossings which are adjacent in the original ordering of N . The chain-homotopy formula that we need in this situation is another application of Proposition 3.2. To see this, consider for example the case that c_1 and c_2 are the first two crossings in a given ordering of N . The construction of the isomorphism in Corollary 3.4 begins with the composite of two chain-maps

$$\mathbf{C}(K, N) \xrightarrow{p} \mathbf{C}(K, N \setminus \{c_1\}) \xrightarrow{q} \mathbf{C}(K, N \setminus \{c_1, c_2\}).$$

If we switch the order, we have a different composite, going via a different middle stage:

$$\mathbf{C}(K, N) \xrightarrow{r} \mathbf{C}(K, N \setminus \{c_2\}) \xrightarrow{s} \mathbf{C}(K, N \setminus \{c_1, c_2\}).$$

The induced maps in homology are the same up to sign, because qp is chain-homotopic to $-sr$. The chain homotopy is the map $\mathbf{C}(K, N) \rightarrow \mathbf{C}(K, N \setminus \{c_1, c_2\})$ obtained as the sum of appropriate terms f_{vu} .

Via the isomorphism of Corollary 3.4, the group $I^\sharp(K)$ obtains from $\mathbf{C}(K, N)$ a decreasing q -filtration. We can interpret the above results as telling us that this filtration is independent of some of the choices involved:

Proposition 5.2. *Let K be a link in \mathbb{R}^3 , and let N be a collection of crossings such that (K, N) is a pseudo-diagram with small self-intersection numbers. Then, via the isomorphism with $H(\mathbf{C}(K, N))$, the instanton homology group $I^\sharp(K)$ obtains a quantum filtration that does not depend on an ordering of N . Furthermore, this filtration of $I^\sharp(K)$ is natural for isotopies of K rel $B(N)$.*

6. Dropping a crossing

We continue to suppose that (K, N) is a pseudo-diagram with small self-intersection numbers. The isomorphism class in \mathfrak{C}_q of the filtered complex obtained from the cube of resolutions is independent of the choice of good auxiliary data by the previous proposition, but we must address the question of whether it depends on N .

We begin investigating this question by considering the situation in which $N' \subset N$ is obtained by dropping a single crossing. We suppose that good auxiliary data is chosen for both (K, N) and (K, N') so that we have complexes $\mathbf{C}(N)$ and $\mathbf{C}(N')$. For the moment, we do not consider the q -filtrations on these. Let us recall from [5] the proof that the homologies of the two cubes $\mathbf{C}(N)$ and $\mathbf{C}(N')$ are isomorphic (the basic case of Theorem 3.3). Let $c_* \in N$ be the distinguished crossing that does not belong to N' . Write

$$\mathbf{C}(N) = \mathbf{C}_1 \oplus \mathbf{C}_0$$

where

$$\mathbf{C}_i = \bigoplus_{\substack{v(c_*)=i \\ v(c') \in \{0,1\} \text{ if } c' \neq c_*}} \mathbf{C}_v. \quad (14)$$

The differential $\mathbf{F}(N)$ on $\mathbf{C}(N)$ then takes the form

$$\mathbf{F}(N) = \begin{bmatrix} \mathbf{F}_{11} & 0 \\ \mathbf{F}_{10} & \mathbf{F}_{00} \end{bmatrix}. \quad (15)$$

We extend the notation \mathbf{C}_i as just defined to all i in \mathbb{Z} . Whenever $i > j$ and $|i - j| = n \leq 3$, we have a map

$$\mathbf{F}_{ij}: \mathbf{C}_i \rightarrow \mathbf{C}_j$$

given as the sum of maps f_{vu} , where each pair (v, u) has type n . Similarly, we have $\mathbf{F}_{ii}: \mathbf{C}_i \rightarrow \mathbf{C}_i$, which is a sum of maps indexed by pairs of type 0 and 1.

The 3-step periodicity means that the complexes \mathbf{C}_2 and \mathbf{C}_{-1} are obtained from the same $(|N| - 1)$ -dimensional cube of resolutions of K . But the complexes $(\mathbf{C}_2, \mathbf{F}_{2,2})$ and $(\mathbf{C}_{-1}, \mathbf{F}_{-1,-1})$ may differ because of the different choices of auxiliary data. (We are free to arrange that the choices of metrics, perturbations and base-points so as to respect the periodicity, but not the choices involved in orienting the moduli spaces.) Nevertheless, because the links K_v that are involved are the same, there are canonical cylindrical cobordisms which give rise to a chain-homotopy equivalences

$$\mathbf{T}_{2,-1}: (\mathbf{C}_2, \mathbf{F}_{2,2}) \longrightarrow (\mathbf{C}_{-1}, \mathbf{F}_{-1,-1}). \quad (16)$$

as in the previous section. Indeed, both of these chain complexes are canonically chain-homotopy equivalent to $(\mathbf{C}(N'), \pm \mathbf{F}(N'))$. Thus, the summand $\mathbf{C}_{v'} \subset \mathbf{C}(N')$ corresponding to $v': N' \rightarrow \{0, 1\}$ is identified with the summand $\mathbf{C}_v \subset \mathbf{C}_2$, where $v: N \rightarrow \mathbb{Z}$ is obtained by extending v' to N with $v(c_*) = 2$, and the cylindrical cobordisms give a chain-homotopy equivalence

$$(\mathbf{C}(N'), \mathbf{F}(N')) \longrightarrow (\mathbf{C}_2, \mathbf{F}_{2,2}).$$

To show that $\mathbf{C}(N)$ and $\mathbf{C}(N')$ are homotopy-equivalent, we therefore need to provide an equivalence

$$\Psi: \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}_{-1}. \quad (17)$$

This is done in [5], where it is shown that such a chain-homotopy equivalence is provided by the map

$$\Psi = [\mathbf{F}_{1,-1} \quad \mathbf{F}_{0,-1}]. \quad (18)$$

We recall the proof from [5] that this map is invertible. We consider the maps

$$\begin{aligned} \Phi_2: \mathbf{C}_2 &\longrightarrow \mathbf{C}_1 \oplus \mathbf{C}_0, \\ \Phi_{-1}: \mathbf{C}_{-1} &\longrightarrow \mathbf{C}_{-2} \oplus \mathbf{C}_{-3} \end{aligned} \quad (19)$$

given by

$$\Phi_2 = \begin{bmatrix} \mathbf{F}_{21} \\ \mathbf{F}_{20} \end{bmatrix} \quad (20)$$

with Φ_{-1} defined similarly, shifting the indices down by 3. We will show that both composites $\Psi \circ \Phi_2$ and $\Phi_{-1} \circ \Psi$ are chain-homotopy equivalences, from which it will follow that Ψ is a chain-homotopy equivalence.

The composite chain-map $\Psi \circ \Phi_2$ is the map

$$(\mathbf{F}_{1,-1}\mathbf{F}_{21} + \mathbf{F}_{0,-1}\mathbf{F}_{20}): \mathbf{C}_2 \longrightarrow \mathbf{C}_{-1}. \quad (21)$$

A version of Proposition 3.2 for type 3 cobordisms (essentially equation (43) of [5]) gives an identity

$$\mathbf{F}_{2,-1}\mathbf{F}_{2,2} + \mathbf{F}_{1,-1}\mathbf{F}_{2,1} + \mathbf{F}_{0,-1}\mathbf{F}_{2,0} + \mathbf{F}_{-1,-1}\mathbf{F}_{2,-1} = \mathbf{T}_{2,-1} + \mathbf{N}_{2,-1} \quad (22)$$

for a certain map $\mathbf{N}_{2,-1}$. Here $\mathbf{T}_{2,-1}$ is again the chain map $\mathbf{C}_2 \rightarrow \mathbf{C}_{-1}$ arising from the cylindrical cobordisms.¹ We can interpret the above equation as saying that the map (21) is chain-homotopic to $\mathbf{T}_{2,-1} + \mathbf{N}_{2,-1}$, and that the chain homotopy is the map $\mathbf{F}_{2,-1}$. Since $\mathbf{T}_{2,-1}$ is an equivalence, it then remains to show that $\mathbf{N}_{2,-1}$ is chain-homotopic to zero. In [5] this null-homotopy is exhibited as a map

$$\mathbf{H}_{2,-1}: \mathbf{C}_2 \longrightarrow \mathbf{C}_{-1} \quad (23)$$

whose matrix entries h_{vu} are defined by counting instantons over a S_{vu} for a family of metrics of dimension

$$\sum_{c \neq c_*} (v(c) - u(c)).$$

For the other composite $\Phi_{-1} \circ \Psi$ the story is similar. We may write the composite as

$$\Phi_{-1} \circ \Psi = \begin{bmatrix} \mathbf{F}_{-1,-2}\mathbf{F}_{1,-1} & \mathbf{F}_{-1,-2}\mathbf{F}_{0,-1} \\ \mathbf{F}_{-1,-3}\mathbf{F}_{1,-1} & \mathbf{F}_{-1,-3}\mathbf{F}_{0,-1} \end{bmatrix}.$$

¹In [5], the authors wrote this as $\pm \mathbf{Id}$, assuming that the perturbations and orientations were chosen so that \mathbf{C}_2 and \mathbf{C}_{-1} were the same complex. In fact, it may not be possible to choose the signs consistently so that this is the case: the best one can do is arrange that $\mathbf{T}_{2,-1}$ is diagonal in a standard basis with diagonal entries ± 1 . This point does not affect the arguments of [5].

We modify this by the chain-homotopy

$$\mathbf{L} = \begin{bmatrix} \mathbf{F}_{1,-2} & \mathbf{F}_{0,-2} \\ \mathbf{F}_{1,-3} & \mathbf{F}_{0,-3} \end{bmatrix} \quad (24)$$

and see that $\Phi_{-1} \circ \Psi$ is chain-homotopic to a map of the form

$$\begin{bmatrix} \mathbf{T}_{1,-2} + \mathbf{N}_{1,-2} & 0 \\ \mathbf{Y} & \mathbf{T}_{0,-3} + \mathbf{N}_{0,-3} \end{bmatrix} : \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}_{-2} \oplus \mathbf{C}_{-3}.$$

Here $\mathbf{N}_{1,-2}$ and $\mathbf{N}_{0,-3}$ are similar to $\mathbf{N}_{2,-1}$ above, and \mathbf{Y} is an unidentified term involving cobordisms of type 4. If we apply a second chain-homotopy of the form

$$\begin{bmatrix} \mathbf{H}_{1,-2} & 0 \\ 0 & \mathbf{H}_{0,-3} \end{bmatrix} : \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}_{-2} \oplus \mathbf{C}_{-3} \quad (25)$$

where $\mathbf{H}_{1,-2}$ and $\mathbf{H}_{0,-3}$ are defined in the same way as $\mathbf{H}_{2,-1}$ above, then we find that $\Phi_{-1} \circ \Psi$ is chain-homotopic to

$$\begin{bmatrix} \mathbf{T}_{1,-2} & 0 \\ \mathbf{X} & \mathbf{T}_{0,-3} \end{bmatrix} : \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}_{-2} \oplus \mathbf{C}_{-3}.$$

This lower-triangular map is a chain-homotopy equivalence because the diagonal entries $\mathbf{T}_{i,i-3}$ are.

We now introduce the quantum filtrations. For this, we suppose that both (K, N) and (K, N') are pseudo-diagrams with small self-intersection numbers; see Definition 4.5. We wish to see whether these chain maps and chain homotopies respect the quantum filtrations. For this, we need to strengthen our hypotheses once more.

Definition 6.1. Let N be a set of crossings and let N' be a subset obtained by forgetting one crossing. We say that the pair (N, N') is *admissible* if the following holds. First, we require that both (K, N) and (K, N') are pseudo-diagrams. In addition, we require that whenever (v, u) is a pair of points in \mathbb{Z}^N of type at most 3 with $v(N'), u(N') \subset \{0, 1\}$, we have $S_{vu} \leq 6$.

When (N, N') is admissible in this sense, then both (K, N) and (K, N') have small self-intersection numbers. The definition is set up so that it applies to all the pairs (v, u) corresponding to the maps f_{vu} which are involved in the chain-maps Ψ and Φ_i defined above, for these chain maps have matrix entries \mathbf{F}_{ij} with $i - j = 0, 1$ or 2 , so they ultimately rest on cobordisms S_{vu} with (v, u) of type at most 2. Furthermore, the chain-homotopies such as (23), (24) and (25) involve only cobordisms S_{vu} of type at most 3, so the condition in the definition covers them also. The q -grading, defined using the crossing-set N , is defined on all C_v for which K_v is an unlink. In particular, q is defined on each C_i .

Proposition 6.2. *If (N, N') is admissible in the sense of Definition 6.1, then the chain map*

$$\Psi: \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}_{-1}$$

has order ≤ 0 with respect to the quantum filtration and is an isomorphism in the category \mathfrak{C}_q .

Proof. We note first that the maps $\mathbf{T}_{2,-1}: \mathbf{C}_2 \rightarrow \mathbf{C}_{-1}$ etc. given by the cylindrical cobordisms are isomorphisms in \mathfrak{C}_q . This is a corollary of Proposition 5.1 and the 3-periodicity of the q -grading described in Lemma 4.3. The question of whether Ψ , Φ_2 and Φ_{-1} preserve the filtrations is just the question of whether the maps \mathbf{F}_{ij} that appear as components of the maps (18) and (20) respect the filtration defined by q . The proof is exactly the same as the proof of Proposition 4.6. Similarly, the chain homotopies (23), (24) and (25) preserve the filtration. So in the category \mathfrak{C}_q , we have

$$\begin{aligned} \Psi \circ \Phi_2 &\simeq \mathbf{T}_{2,-1} \\ \Phi_{-1} \circ \Psi &\simeq \begin{bmatrix} \mathbf{T}_{1,-2} & 0 \\ \mathbf{X} & \mathbf{T}_{0,-3} \end{bmatrix} \end{aligned}$$

and the maps on the right are isomorphisms in \mathfrak{C}_q . □

Just as the quantum grading q is defined on $\mathbf{C}(N)$, so we define a quantum grading q' on $\mathbf{C}(N')$. The complexes $\mathbf{C}(N')$ and \mathbf{C}_2 are chain-homotopy equivalent, as noted above, but the gradings q' and q may apparently differ, because the correction terms involved in the definition depend on the original choice of a set of crossings. In fact, however, the gradings do coincide.

Lemma 6.3. *With the quantum gradings q and q' corresponding to the crossing-sets N and N' respectively, the complexes \mathbf{C}_2 , \mathbf{C}_{-1} and $\mathbf{C}(N')$ are isomorphic in the category \mathfrak{C}_q . The isomorphisms are the maps \mathbf{T} given by the cylindrical cobordisms.*

Proof. We have already noted the isomorphism between \mathbf{C}_2 and \mathbf{C}_{-1} in the proof of the previous proposition. For $\mathbf{C}(N')$, by an application of Proposition 5.1, we may reduce to the case that the auxiliary data for (K, N') is chosen so that $\mathbf{C}(N')$ and \mathbf{C}_2 coincide as complexes and the map \mathbf{T} is the identity map. We must then compare the definitions of the quantum gradings. Choose any $v': N' \rightarrow \{0, 1\}$ and let v denote the function $v(c) = v'(c)$ for $c \in N'$ and $v(c^*) = 2$. Let ϵ denote the sign of c^* . For a generator in $\mathbf{C}_2 \cong \mathbf{C}(N')$ the two Q -gradings agree so the difference of the q -grading is

$$\begin{aligned} q - q' &= -v(c^*) + \frac{3}{2}(\sigma(v, o) - \sigma(v', o')) - (n_+ - n'_+) + 2(n_- - n'_-) \\ &= -2 + \frac{3}{2}(\sigma(v, o) - \sigma(v', o')) + \frac{3}{2}(1 - \epsilon) - 1 \end{aligned}$$

(cf. equation (11)). If c^* is a positive crossing then the oriented resolution of K has $o(c^*) = 0$ while if c^* is a negative crossing then the oriented resolution has $o(c^*) = 1$. Thus for a positive crossing we have $\sigma(v, o) - \sigma(v', o') = 2$ and $1 - \epsilon = 0$ while for a negative crossing we have $\sigma(v, o) - \sigma(v', o') = 0$ and $1 - \epsilon = 2$. In either case above difference is zero. \square

Putting together this lemma and the previous proposition, we have the following result.

Proposition 6.4. *If (N, N') is admissible in the sense of Definition 6.1, then $\mathbf{C}(N)$ and $\mathbf{C}(N')$ are isomorphic in \mathfrak{C}_q .*

7. Dropping two crossings

Consider again a collection of crossings N for $K \subset \mathbb{R}^3$, and let N' and N'' be obtained from N by dropping first one and then a second crossing:

$$N = N'' \cup \{c_1, c_2\},$$

$$N' = N'' \cup \{c_2\}.$$

Lemma 7.1. *Suppose that all the links K_v corresponding to vertices of $\mathbf{C}(N)$, $\mathbf{C}(N')$, and $\mathbf{C}(N'')$ are unlinks. Suppose also that for all pairs (v, u) in $\{0, 1\}^N$ the corresponding cobordism S_{vu} is orientable. Then the pairs (N, N') and (N', N'') are both admissible in the sense of Definition 6.1.*

Remark. The orientability hypothesis in the lemma is equivalent to asking that the cobordism from $\{1\}^N$ to $\{0\}^N$ is orientable, since all the others are contained in this one.

Proof of the lemma. Let $o \in \{0, 1\}^N$ be the oriented resolution (or indeed any chosen point in this cube). Consider $\sigma(o, v)$ as a function of v . It is well-defined on all v with $v(N'') \subset \{0, 1\}$, because the corresponding links K_v are unlinks. For $v \in \{0, 1\}^N$ we have $\sigma(o, v) = 0$, because S_{ov} is orientable.

If $v' \in \mathbb{Z}^N$ has $v'(N') \subset \{0, 1\}$ while $v'(c_1) = -1$, then $\sigma(o, v') = 0$ or 2 . To see this, consider because the unique $v \in \{0, 1\}^N$ with $v|_{N'} = v'|_{N'}$ $v(c_1) = 0$. By additivity, we have $\sigma(o, v') = \sigma(v, v')$. On the other hand, $\sigma(v, v') = 0$ or 2 , according as the cobordism $S_{vv'}$ from the unlink K_v to the unlink $K_{v'}$ is orientable or not.

Similarly, if $v'' \in \mathbb{Z}^N$ has $v''(N'') \subset \{0, 1\}$ while $v''(c_1) = v''(c_2) = -1$, then $\sigma(o, v'')$ is either 0 , 2 or 4 , as we see by comparing $\sigma(o, v'')$ to $\sigma(o, v')$ where v' is an adjacent element with $v'(c_2) = 0$.

With these observations in place, we can verify that (N, N') and (N', N'') are both admissible. For example, to show that (N', N'') is admissible, consider $v' \geq u'$

in $\mathbb{Z}^{N'}$ of type at most 3. We may regard v' and u' as elements of \mathbb{Z}^N by extending them with $v'(c_1) = u'(c_1) = -1$. Consider (as an illustrative case) the situation in which $v'(c_2) = 2$ and $u'(c_2) = -1$. Let \tilde{v}' be the element with $\tilde{v}'|_{N'} = v'|_{N'}$ and $\tilde{v}'(c_2) = -1$. By Lemma 4.2, we have $\sigma(v', \tilde{v}') = 2$, while the observations in the previous paragraph give $\sigma(\tilde{v}', v') \leq 4$. So $\sigma(v', u') \leq 6$, as required for admissibility. \square

Corollary 7.2. *Let N be the set of all crossings for a link K arising from a given plane diagram $D(K)$. Let N' and N'' be obtained by dropping first a crossing c_1 and then a pair of crossings $\{c_1, c_2\}$. Suppose that c_1 and c_2 are adjacent crossings along a parametrization of some component of K , and are either both over-crossings or under-crossings along this component. Then the pairs (N, N') and (N', N'') are admissible.*

Proof. If we take the 0- or 1-resolution at every crossing in $D(K)$, we obtain in every case an unlink, and the cobordisms between them are orientable. If we resolve all crossings except c_1 , then we obtain a diagram of a link with only one crossing, so this is again an unlink. If we resolve all crossings except c_1 and c_2 then we obtain a 2-crossing diagram. This additional hypothesis in the statement of the Corollary ensures that this 2-crossing diagram is not alternating. This guarantees that the diagram is still an unlink. Thus all the conditions of the previous lemma are satisfied. \square

Corollary 7.3. *As in the previous corollary, let N be the set of all crossings arising from a plane diagram $D(K)$, let $\{c_1, c_2\}$ be a pair of these crossings that are adjacent and are either both over-crossings or both under-crossings. Again let $N' = N \setminus \{c_1\}$ and $N'' = N \setminus \{c_1, c_2\}$. Then there are quantum-filtration-preserving chain maps*

$$\mathbf{C}(N) \xrightarrow{\Psi} \mathbf{C}(N') \xrightarrow{\Psi'} \mathbf{C}(N'')$$

and

$$\mathbf{C}(N'') \xrightarrow{\Phi'} \mathbf{C}(N') \xrightarrow{\Phi} \mathbf{C}(N),$$

such that Φ is the inverse of Ψ and Φ' is the inverse of Ψ' in the category \mathcal{C}_q . In particular, $\mathbf{C}(N)$, $\mathbf{C}(N')$ and $\mathbf{C}(N'')$ are isomorphic in \mathcal{C}_q .

8. Reidemeister moves

We now specialize to the crossing-sets that arise from a plane diagram of our knot or link K . We regard a diagram as being an image of a generic projection of K into \mathbb{R}^2 , with a labeling of each crossing as ‘‘under’’ or ‘‘over’’. As we have mentioned, the set of crossings N coming from a diagram D of K fulfills the conditions of Definition 4.1; that is, (K, N) is a pseudo-diagram. Furthermore, (K, N) has small

self-intersection numbers in the sense of Definition 4.5. Indeed, all the cobordisms S_{vu} are orientable, and therefore have $S_{vu} \cdot S_{vu} = 0$. After a choice of good auxiliary data, we therefore arrive at a filtered complex $\mathbf{C}(K, N)$, an object of our category \mathfrak{C}_q .

It follows from Proposition 5.1 that the isomorphism class of $\mathbf{C}(K, N)$ in \mathfrak{C}_q depends at most on the isotopy class of the diagram D : isotopic diagrams, with different choices of auxiliary data, will yield isomorphic objects. Our task is to show that the isomorphism class of $\mathbf{C}(K, N)$ is independent of the diagram altogether, and depend only on the link type of K . For this, we need to consider Reidemeister moves.

So let D_1 and D_2 be two plane diagrams for the same link type, differing by a single Reidemeister move. That is, we suppose that D_1 and D_2 coincide outside a standard disk in the plane, and that inside the disk they have the standard form corresponding to a Reidemeister move of type I, II or III. Let K_1 and K_2 be the (isotopic) oriented links in \mathbb{R}^3 whose projections are D_1 and D_2 , and let N_1 and N_2 be their crossing-sets. Associated to these links and crossing-sets, after choices of good auxiliary data, we have cubes $\mathbf{C}(K_1, N_1)$ and $\mathbf{C}(K_2, N_2)$, two objects of \mathfrak{C}_q .

Proposition 8.1. *When the diagrams differ by a Reidemeister move as above, the filtered cubes $\mathbf{C}(K_1, N_1)$ and $\mathbf{C}(K_2, N_2)$ are isomorphic in the category \mathfrak{C}_q .*

Proof. This is a straightforward consequence of Corollary 7.3. Thus, for example, in the case that D_1 and D_2 differ by a Reidemeister move of type III as shown in Figure 2, then we pass from (K_1, N_1) to (K_2, N_2) in three steps, as follows.

- (1) First, drop the two over-crossings c_1 and c_2 from the crossing set N_1 to obtain a smaller set of crossings N'' .
- (2) Second, apply an isotopy to K_1 relative to $B(N'')$.
- (3) Third, introduce two new crossings to N'' to obtain the crossing-set N_2 .

Corollary 7.3 applies to the first and third steps, while Proposition 5.1 applies to the middle step. The other types of Reidemeister moves are treated the same way. \square

Corollary 8.2. *Let K_1 and K_2 be isotopic oriented links in \mathbb{R}^3 and let N_1 and N_2 be crossing-sets arising from diagrams D_1 and D_2 for these links. After choices of good auxiliary data, let $\mathbf{C}(K_i, N_i)$ be the corresponding cubes ($i = 1, 2$). Then, with the filtrations defined by q , these cubes define isomorphic objects in the category \mathfrak{C}_q .*

9. The h -filtration on cubes

We now turn to the cohomological filtration h . Although there are a few important differences, we can for the most part adapt the sequence of arguments that we have used for the q filtration. This will lead us (for example) to a version of Corollary 8.2 for h .

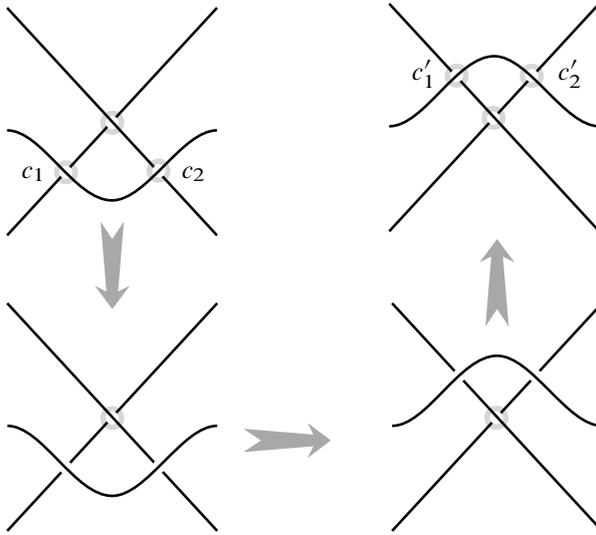


Figure 2. Dropping two crossings, isotoping a strand, and adding two crossings, to achieve Reidemeister III..

To begin, we suppose that (K, N) is a pseudo-diagram (Definition 4.1). On the cube $\mathbf{C}(N)$, we define the h -grading by declaring that the summand C_v has grading

$$h|_{C_v} = -\left(\sum_{c \in N} v(c)\right) + \frac{1}{2}\sigma(v, o) + n_-. \quad (26)$$

Here, as in equation (11) where we defined the q grading, the term n_- denotes the number of negative crossings and o denotes the oriented resolution. As with q , we can extend the definition of h beyond the cube $\{0, 1\}^N$ to all $v \in \mathbb{Z}^N$ for which K_v is an unlink, because $\sigma(v, o)$ can be defined for all such v .

Unlike q , the grading h does not have period 3 in each coordinate; see Lemma 4.3. Instead we have the following computation.

Lemma 9.1. *Suppose v is such that K_v is an unlink, and suppose $v - u$ is divisible by 3 in \mathbb{Z}^N so that we have an isomorphism $\beta: C_v \rightarrow C_u$ as abelian groups. Then the h -gradings on C_v and C_u are related by*

$$h|_{C_v} - h|_{C_u} = -\frac{2}{3} \sum_{c \in N} (v(c) - u(c))$$

Proof. This is immediate from Lemma 4.2 and the formula for h . □

The q -grading has constant parity on \mathbf{C} , but the h -grading does not. The next lemma shows that (\mathbf{C}, \mathbf{F}) can be regarded as a $\mathbb{Z}/2$ -graded complex with grading given by h .

Lemma 9.2. *The differential \mathbf{F} on \mathbf{C} has odd degree with respect to the $\mathbb{Z}/2$ grading defined by $h \bmod 2$.*

Proof. For $v \geq u$ in $\{0, 1\}^N$, the corresponding component f_{vu} of \mathbf{F} is obtained by counting instantons in 0-dimensional moduli spaces $M_{vu}(\beta, \alpha)$ parametrized by a family of metrics \check{G}_{vu} of dimension

$$\begin{aligned} \dim \check{G}_{vu} &= -\chi(S_{vu}) - 1 \\ &= \sum_c (v(c) - u(c)) - 1. \end{aligned}$$

The links K_v and K_u are unlinks and the perturbations are chosen so that the critical points all have the same index mod 2. So the fiber dimension of $M_{vu}(\beta, \alpha) \rightarrow \check{G}_{vu}$ is independent of β and $\alpha \bmod 2$, and is given by (7). Taking account of the dimension of the \check{G}_{vu} , we can therefore write

$$\begin{aligned} \dim M_{vu}(\beta, \alpha) &= 8\kappa + \chi(\bar{S}_{vu}) + \frac{1}{2}(\bar{S}_{vu} \cdot \bar{S}_{vu}) + \dim \check{G}_{vu} \\ &= 8\kappa + \chi(\bar{S}_{vu}) + \frac{1}{2}(\bar{S}_{vu} \cdot \bar{S}_{vu}) - \chi(S_{vu}) - 1 \end{aligned}$$

mod 2. For any closed surface \bar{S} in \mathbb{R}^4 , we have $\chi(\bar{S}) = (1/2)(\bar{S} \cdot \bar{S}) \bmod 2$. We also have $\kappa = \frac{1}{16}(S \cdot S)$ modulo $\frac{1}{2}\mathbb{Z}$. So the above formula can be written

$$\dim M_{vu}(\beta, \alpha) = \frac{1}{2}(S_{vu} \cdot S_{vu}) - \chi(S_{vu}) - 1$$

mod 2. So the component f_{vu} can be non-zero only when $\frac{1}{2}(S_{vu} \cdot S_{vu}) + \chi(S_{vu})$ is odd. On the other hand, $\frac{1}{2}(S_{vu} \cdot S_{vu}) - \chi(S_{vu})$ is precisely the difference in $h \bmod 2$, between C_v and C_u . \square

As well as having odd degree, the differential \mathbf{F} preserves the decreasing filtration defined by h .

Proposition 9.3. *If (K, N) is a pseudo-diagram, then the differential $\mathbf{F}: \mathbf{C} \rightarrow \mathbf{C}$ has order ≥ 1 with respect to the decreasing filtration defined by h .*

Proof. This follows from Lemma 4.7, as did the corresponding statement for the q -filtration, Proposition 4.6. Indeed, for any $v \geq u$ in the cube $\{0, 1\}^N$, the corresponding map $f_{vu}: C_v \rightarrow C_u$ of \mathbf{F} satisfies

$$\begin{aligned} \text{ord}_h f_{vu} &\geq h|_{C_u} - h|_{C_v} \\ &= \left(\sum_{c \in N} (v(c) - u(c)) \right) - \frac{1}{2}\sigma(v, u). \end{aligned}$$

So we obtain $\text{ord}_h f_{vu} \geq 0$ directly from Lemma 4.7. To actually obtain $\text{ord}_h f_{vu} \geq 1$, which is the desired result, we appeal to the fact that \mathbf{F} has odd degree with respect to the mod-2 grading, as we have seen in the previous lemma. \square

Remark. In contrast to the corresponding result for the q -filtration (Proposition 4.6), the proposition above does not require the hypothesis that (K, N) has small self-intersection numbers.

Let us now introduce the category \mathfrak{C}_h , whose objects are filtered, finitely-generated abelian groups with a differential of order ≥ 1 , and whose morphisms are chain maps of order ≥ 0 modulo chain-homotopies of order ≥ -1 . When equipped with the filtration defined by h , the complex $\mathbf{C} = \mathbf{C}(N)$, with its differential \mathbf{F} , defines an object in the category \mathfrak{C}_h , by the proposition above, though this object is dependent on the set of crossings N , and the choice of good auxiliary data. As in the case of the q -filtration, we now wish to show that different choices lead to isomorphic objects in this category; and as before, the essential step is to compare $\mathbf{C}(N)$ to $\mathbf{C}(N')$, where N' is obtained from N by “forgetting” a crossing.

So we suppose again that $N' = N \setminus c^*$, and that both (K, N) and (K, N') are pseudo-diagrams. (We do not need to assume that these pseudo-diagrams have small self-intersection numbers.) The cubes $\mathbf{C}(N)$ and $\mathbf{C}(N')$ have gradings which we call h and h' respectively. On the other hand, we can identify $\mathbf{C}(N')$ as before with \mathbf{C}_2 , where \mathbf{C}_i for $i \in \mathbb{Z}$ is defined by (14) using the crossing-set N . In this way, we can regard the grading h as being defined also on $\mathbf{C}(N')$. The following two lemmas are the counterpart of Lemma 6.3 for the h -grading.

Lemma 9.4. *With the cohomological gradings h and h' corresponding to the crossing-sets N and N' , the filtered complexes \mathbf{C}_2 and $\mathbf{C}(N')[1]$ are isomorphic in \mathfrak{C}_h . Here the notation $\mathbf{C}[n]$ denotes the filtered complex obtained from \mathbf{C} by shifting the filtration down by n so that the map $\mathbf{C}[n] \rightarrow \mathbf{C}$ has order $\geq n$.*

Proof. The lemma can again be reduced to the case that $\mathbf{C}(N')$ and \mathbf{C}_2 are the same complex. So we need to prove that $h' = h + 1$. As in the proof of Lemma 6.3, we write ϵ for the sign of the crossing c^* , and we have

$$\begin{aligned} h - h' &= -v(c^*) + \frac{1}{2}(\sigma(v, o) - \sigma(v', o')) + n_- - n'_- \\ &= -2 + \frac{1}{2}(\sigma(v, o) - \sigma(v', o')) + \frac{1}{2}(1 - \epsilon). \end{aligned}$$

Once again, we have $o(c^*) = 0$ or 1 according as ϵ is 1 or -1 respectively. Thus for a positive crossing we have $\sigma(v, o) - \sigma(v', o') = 2$ and $1 - \epsilon = 0$, while for a negative crossing we have $\sigma(v, o) - \sigma(v', o') = 0$ and $1 - \epsilon = 2$. In either case the difference is -1 . \square

Since the grading h is not 3-periodic, we get a different result when we compare $\mathbf{C}(N')$ with \mathbf{C}_{-1} instead of \mathbf{C}_2 .

Lemma 9.5. *With the cohomological gradings h and h' corresponding to the crossing-sets N and N' , the filtered complexes \mathbf{C}_{-1} and $\mathbf{C}(N')[-1]$ are isomorphic in \mathfrak{C}_h .*

Proof. This follows from the previous lemma and Lemma 9.1. \square

We can now state our main result about the effect of dropping a crossing, for the h -filtration.

Proposition 9.6. *Suppose that $N' = N \setminus \{c^*\}$ and that both (K, N) and (K, N') are pseudo-diagrams. Then the complexes $\mathbf{C}(N)$ and $\mathbf{C}(N')$, equipped with the filtrations defined by h and h' , are isomorphic in the category \mathfrak{C}_h .*

Proof. We consider again the maps (17) and (19):

$$\begin{aligned}\Psi &: \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}_{-1}, \\ \Phi_2 &: \mathbf{C}_2 \longrightarrow \mathbf{C}_1 \oplus \mathbf{C}_0.\end{aligned}$$

With respect to the grading h on \mathbf{C}_i , $i \in \mathbb{Z}$, the maps Ψ and Φ_2 have order ≥ 0 , just as in the proof of Proposition 9.3; and once again, because these maps have odd degree with respect to $h \bmod 2$, it actually follows that Ψ and Φ_2 have order ≥ 1 . Via the isomorphisms in the previous two lemmas, the maps Ψ and Φ_2 therefore become maps

$$\begin{aligned}\Psi' &: \mathbf{C}_1 \oplus \mathbf{C}_0 \longrightarrow \mathbf{C}(N'), \\ \Phi'_2 &: \mathbf{C}(N') \longrightarrow \mathbf{C}_1 \oplus \mathbf{C}_0.\end{aligned}$$

of order ≥ 0 . They therefore define morphisms in \mathfrak{C}_h .

If we look at the composite $\Psi \circ \Phi_2 : \mathbf{C}_2 \rightarrow \mathbf{C}_{-1}$, we see from the formula (21) and the arguments below it that this chain map is chain-homotopic to the isomorphism $\mathbf{T}_{2,-1}$ given by the cylindrical cobordisms. The chain-homotopies are the maps

$$\mathbf{F}_{2,-1} : \mathbf{C}_2 \longrightarrow \mathbf{C}_{-1}$$

and

$$\mathbf{H}_{2,-1} : \mathbf{C}_2 \longrightarrow \mathbf{C}_{-1}.$$

Once again, with respect to the filtrations defined by h , these maps have order ≥ 0 , and therefore order ≥ 1 because of the mod 2 grading. Using the isomorphisms of the previous two lemmas again, we obtain chain-homotopies

$$\mathbf{F}'_{2,-1} : \mathbf{C}(N') \longrightarrow \mathbf{C}(N')$$

and

$$\mathbf{H}'_{2,-1} : \mathbf{C}(N') \longrightarrow \mathbf{C}(N').$$

of order ≥ -1 . So $\Psi' \circ \Phi'_2$ is the identity morphism in the category \mathfrak{C}_h . The argument for $\Phi'_{-1} \circ \Psi'$ is very similar. \square

With the above result about forgetting a single crossing, we can now continue just as in the case of the q -filtration, to prove invariance under Reidemeister moves. So we arrive at the following statement, exactly analogous to Corollary 8.2 above.

Corollary 9.7. *Let K_1 and K_2 be isotopic oriented links in \mathbb{R}^3 and let N_1 and N_2 be crossing-sets arising from diagrams D_1 and D_2 for these links. After choices of good auxiliary data, let $\mathbf{C}(K_i, N_i)$ be the corresponding cubes ($i = 1, 2$). Then, with the decreasing filtrations defined by h , these cubes define isomorphic objects in the category \mathcal{C}_h .*

10. Proof of the main theorems

The proofs of Theorem 1.1, Theorem 1.2 and Proposition 1.5 are now just a matter of pulling together the above material with the results of [5], as we now explain.

Given a diagram D for an oriented link K , we obtain from it a set of crossings N . The pair (K, N) is a pseudo-diagram, and after a choice of perturbations we obtain a complex $\mathbf{C} = \mathbf{C}(K, N)$ which carries a filtration by $\mathbb{Z} \times \mathbb{Z}$ arising from the gradings h and q . We have seen that the differential \mathbf{F} on \mathbf{C} has order $\geq (1, 0)$, meaning that it has order ≥ 1 with respect to h and ≥ 0 with respect to q (Propositions 9.3 and 4.6 respectively).

The h and q gradings decompose \mathbf{C} as

$$\mathbf{C} = \bigoplus_{i,j} \mathbf{C}^{i,j}$$

(where j corresponds to the q grading). Let us write \mathbf{F}^0 for the sum of those terms of \mathbf{F} which preserve q : so for each j we have

$$\mathbf{F}^0 : \bigoplus_i \mathbf{C}^{i,j} \rightarrow \bigoplus_i \mathbf{C}^{i,j}.$$

Lemma 10.1. *The map \mathbf{F}^0 shifts the h grading by exactly 1, so that*

$$\mathbf{F}^0(\mathbf{C}^{i,j}) \subset \mathbf{C}^{i+1,j}$$

for all i, j .

Proof. We refer to the equations (12) and (26). Because our collection of crossings comes from a diagram, all self-intersection numbers are zero, and the formula can therefore be rewritten

$$\text{ord}_q f_{vu} \geq h|_{C_u} - h|_{C_v} - 1. \quad (27)$$

So for a non-zero map f_{vu} that contributes to \mathbf{F}^0 , we have

$$0 \geq h|_{C_u} - h|_{C_v} - 1,$$

which implies that the difference in the h -gradings is exactly 1, because we always have the reverse inequality. \square

As the leading term of \mathbf{F} with respect to the q -filtration, \mathbf{F}^0 has square zero. So $(\mathbf{C}, \mathbf{F}^0)$ is a bigraded complex whose differential has bidegree $(1, 0)$.

Proposition 10.2. *With its bigrading supplied by h and q , the bigraded complex $(\mathbf{C}, \mathbf{F}^0)$ is isomorphic to Khovanov's complex $(\mathbf{C}(D^\dagger), d_{\text{Kh}})$ associated to the diagram D^\dagger for K^\dagger (obtained from the diagram D for K by changing all over-crossings to under-crossings and vice versa).*

Proof. Recall that the complex $\mathbf{C} = \mathbf{C}(N)$ has summands C_v indexed by $v \in \{0, 1\}^N$, and that C_v is the chain complex that computes $I^\#$ for the unlink K_v . We have chosen perturbations so that the differential on C_v is zero. So if K_v has $p(v)$ components, then (as in Section 2) we have

$$\begin{aligned} C_v &= I^\#(K_v) \\ &= H_*(S^2)^{\otimes p(v)} \end{aligned}$$

by the isomorphism (2). The isomorphism depends on a choice of meridians, as well as a choice of a standard Morse function on the product of S^2 's and a choice of perturbations for the instanton equations. Via the identification

$$s: H_*(S^2) \longrightarrow V$$

of $H_*(S^2)$ with the rank-2 group $V = \langle \mathbf{v}_+, \mathbf{v}_- \rangle$, this becomes an isomorphism

$$s^{\otimes p} \circ \beta: C_v \longrightarrow V^{\otimes p(v)}$$

Allowing for the change in orientation convention, this gives an isomorphism of groups

$$\beta: \mathbf{C} \longrightarrow \mathbf{C}(D^\dagger) \tag{28}$$

We also see that the formulae for the h -grading and q -grading on \mathbf{C} , given by (26) and (11) respectively, coincide with Khovanov's cohomological and quantum grading as defined in [3]. (The terms $\sigma(v, o)$ in those formulae are absent in the case that the cube arises from a diagram, because all the self-intersection numbers are zero.)

On the other hand, it is shown in [5] that the spectral sequence associated to the filtered complex (\mathbf{C}, \mathbf{F}) , with its filtration by h , has E_1 page isomorphic to Khovanov's complex: that is, there is an isomorphism

$$(E_1, d_1) \xrightarrow{\cong} (\mathbf{C}(D^\dagger), d_{\text{Kh}})$$

as groups with differential. With our choice of perturbations, the elements in a given h -grading all have the same degree mod 2, so the d_0 differential is absent and we have

$E_1 = \mathbf{C}$. Furthermore, the d_1 differential is precisely the part of \mathbf{F} that increases h -grading by exactly 1. Let us call this \mathbf{F}_e . So we have

$$\gamma: (\mathbf{C}, \mathbf{F}_e) \xrightarrow{\cong} (\mathbf{C}(D^\dagger), d_{\text{Kh}}). \quad (29)$$

The isomorphism $\gamma: \mathbf{C} \rightarrow \mathbf{C}(D^\dagger)$ in (29) is not the same isomorphism as the map β in (28). Rather, on each component C_v , it is defined by the map γ from (1). As explained in Section 2, the isomorphism γ and β may differ. Both β and γ respect the homological grading h , but only β respects the q -grading. The two isomorphisms differ by terms that strictly increase q , for on each C_v we have

$$\gamma - s^{\otimes p} \circ \beta = (\epsilon_1 + \dots) \circ \beta$$

where ϵ_1 etc. are as in (4). In particular then, the map γ gives rise to a map on the associated graded objects with respect to the q filtration, which we write as

$$\text{gr}(\gamma): \text{gr}(\mathbf{C}, \mathbf{F}_e) \longrightarrow (\mathbf{C}(D^\dagger), d_{\text{Kh}}). \quad (30)$$

On the left, we can identify the bigraded complex $\text{gr}(\mathbf{C}, \mathbf{F}_e)$ with $(\mathbf{C}, \mathbf{F}')$, where \mathbf{F}' is obtained from \mathbf{F}_e by keeping only those summands that preserve q . This differential \mathbf{F}' coincides with \mathbf{F}^0 by Lemma 10.1. So the map (30) can be interpreted as a map

$$\text{gr}(\gamma): (\mathbf{C}, \mathbf{F}^0) \longrightarrow (\mathbf{C}(D^\dagger), d_{\text{Kh}})$$

which respects the bigrading. The map $\mathbf{C} \rightarrow \mathbf{C}(D^\dagger)$ which appears here is the associated graded map arising from γ , and is therefore equal to β . Thus β in (28) intertwines the differential \mathbf{F}^0 with d_{Kh} . \square

Theorem 1.1 is a rewording of the above proposition: we identify \mathbf{C} with $\mathbf{C}(D^\dagger)$ by the isomorphism (28) and write the differential \mathbf{F} as d_{\sharp} .

Proof of Theorem 1.2. The theorem asserts that (\mathbf{C}, \mathbf{F}) is independent of the choices made, up to isomorphism in the category \mathcal{C} . The choices here are again a diagram for K with its associated collection of crossings, and a choice of good auxiliary data. Given two sets of choices and two corresponding objects $(\mathbf{C}_1, \mathbf{F}_1)$ and $(\mathbf{C}_2, \mathbf{F}_2)$ in the category \mathcal{C} , we have seen already that these objects are isomorphic in both the category \mathcal{C}_h and \mathcal{C}_q (Corollaries 9.7 and 8.2 respectively). An examination of the proofs shows that these objects are isomorphic also in \mathcal{C} . Indeed, the proofs were obtained by exhibiting chain maps

$$\Phi: (\mathbf{C}_1, \mathbf{F}_1) \longrightarrow (\mathbf{C}_2, \mathbf{F}_2),$$

$$\Psi: (\mathbf{C}_2, \mathbf{F}_2) \longrightarrow (\mathbf{C}_1, \mathbf{F}_1)$$

and chain-homotopy formulae

$$\Phi \circ \Psi - \mathbf{1} = \mathbf{F} \circ \Pi + \Pi \circ \mathbf{F},$$

$$\Psi \circ \Phi - \mathbf{1} = \mathbf{F} \circ \Pi' + \Pi' \circ \mathbf{F}.$$

The point here is that the *same* chain-maps and chain-homotopies are used in the proofs of both Corollary 9.7 and Corollary 8.2. Thus from the proof of Corollary 9.7 we learn that with respect to the h -filtration, the chain-maps have order ≥ 0 and the chain-homotopies have order ≥ -1 ; while from the proof of Corollary 8.2 we learn that, with respect to the q -filtration, the chain-maps have order ≥ 0 and the chain-homotopies have order ≥ 0 . It follows that Φ and Ψ are mutually-inverse isomorphisms in the category \mathfrak{C} , where the maps are chain-maps of order $\geq (0, 0)$ up to chain-homotopies of order $\geq (-1, 0)$. This proves Theorem 1.2. \square

Proof of Proposition 1.5. Because the maps $I^\#(S)$ can be computed as composite maps when the cobordism S is a composite, it is sufficient to treat the cases that S corresponds to the addition of a single handle, of index 0, 1, or 2. The cases 0 and 2 correspond to the addition or removal of a single extra unlinked component, and are straightforward. So we consider the case of index 1.

In the index-1 case, we can form a link K_2 with plane diagram having a crossing-set N in such a way that K_1 and K_0 are obtained from (K_2, N) by resolving a distinguished crossing $c_* \in N$ in two different ways. The links K_2, K_1 and K_0 are to be oriented independently and arbitrarily. The set $N' = N \setminus \{c_*\}$ is then the set of crossings for diagrams of both K_1 and K_0 . By a straightforward generalization of the commutativity of the square (13), we know that the map $I^\#(S)$ arises from a chain map

$$\mathbf{F}_{10}: \mathbf{C}_1 \longrightarrow \mathbf{C}_0.$$

We can regard this map as one term in the differential $\mathbf{F}(K_2, N)$ on the cube $\mathbf{C}(K_2, N) = \mathbf{C}_1 \oplus \mathbf{C}_0$ corresponding to (K_2, N) ; see (15). With respect to the h - and q -filtrations which arise from (K_2, N) , we know that \mathbf{F}_{10} has order $\geq (1, 0)$. Let us denote the corresponding h - and q -gradings on the abelian group $\mathbf{C}_1 \oplus \mathbf{C}_0$ by h_2 and q_2 . On \mathbf{C}_1 and \mathbf{C}_0 we also have gradings h_1, q_1 and h_0, q_0 respectively, arising from (K_1, N') and (K_0, N') .

We examine the quantum gradings. With respect to q_2 , the order of \mathbf{F}_{10} is ≤ 0 . On a summand $C_v \subset \mathbf{C}_1$, we have

$$\begin{aligned} q_2 - q_1 &= -v(c_*) - (n_+^2 - n_+^1) + 2(n_-^2 - n_-^1) \\ &= -1 - (n_+^2 - n_+^1) + 2(n_-^2 - n_-^1), \end{aligned}$$

where n_+^2 denotes the number of crossings in N that are positive for the chosen orientation of K_2 and so on. Similarly, on a summand $C_u \subset \mathbf{C}_0$ we have

$$\begin{aligned} q_2 - q_0 &= -u(c_*) - (n_+^2 - n_+^1) + 2(n_-^2 - n_-^1) \\ &= -(n_+^2 - n_+^1) + 2(n_-^2 - n_-^1). \end{aligned}$$

So with respect to the filtrations defined by q_1 and q_0 , the order of \mathbf{F}_{10} is greater than or equal to the difference of the above two expressions, which is

$$-1 + (n_+^1 - n_+^0) - 2(n_-^1 - n_-^0).$$

Since $\chi(S) = -1$, the proposition's assertion about the quantum gradings will be proved if we show that

$$\frac{3}{2}(S \cdot S) = (n_+^1 - n_+^0) - 2(n_-^1 - n_-^0).$$

Since the number of crossings is the same for K_1 and K_0 , this is equivalent to

$$S \cdot S = w_1 - w_0,$$

where $w_i = n_+^i - n_-^i$ is the writhe of the diagram (K_i, N') . At this point we should recall that $S \cdot S$ is defined with respect to framings of the boundaries K_1 and K_0 which have linking numbers zero with K_1 and K_0 . With respect to the blackboard framings of K_1 and K_0 , the self-intersection number of S is zero. The writhe measures the difference between the blackboard framing and the framing with linking-number zero. So the result follows. The calculation for the h -filtration is similar. \square

11. Examples

Simple examples of pseudo-diagrams. To illustrate how one can work with pseudo-diagrams, we consider the two oriented pseudo-diagrams in Figure 3. The first represents a 2-component unlink; the second is a Hopf link. Each has one crossing, and Figure 3 shows the two different resolutions ($v = 1, 0$) in each case.

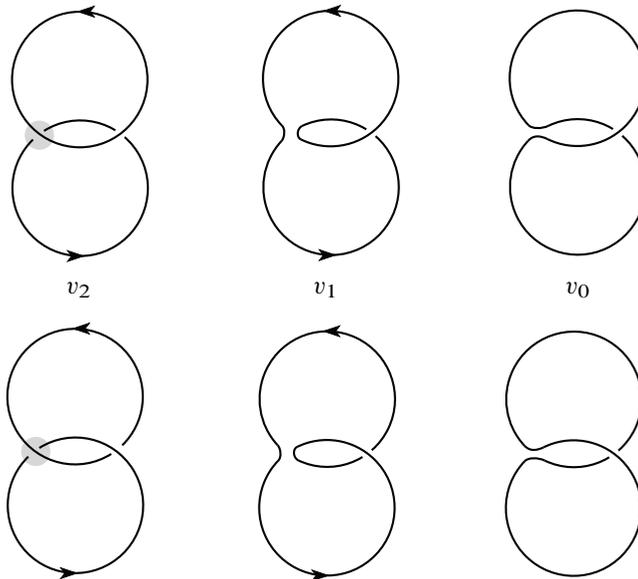
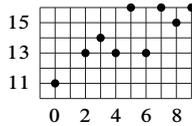


Figure 3. Pseudo-diagrams (left column) for an unlink and a Hopf link, each with one crossing. The resolution with the arrows is the oriented one in each case.

These examples show how the self-intersection number comes into play when there are non-orientable cobordisms involved. In the first diagram we have $n_+ = 0$, $n_- = 1$. (The total number of crossings is 1 in the pseudo-diagram.) The 1-resolution is the oriented resolution. The surface S_{10} has $S_{10} \cdot S_{10} = 2$, and hence $\sigma(1, 1) = 0$ while $\sigma(0, 1) = -2$. Thus we see from the definition of the h and q gradings on the cube \mathbf{C} that the resolution with $v = 1$ contributes two generators to the cube in bi-gradings $(0, 2)$ and $(0, 0)$. From $v = 0$ there are also two contributions, now in bi-gradings $(0, -2)$ and $(0, 0)$. Since the h -gradings are all even, there can be no differential. The resulting rank-4 group with its h - and q -gradings agrees with the Khovanov homology of the unlink.

The Hopf link diagram still has $n_+ = 0$, $n_- = 1$ but now $S_{10} \cdot S_{10} = -2$, and the 1-resolution is still the oriented resolution. From $v = 1$ there are two generators in bi-gradings $(-1, -3)$ and $(-1, -1)$. From $v = 0$ there are also two generators, now with bi-gradings $(1, 1)$ and $(1, 3)$. This reproduces the Khovanov homology of the Hopf link (with the given orientations.)

An example of a non-trivial differential, the (4, 5)-torus knot. We will deal with the reduced case $I^{\natural}(K)$ and work with rational coefficients. For the torus knot $T(4, 5)$, the reduced Khovanov homology is known and has rank 9. Below is a plot indicating with bullet-points where the non-zero groups are. These are plotted in the plane with coordinates i and $j - i$, where i is the h -grading and j is the q -grading.



Starting from any diagram, the reduced Khovanov homology is a page of both the h - and q -spectral sequences converging to $I^{\natural}(K)$, and we can ask where the differentials (if any) may be for $K = T(4, 5)$. The grading $j - i - 1$ on the Khovanov homology reduces to the canonical mod 4 grading on I^{\natural} ; see [5], Section 8.1. From the figure, we read off the Betti numbers for the mod 4 grading on the Khovanov homology as

$$3, \quad 1, \quad 2, \quad 3 \tag{31}$$

in gradings 0, 1, 2 and 3 mod 4 respectively. The differentials in the spectral sequence all have degree $-1 \pmod 4$ with respect to this grading. In these coordinates the higher differentials in the spectral sequence are also constrained by

$$\Delta j - \Delta i \geq -1,$$

where Δi and Δj denote the change in the h and q -gradings, as follows from (12). (The $S \cdot S$ term is absent in (12) because we are dealing with a diagram rather than a pseudo-diagram.)

The generators of the instanton complex that computes $I^{\natural}(K)$ are obtained from the representation variety

$$\mathcal{R}^{\natural}(K, \mathbf{i}) \stackrel{\text{def}}{=} \mathcal{R}(K^{\natural}, \mathbf{i})/SO(3),$$

which can also be regarded as the fiber of the map $\mathcal{R}(K, \mathbf{i}) \rightarrow S^2$ given by the holonomy around a chosen meridian. In the case of $T(4, 5)$, one can show that this representation variety is a union of an isolated non-degenerate point (corresponding to the reducible representation) and four Morse–Bott circles (corresponding to irreducibles). Thus after a small perturbation, we see that the instanton complex that computes $I^{\natural}(K)$ has 9 generators. Since 9 is also the rank of the Khovanov homology, it seems at first possible that the homology groups are isomorphic. But a closer examination shows that the four Morse–Bott circles contribute the same rank to each of the gradings mod 4 in the instanton complex. (To compute the relative Morse index of the 4 circles, one can study the representation varieties $\mathcal{R}^{\natural}(T(4, 5), \alpha)$, defined analogously to $\mathcal{R}^{\natural}(T(4, 5), \mathbf{i})$ but with holonomy around the meridian of the knot given by

$$\exp 2\pi i \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

for $\alpha \in [0, 1/4]$. As α increases, circles of critical points are emitted from or absorbed into the unique reducible. The Morse indices can be read off from the sequence of these events.) So if the instanton homology had rank 9, then the Betti numbers (31) for the Khovanov homology would differ from each other by at most 1 (corresponding to the contribution of the isolated point in the representation variety). Since this is not the case, it follows that the instanton homology has rank at most 7.

On the other hand, according to [5], Proposition 1.4, for a knot K there is an isomorphism between $I^{\natural}(K)$ and the sutured instanton knot Floer homology group $KHI(K)$, respecting the mod 4 grading; and from [4], Corollary 1.2, we know that the total rank of $KHI(K)$ is not less than the sum of the absolute values of the coefficients of the Alexander polynomial, which for $T(4, 5)$ is

$$T^6 - T^5 + T^2 - 1 + T^{-2} - T^{-5} + T^{-6}.$$

This gives a lower bound of 7 for the rank of $I^{\natural}(K)$. We deduce that in fact the rank is exactly 7. So the spectral sequence has a single non-zero differential which must have rank 1.

The results of [4] give a little more information. The coefficients of the Alexander polynomial arise as Euler characteristics of the generalized eigenspaces of an operator μ on $KHI(K)$ which has degree 2 with respect to the mod 4 grading. It follows that the Betti numbers in mod-4 gradings l and $l + 2$ are equal, except for an offset term arising from the eigenvalue 0. The dimension of the generalized 0-eigenspace is 1, because it corresponds to the coefficient of T^0 . Inspecting (31) again, we see that the only possibility is that the Betti numbers of the mod 4 grading on $I^{\natural}(K)$ are

$$2, \quad 1, \quad 2, \quad 2.$$

The 0-eigenspace contributes 1 to the last of these. The differential in the spectral sequence goes from 0 to 3 in the mod 4 grading. In terms of the diagram above, this means that the differential must go from the row $j - i = 13$ to the row $j - i = 16$. The authors do not know in which pages of the two spectral sequences the differential appears. It is also not yet apparent what the ranks of the associated graded groups are, for the h - and q -filtrations on $I^{\natural}(K)$.

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