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On finite groups in which all minimal subgroups are BNA-subgroups

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Abstract – A subgroup H of a group G is said to be a BNA-subgroup of G if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. The purpose of this paper is first to give the best bound for the Fitting height of G if all minimal subgroups of G are BNA-subgroups of G, and next to give an answer to the question of He, Li, and Wang [Rend. Semin. Mat. Univ. Padova 136 (2016), 51–60]. Finally, we use a few BNA-subgroups of prime order to determine the structure of the finite groups. In fact, some new conditions for a finite group to be supersolvable have been given.

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1. Introduction

There has been much interest in investigating the structure of finite groups under the assumption that minimal subgroups of a finite group G have some kinds of properties in G [1, 2, 4, 12]. For example, Ballester-Bolinches and Guo [2] studied the class of finite groups for which every minimal subgroup is complemented. They prove that this class is just the class of all finite supersolvable groups with elementary abelian Sylow subgroups. Itô proved that if p is an odd prime and all minimal subgroups of order p of G are contained in the center of the finite group G, then G is p-nilpotent [9]. Later,

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Buckley [4] proved that if G is a finite group of odd order and all minimal subgroups of G are normal in G, then G is supersolvable. However, if $p = 2$ or G is a group of even order, then the above corresponding results are all not true. Although many interesting results in this line have been given, there are still many issues that people are very interested in, such as the bound for the 2-length if every minimal subgroup of a finite group is normal.

Now recall that a subgroup H of a group G is said to be abnormal if $x \in \langle H, H^x \rangle$ for all $x \in G$. The interesting thing is that a group G is a unique subgroup that is both normal and abnormal in G and that every maximal subgroup of G is either normal or abnormal. Many authors have investigated the structure of finite groups by using the assumption that subgroups of a finite group are either normal or abnormal (for example, see [5, 6, 10]). Recently, He, Li, and Wang [7] introduced a concept about subgroups – called BNA-subgroups. A subgroup H of a finite group G is called a BNA-subgroup of G if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. It is clear that both normal subgroups and abnormal subgroups are BNA-subgroups. Of course, there exist BNA-subgroups which are neither normal subgroups nor abnormal subgroups. For example, every cyclic subgroup of order 4 in S_4, the symmetric group of degree 4, is a BNA-subgroup, but it is neither normal nor abnormal, which means that it is meaningful to investigate the structure of finite groups using BNA-subgroups. In fact, He, Li, and Wang [7, 8] have studied the structure of finite groups with the assumption that all cyclic subgroups of prime power order or all minimal subgroups are BNA-subgroups, and many interesting results have been given. We should mention the following results:

Theorem 1.1 ([7, Theorem 3.3 (5)]). *Suppose that all minimal subgroups of a finite group G are BNA-subgroups of G. Then the Fitting height of G is bounded by* 4.

The authors also asked the following question at the end of the paper:

Question 1.2 ([7, Question, p. 60]). Are there finite groups G such that every minimal subgroup of G is a BNA-subgroup and $l_2(G) \geq 2$?

In the present paper we first prove that the Fitting height of G is bounded by 3 if all minimal subgroups of G are BNA-subgroups of G. Also, we find a finite group G such that the Fitting height of G is just 3 and all minimal subgroups of G are BNA-subgroups of G, which means that this bound 3 is best. Furthermore, this finite group G satisfies $l_2(G) = 2$. So the above question has been answered. In the rest of the paper, we continue to investigate the structure of finite groups by using the minimal subgroups. However, we drop the assumption that every minimal subgroup is a BNA-subgroup of G. We want to use a few BNA-subgroups of prime order to determine the structure of the finite groups. In fact, some new conditions for a finite group to be supersolvable have been given.

2. Preliminary results

In this section we collect some lemmas and some known concepts which will be used frequently in the sequel.

LEMMA 2.1 ([7, Lemma 2.1]). *Let G be a finite group, $H \leq K \leq G$, and $N \trianglelefteq G$. Suppose that H is a BNA-subgroup of G. Then*

(1) *H is a BNA-subgroup of K;*

(2) *HN is a BNA-subgroup of G;*

(3) *HN/N is a BNA-subgroup of G/N.*

LEMMA 2.2 ([7, Lemma 2.2 (2)]). *Let H be a BNA-subgroup of a finite group G. If H is subnormal in G, then H is normal in G.*

LEMMA 2.3 ([9, 9.1 Satz]). *If G is a finite p-supersolvable group, then G' is p-nilpotent. If G is a finite supersolvable group, then G' is nilpotent.*

Let p be a prime and G a finite p-solvable group. Then the upper $p'p$-series

$$1 = P_0 \leq N_0 < P_1 < N_1 < P_2 < \cdots < P_l \leq N_l = G$$

could be inductively defined by the rule that N_k/P_k is the greatest normal p'-subgroup of G/P_k, and P_{k+1}/N_k the greatest normal p-subgroup of G/N_k. The number l, which is the least integer such that $N_l = G$, is called the p-length of G, denoted by $l_p(G)$.

Recall that the product of all the normal p-nilpotent subgroups of a finite group G is clearly $O_{p'p}(G)$: that is, the maximal normal p-nilpotent subgroup of G, which is called the p-Fitting subgroup of G and denoted by $F_p(G)$.

Next we recall the concept of the p-Frattini subgroup. Set

$$S = \{M \text{ is a maximal subgroup in } G \mid [G : M] \text{ is a power of } p\}.$$

Then the p-Frattini subgroup of G, denoted by $\Phi_p(G)$, is defined as

$$\Phi_p(G) = \bigcap_{M \in S} M \quad \text{if } S \text{ is nonempty}$$

and $\Phi_p(G) = G$ if S is empty.

It is clear that $\Phi_p(G)$ is a characteristic subgroup of G and the Frattini subgroup $\Phi(G)$ of G is contained in $\Phi_p(G)$. It is also clear that $O_{p'}(G) \leq \Phi_p(G) \leq F_p(G)$ and $O_{p'}(G)$ is the Hall p'-subgroup of $\Phi_p(G)$ if G is p-solvable. Furthermore, we may prove the following:

Lemma 2.4. *Let p be a prime and let G be a finite p-solvable group. Then* $\Phi_p(G)/O_{p'}(G) = \Phi(G/O_{p'}(G))$.

Proof. It is clear that $\Phi(G/O_{p'}(G)) \leq \Phi_p(G)/O_{p'}(G)$. Conversely, let M be a maximal subgroup of G with $O_{p'}(G) \leq M$. If $\Phi_p(G) \nleq M$, then the maximality of M implies that $\Phi_p(G)M = G$. Thus $[G : M] = [\Phi_p(G) : \Phi_p(G) \cap M]$. Noticing that $O_{p'}(G) \leq M$ and $O_{p'}(G)$ is the Hall p'-subgroup of $\Phi_p(G)$, we see $[G : M]$ is a power of p and therefore $\Phi_p(G) \leq M$, a contradiction. Hence $\Phi_p(G) \leq M$ for every maximal subgroup of G with $O_{p'}(G) \leq M$, and therefore $\Phi_p(G)/O_{p'}(G) \leq \Phi(G/O_{p'}(G))$. The lemma is proved. ∎

3. The Fitting height and the 2-length

In this section we discuss the Fitting height and the 2-length of finite groups in which every minimal subgroup is a BNA-subgroup.

Theorem 3.1. *If all minimal subgroups of a finite group G are BNA-subgroups of G, then the Fitting height of G is bounded by* 3.

Proof. By [7, Theorem 3.3 (3)], G is p-supersolvable for every odd prime p dividing $|G|$ and therefore G' is p-nilpotent by Lemma 2.3 for every odd prime p in $\pi(G)$. Let T_p be the normal p-complement of G'. Then

$$\bigcap_{p \neq 2} T_p$$

is the Sylow 2-subgroup of G', denoted by P. It is clear that the Hall $2'$-subgroup of G' is nilpotent. It follows that

$$P \leq F_1(G), \quad G' \leq F_2(G), \quad G \leq F_3(G),$$

and so the Fitting height of G is bounded by 3. ∎

The following example illustrates that 3 is the best bound for the Fitting height of the kinds of finite groups above, and it also gives an answer to Question 1.2.

Example 3.2. Let $H = \langle c, d \mid c^9 = d^4 = 1,\ c^d = c^{-1} \rangle = \langle c \rangle \rtimes \langle d \rangle$. Then it is clear that $N = \langle c^3 \rangle \langle d^2 \rangle$ is normal in H with $d^2 \in Z(H)$ and $\langle c^3 \rangle \trianglelefteq H$, and that $H/N \simeq S_3$. Also, let Q_8 be a quaternion group of order 8. Since H/N can be seen as a subgroup of S_4 and the automorphism group $\mathrm{Aut}(Q_8)$ of Q_8 is isomorphic to S_4, the symmetric group of degree 4, there exists an action from H to Q_8 such that

$$C_H(Q_8) = \mathrm{Ker}(H \text{ on } Q_8) = N.$$

Now let G be the semidirect product $[Q_8]H$ of Q_8 and H by using the above action. Then $P = Q_8\langle d\rangle \in \mathrm{Syl}_2(G)$ and $R = \langle c\rangle \in \mathrm{Syl}_3(G)$. Furthermore, we may verify that every minimal subgroup of P and R is normal in G and therefore every minimal subgroup of G is normal. In this case, it is clear that

$$F_1(G) = Q_8\langle c^3\rangle\langle d^2\rangle, \quad F_2(G) = Q_8\langle c\rangle\langle d^2\rangle, \quad F_3(G) = G.$$

It is also clear that

$$O_{2'}(G) = \langle c^3\rangle, \quad O_{2'2}(G) = Q_8\langle c^3\rangle\langle d^2\rangle,$$
$$O_{2'22'}(G) = Q_8\langle c\rangle\langle d^2\rangle, \quad O_{2'22'2}(G) = G.$$

Thus the Fitting height of G is 3 and the 2-length of G is 2.

REMARK 3.3. By the above discussion, we see that 3 is the best bound for the Fitting height of finite groups in which very minimal subgroup is normal.

4. New sufficient conditions for p-supersolvability

In this section we use a few BNA-subgroups of prime order to determine the structure of the finite groups. In fact, some new conditions for a finite group to be supersolvable are given.

LEMMA 4.1. *Let p be a prime and G a finite p-solvable group. If H/K is a cyclic group for every p-chief factor H/K of G between $O_{p'}(G)$ and $F_p(G)$, then $G/C_G(F_p(G)/O_{p'}(G))$ is supersolvable.*

PROOF. Let $\overline{G} = G/O_{p'}(G)$. Then $O_p(\overline{G}) = F_p(G)/O_{p'}(G)$. By the hypotheses, we may assume that

$$\overline{1} = \overline{N_0} < \overline{N_1} < \cdots < \overline{N_t} = F_p(G)/O_{p'}(G)$$

is a part of the $\overline{G}$-chief series contained in $F_p(G)/O_{p'}(G)$ with $\overline{N_i}/\overline{N_{i-1}}$ cyclic of order p. It is clear that $t \geq 1$. If $t = 1$, then it follows from $|\overline{N_1}| = p$ that $\overline{G}/C_{\overline{G}}(\overline{N_1})$ is cyclic and therefore $\overline{G}/C_{\overline{G}}(\overline{N_1})$ is supersolvable. Now assume $t > 1$. By induction on t, $\overline{G}/C_{\overline{G}}(\overline{N_{t-1}})$ is supersolvable. Since $\overline{N_t}/\overline{N_1}$ is normal in $\overline{G}/\overline{N_1}$, we may use induction again for $\overline{G}/\overline{N_1}$ and we have that $\overline{G}/C_{\overline{G}}(\overline{N_t}/\overline{N_1})$ is supersolvable.

Set $C = C_{\overline{G}}(\overline{N_{t-1}}) \cap C_{\overline{G}}(\overline{N_t}/\overline{N_1})$. Then $\overline{G}/C$ is supersolvable. Since $\overline{N_t}/\overline{N_{t-1}}$ is cyclic, there exists $n_1 \in \overline{N_t}$ such that $\overline{N_t} = \langle \overline{N_{t-1}}, n_1\rangle$. Then, for any $x \in C$, $u^x = u$

for any $u \in \overline{N_{t-1}}$, and there exists $k(x) \in \overline{N_1}$ such that $n_1^x = n_1 k(x)$. Thus, noticing that $C_{\overline{G}}(\overline{N_{t-1}}) \leq C_{\overline{G}}(\overline{N_1})$, we see

$$n_1 k(xy) = n_1^{xy} = (n_1 k(x))^y = n_1^y k(x) = n_1 k(y) k(x)$$

if $y \in C$. The commutativity of $\overline{N_1}$ implies that $k(xy) = k(x)k(y)$. This means that

$$\mu\colon x^{\mu} = k(x)$$

is a homomorphism from C to $\overline{N_1}$ and the kernel of μ is just $C_{\overline{G}}(\overline{N_t})$. Hence $C/C_{\overline{G}}(\overline{N_t})$ is cyclic, so $\overline{G}/C_{\overline{G}}(\overline{N_t})$ is supersolvable and therefore $G/C_G(F_p(G)/O_{p'}(G))$ is supersolvable. ■

Lemma 4.2. *Let p be an odd prime and let G be a finite p-solvable group. If every minimal subgroup of order p in $F_p(G)/O_{p'}(G)$ is a BNA-subgroup of $G/O_{p'}(G)$, then every p-chief factor of G between $O_{p'}(G)$ and $F_p(G)$ is cyclic.*

Proof. Let $A/O_p(G)$ be a minimal subgroup of order p in $F_p(G)/O_{p'}(G)$. It is clear that $A/O_{p'}(G)$ is subnormal in $G/O_{p'}(G)$. Then, by Lemma 2.2 , $A/O_{p'}(G)$ is normal in $G/O_{p'}(G)$ and therefore $G/C_G(A/O_{p'}(G))$ is cyclic with exponent dividing $p-1$. Now let T be the subgroup of G generated by $O_{p'}(G)$, G', and all elements of the form g^{p-1} in G. Thus $T \leq C_G(A/O_{p'}(G))$. It follows from [9, 5.12 Satz] that for every p'-element x in T, $xO_{p'}(G)$ acts trivially on $F_p(G)/O_{p'}(G)$. Hence, $G/C_G(H/K)$ is abelian with exponent dividing $p-1$ for every p-chief factor H/K of G between $O_{p'}(G)$ and $F_p(G)$, and therefore, by [3, Lemma 1.3], H/K is cyclic for every p-chief factor H/K of G between $O_{p'}(G)$ and $F_p(G)$. ■

Theorem 4.3. *Let p be an odd prime and G a finite p-solvable group. If every minimal subgroup of order p in $F_p(G)/O_{p'}(G)$ is a BNA-subgroup of $G/O_{p'}(G)$, then G is p-supersolvable.*

Proof. Since G is p-solvable, $C_G(F_p(G)/O_{p'}(G)) \leq F_p(G)$ by [11, Theorem 9.3.1]. Then Lemmas 4.2 and 4.1 imply that $G/C_G(F_p(G)/O_{p'}(G))$ is supersolvable and so $G/F_p(G)$ is supersolvable. Clearly $(G/O_{p'}(G))/(F_p(G)/O_{p'}(G)) \cong G/F_p(G)$ and $F_p(G)/O_{p'}(G)$ is a p-group. By the hypothesis, every minimal subgroup of $F_p(G)/O_{p'}(G)$ is a BNA-subgroup of $G/O_{p'}(G)$, and so every minimal subgroup of $F_p(G)/O_{p'}(G)$ is normal in $G/O_{p'}(G)$ by Lemma 2.2. It follows from [13, Corollary 3] that $G/O_{p'}(G)$ is supersolvable. Therefore G is p-supersolvable. ■

Corollary 4.4. *Let p be an odd prime and G a finite p-solvable group. If every minimal subgroup of order p in $F_p(G)$ is a BNA-subgroup of G, then G is p-supersolvable.*

By using the arguments used in the proofs of Lemmas 4.1 and 4.2, we may prove the following results.

LEMMA 4.5. *Let p be a prime and let G be a finite p-solvable group. If H/K is cyclic for every p-chief factor H/K of G between $\Phi_p(G)$ and $F_p(G)$, then $G/C_G(F_p(G)/\Phi_p(G))$ is supersolvable.*

LEMMA 4.6. *Let p be an odd prime and let G be a finite p-solvable group. If every minimal subgroup of order p in $F_p(G)/\Phi_p(G)$ is a BNA-subgroup of $G/\Phi_p(G)$, then every p-chief factor of G between $\Phi_p(G)$ and $F_p(G)$ is cyclic.*

THEOREM 4.7. *Let p be an odd prime and let G be a finite p-solvable group. If every minimal subgroup of order p in $F_p(G)/\Phi_p(G)$ is a BNA-subgroup of $G/\Phi_p(G)$, then G is p-supersolvable.*

PROOF. It is clear that $O_{p'}(G/\Phi_p(G)) = 1$ and $O_p(G/\Phi_p(G)) = F_p(G)/\Phi_p(G) = F_p(G/\Phi_p(G))$. Since G is p-solvable, we have that $C_{G/\Phi_p(G)}(F_p(G/\Phi_p(G))) \leq F_p(G/\Phi_p(G)) = F_p(G)/\Phi_p(G)$ by [11, Theorem 9.3.1]. Clearly,

$$\begin{aligned} C_{G/\Phi_p(G)}(F_p(G/\Phi_p(G))) &= C_{G/\Phi_p(G)}(F_p(G)/\Phi_p(G)) \\ &= C_G(F_p(G)/\Phi_p(G))/\Phi_p(G). \end{aligned}$$

It follows from Lemmas 4.5 and 4.6 that $G/C_G(F_p(G)/\Phi_p(G))$ is supersolvable and so $G/F_p(G)$ is supersolvable. Clearly, $(G/\Phi_p(G))/(F_p(G)/\Phi_p(G)) \cong G/F_p(G)$ and $F_p(G)/\Phi_p(G)$ is a p-group. By the hypothesis, every minimal subgroup of $F_p(G)/\Phi_p(G)$ is a BNA-subgroup of $G/\Phi_p(G)$, and so every minimal subgroup of $F_p(G)/\Phi_p(G)$ is normal in $G/\Phi_p(G)$ by Lemma 2.2. It follows from [13, Corollary 3] that $G/\Phi_p(G)$ is supersolvable. Since $(G/O_{p'}(G))/(\Phi_p(G)/O_{p'}(G)) = (G/O_{p'}(G))/\Phi(G/O_{p'}(G))$, $G/O_{p'}(G)$ is supersolvable. Therefore G is p-supersolvable. ■

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