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Note on the relation between enhanced ind-sheaves and enhanced subanalytic sheaves

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Abstract – In this paper we shall explain a relation between Theorem 9.5.3 by D'Agnolo and Kashiwara [Publ. Math. Inst. Hautes Études Sci. 123 (2016), 69–197] and Theorem 6.3 by Kashiwara [Jpn. J. Math. 11 (2016) 113–149]. For this purpose, we shall prove that there exists a fully faithful functor from the triangulated category of enhanced subanalytic sheaves to that of enhanced ind-sheaves.

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1. Introduction

Kashiwara and Schapira [10] introduced the notion of ind-sheaves and subanalytic sheaves to treat "sheaves" of functions with tempered growth conditions. Ind-sheaves are defined as ind-objects of the category of sheaves of vector spaces with compact support. Subanalytic sheaves are defined as sheaves on subanalytic sites. Moreover, the authors proved that there exists a fully faithful functor from the category of subanalytic sheaves to the category of ind-sheaves, and its essential image is equal to the category of ind-objects of $\mathbb{R}$-constructible sheaves with compact support.

After a groundbreaking development in the theory of irregular meromorphic connections by Kedlaya [14, 15] and Mochizuki [16, 17], D'Agnolo and Kashiwara introduced the notion of enhanced ind-sheaves extending the notion of ind-sheaves and established the Riemann–Hilbert correspondence for analytic irregular holonomic $\mathcal{D}$-modules in [1] as below (see [7] for the algebraic case). Let X be a complex manifold. Then

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there exists a fully faithful functor $\mathrm{Sol}_X^{\mathrm{E}}$ which is called the enhanced solution functor (see [1, Def. 9.1.1] and also Definition 3.31) from the full triangulated subcategory $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ of the derived category of $\mathcal{D}_X$-modules consisting of objects with holonomic cohomologies to the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ of $\mathbb{R}$-constructible enhanced ind-sheaves:

$$\mathrm{Sol}_X^{\mathrm{E}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X). \tag{1.1}$$

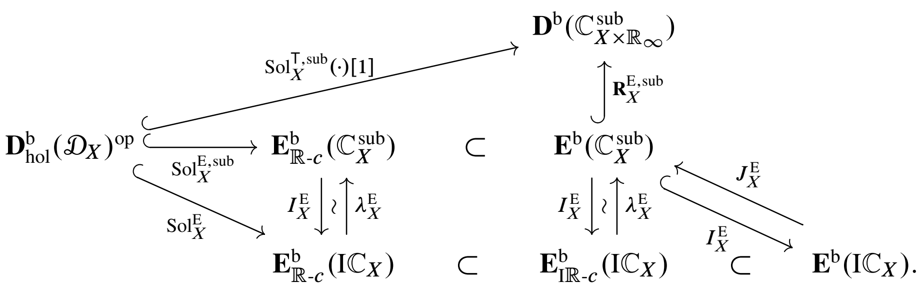
Moreover, Mochizuki characterized its essential image by the curve test [18, 19]. In [4] (see also [5]), the author defined $\mathbb{C}$-constructibility for enhanced ind-sheaves and proved that they are nothing but objects of its essential image. Namely, we obtain an equivalence of categories between the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}-c}(\mathrm{I}\mathbb{C}_X)$ of $\mathbb{C}$-constructible enhanced ind-sheaves:

$$\mathrm{Sol}_X^{\mathrm{E}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}-c}(\mathrm{I}\mathbb{C}_X).$$

At the 16th Takagi Lectures,[1] Kashiwara explained a similar result to (1.1) by using "enhanced subanalytic sheaves" instead of enhanced ind-sheaves as below. We denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$ the derived category of subanalytic sheaves on a bordered space $X \times \mathbb{R}_\infty$; see Section 3.1 for the definition. Then there exists a fully faithful functor $\mathrm{Sol}_X^{\mathsf{T}}$ (see [8, §5.4] and Definition 3.34) from $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ to $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$:

$$\mathrm{Sol}_X^{\mathsf{T}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}). \tag{1.2}$$

In this paper we shall explain the relation between (1.1) and (1.2). For this purpose, we shall prove that there exists a fully faithful functor from the triangulated category of enhanced subanalytic sheaves to that of enhanced ind-sheaves. Although it may be known by experts, it is not in the literature to our knowledge. The main results of this paper are Theorems 3.15, 3.20, 3.38 and 3.39. One can summarize these results in the following commutative diagram:



(1) The 16th Takagi Lectures took place at Graduate School of Mathematical Sciences, The University of Tokyo, on November 28 and 29, 2015.

See Section 2.6 for the definition of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$, Section 3.1 for the definition of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$, Section 3.3 for the definitions of $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X)$, $\mathbf{R}^{\mathrm{E,sub}}_X$, Section 3.4 for the definitions of $\mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)$, I^{E}_X, J^{E}_X, $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, Definition 3.19 for $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)$, Definition 3.31 for $\mathrm{Sol}^{\mathrm{E}}_X$, Definition 3.34 for $\mathrm{Sol}^{\mathrm{T,sub}}_X$ and Definition 3.36 for $\mathrm{Sol}^{\mathrm{E,sub}}_X$.

2. Preliminary notions and results

In this section we briefly recall some basic notions and results which will be used in this paper.

2.1 – *Subanalytic sheaves*

We shall briefly recall the notion of subanalytic sheaves. References are made to [10, §6] and [20].

Let $\Bbbk$ be a field and M a real analytic manifold. We denote by $\mathcal{O}\mathrm{p}^{\mathrm{sub}}_M$ the category of subanalytic open subsets of M. Then we can endow $\mathcal{O}\mathrm{p}^{\mathrm{sub}}_M$ with the following Grothendieck topology: a subset $S \subset \mathcal{O}\mathrm{b}((\mathcal{O}\mathrm{p}^{\mathrm{sub}}_M)_U)$ is a covering of $U \in \mathcal{O}\mathrm{p}^{\mathrm{sub}}_M$ if for any compact subset K of M there exists a finite subset $S_0 \subset S$ of S such that

$$U \cap K = \Big(\bigcup_{V\in S_0} V\Big) \cap K.$$

We denote such a site by M^{sub} and call it the subanalytic site.

A subanalytic sheaf of $\Bbbk$-modules on M is a sheaf of $\Bbbk$-modules on the subanalytic site M^{sub}. We shall write $\mathrm{Mod}(\Bbbk^{\mathrm{sub}}_M)$ instead of $\mathrm{Mod}(\Bbbk_{M^{\mathrm{sub}}})$. Note that it is abelian. Note also that there exists the natural morphism $\rho_M\colon M \to M^{\mathrm{sub}}$ of sites. Then we have a natural left exact embedding $\rho_{M*}\colon \mathrm{Mod}(\mathbb{C}_M) \to \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_M)$ of categories. It has an exact left adjoint ρ_M^{-1} that has in turn an exact fully faithful left adjoint functor $\rho_{M!}$. We denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_M)$ the derived category of $\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_M)$. Note that there exist the six Grothendieck operations[2] $\otimes$, $\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{sub}}$, $\mathrm{R}f_*$, $\mathrm{R}f_{!!}$, f^{-1}, $f^!$ for a real analytic map $f\colon M \to N$. Moreover, we set $\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{sub}} := \rho_M^{-1} \circ \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{sub}}$. Note also that these functors have many properties similar to classical sheaves. We shall skip an explanation of this.

2.2 – *Ind-sheaves*

Let us briefly recall the notion of ind-sheaves. References are made to [10] and [11].

(2) We shall use the symbol $\mathrm{R}f_{!!}$ instead of $\mathrm{R}f_!$.

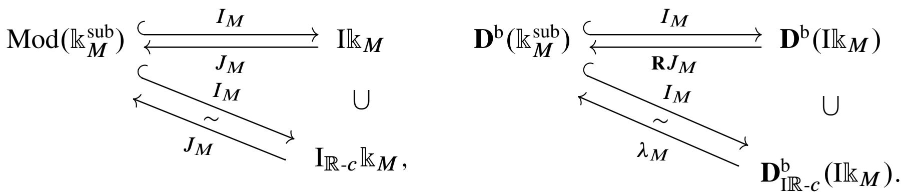
Let $\Bbbk$ be a field and M a good topological space (that is, a topological space which is locally compact, Hausdorff, countable at infinity and has finite flabby dimension). We denote by $\mathrm{Mod}^c(\Bbbk_M)$ the category of sheaves of $\Bbbk$-vector spaces on M with compact support. An ind-sheaf of $\Bbbk$-vector spaces on M is an ind-object of $\mathrm{Mod}^c(\Bbbk_M)$, that is, an inductive limit

$$\text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{F}_i := \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{Mod}^c(\Bbbk_M)}(\,\cdot\,, \mathcal{F}_i)$$

of a small filtrant inductive system $\{\mathcal{F}_i\}_{i\in I}$ in $\mathrm{Mod}^c(\Bbbk_M)$. Let us denote by $\mathrm{I}\Bbbk_M$ the category of ind-sheaves of $\Bbbk$-vector spaces on M. Note that it is abelian. Note also that there exists a natural exact embedding $\iota_M\colon \mathrm{Mod}(\Bbbk_M) \to \mathrm{I}\Bbbk_M$. It has an exact left adjoint α_M, that has in turn an exact fully faithful left adjoint functor β_M. We denote by $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_M)$ the derived category of $\mathrm{I}\Bbbk_M$. Note that there exist the six Grothendieck operations $\otimes$, $\mathbf{R}\mathcal{I}hom$, f^{-1}, $\mathrm{R}f_*$, $f^!$ and $\mathrm{R}f_{!!}$ for a continuous map $f\colon M \to N$. Moreover, we set $\mathbf{R}\mathcal{H}om := \alpha_M \circ \mathbf{R}\mathcal{I}hom$. Note also that these functors have many properties similar to classical sheaves. We shall skip an explanation of this.

2.3 – *Relation between ind-sheaves and subanalytic sheaves*

Let us briefly recall the relation between ind-sheaves and subanalytic sheaves. The results are summarized by the following commutative diagrams:



References are made to [10, §§6.3, 7.1] and [21, §A.2].

Let $\Bbbk$ be a field and M a real analytic manifold. We denote by $\mathrm{Mod}^c_{\mathbb{R}\text{-}c}(\Bbbk_M)$ the abelian category of $\mathbb{R}$-constructible sheaves on M with compact support and denote by $\mathrm{I}_{\mathbb{R}\text{-}c}\Bbbk_M$ the category of ind-objects of $\mathrm{Mod}^c_{\mathbb{R}\text{-}c}(\Bbbk_M)$. Moreover, let us denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_M)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_M)$ consisting of objects whose cohomologies are contained in $\mathrm{I}_{\mathbb{R}\text{-}c}\Bbbk_M$. Then there exists a functor

$$J_M\colon \mathrm{I}\Bbbk_M \to \mathrm{Mod}(\Bbbk_M^{\mathrm{sub}}), \quad \text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{F}_i \mapsto \varinjlim_{i\in I} \rho_{M*}\mathcal{F}_i.$$

Note also that the functor J_M is left exact and admits a left adjoint $I_M\colon \mathrm{Mod}(\Bbbk_M^{\mathrm{sub}}) \to \mathrm{I}\Bbbk_M$ which is fully faithful, exact and commutes with filtrant inductive limits.

Theorem 2.1 ([10, Thm. 6.3.5], see also [21, A.2.1]). *There exists an equivalence of abelian categories:*

$$\mathrm{Mod}(\Bbbk_M^{\mathrm{sub}}) \underset{J_M}{\overset{I_M}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathrm{I}_{\mathbb{R}\text{-}c}\Bbbk_M.$$

Furthermore, there exists an equivalence of triangulated categories:

$$\mathbf{D}^{\mathrm{b}}(\Bbbk_M^{\mathrm{sub}}) \underset{\mathbf{R}J_M}{\overset{I_M}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_M).$$

We shall denote by $\lambda_M\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_M) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\Bbbk_M^{\mathrm{sub}})$ the inverse functor of $I_M\colon \mathbf{D}^{\mathrm{b}}(\Bbbk_M^{\mathrm{sub}}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_M)$.

Let us summarize the commutativity between the various functors and functors I, $\mathbf{R}J$. Let $f\colon M \to N$ be a morphism of real analytic manifolds. Then we have

$$\mathbf{R}J_M \mathbf{R}\mathcal{I}hom(I_M(\cdot), \cdot) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\cdot, \mathbf{R}J_M(\cdot)).$$

and

$$\begin{aligned}
&\alpha_M \circ I_M \simeq \rho_M^{-1}, \\
&I_M \circ \rho_{M!} \simeq \beta_M, \\
&I_M \circ f^{-1} \simeq f^{-1} \circ I_N, \\
&\mathbf{R}f_{!!} \circ I_M \simeq I_N \circ \mathbf{R}f_{!!}, \\
&I_M \circ f^{!} \simeq f^{!} \circ I_N, \\
&I_M(\cdot \otimes \cdot) \simeq I_M(\cdot) \otimes I_M(\cdot),
\end{aligned}
\qquad
\begin{aligned}
&\mathbf{R}J_M \circ \iota_M \simeq \rho_{M*}, \\
&\rho_M^{-1} \circ \mathbf{R}J_M \simeq \alpha_M, \\
&\mathbf{R}J_M \circ \beta_M \simeq \rho_{M!}, \\
&\mathbf{R}f_* \circ \mathbf{R}J_M \simeq \mathbf{R}J_N \circ \mathbf{R}f_*, \\
&\mathbf{R}J_M \circ f^{!} \simeq f^{!} \circ \mathbf{R}J_N,
\end{aligned}
\qquad
\begin{aligned}
&I_M \circ \rho_{M*}^{\mathbb{R}\text{-}c} \simeq \iota_M|_{\mathrm{Mod}_{\mathbb{R}\text{-}c}(\Bbbk_M)}, \\
&\lambda_M \circ f^{-1} \simeq f^{-1} \circ \lambda_N, \\
&\mathbf{R}f_{!!} \circ \lambda_M \simeq \lambda_N \circ \mathbf{R}f_{!!}, \\
&\lambda_M(\cdot \otimes \cdot) \simeq \lambda_M(\cdot) \otimes \lambda_M(\cdot).
\end{aligned}$$

2.4 – *Bordered spaces*

We shall briefly recall the notion of bordered spaces. See [1, §3.2] and [3, §2.1] for the details.

A bordered space is a pair $M_\infty = (M, \check{M})$ of a good topological space $\check{M}$ and an open subset M of $\check{M}$. A morphism $f\colon (M, \check{M}) \to (N, \check{N})$ of bordered spaces is a continuous map $f\colon M \to N$ such that the first projection $\check{M} \times \check{N} \to \check{M}$ is proper on the closure $\overline{\Gamma}_f$ of the graph Γ_f of f in $\check{M} \times \check{N}$. The category of good topological spaces is embedded into that of bordered spaces by the identification $M = (M, M)$. Note that we have the morphism $j_{M_\infty}\colon M_\infty \to \check{M}$ of bordered spaces given by the embedding $M \hookrightarrow \check{M}$. For a locally closed subset $Z \subset M$ of M, we set $Z_\infty := (Z, \overline{Z})$, where $\overline{Z}$ is the closure of Z in $\check{M}$ and denote by $i_{Z_\infty}\colon Z_\infty \to M_\infty$ the morphism of bordered spaces given by the natural embedding $Z \hookrightarrow M$.

2.5 – *Ind-sheaves on bordered spaces*

Let us briefly recall the notion of ind-sheaves on bordered spaces. The results are summarized by the following (non-commutative) diagram:

$$\begin{array}{ccccc}
\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}\setminus M}) & \xrightarrow[\mathbf{R}i_{\check{M}\setminus M!!}]{} & \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}}) & \overset{\mathbf{l}_{M_\infty}}{\underset{\mathbf{r}_{M_\infty}}{\overleftarrow{\underrightarrow{\ \mathbf{q}_{M_\infty}\ }}}} & \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}})/\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}\setminus M}) \\
 & & & & \beta_{M_\infty}\uparrow \ \downarrow\alpha_{M_\infty} \ \uparrow\iota_{M_\infty} \\
 & & & & \mathbf{D}^{\mathrm{b}}(\Bbbk_M) \simeq \mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\Bbbk_{\check{M}})/\mathbf{D}^{\mathrm{b}}(\Bbbk_{\check{M}\setminus M}).
\end{array}$$

References are made to [1, §3].

Let $\Bbbk$ be a field and $M_\infty = (M, \check{M})$ a bordered space. A quotient category

$$\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}})/\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}\setminus M})$$

is called the category of ind-sheaves on M_∞, where $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}\setminus M})$ is identified with its essential image in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}})$ by the fully faithful functor $\mathbf{R}i_{\check{M}\setminus M!!}$. Here, $i_{\check{M}\setminus M}\colon \check{M}\setminus M \to \check{M}$ is the closed embedding. An object of $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is called an ind-sheaf on M_∞. The quotient functor

$$\mathbf{q}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$$

has a left adjoint $\mathbf{l}_{M_\infty}$ and a right adjoint $\mathbf{r}_{M_\infty}$, both fully faithful, defined by

$$\begin{aligned}
\mathbf{l}_{M_\infty}&\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}}), \quad \mathbf{q}F \mapsto \iota_{\check{M}}\Bbbk_M \otimes F,\\
\mathbf{r}_{M_\infty}&\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}}), \quad \mathbf{q}F \mapsto \mathbf{R}\mathcal{I}hom(\iota_{\check{M}}\Bbbk_M, F).
\end{aligned}$$

Note that there exists an embedding functor

$$\iota_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk_M) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), \quad \mathcal{F} \mapsto j_{M_\infty}^{-1}\iota_{\check{M}}j_{M!}\mathcal{F} \ \ (\simeq j_{M_\infty}^{-1}\iota_{\check{M}}\mathbf{R}j_{M*}\mathcal{F}),$$

which has an exact left adjoint

$$\alpha_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk_M), \quad F \mapsto j_M^{-1}\alpha_{\check{M}}\mathbf{R}j_{M_\infty!!}F \ \ (\simeq j_M^{-1}\alpha_{\check{M}}\mathbf{R}j_{M_\infty *}F)$$

that has in turn an exact fully faithful left adjoint functor

$$\beta_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk_M) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), \quad \mathcal{F} \mapsto j_{M_\infty}^{-1}\beta_{\check{M}}j_{M!}\mathcal{F} \ \ (\simeq j_{M_\infty}^{-1}\beta_{\check{M}}\mathbf{R}j_{M*}\mathcal{F}).$$

Note also that there exist the six Grothendieck operations $\otimes$, $\mathbf{R}\mathcal{I}hom$, $\mathrm{R}f_*$, $\mathrm{R}f_{!!}$, f^{-1}, $f^!$ for a morphism $f\colon M_\infty \to N_\infty$ of bordered spaces. Moreover, we set $\mathbf{R}\mathcal{H}om :=$

$\alpha_{M_\infty} \circ \mathbf{R}\mathcal{I}\mathrm{hom}$. Note that these functors have many properties similar to classical sheaves. We shall skip an explanation of this. We just recall that the functor $j_{M_\infty}^{-1}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is isomorphic to the quotient functor and the functor $\mathbf{R}j_{M_\infty !!}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}})$ (resp. $\mathbf{R}j_{M_\infty *}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}})$) is isomorphic to the functor $\mathbf{l}_{M_\infty}$ (resp. $\mathbf{r}_{M_\infty}$).

It is clear that the quotient category

$$\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\Bbbk_{\check{M}})/\mathbf{D}^{\mathrm{b}}(\Bbbk_{\check{M}\setminus M})$$

is equivalent to the derived category $\mathbf{D}^{\mathrm{b}}(\Bbbk_M)$ of the abelian category $\mathrm{Mod}(\Bbbk_M)$. We sometimes write $\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty})$ for $\mathbf{D}^{\mathrm{b}}(\Bbbk_M)$, when considered as a full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$.

2.6 – *Enhanced ind-sheaves*

We shall briefly recall the notion of enhanced ind-sheaves on bordered spaces:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty}) & \overset{\mathbf{L}_{M_\infty}}{\underset{\mathbf{R}_{M_\infty}}{\overset{\xrightarrow{\ \mathbf{Q}_{M_\infty}\ }}{\leftleftarrows}}} & \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})/\pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \\
 & \nwarrow^{\pi^{-1}} & e_{M_\infty}\uparrow\ \downarrow \mathrm{Ish}_{M_\infty} \\
\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}) \simeq \mathbf{D}^{\mathrm{b}}(\Bbbk_M) & \overset{\beta_{M_\infty}}{\underset{\iota_{M_\infty}}{\overset{\xleftarrow{\ \alpha_{M_\infty}\ }}{\rightrightarrows}}} & \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}).
\end{array}$$

References are made to [2, 12]. We also refer to [1, 13] for the notion of enhanced ind-sheaves on good topological spaces.

Let $\Bbbk$ be a field and $M_\infty = (M, \check{M})$ a bordered space. We set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ for $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$, and let $t \in \mathbb{R}$ be the affine coordinate. We consider the morphisms of bordered spaces

$$M_\infty \times \mathbb{R}^2_\infty \xrightarrow{p_1, p_2, \mu} M_\infty \times \mathbb{R}_\infty \xrightarrow{\pi} M_\infty$$

given by the maps $p_1(x, t_1, t_2) := (x, t_1)$, $p_2(x, t_1, t_2) := (x, t_2)$, $\mu(x, t_1, t_2) := (x, t_1 + t_2)$ and $\pi(x, t) := x$. Then the convolution functors for ind-sheaves on $M_\infty \times \mathbb{R}_\infty$,

$$(\cdot) \overset{+}{\otimes} (\cdot)\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty}) \times \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty}),$$
$$\mathbf{R}\mathcal{I}\mathrm{hom}^+(\cdot, \cdot)\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$$

are defined by

$$F_1 \overset{+}{\otimes} F_2 := \mathrm{R}\mu_{!!}(p_1^{-1}F_1 \otimes p_2^{-1}F_2),$$
$$\mathbf{R}\mathcal{I}\mathrm{hom}^+(F_1, F_2) := \mathrm{R}p_{1*}\mathbf{R}\mathcal{I}\mathrm{hom}(p_2^{-1}F_1, \mu^! F_2),$$

where $F_1, F_2 \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty})$. Then the triangulated category of enhanced ind-sheaves on a bordered space M_∞ is defined by

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty})/\pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}).$$

An object of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is called an enhanced ind-sheaf on M_∞. The quotient functor $\mathbf{Q}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ has fully faithful left and right adjoints

$$\begin{aligned}
&\mathbf{L}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty}),\\
&\mathbf{Q}_{M_\infty}(F) \mapsto \iota_{M_\infty \times \mathbb{R}_\infty}(\Bbbk_{\{t \geq 0\}} \oplus \Bbbk_{\{t \leq 0\}}) \overset{+}{\otimes} F,\\
&\mathbf{R}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty}),\\
&\mathbf{Q}_{M_\infty}(F) \mapsto \mathbf{R}\mathcal{I}\mathrm{hom}^{+}(\iota_{M_\infty \times \mathbb{R}_\infty}(\Bbbk_{\{t \geq 0\}} \oplus \Bbbk_{\{t \leq 0\}}), F).
\end{aligned}$$

Here, $\{t \geq 0\}$ stands for $\{(x,t) \in M \times \mathbb{R} \mid t \geq 0\}$ and $\{t \leq 0\}$ is defined similarly.

For a morphism $f\colon M_\infty \to N_\infty$ of bordered spaces, we have the six Grothendieck operations $\overset{+}{\otimes}$, $\mathbf{R}\mathcal{I}\mathrm{hom}^{+}$, $\mathbf{E}f^{-1}$, $\mathbf{E}f_*$, $\mathbf{E}f^{!}$, $\mathbf{E}f_{!!}$ for enhanced subanalytic sheaves on bordered spaces. Note that these functors have many properties similar to classical sheaves. We shall skip an explanation of this. Moreover, we have external hom functors $\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E}}$, $\mathbf{R}\mathcal{H}om^{\mathrm{E}} := \alpha_{M_\infty} \circ \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E}}$ and functors

$$\begin{aligned}
\pi^{-1}(\cdot) \otimes (\cdot)\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \times \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}),\\
\mathbf{R}\mathcal{I}\mathrm{hom}(\pi^{-1}(\cdot), \cdot)\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})^{\mathrm{op}} \times \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})
\end{aligned}$$

which are defined by

$$\begin{aligned}
\pi^{-1}F \otimes K := \mathbf{Q}_{M_\infty}(\pi^{-1}F \otimes \mathbf{L}^{\mathrm{E}}_{M_\infty}K),\\
\mathbf{R}\mathcal{I}\mathrm{hom}(\pi^{-1}F, K) := \mathbf{Q}_{M_\infty}\mathbf{R}\mathcal{I}\mathrm{hom}(\pi^{-1}F, \mathbf{R}^{\mathrm{E}}_{M_\infty}K).
\end{aligned}$$

We set

$$\Bbbk^{\mathrm{E}}_{M_\infty} := \mathbf{Q}_{M_\infty}\mathbf{q}_{M_\infty}\Big(“\varinjlim_{a \to +\infty}” \iota_{\check{M} \times \overline{\mathbb{R}}}\Bbbk_{\{t \geq a\}}\Big) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}).$$

Then we have a natural embedding functor

$$e_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), \quad F \mapsto \Bbbk^{\mathrm{E}}_{M_\infty} \otimes \pi^{-1}F.$$

Let us define $\omega^{\mathrm{E}}_{M_\infty} := e_{M_\infty}(\iota_{M_\infty}\omega_M) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$, where $\omega_M \in \mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty})$ ($\simeq \mathbf{D}^{\mathrm{b}}(\Bbbk_M)$) is the dualizing complex; see [9, Def. 3.1.16] for the details. Then we have the Verdier duality functor for enhanced ind-sheaves on bordered spaces,

$$\mathrm{D}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), \quad K \mapsto \mathbf{R}\mathcal{I}\mathrm{hom}^{+}(K, \omega^{\mathrm{E}}_{M_\infty}).$$

Let $i_0 \colon M_\infty \to M_\infty \times \mathbb{R}_\infty$ be a morphism of bordered spaces induced by $x \mapsto (x, 0)$. We set

$$\begin{aligned} \mathrm{Ish}_{M_\infty} &:= i_0^! \circ \mathbf{R}^{\mathrm{E}}_{M_\infty} \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), \\ \mathrm{sh}_{M_\infty} &:= \alpha_{M_\infty} \circ \mathrm{Ish}_{M_\infty} \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk_M) \end{aligned}$$

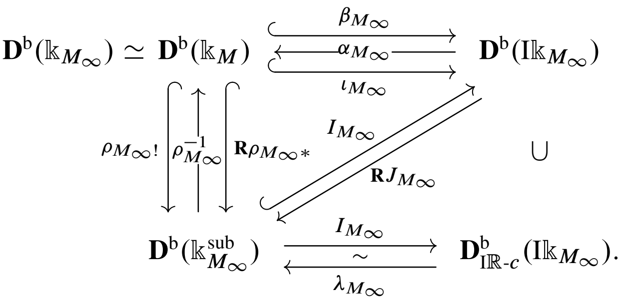
and call them the ind-sheafification functor and the sheafification functor, for enhanced ind-sheaves on bordered spaces, respectively. Note that a pair $(e_{M_\infty}, \mathrm{Ish}_{M_\infty})$ is an adjoint pair and there exist isomorphisms $F \xrightarrow{\sim} \mathrm{Ish}_{M_\infty} e_{M_\infty} F$ for $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_M)$ and $\mathcal{F} \xrightarrow{\sim} \mathrm{sh}_{M_\infty} e_{M_\infty} \iota_{M_\infty} \mathcal{F}$ for $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_M)$. See [3, §3] for details.

3. Main results

The main theorems of this paper are Theorems 3.15, 3.20, 3.38 and 3.39.

3.1 – *Subanalytic sheaves on real analytic bordered spaces*

The notion of subanalytic sheaves on bordered spaces was introduced by Kashiwara. Although it has already been explained in [8, §§3.4–3.7], in this subsection we shall explain it again in detail.[3] The results are summarized by the following (non-commutative) diagram:



A real analytic bordered space is a bordered space $M_\infty = (M, \check{M})$ such that $\check{M}$ is a real analytic manifold and M is an open subanalytic subset. A morphism $f \colon (M, \check{M}) \to (N, \check{N})$ of real analytic bordered spaces is a morphism of bordered spaces such that the

(3) Kashiwara [8] introduced the notion of subanalytic sheaves on subanalytic bordered spaces. In this paper, we shall only consider them on real analytic bordered spaces.

graph Γ_f of f is a subanalytic subset of $\check{M} \times \check{N}$. Note that a morphism $\check{f}\colon \check{M} \to \check{N}$ of real analytic manifolds such that $\check{f}(M) \subset N$ induces a morphism of bordered spaces of real analytic bordered spaces from $(M, \check{M})$ to $(N, \check{N})$. The category of real analytic manifolds is embedded into that of real analytic bordered spaces by the identification $M = (M, M)$.

Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. We denote by $\mathcal{O}\mathrm{p}^{\mathrm{sub}}_{M_\infty}$ the category of open subsets of M which are subanalytic in $\check{M}$. Note that the category $\mathcal{O}\mathrm{p}^{\mathrm{sub}}_{M_\infty}$ can be endowed with the following Grothendieck topology:

a subset $S \subset \mathcal{O}\mathrm{b}((\mathcal{O}\mathrm{p}^{\mathrm{sub}}_{M_\infty})_U)$ is a covering of $U \in \mathcal{O}\mathrm{b}(\mathcal{O}\mathrm{p}^{\mathrm{sub}}_{M_\infty})$ if for any compact subset K of $\check{M}$ there exists a finite subset $S' \subset S$ of S such that $K \cap U = K \cap \bigcup_{V \in S'} V$.

This Grothendieck topology is the one induced from that of $\check{M}$.

Let us denote by M^{sub}_∞ the site $\mathcal{O}\mathrm{p}^{\mathrm{sub}}_{M_\infty}$ with the above Grothendieck topology and denote by $\mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ the category of sheaves of $\Bbbk$-vector spaces on the site M^{sub}_∞. Note that the category $\mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ is abelian. Note also that there exists the natural morphism $\rho_{M_\infty}\colon M \to M^{\mathrm{sub}}_\infty$ of sites. Then we have a natural left exact embedding

$$\rho_{M_\infty *}\colon \mathrm{Mod}(\Bbbk_M) \to \mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

of categories. It has an exact left adjoint $\rho^{-1}_{M_\infty}\colon \mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathrm{Mod}(\Bbbk_M)$, which has in turn an exact fully faithful left adjoint functor $\rho_{M_\infty !}\colon \mathrm{Mod}(\Bbbk_M) \to \mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. Note that the restriction $\rho^{\mathbb{R}\text{-}c}_{M_\infty *}$ of $\rho_{M_\infty *}$ to $\mathrm{Mod}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty})$ is exact.

We denote by $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ the derived category of $\mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. For a morphism $f\colon M_\infty \to N_\infty$ of real analytic bordered spaces, we have the Grothendieck operations[4] $\otimes$, $\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{sub}}$, $\mathrm{R}f_*$, $\mathrm{R}f_{!!}$, f^{-1}, $f^!$ for subanalytic sheaves on bordered spaces and set $\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{sub}} := \rho^{-1}_{M_\infty} \circ \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{sub}}$. Then these functors have many properties similar to classical (subanalytic) sheaves.

Proposition 3.1. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces and $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_M)$, $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_N)$, $\mathcal{J} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty})$.*

(1) $\mathbf{R}\rho_{M_\infty *}\mathcal{H}\mathrm{om}(\rho^{-1}_{M_\infty}\mathcal{F}, \mathcal{K}) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{sub}}(\mathcal{F}, \mathbf{R}\rho_{M_\infty *}\mathcal{K})$,
$\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{sub}}(\rho_{M_\infty !}\mathcal{K}, \mathcal{F}) \simeq \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{K}, \rho^{-1}_{M_\infty}\mathcal{F})$;

(4) We shall use the symbol $\mathrm{R}f_{!!}$ instead of $\mathrm{R}f_!$.

(2) $$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathbf{R}f_{!!}\mathcal{F},\mathcal{G}) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}, f^!\mathcal{G}),$$
$$\mathbf{R}f_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(f^{-1}\mathcal{G},\mathcal{F}) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{G},\mathbf{R}f_*\mathcal{F}),$$
$$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}_1\otimes\mathcal{F}_2,\mathcal{F}) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}_1,\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}_2,\mathcal{F}));$$

(3) $$f^{-1}(\mathcal{F}_1\otimes\mathcal{F}_2) \simeq f^{-1}\mathcal{F}_1\otimes f^{-1}\mathcal{F}_2,$$
$$\mathbf{R}f_{!!}(\mathcal{F}\otimes f^{-1}\mathcal{G}) \simeq \mathbf{R}f_{!!}\mathcal{F}\otimes\mathcal{G},$$
$$f^!\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{G}_1,\mathcal{G}_2) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(f^{-1}\mathcal{G}_1, f^!\mathcal{G}_2);$$

(4) *for a cartesian diagram*

$$\begin{array}{ccc} M'_\infty & \xrightarrow{\ f'\ } & N'_\infty \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ M_\infty & \xrightarrow[\ f\ ]{} & N_\infty, \end{array}$$

we have $g^{-1}\mathbf{R}f_{!!}\mathcal{F} \simeq \mathbf{R}f'_{!!}g'^{-1}\mathcal{F}$, $g^!\mathbf{R}f_*\mathcal{F} \simeq \mathbf{R}f'_*g'^!\mathcal{F}$;

(5) $$f^!(\mathcal{G}\otimes\rho_{N_\infty!}\mathcal{L}) \simeq f^!\mathcal{G}\otimes\rho_{M_\infty!}f^{-1}\mathcal{L},$$
$$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{J},\mathcal{F})\otimes\rho_{M_\infty!}\mathcal{K} \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{J},\mathcal{F}\otimes\rho_{M_\infty!}\mathcal{K});$$

(6) *we have the commutativity of the various functors:*

	$\otimes$	f^{-1}	$\mathbf{R}f_*$	$f^!$	$\mathbf{R}f_{!!}$
$\mathbf{R}\rho_*$	$\times$	$\times$	$\circ$	$\circ$	$\times$
$\rho_*^{\mathbb{R}\text{-}c}$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$
ρ^{-1}	$\circ$	$\circ$	$\circ$	$\times$	$\circ$
$\rho_!$	$\circ$	$\circ$	$\times$	$\times$	$\times$

where "$\circ$" means that the functors commute, and "$\times$" that they do not.

Since the proof of this proposition is similar to the case of classical sheaves, we shall skip the proof.

From now on, we shall describe a relation between ind-sheaves on M_∞ and subanalytic sheaves on M_∞. Let us recall the diagram:

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}(\Bbbk_M^{\mathrm{sub}}) & \underset{\mathbf{R}J_M}{\overset{I_M}{\rightleftarrows}} & \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_M) \\ & \searrow^{I_M}_{\sim}\ \nwarrow_{\lambda_M} & \cup \\ & & \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_M). \end{array}$$

See Section 2.3 for the details. Recall also that $j_{M_\infty}\colon M_\infty \to \check{M}$ is a morphism of real analytic bordered spaces associated to the natural embedding $j_M\colon M \hookrightarrow \check{M}$. Then for any $\mathcal{F} \in \mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $\mathcal{G} \in \mathrm{Mod}(\Bbbk^{\mathrm{sub}}_{\check{M}})$ we have

$$\begin{aligned} &j_{M_\infty}^{-1}\mathbf{R}j_{M_\infty *}\mathcal{F} \xleftarrow{\sim} \mathcal{F}, && j_{M_\infty}^{-1}\mathbf{R}j_{M_\infty !!}\mathcal{F} \xrightarrow{\sim} \mathcal{F},\\ &\mathbf{R}j_{M_\infty !!}j_{M_\infty}^{-1}\mathcal{G} \simeq \rho_{\check{M}*}\Bbbk_M \otimes \mathcal{G}, && \mathbf{R}j_{M_\infty *}j_{M_\infty}^{-1}\mathcal{G} \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\rho_{\check{M}*}\Bbbk_M, \mathcal{G}), \end{aligned}$$

and hence functors $j_{M_\infty}^{-1}, \mathbf{R}j_{M_\infty !!}, \mathbf{R}j_{M_\infty *}$ induce equivalences of categories:

$$\begin{aligned} \tilde{j}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) &\xrightarrow{\sim} \{\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{\check{M}}) \mid \rho_{\check{M}*}\Bbbk_M \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{F}\}\\ &\simeq \{\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{\check{M}}) \mid \mathcal{F} \xrightarrow{\sim} \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\rho_{\check{M}*}\Bbbk_M, \mathcal{F})\}. \end{aligned}$$

Let us consider the following functors:

$$\begin{aligned} I_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), &\quad \mathcal{F} \mapsto \mathbf{q}I_{\check{M}}\mathbf{R}j_{M_\infty !!}\mathcal{F},\\ \mathbf{R}J_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}), &\quad F \mapsto j_{M_\infty}^{-1}\mathbf{R}J_{\check{M}}\mathbf{R}j_{M_\infty *}F, \end{aligned}$$

where $\mathbf{q}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is the quotient functor.

We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ consisting of objects such that $\mathbf{R}j_{M_\infty !!}F \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{\check{M}})$. Then the next lemma follows from [10, Lem. 7.1.3] and the definitions of functors $I_{M_\infty}, \mathbf{R}J_{M_\infty}$. We shall skip the proofs.

Lemma 3.2. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces associated with a morphism $\check{f}\colon \check{M} \to \check{N}$ of real analytic manifolds. The functors below are well defined:*

(1) $\iota_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_M) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(2) $\beta_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk_M) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(3) $(\cdot)\otimes(\cdot)\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \times \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(4) $\mathbf{R}\mathcal{I}hom(\iota_{M_\infty}(\cdot), \cdot)\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty})^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(5) $f^{-1}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(6) $\mathbf{R}f_{!!}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty})$,

(7) $f^{!}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$.

Let us describe the relation between ind-sheaves on M_∞ and subanalytic sheaves on M_∞.

Proposition 3.3. *Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. Then we have*

(1) *a pair* $(I_{M_\infty}, \mathbf{R}J_{M_\infty})$ *is an adjoint pair and there exists a canonical isomorphism* $\mathrm{id} \xrightarrow{\sim} \mathbf{R}J_{M_\infty} \circ I_{M_\infty}$*;*

(2) *there exists an equivalence of triangulated categories:*

$$\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \underset{\mathbf{R}J_{M_\infty}}{\overset{I_{M_\infty}}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).$$

Proof. (1) Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and $G \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(I_{M_\infty}\mathcal{F}, G) &\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(j^{-1}_{M_\infty} I_{\check{M}} \mathbf{R}j_{M_\infty !!}\mathcal{F}, G)\\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathcal{F}, \mathbf{R}j^{-1}_{M_\infty} \mathbf{R}J_{\check{M}} \mathbf{R}j_{M_\infty *}G)\\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathcal{F}, \mathbf{R}J_{M_\infty}G),
\end{aligned}$$

where in the first and third isomorphisms we used the fact that $\mathbf{q} = j^{-1}_{M_\infty}$ and the second isomorphism follows from the pairs $(j^{-1}_{M_\infty}, \mathbf{R}j_{M_\infty *})$, $(I_{\check{M}}, \mathbf{R}J_{\check{M}})$, $(\mathbf{R}j_{M_\infty !!}, j^{-1}_{M_\infty})$ being adjoint pairs. This implies that a pair $(I_{M_\infty}, \mathbf{R}J_{M_\infty})$ is an adjoint pair.

Hence there exists a natural morphism $\mathrm{id} \to \mathbf{R}J_{M_\infty} \circ I_{M_\infty}$ of functors. Moreover, for any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$, we have isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$,

$$\begin{aligned}
(\mathbf{R}J_{M_\infty} \circ I_{M_\infty})(\mathcal{F}) &\simeq j^{-1}_{M_\infty} \mathbf{R}J_{\check{M}} \mathbf{R}j_{M_\infty *} j^{-1}_{M_\infty} I_{\check{M}} \mathbf{R}j_{M_\infty !!}\mathcal{F}\\
&\simeq j^{-1}_{M_\infty} \mathbf{R}J_{\check{M}} \mathbf{R}\mathcal{I}\mathrm{hom}(\iota_{\check{M}}\Bbbk_M, I_{\check{M}}\mathbf{R}j_{M_\infty !!}\mathcal{F})\\
&\simeq j^{-1}_{M_\infty} \mathbf{R}J_{\check{M}} \mathbf{R}\mathcal{I}\mathrm{hom}(I_{\check{M}}\rho_{\check{M}}\Bbbk_M, I_{\check{M}}\mathbf{R}j_{M_\infty !!}\mathcal{F})\\
&\simeq j^{-1}_{M_\infty} \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{sub}}(\rho_{\check{M}}\Bbbk_M, \mathbf{R}J_{\check{M}} I_{\check{M}}\mathbf{R}j_{M_\infty !!}\mathcal{F})\\
&\simeq j^{-1}_{M_\infty} \mathbf{R}j_{M_\infty *} j^{-1}_{M_\infty} \mathbf{R}j_{M_\infty !!}\mathcal{F}\\
&\simeq \mathcal{F},
\end{aligned}$$

where the first isomorphism follows from $\mathbf{q} = j^{-1}_{M_\infty}$, in the second and fifth isomorphisms we used the fact that $\mathbf{R}j_{M_\infty *} j^{-1}_{M_\infty}(\cdot) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}(\iota_{\check{M}}\Bbbk_M, \cdot)$, the third isomorphism follows from $I_{\check{M}} \circ \rho^{\mathbb{R}\text{-}c}_{\check{M}} = \iota_{\check{M}}|_{\mathrm{Mod}_{\mathbb{R}\text{-}c}(\Bbbk_{\check{M}})}$, the fourth isomorphism follows from the adjointness of $(I_{\check{M}}, \mathbf{R}J_{\check{M}})$ and in the fifth we used the fact that $\mathrm{id} \xrightarrow{\sim} \mathbf{R}J_{\check{M}} \circ I_{\check{M}}$.

(2) First, let us prove that for any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ one has $I_{M_\infty}(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. There exist isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}})$,

$$\begin{aligned}
\mathbf{R}j_{M_\infty !!} I_{M_\infty}(\mathcal{F}) \simeq \mathbf{R}j_{M_\infty !!}\mathbf{q} I_{\check{M}} \mathbf{R}j_{M_\infty !!}\mathcal{F} &\simeq \iota_{\check{M}}\Bbbk_M \otimes I_{\check{M}} \mathbf{R}j_{M_\infty !!}\mathcal{F}\\
&\simeq I_{\check{M}}(\rho_{\check{M}}\Bbbk_M \otimes \mathbf{R}j_{M_\infty !!}\mathcal{F}),
\end{aligned}$$

where the third isomorphism follows from $I_{\check{M}} \circ \rho_{\check{M}}^{\mathbb{R}\text{-}c} = \iota_{\check{M}}|_{\mathrm{Mod}_{\mathbb{R}\text{-}c}(\Bbbk_{\check{M}})}$ and the fact that $I_{\check{M}}(\cdot \otimes \cdot) \simeq I_{\check{M}}(\cdot) \otimes I_{\check{M}}(\cdot)$. Since $I_{\check{M}}(\rho_{\check{M}}\Bbbk_M \otimes \mathbf{R}j_{M_\infty !!}\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{\check{M}})$, we have $\mathbf{R}j_{M_\infty !!}I_{M_\infty}(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{\check{M}})$. This implies $I_{M_\infty}(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$.

By (1), the functor I_{M_∞} is fully faithful. Let us prove that the functor I_{M_∞} is essentially surjective. Let $G \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. Then we have $\mathbf{R}j_{M_\infty !!}G \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{\check{M}})$. By Theorem 2.1, there exists $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{\check{M}})$ such that $\mathbf{R}j_{M_\infty !!}G \simeq I_{\check{M}}\mathcal{F}$ and hence we have $G \simeq j_{M_\infty}^{-1} I_{\check{M}}\mathcal{F}$. Moreover, there exist isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$,

$$\begin{aligned} j_{M_\infty}^{-1} I_{\check{M}} \mathbf{R}j_{M_\infty !!} j_{M_\infty}^{-1}\mathcal{F} \simeq j_{M_\infty}^{-1}(\iota_{\check{M}}\Bbbk_M \otimes I_{\check{M}}\mathcal{F}) &\simeq j_{M_\infty}^{-1}\mathbf{R}j_{M_\infty !!} j_{M_\infty}^{-1} I_{\check{M}}\mathcal{F} \\ &\simeq j_{M_\infty}^{-1} I_{\check{M}}\mathcal{F}, \end{aligned}$$

where in the first isomorphism we used

$$(\mathbf{R}j_{M_\infty !!} \circ j_{M_\infty}^{-1})(\cdot) \simeq \rho_{\check{M}*}\Bbbk_M \otimes (\cdot), \quad I_{\check{M}} \circ \rho_{\check{M}}^{\mathbb{R}\text{-}c} = \iota_{\check{M}}|_{\mathrm{Mod}_{\mathbb{R}\text{-}c}(\Bbbk_{\check{M}})}$$

and the fact that $I_{\check{M}}(\cdot \otimes \cdot) \simeq I_{\check{M}}(\cdot) \otimes I_{\check{M}}(\cdot)$ and in the second isomorphism we used the fact that $(\mathbf{R}j_{M_\infty !!} \circ j_{M_\infty}^{-1})(\cdot) \simeq \iota_{\check{M}}\Bbbk_M \otimes (\cdot)$. Hence we have

$$G \simeq j_{M_\infty}^{-1} I_{\check{M}}\mathcal{F} \simeq j_{M_\infty}^{-1} I_{\check{M}} \mathbf{R}j_{M_\infty !!} j_{M_\infty}^{-1}\mathcal{F} \simeq I_{M_\infty}(j_{M_\infty}^{-1}\mathcal{F}).$$

This implies that the functor I_{M_∞} is essentially surjective.

Therefore, the proof is completed. ■

We shall denote by

$$\lambda_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

the inverse functor of $I_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$.

Proposition 3.4. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces associated with a morphism $\check{f}\colon \check{M} \to \check{N}$ of real analytic manifolds. Then we have the following:*

(1) $\mathbf{R}J_{M_\infty}\mathbf{R}\mathcal{I}hom(I_{M_\infty}(\cdot), \cdot) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\cdot, \mathbf{R}J_{M_\infty}(\cdot))$.

(2) *For any $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_M)$, any $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, we have*

(i) $\alpha_{M_\infty} I_{M_\infty}\mathcal{F} \simeq \rho_{M_\infty}^{-1}\mathcal{F}$,

(ii) $I_{M_\infty}\rho_{M_\infty !}\mathcal{L} \simeq \beta_{M_\infty}\mathcal{L}$,

(iii) $I_{M_\infty} f^{-1}\mathcal{G} \simeq f^{-1} I_{N_\infty}\mathcal{G}$,

(iv) $\mathbf{R}f_{!!} I_{M_\infty}\mathcal{F} \simeq I_{N_\infty}\mathbf{R}f_{!!}\mathcal{F}$,

(v) $I_{M_\infty} f^! \mathcal{G} \simeq f^! I_{N_\infty} \mathcal{G}$,

(vi) $I_{M_\infty}(\mathcal{F}_1 \otimes \mathcal{F}_2) \simeq I_{M_\infty}(\mathcal{F}_1) \otimes I_{M_\infty}(\mathcal{F}_2)$.

(3) *For any* $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_M)$, *any* $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ *and any* $G \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{N_\infty})$, *we have*

(i) $\mathbf{R}J_{M_\infty} \iota_{M_\infty} \mathcal{L} \simeq \rho_{M_\infty *} \mathcal{L}$,

(ii) $\rho_{M_\infty}^{-1} \mathbf{R}J_{M_\infty} F \simeq \alpha_{M_\infty} F$,

(iii) $\mathbf{R}J_{M_\infty} \beta_{M_\infty} \mathcal{L} \simeq \rho_{M_\infty !} \mathcal{L}$,

(iv) $\mathbf{R}f_* \mathbf{R}J_{M_\infty} F \simeq \mathbf{R}J_{N_\infty} \mathbf{R}f_* F$,

(v) $\mathbf{R}J_{M_\infty} f^! G \simeq f^! \mathbf{R}J_{N_\infty} G$.

(4) *For any* $F, F_1, F_2 \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$, *any* $G \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty})$ *and any* $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty})$, *we have*

(i) $I_{M_\infty} \rho^{\mathbb{R}\text{-}c}_{M_\infty *} \mathcal{L} \simeq \iota_{M_\infty} \mathcal{L}$,

(ii) $\lambda_{M_\infty} f^{-1} G \simeq f^{-1} \lambda_{N_\infty} G$,

(iii) $\mathbf{R}f_{!!} \lambda_{M_\infty} F \simeq \lambda_{N_\infty} \mathbf{R}f_{!!} F$,

(iv) $\lambda_{M_\infty}(F_1 \otimes F_2) \simeq \lambda_{M_\infty}(F_1) \otimes \lambda_{M_\infty}(F_2)$.

Proof. Since the proofs of the assertions in the proposition are similar, we only prove (2)(i). Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. Then we have

$$
\begin{aligned}
\alpha_{M_\infty} I_{M_\infty} \mathcal{F} &\simeq j_M^{-1} \alpha_{\check{M}} \mathbf{R} j_{M_\infty !!} j_{M_\infty}^{-1} I_{\check{M}} \mathbf{R} j_{M_\infty !!} \mathcal{F} \\
&\simeq j_M^{-1} \alpha_{\check{M}} (\iota_{\check{M}} \Bbbk_M \otimes I_{\check{M}} \mathbf{R} j_{M_\infty !!} \mathcal{F}) \\
&\simeq j_M^{-1} \alpha_{\check{M}} \iota_{\check{M}} \Bbbk_M \otimes j_M^{-1} \alpha_{\check{M}} I_{\check{M}} \mathbf{R} j_{M_\infty !!} \mathcal{F} \\
&\simeq j_M^{-1} \rho_{\check{M}}^{-1} \mathbf{R} j_{M_\infty !!} \mathcal{F} \\
&\simeq \rho_{M_\infty}^{-1} \mathcal{F},
\end{aligned}
$$

where in the second isomorphism we used the fact that $\mathbf{R} j_{M_\infty !!} j_{M_\infty}^{-1} \simeq \iota_{\check{M}} \Bbbk_M \otimes (\cdot)$ and the fourth isomorphism follows from $\alpha_{\check{M}} \circ I_{\check{M}} \simeq \rho_{\check{M}}^{-1}$. ■

3.2 – *Convolutions for subanalytic sheaves on real analytic bordered spaces*

In this subsection, let us define convolution functors for subanalytic sheaves on real analytic bordered spaces. Although it has already been explained in [8, §5.1], in this subsection we shall explain it again in detail.[5]

([5]) Kashiwara [8] introduced convolution functors for subanalytic sheaves on subanalytic bordered spaces. In this paper, we shall only consider them on real analytic bordered spaces.

Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. We set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ for $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$, and let $t \in \mathbb{R}$ be the affine coordinate. We consider the morphisms of real analytic bordered spaces

$$M_\infty \times \mathbb{R}^2_\infty \xrightarrow{p_1, p_2, \mu} M_\infty \times \mathbb{R}_\infty \xrightarrow{\pi} M_\infty$$

given by the maps $p_1(x,t_1,t_2) := (x,t_1)$, $p_2(x,t_1,t_2) := (x,t_2)$, $\mu(x,t_1,t_2) := (x,t_1 + t_2)$ and $\pi(x,t) := x$.

Then the convolution functors for subanalytic sheaves on $M_\infty \times \mathbb{R}_\infty$,

$$(\cdot) \overset{+}{\otimes} (\cdot) \colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \times \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}),$$
$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\cdot, \cdot) \colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$$

are defined by

$$\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2 := \mathrm{R}\mu_{!!}(p_1^{-1}\mathcal{F}_1 \otimes p_2^{-1}\mathcal{F}_2),$$
$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1, \mathcal{F}_2) := \mathrm{R}p_{1*}\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(p_2^{-1}\mathcal{F}_1, \mu^!\mathcal{F}_2),$$

for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$. Note that for any $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$ there exist isomorphisms

$$\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2 \simeq \mathcal{F}_2 \overset{+}{\otimes} \mathcal{F}_1,$$
$$\mathcal{F}_1 \overset{+}{\otimes} (\mathcal{F}_2 \overset{+}{\otimes} \mathcal{F}_3) \simeq (\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2) \overset{+}{\otimes} \mathcal{F}_3$$

and

$$\Bbbk_{\{t=0\}} \overset{+}{\otimes} \mathcal{F} \simeq \mathcal{F} \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\Bbbk_{\{t=0\}}, \mathcal{F}),$$

where $\{t = 0\}$ stands for $\{(x,t) \in M \times \mathbb{R} \mid t = 0\}$. Hence, the category $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$ has the structure of a commutative tensor category with $\overset{+}{\otimes}$ as the tensor product functor and $\Bbbk_{\{t=0\}}$ as the unit object.

The convolution functors have several properties similar to the tensor product functor and the internal hom functor. For a morphism of real analytic bordered spaces $f \colon M_\infty \to N_\infty$, let us denote by $f_{\mathbb{R}_\infty} \colon M_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty$ the morphism $f \times \mathrm{id}_{\mathbb{R}_\infty}$ of real analytic bordered spaces.

Proposition 3.5. *Let $f \colon M_\infty \to \mathcal{N}_\infty$ be a morphism of real analytic bordered spaces, $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$ and $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2, \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty \times \mathbb{R}_\infty})$. There exist isomorphisms*

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2, \mathcal{F}) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_2, \mathcal{F})),$$
$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2, \mathcal{F}) \simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})}(\mathcal{F}_1, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_2, \mathcal{F})),$$

$$\mathbf{R}f_{\mathbb{R}_\infty *}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(f_{\mathbb{R}_\infty}^{-1}\mathcal{G},\mathcal{F}) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G},\mathbf{R}f_{\mathbb{R}_\infty *}\mathcal{F}),$$

$$f_{\mathbb{R}_\infty}^{-1}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2) \simeq f_{\mathbb{R}_\infty}^{-1}\mathcal{F}_1 \overset{+}{\otimes} f_{\mathbb{R}_\infty}^{-1}\mathcal{F}_2,$$

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathbf{R}f_{\mathbb{R}_\infty !!}\mathcal{F},\mathcal{G}) \simeq \mathbf{R}f_{\mathbb{R}_\infty *}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, f_{\mathbb{R}_\infty}^{!}\mathcal{G}),$$

$$\mathbf{R}f_{\mathbb{R}_\infty !!}(\mathcal{F} \overset{+}{\otimes} f_{\mathbb{R}_\infty}^{-1}\mathcal{G}) \simeq \mathbf{R}f_{\mathbb{R}_\infty !!}\mathcal{F} \overset{+}{\otimes} \mathcal{G},$$

$$f_{\mathbb{R}_\infty}^{!}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G}_1,\mathcal{G}_2) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(f_{\mathbb{R}_\infty}^{-1}\mathcal{G}_1, f_{\mathbb{R}_\infty}^{!}\mathcal{G}_2).$$

Proof. First, let us prove the second isomorphism. By using the adjointness, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2, \mathcal{F})\\
&\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{R}\mu_{!!}(p_1^{-1}\mathcal{F}_1 \otimes p_2^{-1}\mathcal{F}_2), \mathcal{F})\\
&\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathcal{F}_1, \mathbf{R}p_{1*}\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(p_2^{-1}\mathcal{F}_2, \mu^{!}\mathcal{F}))\\
&\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathcal{F}_1, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_2, \mathcal{F})).
\end{aligned}$$

Let us prove the first isomorphism. By using the second isomorphism, for any $\mathcal{F}_0 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$, we have an isomorphism

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathcal{F}_0, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2, \mathcal{F}_3))\\
&\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}\big(\mathcal{F}_0, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_2, \mathcal{F}_3))\big).
\end{aligned}$$

Hence, we have

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2, \mathcal{F}_3) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_1, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}_2, \mathcal{F}_3)).$$

Let us denote by $\tilde{f}_{\mathbb{R}_\infty}\colon M_\infty \times \mathbb{R}^2_\infty \to N_\infty \times \mathbb{R}^2_\infty$ the morphism $f \times \mathrm{id}_{\mathbb{R}^2_\infty}$ of real analytic bordered spaces. Then there exist cartesian diagrams,

$$\begin{array}{ccccc}
M_\infty \times \mathbb{R}^2_\infty & \xrightarrow{\ \bigstar\ } & M_\infty \times \mathbb{R}_\infty & \xrightarrow{\ \pi\ } & M_\infty\\
{\scriptstyle \tilde{f}_{\mathbb{R}_\infty}}\downarrow & \square & {\scriptstyle f_{\mathbb{R}_\infty}}\downarrow & \square & {\scriptstyle f}\downarrow\\
N_\infty \times \mathbb{R}^2_\infty & \xrightarrow[\ \bigstar\ ]{} & N_\infty \times \mathbb{R}_\infty & \xrightarrow[\ \pi\ ]{} & N_\infty,
\end{array}$$

where $\bigstar = p_1, p_2, \mu$, respectively. Hence, we have

$$\begin{aligned}
f_{\mathbb{R}_\infty}^{-1}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2) &\simeq f_{\mathbb{R}_\infty}^{-1}\mathbf{R}\mu_{!!}(p_1^{-1}\mathcal{F}_1 \overset{+}{\otimes} p_2^{-1}\mathcal{F}_2)\\
&\simeq \mathbf{R}\mu_{!!}\tilde{f}_{\mathbb{R}_\infty}^{-1}(p_1^{-1}\mathcal{F}_1 \overset{+}{\otimes} p_2^{-1}\mathcal{F}_2)
\end{aligned}$$

$$\simeq \mathbf{R}\mu_{!!}(\tilde{f}_{\mathbb{R}_\infty}^{-1} p_1^{-1}\mathcal{F}_1 \overset{+}{\otimes} \tilde{f}_{\mathbb{R}_\infty}^{-1} p_2^{-1}\mathcal{F}_2)$$
$$\simeq \mathbf{R}\mu_{!!}(p_1^{-1} f_{\mathbb{R}_\infty}^{-1}\mathcal{F}_1 \overset{+}{\otimes} p_2^{-1} f_{\mathbb{R}_\infty}^{-1}\mathcal{F}_2)$$
$$\simeq f_{\mathbb{R}_\infty}^{-1}\mathcal{F}_1 \overset{+}{\otimes} f_{\mathbb{R}_\infty}^{-1}\mathcal{F}_2,$$

where in the second isomorphism we used Proposition 3.1(4) and in the third isomorphism we used Proposition 3.1(3). The remaining assertions can be proved in the similar way. We shall skip the proofs. ■

Proposition 3.6. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ and $\mathcal{F}_0 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. Then there exist isomorphisms*

$$\pi^{-1}\mathcal{F}_0 \otimes (\mathcal{F} \overset{+}{\otimes} \mathcal{G}) \simeq (\pi^{-1}\mathcal{F}_0 \otimes \mathcal{F}) \overset{+}{\otimes} \mathcal{G},$$
$$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}_0, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, \mathcal{G})) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\pi^{-1}\mathcal{F}_0 \otimes \mathcal{F}, \mathcal{G})$$
$$\simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}_0, \mathcal{G})),$$
$$\mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F} \overset{+}{\otimes} \mathcal{G}, \mathcal{H}) \simeq \mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G}, \mathcal{H})).$$

Proof. First let us remark that $\pi \circ p_1 = \pi \circ p_2 = \pi \circ \mu$.

By using Proposition 3.1(3), we have

$$\pi^{-1}\mathcal{F}_0 \otimes (\mathcal{F} \overset{+}{\otimes} \mathcal{G}) \simeq \pi^{-1}\mathcal{F}_0 \otimes \mathbf{R}\mu_{!!}(p_1^{-1}\mathcal{F} \otimes p_2^{-1}\mathcal{G})$$
$$\simeq \mathbf{R}\mu_{!!}(\mu^{-1}\pi^{-1}\mathcal{F}_0 \otimes (p_1^{-1}\mathcal{F} \otimes p_2^{-1}\mathcal{G}))$$
$$\simeq \mathbf{R}\mu_{!!}(p_1^{-1}\pi^{-1}\mathcal{F}_0 \otimes (p_1^{-1}\mathcal{F} \otimes p_2^{-1}\mathcal{G}))$$
$$\simeq \mathbf{R}\mu_{!!}(p_1^{-1}(\pi^{-1}\mathcal{F}_0 \otimes \mathcal{F}) \otimes p_2^{-1}\mathcal{G})$$
$$\simeq (\pi^{-1}\mathcal{F}_0 \otimes \mathcal{F}) \overset{+}{\otimes} \mathcal{G}.$$

The second assertion can be proved in the similar way. We shall skip the proof.

Let us prove the last assertion. For any $\mathcal{G}_0 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, there exist isomorphisms

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathcal{G}_0, \mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F} \overset{+}{\otimes} \mathcal{G}, \mathcal{H}))$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\pi^{-1}\mathcal{G}_0, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F} \overset{+}{\otimes} \mathcal{G}, \mathcal{H}))$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\pi^{-1}\mathcal{G}_0 \otimes (\mathcal{F} \overset{+}{\otimes} \mathcal{G}), \mathcal{H})$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}((\pi^{-1}\mathcal{G}_0 \otimes \mathcal{F}) \overset{+}{\otimes} \mathcal{G}, \mathcal{H})$$

$$\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\pi^{-1}\mathcal{G}_0\otimes\mathcal{F}, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G},\mathcal{H}))$$

$$\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}\big(\mathcal{G}_0, \mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G},\mathcal{H}))\big),$$

where in the third (resp. fourth) isomorphism we used the first assertion (resp. Proposition 3.5). Hence, we have an isomorphism

$$\mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}\overset{+}{\otimes}\mathcal{G},\mathcal{H}) \simeq \mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G},\mathcal{H})). \qquad \blacksquare$$

Lemma 3.7. *For any $\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ and any $\mathcal{G}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, we have*

$$\pi^{-1}\mathcal{G}\overset{+}{\otimes}\mathcal{F} \simeq \pi^{-1}(\mathcal{G}\otimes\mathbf{R}\pi_{!!}\mathcal{F}),$$
$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\pi^{-1}\mathcal{G},\mathcal{F}) \simeq \pi^{!}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{G},\mathbf{R}\pi_*\mathcal{F}),$$
$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F},\pi^{!}\mathcal{G}) \simeq \pi^{!}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathbf{R}\pi_{!!}\mathcal{F},\mathcal{G}).$$

Proof. Note that there exist cartesian diagrams ($i = 1, 2$):

$$\begin{array}{ccccc} M_\infty\times\mathbb{R}^2_\infty & \xrightarrow{\ \mu\ } & M_\infty\times\mathbb{R}_\infty & \xleftarrow{\ p_1\ } & M_\infty\times\mathbb{R}^2_\infty \\ {\scriptstyle p_i}\downarrow & \square & {\scriptstyle \pi}\downarrow & \square & {\scriptstyle p_2}\downarrow \\ M_\infty\times\mathbb{R}_\infty & \xrightarrow[\pi]{} & M_\infty & \xleftarrow[\pi]{} & M_\infty\times\mathbb{R}_\infty. \end{array}$$

Then by using Proposition 3.1(3), (4), we have

$$\begin{aligned} \pi^{-1}\mathcal{G}\overset{+}{\otimes}\mathcal{F} &\simeq \mathbf{R}\mu_{!!}(p_1^{-1}\pi^{-1}\mathcal{G}\otimes p_2^{-1}\mathcal{F}) \\ &\simeq \mathbf{R}\mu_{!!}(\mu^{-1}\pi^{-1}\mathcal{G}\otimes p_2^{-1}\mathcal{F}) \\ &\simeq \pi^{-1}\mathcal{G}\otimes\mathbf{R}\mu_{!!}p_2^{-1}\mathcal{F} \\ &\simeq \pi^{-1}\mathcal{G}\otimes\pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F} \\ &\simeq \pi^{-1}(\mathcal{F}\otimes\mathbf{R}\pi_{!!}\mathcal{F}). \end{aligned}$$

The remaining assertions can be proved in the similar way. We shall skip the proofs. $\blacksquare$

Since $\mathbf{R}\pi_{!!}\Bbbk_{\{t\leq 0\}} \simeq \mathbf{R}\pi_{!!}\Bbbk_{\{t\geq 0\}} \simeq 0$ and $\pi^{-1}\Bbbk_M \simeq \Bbbk_{M\times\mathbb{R}}$, we have the following corollary:

Corollary 3.8. *For any $\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ and any $\mathcal{G}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, we have*

$$\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\pi^{-1}\mathcal{G} \simeq 0$$

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \pi^{-1}\mathcal{G}) \simeq 0,$$

$$\rho_{M_\infty\times\mathbb{R}_\infty *}\Bbbk_{M\times\mathbb{R}} \overset{+}{\otimes} \mathcal{F} \simeq \pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F} \ \ (\simeq \pi^{!}\mathbf{R}\pi_{!!}\mathcal{F}[-1]),$$

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}\Bbbk_{M\times\mathbb{R}}, \mathcal{F}) \simeq \pi^{!}\mathbf{R}\pi_{*}\mathcal{F} \ \ (\simeq \pi^{-1}\mathbf{R}\pi_{*}\mathcal{F}[1]).$$

At the end of this subsection, we shall prove the following proposition:

Proposition 3.9. *Let $M_\infty = (M, \check{M})$ be a real analytic bordered space.*

(1) *For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$:*

$$I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_1 \overset{+}{\otimes} I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_2 \simeq I_{M_\infty\times\mathbb{R}_\infty}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2).$$

(2) *For any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ and any $G \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$:*

$$\mathbf{R}J_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}, G) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, \mathbf{R}J_{M_\infty\times\mathbb{R}_\infty}G).$$

(3) *The functor $(\cdot)\overset{+}{\otimes}(\cdot)\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})\times\mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})\to\mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ is well defined.*

(4) *For any $F_1, F_2 \in \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$:*

$$\lambda_{M_\infty\times\mathbb{R}_\infty}F_1 \overset{+}{\otimes} \lambda_{M_\infty\times\mathbb{R}_\infty}F_2 \simeq \lambda_{M_\infty\times\mathbb{R}_\infty}(F_1 \overset{+}{\otimes} F_2).$$

Proof. (1) Let us denote by S the closure of $\{(t_1,t_2,t_3)\in\mathbb{R}^3 \mid t_1+t_2+t_3=0\}$ in $\overline{\mathbb{R}}^3$, and consider the morphisms $\tilde{p}_1, \tilde{p}_2, \tilde{\mu}\colon S\to\overline{\mathbb{R}}$ given by $\tilde{p}_1(t_1,t_2,t_3)=t_1$, $\tilde{p}_2(t_1,t_2,t_3)=t_2$, $\tilde{\mu}(t_1,t_2,t_3)=t_1+t_2=-t_3$. We shall denote the corresponding morphisms $\check{M}\times S\to\check{M}\times\overline{\mathbb{R}}$ by the same symbols. Then there exists a commutative diagram

$$\begin{array}{ccc} M_\infty\times\mathbb{R}^2_\infty & \xrightarrow{\ k\ } & \check{M}\times S \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{u}} \\ M_\infty\times\mathbb{R}_\infty & \xrightarrow[j_{M_\infty\times\mathbb{R}_\infty}]{} & \check{M}\times\overline{\mathbb{R}}, \end{array}$$

where $u = p_1, p_2, \mu$, and k is the morphism associated to the embedding $\mathbb{R}^2\hookrightarrow S$, $(t_1,t_2)\mapsto(t_1,t_2,-t_1-t_2)$. Note that for any $F_1, F_2\in\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ there exists an isomorphism

$$F_1\overset{+}{\otimes}F_2 \simeq j^{-1}_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\tilde{\mu}_{!!}(\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}F_1\otimes\tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}F_2).$$

This assertion can be proved similarly to [1, Lem. 4.3.9]. Then we have an isomorphism

$$I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_1 \overset{+}{\otimes} I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_2 \simeq j^{-1}_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\tilde{\mu}_{!!}(\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_1 \otimes \tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_2).$$

Moreover, we have isomorphisms

$$\begin{aligned}
\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_1 &\simeq \tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}j^{-1}_{M_\infty\times\mathbb{R}_\infty}I_{\check{M}\times\bar{\mathbb{R}}}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1\\
&\simeq \tilde{p}_1^{-1}(\iota_{\check{M}\times\bar{\mathbb{R}}}\Bbbk_{M\times\mathbb{R}} \otimes I_{\check{M}\times\bar{\mathbb{R}}}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1)\\
&\simeq \tilde{p}_1^{-1}(I_{\check{M}\times\bar{\mathbb{R}}}\rho_{\check{M}\times\bar{\mathbb{R}}*}\Bbbk_{M\times\mathbb{R}} \otimes I_{\check{M}\times\bar{\mathbb{R}}}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1)\\
&\simeq \tilde{p}_1^{-1}I_{\check{M}\times\bar{\mathbb{R}}}(\rho_{\check{M}\times\bar{\mathbb{R}}*}\Bbbk_{M\times\mathbb{R}} \otimes \mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1)\\
&\simeq I_{\check{M}\times S}\tilde{p}_1^{-1}(\rho_{\check{M}\times\bar{\mathbb{R}}*}\Bbbk_{M\times\mathbb{R}} \otimes \mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1)\\
&\simeq I_{\check{M}\times S}\tilde{p}_1^{-1}(\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}j^{-1}_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1)\\
&\simeq I_{\check{M}\times S}(\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1),
\end{aligned}$$

where in the second isomorphism we used the fact that

$$\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}j^{-1}_{M_\infty\times\mathbb{R}_\infty} \simeq \iota_{\check{M}\times\bar{\mathbb{R}}}\Bbbk_{M\times\mathbb{R}} \otimes (\cdot)$$

and in the sixth isomorphism that

$$\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}j^{-1}_{M_\infty\times\mathbb{R}_\infty} \simeq \rho_{\check{M}\times\bar{\mathbb{R}}*}\Bbbk_{M\times\mathbb{R}} \otimes (\cdot).$$

See also the end of Section 2.3. In a similar way, we have an isomorphism

$$\tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_2 \simeq I_{\check{M}\times S}(\tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_2).$$

Hence, there exist isomorphisms

$$\begin{aligned}
&I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_1 \overset{+}{\otimes} I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}_2\\
&\simeq j^{-1}_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\tilde{\mu}_{!!}I_{\check{M}\times S}(\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1 \otimes \tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_2)\\
&\simeq j^{-1}_{M_\infty\times\mathbb{R}_\infty}I_{\check{M}\times\bar{\mathbb{R}}}\mathbf{R}\tilde{\mu}_{!!}(\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1 \otimes \tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_2)\\
&\simeq I_{M_\infty\times\mathbb{R}_\infty}j^{-1}_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\tilde{\mu}_{!!}(\tilde{p}_1^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_1 \otimes \tilde{p}_2^{-1}\mathbf{R}j_{M_\infty\times\mathbb{R}_\infty !!}\mathcal{F}_2)\\
&\simeq I_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\mu_{!!}k^{-1}(\mathbf{R}k_{!!}p_1^{-1}\mathcal{F}_1 \otimes \mathbf{R}k_{!!}p_2^{-1}\mathcal{F}_2)\\
&\simeq I_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\mu_{!!}(p_1^{-1}\mathcal{F}_1 \otimes p_2^{-1}\mathcal{F}_2)\\
&\simeq I_{M_\infty\times\mathbb{R}_\infty}(\mathcal{F}_1 \overset{+}{\otimes} \mathcal{F}_2).
\end{aligned}$$

(2) By using Proposition 3.3(1), 3.5 and assertion (1), for any $\mathcal{F}_0 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$, we have the isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})}(\mathcal{F}_0, \mathbf{R}J_{M_\infty \times \mathbb{R}_\infty} \mathbf{R}\mathcal{I}hom^{+}(I_{M_\infty \times \mathbb{R}_\infty}\mathcal{F}, G))\\ &\quad \simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})}(\mathcal{F}_0, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, \mathbf{R}J_{M_\infty \times \mathbb{R}_\infty} G)).\end{aligned}$$

Hence we have

$$\mathbf{R}J_{M_\infty \times \mathbb{R}_\infty} \mathbf{R}\mathcal{I}hom^{+}(I_{M_\infty \times \mathbb{R}_\infty}\mathcal{F}, G) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, \mathbf{R}J_{M_\infty \times \mathbb{R}_\infty} G).$$

(3) Let $F_1, F_2 \in \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty})$. By Proposition 3.3(2), there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ such that $F_1 \simeq I_{M_\infty \times \mathbb{R}_\infty}\mathcal{F}_1$, $F_2 \simeq I_{M_\infty \times \mathbb{R}_\infty}\mathcal{F}_2$. Moreover, by using the first assertion, we have

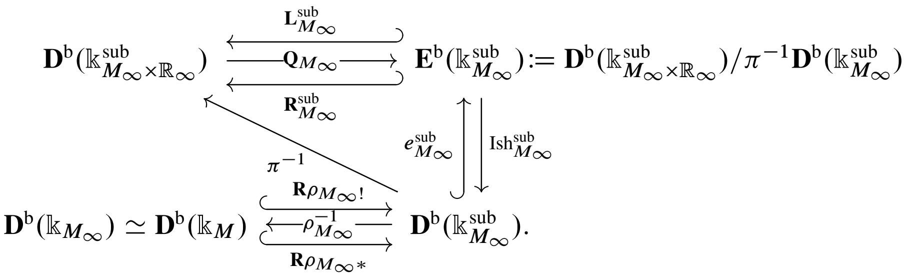
$$F_1 \overset{+}{\otimes} F_2 \simeq I_{M_\infty \times \mathbb{R}_\infty}\mathcal{F}_1 \overset{+}{\otimes} I_{M_\infty \times \mathbb{R}_\infty}\mathcal{F}_1 \simeq I_{M_\infty \times \mathbb{R}_\infty}(\mathcal{F}_1 \otimes \mathcal{F}_2).$$

This implies that $F_1 \overset{+}{\otimes} F_2 \in \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty \times \mathbb{R}_\infty})$. The proof is completed.

(4) This assertion follows from Propositions 3.3(2) and assertion (1). ■

3.3 – *Enhanced subanalytic sheaves*

In this subsection, let us define enhanced subanalytic sheaves similarly to the definition of enhanced ind-sheaves. The results are summarized by the following (non-commutative) diagram. Functors e^{sub} and $\mathrm{Ish}^{\mathrm{sub}}$ will be defined in Section 3.4:



Let $M_\infty = (M, \check{M})$ be a real analytic bordered space and set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ for $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$. Let us set

$$\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})/\pi^{-1}\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

and we shall call an object of $\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ an enhanced subanalytic sheaf[6] on M_∞. The category $\pi^{-1}\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ can be characterized as follows:

Lemma 3.10. *For $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$, the following five conditions are equivalent:*

(i) $\mathcal{F} \in \pi^{-1}\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$,

(ii) $\mathcal{F} \xrightarrow{\sim} \Bbbk_{M\times\mathbb{R}}[1] \overset{+}{\otimes} \mathcal{F}$,

(iii) $\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\Bbbk_{M\times\mathbb{R}}[1], \mathcal{F}) \xrightarrow{\sim} \mathcal{F}$,

(iv) $(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}) \overset{+}{\otimes} \mathcal{F} \simeq 0$,

(v) $\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}, \mathcal{F}) \simeq 0$.

Proof. By using a distinguished triangle

$$\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}} \longrightarrow \Bbbk_{\{t=0\}} \longrightarrow \Bbbk_{M\times\mathbb{R}}[1] \xrightarrow{+1}$$

and the fact that $\Bbbk_{\{t=0\}} \overset{+}{\otimes} \mathcal{F} \simeq \mathcal{F}$ (resp. $\mathcal{F} \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\Bbbk_{\{t=0\}}, \mathcal{F})$), we have that condition (ii) (resp. (iii)) is equivalent to condition (iv) (resp. (v)).

Let us prove that the three conditions (i), (ii), (iii) are equivalent. Let us assume that condition (ii) (resp. (iii)) is satisfied. Then we have

$$\mathcal{F} \xrightarrow{\sim} \Bbbk_{M\times\mathbb{R}} \overset{+}{\otimes} \mathcal{F}[1] \quad (\text{resp. } \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\Bbbk_{M\times\mathbb{R}}[1], \mathcal{F}) \xrightarrow{\sim} \mathcal{F}).$$

By Corollary 3.8, $\mathcal{F} \xrightarrow{\sim} \pi^{-1}\mathbf{R}\pi_{!!}[1]\mathcal{F}$ (resp. $\pi^{-1}\mathbf{R}\pi_*\mathcal{F} \xleftarrow{\sim} \mathcal{F}$). Hence, condition (i) is satisfied.

Let us assume that condition (i) is satisfied. By using Proposition 3.4(2)(iii), we have $I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F} \in \pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$, and hence by [1, Lem. 4.4.3] we have

$$\begin{aligned} \pi^{-1}\mathbf{R}\pi_* I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F} &\xrightarrow{\sim} I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}, \\ I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F} &\xrightarrow{\sim} \pi^{!}\mathbf{R}\pi_{!!} I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}. \end{aligned}$$

By using Proposition 3.4(3)(iv), (v) (resp. (2)(iv), (v)) and Proposition 3.3(2), we have

$$\pi^{-1}\mathbf{R}\pi_*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \quad (\text{resp. } \mathcal{F} \xrightarrow{\sim} \pi^{!}\mathbf{R}\pi_{!!}\mathcal{F}).$$

By Corollary 3.8, this implies that condition (iii) (resp. (ii)) is satisfied.

Therefore, the proof is completed. ∎

(6) In [8], it seems that an object of $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ is called an enhanced subanalytic sheaf.

Let us prove that the quotient functor

$$\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

has fully faithful left and right adjoints. By Corollary 3.8, the two functors

$$\begin{aligned}
&\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}),\\
&\mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{F}) \mapsto \rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}) \overset{+}{\otimes} \mathcal{F},\\
&\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}),\\
&\mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{F}) \mapsto \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{F})
\end{aligned}$$

are well defined.

Lemma 3.11. *The functors $\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}$, $\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}$ induce equivalences of categories*

$$\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \xrightarrow{\sim} \{\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}) \mid \rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}) \overset{+}{\otimes} \mathcal{F} \xrightarrow{\sim} \mathcal{F}\},$$

$$\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \xrightarrow{\sim} \{\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}) \mid \mathcal{F} \xrightarrow{\sim} \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{F})\},$$

respectively.

Moreover, the quotient functor admits a left (resp. right) adjoint $\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}$ (resp. $\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}$).

Proof. By Corollary 3.8 and the fact that for any $K\in\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ there exists an isomorphism

$$\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}) \overset{+}{\otimes} \mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K \xrightarrow{\sim} \mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K,$$

the functor

$$\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \{\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}) \mid \rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}) \overset{+}{\otimes} \mathcal{F} \xrightarrow{\sim} \mathcal{F}\}$$

is well defined. Let $K\in\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. Then there exists $\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ such that $K\simeq\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}$. Let us prove that

$$\mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}) \overset{+}{\otimes} \mathcal{F}) \simeq \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{F}).$$

Since there exists a distinguished triangle

$$\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}} \longrightarrow \Bbbk_{\{t=0\}} \longrightarrow \Bbbk_{M\times\mathbb{R}}[1] \xrightarrow{+1},$$

it is enough to show that $\mathbf{Q}^{\rm sub}_{M_\infty}(\rho_{M_\infty\times\mathbb{R}_\infty *}\Bbbk_{M\times\mathbb{R}}[1]\overset{+}{\otimes}\mathcal{F})\simeq 0$. This assertion follows from Corollary 3.8, so that we have

$$\mathbf{Q}^{\rm sub}_{M_\infty}\mathbf{L}^{\rm E,sub}_{M_\infty}K\simeq\mathbf{Q}^{\rm sub}_{M_\infty}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathcal{F})\simeq\mathbf{Q}^{\rm sub}_{M_\infty}(\mathcal{F})\simeq K.$$

Hence we have $\mathbf{Q}^{\rm sub}_{M_\infty}\circ\mathbf{L}^{\rm E,sub}_{M_\infty}\simeq\mathrm{id}$. Moreover, it is clear that for any

$$\mathcal{G}_1\in\{\mathcal{F}\in\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})\mid\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathcal{F}\xrightarrow{\sim}\mathcal{F}\},$$

we have $\mathbf{L}^{\rm E,sub}_{M_\infty}\mathbf{Q}^{\rm sub}_{M_\infty}(\mathcal{G}_1)\simeq\mathcal{G}_1$. Therefore, there exist equivalences of categories

$$\mathbf{L}^{\rm E,sub}_{M_\infty}\colon\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})\xrightarrow{\sim}\{\mathcal{F}\in\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})\mid\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathcal{F}\xrightarrow{\sim}\mathcal{F}\}.$$

Using a similar method, we have an equivalence of category

$$\mathbf{R}^{\rm E,sub}_{M_\infty}\colon\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})\xrightarrow{\sim}\{\mathcal{F}\in\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})\mid\mathcal{F}\xrightarrow{\sim}\mathbf{R}\mathcal{I}hom^{+,\rm sub}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),\mathcal{F})\}.$$

Let us prove that the quotient functor admits a left (resp. right) adjoint $\mathbf{L}^{\rm E,sub}_{M_\infty}$ (resp. $\mathbf{R}^{\rm E,sub}_{M_\infty}$). Let $\mathcal{F}\in\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})$ and $K\in\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})$. Since functors $\mathbf{L}^{\rm E,sub}_{M_\infty}$, $\mathbf{R}^{\rm E,sub}_{M_\infty}$ induce fully faithful functors

$$\mathbf{L}^{\rm E,sub}_{M_\infty}\colon\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})\hookrightarrow\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty}),\quad\mathbf{R}^{\rm E,sub}_{M_\infty}\colon\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})\hookrightarrow\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})$$

and Proposition 3.5, there exist isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})}(\mathbf{Q}^{\rm sub}_{M_\infty}\mathcal{F},K)\\
&\quad\simeq\mathrm{Hom}_{\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{L}^{\rm E,sub}_{M_\infty}\mathbf{Q}^{\rm sub}_{M_\infty}\mathcal{F},\mathbf{L}^{\rm E,sub}_{M_\infty}K)\\
&\quad\simeq\mathrm{Hom}_{\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathcal{F},\mathbf{L}^{\rm E,sub}_{M_\infty}K)\\
&\quad\simeq\mathrm{Hom}_{\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})}\big(\mathcal{F},\mathbf{R}\mathcal{I}hom^{+}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),\mathbf{L}^{\rm E,sub}_{M_\infty}K)\big),\\
&\mathrm{Hom}_{\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})}(K,\mathbf{Q}^{\rm sub}_{M_\infty}\mathcal{F})\\
&\quad\simeq\mathrm{Hom}_{\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{R}^{\rm E,sub}_{M_\infty}K,\mathbf{R}^{\rm E,sub}_{M_\infty}\mathbf{Q}^{\rm sub}_{M_\infty}\mathcal{F})\\
&\quad\simeq\mathrm{Hom}_{\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})}\big(\mathbf{R}^{\rm E,sub}_{M_\infty}K,\mathbf{R}\mathcal{I}hom^{+}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),\mathcal{F})\big)\\
&\quad\simeq\mathrm{Hom}_{\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty\times\mathbb{R}_\infty})}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathbf{R}^{\rm E,sub}_{M_\infty}K,\mathcal{F}).
\end{aligned}$$

Let us prove that there exist isomorphisms in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$:

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K)\simeq\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K,$$
$$\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K\simeq\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K.$$

Since $K\in\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, there exists $\mathcal{F}_0\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$ such that $K\simeq\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_0$. Moreover, there exists $F_0\in\mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ such that $\mathcal{F}_0\simeq\lambda_{M_\infty\times\mathbb{R}_\infty}F_0$ by Proposition 3.3(2). Then we have isomorphisms

$$\begin{aligned}
&\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathcal{F}_0)\\
&\quad\simeq\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),\\
&\qquad\qquad\lambda_{M_\infty\times\mathbb{R}_\infty}\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\lambda_{M_\infty\times\mathbb{R}_\infty}F_0)\\
&\quad\simeq\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}\big(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),\\
&\qquad\qquad\lambda_{M_\infty\times\mathbb{R}_\infty}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}F_0)\big)\\
&\quad\simeq\mathbf{R}J_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),\\
&\qquad\qquad\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}F_0)\\
&\quad\simeq\mathbf{R}J_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), F_0)\\
&\quad\simeq\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \lambda_{M_\infty\times\mathbb{R}_\infty}F_0),
\end{aligned}$$

where in the first isomorphism we used Proposition 3.4(3)(i), in the second isomorphism we used Proposition 3.9(4) and in the third and fifth isomorphisms we used Propositions 3.4(4)(i) and 3.9(2). In the fourth isomorphism we used the fact that for any $G\in\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$:

$$\begin{aligned}
&\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}G)\\
&\quad\simeq\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), G).
\end{aligned}$$

This assertion can be proved similarly to [1, Cor. 4.3.11]. Hence there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$:

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K)\simeq\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K.$$

Moreover, we have isomorphisms

$$\begin{aligned}
&\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{F}_0)\\
&\quad\simeq\lambda_{M_\infty\times\mathbb{R}_\infty}\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\\
&\quad\ \overset{+}{\otimes}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \lambda_{M_\infty\times\mathbb{R}_\infty}F_0)
\end{aligned}$$

$$
\begin{aligned}
&\simeq \lambda_{M_\infty\times\mathbb{R}_\infty}\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\\
&\quad\overset{+}{\otimes}\lambda_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),F_0)\\
&\simeq \lambda_{M_\infty\times\mathbb{R}_\infty}\big(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\\
&\qquad\qquad\overset{+}{\otimes}\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),F_0)\big)\\
&\simeq \lambda_{M_\infty\times\mathbb{R}_\infty}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}F_0)\\
&\simeq \lambda_{M_\infty\times\mathbb{R}_\infty}\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\lambda_{M_\infty\times\mathbb{R}_\infty}F_0,
\end{aligned}
$$

where in the first isomorphism we used Proposition 3.4(3)(i), in the second isomorphisms we used Proposition 3.9(2) and in the third and fifth isomorphisms we used Proposition 3.9(4). In the fourth isomorphism we used the fact that for any $G\in\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$:

$$
\begin{aligned}
&\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathbf{R}\mathcal{I}hom^{+}(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}),G)\\
&\quad\simeq \iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}G.
\end{aligned}
$$

This assertion can be proved similarly to [1, Cor. 4.3.11]. Hence there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$:

$$
\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}})\overset{+}{\otimes}\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K\simeq\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K.
$$

Therefore, we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F},\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K)&\simeq\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathcal{F},\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K),\\
\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K,\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F})&\simeq\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K,\mathcal{F}),
\end{aligned}
$$

and hence there exist isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F},K)&\simeq\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathcal{F},\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}K),\\
\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(K,\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F})&\simeq\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K,\mathcal{F}).
\end{aligned}
$$

Therefore, the quotient functor admits a left (resp. right) adjoint $\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}$ (resp. $\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}$). ■

We sometimes denote $\mathbf{Q}^{\mathrm{sub}}_{M_\infty}$ (resp. $\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}$, $\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}$) by $\mathbf{Q}^{\mathrm{sub}}$ (resp. $\mathbf{L}^{\mathrm{E,sub}}$, $\mathbf{R}^{\mathrm{E,sub}}$) for short. Let us set

$$
\begin{aligned}
\mathbf{E}^{\leq 0}(\Bbbk^{\mathrm{sub}}_{M_\infty})&=\{K\in\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})\mid\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K\in\mathbf{D}^{\leq 0}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})\},\\
\mathbf{E}^{\geq 0}(\Bbbk^{\mathrm{sub}}_{M_\infty})&=\{K\in\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})\mid\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K\in\mathbf{D}^{\geq 0}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})\},
\end{aligned}
$$

where $(\mathbf{D}^{\leq 0}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}),\mathbf{D}^{\geq 0}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}))$ is the standard t-structure on $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$.

Proposition 3.12. *A pair* $(\mathbf{E}^{\leq 0}(\Bbbk^{\rm sub}_{M_\infty}), \mathbf{E}^{\geq 0}(\Bbbk^{\rm sub}_{M_\infty}))$ *is a t-structure on* $\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})$.

Proof. It is enough to show that for any $K \in \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})$ there exists a distinguished triangle

$$K_1 \longrightarrow K \longrightarrow K_2 \xrightarrow{+1},$$

with $K_1 \in \mathbf{E}^{\leq 0}(\Bbbk^{\rm sub}_{M_\infty})$, $K_2 \in \mathbf{E}^{\geq 1}(\Bbbk^{\rm sub}_{M_\infty})$.

Let $k \in \mathbf{E}^{\rm b}(\Bbbk_{M_\infty})$. Then there exists a distinguished triangle

$$\mathcal{F}_1 \longrightarrow \mathbf{L}^{\rm E,sub}_{M_\infty} K \longrightarrow \mathcal{F}_2 \xrightarrow{+1}$$

in $\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$ with $\mathcal{F}_1 \in \mathbf{D}^{\leq 0}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$, $\mathcal{F}_2 \in \mathbf{D}^{\geq 1}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$. By Corollary 3.8 and Lemma 3.11, we have an isomorphism $\pi^{-1}\mathbf{R}\pi_{!!}\mathbf{L}^{\rm E,sub}_{M_\infty} K \simeq 0$ in $\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$ and hence we have an isomorphism

$$\pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F}_1[1] \simeq \pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F}_2$$

in $\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$. Since functors $\mathbf{R}\pi_{!!}$ and π^{-1} are left t-exact with respect to standard t-structures, we have $\pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F}_2 \in \mathbf{D}^{\geq 1}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$. Since functors $\mathbf{R}\pi_{!!}[1]$ and π^{-1} are right t-exact with respect to standard t-structures, we have $\pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F}_1[1] \in \mathbf{D}^{\leq 0}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$. Therefore, we have

$$\pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F}_1[1] \simeq \pi^{-1}\mathbf{R}\pi_{!!}\mathcal{F}_2 \simeq 0$$

and hence there exist isomorphisms

$$\mathbf{L}^{\rm E,sub}_{M_\infty}\mathbf{Q}^{\rm sub}(\mathcal{F}_1) \simeq \mathcal{F}_1, \quad \mathbf{L}^{\rm E,sub}_{M_\infty}\mathbf{Q}^{\rm sub}(\mathcal{F}_2) \simeq \mathcal{F}_2$$

in $\mathbf{D}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty \times \mathbb{R}_\infty})$ and a distinguished triangle

$$\mathbf{Q}^{\rm sub}\mathcal{F}_1 \longrightarrow K \longrightarrow \mathbf{Q}^{\rm sub}\mathcal{F}_2 \xrightarrow{+1}$$

in $\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})$.

The proof is completed. ■

We denote by

$$\mathcal{H}^n \colon \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty}) \to \mathbf{E}^{0}(\Bbbk^{\rm sub}_{M_\infty})$$

the n-th cohomology functor, where we set $\mathbf{E}^{0}(\Bbbk^{\rm sub}_{M_\infty}) := \mathbf{E}^{\leq 0}(\Bbbk^{\rm sub}_{M_\infty}) \cap \mathbf{E}^{\geq 0}(\Bbbk^{\rm sub}_{M_\infty})$.

By Proposition 3.6, the convolution functors can be lifted to the triangulated category $\mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})$. We denote them by the same symbols $\overset{+}{\otimes}$, $\mathbf{R}\mathcal{I}hom^{+,\rm sub}$. Namely, we obtain functors

$$(\cdot) \overset{+}{\otimes} (\cdot) \colon \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty}) \times \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty}) \to \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty}),$$
$$\mathbf{R}\mathcal{I}hom^{+,\rm sub}(\cdot,\cdot) \colon \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})^{\rm op} \times \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty}) \to \mathbf{E}^{\rm b}(\Bbbk^{\rm sub}_{M_\infty})$$

which are defined by

$$\mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{F}) \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{G}) := \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{F} \overset{+}{\otimes} \mathcal{G}),$$
$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{F}), \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathcal{G})) := \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathcal{F}, \mathcal{G})),$$

for $\mathcal{F}, \mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$. Moreover, by Proposition 3.1(4), for a morphism $f: M_\infty \to N_\infty$ of real analytic bordered spaces, the following functors are well defined:

$$\mathbf{E}f_*: \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}), \quad \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F} \mapsto \mathbf{Q}^{\mathrm{sub}}_{N_\infty}(\mathbf{R}f_{\mathbb{R}_\infty *}\mathcal{F})$$
$$\mathbf{E}f^{-1}: \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}), \quad \mathbf{Q}^{\mathrm{sub}}_{N_\infty}\mathcal{G} \mapsto \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(f^{-1}_{\mathbb{R}_\infty}\mathcal{G}),$$
$$\mathbf{E}f_{!!}: \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}), \quad \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F} \mapsto \mathbf{Q}^{\mathrm{sub}}_{N_\infty}(\mathbf{R}f_{\mathbb{R}_\infty !!}\mathcal{F}),$$
$$\mathbf{E}f^{!}: \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}), \quad \mathbf{Q}^{\mathrm{sub}}_{N_\infty}\mathcal{G} \mapsto \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(f^{!}_{\mathbb{R}_\infty}\mathcal{G}).$$

Let us define external hom functors

$$\mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(\cdot,\cdot): \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}),$$
$$\mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\cdot,\cdot): \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk_{M}),$$
$$\mathbf{R}\mathrm{Hom}^{\mathrm{E,sub}}(\cdot,\cdot): \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk),$$

by

$$\mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2) := \mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}_1, \mathcal{F}_2),$$
$$\mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2) := \rho_{M_\infty *}\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2),$$
$$\mathbf{R}\mathrm{Hom}^{\mathrm{E,sub}}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2) := \mathbf{R}\Gamma(M; \mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2)),$$

for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$. Note that for any $K_1, K_2 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, we have

$$\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(K_1, K_2) \simeq \mathcal{H}^0\mathbf{R}\mathrm{Hom}^{\mathrm{E,sub}}(K_1, K_2).$$

Moreover, for $\mathcal{F}_0 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$, the objects

$$\pi^{-1}\mathcal{F}_0 \otimes \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F} := \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\pi^{-1}\mathcal{F}_0 \otimes \mathcal{F}),$$
$$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}_0, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}) := \mathbf{Q}^{\mathrm{sub}}_{M_\infty}(\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}_0, \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F}))$$

are well defined and hence the following functors are well defined:

$$\pi^{-1}(\cdot) \otimes (\cdot): \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \times \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}),$$
$$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}(\cdot), \cdot): \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}).$$

At the end of this subsection, let us prove that these functors have several properties similar to classical sheaves.

Proposition 3.13. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces.*

(1) (i) *For any $K_1, K_2, K_3 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, one has*

$$\mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_1, \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_2, K_3)),$$
$$\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E},\mathrm{sub}}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E},\mathrm{sub}}(K_1, \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_2, K_3)),$$
$$\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{E},\mathrm{sub}}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{E},\mathrm{sub}}(K_1, \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_2, K_3)),$$
$$\mathbf{R}\mathrm{Hom}^{\mathrm{E},\mathrm{sub}}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{E},\mathrm{sub}}(K_1 \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_2, K_3)),$$
$$\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(K_1, \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K_2, K_3)).$$

(ii) *For any $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, one has*

$$\mathbf{E}f_*\mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(\mathbf{E}f^{-1}L, K) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(L, \mathbf{E}f_*K),$$
$$\mathbf{R}f_*\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E},\mathrm{sub}}(\mathbf{E}f^{-1}L, K) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E},\mathrm{sub}}(L, \mathbf{E}f_*K),$$
$$\mathbf{R}f_*\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{E},\mathrm{sub}}(\mathbf{E}f^{-1}L, K) \simeq \mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{E},\mathrm{sub}}(L, \mathbf{E}f_*K),$$
$$\mathbf{R}\mathrm{Hom}^{\mathrm{E},\mathrm{sub}}(\mathbf{E}f^{-1}L, K) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{E},\mathrm{sub}}(L, \mathbf{E}f_*K),$$
$$\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathbf{E}f^{-1}L, K) \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})}(L, \mathbf{E}f_*K).$$

(iii) *For any $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, one has*

$$\mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(\mathbf{E}f_{!!}K, L) \simeq \mathbf{E}f_*\mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(K, \mathbf{E}f^!L),$$
$$\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E},\mathrm{sub}}(\mathbf{E}f_{!!}K, L) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{I}\mathrm{hom}^{\mathrm{E},\mathrm{sub}}(K, \mathbf{E}f^!L),$$
$$\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{E},\mathrm{sub}}(\mathbf{E}f_{!!}K, L) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}\mathrm{om}^{\mathrm{E},\mathrm{sub}}(K, \mathbf{E}f^!L),$$
$$\mathbf{R}\mathrm{Hom}^{\mathrm{E},\mathrm{sub}}(\mathbf{E}f_{!!}K, L) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{E},\mathrm{sub}}(K, \mathbf{E}f^!L),$$
$$\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})}(\mathbf{E}f_{!!}K, L) \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(K, \mathbf{E}f^!L,).$$

(2) *For any $K, K_1, K_2 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $L, L_1, L_2 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, one has*

$$\mathbf{E}f^{-1}(K_1 \overset{+}{\otimes} K_2) \simeq \mathbf{E}f^{-1}K_1 \overset{+}{\otimes} \mathbf{E}f^{-1}K_2,$$
$$\mathbf{E}f_{!!}(K \overset{+}{\otimes} \mathbf{E}f^{-1}L) \simeq \mathbf{E}f_{!!}K \overset{+}{\otimes} L,$$
$$\mathbf{E}f^!\mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(L_1, L_2) \simeq \mathbf{R}\mathcal{I}\mathrm{hom}^{+,\mathrm{sub}}(\mathbf{E}f^{-1}L_1, \mathbf{E}f^!L_2).$$

(3) *For a cartesian diagram*

$$\begin{array}{ccc} M'_\infty & \xrightarrow{f'} & N'_\infty \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ M_\infty & \xrightarrow[f]{} & N_\infty \end{array}$$

and any $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$*, one has*

$$\mathbf{E}g^{-1}\mathbf{E}f_{!!}K \simeq \mathbf{E}f'_{!!}\mathbf{E}g'^{-1}K, \quad \mathbf{E}g^{!}\mathbf{E}f_{*}K \simeq \mathbf{E}f'_{*}\mathbf{E}g'^{!}K.$$

(4) (i) *For any* $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and any* $K_1, K_2 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$*, one has*

$$\begin{aligned} &\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\pi^{-1}\mathcal{F}\otimes K_1, K_2)\\ &\quad\simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K_1, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}, K_2))\\ &\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}, \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K_1,K_2)),\end{aligned}$$

$$\begin{aligned} &\mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(\pi^{-1}\mathcal{F}\otimes K_1, K_2)\\ &\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K_1, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}, K_2))\\ &\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K_1, K_2)),\end{aligned}$$

$$\begin{aligned} &\mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\pi^{-1}\mathcal{F}\otimes K_1, K_2)\\ &\quad\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(K_1, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}, K_2))\\ &\quad\simeq \mathbf{R}\mathcal{H}om^{\mathrm{sub}}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K_1, K_2)),\end{aligned}$$

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{E,sub}}(\pi^{-1}\mathcal{F}\otimes K_1, K_2)\\ &\quad\simeq \mathbf{R}\mathrm{Hom}^{\mathrm{E,sub}}(K_1, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}, K_2))\\ &\quad\simeq \mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K_1, K_2)),\end{aligned}$$

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\pi^{-1}\mathcal{F}\otimes K_1, K_2)\\ &\quad\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(K_1, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F}, K_2))\\ &\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathcal{F}, \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K_1, K_2)),\end{aligned}$$

(ii) *For any* $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$*, any* $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$*, any* $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and any* $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$*, one has*

$$\begin{aligned} \mathbf{E}f^{-1}(\pi^{-1}\mathcal{F}\otimes L) &\simeq \pi^{-1}f^{-1}\mathcal{F}\otimes \mathbf{E}f^{-1}L,\\ \mathbf{E}f_{!!}(\pi^{-1}\mathcal{F}\otimes \mathbf{E}f^{-1}L) &\simeq \pi^{-1}\mathbf{R}f_{!!}\mathcal{F}\otimes L,\end{aligned}$$

$$\mathbf{E}f_{!!}(\pi^{-1}f^{-1}\mathcal{F}\otimes K)\simeq\pi^{-1}\mathcal{F}\otimes\mathbf{E}f_{!!}K,$$
$$\mathbf{E}f^{!}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\pi^{-1}\mathcal{G},L)\simeq\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\pi^{-1}f^{-1}\mathcal{G},\mathbf{E}f^{!}L).$$

(iii) *For any* $\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and any* $K,L\in\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$*, one has*

$$\pi^{-1}\mathcal{F}\otimes(K\overset{+}{\otimes}L)\simeq(\pi^{-1}\mathcal{F}\otimes K)\overset{+}{\otimes}L,$$
$$\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F},\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K,L))\simeq\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\pi^{-1}\mathcal{F}\otimes K,L)$$
$$\simeq\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K,\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\pi^{-1}\mathcal{F},L)),$$
$$\mathbf{R}\mathcal{I}hom^{\mathbf{E},\mathrm{sub}}(K\overset{+}{\otimes}L,\mathcal{F})\simeq\mathbf{R}\mathcal{I}hom^{\mathbf{E},\mathrm{sub}}(K,\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(L,\mathcal{F})).$$

Proof. (1)(i) The first (resp. second) assertion follows from Proposition 3.5 (resp. Proposition 3.1(2)). The third (resp. fourth, fifth) assertion follows from the second (resp. third, fourth) one.

(ii) The first (resp. second) assertion follows from Proposition 3.5 (resp. Proposition 3.1(2)). The third (resp. fourth, fifth) assertion follows from the second (resp. third, fourth) one.

(iii) The first (resp. second) assertion follows from Proposition 3.5 (resp. Proposition 3.1(2)). The third (resp. fourth, fifth) assertion follows from the second (resp. third, fourth) one.

(2) The three assertions follow from Proposition 3.5.

(3) The two assertions follow from Proposition 3.1(4).

(4)(i) The first assertion of (i) follows from Proposition 3.6.

The second assertion of (i) follows from Proposition 3.1(2).

The third (resp. fourth, fifth) assertion of (i) follows from the second (resp. third, fourth) one.

(ii) These assertions follow from Proposition 3.1(3), (4).

(iii) These assertions follow from Proposition 3.6. ∎

3.4 – *Relation between enhanced ind-sheaves and enhanced subanalytic sheaves*

In this subsection we shall explain the relation between enhanced subanalytic sheaves and enhanced ind-sheaves. The results are summarized by the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \underset{\mathbf{R}J_{M_\infty}}{\overset{I_{M_\infty}}{\rightleftarrows}} & \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \\
\mathrm{sh}^{\mathrm{sub}}_{M_\infty}\uparrow\;\downarrow e^{\mathrm{sub}}_{M_\infty} & & \mathrm{Ish}_{M_\infty}\uparrow\;\downarrow e_{M_\infty} \\
\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \underset{J^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\rightleftarrows}} & \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \\
\cup & \searrow I^{\mathrm{E}}_{M_\infty}\ \sim\ \lambda^{\mathrm{E}}_{M_\infty}\nwarrow & \cup \\
\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & & \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \\
 & \searrow I^{\mathrm{E}}_{M_\infty}\ \sim\ \lambda^{\mathrm{E}}_{M_\infty}\nwarrow & \cup \\
 & & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).
\end{array}
$$

Theorems 3.15 and 3.20 are two of the main results of this subsection and this paper.

Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. Let us consider a quotient category

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) := \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})/\pi^{-1}\mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).$$

Note that this is a full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ by using $\pi^{-1}\mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) = \pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \cap \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ and [9, Prop. 1.6.10]. Note also that $\Bbbk^{\mathrm{E}}_{M_\infty} \in \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. Moreover, Proposition 3.14 below follows from Lemma 3.2 and Proposition 3.9(3).

Proposition 3.14. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces associated with a morphism $\check{f}\colon \check{M} \to \check{N}$ of real analytic manifolds. The functors below are well defined:*

(1) $e_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(2) $(\cdot)\overset{+}{\otimes}(\cdot)\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \times \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(3) $\mathbf{E}f^{-1}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(4) $\mathbf{E}f_{!!}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty})$,

(5) $\mathbf{E}f^{!}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$.

By Proposition 3.4(2)(iii), (3)(v), the following functors are well defined:

$$I^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}),\quad \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathcal{F} \mapsto \mathbf{Q}_{M_\infty} I_{M_\infty\times\mathbb{R}_\infty}\mathcal{F},$$
$$J^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}),\quad \mathbf{Q}_{M_\infty} F \mapsto \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\mathbf{R}J_{M_\infty\times\mathbb{R}_\infty}F.$$

Theorem 3.15. *Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. Then we have*

(1) *a pair $(I^{\mathrm{E}}_{M_\infty}, J^{\mathrm{E}}_{M_\infty})$ is an adjoint pair and there exists a canonical isomorphism $\mathrm{id} \xrightarrow{\sim} J^{\mathrm{E}}_{M_\infty}\circ I^{\mathrm{E}}_{M_\infty}$,*

(2) *there exists an equivalence of triangulated categories:*

$$\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \underset{J^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).$$

Proof. (1) This assertion follows from Proposition 3.3 and Lemma 3.11.

(2) Since the functor $I_{M_\infty\times\mathbb{R}_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ is well defined, for any $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ we have

$$I^{\mathrm{E}}_{M_\infty}K = \mathbf{Q}_{M_\infty} I_{M_\infty\times\mathbb{R}_\infty}\mathbf{L}^{\mathrm{E,sub}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).$$

Let $L \in \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. Then there exists $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ such that $L \simeq \mathbf{Q}_{M_\infty}G$ and hence

$$I^{\mathrm{E}}_{M_\infty}\mathbf{R}J^{\mathrm{E}}_{M_\infty}L \simeq \mathbf{Q}_{M_\infty}I_{M_\infty\times\mathbb{R}_\infty}\mathbf{R}J_{M_\infty\times\mathbb{R}_\infty}G \simeq \mathbf{Q}_{M_\infty}G \simeq L,$$

where in the second isomorphism we used Proposition 3.3(2). Therefore, the proof is completed. ■

We shall denote by

$$\lambda^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

the inverse functor of $I^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. There exists a commutativity between the various functors and functors I^{E}, J^{E}, λ^{E} as below.

Proposition 3.16. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces associated with a morphism $\check{f}\colon \check{M} \to \check{N}$ of real analytic manifolds.*

(1) *For any $K, K_1, K_2 \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $L \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$, we have*

(i) $I^{\mathrm{E}}_{M_\infty}(K_1 \overset{+}{\otimes} K_2) \simeq I^{\mathrm{E}}_{M_\infty}K_1 \overset{+}{\otimes} I^{\mathrm{E}}_{M_\infty}K_2$,

(ii) $J^{\mathrm{E}}_{M_\infty}\mathbf{R}\mathcal{I}hom^{+}(I^{\mathrm{E}}_{M_\infty}K, L) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K, J^{\mathrm{E}}_{M_\infty}L)$,

(iii) $\mathbf{R}J_{M_\infty}\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(I^{\mathrm{E}}_{M_\infty}K, L) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K, J^{\mathrm{E}}_{M_\infty}L)$.

(2) *For any* $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and any* $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$*, we have*

(i) $I^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{-1}L \simeq \mathbf{E}f^{-1}I^{\mathrm{E}}_{N_\infty}L$,

(ii) $\mathbf{E}f_{!!}I^{\mathrm{E}}_{M_\infty}K \simeq I^{\mathrm{E}}_{N_\infty}\mathbf{E}f_{!!}K$,

(iii) $I^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{!}L \simeq \mathbf{E}f^{!}I^{\mathrm{E}}_{N_\infty}L$.

(3) *For any* $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ *and any* $L \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{N_\infty})$*, we have*

(i) $\mathbf{E}f_{*}J^{\mathrm{E}}_{M_\infty}K \simeq J^{\mathrm{E}}_{N_\infty}\mathbf{E}f_{*}K$,

(ii) $J^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{!}L \simeq \mathbf{E}f^{!}J^{\mathrm{E}}_{N_\infty}L$.

(4) *For any* $K \in \mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$ *and any* $L \in \mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty})$*, we have*

(i) $\lambda^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{-1}L \simeq \mathbf{E}f^{-1}\lambda^{\mathrm{E}}_{N_\infty}L$,

(ii) $\mathbf{E}f_{!!}\lambda^{\mathrm{E}}_{M_\infty}K \simeq \lambda^{\mathrm{E}}_{N_\infty}\mathbf{E}f_{!!}K$,

(iii) $\lambda^{\mathrm{E}}_{M_\infty}(K_1 \overset{+}{\otimes} K_2) \simeq \lambda^{\mathrm{E}}_{M_\infty}K_1 \overset{+}{\otimes} \lambda^{\mathrm{E}}_{M_\infty}K_2$.

Proof. Let us denote by $f_{\mathbb{R}_\infty}\colon M_\infty \times \mathbb{R}_\infty \to N_\infty \times \mathbb{R}_\infty$ the morphism $f \times \mathrm{id}_{\mathbb{R}_\infty}$ of real analytic bordered spaces.

The assertions of (1), (2) and (3) follow from Propositions 3.4, 3.9. Let us skip the details. Since the proofs of (4)(ii), (iii) are similar, we shall only prove (4)(i). Let $K \in \mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$, $L \in \mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty})$. Using (2)(i) and Theorem 3.15(2), we have

$$\begin{aligned}\lambda^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{-1}L \simeq \lambda^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{-1}I^{\mathrm{E}}_{N_\infty}\lambda^{\mathrm{E}}_{N_\infty}L \simeq \lambda^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}\mathbf{E}f^{-1}\lambda^{\mathrm{E}}_{N_\infty}L\\ \simeq \mathbf{E}f^{-1}\lambda^{\mathrm{E}}_{N_\infty}L.\end{aligned}$$ ∎

Let us prove that the functors $I^{\mathrm{E}}_{M_\infty}$, $J^{\mathrm{E}}_{M_\infty}$ preserve the $\mathbb{R}$-constructibility. Let us recall that an enhanced ind-sheaf K on M_∞ is $\mathbb{R}$-constructible if for any open subset U of M which is subanalytic and relatively compact in $\check{M}$ there exists $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty\times\mathbb{R}_\infty})$ such that $\mathbf{E}i^{-1}_{U_\infty}K \simeq \Bbbk^{\mathrm{E}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}_{U_\infty}\iota_{U_\infty\times\mathbb{R}_\infty}\mathcal{F}$. We denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$ the category of $\mathbb{R}$-constructible enhanced ind-sheaves. See [2, §3.3] for the details.

We shall set

$$\Bbbk^{\mathrm{E,sub}}_{M_\infty} := \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\Big(\varinjlim_{a\to+\infty}\rho_{M_\infty\times\mathbb{R}_\infty *}\Bbbk_{\{t\geq a\}}\Big) \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}).$$

Lemma 3.17. *There exist an isomorphism* $I^{\mathrm{E}}_{M_\infty}\Bbbk^{\mathrm{E,sub}}_{M_\infty} \simeq \Bbbk^{\mathrm{E}}_{M_\infty}$ *in* $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ *and an isomorphism* $J^{\mathrm{E}}_{M_\infty}\Bbbk^{\mathrm{E}}_{M_\infty} \simeq \Bbbk^{\mathrm{E,sub}}_{M_\infty}$ *in* $\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$.

Proof. By the definition of $\Bbbk^{\mathrm{E}}_{M_\infty}$, we have

$$\Bbbk^{\mathrm{E}}_{M_\infty} \simeq \mathbf{Q}_{M_\infty} j^{-1}_{M_\infty\times\mathbb{R}_\infty}\big(\text{“}\varinjlim_{a\to+\infty}\text{”}\,\iota_{\check{M}\times\bar{\mathbb{R}}}\Bbbk_{\{t\geq a\}}\big).$$

Since $I_{\check{M}\times\bar{\mathbb{R}}}\circ\rho_{\check{M}\times\bar{\mathbb{R}}*}\simeq\iota_{\check{M}\times\bar{\mathbb{R}}}$ and the fact that a functor I commutes with the filtrant inductive limits, there exist isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\check{M}\times\bar{\mathbb{R}}})$:

$$\text{“}\varinjlim_{a\to+\infty}\text{”}\,\iota_{\check{M}\times\bar{\mathbb{R}}}\Bbbk_{\{t\geq a\}} \simeq I_{\check{M}\times\bar{\mathbb{R}}}\big(\varinjlim_{a\to+\infty}\rho_{\check{M}\times\bar{\mathbb{R}}*}\Bbbk_{\{t\geq a\}}\big).$$

Hence, we have

$$\Bbbk^{\mathrm{E}}_{M_\infty} \simeq \mathbf{Q}_{M_\infty} j^{-1}_{M_\infty\times\mathbb{R}_\infty} I_{\check{M}\times\bar{\mathbb{R}}}\big(\varinjlim_{a\to+\infty}\rho_{\check{M}\times\bar{\mathbb{R}}*}\Bbbk_{\{t\geq a\}}\big) \simeq I^{\mathrm{E}}_{M_\infty}\Bbbk^{\mathrm{E,sub}}_{M_\infty}.$$

The second assertion follows from the first assertion and Theorem 3.15(1). ∎

Proposition 3.18. *The triangulated category* $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$ *is a full triangulated subcategory of* $\mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$.

Proof. Let $K\in\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. Since a pair $(I^{\mathrm{E}}_{M_\infty}, J^{\mathrm{E}}_{M_\infty})$ is an adjoint pair, we have a morphism $I^{\mathrm{E}}_{M_\infty}J^{\mathrm{E}}_{M_\infty}K\to K$ in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$. Since K is $\mathbb{R}$-constructible, for any open subset U of M which is subanalytic and relatively compact in $\check{M}$ there exists $\mathcal{F}^U\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty\times\mathbb{R}_\infty})$ such that $\mathbf{E}i^{-1}_{U_\infty}K\simeq\Bbbk^{\mathrm{E}}_{U_\infty}\overset{+}{\otimes}\mathbf{Q}_{U_\infty}\iota_{U_\infty\times\mathbb{R}_\infty}\mathcal{F}^U$. By Proposition 3.4(4)(i), Theorem 3.15(2), Proposition 3.16(1)(i) and Lemma 3.17, we have

$$\Bbbk^{\mathrm{E}}_{U_\infty}\overset{+}{\otimes}\mathbf{Q}_{U_\infty}\iota_{U_\infty\times\mathbb{R}_\infty}\mathcal{F}^U\simeq I^{\mathrm{E}}_{U_\infty}(\Bbbk^{\mathrm{E,sub}}_{U_\infty}\overset{+}{\otimes}\mathbf{Q}^{\mathrm{sub}}_{U_\infty}\rho_{U_\infty\times\mathbb{R}_\infty*}\mathcal{F}^U)\in\mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{U_\infty}).$$

This implies that for any open subset U of M which is subanalytic and relatively compact in $\check{M}$, $\mathbf{E}i^{-1}_{U_\infty}K\in\mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{U_\infty})$. Hence, by Proposition 3.16(2)(i), (3)(ii) there exist isomorphisms

$$(I^{\mathrm{E}}_{M_\infty}J^{\mathrm{E}}_{M_\infty}K)|_{U_\infty}\simeq I^{\mathrm{E}}_{U_\infty}J^{\mathrm{E}}_{U_\infty}(K|_{U_\infty})\simeq K|_{U_\infty}$$

for any open subset U of M which is subanalytic and relatively compact in $\check{M}$. This implies that $I^{\mathrm{E}}_{M_\infty}J^{\mathrm{E}}_{M_\infty}K\xrightarrow{\sim}K$ and hence $K\in\mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. ∎

Definition 3.19. Let $M_\infty=(M,\check{M})$ be a real analytic bordered space. We say that an enhanced subanalytic sheaf K is $\mathbb{R}$-constructible if for any open subset U of M which is subanalytic and relatively compact in $\check{M}$ there exists $\mathcal{F}\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty\times\mathbb{R}_\infty})$ such that

$$\mathbf{E}i^{-1}_{U_\infty}K\simeq\Bbbk^{\mathrm{E,sub}}_{U_\infty}\overset{+}{\otimes}\mathbf{Q}^{\mathrm{sub}}_{U_\infty}\rho_{U_\infty\times\mathbb{R}_\infty*}\mathcal{F}.$$

Let us denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ the category of $\mathbb{R}$-constructible enhanced subanalytic sheaves.

Theorem 3.20. *Let M_∞ be a real analytic bordered space. Then the functors $I^{\mathrm{E}}_{M_\infty}$, $J^{\mathrm{E}}_{M_\infty}$ induce an equivalence of categories*

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \underset{J^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).$$

Proof. It is enough to show that the following functors are well defined:

$$I^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}), \quad J^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}).$$

Since the proofs of them are similar, we shall only prove that $I^{\mathrm{E}}_{M_\infty}$ is well defined.

Let $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and U be an open subset of M which is subanalytic and relatively compact in $\check{M}$. Then there exists $\mathcal{F}^U \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty\times\mathbb{R}_\infty})$ such that $\mathbf{E}i^{-1}_{U_\infty}K \simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\rho_{M_\infty\times\mathbb{R}_\infty *}\mathcal{F}^U$. By Propositions 3.4(4)(i), 3.16(1)(i), (2)(i) and Lemma 3.17, there exist isomorphisms

$$\begin{aligned}\mathbf{E}i^{-1}_{U_\infty} I^{\mathrm{E}}_{M_\infty} K &\simeq I^{\mathrm{E}}_{U_\infty}\mathbf{E}i^{-1}_{U_\infty} K \simeq I^{\mathrm{E}}_{U_\infty}(\Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{M_\infty}\rho_{M_\infty\times\mathbb{R}_\infty *}\mathcal{F}^U)\\ &\simeq \Bbbk^{\mathrm{E}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}_{M_\infty}\iota_{M_\infty\times\mathbb{R}_\infty}\mathcal{F}^U.\end{aligned}$$

Therefore, we have $I^{\mathrm{E}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. ■

Let us summarize the results of Proposition 3.18 and Theorems 3.15, 3.20 in the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \underset{J^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\hookrightarrow\!\!\!\leftarrow}} & \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})\\
\cup & \underset{\lambda^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\searrow\!\!\!\nwarrow}}\ \sim & \cup\\
\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & & \mathbf{E}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})\\
 & \underset{\lambda^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\searrow\!\!\!\nwarrow}}\ \sim & \cup\\
 & & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}).
\end{array}$$

At the end of this subsection, let us consider an embedding functor from $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ to $\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and a Verdier duality functor for enhanced subanalytic sheaves.

Let us define a functor

$$e^{\mathrm{sub}}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}), \quad \mathcal{F} \mapsto \pi^{-1}\mathcal{F} \otimes \Bbbk^{\mathrm{E,sub}}_{M_\infty}.$$

By Proposition 3.21 below, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \xrightarrow{e^{\mathrm{sub}}_{M_\infty}} & \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \\ {\scriptstyle I_{M_\infty}}\downarrow & & \downarrow{\scriptstyle I^{\mathrm{E}}_{M_\infty}} \\ \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) & \xrightarrow[e_{M_\infty}]{} & \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}), \end{array} \qquad \begin{array}{ccc} \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) & \xrightarrow{e_{M_\infty}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty}) \\ {\scriptstyle \lambda_{M_\infty}}\downarrow & & \downarrow{\scriptstyle \lambda^{\mathrm{E}}_{M_\infty}} \\ \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \xrightarrow[e^{\mathrm{sub}}_{M_\infty}]{} & \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}). \end{array}$$

Proposition 3.21. *For any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$, we have*

$$I^{\mathrm{E}}_{M_\infty} e^{\mathrm{sub}}_{M_\infty}\mathcal{F} \simeq e_{M_\infty} I_{M_\infty}\mathcal{F}, \quad e^{\mathrm{sub}}_{M_\infty}\lambda_{M_\infty}F \simeq \lambda^{\mathrm{E}}_{M_\infty} e_{M_\infty}F.$$

Moreover, the functor $e^{\mathrm{sub}}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ is fully faithful.

Proof. Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. By Proposition 3.4(2)(iii), (vi) and Lemma 3.17, there exist isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$:

$$I^{\mathrm{E}}_{M_\infty} e^{\mathrm{sub}}_{M_\infty}\mathcal{F} \simeq I^{\mathrm{E}}_{M_\infty}\Bbbk^{\mathrm{E,sub}}_{M_\infty} \otimes \pi^{-1}I_{M_\infty}\mathcal{F} \simeq \Bbbk^{\mathrm{E}}_{M_\infty} \otimes \pi^{-1}I_{M_\infty}\mathcal{F} \simeq e_{M_\infty}I_{M_\infty}\mathcal{F}.$$

Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{IR}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. By Proposition 3.4(4)(ii), (iv) and Lemma 3.17, there exist isomorphisms in $\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$:

$$e^{\mathrm{sub}}_{M_\infty}\lambda_{M_\infty}F \simeq \Bbbk^{\mathrm{E,sub}}_{M_\infty} \otimes \pi^{-1}\lambda_{M_\infty}F \simeq \lambda^{\mathrm{E}}_{M_\infty}\Bbbk^{\mathrm{E}}_{M_\infty} \otimes \pi^{-1}\lambda_{M_\infty}F \simeq \lambda^{\mathrm{E}}_{M_\infty}e_{M_\infty}F.$$

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. By Proposition 3.3, the functor $I_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is fully faithful and hence there exists an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(I_{M_\infty}\mathcal{F}_1, I_{M_\infty}\mathcal{F}_2).$$

Since the functor $e_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is fully faithful, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(I_{M_\infty}\mathcal{F}_1, I_{M_\infty}\mathcal{F}_2) \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(e_{M_\infty}I_{M_\infty}\mathcal{F}_1, e_{M_\infty}I_{M_\infty}\mathcal{F}_2).$$

Moreover, by the first assertion, we have

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(e_{M_\infty}I_{M_\infty}\mathcal{F}_1, e_{M_\infty}I_{M_\infty}\mathcal{F}_2) \\ &\qquad \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(I^{\mathrm{E}}_{M_\infty}e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, I^{\mathrm{E}}_{M_\infty}e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2). \end{aligned}$$

By Theorem 3.15, the functor $I^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ is fully faithful, and hence

$$\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(I^{\mathrm{E}}_{M_\infty} e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, I^{\mathrm{E}}_{M_\infty} e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2) \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2).$$

Therefore, we have

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})}(e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1, e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2).$$

This implies that the functor $e^{\mathrm{sub}}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ is fully faithful. ∎

The functor e^{sub} commutes with several functors as below.

Proposition 3.22. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces. For any $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and any $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, we have*

$$\begin{aligned} e^{\mathrm{sub}}_{M_\infty}(\mathcal{F}_1 \otimes \mathcal{F}_2) &\simeq e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1 \overset{+}{\otimes} e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2,\\ \mathbf{E}f_{!!}e^{\mathrm{sub}}_{M_\infty}\mathcal{F} &\simeq e^{\mathrm{sub}}_{N_\infty}\mathbf{R}f_{!!}\mathcal{F},\\ \mathbf{E}f^{-1}e^{\mathrm{sub}}_{N_\infty}\mathcal{G} &\simeq e^{\mathrm{sub}}_{M_\infty}f^{-1}\mathcal{G},\\ \mathbf{E}f^{!}e^{\mathrm{sub}}_{N_\infty}\mathcal{G} &\simeq e^{\mathrm{sub}}_{M_\infty}f^{!}\mathcal{G}. \end{aligned}$$

Proof. By Proposition 3.13(4)(iii) and the fact that $\Bbbk^{\mathrm{E,sub}}_{M_\infty} \overset{+}{\otimes} \Bbbk^{\mathrm{E,sub}}_{M_\infty} \simeq \Bbbk^{\mathrm{E,sub}}_{M_\infty}$, we have

$$\begin{aligned} e^{\mathrm{sub}}_{M_\infty}(\mathcal{F}_1 \otimes \mathcal{F}_2) &\simeq \Bbbk^{\mathrm{E,sub}}_{M_\infty} \otimes \pi^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_2)\\ &\simeq (\Bbbk^{\mathrm{E,sub}}_{M_\infty} \overset{+}{\otimes} \Bbbk^{\mathrm{E,sub}}_{M_\infty}) \otimes (\pi^{-1}\mathcal{F}_1 \otimes \pi^{-1}\mathcal{F}_2)\\ &\simeq (\Bbbk^{\mathrm{E,sub}}_{M_\infty} \otimes \pi^{-1}\mathcal{F}_1) \overset{+}{\otimes} (\Bbbk^{\mathrm{E,sub}}_{M_\infty} \otimes \pi^{-1}\mathcal{F}_2)\\ &\simeq e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_1 \overset{+}{\otimes} e^{\mathrm{sub}}_{M_\infty}\mathcal{F}_2. \end{aligned}$$

The second and third assertions follow from Proposition 3.13(4)(ii) and the fact that $\mathbf{E}f^{-1}\Bbbk^{\mathrm{E,sub}}_{N_\infty} \simeq \Bbbk^{\mathrm{E,sub}}_{M_\infty}$.

Let us prove the last assertion. By Propositions 3.3, 3.16(3)(ii) and 3.21, we have isomorphisms in $\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$:

$$\mathbf{E}f^{!}e^{\mathrm{sub}}_{N_\infty}\mathcal{G} \simeq \mathbf{E}f^{!}e^{\mathrm{sub}}_{N_\infty}\lambda_{N_\infty}I_{N_\infty}\mathcal{G} \simeq e^{\mathrm{sub}}_{M_\infty}\lambda^{\mathrm{E}}_{M_\infty}I_{M_\infty}f^{!}\mathcal{G} \simeq e^{\mathrm{sub}}_{M_\infty}f^{!}\mathcal{G}.$$ ∎

We set

$$\mathrm{sh}^{\mathrm{sub}}_{M_\infty} := i_0^{!} \circ \mathbf{R}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

and call it the subanalytic sheafification functor for enhanced subanalytic sheaves on real analytic bordered spaces.

Proposition 3.23. *The subanalytic sheafification functor* $\mathrm{sh}^{\mathrm{sub}}_{M_\infty}$ *has the following properties:*

(1) *A pair* $(e^{\mathrm{sub}}_{M_\infty}, \mathrm{sh}^{\mathrm{sub}}_{M_\infty})$: $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \underset{\mathrm{sh}^{\mathrm{sub}}_{M_\infty}}{\overset{e^{\mathrm{sub}}_{M_\infty}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *is an adjoint pair.*

(2) *For any* $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, *one has* $\mathcal{F} \xrightarrow{\sim} \mathrm{sh}^{\mathrm{sub}}_{M_\infty} e^{\mathrm{sub}}_{M_\infty} \mathcal{F}$. *Namely, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \xrightarrow{e^{\mathrm{sub}}_{M_\infty}} & \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \\ & {\scriptstyle\mathrm{id}}\searrow & \downarrow{\scriptstyle \mathrm{sh}^{\mathrm{sub}}_{M_\infty}} \\ & & \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}). \end{array}$$

(3) *For any* $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$, *one has* $\mathbf{R}J_{M_\infty}\mathrm{Ish}_{M_\infty}K \simeq \mathrm{sh}^{\mathrm{sub}}_{M_\infty} J^{\mathrm{E}}_{M_\infty} K$. *Namely, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) & \xrightarrow{J^{\mathrm{E}}_{M_\infty}} & \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \\ {\scriptstyle \mathrm{Ish}_{M_\infty}}\downarrow & & \downarrow{\scriptstyle \mathrm{sh}^{\mathrm{sub}}_{M_\infty}} \\ \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty}) & \xrightarrow[\mathbf{R}J_{M_\infty}]{} & \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}). \end{array}$$

(4) *Let* $f\colon M_\infty \to N_\infty$ *be a morphism of real analytic bordered spaces. For any* $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and any* $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, *one has*

$$\mathbf{R}f_*\mathrm{sh}^{\mathrm{sub}}_{M_\infty}K \simeq \mathrm{sh}^{\mathrm{sub}}_{N_\infty}\mathbf{E}f_*K, \quad f^!\mathrm{sh}^{\mathrm{sub}}_{N_\infty}L \simeq \mathrm{sh}^{\mathrm{sub}}_{M_\infty}\mathbf{E}f^!L.$$

Namely, the following diagrams are commutative:

$$\begin{array}{ccc} \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \xrightarrow{\mathbf{E}f_*} & \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}) \\ {\scriptstyle \mathrm{sh}^{\mathrm{sub}}_{M_\infty}}\downarrow & & \downarrow{\scriptstyle \mathrm{sh}^{\mathrm{sub}}_{N_\infty}} \\ \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) & \xrightarrow[\mathbf{R}f_*]{} & \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}), \end{array} \qquad \begin{array}{ccc} \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}) & \xrightarrow{\mathbf{E}f^!} & \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \\ {\scriptstyle \mathrm{sh}^{\mathrm{sub}}_{N_\infty}}\downarrow & & \downarrow{\scriptstyle \mathrm{sh}^{\mathrm{sub}}_{M_\infty}} \\ \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty}) & \xrightarrow[f^!]{} & \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}). \end{array}$$

Proof. (3) Let $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$. Then there exists $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty\times\mathbb{R}_\infty})$ such that $K \simeq \mathbf{Q}_{M_\infty}(F)$. Then we have

$$\begin{aligned} &\mathbf{R}J_{M_\infty}\mathrm{Ish}_{M_\infty}K \\ &\quad\simeq \mathbf{R}J_{M_\infty} i_0^! \mathbf{R}^{\mathrm{E}}_{M_\infty} K \\ &\quad\simeq i_0^! \mathbf{R}J_{M_\infty\times\mathbb{R}_\infty} \mathbf{R}\mathcal{I}\mathrm{hom}^+(\iota_{M_\infty\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}), F) \end{aligned}$$

$$\simeq i_0^! \mathbf{R} J_{M_\infty \times \mathbb{R}_\infty} \mathbf{R}\mathcal{I}hom^+(I_{M_\infty \times \mathbb{R}_\infty} \rho_{M_\infty \times \mathbb{R}_\infty *}(\Bbbk_{\{t \geq 0\}} \oplus \Bbbk_{\{t \leq 0\}}), F)$$
$$\simeq i_0^! \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty \times \mathbb{R}_\infty *}(\Bbbk_{\{t \geq 0\}} \oplus \Bbbk_{\{t \leq 0\}}), \mathbf{R} J_{M_\infty \times \mathbb{R}_\infty} F)$$
$$\simeq \mathrm{sh}_{M_\infty}^{\mathrm{sub}} J_{M_\infty}^{\mathrm{E}} K,$$

where in the second (resp. fourth, fifth) isomorphism we used Proposition 3.4(3)(v) (resp. (4)(i), Proposition 3.9(2)).

(1) Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})$ and $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})$. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(e_{M_\infty}^{\mathrm{sub}} \mathcal{F}, K) &\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(I_{M_\infty}^{\mathrm{E}} e_{M_\infty}^{\mathrm{sub}} \mathcal{F}, I_{M_\infty}^{\mathrm{E}} K) \\
&\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})}(e_{M_\infty} I_{M_\infty} \mathcal{F}, I_{M_\infty}^{\mathrm{E}} K) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(\mathcal{F}, \mathbf{R} J_{M_\infty} \mathrm{Ish}_{M_\infty} I_{M_\infty}^{\mathrm{E}} K) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(\mathcal{F}, \mathrm{sh}_{M_\infty}^{\mathrm{sub}} J_{M_\infty}^{\mathrm{E}} I_{M_\infty}^{\mathrm{E}} K) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(\mathcal{F}, \mathrm{sh}_{M_\infty}^{\mathrm{sub}} K),
\end{aligned}$$

where in the first and last isomorphisms (resp. second isomorphism) we used Theorem 3.15 (resp. Proposition 3.21), in the third isomorphism we used the fact that $(e_{M_\infty}, \mathrm{Ish}_{M_\infty})$ is an adjoint pair and Proposition 3.3(1), and the fourth isomorphism follows from assertion (3). This implies that a pair $(e_{M_\infty}^{\mathrm{sub}}, \mathrm{sh}_{M_\infty}^{\mathrm{sub}})$ is an adjoint pair.

(2) Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})$. By assertion (1), there exists a canonical morphism $\mathcal{F} \to \mathrm{sh}_{M_\infty}^{\mathrm{sub}} e_{M_\infty}^{\mathrm{sub}} \mathcal{F}$. Moreover, we have

$$\mathrm{sh}_{M_\infty}^{\mathrm{sub}} e_{M_\infty}^{\mathrm{sub}} \mathcal{F} \simeq \mathrm{sh}_{M_\infty}^{\mathrm{sub}} J_{M_\infty}^{\mathrm{E}} I_{M_\infty}^{\mathrm{E}} e_{M_\infty}^{\mathrm{sub}} \mathcal{F} \simeq \mathbf{R} J_{M_\infty} \mathrm{Ish}_{M_\infty} e_{M_\infty} I_{M_\infty} \mathcal{F} \simeq \mathcal{F},$$

where in the first isomorphism we used Theorem 3.15(1), in the second isomorphism we used assertion (3) and Proposition 3.21, and in the last isomorphism we used the fact that $\mathrm{Ish}_{M_\infty} \circ e_{M_\infty} \simeq \mathrm{id}$ and Proposition 3.3(1).

(4) Let $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})$. For any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\Bbbk_{N_\infty}^{\mathrm{sub}})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{N_\infty}^{\mathrm{sub}})}(\mathcal{F}, \mathbf{R} f_* \mathrm{sh}_{M_\infty}^{\mathrm{sub}} K) &\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(e_{M_\infty}^{\mathrm{sub}} f^{-1} \mathcal{F}, K) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(\mathbf{E} f^{-1} e_{N_\infty}^{\mathrm{sub}} \mathcal{F}, K) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty}^{\mathrm{sub}})}(\mathcal{F}, \mathrm{sh}_{N_\infty}^{\mathrm{sub}} \mathbf{E} f_* K),
\end{aligned}$$

where in the first and last isomorphisms we used the fact that a pair $(f^{-1}, \mathbf{R} f_*)$ is an adjoint pair, assertion (1) and Proposition 3.13(1)(ii), and in the second isomorphism we used Proposition 3.22. This implies that there exists an isomorphism $\mathbf{R} f_* \mathrm{sh}_{M_\infty}^{\mathrm{sub}} K \simeq \mathrm{sh}_{N_\infty}^{\mathrm{sub}} \mathbf{E} f_* K$.

Let $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$. Then there exists $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty\times\mathbb{R}_\infty})$ such that $L \simeq \mathbf{Q}_{N_\infty}\mathcal{G}$. We shall denote by $f_{\mathbb{R}_\infty}\colon M_\infty \times \mathbb{R}_\infty \to N_\infty \times \mathbb{R}_\infty$ the morphism $f \times \mathrm{id}_{\mathbb{R}_\infty}$ of real analytic bordered spaces. Then we have isomorphisms in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$:

$$\begin{aligned} f^!\mathrm{sh}^{\mathrm{sub}}_{N_\infty} L &\simeq f^! i_0^! \mathbf{R}^{\mathrm{E,sub}}_{N_\infty} L \simeq i_0^! f^!_{\mathbb{R}_\infty} \mathbf{R}^{\mathrm{E,sub}}_{N_\infty} L \\ &\qquad\simeq i_0^! f^!_{\mathbb{R}_\infty} \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}), \mathcal{G}). \end{aligned}$$

Moreover, by Proposition 3.5, we have

$$\begin{aligned} f^!\mathrm{sh}^{\mathrm{sub}}_{N_\infty} L &\simeq i_0^! f^!_{\mathbb{R}_\infty} \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}), \mathcal{G}) \\ &\simeq i_0^! \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(f^{-1}_{\mathbb{R}_\infty} \rho_{N_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}), f^!_{\mathbb{R}_\infty}\mathcal{G}) \\ &\simeq i_0^! \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{M_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}), f^!_{\mathbb{R}_\infty}\mathcal{G}) \\ &\simeq i_0^! \mathbf{R}^{\mathrm{E,sub}}_{M_\infty} \mathbf{Q}^{\mathrm{sub}}_{M_\infty} f^!_{\mathbb{R}_\infty}\mathcal{G} \\ &\simeq \mathrm{sh}^{\mathrm{sub}}_{M_\infty} \mathbf{E} f^! L. \end{aligned}$$

∎

Let us set

$$\omega^{\mathrm{E,sub}}_{M_\infty} := e^{\mathrm{sub}}_{M_\infty}(\rho_{M_\infty *}\omega_M) \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}),$$

where $\omega_M \in \mathbf{D}^{\mathrm{b}}(\Bbbk_{M_\infty})$ $(\simeq \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M))$ is the dualizing complex; see [9, Def. 3.1.16(i)] for the details. Note that since $\omega_M \simeq j_M^{-1}\omega_{\check{M}}$, we have $\omega_M \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty})$. We shall define a functor

$$\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty}), \quad K \mapsto \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K, \omega^{\mathrm{E,sub}}_{M_\infty}).$$

Lemma 3.24. *There exist an isomorphism* $I^{\mathrm{E}}_{M_\infty}\omega^{\mathrm{E,sub}}_{M_\infty} \simeq \omega^{\mathrm{E}}_{M_\infty}$ *in* $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$ *and an isomorphism* $J^{\mathrm{E}}_{M_\infty}\omega^{\mathrm{E}}_{M_\infty} \simeq \omega^{\mathrm{E,sub}}_{M_\infty}$ *in* $\mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$.

Proof. Since ω_M is $\mathbb{R}$-constructible, there exists an isomorphism $\iota_{M_\infty}\omega_M \simeq I_{M_\infty}\rho_{M_\infty *}\omega_M$ in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{M_\infty})$. Hence, we have

$$\begin{aligned} \omega^{\mathrm{E}}_{M_\infty} := e_{M_\infty}(\iota_{M_\infty}\omega_M) \simeq e_{M_\infty}(I_{M_\infty}\rho_{M_\infty *}\omega_M) &\simeq I^{\mathrm{E}}_{M_\infty} e^{\mathrm{sub}}_{M_\infty}(\rho_{M_\infty *}\omega_M) \\ &\simeq I^{\mathrm{E}}_{M_\infty}\omega^{\mathrm{E,sub}}_{M_\infty}, \end{aligned}$$

where in the third isomorphism we used Proposition 3.21.

The second assertion follows from the first assertion and Theorem 3.15(1). ∎

Proposition 3.25. *Let* $f\colon M_\infty \to N_\infty$ *be a morphism of real analytic bordered spaces.*

(1) *For any* $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and any* $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$*, one has*

$$\mathbf{E}f^!\mathbf{D}^{\mathrm{E,sub}}_{N_\infty}L \simeq \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\mathbf{E}f^{-1}L,$$
$$\mathbf{E}f_*\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K \simeq \mathbf{D}^{\mathrm{E,sub}}_{N_\infty}\mathbf{E}f_{!!}K,$$
$$J^{\mathrm{E}}_{M_\infty}\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \simeq \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K.$$

(2) *For any* $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$*, we have* $\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ *and* $\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K \simeq K$. *In particular, there exists an equivalence of categories:*

$$\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}).$$

Proof. (1) First, let us prove that $\mathbf{E}f^!\omega^{\mathrm{E,sub}}_{N_\infty} \simeq \omega^{\mathrm{E,sub}}_{M_\infty}$. By Proposition 3.22 and the fact that $f^!\omega_N \simeq \omega_M$, we have

$$\mathbf{E}f^!\omega^{\mathrm{E,sub}}_{N_\infty} \simeq \mathbf{E}f^!e^{\mathrm{E,sub}}_{N_\infty}(\rho_{N_\infty *}\omega_N) \simeq e^{\mathrm{sub}}_{M_\infty}(\rho_{M_\infty *}f^!\omega_N) \simeq e^{\mathrm{sub}}_{M_\infty}(\rho_{M_\infty *}\omega_M) \simeq \omega^{\mathrm{E,sub}}_{M_\infty}.$$

The proofs of assertions in (1) are similar, so we shall only prove the first assertion. Let $K \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and $L \in \mathbf{E}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{N_\infty})$. By Proposition 3.13(2), we have

$$\mathbf{E}f^!\mathbf{D}^{\mathrm{E,sub}}_{N_\infty}L \simeq \mathbf{E}f^!\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(L,\omega^{\mathrm{E,sub}}_{N_\infty}) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathbf{E}f^{-1}L,\mathbf{E}f^!\omega^{\mathrm{E,sub}}_{N_\infty}) \simeq \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\mathbf{E}f^{-1}L.$$

(2) Let $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. By Theorem 3.20 we have $I^{\mathrm{E}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. Since

$$\mathbf{D}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$$

(see [2, Prop. 3.3.3(ii)]), we have $\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$. Hence, by Theorem 3.20, we have $J^{\mathrm{E}}_{M_\infty}\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. By assertion (1), we have $J^{\mathrm{E}}_{M_\infty}\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \simeq \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K$, and hence $\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$.

Moreover, since $I^{\mathrm{E}}_{M_\infty}J^{\mathrm{E}}_{M_\infty}\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \simeq \mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K$ we have

$$\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K \simeq J^{\mathrm{E}}_{M_\infty}\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}J^{\mathrm{E}}_{M_\infty}\mathbf{D}^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \simeq K,$$

where the second isomorphism follows from [2, Prop. 3.3.3(ii)] and Theorem 3.15(1). ∎

Several operations preserve the $\mathbb{R}$-constructibility as below. Let us recall that a morphism $f\colon (M,\check{M}) \to (N,\check{N})$ of real analytic bordered spaces is called semi-proper if the second projection $\check{M}\times\check{N} \to \check{N}$ is proper on the closure $\overline{\Gamma}_f$ of the graph Γ_f of f in $\check{M}\times\check{N}$.

Proposition 3.26. *Let $f\colon M_\infty \to N_\infty$ be a morphism of real analytic bordered spaces associated with a morphism $\check{f}\colon \check{M} \to \check{N}$ of real analytic manifolds. The functors below are well defined:*

(1) $e^{\mathrm{sub}}_{M_\infty} \circ \rho_{M_\infty *}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$,

(2) $\mathbf{E}f_*\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, *if f is semi-proper,*

(3) $\mathbf{E}f^{-1}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{N_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$,

(4) $\mathbf{E}f_{!!}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{N_\infty})$, *if f is semi-proper,*

(5) $\mathbf{E}f^{!}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{N_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$.

Proof. (1) Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{M_\infty})$ and U be an open subset of M which is subanalytic and relatively compact in $\check{M}$. We set $\mathcal{F}^U := \Bbbk_{\{t=0\}} \otimes \pi^{-1}\mathcal{F}|_U$. By Propositions 3.1(6) and 3.22, we have

$$\mathbf{E}i^{-1}_{U_\infty}(e^{\mathrm{sub}}_{M_\infty}\rho_{M_\infty *}\mathcal{F}) \simeq e^{\mathrm{sub}}_{U_\infty}\rho_{U_\infty *}(\mathcal{F}|_U) \simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \otimes \pi^{-1}\rho_{U_\infty *}(\mathcal{F}|_U).$$

Since $\Bbbk^{\mathrm{E,sub}}_{U_\infty} \simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \rho_{U_\infty\times\mathbb{R}_\infty *}\Bbbk_{\{t=0\}}$, there exist isomorphisms

$$\begin{aligned}\Bbbk^{\mathrm{E,sub}}_{U_\infty} \otimes \pi^{-1}\rho_{U_\infty *}(\mathcal{F}|_U) &\simeq (\Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \rho_{U_\infty\times\mathbb{R}_\infty *}\Bbbk_{\{t=0\}}) \otimes \rho_{U_\infty\times\mathbb{R}_\infty *}\pi^{-1}\mathcal{F}|_U\\ &\simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \rho_{U_\infty\times\mathbb{R}_\infty *}(\Bbbk_{\{t=0\}} \otimes \pi^{-1}\mathcal{F}|_U)\\ &\simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \rho_{U_\infty\times\mathbb{R}_\infty}(\mathcal{F}^U),\end{aligned}$$

where in the first isomorphism we used Proposition 3.13(4)(iii). Since $\mathcal{F}$ is $\mathcal{R}$-constructible, we have $\mathcal{F}^U \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty})$. This implies that $e^{\mathrm{sub}}_{M_\infty}\rho_{M_\infty *}\mathcal{F}$ is $\mathbb{R}$-constructible.

(2) Let $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. By Theorem 3.15 and Proposition 3.16(3)(i), we have isomorphisms

$$\mathbf{E}f_*K \simeq \mathbf{E}f_*\lambda^{\mathrm{E}}_{M_\infty}I^{\mathrm{E}}_{M_\infty}K \simeq \lambda^{\mathrm{E}}_{N_\infty}\mathbf{E}f_*I^{\mathrm{E}}_{M_\infty}K.$$

Since K is $\mathbb{R}$-constructible, we have $I^{\mathrm{E}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{M_\infty})$ by Theorem 3.20, and hence, by [2, Prop. 3.3.3(iv)], we have $\mathbf{E}f_*I^{\mathrm{E}}_{M_\infty}K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\Bbbk_{N_\infty})$. This implies that $\mathbf{E}f_*K \simeq \lambda^{\mathrm{E}}_{N_\infty}\mathbf{E}f_*I^{\mathrm{E}}_{M_\infty}K$ is $\mathbb{R}$-constructible by Theorem 3.20.

(3) Since this assertion can be proved in the similar way as (2), we shall skip the proof.

(4) Let $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$. Then we have

$$\mathbf{E}f_{!!}K \simeq \mathbf{E}f_{!!}\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K \simeq \mathbf{D}^{\mathrm{E,sub}}_{N_\infty}\mathbf{E}f_*\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K$$

by Proposition 3.25. This implies that $\mathbf{E} f_{!!} K$ is $\mathbb{R}$-constructible by assertion (2) and Proposition 3.25(2).

(5) Since this assertion can be proved similarly to (4), we shall skip the proof. ■

Moreover, convolution functors preserve the $\mathbb{R}$-constructibility as below.

Proposition 3.27. *We have the following:*

(1) *The functors*

$$(\cdot) \overset{+}{\otimes} (\cdot)\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \times \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}),$$
$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\cdot,\cdot)\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$$

are well defined.

(2) *For any $K, L \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$, one has*

(i) $\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}(K \overset{+}{\otimes} L) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K, \mathbf{D}^{\mathrm{E,sub}}_{M_\infty} L)$,

(ii) $\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K, L) \simeq K \overset{+}{\otimes} \mathbf{D}^{\mathrm{E,sub}}_{M_\infty} L$,

(iii) $\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K, L) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathbf{D}^{\mathrm{E,sub}}_{M_\infty} L, \mathbf{D}^{\mathrm{E,sub}}_{M_\infty} K)$,

(iv) $\mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(K, L) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(\mathbf{D}^{\mathrm{E,sub}}_{M_\infty} L, \mathbf{D}^{\mathrm{E,sub}}_{M_\infty} K)$,

(v) $\mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(K, L) \simeq \mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\mathbf{D}^{\mathrm{E,sub}}_{M_\infty} L, \mathbf{D}^{\mathrm{E,sub}}_{M_\infty} K)$.

Proof. (1) Let $K, L \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk^{\mathrm{sub}}_{M_\infty})$ and U be an open subset of M which is subanalytic and relatively compact in $\check{M}$. Then there exist $\mathcal{F}, \mathcal{G} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty \times \mathbb{R}_\infty})$ such that $\mathbf{E}i^{-1}_{U_\infty} K \simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{U_\infty} \rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{F}$, $\mathbf{E}i^{-1}_{U_\infty} L \simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{U_\infty} \rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{G}$. Hence ,we have

$$\begin{aligned}\mathbf{E}i^{-1}_{U_\infty}(K \overset{+}{\otimes} L) &\simeq \mathbf{E}i^{-1}_{U_\infty} K \overset{+}{\otimes} \mathbf{E}i^{-1}_{U_\infty} L \\ &\simeq (\Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{U_\infty} \rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{F}) \overset{+}{\otimes} (\Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{U_\infty} \rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{G}) \\ &\simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \mathbf{Q}^{\mathrm{sub}}_{U_\infty} (\rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{F} \overset{+}{\otimes} \rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{G}),\end{aligned}$$

where in the first isomorphism we used Proposition 3.13(2) and in the last isomorphism we used $\Bbbk^{\mathrm{E,sub}}_{U_\infty} \overset{+}{\otimes} \Bbbk^{\mathrm{E,sub}}_{U_\infty} \simeq \Bbbk^{\mathrm{E,sub}}_{U_\infty}$. Since $\mathcal{F}, \mathcal{G} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty \times \mathbb{R}_\infty})$, there exists an isomorphism $\rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{F} \overset{+}{\otimes} \rho_{U_\infty \times \mathbb{R}_\infty *}\mathcal{G} \simeq \rho_{U_\infty \times \mathbb{R}_\infty *}(\mu_{!!}(p_1^{-1}\mathcal{F} \otimes p_2^{-1}\mathcal{G}))$ and $\mu_{!!}(p_1^{-1}\mathcal{F} \otimes p_2^{-1}\mathcal{G}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\Bbbk_{U_\infty \times \mathbb{R}_\infty})$. Therefore, $K \overset{+}{\otimes} L$ is $\mathbb{R}$-constructible.

Moreover, by using assertions (2)(i), (iii), there exist isomorphisms

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K,L) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}L, \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}K) \simeq \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}(\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}L \overset{+}{\otimes} K),$$

and hence by Proposition 3.25(2) and the first assertion of (1), $\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K,L)$ is $\mathbb{R}$-constructible.

(2) Since the proofs of the assertions in (2) are similar, we shall only prove (i). By Proposition 3.13(1)(i), we have

$$\mathbf{D}^{\mathrm{E,sub}}_{M_\infty}(K \overset{+}{\otimes} L) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K \overset{+}{\otimes} L, \omega^{\mathrm{E,sub}}_{M_\infty}) \simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(K, \mathbf{D}^{\mathrm{E,sub}}_{M_\infty}L). \quad \blacksquare$$

3.5 – *Irregular Riemann–Hilbert correspondence and enhanced subanalytic sheaves*

In this subsection we shall explain the relation between [1, Thm. 9.5.3] and [8, Thm. 6.3]. Theorems 3.38 and 3.39 are two of the main results of this subsection and this paper.

3.5.1. Main results of [1] and [8]. The aim of this subsection is to introduce the main results of [1, 8]. The results are summarized by the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}) \\
 & \nearrow^{\mathrm{Sol}^{\mathrm{T,sub}}_X(\cdot)[1]} & & & & & \uparrow \mathbf{R}^{\mathrm{E,sub}}_X \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} & \xrightarrow[\mathrm{Sol}^{\mathrm{E}}_X]{} & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) & \xrightarrow[I^{\mathrm{E}}_X]{\sim} & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X) & \subset & \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X).
\end{array}$$

Let X be a complex manifold and denote by $X_{\mathbb{R}}$ the underlying real analytic manifold of X. We denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the sheaves of holomorphic functions and holomorphic differential operators on X, respectively. Let $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ be the bounded derived category of left $\mathcal{D}_X$-modules. Moreover, we denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ the full triangulated subcategories of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with coherent, holonomic and regular holonomic cohomologies, respectively. For a morphism $f\colon X \to Y$ of complex manifolds, denote by $\overset{D}{\otimes}$, $\mathbf{D}f_*$, $\mathbf{D}f^*$, $\mathbb{D}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the standard operations for $\mathcal{D}$-modules.

Let M be a real analytic manifold of dimension n. We denote by $\mathcal{C}^\infty_M$ the sheaf of complex functions of class $\mathcal{C}^\infty$ on M and by $\mathcal{D}b_M$ the sheaf of Schwartz distributions on M.

Definition 3.28 ([10, Def. 7.2.3]). Let U be an open subset of M.

(1) One can say that $f \in \mathcal{C}_M^\infty(U)$ has polynomial growth at $p \in M$ if for a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \operatorname{dist}(x, K \backslash U)^N |f(x)| < +\infty.$$

A function $f \in \mathcal{C}_M^\infty(U)$ is said to be tempered at $p \in M$ if all its derivatives have polynomial growth at p, and is said to be tempered if tempered at any point of M. Let us denote by $\mathcal{C}_M^{\infty,\mathrm{t}}(U)$ the subset of $\mathcal{C}_M^\infty(U)$ consisting of functions which are tempered.

(2) We set $\mathcal{D}b_M^{\mathrm{t}}(U) := \operatorname{Im}(\Gamma(M; \mathcal{D}b_M) \to \Gamma(U; \mathcal{D}b_M))$.

Note that subanalytic presheaves $U \mapsto \mathcal{C}_M^{\infty,\mathrm{t}}(U)$ and $U \mapsto \mathcal{D}b_M^{\mathrm{t}}(U)$ are subanalytic sheaves; see [10, §7.2] and also [20, §3.3].

We shall write $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}})$, $\mathbf{E}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}})$, $\mathbf{E}_{\mathbb{R}\text{-}c}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}})$ instead of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_\mathbb{R}}^{\mathrm{sub}})$, $\mathbf{E}^{\mathrm{b}}(\mathbb{C}_{X_\mathbb{R}}^{\mathrm{sub}})$, $\mathbf{E}_{\mathbb{R}\text{-}c}^{\mathrm{b}}(\mathbb{C}_{X_\mathbb{R}}^{\mathrm{sub}})$, respectively.

Definition 3.29 ([10, §7.3], [1, §5.2] and also [20, §3.3]). Let us denote by X^c the complex conjugate manifold of X. An object $\mathcal{O}_X^{\mathrm{t}} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}})$ is defined by

$$\mathcal{O}_X^{\mathrm{t}} := \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\rho_X!\mathcal{D}_{X^c}}(\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{C}_{X_\mathbb{R}}^{\infty,\mathrm{t}}) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\rho_X!\mathcal{D}_{X^c}}(\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_\mathbb{R}}^{\mathrm{t}})$$

and is called the subanalytic sheaf of tempered holomorphic functions on X.

Moreover, the tempered solution functor is defined by

$$\mathrm{Sol}_X^{\mathrm{t}} \colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), \quad \mathcal{M} \mapsto \mathbf{R}\mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, I_X\mathcal{O}_X^{\mathrm{t}}).$$

Note that an ind-sheaf $I_X\mathcal{O}_X^{\mathrm{t}}$ is nothing but the ind-sheaf of tempered holomorphic functions on X which is denoted by $\mathcal{O}_X^{\mathrm{t}}$ in [10, §7.3]. Note also that there exist isomorphisms $\rho_X^{-1}\mathcal{O}_X^{\mathrm{t}} \simeq \alpha_X I_X\mathcal{O}_X^{\mathrm{t}} \simeq \mathcal{O}_X$ and hence we have $\alpha_X\mathrm{Sol}_X^{\mathrm{t}}(\mathcal{M}) \simeq \mathrm{Sol}_X(\mathcal{M})$ for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$.

Definition 3.30 ([1, Def. 8.1.1] and [13, Def. 7.2.1]). Let $\tilde{k}_M \colon M \times \mathbb{R}_\infty \to M \times \mathbb{P}^1\mathbb{R}$ be the natural morphism of real analytic bordered spaces, where $\mathbb{P}^1\mathbb{R}$ is the real projective line. An object $\mathcal{D}b_{M\times\mathbb{R}_\infty}^{\mathrm{t}} \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$ is defined by

$$\mathcal{D}b_{M\times\mathbb{R}_\infty}^{\mathrm{t}} := \tilde{k}_M^! I_{M\times\mathbb{P}^1\mathbb{R}}\mathcal{D}b_{M\times\mathbb{P}^1\mathbb{R}}^{\mathrm{t}} \simeq I_{M\times\mathbb{R}_\infty}\tilde{k}_M^!\mathcal{D}b_{M\times\mathbb{P}^1\mathbb{R}}^{\mathrm{t}},$$

and $\mathcal{D}b_M^{\mathsf{T}} \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$ is defined by the complex, concentrated in -1 and 0,

$$\mathcal{D}b_{M\times\mathbb{R}_\infty}^{\mathrm{t}} \xrightarrow{\partial_t - 1} \mathcal{D}b_{M\times\mathbb{R}_\infty}^{\mathrm{t}}.$$

Moreover, we set $\mathcal{D}b_M^{\mathrm{E}} := \mathbf{Q}_M(\mathcal{D}b_M^{\mathsf{T}}) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$ and call it the enhanced ind-sheaf of tempered distributions.

Note that we have $\mathcal{H}^i(\mathcal{D}\mathrm{b}_M^{\mathsf{T}}) = 0$ for any $i \neq -1$ and hence there exists an isomorphism $\mathcal{D}\mathrm{b}_M^{\mathsf{T}} \simeq \mathrm{Ker}(\partial_t - 1)[1]$ in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$.

Definition 3.31 ([1, Def. 8.2.1] and [13, Def. 7.2.3]). Let $\tilde{i}\colon X\times\mathbb{R}_\infty \to X\times\mathbb{P}^1\mathbb{C}$ be the natural morphism of bordered spaces and $\tau\in\mathbb{C}\subset\mathbb{P}^1\mathbb{C}$ the affine coordinate such that $t = \tau|_{\mathbb{R}}$, where t is a coordinate of $\mathbb{R}$ and $\mathbb{P}^1\mathbb{C}$ is the complex projective line. An object $\mathcal{O}_X^{\mathrm{E}} \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ is defined by

$$\begin{aligned}\mathcal{O}_X^{\mathrm{E}} &:= \mathbf{Q}_X\mathbf{R}\mathcal{I}hom_{\pi_{X^c}^{-1}\beta_{X^c}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\beta_{X^c}\mathcal{O}_{X^c}, \mathcal{D}\mathrm{b}_{X_{\mathbb{R}}}^{\mathsf{T}})\\ &\simeq \mathbf{Q}_X\tilde{i}^!\mathbf{R}\mathcal{I}hom_{p^{-1}\beta_{\mathbb{P}^1\mathbb{C}}\mathcal{D}_{\mathbb{P}^1\mathbb{C}}}(p^{-1}\beta_{\mathbb{P}}\mathcal{E}_{\mathbb{C}|\mathbb{P}^1\mathbb{C}}^{\tau}, I_{X\times\mathbb{P}^1\mathbb{C}}\mathcal{O}_{X\times\mathbb{P}^1\mathbb{C}}^{\mathrm{t}})[2],\end{aligned}$$

where $\mathcal{E}_{\mathbb{C}|\mathbb{P}}^{\tau}$ is the meromorphic connection associated to $d + d\tau$, $p\colon X\times\mathbb{P}^1\mathbb{C}\to\mathbb{P}^1\mathbb{C}$ is the projection and $\pi_{X^c}\colon X^c\times\mathbb{R}_\infty \to X^c$ is the morphisms of bordered spaces associated with the projection $X^c\times\mathbb{R}\to X^c$.

It is called the enhanced ind-sheaf of tempered holomorphic functions on X.

Moreover, the enhanced solution functor is defined by

$$\mathrm{Sol}_X^{\mathrm{E}}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), \quad \mathcal{M} \mapsto \mathbf{R}\mathcal{I}hom_{\pi^{-1}\beta_X\mathcal{D}_X}(\pi^{-1}\beta_X\mathcal{M}, \mathcal{O}_X^{\mathrm{E}}),$$

where $\pi\colon X\times\mathbb{R}_\infty\to X$ is the morphism of bordered spaces associated with the first projection $X\times\mathbb{R}\to X$.

Note that $\mathcal{O}_X^{\mathrm{E}}$ is isomorphic to the enhanced ind-sheaf induced by the Dolbeault complex with coefficients in $\mathcal{D}\mathrm{b}_{X_{\mathbb{R}}}^{\mathsf{T}}[-1]$,

$$\mathcal{D}\mathrm{b}_{X_{\mathbb{R}}}^{\mathsf{T}}[-1] \xrightarrow{\bar\partial} \Omega_{X^c}^1 \otimes_{\mathcal{O}_{X^c}} \mathcal{D}\mathrm{b}_{X_{\mathbb{R}}}^{\mathsf{T}}[-1] \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega_{X^c}^{d_X} \otimes_{\mathcal{O}_{X^c}} \mathcal{D}\mathrm{b}_{X_{\mathbb{R}}}^{\mathsf{T}}[-1],$$

where $\Omega_{X^c}^p$ is the sheaf of p-differential forms with coefficients in $\mathcal{O}_{X^c}$ and d_X is the complex dimension of X.

Note that $\mathrm{Ish}_X\mathcal{O}_X^{\mathrm{E}} \simeq I_X\mathcal{O}_X^{\mathrm{t}}$ and hence there exists an isomorphism $\mathrm{Ish}_X\mathrm{Sol}_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathrm{Sol}_X^{\mathrm{t}}(\mathcal{M})$ for any $\mathcal{M}\in\mathbf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X)$.

Let us recall the main results of [1].

Theorem 3.32 ([1, Thm. 9.5.3[7] and 9.6.1]). *The enhanced solution functor induces an embedding functor*

$$\mathrm{Sol}_X^{\mathrm{E}}\colon \mathbf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{E}_{\mathbb{R}\text{-}c}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X).$$

(7) Although [1, Thm. 9.5.3] was stated using the enhanced de Rham functor, we can obtain a similar statement using the enhanced solution functor.

Moreover, for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *there exists an isomorphism in* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$*,*

$$\mathcal{M} \xrightarrow{\sim} \mathrm{RH}^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}),$$

where $\mathrm{RH}^{\mathrm{E}}_X(K) := \mathbf{R}\mathcal{H}om^{\mathrm{E}}(K, \mathcal{O}^{\mathrm{E}}_X)$.

Let us recall the main results of [8].

Definition 3.33 ([8, §5.2]). Let $\tilde{k}_M \colon M \times \mathbb{R}_\infty \to M \times \mathbb{P}^1\mathbb{R}$ be the natural morphism of real analytic bordered spaces. An object $\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_M \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$ is defined by the complex, concentrated in -1 and 0,

$$\tilde{k}^! \mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{P}^1\mathbb{R}} \xrightarrow{\partial_t - 1} \tilde{k}^! \mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{P}^1\mathbb{R}}.$$

Note that we have $\mathcal{H}^i(\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_M) = 0$ for any $i \neq -1$ and hence there exists an isomorphism $\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_M \simeq \mathrm{Ker}(\partial_t - 1)[1]$ in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$. Remark that the notion $\mathcal{D}b^{\mathsf{T}}$ in [8, §5.2] is equal to $\mathcal{D}b^{\mathsf{T},\mathrm{sub}}[-1]$ in our notion.

Definition 3.34 ([8, §§5.3, 5.4]). An object $\mathcal{O}^{\mathsf{T},\mathrm{sub}}_X \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$ is defined by the Dolbeault complex with coefficients in $\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_{X_\mathbb{R}}[-1]$,

$$\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_{X_\mathbb{R}}[-1] \xrightarrow{\bar\partial} \Omega^1_{X^c} \otimes_{\mathcal{O}_{X^c}} \mathcal{D}b^{\mathsf{T},\mathrm{sub}}_{X_\mathbb{R}}[-1] \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{d_X}_{X^c} \otimes_{\mathcal{O}_{X^c}} \mathcal{D}b^{\mathsf{T},\mathrm{sub}}_{X_\mathbb{R}}[-1].$$

Moreover, we set

$$\mathrm{Sol}^{\mathsf{T},\mathrm{sub}}_X \colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}), \quad \mathcal{M} \mapsto \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}(\pi^{-1}\rho_{X!}\mathcal{M}, \mathcal{O}^{\mathsf{T},\mathrm{sub}}_X).$$

Note also that there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\Bbbk^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$:

$$\mathcal{O}^{\mathsf{T},\mathrm{sub}}_X \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}_{X^c}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi^{-1}_{X^c}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b^{\mathsf{T},\mathrm{sub}}_{X_\mathbb{R}}[-1]).$$

Theorem 3.35 ([8, Thms. 6.2 and 6.3[8]]). *There exists an embedding functor*

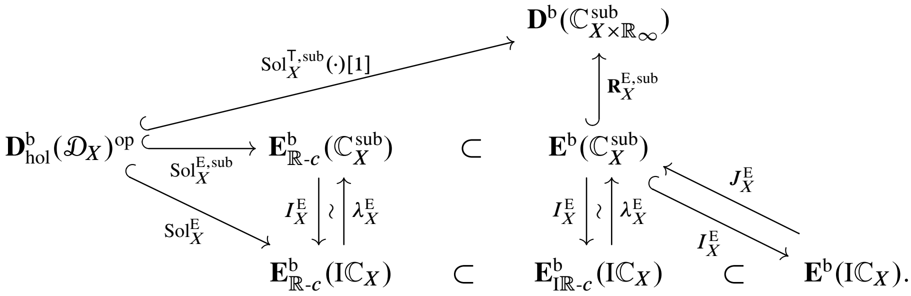
$$\mathrm{Sol}^{\mathsf{T},\mathrm{sub}}_X \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}).$$

Moreover, for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *there exists an isomorphism in* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$*:*

$$\mathcal{M} \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^{\mathrm{E},\mathrm{sub}}(\mathrm{Sol}^{\mathsf{T},\mathrm{sub}}_X(\mathcal{M}), \mathcal{O}^{\mathsf{T},\mathrm{sub}}_X).$$

(8) Although [8, Thm. 6.3] was stated using the enhanced de Rham functor, we can obtain a similar statement using the enhanced solution functor.

3.5.2. Relation between [1, Thm. 9.5.3] and [8, Thm. 6.3]. Let us explain the relation between [1, Thm. 9.5.3] and [8, Thm. 6.3]. The results are summarized by the following commutative diagram:



Definition 3.36. Let us define $\mathcal{O}^{\mathsf{E},\mathrm{sub}}_X := \mathbf{Q}^{\mathrm{sub}}_X(\mathcal{O}^{\mathsf{T},\mathrm{sub}}_X[1]) \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X)$ and set

$$\mathrm{Sol}^{\mathsf{E},\mathrm{sub}}_X\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X), \quad \mathcal{M} \mapsto \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}(\pi^{-1}\rho_{X!}\mathcal{M}, \mathcal{O}^{\mathsf{E},\mathrm{sub}}_X).$$

By the definition, it is clear that

$$\mathcal{O}^{\mathsf{E},\mathrm{sub}}_X \simeq \mathbf{Q}^{\mathrm{sub}}_X \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}_{X^c}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi^{-1}_{X^c}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b^{\mathsf{T},\mathrm{sub}}_{X_\mathbb{R}}),$$

and for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ one has

$$\mathrm{Sol}^{\mathsf{E},\mathrm{sub}}_X(\mathcal{M}) \simeq \mathbf{Q}^{\mathrm{sub}}_X(\mathrm{Sol}^{\mathsf{T},\mathrm{sub}}_X(\mathcal{M}))[1].$$

Lemma 3.37. *We have the following:*

(1) *There exists an isomorphism* $\mathcal{D}b^{\mathsf{T}}_M \simeq I_{M\times\mathbb{R}_\infty}\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_M$ *in* $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$.

(2) *There exists an isomorphism* $\mathcal{O}^{\mathsf{E},\mathrm{sub}}_X \simeq J^{\mathsf{E}}_X\mathcal{O}^{\mathsf{E}}_X$ *in* $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X)$.

(3) *For any* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, *there exists an isomorphism in* $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X)$:

$$\mathrm{Sol}^{\mathsf{E},\mathrm{sub}}_X(\mathcal{M}) \simeq J^{\mathsf{E}}_X\mathrm{Sol}^{\mathsf{E}}_X(\mathcal{M}).$$

(4) *For any* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, *there exists an isomorphism in* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$:

$$\mathrm{Sol}^{\mathsf{T},\mathrm{sub}}_X(\mathcal{M})[1] \simeq \mathbf{R}^{\mathsf{E},\mathrm{sub}}_X\mathrm{Sol}^{\mathsf{E},\mathrm{sub}}_X(\mathcal{M}).$$

Proof. (1) Since the functor $I_{M\times\mathbb{R}_\infty}$ is exact, we have isomorphisms

$$\begin{aligned}
\mathcal{D}b^{\mathsf{T}}_M &\simeq \mathrm{Ker}(\partial_t - 1\colon \mathcal{D}b^{\mathsf{t}}_{M\times\mathbb{R}_\infty} \to \mathcal{D}b^{\mathsf{t}}_{M\times\mathbb{R}_\infty})\\
&\simeq \mathrm{Ker}(\partial_t - 1\colon I_{M\times\mathbb{R}_\infty}\tilde{k}^!_M\mathcal{D}b^{\mathsf{t}}_{M\times\mathbb{P}^1\mathbb{R}} \to I_{M\times\mathbb{R}_\infty}\tilde{k}^!_M\mathcal{D}b^{\mathsf{t}}_{M\times\mathbb{P}^1\mathbb{R}})\\
&\simeq I_{M\times\mathbb{R}_\infty}\,\mathrm{Ker}(\partial_t - 1\colon \tilde{k}^!_M\mathcal{D}b^{\mathsf{t}}_{M\times\mathbb{P}^1\mathbb{R}} \to \tilde{k}^!_M\mathcal{D}b^{\mathsf{t}}_{M\times\mathbb{P}^1\mathbb{R}})\\
&\simeq I_{M\times\mathbb{R}_\infty}\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_M;
\end{aligned}$$

see Definition 3.30 for the details of $\mathcal{D}b^{\mathsf{T}}_M$ and Definition 3.33 for the details of $\mathcal{D}b^{\mathsf{T},\mathrm{sub}}_M$.

(2) We have isomorphisms

$$\begin{aligned}
J_X^{\mathsf{E}}\mathcal{O}_X^{\mathsf{E}} &\simeq \mathbf{Q}_X \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom_{\pi_{X^c}^{-1}\beta_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\beta_{X^c}\mathcal{O}_{X^c}, \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}})\\
&\simeq \mathbf{Q}_X \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom_{\pi_{X^c}^{-1}\beta_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}I_{X^c}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}})\\
&\simeq \mathbf{Q}_X \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom_{\pi_{X^c}^{-1}\beta_{X^c!}\mathcal{D}_{X^c}}(I_{X^c\times\mathbb{R}_\infty}\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}})\\
&\simeq \mathbf{Q}_X \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c}, \mathbf{R}J_{X_\mathbb{R}\times\mathbb{R}_\infty}\mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}})\\
&\simeq \mathbf{Q}_X \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}})\\
&\simeq \mathcal{O}_X^{\mathsf{E},\mathrm{sub}},
\end{aligned}$$

where in the second isomorphism we used $\beta_{X^c} \simeq I_{X^c}\circ\rho_{X^c!}$, in the third isomorphism we used $\pi_{X^c}^{-1}\circ I_{X^c} \simeq I_{X^c\times\mathbb{R}_\infty}\circ\pi_{X^c}^{-1}$, in the fourth isomorphism we used Proposition 3.4(1) and in the fifth isomorphism we used assertion (1) and Proposition 3.3(1).

(3) Let $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$. By the fact that $\beta_X \simeq I_X\circ\rho_{X!}$ and assertion (2), there exist isomorphisms

$$J_X^{\mathsf{E}}\mathrm{Sol}_X^{\mathsf{E}}(\mathcal{M}) \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}(\pi^{-1}\rho_{X!}\mathcal{M}, J_X^{\mathsf{E}}\mathcal{O}_X^{\mathsf{E}}) \simeq \mathrm{Sol}_X^{\mathsf{E},\mathrm{sub}}(\mathcal{M}).$$

(4) First let us prove that

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}}) \simeq \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}}.$$

By assertion (1) and Proposition 3.16(1)(ii),

$$\begin{aligned}
&\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}})\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}})\\
&\quad\simeq \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(I_{X\times\mathbb{R}_\infty}\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}})\\
&\quad\simeq \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(\iota_{X\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}}).
\end{aligned}$$

Moreover, by using the fact that $\mathbf{R}\mathcal{I}hom^{+}(\iota_{X\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}}) \simeq \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}}$ (see e.g. [1, Prop. 8.1.3]), we have

$$\mathbf{R}J_{X\times\mathbb{R}_\infty}\mathbf{R}\mathcal{I}hom^{+}(\iota_{X\times\mathbb{R}_\infty}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}}) \simeq \mathbf{R}J_{X\times\mathbb{R}_\infty}\mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T}} \simeq \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}}.$$

Hence, we proved $\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}}) \simeq \mathcal{D}b_{X_\mathbb{R}}^{\mathsf{T},\mathrm{sub}}$.

Next we shall prove that

$$\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus\Bbbk_{\{t\leq 0\}}, \mathcal{O}_X^{\mathsf{T},\mathrm{sub}})) \simeq \mathcal{O}_X^{\mathsf{T},\mathrm{sub}}.$$

By the fact that $\mathcal{O}_X^{\mathsf{T},\mathrm{sub}} \simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_{\mathbb{R}}}^{\mathsf{T},\mathrm{sub}}[-1])$, we have

$$\begin{aligned}
&\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}, \mathcal{O}_X^{\mathsf{T},\mathrm{sub}}))\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}),\\
&\qquad\qquad \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_{\mathbb{R}}}^{\mathsf{T},\mathrm{sub}}))[-1]\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{D}_{X^c}}\big(\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c},\\
&\qquad\qquad \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}}\oplus \Bbbk_{\{t\leq 0\}}), \mathcal{D}b_{X_{\mathbb{R}}}^{\mathsf{T},\mathrm{sub}})\big)[-1]\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{D}_{X^c}}(\pi_{X^c}^{-1}\rho_{X^c!}\mathcal{O}_{X^c}, \mathcal{D}b_{X_{\mathbb{R}}}^{\mathsf{T},\mathrm{sub}})[-1]\\
&\quad\simeq \mathcal{O}_X^{\mathsf{T},\mathrm{sub}}.
\end{aligned}$$

By the definition, it is clear that

$$\begin{aligned}
&\mathbf{R}_X^{\mathrm{E},\mathrm{sub}}\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}}(\mathcal{M})\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}),\\
&\qquad\qquad \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}(\pi^{-1}\rho_{X!}\mathcal{M}, \mathcal{O}_X^{\mathsf{T},\mathrm{sub}}))[1]\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}\big(\pi^{-1}\rho_{X!}\mathcal{M},\\
&\qquad\qquad \mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}(\rho_{X\times\mathbb{R}_\infty *}(\Bbbk_{\{t\geq 0\}} \oplus \Bbbk_{\{t\leq 0\}}), \mathcal{O}_X^{\mathsf{T},\mathrm{sub}})\big)[1]\\
&\quad\simeq \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}(\pi^{-1}\rho_{X!}\mathcal{M}, \mathcal{O}_X^{\mathsf{T},\mathrm{sub}})[1]\\
&\quad\simeq \mathrm{Sol}_X^{\mathsf{T},\mathrm{sub}}(\mathcal{M})[1].
\end{aligned}$$

■

THEOREM 3.38. *The functor* $\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}}$ *induces an embedding functor*

$$\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}}).$$

Moreover, for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *there exists an isomorphism in* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$,

$$\mathcal{M} \xrightarrow{\sim} \mathrm{RH}_X^{\mathrm{E},\mathrm{sub}}\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}}(\mathcal{M}),$$

where $\mathrm{RH}_X^{\mathrm{E},\mathrm{sub}}(K) := \mathbf{R}\mathcal{H}om^{\mathrm{E},\mathrm{sub}}(K, \mathcal{O}_X^{\mathrm{E},\mathrm{sub}})$.

PROOF. First, let us prove that $\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}}(\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$ for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. By Theorem 3.32, we have $\mathrm{Sol}_X^{\mathrm{E}}(\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ and hence by Theorem 3.20 we have $J_X^{\mathrm{E}}\mathrm{Sol}_X^{\mathrm{E}}(\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$. This implies that $\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}}(\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$ by Lemma 3.37(3). Hence, a functor

$$\mathrm{Sol}_X^{\mathrm{E},\mathrm{sub}} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$$

is well defined.

For any $\mathcal{M}, \mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there exist isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)}(\mathcal{M}, \mathcal{N}) &\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}))\\ &\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)}(J^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), J^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}))\\ &\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M})),\end{aligned}$$

where in the first (resp. second, third) isomorphism we used Theorem 3.32 (resp. Theorem 3.20, Lemma 3.37(3)). This implies that the functor $\mathrm{Sol}^{\mathrm{E,sub}}_X : \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)$ is fully faithful.

Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. By using the adjointness, there exist isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)}(\mathcal{M}, \mathrm{RH}^{\mathrm{E,sub}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}))\\ &\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)}\big(\mathcal{M}, \mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E,sub}}_X)\big)\\ &\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)}\big(\mathcal{M}, \rho_X^{-1}\mathbf{R}\pi_*\mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E,sub}}_X)\big)\\ &\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)}\big(\pi^{-1}\rho_{X!}\mathcal{M}, \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E,sub}}_X)\big)\\ &\quad\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathbf{R}\mathcal{I}hom^{\mathrm{sub}}_{\pi^{-1}\rho_{X!}\mathcal{D}_X}(\pi^{-1}\rho_{X!}\mathcal{M}, \mathcal{O}^{\mathrm{E,sub}}_X))\\ &\quad\simeq \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X)}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}))\\ &\quad\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)}(\mathcal{M}, \mathcal{M}).\end{aligned}$$

Hence, there exists a canonical morphism

$$\mathcal{M} \to \mathrm{RH}^{\mathrm{E,sub}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M})$$

which is given by the identity map $\mathrm{id}_{\mathcal{M}}$ of $\mathcal{M}$. Let us prove that it is an isomorphism.

By Lemma 3.37(2), we have isomorphisms

$$\begin{aligned}\mathrm{RH}^{\mathrm{E,sub}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}) &\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E,sub}}_X)\\ &\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), J^{\mathrm{E}}_X\mathcal{O}^{\mathrm{E}}_X),\end{aligned}$$

and by Proposition 3.16(1)(iii) we have

$$\begin{aligned}\mathbf{R}\mathcal{H}om^{\mathrm{E,sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), J^{\mathrm{E}}_X\mathcal{O}^{\mathrm{E}}_X) &\simeq \rho^{-1}\mathbf{R}\mathcal{I}hom^{\mathrm{E,sub}}(\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), J^{\mathrm{E}}_X\mathcal{O}^{\mathrm{E}}_X)\\ &\simeq \rho^{-1}\mathbf{R}J_X\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(I^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E}}_X).\end{aligned}$$

By Proposition 3.4(3)(ii) and Lemma 3.37(3), there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$,

$$\begin{aligned}\rho^{-1}\mathbf{R}J_X\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(I^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E}}_X) &\simeq \alpha_X\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(I^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E}}_X)\\ &\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E}}(I^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E}}_X)\\ &\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E}}(I^{\mathrm{E}}_X J^{\mathrm{E}}_X\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E}}_X).\end{aligned}$$

Since $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, $\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})$ is $\mathbb{R}$-constructible by the first assertion and hence there exists an isomorphism $I^{\mathrm{E}}_X J^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})$ by Theorem 3.20. By Theorem 3.32 we have

$$\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}), \mathcal{O}^{\mathrm{E}}_X) \simeq \mathrm{RH}^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathcal{M}.$$

Therefore, there exists an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathrm{RH}^{\mathrm{E,sub}}_X \mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M})$. ■

Theorem 3.39. *We have the following:*

(1) *For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$:*

$$\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) \simeq I^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}).$$

Namely, there exists a commutative diagram:

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} & \xhookrightarrow{\mathrm{Sol}^{\mathrm{E,sub}}_X} & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X) \\ & \underset{\mathrm{Sol}^{\mathrm{E}}_X}{\searrow} & \downarrow{\scriptstyle I^{\mathrm{E}}_X}\ \wr \\ & & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X). \end{array}$$

(2) *For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$:*

$$\mathrm{Sol}^{\mathrm{T,sub}}_X(\mathcal{M})[1] \simeq \mathbf{R}^{\mathrm{E,sub}}_X J^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}).$$

Moreover, there exists a commutative diagram:

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} & \xhookrightarrow{\mathrm{Sol}^{\mathrm{T,sub}}_X(\cdot)[1]} & \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}) \\ & \underset{\mathrm{Sol}^{\mathrm{E}}_X}{\searrow} & \uparrow{\scriptstyle \mathbf{R}^{\mathrm{E,sub}}_X \circ J^{\mathrm{E}}_X} \\ & & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X). \end{array}$$

Proof. (1) Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Since $\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})$ is $\mathbb{R}$-constructible, there exists an isomorphism $I^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E,sub}}_X(\mathcal{M}) \simeq I^{\mathrm{E}}_X J^{\mathrm{E}}_X \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})$ by Theorem 3.20 and Lemma 3.37(3).

(2) This follows from Lemma 3.37(3), (4). ■

Remark 3.40. One can consider $\mathbb{C}$-constructibility for enhanced subanalytic sheaves similar to [4, Def. 3.19]. See [6] for the details.

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