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On finite groups in which every maximal subgroup of order divisible by p is nilpotent (or abelian)

Jiangtao Shi (*) – Yunfeng Tian (**)

Abstract – We obtain a complete classification of a finite group G in which every maximal subgroup of order divisible by p is nilpotent (or abelian) for any fixed prime divisor p of $|G|$ and our results generalize some known results.

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1. Introduction

In this paper, all groups are assumed to be finite. It is known that if every maximal subgroup of a group G is nilpotent then G is called a Schmidt group which is either a nilpotent group or a minimal non-nilpotent group, and Rédei [5] gave a complete description of the structure of Schmidt groups. Moreover, Itô [2, Proposition 2] characterized the structure of minimal non-p-nilpotent groups. A theorem of Thompson [6, Theorem 10.4.2] shows that a group having a nilpotent maximal subgroup of odd order is solvable. Moreover, Rose [7, Theorem 1] gave a characterization of the structure of non-solvable groups having a nilpotent maximal subgroup of even order. Recently, as a generalization of the above literature, Deng, Meng and Lu [1, Theorem 3.1] characterized the structure of a group of even order in which all maximal subgroups of even order are nilpotent.

(*) *Indirizzo dell'A.*: School of Mathematics and Information Sciences, Yantai University, Qingquan Road 30, Laishan District, 264005 Yantai, P. R. China; shijt2005@pku.org.cn

(**) *Indirizzo dell'A.*: School of Mathematics and Information Sciences, Yantai University, Qingquan Road 30, Laishan District, 264005 Yantai, P. R. China; m18863093906@163.com

In [8, Theorem 1.1], Shi, Li and Shen showed that any non-solvable group G must have a non-nilpotent maximal subgroup of order divisible by p for any fixed prime divisor p of $|G|$. In this paper, as a further complete generalization of minimal non-nilpotent groups and [1, Theorem 3.1], considering any fixed prime divisor p of the order of a group G, we obtain the following result whose proof is given in Section 2.

THEOREM 1.1. *Suppose that G is a group and p is any fixed prime divisor of $|G|$, then every maximal subgroup of G of order divisible by p is nilpotent if and only if one of the following statements holds:*

(1) *G is a nilpotent group;*

(2) *$G = P \rtimes Q$ is a minimal non-nilpotent group, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, P is normal in G, $p \neq q$;*

(3) *$G = Q \rtimes P$ is a minimal non-nilpotent group, where $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$, Q is normal in G, $q \neq p$;*

(4) *$G = Z_p \times K$, where K is a minimal non-nilpotent group and $(p, |K|) = 1$.*

REMARK 1.2. *It is easy to see that the group in Theorem 1.1 is solvable. Therefore,* [8, Theorem 1.1] *can be seen as a corollary of the above theorem.*

It is known that [3] and [4] characterized the structure of minimal non-abelian groups (that is, groups in which every proper subgroup is abelian). As a generalization of [3] and [4], based on Theorem 1.1, we can easily get the following result whose proof is given in Section 3.

THEOREM 1.3. *Suppose that G is a group and p is any fixed prime divisor of $|G|$, then every maximal subgroup of G of order divisible by p is abelian if and only if one of the following statements holds:*

(1) *G is an abelian group;*

(2) *$G = Z_p \times Q$, where Q is a minimal non-abelian group of prime-power order and $(p, |Q|) = 1$;*

(3) *$G = P \times L$, where $P \in \mathrm{Syl}_p(G)$ and G is a minimal non-abelian p-group, L is an abelian group;*

(4) *$G = P \rtimes Q$ is a minimal non-abelian group, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, P is normal in G, $p \neq q$;*

(5) *$G = Q \rtimes P$ is a minimal non-abelian group, where $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$, Q is normal in G, $q \neq p$;*

(6) *$G = Q \rtimes Z_p$ is a minimal non-nilpotent group with Q being the unique non-abelian maximal subgroup, where $Q \in \mathrm{Syl}_q(G)$ and Q is normal in G, $q \neq p$;*

(7) *$G = Z_p \times K$, where K is a minimal non-abelian group of non-prime-power order and $(p, |K|) = 1$.*

2. Proof of Theorem 1.1

The sufficiency part is evident, we only need to prove the necessity part. We first prove that G has a normal Sylow subgroup. By the hypothesis, every maximal subgroup of G is p-nilpotent.

Case 1. Suppose that G is not p-nilpotent, then G is a minimal non-p-nilpotent group. It implies that the Sylow p-subgroup of G is normal.

Case 2. Suppose that G is p-nilpotent, then G has a normal p-complement M. Let $P \in \mathrm{Syl}_p(G)$, one has $G = P \ltimes M$, where M is normal in G.

(i) Assume that M is nilpotent. Let $Q \in \mathrm{Syl}_q(M)$, where $q \neq p$. It is clear that Q is a normal Sylow subgroup of G.

(ii) Assume that M is non-nilpotent. Since every maximal subgroup of G of order divisible by p is nilpotent, M must be maximal in G. Moreover, M is the unique non-nilpotent maximal subgroup of G.

If there exists a Sylow subgroup of M which is also a Sylow subgroup of G and is normal in G, then the proof is complete.

In the following, we assume that every Sylow q-subgroup Q of M is not normal in G, that is, $N_G(Q) < G$. Since M is the unique non-nilpotent maximal subgroup of G, one has $N_G(Q) \nleq M$ by the Frattini argument. Then $N_G(Q)$ must be contained in some nilpotent maximal subgroup of G. Note that $p \nmid |M|$. For any nilpotent maximal subgroup L of G, we must have $p \mid |L|$. Moreover, let $P_L \in \mathrm{Syl}_p(L)$, then P_L is also a Sylow p-subgroup of G.

If G has at least two nilpotent maximal subgroups, assume that L_1 and L_2 are two distinct nilpotent maximal subgroups of G. Let $P_1 \in \mathrm{Syl}_p(L_1)$ and $P_2 \in \mathrm{Syl}_p(L_2)$. If P_1 is normal in G, then $P_1 = P_2$ is a normal Sylow subgroup of G. If $P_1 \neq P_2$, then $N_G(P_1) < G$ and $N_G(P_2) < G$. Moreover, there exists a $g \in G$ such that $P_2 = {P_1}^g$ by Sylow's theorem. Note that $L_1 \leq N_G(P_1) < G$ and $L_2 \leq N_G(P_2) < G$. As both L_1 and L_2 being maximal in G, one has $N_G(P_1) = L_1$ and $N_G(P_2) = L_2$. It follows that $N_G({P_1}^g) = L_2$. That is, $(N_G(P_1))^g = L_2$. Then $L_2 = {L_1}^g$. It implies that all nilpotent maximal subgroups of G are conjugate in G. Note that any nilpotent maximal subgroup of G contains a Sylow p-subgroup of G, and any Sylow q-subgroup of G where $q \neq p$ is contained in some nilpotent maximal subgroup of G. This implies that the order of a nilpotent maximal subgroup of G is equal to the order of G, a contradiction.

Thus, if L is the unique nilpotent maximal subgroup of G, then L is normal in G. It follows that P is a normal Sylow subgroup of G.

All the above arguments show that G has a normal Sylow subgroup.

Let $P_1, \ldots, P_s$ be all normal Sylow subgroups of G. Let $N = P_1 \times \cdots \times P_s$, then N is a normal Hall-subgroup of G. By Schur–Zassenhaus's theorem, there exists a proper subgroup K of G such that $G = N \rtimes K$, where K is also a Hall-subgroup of G.

For K, we divide our discussions into two cases.

Case 1. Suppose $p \mid |K|$. By the hypothesis, K is nilpotent. Let P be a Sylow p-subgroup of K that is also a Sylow p-subgroup of G, one has that P is normal in K but P is not normal in G. Consider the subgroup $N \rtimes P$. If $N \rtimes P < G$, then $N \rtimes P$ is nilpotent by the hypothesis. One has that P is normal in $N \rtimes P$. It follows that P is normal in $N \rtimes K = G$, a contradiction. Therefore, $N \rtimes P = G$.

If P has at least two maximal subgroups. Let P' and P'' be two distinct maximal subgroups of P, then $N \rtimes P'$ and $N \rtimes P''$ are two maximal subgroups of G of order divisible by p. It follows that both $N \rtimes P'$ and $N \rtimes P''$ are nilpotent by the hypothesis. It implies that $P = P'P''$ is normal in $N \rtimes P = G$, a contradiction. Thus, P has exactly one maximal subgroup and then P is cyclic.

For $N = P_1 \times \cdots \times P_s$. If $s > 1$, then $P_i \rtimes P$ is a proper subgroup of G of order divisible by p for each $1 \leq i \leq s$, by the hypothesis $P_i \rtimes P$ is nilpotent for each $1 \leq i \leq s$. It follows that P is normal in $G = N \rtimes P = (P_1 \times \cdots \times P_s) \rtimes P$, a contradiction. Thus, $s = 1$.

One has $G = P_1 \rtimes P$, where P is cyclic. It is easy to see that every maximal subgroup of G is nilpotent. Then G is a minimal non-nilpotent group.

Case 2. Suppose $p \nmid |K|$, then $p \mid |N|$. For any maximal subgroup K_0 of K, $N \rtimes K_0$ is a maximal subgroup of G of order divisible by p. One has that $N \rtimes K_0$ is nilpotent by the hypothesis. Then every maximal subgroup of K is nilpotent. It follows that K is either nilpotent or minimal non-nilpotent.

If K is a minimal non-nilpotent group, assume $K = Q_1 \rtimes Q_2$, where Q_1 and Q_2 are two distinct Sylow subgroups of K. It is clear that $N \rtimes Q_1$ is a proper subgroup of G of order divisible by p. Then $N \rtimes Q_1$ is nilpotent. It follows that Q_1 is normal in $G = N \rtimes K$, a contradiction. Therefore, K is nilpotent.

Let $K = Q_1 \times Q_2 \times \cdots \times Q_t$, where Q_i is a Sylow subgroup of K for each $1 \leq i \leq t$. If $t > 1$, it follows that $N \rtimes Q_1$ is a proper subgroup of G of order divisible by p and then $N \rtimes Q_1$ is nilpotent. It is easy to see that Q_1 is normal in $G = N \rtimes K$, a contradiction. Thus, $t = 1$.

If Q_1 has two distinct maximal subgroups Q_{01} and Q_{02}, one has that both $N \rtimes Q_{01}$ and $N \rtimes Q_{02}$ are nilpotent. It follows that $Q_1 = Q_{01}Q_{02}$ is normal in $G = N \rtimes Q_1$, a contradiction. Therefore, Q_1 has exactly one maximal subgroup and then Q_1 is cyclic.

For $N = P_1 \times \cdots \times P_s$, let $P_1 = P \in \mathrm{Syl}_p(N)$.

If $s = 1$, then $G = P \rtimes Q_1$, where Q_1 is cyclic. It is easy to see that G is a minimal non-nilpotent group.

If $s > 1$, we can easily get that $(P_2 \times \cdots \times P_s) \rtimes Q_1$ is the unique non-nilpotent maximal subgroup of G. Let $L = (P_2 \times \cdots \times P_s) \rtimes Q_1$, then $G = P \times L$, where $P \cong Z_p$. It is easy to see that every maximal subgroup of L is nilpotent by the hypothesis. Since L is non-nilpotent, one has that L is a minimal non-nilpotent group. ■

3. Proof of Theorem 1.3

The sufficiency part is evident, we also only need to prove the necessity part.

Case 1. Suppose that G is nilpotent. If G is abelian, then G obviously satisfies the hypothesis. Next, assume that G is non-abelian. Let $P \in \mathrm{Syl}_p(G)$, then there exists a nilpotent p'-Hall subgroup H of G such that $G = P \times H$.

(1) Assume $|P| > p$, let P_0 be any maximal subgroup of P. It follows that $P_0 \times H$ is a maximal subgroup of G of order divisible by p. By the hypothesis, $P_0 \times H$ is abelian, which implies that H is abelian and P is a minimal non-abelian p-group since G is non-abelian. Furthermore, one has $H = 1$ by the hypothesis. It follows that $G = P$ is a minimal non-abelian p-group.

(2) Assume $|P| = p$, let H_0 be any maximal subgroup of H. Then $P \times H_0$ is a maximal subgroup of G of order divisible by p. It follows that $P \times H_0$ is abelian by the hypothesis. One has that H_0 is abelian and then H is a minimal non-abelian group of prime-power order since G is non-abelian and H is nilpotent.

Case 2. Suppose that G is non-nilpotent.

Since every maximal subgroup of G of order divisible by p is abelian, it is obvious that every maximal subgroup of G of order divisible by p is nilpotent. By Theorem 1.1, one has that G is one of the following groups:

(1) $G = P \rtimes Q$ is a minimal non-nilpotent group, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, P is normal in G, $p \neq q$;

(2) $G = Q \rtimes P$ is a minimal non-nilpotent group, where $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$, Q is normal in G, $q \neq p$;

(3) $G = Z_p \times K$, where K is a minimal non-nilpotent group and $(p, |K|) = 1$.

When $G = P \rtimes Q$ is a minimal non-nilpotent group, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, P is normal in G, $p \neq q$. By the hypothesis, one can easily have that $G = P \rtimes Q$ is a minimal non-abelian group.

When $G = Q \rtimes P$ is a minimal non-nilpotent group, where $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$, Q is normal in G, $q \neq p$.

(i) Assume $|P| > p$. Then one can easily get that $G = Q \rtimes P$ is a minimal non-abelian group by the hypothesis.

(ii) Assume $|P| = p$. If Q is abelian, then $G = Q \rtimes P$ is a minimal non-abelian group by the hypothesis. If Q is non-abelian, then Q must be the unique non-abelian maximal subgroup of G.

When $G = Z_p \times K$, where K is a minimal non-nilpotent group and $(p, |K|) = 1$. Since G is non-nilpotent, one has that K is a minimal non-abelian group of non-prime-power order by the hypothesis.

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