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Certain homological properties of triangular matrix rings

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Abstract – In this paper, we study the (Gorenstein) flat dimension of modules over triangular matrix rings, and give estimations of the (Gorenstein) weak global dimension and the supremum of flat lengths of injective modules over triangular matrix rings. As applications, some new Ding–Chen rings are constructed.

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1. Introduction

Gorenstein flat modules were first introduced by Enochs, Jenda and Torrecillas in the 1990s [14] as generalizations of the classical flat modules. Such modules over Gorenstein rings were shown to share many nice properties similar to those of the classical flat modules over general rings [13]. Recently, Gorenstein flat modules over more general rings have been studied by many authors (see [2, 5, 10, 20, 21, 24, 26, 28]), and Gorenstein flatness of objects in other categories has also been extensively studied (see [7, 29]). Generally speaking, extension closed properties of this kind of module were studied by Holm [20] over coherent rings, by Bennis [2] over GF-closed rings, and finally by Šaroch and Šťovíček [26], who proved that the class of Gorenstein flat modules is closed under extensions, or equivalently, that the class is resolving, over an arbitrary ring. The size of the Gorenstein flat dimension measures how far

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a module is from being Gorenstein flat, which was investigated by Christensen et al. [4, 5], Emmanouil [10], Iacob [21], Wang and Zhang [27] and others. It turns out that such a dimension has an important application in representation theory of groups and rings, and is closely related to Auslander categories and Tate homologies.

Let A and B be rings and U be a (B, A)-bimodule. Then $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ forms a triangular matrix ring with the usual matrix addition and multiplication. Triangular matrix rings play an important role in ring theory and the representation theory of algebra. Mainly because this kind of ring is generally noncommutative and left-right nonsymmetric, they are initially used to construct counterexamples, which make the theory of rings and modules more abundant and concrete. So the properties of triangular matrix rings and modules over them have attracted great interest from scholars (see [1, 12, 17–19, 22, 24, 25, 28, 31]) since classical results were established by Green [15]. In particular, Zhu, Liu and Wang [31] characterized Gorenstein flat modules over a triangular matrix ring T under the condition that T is a Gorenstein ring, ${}_BU$ and U_A are finitely generated and have finite projective dimension. Mao [24] improved this characterization from the Gorensteinness assumption to the one of coherence, and proved the following results.

Theorem A ([24, Theorems 2.6 and 2.7]). *Let $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ be a right coherent triangular matrix ring with U a flat B-module. The following statements hold:*

(1) *If $\mathrm{Gwgldim}(B) < \infty$, U has a finite flat or injective dimension as an A^{o}-module and $X = \left(\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right)_{\varphi^X}$ is a T-module, then*

$$\max\{\mathrm{Gfd}_A(X_1), \mathrm{Gfd}_B(X_2)\} \le \mathrm{Gfd}_T(X) \le \max\{\mathrm{Gfd}_A(X_1) + 1, \mathrm{Gfd}_B(X_2)\}.$$

(2) *If U has a finite flat or injective dimension as an A^{o}-module, then*

$$\begin{aligned}\max\{\mathrm{Gwgldim}(A), \mathrm{Gwgldim}(B)\} &\le \mathrm{Gwgldim}(T)\\ &\le \max\{\mathrm{Gwgldim}(A) + 1, \mathrm{Gwgldim}(B)\}.\end{aligned}$$

This paper is intended to show that the result in Theorem A can be obtained in a more general setting. To be exact, using a more direct method, by Proposition 4.3 and Theorem 4.4 we remove the coherence assumption of T in Theorem A and then obtain Theorem B below. We also note that the condition $\mathrm{Gwgldim}(B) < \infty$ implies $\mathrm{sfli}(B^{\mathrm{o}}) < \infty$ (see Lemma 2.1), but the converse is not true in general (see [23, Example 2.5]).

Theorem B. *Let $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ be a triangular matrix ring with U a flat B-module. The following statements hold:*

(1) *If* $\mathrm{sfli}(B^{\mathrm{o}}) < \infty$, *$U$ has a finite flat or injective dimension as an A^{o}-module and $X = \binom{X_1}{X_2}_{\varphi^X}$ is a T-module, then*

$$\max\{\mathrm{Gfd}_A(X_1), \mathrm{Gfd}_B(X_2)\} \le \mathrm{Gfd}_T(X) \le \max\{\mathrm{Gfd}_A(X_1) + 1, \mathrm{Gfd}_B(X_2)\}.$$

(2) *If U has a finite flat or injective dimension as an A^{o}-module, then*

$$\begin{aligned}\max\{\mathrm{Gwgldim}(A), \mathrm{Gwgldim}(B)\} &\le \mathrm{Gwgldim}(T)\\ &\le \max\{\mathrm{Gwgldim}(A) + 1, \mathrm{Gwgldim}(B)\}.\end{aligned}$$

As an application, the Gorenstein weak global dimension of higher-order triangular matrix rings over a ring R is characterized and some new Ding–Chen rings are constructed.

The paper is organized as follows. In Section 2 we give some definitions and notation for use in the paper. In Section 3 we first focus on investigating the classical flat dimension of T-modules, and give estimations of the homological invariants, $\mathrm{wgldim}(T)$, the weak global dimension of T, and $\mathrm{sfli}(T)$, the supremum of the flat lengths (dimension) of injective modules over T, where $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is a triangular matrix ring (see Proposition 3.3, Theorems 3.4 and 3.6). Sections 4 and 5 are devoted to obtaining Theorem B and its applications.

2. Preliminaries

Throughout the paper, all rings R under consideration have an identity and all modules are unitary. An R-module will mean a left R-module, unless stated otherwise. We also refer to right R-modules as modules over the opposite ring R^{o} of R. As usual, the symbols $\mathrm{fd}_R(-)$ and $\mathrm{Gfd}_R(-)$ stand for the flat and the Gorenstein flat dimension, respectively.

In the following, we give some definitions and notation for use later.

Gorenstein flat module. Recall that an R-module G is *Gorenstein flat* [14] if there is an exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F_{-1} \xrightarrow{f_{-1}} F_{-2} \longrightarrow \cdots$$

of flat R-modules with $G = \mathrm{Ker}(f_{-1})$, which remains exact after applying $I \otimes_R -$ for any injective R^{o}-module I.

Gorenstein flat dimension. The *Gorenstein flat dimension* of an R-module M, $\mathrm{Gfd}_R(M)$, is the smallest nonnegative integer m for which there is an exact sequence

$$0 \longrightarrow G_m \longrightarrow G_{m-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with each G_i Gorenstein flat. If no such integer exists, then we set $\mathrm{Gfd}_R(M) = \infty$.

Global dimensions. Let R be a ring and M be an R-module. Then homological algebra attaches kinds of dimensions to M, such as projective, injective and flat dimensions, respectively. By taking the supremum of one of these dimensions as M ranges over specified R-modules, one obtains various left (relative) global dimensions of the ring R ([5, 9–11, 20]). For example, the left weak global dimension of R,

$$\mathrm{wgldim}(R) = \sup\{\mathrm{fd}_R(M) \mid M \text{ is an } R\text{-module}\},$$

and the left Gorenstein weak global dimension of R

$$\mathrm{Gwgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Emmanouil [11] introduced another relative global dimension $\mathrm{sfli}(R)$, which is defined as

$$\mathrm{sfli}(R) = \sup\{\mathrm{fd}_R(I) \mid I \text{ is an injective } R\text{-module}\}.$$

The corresponding right (relative) global dimensions of a ring R can be defined similarly.

It is well known that the weak global dimension of a ring is a left-right symmetric invariant, that is, $\mathrm{wgldim}(R) = \mathrm{wgldim}(R^{\mathrm{o}})$. Recently, Christensen, Estrada and Thompson [5, Corollary 2.5] have proved the Gorenstein version of the equality. Furthermore, by [5, Theorem 2.4, Corollary 2.5] and [10, Theorem 5.3], one has the following result for the Gorenstein weak global dimension.

Lemma 2.1. *The following equality holds for any ring R:*

$$\mathrm{Gwgldim}(R) = \mathrm{Gwgldim}(R^{\mathrm{o}}) = \max\{\mathrm{sfli}(R), \mathrm{sfli}(R^{\mathrm{o}})\}.$$

3. Flat dimensions over triangular matrix rings

In this section, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ always stands for a triangular matrix ring, where A and B are rings and U is a (B, A)-module. It follows from Green [15] that a T-module X can be viewed as a triplet $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi}$, where X_1 is an A-module, X_2 is a B-module and $\varphi: U \otimes_A X_1 \to X_2$ is a B-homomorphism. A T-homomorphism from $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi}$ to $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\psi}$ is a pair $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where $f_1: X_1 \to Y_1$ is an A-homomorphism and $f_2: X_2 \to Y_2$ is a B-homomorphism such that $\psi(1 \otimes f_1) = f_2\varphi$. Given such a triplet $\binom{X_1}{X_2}_\varphi$, we shall denote by $\tilde{\varphi}$ the morphism from X_1 to $\mathrm{Hom}_B(U, X_2)$ given by $\tilde{\varphi}(x)(u) = \varphi(u \otimes x)$ for each $u \in U$ and $x \in X_1$.

We notice that a sequence of T-modules

$$0 \longrightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_\varphi \longrightarrow \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_\psi \longrightarrow \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_\phi \longrightarrow 0$$

is exact if and only if the two sequences of A-modules or of B-modules

$$0 \longrightarrow X_1 \longrightarrow Y_1 \longrightarrow Z_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X_2 \longrightarrow Y_2 \longrightarrow Z_2 \longrightarrow 0$$

are both exact.

The following lemma is well known; compare to [18, Theorem 3.1].

LEMMA 3.1. *Let* $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ *be a triangular matrix ring. Then the following statements hold:*

(1) *A left T-module* $\binom{X_1}{X_2}_\varphi$ *is injective if and only if X_2 is an injective B-module and the morphism $\tilde{\varphi}$ is surjective with kernel injective over A.*

(2) *A left T-module* $\binom{X_1}{X_2}_\varphi$ *is flat if and only if X_1 is a flat A-module and the morphism φ is injective with cokernel flat over B.*

Recall from [13] that an R-module C is *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for any flat R-module F. In order to estimate the flat dimension of T-modules, we need the following lemma.

LEMMA 3.2. *Let* $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ *be a triangular matrix ring with U a flat B-module, and* $X = \binom{X_1}{X_2}_\varphi$ *a T-module. If there are two exact sequences*

$$0 \longrightarrow K_1 \xrightarrow{\rho_1} P_0 \xrightarrow{f_0} X_1 \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow C_1 \longrightarrow F_0 \xrightarrow{g_0} X_2 \longrightarrow 0$$

of A-modules and B-modules, respectively, with P_0, F_0 flat, and C_1 cotorsion, then we have the following exact sequence of T-modules:

$$(*) \quad 0 \longrightarrow \begin{pmatrix} K_1 \\ (U \otimes_A P_0) \oplus C_1 \end{pmatrix}_\phi \longrightarrow \begin{pmatrix} P_0 \\ (U \otimes_A P_0) \oplus F_0 \end{pmatrix} \xrightarrow{\binom{f_0}{h_0}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_\varphi \longrightarrow 0,$$

where $h_0 = (\varphi \circ (\mathrm{Id} \otimes f_0), g_0)$.

Proof. By using the tensor functor $U \otimes_A -$, we have an exact sequence

$$U \otimes_A K_1 \xrightarrow{\mathrm{Id}\otimes\rho_1} U \otimes_A P_0 \xrightarrow{\mathrm{Id}\otimes f_0} U \otimes_A X_1 \longrightarrow 0.$$

If we take into account $h_0 = (\varphi \circ (\mathrm{Id} \otimes f_0), g_0)\colon (U \otimes_A P_0) \oplus F_0 \to X_2$, then we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & U \otimes_A K_1 & \xrightarrow{\mathrm{Id}\otimes\rho_1} & U \otimes_A P_0 & \xrightarrow{\mathrm{Id}\otimes f_0} & U \otimes_A X_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\binom{\mathrm{Id}}{0}} & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & C & \xrightarrow{\binom{\alpha}{\beta}} & (U \otimes_A P_0) \oplus F_0 & \xrightarrow{h_0} & X_2 & \longrightarrow & 0, \end{array}$$

where C is some B-module such that the second row is exact. Thus we get the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C & \xrightarrow{\alpha} & U \otimes_A P_0 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle -\beta} & & \downarrow{\scriptstyle \varphi\circ(\mathrm{Id}\otimes f_0)} & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & F_0 & \xrightarrow{g_0} & X_2 & \longrightarrow & 0. \end{array}$$

Noting that $U \otimes_A P_0$ is flat and C_1 is cotorsion under our assumption, we get that the first row in the above pullback diagram splits, and so $C \cong (U \otimes_A P_0) \oplus C_1$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & U \otimes_A K_1 & \xrightarrow{\mathrm{Id}\otimes\rho_1} & U \otimes_A P_0 & \xrightarrow{\mathrm{Id}\otimes f_0} & U \otimes_A X_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\binom{\mathrm{Id}}{0}} & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & (U \otimes_A P_0) \oplus C_1 & \longrightarrow & (U \otimes_A P_0) \oplus F_0 & \xrightarrow{h_0} & X_2 & \longrightarrow & 0, \end{array}$$

which yields the exact sequence (∗), where $\phi\colon U \otimes_A K_1 \to (U \otimes_A P_0) \oplus C_1$ is a morphism such that the left square in the above diagram commutes. ■

Now we give an estimation of the flat dimension of T-modules, which supplements Hirano's [19, Lemma 2.7] and Mao's [25, Theorem 2.4].

Proposition 3.3. *Let $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ be a triangular matrix ring with U a flat B-module. Then there are inequalities for any T-module $X = \binom{X_1}{X_2}_{\varphi^X}$:*

$$\max\{\mathrm{fd}_A(X_1), \mathrm{fd}_B(X_2)\} \le \mathrm{fd}_T(X) \le \max\{\mathrm{fd}_A(X_1) + 1, \mathrm{fd}_B(X_2)\}.$$

Proof. We first prove that $\mathrm{fd}_T(X) \le \max\{\mathrm{fd}_A(X_1) + 1, \mathrm{fd}_B(X_2)\}$. Without loss of generality, we assume that $\max\{\mathrm{fd}_A(X_1) + 1, \mathrm{fd}_B(X_2)\} = n < \infty$. Then there are exact sequences

$$0 \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_{n-2} \xrightarrow{f_{n-2}} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X_1 \longrightarrow 0$$

and

$$0 \longrightarrow F_n \xrightarrow{g_n} F_{n-1} \xrightarrow{g_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{g_1} F_0 \xrightarrow{g_0} X_2 \longrightarrow 0$$

such that P_i and F_i are flat A-modules and B-modules, respectively, for $0 \leq i \leq n$, and such that $C_i = \mathrm{Ker}(g_{i-1})$ are cotorsion B-modules for $1 \leq i \leq n$. Note that the second exact sequence exists because of any module admitting a flat cover (see [3]).

Write $K_0 = X_1$ (resp. $C_0 = X_2$) and $K_i = \mathrm{Ker}(f_{i-1})$ (resp. $C_i = \mathrm{Ker}(g_{i-1})$) for $1 \leq i \leq n$.

Consider the following short exact sequences:

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow K_0 \longrightarrow 0$$

and

$$0 \longrightarrow C_1 \longrightarrow F_0 \longrightarrow C_0 \longrightarrow 0\ .$$

In view of Lemma 3.2, they induce an exact sequence

$$0 \longrightarrow \begin{pmatrix} K_1 \\ (U \otimes_A P_0) \oplus C_1 \end{pmatrix} \longrightarrow \begin{pmatrix} P_0 \\ (U \otimes_A P_0) \oplus F_0 \end{pmatrix} \longrightarrow \begin{pmatrix} K_0 \\ C_0 \end{pmatrix}_{\varphi} \longrightarrow 0.$$

It follows easily from Lemma 3.1 that the middle term module is flat. Applying Lemma 3.2 continually to the exact sequences

$$0 \longrightarrow K_{i+1} \longrightarrow P_i \longrightarrow K_i \longrightarrow 0$$

and

$$0 \longrightarrow C_{i+1} \longrightarrow (U \otimes_A P_{i-1}) \oplus F_i \longrightarrow (U \otimes_A P_{i-1}) \oplus C_i \longrightarrow 0,$$

as well as

$$0 \longrightarrow \begin{pmatrix} K_{i+1} \\ (U \otimes_A P_i) \oplus C_{i+1} \end{pmatrix} \longrightarrow \begin{pmatrix} P_i \\ (U \otimes_A P_i) \oplus (U \otimes_A P_{i-1}) \oplus F_i \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} K_i \\ (U \otimes_A P_{i-1}) \oplus C_i \end{pmatrix} \longrightarrow 0,$$

with the middle term module flat for $i = 1, 2, \ldots, n-1$, we have a long exact sequence

$$0 \longrightarrow \begin{pmatrix} 0 \\ (U \otimes_A P_{n-1}) \oplus F_n \end{pmatrix} \rightarrow \begin{pmatrix} P_{n-1} \\ \bigoplus_{i=n-2}^{n-1} (U \otimes_A P_i) \oplus F_{n-1} \end{pmatrix} \rightarrow \cdots$$

$$\begin{pmatrix} P_1 \\ \bigoplus_{i=0}^{1} (U \otimes_A P_i) \oplus F_1 \end{pmatrix} \longrightarrow \begin{pmatrix} P_0 \\ (U \otimes_A P_0) \oplus F_0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi} \longrightarrow 0,$$

which implies that $\mathrm{fd}_T \binom{X_1}{X_2}_{\varphi} \leq n$ since

$$\begin{pmatrix} 0 \\ (U \otimes_A P_{n-1}) \oplus F_n \end{pmatrix}, \quad \begin{pmatrix} P_0 \\ (U \otimes_A P_0) \oplus F_0 \end{pmatrix}$$

and

$$\begin{pmatrix} P_i \\ (U \otimes_A P_{i-1}) \oplus (U \otimes_A P_i) \oplus F_i \end{pmatrix}$$

are flat for $i = 1, 2, \ldots, n-1$. This implies that

$$\mathrm{fd}_T(X) \leq \max\big\{\mathrm{fd}_A(X_1) + 1, \mathrm{fd}_B(X_2)\big\}.$$

Next we prove that $\max\{\mathrm{fd}_A(X_1), \mathrm{fd}_B(X_2)\} \leq \mathrm{fd}_T(X)$. We let $\mathrm{fd}_T(X) = n < \infty$. So there is an exact sequence of T-modules

$$0 \longrightarrow \begin{pmatrix} F_n \\ F'_n \end{pmatrix}_{\varphi_n} \xrightarrow{\binom{f_n}{f'_n}} \begin{pmatrix} F_{n-1} \\ F'_{n-1} \end{pmatrix}_{\varphi_{n-1}} \longrightarrow \cdots \longrightarrow \begin{pmatrix} F_0 \\ F'_0 \end{pmatrix}_{\varphi_0} \xrightarrow{\binom{f_0}{f'_0}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi} \longrightarrow 0,$$

with each $\binom{F_i}{F'_i}_{\varphi_i}$ flat for $i = 0, 1, \ldots, n$. By Lemma 3.1, all modules F_i and $\mathrm{Coker}(\varphi_i)$ are flat, and each φ_i is injective. Since U is flat as a B-module by our assumption, we have each B-module $U \otimes_A F_i$ is flat for $i = 0, 1, \ldots, n$. Hence, the exactness of the sequence

$$0 \longrightarrow U \otimes_A F_i \xrightarrow{\varphi_i} F'_i \longrightarrow \mathrm{Coker}(\varphi_i) \longrightarrow 0$$

implies that each F'_i is also flat for $i = 0, 1, \ldots, n$. Since there are exact sequences

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{f_0} X_1 \longrightarrow 0$$

and

$$0 \longrightarrow F'_n \xrightarrow{f'_n} F_{n-1'} \longrightarrow \cdots \longrightarrow F'_0 \xrightarrow{f'_0} X_2 \longrightarrow 0,$$

we have $\mathrm{fd}_A(X_1) \le n$ and $\mathrm{fd}_B(X_2) \le n$. In other words, $\max\{\mathrm{fd}_A(X_1), \mathrm{fd}_B(X_2)\} \le \mathrm{fd}_T(X)$. ■

The following theorem gives an estimation of the homological invariant, $\mathrm{wgldim}(T)$, the weak global dimension of a triangular matrix ring T.

THEOREM 3.4. *Let $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ be a triangular matrix ring with U a flat B-module. Then*

$$\max\bigl\{\mathrm{wgldim}(A), \mathrm{wgldim}(B)\bigr\} \le \mathrm{wgldim}(T) \le \max\bigl\{\mathrm{wgldim}(A) + 1, \mathrm{wgldim}(B)\bigr\}.$$

PROOF. We first prove that $\max\{\mathrm{wgldim}(A), \mathrm{wgldim}(B)\} \le \mathrm{wgldim}(T)$. We let $\mathrm{wgldim}(T) = n < \infty$.

For any A-module X and any B-module Y, one has $\mathrm{fd}_T\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right) \le n$ and $\mathrm{fd}_T\left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right) \le n$ by the assumption. So by Proposition 3.3 we have

$$\mathrm{fd}_A(X) \le \mathrm{fd}_T\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right) \le n \quad \text{and} \quad \mathrm{fd}_B(Y) \le \mathrm{fd}_T\left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right) \le n,$$

which yields

$$\max\{\mathrm{wgldim}(A), \mathrm{wgldim}(B)\} \le n.$$

Next we prove that

$$\mathrm{wgldim}(T) \le \max\bigl\{\mathrm{wgldim}(A) + 1, \mathrm{wgldim}(B)\bigr\}.$$

We may assume that $\max\{\mathrm{wgldim}(A) + 1, \mathrm{wgldim}(B)\} = m + 1 < \infty$. Then we have $\mathrm{wgldim}(A) \le m$ and $\mathrm{wgldim}(B) \le m + 1$. Let $Z = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\varphi$ be any T-module. It follows from Proposition 3.3 that

$$\mathrm{fd}_T(Z) \le \max\bigl\{\mathrm{fd}_A(X) + 1, \mathrm{fd}_B(Y)\bigr\} \le m + 1,$$

and hence one has $\mathrm{wgldim}(T) \le m + 1$. This completes the proof. ■

As an application of theorem above, we give a characterization of the von Neumann regular triangular matrix ring.

PROPOSITION 3.5. *Let $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ be a triangular matrix ring. Then T is von Neumann regular if and only if A and B are von Neumann regular, and $U = 0$.*

PROOF. The sufficiency is plain. For the necessity, we let T be a von Neumann regular ring. Then $\mathrm{wgldim}(T) = 0$, that is, any T-module is flat. In particular, $\left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right)_\varphi$ is flat. Hence, by Lemma 3.1, $\varphi\colon U \otimes_A A \to 0$ is injective, and so $U \cong U \otimes_A A = 0$. Therefore, by Theorem 3.4, it is now easily seen that $\mathrm{wgldim}(A) = \mathrm{wgldim}(B) = 0$, as desired. ■

If we restrict all modules to injectives, then we have an estimation of the homological invariant, $\mathrm{sfli}(T)$, the supremum of the flat lengths of injective modules over a triangular matrix ring T.

Theorem 3.6. *Let $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ be a triangular matrix ring with U a flat B-module. Then*

$$\max\{\mathrm{sfli}(A), \mathrm{sfli}(B)\} \leq \mathrm{sfli}(T).$$

Moreover, if U is a flat A^{o}-module, then

$$\max\{\mathrm{sfli}(A), \mathrm{sfli}(B)\} \leq \mathrm{sfli}(T) \leq \max\{\mathrm{sfli}(A)+1, \mathrm{sfli}(B)\}.$$

Proof. We first prove that $\max\{\mathrm{sfli}(A), \mathrm{sfli}(B)\} \leq \mathrm{sfli}(T)$. We may assume that $\mathrm{sfli}(T) < \infty$.

Let X be an injective A-module and Y an injective B-module. Then $\begin{pmatrix} X \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \mathrm{Hom}_B(U,Y) \\ Y \end{pmatrix}$ are injective T-modules by Lemma 3.1. In view of Proposition 3.3, we have

$$\mathrm{fd}_A(X) \leq \mathrm{fd}_T\begin{pmatrix} X \\ 0 \end{pmatrix} \leq \mathrm{sfli}(T) \quad \text{and} \quad \mathrm{fd}_B(Y) \leq \mathrm{fd}_T\begin{pmatrix} \mathrm{Hom}_B(U,Y) \\ Y \end{pmatrix} \leq \mathrm{sfli}(T).$$

Thus $\max\{\mathrm{sfli}(A), \mathrm{sfli}(B)\} \leq \mathrm{sfli}(T)$.

Next we prove that $\mathrm{sfli}(T) \leq \max\{\mathrm{sfli}(A)+1, \mathrm{sfli}(B)\}$ under the additional condition that U is a flat A^{o}-module. We may assume that $\max\{\mathrm{sfli}(A)+1, \mathrm{sfli}(B)\} < \infty$. Then $\mathrm{sfli}(A)$ and $\mathrm{sfli}(B)$ are both finite. Let $Z = \begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}$ be any injective T-module. Then by Lemma 3.1, Y is an injective B-module, $\mathrm{Ker}(\tilde{\varphi})$ is an injective A-module, and $\tilde{\varphi}$ is surjective. Since U is a flat A^{o}-module, it follows from [13, Therorem 3.2.9] that $\mathrm{Hom}_B(U,Y)$ is an injective A-module. Therefore, the exactness of the sequence

$$0 \longrightarrow \mathrm{Ker}(\tilde{\varphi}) \longrightarrow X \xrightarrow{\tilde{\varphi}} \mathrm{Hom}_B(U,Y) \longrightarrow 0$$

implies that X is an injective A-module. It follows from Proposition 3.3 that

$$\mathrm{fd}_T(Z) \leq \max\{\mathrm{fd}_A(X)+1, \mathrm{fd}_B(Y)\} \leq \max\{\mathrm{sfli}(A)+1, \mathrm{sfli}(B)\},$$

and so we obtain $\mathrm{sfli}(T) \leq \max\{\mathrm{sfli}(A)+1, \mathrm{sfli}(B)\}$.

Now we are done. ∎

Let us give a particular characterization of $\mathrm{sfli}(T) = 0$ of a triangular matrix ring T; such a ring is usually called *IF* (see [6]).

Proposition 3.7. *Let $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ be a triangular matrix ring with U a flat A^{o}-module. Then T is IF if and only if A and B are IF, and $U = 0$.*

Proof. The sufficiency is trivial. For the necessity, we let T be an IF ring. Then by Lemma 3.1, $\binom{E}{0}_{\varphi}$ is an injective T-module with E an injective A-module, and so it is flat. Thus, by Lemma 3.1, $\varphi\colon U \otimes_A E \to 0$ is injective; this implies $U \otimes_A E = 0$. Now consider the exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow C \longrightarrow 0,$$

with E an injective A-module. Noting that U is a flat A°-module, we have an exact sequence

$$0 \longrightarrow U \otimes_A A \longrightarrow U \otimes_A E \longrightarrow U \otimes_A C \longrightarrow 0.$$

This implies that $U \cong U \otimes_A A = 0$. Hence, by Theorem 3.6, we get that A and B are IF, as desired. ■

Corollary 3.8. *Let $T = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ be a triangular matrix ring. Then the following statements hold:*

(1) $\mathrm{wgldim}(T) = \mathrm{wgldim}(R) + 1$.

(2) $\mathrm{sfli}(T) = \mathrm{sfli}(R) + 1$.

Proof. (1) By Theorem 3.4, $\mathrm{wgldim}(R) \leq \mathrm{wgldim}(T) \leq \mathrm{wgldim}(R) + 1$. We assume that $\mathrm{wgldim}(R) = n$. If $n = 0$, then by Proposition 3.5, one has $\mathrm{wgldim}(T) \geq 1$ since $U = R \neq 0$, and so $\mathrm{wgldim}(T) = \mathrm{wgldim}(R) + 1$. If $n > 0$, and letting X be an R-module with $\mathrm{fd}_R(X) = n$, then there exists an exact sequence of R-modules

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \longrightarrow F_0 \xrightarrow{f_0} X \longrightarrow 0$$

with each F_i flat, and $K_{n-1} = \mathrm{Ker}(f_{n-2})$ not flat. Now, by using Lemma 3.2, we may construct the following exact sequence of T-modules:

$$0 \longrightarrow \begin{pmatrix} 0 \\ F_n \end{pmatrix} \longrightarrow \begin{pmatrix} F_n \\ F_n \oplus F_{n-1} \end{pmatrix} \longrightarrow \begin{pmatrix} F_{n-1} \\ F_{n-1} \oplus F_{n-2} \end{pmatrix} \longrightarrow \cdots$$

$$\cdots \longrightarrow \begin{pmatrix} F_1 \\ F_1 \oplus F_0 \end{pmatrix} \longrightarrow \begin{pmatrix} F_0 \\ F_0 \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ 0 \end{pmatrix} \longrightarrow 0,$$

with $\binom{0}{F_n}$, $\binom{F_0}{F_0}$ and each $\binom{F_i}{F_i \oplus F_{i-1}}$ flat for $i = 1, 2, \ldots, n$. Thus the above sequence is a flat resolution of $\binom{X}{0}$. Since $\mathrm{Ker}\binom{f_{n-1}}{h_{n-1}} = \binom{F_n}{F_{n-1}}_{f_n}$, the $(n-1)$th yoke of $\binom{X}{0}$, is not flat by Lemma 3.1, we have $\mathrm{fd}_T\binom{X}{0} = n + 1$. This proves $\mathrm{wgldim}(T) \geq \mathrm{wgldim}(R) + 1$. Hence, we have $\mathrm{wgldim}(T) = \mathrm{wgldim}(R) + 1$.

(2) The proof is similar to the arguments that used in (1). ■

4. Gorenstein weak global dimension of triangular matrix rings

In this section, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ always stands for a triangular matrix ring, where A and B are rings and U is a (B, A)-module, unless stated specifically. Note that a right T-module is a triple $(Y_1, Y_2)_\varphi$, where Y_1 is an A°-module, Y_2 is a B°-module and $\varphi: Y_2 \otimes_B U \to Y_1$ is an A°-homomorphism. A T°-homomorphism from $(Y_1, Y_2)_\varphi$ to $(Z_1, Z_2)_\phi$ is a pair (f_1, f_2) such that $\phi(f_2 \otimes 1) = f_1\varphi$, where $f_1: Y_1 \to Z_1$ is an A°-homomorphism and $f_2: Y_2 \to Z_2$ is a B°-homomorphism. For a given T°-module $(Y_1, Y_2)_\varphi$, we shall denote by $\tilde{\varphi}$ the morphism from Y_2 to $\mathrm{Hom}_A(U, Y_1)$ given by $\tilde{\varphi}(y)(u) = \varphi(y \otimes u)$ for each $u \in U$ and $y \in Y_2$. By [1] and [17, Proposition 5.1], it follows that a T°-module $(Y_1, Y_2)_\varphi$ is injective if and only if Y_1 is an injective A°-module, $\mathrm{Ker}\,\tilde{\varphi}$ is an injective B°-module and $\tilde{\varphi}$ is an epimorphism. Also it follows from [19, Lemma 2.6] that a T°-module $(Y_1, Y_2)_\varphi$ is flat if and only if Y_2 is a flat B°-module, $\mathrm{Coker}\,\varphi$ is a flat A°-module and φ is a monomorphism.

In the following, we shall focus on investigating Gorenstein flat dimensions of modules, and give estimations of the homological invariants, $\mathrm{Gwgldim}(T)$, the Gorenstein weak global dimension of a triangular matrix ring T. Before doing this, we also need to give a characterization of Gorenstein flat modules over triangular matrix rings.

In view of [26, Corollary 4.12], we know that over any ring R the subcategory of all Gorenstein flat R-modules is closed under extensions. This result was previously shown over a right coherent ring by Holm [20, Theorem 3.7]. Based on this fact, we can improve [24, Theorem 2.3] from a right coherent ring to a general one with a similar proof.

Lemma 4.1. *Assume that U has finite flat dimension as a B-module, U has finite flat or injective dimension as an A°-module and $M = \binom{M_1}{M_2}_\varphi$ is a T-module. Then the following conditions are equivalent:*

(1) *M is a Gorenstein flat T-module.*

(2) *M_1 is a Gorenstein flat A-module,* $\mathrm{Coker}\,\varphi$ *is a Gorenstein flat B-module and φ is a monomorphism.*

In this case, $U \otimes_A M_1$ is Gorenstein flat if and only if M_2 is Gorenstein flat.

PROOF. It is similar to [24, Theorem 2.3]. ■

By Lemma 2.1, it follows that if $\mathrm{Gwgldim}(B) < \infty$ then $\mathrm{sfli}(B^{\mathrm{o}}) < \infty$, but the converse is not true in general (see [23, Example 2.5]). The following results improve [24, Lemma 2.5, Theorems 2.6 and 2.7], which are proved under the condition that $\mathrm{Gwgldim}(B) < \infty$.

LEMMA 4.2. *Let* $\mathrm{sfli}(B^{\mathrm{o}}) < \infty$*,* U *be a flat* B*-module and* U *have finite flat or injective dimension as an* A^{o}*-module. If* X *is a Gorenstein flat* A*-module, then* $U \otimes_A X$ *is a Gorenstein flat* B*-module.*

PROOF. Since $\mathrm{sfli}(B^{\mathrm{o}}) < \infty$, it is easy to check that any acyclic complex

$$\cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F_{-1} \xrightarrow{f_{-1}} F_{-2} \longrightarrow \cdots$$

of flat B-modules remains exact after applying the functor $I \otimes_B -$ for any injective B^{o}-module I. Then the proof is similar to that of [24, Lemma 2.5]. ■

PROPOSITION 4.3. *Let* $\mathrm{sfli}(B^{\mathrm{o}}) < \infty$*,* U *be a flat* B*-module,* U *have finite flat or injective dimension as an* A^{o}*-module and* $X = \binom{X_1}{X_2}_{\varphi^X}$ *be a* T*-module. Then*

$$\max\{\mathrm{Gfd}_A(X_1), \mathrm{Gfd}_B(X_2)\} \le \mathrm{Gfd}_T(X) \le \max\{\mathrm{Gfd}_A(X_1) + 1, \mathrm{Gfd}_B(X_2)\}.$$

PROOF. By [26, Corollary 4.12], it follows that any module M admits a Gorenstein flat cover, and so there is an exact sequence $0 \to C \to G \to M \to 0$ such that G is Gorenstein flat and C is Gorenstein flat cotorsion. Now one can prove the result similarly to Proposition 3.3 by using Lemmas 4.1 and 4.2. ■

We can prove the following result, which gives an estimate of the Gorenstein weak global dimension of a triangular matrix ring.

THEOREM 4.4. *Let* U *be a flat* B*-module,* U *have finite flat or injective dimension as an* A^{o}*-module. Then*

$$\begin{aligned}\max\{\mathrm{Gwgldim}(A), \mathrm{Gwgldim}(B)\} &\le \mathrm{Gwgldim}(T)\\ &\le \max\{\mathrm{Gwgldim}(A) + 1, \mathrm{Gwgldim}(B)\}.\end{aligned}$$

PROOF. The proof is similar to that of Theorem 3.4 by using Proposition 4.3. ■

Recall that a ring R is *Ding–Chen* if R is an n-FC ring for some nonnegative integer n, that is, is a two-sided coherent ring with finite FP-injective dimension at most n on both sides. In particular, 0-FC rings are just *FC* rings.

Note that a ring R is 0-FC if and only if $\mathrm{Gwgldim}(R) = 0$ (see [8, Theorem 6] or [27, Lemma 1.2]). The next result can be viewed as a Gorenstein analogue of Proposition 3.5.

Proposition 4.5. *Let $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ be a triangular matrix ring with U a flat A^{o}-module and a flat B-module. Then T is 0-FC if and only if A and B are 0-FC, and* $U = 0$.

Proof. The sufficiency is obvious. For the necessity, we let T be 0-FC. It happens if and only if $\mathrm{Gwgldim}(T) = 0$, that is, any T-module is Gorenstein flat. In particular, $\left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right)_{\varphi}$ is Gorenstein flat. Hence, we can infer from Lemma 4.1 that $\varphi\colon U \otimes_A A \to 0$ is a monomorphism, and so $U \cong U \otimes_A A = 0$. Therefore, it follows by Theorem 4.4 that $\mathrm{Gwgldim}(A) = \mathrm{Gwgldim}(B) = 0$, as desired. ■

Proposition 4.6. *Let $T = \left(\begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix}\right)$ be a triangular matrix ring. Then*

$$\mathrm{Gwgldim}(T) = \mathrm{Gwgldim}(R) + 1.$$

Proof. Combining Lemma 4.1 and Theorem 4.4 with Proposition 4.5, we can show the result similarly to Corollary 3.8. ■

Corollary 4.7. *Let $T = \left(\begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix}\right)$ be a triangular matrix ring and n a nonnegative integer. Then T is $n+1$-FC if and only if R is n-FC. In particular, T is Ding–Chen if and only if R is Ding–Chen.*

Proof. According to [16, Corollary 4.5], we know that R is two-sided coherent if and only if T is too. Now the result follows by [27, Lemma 1.2] and Proposition 4.6. ■

5. An application to $T_n(R)$

As another application, we give an explicit description of the Gorenstein flat $T_n(R)$-modules, where

$$T_n(R) = \begin{pmatrix} R & 0 & 0 & \cdots & 0 \\ R & R & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ R & \cdots & R & R & 0 \\ R & \cdots & R & R & R \end{pmatrix}$$

is the $n \times n$ lower-triangular matrix ring of order $n \geq 2$.

If we put

$$U=\begin{pmatrix} R\\ R\\ \vdots\\ R \end{pmatrix}_{(n-1)\times 1}$$

then we may regard U as a natural $(T_{n-1}(R), R)$-bimodule via multiplication of matrices, and $T_n(R)=\left(\begin{smallmatrix} R & 0\\ U & T_{n-1}(R)\end{smallmatrix}\right)$. By [30, Lemma 1.3], a left $T_n(R)$-module is an n-tuple

$$\begin{pmatrix} X_1\\ X_2\\ \vdots\\ X_n \end{pmatrix}_{(\phi_i)},$$

where the X_i are R-modules for $1\le i\le n$, and $\phi_i\colon X_i\to X_{i+1}$ are R-homomorphisms for $1\le i\le n-1$. A $T_n(R)$-homomorphism

$$\begin{pmatrix} X_1\\ X_2\\ \vdots\\ X_n \end{pmatrix}_{(\phi_i)} \longrightarrow \begin{pmatrix} Y_1\\ Y_2\\ \vdots\\ Y_n \end{pmatrix}_{(\varphi_i)} \quad \text{is an } n\text{-tuple} \begin{pmatrix} f_1\\ f_2\\ \vdots\\ f_n \end{pmatrix},$$

where all $f_i\colon X_i\to Y_i$ are R-homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\phi_1} & X_2 & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_{n-1}} & X_n\\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n}\\ Y_1 & \xrightarrow{\varphi_1} & Y_2 & \xrightarrow{\varphi_2} & \cdots & \xrightarrow{\varphi_{n-1}} & Y_n. \end{array}$$

It follows from [12, Proposition 2.4] that a $T_n(R)$-module

$$\begin{pmatrix} E_1\\ E_2\\ \vdots\\ E_n \end{pmatrix}_{(\phi_i)}$$

is injective if and only if the E_i are injective R-modules, $\phi_i\colon E_i\to E_{i+1}$ are epimorphisms and $\operatorname{Ker}\phi_i$ are injective R-modules, for all i. By [12, Theorem 2.5], a $T_n(R)$-module

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}_{(\phi_i)}$$

is flat if and only if F_i are flat R-modules, $\phi_i\colon F_i \to F_{i+1}$ are monomorphisms and $\operatorname{Coker}\phi_i$ are flat R-modules, for all i. Similarly, characterizations of an injective and a flat right $T_n(R)$-module can be given.

Proposition 5.1. *Let $T_n(R)$ be a triangular matrix ring of order $n \geq 2$. Then* $\operatorname{Gwgldim}(T_n(R)) = \operatorname{Gwgldim}(R) + 1$.

Proof. Note that $\operatorname{Gwgldim}(T_2(R)) = \operatorname{Gwgldim}(R) + 1$ by Proposition 4.6. If we put $T_n(R) = \left(\begin{smallmatrix} R & 0 \\ U & T_{n-1}(R) \end{smallmatrix}\right)$ as above, then U is flat as both a left $T_{n-1}(R)$-module and a right R-module, and so the assertion follows inductively from Theorem 4.4. ■

Corollary 5.2. *Let $T_n(R)$ be a triangular matrix ring of order $n \geq 2$. Then R is Ding–Chen if and only if so is $T_n(R)$.*

Proof. According to [16, Corollary 4.5], we know that R is two-sided coherent if and only if $T_n(R)$ is too. Now the result follows by [27, Lemma 1.2] and Proposition 5.1. ■

We end the paper by giving an explicit description of Gorenstein flat $T_n(R)$-modules.

Proposition 5.3. *A $T_n(R)$-module*

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}_{(\phi_i)}$$

is Gorenstein flat if and only if the following conditions hold:

(1) *All X_i are Gorenstein flat R-modules for $1 \leq i \leq n$.*

(2) *All $\phi_i\colon X_i \to X_{i+1}$ are monomorphisms for $1 \leq i \leq n-1$.*

(3) *All* $\operatorname{Coker}\phi_i$ *are Gorenstein flat R-modules for $1 \leq i \leq n-1$.*

PROOF. We prove the assertion by induction on n. If $n = 2$, then the result follows from Lemma 4.1.

Now we assume that the result holds for $n-1$ with $n > 2$. Set $T_n(R) = \left(\begin{smallmatrix} R & 0 \\ U & T_{n-1}(R) \end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix} X_1 \\ X' \end{smallmatrix}\right)_{\varphi}$, where

$$X' = \begin{pmatrix} X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}_{(\phi_i)} \quad \text{and} \quad \varphi = \begin{pmatrix} \phi_1 \\ \phi_2\phi_1 \\ \vdots \\ \phi_{n-1}\cdots\phi_2\phi_1 \end{pmatrix}.$$

Then U is flat as both a left $T_{n-1}(R)$-module and a right R-module.

Let X be a Gorenstein flat $T_n(R)$-module. Then it follows from Lemma 4.1 that the sequence

$$\dagger \qquad 0 \longrightarrow U \otimes_R X_1 \xrightarrow{\varphi} X' \longrightarrow \mathrm{Coker}(\varphi) \longrightarrow 0$$

is exact, X_1 is a Gorenstein flat R-module, and $\mathrm{Coker}(\varphi)$ is a Gorenstein flat $T_{n-1}(R)$-module. Thus, by our assumption on induction, one gets that

$$U \otimes_R X_1 = \begin{pmatrix} X_1 \\ X_1 \\ \vdots \\ X_1 \end{pmatrix}$$

is a Gorenstein flat $T_{n-1}(R)$-module, and so is X', since the class of all Gorenstein flat modules is closed under extensions. Now conditions (1), (2) and (3) are easily seen to be satisfied.

Conversely, if conditions (1), (2) and (3) hold, then we have the exact sequence (†) as above, that is, φ is monomorphic. If we set $Y_{i+1} =: \mathrm{Coker}\,\phi_i\phi_{i-1}\cdots\phi_1$ for $1 \le i \le n-1$, then

$$Y =: \mathrm{Coker}\,\varphi = \begin{pmatrix} Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix}_{(\psi_i)},$$

where $\psi_i\colon Y_i \to Y_{i+1}$ are R-morphisms with $2 \le i \le n-1$. In particular, one has $Y_2 = \mathrm{Coker}\,\phi_1$ Gorenstein as an R-module by condition (3). One gets from condition (2) that the following diagrams with exact rows and columns are commutative for $2 \leq i \leq n-1$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \xrightarrow{\phi_{i-1}\cdots\phi_2\phi_1} & X_i & \longrightarrow & Y_i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \psi_i} & & \\
0 & \longrightarrow & X_1 & \xrightarrow{\phi_i\cdots\phi_2\phi_1} & X_{i+1} & \longrightarrow & Y_{i+1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Coker}\phi_i & \xrightarrow{\cong} & \operatorname{Coker}\psi_i & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Successively, one can infer from conditions (1) and (3) that all Y_i are Gorenstein flat R-modules. By our assumption on induction, we get that $Y =: \operatorname{Coker}\varphi$ is a Gorenstein flat $T_{n-1}(R)$-module, and so X is a Gorenstein flat $T_n(R)$-module by Lemma 4.1. This completes the proof. ■

Remark 5.4. Let R be a ring, and $n \geq 2$ a positive integer. We note that the ring $T_n(R)$ is defined by the following quiver Q (i.e., if there is a path from i to j, then the (j,i)-entry is R, or else the (j,i)-entry is 0):

$$Q =: 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n.$$

In a sense, Gorenstein flat $T_n(R)$-modules are exactly Gorenstein flat objects in the category $\operatorname{Rep}(Q,R)$ of representations of Q by R-modules; see [7, Theorem A].

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