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Subellipticity, compactness, H^ϵ estimates and regularity for $\bar\partial$ on weakly q-pseudoconvex/concave domains

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Abstract – For a weakly q-pseudoconvex (resp. q-pseudoconcave) domain Ω in a Stein manifold X of dimension n, we give a sufficient condition for subelliptic estimates for the $\bar\partial$-Neumann problem. Moreover, we study the compactness of the $\bar\partial$-Neumann operator N on Ω. Such compactness estimates immediately lead to smoothness of solutions, the closed range property, the L^2-setting and the Sobolev estimates of N on Ω for any $\bar\partial$-closed (r,k)-form with $k \geqslant q$ (resp. $k \leqslant q$). Furthermore, we study the $\bar\partial$-problem with support conditions in Ω for forms of type (r,k), with values in a holomorphic vector bundle. Applications to the $\bar\partial_b$-problem for smooth forms on boundaries of Ω are given.

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1. Introduction and main results

The $\bar{\partial}$-Neumann problem in strongly pseudoconvex domains was solved by Kohn [16, 17], and the technique there showed that the existence of a subelliptic estimate will give the solution. The condition is simply counting the number of positive or negative eigenvalues of the Levi form. In [18], he gave a sufficient condition for subellipticity over pseudoconvex domains with real analytic boundary by introducing a sequence of ideals of subelliptic multipliers; see also [19]. The most general result concerning subelliptic estimates for the $\bar{\partial}$-Neumann problem was obtained by Catlin [6–8]. He proved [6] that subelliptic estimates hold for k-forms at z_0 on any smoothly bounded, pseudoconvex domain, which is of finite type in the sense of D'Angelo [10]. Herbig [12] extended Catlin's sufficiency result by replacing the boundedness condition on the weight functions with that of self-bounded complex gradient, a weaker condition which allows unbounded families of functions. This notion was introduced by McNeal [20]. However, in the case when the domain is not necessarily pseudoconvex the results are related to the celebrated $Z(k)$ condition which characterizes the existence of subelliptic estimates of index $\epsilon = \frac{1}{2}$ according to Hörmander [13] and Folland–Kohn [11]. Hörmander [13] proved the necessary and sufficient conditions for the $\frac{1}{2}$ estimate on non-pseudoconvex domains. For similar results see [1, 2, 5, 9, 21–26].

The first purpose of this paper is to give a sufficient condition for subelliptic estimates for the $\bar{\partial}$-Neumann problem on a smoothly bounded, weakly q-pseudoconvex (resp. q-pseudoconcave) domain in a Stein manifold X for forms of type (r, k), with $k \geqslant q$ (resp. $k \leqslant q$) with values in a holomorphic vector bundle. Second, we study the compactness and the Sobolev estimates of the $\bar{\partial}$-Neumann operator N. Such compactness estimates immediately lead to very important qualitative properties of the $\bar{\partial}$-operator, such as smoothness of solutions, the closed range property, the L^2-setting and the Sobolev estimates of N on Ω for any $\bar{\partial}$-closed (r, k)-form with $k \geqslant q$ (resp. $k \leqslant q$). The main results generalizes Khanh and Zampieri [14, 15] results to forms with values in a holomorphic vector bundle. The proof starts with the known estimate on scalar differential forms and then obtains a similar estimate locally on bundle-valued forms using a local frame. Then, by using a partition of unity, we globalize this estimate at the cost of the constants.

On the other hand, we study the $\bar{\partial}$-equation, $\bar{\partial}u = f$, with support conditions in a weakly q-pseudoconvex domain Ω in a Stein manifold X for forms of type (r, k), with $k \geqslant q$ with values in a holomorphic vector bundle. Similar results for this problem were considered in [3, 4, 15, 27–43]. Applications to the $\bar{\partial}_b$-problem for smooth forms on boundaries of Ω are given.

2. Preliminaries

2.1 – *Notation in* $\mathbb{C}^n$

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $L^2(\Omega)$ denote the space of square integrable functions on Ω. We can write an (r,k)-form f as

$$f = \sum_{I,J}{}' f_{I,J}\, dz_I \wedge d\bar z_J,$$

where $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_k)$ are multiindices and $dz_I = dz_1 \wedge \cdots \wedge dz_r$, $d\bar z_J = d\bar z_1 \wedge \cdots \wedge d\bar z_k$. The notation $\sum'$ means summation over strictly increasing multiindices and $L^2_{r,k}(\Omega)$ denotes the space of (r,k)-forms whose coefficients are in $L^2(\Omega)$. The norm $L^2_{r,k}(\Omega)$ is defined by

$$\|f\|^2 = \int_\Omega |f|^2\, dV \quad \text{and} \quad |f|^2 = \sum_{I,J}{}' f_{I,J}\, \overline{f_{I,J}},$$

for $f \in L^2_{r,k}(\Omega)$ and $dV = i^n dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n$. Denote by $C^\infty_{r,k}(\bar\Omega)$ the space of complex-valued differential forms of class C^∞ and of type (r,k) on Ω that are smooth up to the boundary, and denote by $\mathcal{D}_{r,k}(U)$ the elements in $C^\infty_{r,k}(\bar\Omega)$ that are compactly supported in $U \cap \bar\Omega$. For $0 \leqslant q \leqslant n$, the Cauchy–Riemann operator, or simply the D-bar operator, $\bar\partial\colon L^2_{r,k}(\Omega) \to L^2_{r,k+1}(\Omega)$ is defined by

$$\bar\partial\Big(\sum_{I,J}{}' f_{I,J}\, dz_I \wedge d\bar z_J\Big) = \sum_{j=1}^n \sum_{I,J}{}' \frac{\partial f_{I,J}}{\partial \bar z_j}\, d\bar z_j \wedge dz_I \wedge d\bar z_J,$$

with $\operatorname{dom}\bar\partial = \{f \in L^2_{r,k}(\Omega) : \bar\partial f \in L^2_{r,k+1}(\Omega)\}$. Here, the derivatives are taken in the distributional sense. The operator $\bar\partial$ is linear, closed and densely defined. The Hilbert space theory of unbounded operators gives that the adjoint of $\bar\partial$, which we denote by $\bar\partial^*$, is also linear, closed and densely defined. The Laplace–Beltrami operator $\Box$ is defined by $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial\colon \operatorname{dom}\Box \to L^2_{r,k}(\Omega)$ and $\ker\Box = \{f \in \operatorname{dom}\bar\partial \cap \operatorname{dom}\bar\partial^*\colon \bar\partial f = 0 \text{ and } \bar\partial^* f = 0\}$ is its kernel. One defines the $\bar\partial$-Neumann operator $N\colon L^2_{r,k}(\Omega) \to L^2_{r,k}(\Omega)$ as the inverse of the restriction of $\Box$ to $(\ker\Box)^\perp$.

We define the Fourier transform $\hat u$ of u as

$$\hat u(\xi) = \int_{\mathbb{R}^n} e^{-ix.\xi} f(x)\, dx,$$

where $dx = dx_1 \dots dx_n$ and $x.\xi = \sum_{j=1}^n x_j \xi_j$. For each $\epsilon \geqslant 0$ and for any $u \in \mathcal{D}_{r,k}(\mathbb{R}^n)$, the Sobolev norm is given by

$$\|f\|^2_{H^\epsilon_{r,k}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^\epsilon |\hat u(\xi)|^2\, d\xi.$$

Thus, $H^{\epsilon}_{r,k}(\Omega)$ can be defined as the completion of $\mathcal{D}_{r,k}(\mathbb{R}^n)$ with respect to the norm $\|f\|^2_{H^{\epsilon}_{r,k}(\mathbb{R}^n)}$. Moreover, let us restrict to the domain $\Omega \subset \mathbb{R}^n$. Then

$$\|f\|^2_{H^{\epsilon}_{r,k}(\Omega)} = \inf \|F\|^2_{H^{\epsilon}_{r,k}(\mathbb{R}^n)},$$

where $F \in H^{\epsilon}_{r,k}(\mathbb{R}^n)$ and $F|_{\Omega} = f$.

Definition 1. The $\bar{\partial}$-Neumann problem is said to satisfy a subelliptic estimate of order $\epsilon > 0$ at $z_0 \in \bar{\Omega}$ on k-forms if there exist a positive constant c and a neighborhood $V \ni z_0$ such that

$$\|f\|^2_{H^{\epsilon}} \leqslant c(\|\bar{\partial} f\|^2_{H^0} + \|\bar{\partial}^* f\|^2_{H^0} + \|f\|^2_{H^0}).$$

2.2 – *Notation in complex manifolds*

Let X be an n-dimensional Stein manifold with a Hermitian metric ω and Ω be a relatively compact domain in X. Let E be a holomorphic vector bundle, of rank p, over X with a Hermitian metric h and E^* its dual. An E-valued differential (r,k)-form u on X is given locally by a column vector $u^{\top} = (u^1, u^2, \dots, u^p)$, where u^a, $1 \leqslant a \leqslant p$, are $\mathbb{C}$-valued differential forms of type (r,k) on X. For integers $r, k \geqslant 0$, $C^{\infty}_{r,k}(\Omega, E)$ is the complex vector space of E-valued differential forms of class C^{∞} and of type (r,k) on Ω and $C^{\infty}_{r,k}(\bar{\Omega}, E)$ is the subspace of $C^{\infty}_{r,k}(\bar{\Omega}, E)$ whose elements can be extended smoothly up to $b\Omega$. Let $\mathcal{D}_{r,k}(\Omega, E)$ be the space of E-valued differential forms of type (r,k) with compact support in Ω. Let $\#_E : C^{\infty}_{r,k}(X, E) \to C^{\infty}_{k,r}(X, E^*)$ be the operator defined by $\#_E\, u = \bar{h}\bar{u}$, which commutes with the Hodge star operator. The corresponding operator $\#_{E^*} : C^{\infty}_{r,k}(X, E^*) \to C^{\infty}_{k,r}(X, E)$ is defined by $\#_{E^*}\, u = h^{-1}\bar{u} = {\#_E}^{-1}\, u$.

For $f, u \in C^{\infty}_{r,k}(X, E)$, a global inner product $\langle f, u\rangle_{\Omega}$ and the norm $\|\cdot\|_{\Omega}$, with respect to ω and h, are defined by

$$\langle f, u\rangle_{\Omega} = \int_{\Omega} f \wedge \star \#_E\, u,$$
$$\|f\|^2 = \langle f, f\rangle.$$

For $f \in C^{\infty}_{r,k}(\Omega, E)$ and $\eta \in \mathcal{D}_{r,k-1}(\Omega, E)$, the formal adjoint operator ϑ of the operator $\bar{\partial} : C^{\infty}_{r,k-1}(\Omega, E) \to C^{\infty}_{r,k}(\Omega, E)$ is defined by

$$\langle \vartheta f, \eta\rangle = \langle f, \bar{\partial}\eta\rangle, \qquad \vartheta = -\#_{E^*} \star \bar{\partial} \star \#_E\,. \tag{2.1}$$

Write

$$Q(u,u) = \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \|f\|^2$$

and

$$\mathcal{B}_{r,k}(\bar\Omega, E) = \{f \in C^\infty_{r,k}(\bar\Omega, E),\ \star\#_E\, f|_{b\Omega} = 0\}.$$

Let $L^2_{r,k}(\Omega, E)$ be the Hilbert space obtained by completing $C^\infty_{r,k}(\bar\Omega, E)$ under the norm $\|f\|^2$. The operators $\bar\partial$, $\bar\partial^*$, $\Box$ and N are defined for E-valued differential (r,k)-forms as in the case of $\mathbb{C}$-valued differential (r,k)-forms.

Definition 2. A form $u \in L^2_{r,k}(\Omega, E)$ is supported in $\bar\Omega$ ($\operatorname{supp} u \subset \bar\Omega$) or u vanishes to infinite order at the boundary of Ω if u vanishes on $b\Omega$.

Choose a finite covering $\{U_j\}_{j\in J}$ by domains of the charts $\eta_j: U_j \to V_j$ and let $\varphi_j: E|_{U_j} \to V_j$ be a collection of trivializations. Let φ_j^* be an induced map $\varphi_j^*\xi = \varphi_j \circ \xi \circ \eta_j^{-1}$ acting from $C^\infty(U_j, E|_{U_j})$ to $C^\infty(V_j, \mathbb{C}^p)$ which can be identified with $C^\infty(V_j)^p$. Let $(\rho_j)_j$ be a smooth partition of unity subordinate to $\{f_j\}_{j\in J}$ and put

$$\|f\|^2_{H^\epsilon(X,E)} = \sum_j \|\varphi_j^*\rho_j f\|^2_{H^\epsilon(\mathbb{R}^n)}, \tag{2.2}$$

where the right-hand side is the usual Sobolev ϵ-norm defined as in the Euclidean case. Thus, $H^\epsilon_{r,k}(\Omega, E)$ can be defined as the completion of $\mathcal{D}_{r,k}(X, E)$ with respect to the norm (2.2). Moreover, let us restrict to the domain $\Omega \subset X$. Then

$$\|f\|^2_{H^\epsilon_{r,k}(\Omega,E)} = \inf \|F\|^2_{H^\epsilon_{r,k}(X,E)},$$

where $F \in H^\epsilon_{r,k}(X, E)$ and $F|_\Omega = f$.

2.3 – *q-pseudoconvexity and the $(q-P)$ property*

Let Ω be a domain of a Stein manifold X defined by $\rho < 0$ with $d\rho \neq 0$ on the boundary $b\Omega$. Let $T^{\mathbb{C}}b\Omega$ be the complex tangent bundle to the boundary $b\Omega$. For a given boundary point $z_0 \in b\Omega$, we consider a boundary complex frame which means an orthonormal basis $\omega^1, \dots, \omega^n = d\rho$ of $(1,0)$-forms with C^∞ coefficients on a small neighborhood U of z_0. We denote by $(\rho(z))_{i,j=1}^{n-1}$, the matrix of the Levi form $\partial\bar\partial\rho(z)$ in the complex tangential direction at z with respect to the basis $\omega^1, \dots, \omega^n$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_{n-1}$ be the eigenvalues of $(\rho(z))_{i,j=1}^{n-1}$ and denote by $s^+_{b\Omega}(z)$, $s^-_{b\Omega}(z)$, $s^0_{b\Omega}(z)$ their number according to the different sign. Take a pair of indices $1 \leqslant q \leqslant n-1$ and $0 \leqslant q_0 \leqslant n-1$ such that $q \neq q_0$. We assume that there is a bundle $\mathcal{V}^{q_0} \subset T^{1,0}b\Omega$ of rank q_0 with smooth coefficients in a neighborhood V of z_0, say the bundle spanned by $\{L_1, L_2, \dots, L_{q_0}\}$, such that

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} \rho_{jj} \geqslant 0 \quad \text{on } b\Omega \cap V. \tag{2.3}$$

Definition 3. We have the following:

(i) If $q > q_0$, Ω is said to be weakly q-pseudoconvex at z_0.

(ii) If $q < q_0$, Ω is said to be weakly q-pseudoconcave at z_0.

Since $\sum_{j=1}^{q_0} \rho_{jj}$ is the trace of the restricted form $(\rho_{jj})|_V$, then Definition 3 depends only on the choice of the bundle V, not on its basis. Condition (2.3) is equivalent to

$$\sum_{\substack{|I|=r\\|K|=k-1}}' \sum_{i,j=1}^{n-1} \rho_{ij} f_{I,iK} \bar{u}_{I,jK} - \sum_{j=1}^{q_0} \rho_{jj} |u|^2 \geqslant 0, \tag{2.4}$$

for any (r,k)-form u with $k \geqslant q$. In this form (2.3) will be applied. For some cases, instead of (2.4), it is better to consider the variant

$$\sum_{\substack{|I|=r\\|K|=k-1}}' \sum_{i,j=1}^{n-1} \rho_{ij} f_{I,iK} \bar{u}_{I,jK} - \sum_{\substack{|I|=r\\|K|=k-1}}' \sum_{j=1}^{q_0} \rho_{jj} |f_{I,jK}|^2 \geqslant 0. \tag{2.5}$$

It is obvious that if $L_{b\Omega}|_V$ is assumed to be diagonal, instead of $\leqslant 0$, the left-hand side of (2.5) equals

$$\sum_{\substack{|I|=r\\|K|=k-1}}' \sum_{i,j=q_0+1}^{n-1} \rho_{ij} f_{I,iK} \bar{u}_{I,jK};$$

thus, if U is the Levi-orthogonal complement of V, then (2.5) is equivalent to $L_{b\Omega}|_U \geqslant 0$.

Remark 1. Definition 3 is a generalization of the usual pseudoconvexity or pseudoconcavity, as well as of the celebrated condition $Z(q)$ (cf. [3, 14, 15, 27]) (that is, the Levi form has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues). Note that a pseudoconvex domain is characterized by $s^-(z) = 0$; thus, it is 1-pseudoconvex in our terminology.

Example 1. Let $s^-(z)$ be constant for $z \in \Omega$ close to z_0. Then (2.3) holds for $q_0 = s^-$ and $q_0 = s^- + 1$. In fact, we have $\lambda_{s^-} < 0 < \lambda_{s^-+1}$, and thus the negative eigenvectors span a bundle $\mathcal{V}^{q_0}$ for $q_0 = s^-$, which, identified to the span of $L_1, L_2, \ldots, L_{q_0}$, yields $\sum_{j=1}^{q} \lambda_j \geqslant \sum_{j=1}^{q_0} \rho_{jj}$.

Example 2. Let Ω satisfy the $Z(q)$ condition at z_0, that is, $s^+(z) \geqslant n-q$ or $s^-(z) \leqslant q+1$ for $z \in V \cap b\Omega$. Thus Ω is strongly q-pseudoconvex or strongly q-pseudoconcave at z_0.

Example 3. It is readily seen that for $q_0 = s^- + s^0$ and for any $q > q_0$ (resp. $q_0 = s^-$ and any $q < q_0$), (2.3) is satisfied in a suitable local boundary frame. Thus any index $q \notin [s^-, s^- + s^0]$ satisfies (2.3) for either choice of q_0. Note that $s^- + s^0 = n - 1 - s^+$ and thus $q \notin [s^-, n - 1 - s^+]$ coincides, in the terminology of [12], with condition $Z(q)$. Instead, we use the terminology of strong q-pseudoconvexity (concavity) when $q > n - 1 - s^+$ (resp. $q < s^-$) because this is the same as asking that (2.3) holds with strict inequality.

The condition in Definition 4 below generalizes to domains which are not necessarily pseudoconvex, the celebrated P property by Catlin [6, 8].

Definition 4. A boundary $b\Omega$ is said to have the $(q - P)$ property in V if for every positive number M there is a function $\phi^M \in C^\infty(\bar\Omega \cap V)$ with $|\phi^M| \leqslant 1$ on Ω and such that, if we denote by $\lambda_1^{\phi^M} \leqslant \lambda_2^{\phi^M} \leqslant \cdots \leqslant \lambda_{n-1}^{\phi^M}$ the ordered eigenvalues of the Levi form (ϕ^M_{ij}), we have

$$(2.6)\qquad \sum_{j=1}^{q} \lambda_j^{\phi^M} - \sum_{j=1}^{q_0} \phi^M_{jj} \geq c\left(M^{-2\epsilon} + \sum_{j=1}^{q_0} |\phi_j|^2\right) \quad \text{on } \bar\Omega \cap V,$$

and

$$\sum_{j=1}^{q} \lambda_j^{\phi^M} - \sum_{j=1}^{q_0} \phi^M_{jj} \geq M \qquad \text{on } b\Omega \cap V,$$

where $\epsilon > 0$ and the constant $c > 0$ does not depend on M.

Remark 2. We notice that if condition (2.6) holds for forms in some degree $q > q_0$ (resp. $q < q_0$), then it also holds in any degree $k \geqslant q$ (resp. $k \leqslant q$). In fact, (2.6) forces $\lambda_q^\phi \geqslant 0$ (resp. $\lambda_q^\phi \leqslant 0$) which implies $\lambda_q^\phi \geqslant 0$ for any $k \geqslant q$ (resp. $\lambda_q^\phi \leqslant 0$ for any $k \leqslant q$).

Now we give a geometric condition on $b\Omega$ which implies that $b\Omega$ has property $(q - P)$.

Definition 5. Let Ω be a smoothly bounded, weakly q-pseudoconvex domain in a Stein manifold and let V be the bundle which occurs in the above definitions. The boundary $b\Omega$ is said to be weakly q regular at z_0 if there exist a neighborhood V of z_0 and a finite number of relatively compact subsets S_i of $b\Omega \cap V$, $i = 0, 1, \dots, N$, such that

(1) $\varnothing = S_N \subset S_{N-1} \subset \cdots \subset S_1 \subset S_0 = b\Omega \cap V$;

(2) if $z \in S_i \setminus S_{i+1}$, then there are a neighborhood $V' \Subset V$ and a CR submanifold D of $b\Omega \cap V'$ of CR dimension q with $z \in D$ which satisfy

(a) $S_i \cap V' \subset D$,

(b) $V_{z_0}|_D \subset T^{1,0}_{z_0} D$,

(c) if $\lambda_1^D \leqslant \cdots \leqslant \lambda_q^D$ are the ordered eigenvalues of the restricted Levi form $(\rho_{ij})|_{T^{\mathbb{C}}D}$, then

$$\sum_{j=1}^{q} \lambda_j^D - \sum_{j=1}^{q_0} \rho_{jj} > 0, \quad z \in D.$$

This generalizes the notion of weak regularity of a pseudoconvex boundary introduced by Catlin [6].

Proposition 2.1. *If the boundary of Ω is weakly q-regular, then it satisfies property $(q - P)$.*

Proof. The proof is the same as in [14, Theorem 2.1]. ∎

Let Ω be a domain of a Stein manifold defined by $\rho < 0$ with $d\rho \neq 0$ on the boundary $b\Omega$. Let V be a subbundle of $T^{1,0}b\Omega$ of rank, say, q_0, $\mathcal{W}$ a bundle contained in $V^{\perp}$, the orthogonal to V in $T^{1,0}b\Omega$ with respect to $\partial\bar{\partial}\rho$, such that $\mathcal{W} \cap V = \{0\}$ and $V + \mathcal{W} = T^{1,0}b\Omega$. The boundary $b\Omega$, as well as the bundles V and $\mathcal{W}$, is supposed to be real analytic.

Proposition 2.2. *Assume that in a neighborhood V of a boundary point z_0, in addition to the above hypotheses, the following two conditions are also fulfilled:*

$$\partial\bar{\partial}\rho|_V \leqslant 0, \quad \partial\bar{\partial}\rho|_{\mathcal{W}} \geqslant 0, \tag{2.7}$$

and $\mathcal{W} \cap \ker L_{b\Omega}$ is involuting and that there is no complex curve $\gamma \subset b\Omega$ such that $T_\gamma \subset \mathcal{W}$. Then $b\Omega$ satisfies the $(q - P)$ property.

Proof. It is obvious that (2.7), combined with $\mathcal{W} \subset V^{\perp}$, implies (2.3); in fact, if V is the bundle of the first q_0 eigenvalues and if λ is the minimum eigenvalue of $\partial\bar{\partial}\rho|_{\mathcal{W}}$, which is supposed to be non-negative, then

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} \rho_{jj} \geqslant \lambda.$$

Thus, Ω is q-pseudoconvex for $q = q_0 + 1$. Then, from Proposition 2.1, $b\Omega$ satisfies the $(q - P)$ property. ∎

3. Subellipticity of the $\bar\partial$-Neumann problem

THEOREM 3.1. *Let Ω be a weakly q-pseudoconvex (resp. q-pseudoconcave) domain with smooth boundary in $\mathbb{C}^n$. Suppose that $b\Omega$ has property $(q-P)$. Then ϵ-subelliptic estimates at z_0 hold for forms of degree $k \geqslant q$ (resp. $k \leqslant q$). More precisely, there exist a positive constant ϵ and a positive constant c such that*

$$\|u\|^2_{H^\epsilon} \leqslant cQ(u,u), \tag{3.1}$$

for $u \in \mathcal{D}_{r,k}(\Omega)$.

PROOF. Let $S_\delta = \{z \in \Omega : -\delta < \rho(z) \leqslant 0\}$ be a strip, where δ is a small enough positive number. Following Khanh and Zampieri [15],

$$\|u\|^2_{H^\epsilon} \leqslant cQ(u,u), \tag{3.2}$$

for $u \in \mathcal{D}_{r,k}(S_\delta \cap \Omega)$ with $k \geqslant q$ (resp. $k \leqslant q$). Since $b\Omega$ is compact, by a finite covering $\{U_\nu\}_{\nu=1}$ of $b\Omega$ by neighborhoods U_ν as in (3.2), we have

$$\|u\|^2_{H^\epsilon} \leqslant cQ(u,u), \tag{3.3}$$

when u is supported in the strip S_δ.

Now we estimate the integral over $\Omega \setminus S_\delta$. Choose $\gamma_\delta \in \mathcal{D}(\Omega)$ such that $\gamma_\delta(z) = 1$ whenever $\rho(z) \leqslant -\delta$ and $z \in \Omega \setminus S_\delta$. Using (3.3), one obtains

$$\begin{aligned}\|u\|^2_{H^\epsilon} &\leqslant \int_{S_\delta} |u|^2\, dV + \|\gamma_\delta u\|^2_{H^\epsilon} \\ &\leqslant c_1 Q(u,u) + \|u\|^2_{H^\epsilon} \\ &= (c_1 + c')Q(u,u),\end{aligned}$$

for some c_1, c'. Thus, we obtain the subelliptic estimate

$$\|u\|^2_{H^\epsilon} \leqslant cQ(u,u),$$

for $u \in \mathcal{D}_{r,k}(\Omega)$. Thus the proof follows. ∎

THEOREM 3.2. *Let X be a Stein manifold of complex dimension n. Let Ω be a weakly q-pseudoconvex (resp. q-pseudoconcave) domain with smooth boundary in X. Suppose that $b\Omega$ has the property $(q-P)$. Let E be a holomorphic vector bundle, of rank p, on X. Then, for $k \geqslant q$ (resp. $k \leqslant q$), the ϵ-subelliptic estimates at z_o for E-valued (r,k)-forms hold for the $\bar\partial$-Neumann problem on Ω. More precisely, there exists a positive constant c such that*

$$\|u\|^2_{H^\epsilon(E)} \leqslant cQ(u,u), \tag{3.4}$$

for $u \in \mathcal{D}_{r,k}(\Omega, E)$.

Proof. Let $\{f_j\}_{j=1}^{N}$ be a finite covering of $b\Omega$ by a local patching. Extend the subelliptic estimate (3.1) to E-valued forms. Let $e_1, e_2, \dots, e_p$ be an orthonormal basis on $E_z = \pi^{-1}(z)$, for every $z \in f_j$, $j \in J$. Thus every E-valued differential (r,k)-form u on X can be written locally, on f_j, as $u(z) = \sum_{a=1}^{p} u^a(z) e_a(z)$, where u^a are the components of the restriction of u on f_j. Since $b\Omega$ is compact, there exist a finite number of elements of the covering $\{f_j\}$, say, f_j, $j = 1, 2, \dots, m$, such that $\bigcup_{\nu=1}^{m} f_{j_\nu}$ cover $b\Omega$. Let $\{\zeta_j\}_{j=0}^{m}$ be a partition of unity such that $\zeta_0 \in \mathcal{D}_{r,k}(\Omega)$, $\zeta_j \in \mathcal{D}_{r,k}(f_j)$, $j = 1, 2, \dots, m$, and $\sum_{j=0}^{m} \zeta_j^2 = 1$ on $\overline{\Omega}$, where $\{U_j\}_{j=1,\dots,m}$ is a covering of $b\Omega$.

Let U be a small neighborhood of a given boundary point $\xi_0 \in b\Omega$ such that $U \Subset V \Subset f_{j_\nu}$, for a certain $j_\nu \in I$. If $u \in \mathcal{D}_{r,k}(\Omega, E)$, $0 \leqslant r \leqslant n$, $q \leqslant k \leqslant n-2$, on applying the subelliptic estimate (3.1) to each u^a and adding for $a = 1, \dots, p$, we get the subelliptic estimate for $u|_{\Omega \cap U}$,

$$\|\zeta_0 u\|_{H^\epsilon}^2 \lesssim c\,Q(\zeta_0 u, \zeta_0 u) \lesssim \epsilon\,Q(u,u).$$

Similarly, for $j = 1, \dots, m$, we get subelliptic estimate for $u|_{\Omega \cap U_j}$

$$\|\zeta_j u\|_{H^\epsilon}^2 \lesssim c\,Q(\zeta_j u, \zeta_j u) \lesssim \epsilon\,Q(u,u).$$

Summing over j, we get

$$\|u\|_{H^\epsilon(E)}^2 \leqslant c\,Q(u,u).$$

Thus the proof follows. ∎

4. Compactness, H^ϵ estimates and regularity for $\bar\partial$

As a consequence of the subelliptic estimate (3.4), one first proves the L^2 existence theorem of the $\bar\partial$-Neumann operator.

Theorem 4.1. *Let X be a Stein manifold of complex dimension n. Let Ω be a weakly q-pseudoconvex (resp. q-pseudoconcave) domain with smooth boundary in X. Suppose that $b\Omega$ has the property $(q-P)$. Let E be a holomorphic vector bundle, of rank p, on X. Then, for $k \geqslant q$ (resp. $k \leqslant q$),* $\ker(\Box, E)$ *is finite-dimensional and the range of $\Box$ is closed in $L^2_{r,k}(\Omega, E)$, and there exists a bounded linear operator $N\colon L^2_{r,k}(\Omega, E) \to L^2_{r,k}(\Omega, E)$ satisfying the following properties:*

(i) $\mathcal{R}\mathrm{an}(N, E) \subset \mathrm{dom}(\Box, E)$; $N\Box = I - \mathbb{H}$ *on* $\mathrm{dom}(\Box, E)$;

(ii) *for* $f \in L^2_{r,k}(\Omega, E)$, *we have* $f = \bar\partial\bar\partial^* N f \oplus \bar\partial^*\bar\partial N f \oplus \mathbb{H} f$;

(iii) $N\bar\partial = \bar\partial N$ *on* $\mathrm{dom}(\bar\partial, E)$ *and* $N\bar\partial^* = \bar\partial^* N$ *on* $\mathrm{dom}(\bar\partial^*, E)$;

(iv) *for all* $f \in L^2_{r,k}(\Omega, E)$*, we have the estimates*

$$\|Nf\| \le c\|f\|,$$
$$\|\bar\partial Nf\| + \|\bar\partial^* Nf\| \le \sqrt{c}\|f\|;$$

(v) *if* $f \in L^2_{r,k}(\Omega, E) \cap \ker(\bar\partial, E)$*, then* $\bar\partial^* Nf$ *gives the solution* u *to the equation* $\bar\partial u = f$ *of minimal* $L^2_{r,k-1}(\Omega, E)$*-norm;*

(vi) *if* $f \in L^2_{r,k}(\Omega, E) \cap \ker(\bar\partial^*, E)$*, then* $\bar\partial Nf$ *gives the solution* u *to the equation* $\bar\partial^* u = f$ *of minimal* $L^2_{r,k+1}(\Omega, E)$*-norm.*

Proof. From (3.4) at $\epsilon = \frac{1}{2}$, and for $f \in \mathrm{dom}(\bar\partial, E) \cap \mathrm{dom}(\bar\partial^*, E)$, we have

$$\|f\|^2_{H^{1/2}(E)} \leqslant C(\|\bar\partial f\|^2_{H^0(E)} + \|\bar\partial^* f\|^2_{H^0(E)} + \|f\|^2_{H^0(E)}). \tag{4.1}$$

This gives the existence of the $\bar\partial$-Neumann operator $N_{r,k}\colon L^2_{r,k}(\Omega, E) \to H^1_{r,k}(\Omega, E)$. Inequality (4.1) implies that from every sequence $\{f_\nu\}_{\nu=1}^\infty$ in $\mathrm{dom}(\bar\partial, E) \cap \mathrm{dom}(\bar\partial^*, E)$ with $\|f_\nu\|$ bounded, $\bar\partial f_\nu \to 0$ in $L^2_{r,k+1}(\Omega, E)$, and $\bar\partial^* f_\nu \to 0$ in $L^2_{r,k-1}(\Omega, E)$ as $\nu \to \infty$. Then (4.1) implies that $\|f_\nu\|^2_{H^{1/2}(E)} \leqslant C$ for some constant C. By the Rellich lemma, the inclusion map $i_\Omega\colon H^{1/2}_{r,k}(\Omega, E) \to L^2_{r,k}(\Omega, E)$ is compact, and one can extract a subsequence of the sequence f_ν which converges in the $L^2_{r,k}(\Omega, E)$-norm. Thus, for $f \in \mathrm{dom}(\bar\partial, E) \cap \mathrm{dom}(\bar\partial^*, E)$, $f \perp \ker(\Box, E)$, the hypotheses of [13, Theorem 1.1.3] are satisfied and so this theorem implies that $\ker(\Box, E)$ is finite-dimensional and the estimate

$$\|f\|^2_{H^0(E)} \leqslant C(\|\bar\partial f\|^2_{H^0(E)} + \|\bar\partial^* f\|^2_{H^0(E)}) \tag{4.2}$$

holds, and hence

$$\|f\|^2_{H^0(E)} \leqslant C\|\Box f\|^2_{H^0(E)}, \quad \text{for } f \in \mathrm{dom}(\Box, E),\ f \perp \ker(\Box, E). \tag{4.3}$$

Thus the closed graph theorem (cf. [13, Theorem 1.1.1]) implies that $\Box$ has closed range and forces, since $\Box$ is self-adjoint, the strong Hodge decomposition

$$\begin{aligned} L^2_{r,k}(\Omega, E) &= \mathcal{R}\mathrm{an}(\Box, E) \oplus \ker(\Box, E) \\ &= \bar\partial\bar\partial^* \,\mathrm{dom}(\Box, E) \oplus \bar\partial^*\bar\partial \,\mathrm{dom}(\Box, E) \oplus \ker(\Box, E). \end{aligned}$$

Estimate (4.3) also implies the existence of N as a unique bounded operator on $L^2_{r,k}(\Omega, E)$ that inverts $\Box$ on $\ker(\Box, E)^\perp$. We extend N to the whole $L^2_{r,k}(\Omega, E)$ space by setting $N = 0$ on $\ker(\Box, E)$. The properties follow directly, as in the proof of [9, Theorem 3.1.14]. ∎

Theorem 4.2. *Under the same assumptions as Theorem 4.1, for $k \geqslant q$ (resp. $k \leqslant q$), the $\bar\partial$-Neumann operator $N\colon L^2_{r,k}(\Omega, E) \to L^2_{r,k}(\Omega, E)$ is compact.*

Proof. To show that N is compact, it suffices to show compactness on $\ker(\Box, E)^\perp$ (since $N = 0$ on $\ker(\Box, E)$). When $f \in \ker(\Box, E)^\perp$ (and hence $Nf \in \ker(\Box, E)^\perp$), integration by parts and the Cauchy–Schwarz inequality imply

$$\begin{aligned}\|\bar\partial N f\|^2_{H^0(E)} + \|\bar\partial^* N f\|^2_{H^0(E)} &= \langle f, Nf\rangle_{H^0(E)} \\ &\leqslant \|f\|_{H^0(E)} \|Nf\|_{H^0(E)} \leqslant \|f\|^2_{H^0(E)}.\end{aligned}$$

The last inequality follows from (4.3) for Nf. Applying (4.1) and then (4.3) to Nf, one obtains

$$\|Nf\|^2_{H^{1/2}(E)} \leqslant C(\|\bar\partial Nf\|^2_{H^0(E)} + \|\bar\partial^* Nf\|^2_{H^0(E)} + \|Nf\|^2_{H^0(E)}) \leqslant K\|f\|^2_{H^0(E)},$$

where K is a positive constant. Thus N is compact on $L^2_{r,k}(\Omega, E)$ by the Rellich lemma (the embedding of $H^{1/2}_{r,k}(\Omega, E)$ into $L^2_{r,k}(\Omega, E)$ is compact). ∎

Theorem 4.3. *Under the same assumptions as Theorem 4.1, the following are equivalent:*

(1) *the compactness of the $\bar\partial$-Neumann operators N;*

(2) *the compactness of the embedding $j_{r,k}$ of the space* $\mathrm{dom}(\bar\partial, E) \cap \mathrm{dom}(\bar\partial^*, E)$, *provided with the graph norm $\|f\| + \|\bar\partial u\| + \|\bar\partial^\star u\|$, into $C^\infty_{r,k}(\bar\Omega, E)$;*

(3) *for every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that, for $u \in \mathcal{D}^{p,q}$, $0 < q < n-1$, we have*

$$\|u\|^2 \leqslant \varepsilon Q(u,u) + C_\varepsilon \|u\|^2_{H^{-1}(E)}. \tag{4.4}$$

Proof. First we prove (3) ⇒ (1). We want to prove that for any bounded sequence $\{u_n\} \subset C^\infty_{r,k}(\bar\Omega, E)$, the sequence $\{N_{r,k}(u_n)\}$ admits a convergent subsequence. Since $N_{r,k}$ is a bounded operator in $C^\infty_{r,k}(\bar\Omega, E)$, we observe that $\{N_{r,k}(u_n)\}$ is a bounded sequence in $C^\infty_{r,k}(\bar\Omega, E)$. According to the Bolzano–Weierstrass theorem, every bounded sequence has a convergent subsequence. Hence there exists a subsequence $v_j = f_{n_j}$ such that $\{N(v_j)\}$ converges in $H^{-1}_{r,k}(\Omega, E)$ since $C^\infty_{r,k}(\bar\Omega, E)$ is compactly embedded in $H^{-1}_{r,k}(\Omega, E)$. To conclude that, it is sufficient to prove that $\{N_{r,k}(v_j)\}$ is a Cauchy sequence. We observe that estimate (3.1) applied to $\{N_{r,k}(v_j)\}$ gives

$$\|N_{r,k}(v_j - v_l)\| \leq \varepsilon \|v_j - v_l\| + C_\varepsilon \|N_{r,k}(v_j - v_l)\|_{H^{-1}(E)}.$$

For fixed $\delta > 0$, we get the conclusion choosing ε such that $\varepsilon\|v_j - v_l\| \leq \frac{\delta}{2}$ for any j, l and $M \in \mathbb{N}$ such that $\varepsilon\|N_{r,k}(v_j - v_l)\| \leqslant \frac{\delta}{2}$ for any $j, l \geq M$. Then $\{N_{r,k}(v_j)\}$

is a Cauchy sequence. Similarly, one obtains that $\{\Box N_{r,k}(v_j)\}$ is a Cauchy sequence which implies that $\{v_j\}$ is a Cauchy sequence.

Now we prove (1) $\Rightarrow$ (2). It is easy to observe that $N_{r,k} = j_{r,k}$, when the range $\mathrm{dom}(\bar{\partial}, E) \cap \mathrm{dom}(\bar{\partial}^*, E)$ is endowed with the graph norm. On the other hand, compactness is stable under adjunction.

Finally, we prove (2) $\Rightarrow$ (3). If the compactness estimate does not hold we can choose a sequence $\{u_n\}$ such that $Q(u_n, u_n) = 1$ and

$$1 \geq \|u_n\|^2 \geq \epsilon + n\|u_n\|^2_{H^{-1}(E)} \tag{4.5}$$

for any $n \in \mathbb{N}$. By compactness of the embedding there exists a subsequence $v_j = f_{n_j}$ which converges in $C^{\infty}_{r,k}(\bar{\Omega}, E)$ and hence also in $H^{-1}_{r,k}(\Omega, E)$. From (4.5) the common limit is 0. But this contradicts the fact that, again by (4.5), $\|u_n\| \geqslant \epsilon$. ∎

Now we briefly review classical results about Sobolev estimates for $\Box$.

Theorem 4.4. *Let X be a Stein manifold of complex dimension n. Let Ω be a weakly q-pseudoconvex (resp. q-pseudoconcave) domain with smooth boundary in X. Suppose that $b\Omega$ has the property $(q - P)$. Let E be a holomorphic vector bundle, of rank p, on X. Then*

$$\|f\|^2_{H^{\epsilon}(E)} \lesssim \|\Box f\|^2_{H^{\epsilon}(E)}, \quad \textit{for } f \in \mathcal{D}_{r,k}(\Omega, E) \cap \mathrm{dom}(\Box, E).$$

Proof. Let $\{f_j\}_{j=1}^{N}$ be a finite covering of $b\Omega$ by a local patching. Extend the subelliptic estimate

$$\|f\|^2_{H^{\epsilon}} \lesssim \|\Box f\|^2_{H^{\epsilon}}, \quad \text{for } f \in \mathcal{D}_{r,k}(\Omega) \cap \mathrm{dom}(\Box)$$

to E-valued forms (for the proof see [34, Theorem 3.1]). Thus the proof follows as in Theorem 3.2. ∎

Theorem 4.5 (Cf. [23]). *Let X be an n-dimensional complex manifold of complex dimension n and Ω be a bounded domain of X. Let $\Omega \Subset X$ be a submanifold with smooth boundary. Suppose the compactness estimate (4.4) holds on Ω. Let E be a holomorphic vector bundle, of rank p, on X. Suppose further that the $\bar{\partial}$-closed (r,k)-form α is in $H^m(\Omega, E)$ and $\alpha \perp \ker(\Box, E)$. Then there is a constant C_m such that the canonical solution u of $\bar{\partial}u = \alpha$, with $u \perp \ker \bar{\partial}$, satisfies*

$$\|f\|_{H^m(E)} \leqslant C_m(\|\alpha\|_{H^m(E)} + \|f\|).$$

Since $C^{\infty}(\bar{\Omega}, E) = \bigcap_{m=0}^{\infty} H^m(\Omega, E)$, it follows that if $\alpha \in C^{\infty}_{r,k}(\bar{\Omega}, E)$, then $u \in C^{\infty}_{r,k-1}(\bar{\Omega}, E)$.

Theorem 4.6. *Let X be a Stein manifold of complex dimension n. Let Ω be a weakly q-pseudoconvex (resp. q-pseudoconcave) domain with smooth boundary in X. Suppose that $b\Omega$ has the property $(q-P)$. Let E be a holomorphic vector bundle, of rank p, on X. Then, for $\alpha \in C^{\infty}_{r,k}(\overline{\Omega}, E)$, $q \leqslant k \leqslant n-2$, satisfying $\bar{\partial} f = 0$ in the distribution sense in X, there exists $u \in C^{\infty}_{r,k-1}(\overline{\Omega}, E)$, such that $\bar{\partial} u = f$ in the distribution sense in X.*

5. Closed range property

Closed range properties of the $\bar{\partial}$-operator are closely connected to the existence of the $\bar{\partial}$-Neumann operator.

Theorem 5.1. *Under the same assumptions as Theorem 4.1, for $k \geqslant q$ (resp. $k \leqslant q$), we have*

(i) *the operator $\bar{\partial}$ has closed range in $L^2_{r,k}(\Omega, E)$ and $L^2_{r,k+1}(\Omega, E)$,*

(ii) *the operator $\bar{\partial}^*$ has closed range in $L^2_{r,k}(\Omega, E)$ and $L^2_{r,k-1}(\Omega, E)$.*

Proof. Using (4.2) and by using [13, Theorems 1.1.3 and 1.1.2], we obtain that $\bar{\partial}\colon L^2_{r,k}(\Omega, E) \to L^2_{r,k+1}(\Omega, E)$ and $\bar{\partial}^*\colon L^2_{r,k}(\Omega, E) \to L^2_{r,k-1}(\Omega, E)$ have closed range. ∎

Theorem 5.2. *Under the same assumptions as Theorem 4.1, for $k \geqslant q$ (resp. $k \leqslant q$), the $\bar{\partial}$-Neumann operator N exists and $N\colon L^2_{r,k}(\Omega, E) \to H^1_{r,k}(\Omega, E)$.*

Proof. From (4.1),

$$\|f\|^2_{H^{1/2}(E)} \leqslant C(\|\bar{\partial} f\|^2_{H^0(E)} + \|\bar{\partial}^* f\|^2_{H^0(E)} + \|f\|^2_{H^0(E)}),$$

for $f \in \mathcal{D}\mathrm{om}(\bar{\partial}, E) \cap \mathcal{D}\mathrm{om}(\bar{\partial}^*, E)$. Since there exists an interpolation between $H^1_{r,k}(\Omega, E)$ and $L^2_{r,k}(\Omega, E)$ as illustrated in [9, Appendix B and Theorem B.3] and by using the procedure of [9, Theorem 5.2.1], we conclude that the range of N is $H^1_{r,k}(\Omega, E)$. This gives the existence of the $\bar{\partial}$-Neumann operator $N\colon L^2_{r,k}(\Omega, E) \to H^1_{r,k}(\Omega, E)$. ∎

Theorem 5.3. *Under the same assumptions as Theorem 4.1, for $k \geqslant q$ (resp. $k \leqslant q$), there exists $u \in H^{1/2}_{r,k}(\Omega, E)$ with $\bar{\partial} u = f$.*

Proof. From Theorem 4.1, for any $f \in L^2_{r,k}(\Omega, E) \cap \ker(\bar{\partial}, E)$ and $f \perp \ker(\Box, E)$, there exists a $u \in H^{1/2}_{r,k}(\Omega, E)$ with $\bar{\partial} u = f$. ∎

6. The $\bar\partial$ problem with support condition

By applying the duality of the $\bar\partial$-Neumann problem one obtains a solution to the $\bar\partial$-equation with exact support on such domains.

THEOREM 6.1. *Let X be a Stein manifold of complex dimension n. Let Ω be a weakly q-pseudoconvex domain with smooth boundary in X. Suppose that $b\Omega$ has the property $(q-P)$. Let E be a holomorphic vector bundle, of rank p, on X. Then, for $q \leqslant k \leqslant n-q$ and for $f \in L^2_{r,k}(X, E)$, supp $f \subset \bar\Omega$ with $\bar\partial f = 0$ in the distribution sense in X, there exists $u \in L^2_{r,k-1}(X, E)$, supp $u \subset \bar\Omega$ such that $\bar\partial u = f$ in the distribution sense in X.*

PROOF. Let $f \in L^2_{r,k}(X, E)$ and supp $f \subset \bar\Omega$. Then $f \in L^2_{r,k}(\Omega, E)$. From Theorem 4.1, $N_{n-r,n-k}$ exists for $n-k \geqslant q$, and since $N_{n-r,n-k} = (\Box_{n-r,n-k})^{-1}$ on $\mathcal{R}\mathrm{an}(\Box_{n-r,n-k}, E^*)$ and $\mathcal{R}\mathrm{an}(N_{n-r,n-k}, E^*) \subset \mathrm{dom}(\Box_{n-r,n-k}, E^*)$, we have $N_{n-r,n-k}\,\#_E \star f \in \mathrm{dom}(\Box_{n-r,n-k}, E^*) \subset L^2_{n-r,n-k}(\Omega, E^*)$, for $k \leqslant n-q$. Thus, one can define $u \in L^2_{r,k-1}(\Omega, E)$ by

$$u = -\star \#_{E^*}\,\bar\partial N_{n-r,n-k}\,\#_E \star f. \tag{6.1}$$

Extend u to X by defining $u = 0$ in $X \setminus \bar\Omega$. That the extended form u satisfies the equation $\bar\partial u = f$ in the distribution sense in X is proved as follows. We shall first prove that $\bar\partial u = f$ in the distribution sense in Ω.

For $\eta \in \mathrm{dom}(\bar\partial, E^*) \subset L^2_{n-r,n-k-1}(\Omega, E^*)$, one obtains

$$\begin{aligned}\langle\bar\partial\eta, \#_E \star f\rangle_\Omega &= \int_\Omega \bar\partial\eta \wedge \star(\#_{E^*}\#_{E^*}\star f) = (-1)^{r+k}\int_\Omega \bar\partial\eta \wedge f\\ &= (-1)^{(r+k)(r+k-1)}\int_\Omega f \wedge \bar\partial\eta = (-1)^{r+k}\langle f, \#_{E^*}\star\bar\partial\eta\rangle_\Omega.\end{aligned}$$

Since $\vartheta = \bar\partial^*$ on $\mathcal{B}_{r,k}(\bar\Omega, E)$, when ϑ acts in the distribution sense and $\mathcal{B}_{r,k}(\bar\Omega, E)$ is dense in $\mathrm{dom}(\bar\partial, E) \cap \mathrm{dom}(\bar\partial^*, E)$ in the graph norm, from [13, Proposition 2.1.1], then from (2.1) one obtains

$$\langle\bar\partial\eta, \#_E \star f\rangle_\Omega = \langle f, \bar\partial^*\,\#_{E^*}\star\eta\rangle_\Omega.$$

Since supp $f \subset \bar\Omega$, then we obtain

$$\langle\bar\partial\eta, \#_{E^*}\star f\rangle_\Omega = \langle f, \bar\partial^*\,\#_{E^*}\star\eta\rangle_\Omega = \langle\bar\partial f, \#_{E^*}\star\eta\rangle_X = 0.$$

It follows that $\bar\partial^*(\#_E \star f) = 0$ on Ω. Using Theorem 4.1 (iii), we have

$$\bar\partial^* N_{n-r,n-k}(\#_E \star f) = N_{n-r,n-s-1}\bar\partial^*(\#_E \star f) = 0. \tag{6.2}$$

Thus, from (2.1), (6.1) and (6.2), we obtain

$$\begin{aligned}\bar{\partial}u &= -\bar{\partial} \star \#_{E^*} \bar{\partial} N_{n-r,n-k} \#_E \star f \\ &= (-1)^{r+k} \star \#_{E^*} \bar{\partial}^* \bar{\partial} N_{n-r,n-k} \#_E \star f \\ &= (-1)^{r+k} \star \#_{E^*} (\bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*) N_{n-r,n-k} \#_E \star f \\ &= (-1)^{r+k} \star \#_{E^*} \#_E \star f \\ &= f\end{aligned}$$

in the distribution sense in Ω. Since $u = 0$ in $X \setminus \Omega$, then for $\eta \in \operatorname{dom}(\bar{\partial}^*, E) \subset L^2_{r,k}(X, E)$, we obtain

$$\langle u, \bar{\partial}^* \eta \rangle_X = \langle u, \bar{\partial}^* \eta \rangle_\Omega = \langle \#_E \star \bar{\partial}^* \eta, \#_E \star u \rangle_\Omega.$$

Thus, from (2.1), we obtain

$$\langle u, \bar{\partial}^* \eta \rangle_X = (-1)^{r+k} \langle \bar{\partial} \#_E \star \eta, \#_E \star u \rangle_\Omega = \langle \#_E \star \eta, \#_E \star \bar{\partial} u \rangle_\Omega = \langle \bar{\partial} u, \eta \rangle_\Omega, \tag{6.3}$$

where the second equality holds since

$$\#_E \star u = (-1)^{r+k+1} \bar{\partial} N_{n-r,n-k} \#_E \star f \in \operatorname{dom}(\bar{\partial}^*, E^*).$$

Thus, from (6.3), we obtain

$$\langle u, \bar{\partial}^* \eta \rangle_X = \langle f, \eta \rangle_\Omega = \langle f, \eta \rangle_X.$$

Thus $\bar{\partial} u = f$ in the distribution sense in X. Thus the proof follows. ■

This gives the extension of $\bar{\partial}_b$ closed forms from the boundary of such domains, where $\bar{\partial}_b$ is the tangential Cauchy Riemann operator.

Theorem 6.2. *Under the same assumptions as Theorem 6.1, for $f \in C^\infty_{r,k}(b\Omega, E)$, $q \leqslant k \leqslant n-q-1$, satisfying $\bar{\partial}_b f = 0$, there exists $F \in C^\infty_{r,k}(\bar{\Omega}, E)$ such that $F|_{b\Omega} = f$ and $\bar{\partial} F = 0$.*

Proof. Let $f \in C^\infty_{r,k}(b\Omega, E)$, $q \leqslant k \leqslant n-q-1$, with $\bar{\partial}_b f = 0$. Then there exists $f' \in C^\infty_{r,k}(\bar{\Omega}, E)$ such that $f'|_{b\Omega} = f$ and $\bar{\partial} f'$ vanishes to infinite order on $b\Omega$. Following Theorem 6.1, there exists $f \in C^\infty_{r,k}(\bar{\Omega}, E)$ with supp $f \subset \bar{\Omega}$ such that $\bar{\partial} f = \bar{\partial} f'$. Then the form $F = f' - f$ satisfies $F \in C^\infty_{r,k}(\bar{\Omega}, E)$, $F|_{b\Omega} = f$ and $\bar{\partial} F = 0$. ■

Theorem 6.3. *Under the same assumptions as Theorem 6.1, if $f \in C^\infty_{r,k}(b\Omega, E)$, $q \leq k \leq n-q-1$, with $\bar{\partial}_b f = 0$, there exists $u \in C^\infty_{r,k-1}(b\Omega, E)$, such that $\bar{\partial}_b u = f$.*

Proof. Let $f \in C^\infty_{r,k}(b\Omega, E)$, $q \leqslant k \leqslant n-q-1$, with $\bar\partial_b f = 0$. Then from Theorem 6.2 there exists $F \in C^\infty_{r,k}(\bar\Omega, E)$ such that $F|_{b\Omega} = f$ and $\bar\partial F = 0$. Following Theorem 6.1, there exists $U \in C^\infty_{r,k-1}(\bar\Omega, E)$ satisfying $\bar\partial U = f$ in Ω. Then $u = U|_{b\Omega}$ satisfies $\bar\partial_b u = f$. ■

The necessary and sufficient condition on $f \in H^{1/2}_{r,k}(b\Omega, E)$ to have a $\bar\partial$-closed extension F on Ω is summarized as follows. In what follows, differentiation is always taken in the distribution sense.

Theorem 6.4. *Under the same assumptions as Theorem 6.1, for $f \in H^{1/2}_{r,k}(b\Omega, E)$, $q \le k \le n-q-1$, we assume that*

$$\bar\partial_b f = 0. \tag{6.4}$$

Then there exists $F \in L^2_{r,k-1}(\Omega, E)$ such that $F = f$ on $b\Omega$ and $\bar\partial F = 0$ in Ω.

Proof. Since $f \in H^{1/2}_{r,k}(b\Omega, E)$ is a form, one can extend f componentwise to Ω such that each component is in $H^1(\Omega, E)$. For the detailed construction of such an extension, see e.g. [7, Lemma 9.3.3]. Let f' be an arbitrary extension of f with $f' \in H^1_{r,k}(\Omega, E)$.

From (6.4), we can require that $f_1 = \bar\partial_b f' = 0$ in $b\Omega$. If we extend f_1 to be zero outside $\bar\Omega$, we get $\bar\partial f_1 = 0$ in X in the distribution sense. We set

$$v = -\star \#_{E^*} \bar\partial N \#_E \star f.$$

From Theorem 6.1 and its proof, we have $v \in L^2_{r,k}(\Omega, E)$. We set $v = 0$ outside Ω. Then $\bar\partial v = f_1$ in the distribution sense on X with v supported in $\bar\Omega$.

Setting $F = f' - v$ in Ω, we have $F = f$ on $b\Omega$ and $\bar\partial F = \bar\partial f' - \bar\partial v = f_1 - f_1 = 0$ on Ω. ■

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