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A note on vector valued maximal inequalities in Sobolev multiplier spaces

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Abstract – We address some vector valued maximal inequalities in a type of Sobolev multiplier spaces associated with Bessel capacities. We show that the vector valued maximal inequalities are of local type.

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1. Introduction and statements of the main results

Let $n \in \mathbb{N}$ and $1 < r, p < \infty$. Denote by $\mathbf{M}$ the standard Hardy–Littlewood maximal function on $\mathbb{R}^n$ such that

$$\mathbf{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where $B_r(x)$ is the ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a locally integrable function. Fefferman and Stein [4] proved the following vector valued maximal inequality:

$$\Big\| \Big(\sum_j (\mathbf{M}f_j)^r \Big)^{\frac{1}{r}} \Big\|_{L^p(\mathbb{R}^n)} \leq C_{n,p,r} \Big\| \Big(\sum_j |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(\mathbb{R}^n)}, \tag{1.1}$$

where $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $j \in \mathbb{N}$, and $C_{n,p,r} > 0$ is a constant depending only on n, p, r.

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The present paper addresses the corresponding vector valued inequalities in a type of Sobolev multiplier spaces. Let $0<\alpha<\infty$ and $1<s\leq n/\alpha$. Recall the definition of Bessel capacities $\mathrm{Cap}_{\alpha,s}(\cdot)$,

$$\mathrm{Cap}_{\alpha,s}(E)=\inf\left\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s:\varphi\geq 0,\ G_\alpha*\varphi\geq 1\text{ on }E\right\},$$

where $E\subseteq\mathbb{R}^n$ is an arbitrary set, $G_\alpha(x)=\mathcal{F}^{-1}[(1+|\cdot|^2)^{-\alpha/2}](x)$, $x\in\mathbb{R}^n$ are the Bessel kernels, and $\mathcal{F}^{-1}$ is the inverse distributional Fourier transform on $\mathbb{R}^n$. The Sobolev multiplier spaces $M_p^{\alpha,s}$ associated with Bessel capacities are then defined to be the subclass of the locally p-integrable space $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^{\alpha,s}}=\sup_K\left(\frac{\int_K|f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}}<\infty$$

whenever $f\in M_p^{\alpha,s}$. Here, the supremum is taken over all compact sets $K\subseteq\mathbb{R}^n$ with non-zero capacities. It is shown in [7, Theorem 3.1.4] that the quantity $\|f\|_{M_p^{\alpha,s}}$ is equivalent to the best constant $C>0$ making the following trace inequality hold:

$$\left(\int_{\mathbb{R}^n}|(G_\alpha*\varphi)(x)|^s|f(x)|^p dx\right)^{\frac{1}{p}}\leq C\|\varphi\|_{L^s(\mathbb{R}^n)}^{\frac{s}{p}}$$

for all $\varphi\in L^s(\mathbb{R}^n)$. Several characterizations of the Sobolev multiplier spaces $M_p^{\alpha,s}$ are studied in [8]. I. E. Verbitsky showed in [6, Theorem, Section 2.6.3] that

$$\|\mathbf{M}f\|_{M_p^{\alpha,s}}\leq C_{n,\alpha,s,p}\|f\|_{M_p^{\alpha,s}} \tag{1.2}$$

(see also Proposition 2.2 below). Motivated by (1.1) and (1.2), one may reasonably suspect if the following Fefferman–Stein type vector valued inequality

$$\left\|\left(\sum_j(\mathbf{M}f_j)^r\right)^{\frac{1}{r}}\right\|_{M_p^{\alpha,s}}\leq C_{n,\alpha,s,p,r}\left\|\left(\sum_j|f_j|^r\right)^{\frac{1}{r}}\right\|_{M_p^{\alpha,s}} \tag{1.3}$$

holds. Unfortunately, it turns out that (1.3) does not hold, which we will show in the proof of Theorem 1.1 in Section 2. Nevertheless, the following local type of vector valued inequality is true.

Theorem 1.1. *Let $n\in\mathbb{N}$, $0<\alpha<\infty$, $1<s\leq n/\alpha$, and $1<p,r<\infty$. Define the local Hardy–Littlewood maximal function $\mathbf{M}^{\mathrm{loc}}$ by*

$$\mathbf{M}^{\mathrm{loc}}f(x)=\sup_{0<r\leq 1}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|dy,\quad x\in\mathbb{R}^n,$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*. Then the vector valued inequality*

$$\Big\| \Big(\sum_j \big(\mathbf{M}^{\mathrm{loc}} f_j \big)^r \Big)^{\frac{1}{r}} \Big\|_{M_p^{\alpha,s}} \le C_{n,\alpha,s,p,r} \Big\| \Big(\sum_j |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{M_p^{\alpha,s}} \tag{1.4}$$

holds for any sequence $\{f_j\}$ *of functions such that* $(\sum_j |f_j|^r)^{\frac{1}{r}} \in M_p^{\alpha,s}$*. The vector valued inequality* (1.4) *does not hold if* $\mathbf{M}^{\mathrm{loc}}$ *is replaced by* $\mathbf{M}$*.*

Proposition 1.2. *The vector valued inequality* (1.4) *fails for* $r = 1$*. In other words, there is a sequence* $\{f_j\}$ *of functions such that*

$$\Big\| \sum_j |f_j| \Big\|_{M_p^{\alpha,s}} < \infty \quad but \quad \Big\| \sum_j \mathbf{M}^{\mathrm{loc}} f_j \Big\|_{M_p^{\alpha,s}} = \infty.$$

The following theorem gives another localized version of the Fefferman–Stein type vector valued inequality in terms of $\mathbf{M}$.

Theorem 1.3. *Let* $n \in \mathbb{N}$*,* $0 < \alpha < \infty$*,* $1 < s \le n/\alpha$*,* $1 < p, r < \infty$*,* $x_0 \in \mathbb{R}^n$*, and* $R_0 > 0$*. Then the vector valued inequality*

$$\Big\| \Big(\sum_j (\mathbf{M} f_j)^r \Big)^{\frac{1}{r}} \cdot \chi_{B_{R_0}(x_0)} \Big\|_{M_p^{\alpha,s}} \le C_{n,\alpha,s,p,r,R_0} \Big\| \Big(\sum_j |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{M_p^{\alpha,s}} \tag{1.5}$$

holds true for any sequence $\{f_j\}$ *of functions such that* $(\sum_j |f_j|^r)^{\frac{1}{r}} \in M_p^{\alpha,s}$*, where* $\mathrm{supp}(f_j) \subseteq B_{R_0}(x_0)$*,* $j \in \mathbb{N}$*, and the constant* $C_{n,\alpha,s,p,r,R_0} > 0$ *depends only on the parameters* n, α, s, p, r, R_0 *but not on* x_0*. The vector valued inequality* (1.5) *fails for* $r = 1$*.*

Assume further that $1 < s < n/\alpha$*. Then, the vector valued inequality*

$$\Big\| \Big(\sum_j (\mathbf{M} f_j)^r \Big)^{\frac{1}{r}} \Big\|_{M_p^{\alpha,s}} \le C_{n,\alpha,s,p,r,R_0} \Big\| \Big(\sum_j |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{M_p^{\alpha,s}} \tag{1.6}$$

holds true for any sequence $\{f_j\}$ *of functions such that* $(\sum_j |f_j|^r)^{\frac{1}{r}} \in M_p^{\alpha,s}$*, where* $\mathrm{supp}(f_j) \subseteq B_{R_0}(x_0)$ *and* $j \in \mathbb{N}$*.*

A function space X is said to be a predual of $M_p^{\alpha,s}$ if the dual space X^* is isomorphic to $M_p^{\alpha,s}$. In other words, for any continuous linear functional $\mathcal{L} \in X^*$, there exists a unique function $g \in M_p^{\alpha,s}$, such that

$$\mathcal{L}(f) = \int_{\mathbb{R}^n} f(x) g(x) dx, \quad f \in X$$

and the operator norm $\|\mathcal{L}\|$ satisfies

$$C^{-1}\|g\|_{M_p^{\alpha,s}} \le \|\mathcal{L}\| \le C\|g\|_{M_p^{\alpha,s}}$$

for some constant $C > 0$. Several preduals of $M_p^{\alpha,s}$ have been characterized in [8], one of which is the space $N_q^{\alpha,s}$, $q = p/(p-1)$, which consists of functions f with the finite quantity

$$\|f\|_{N_q^{\alpha,s}} = \inf_{\omega}\left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)^{1-q} dx\right)^{\frac{1}{q}},$$

where the infimum is taken over all weights $\omega \ge 0$ such that

$$\|\omega\|_{L^1(\mathrm{Cap}_{\alpha,s})} = \int_0^\infty \mathrm{Cap}_{\alpha,s}\big(\{x \in \mathbb{R}^n : \omega(x) > t\}\big)dt \le 1.$$

It is shown in [8, Theorem 1.10] that

$$\|\mathbf{M}^{\mathrm{loc}} f\|_{N_q^{\alpha,s}} \le C_{n,\alpha,s,q}\|f\|_{N_q^{\alpha,s}} \tag{1.7}$$

and the inequality (1.7) cannot be improved by replacing $\mathbf{M}^{\mathrm{loc}}$ with $\mathbf{M}$ since $N_q^{\alpha,s}$ is canonically embedded into $L^1(\mathbb{R}^n)$ (see Proposition 2.1 below) and it is known that $\mathbf{M}$ is unbounded on $L^1(\mathbb{R}^n)$. The vector valued analogue of (1.7) is given by the following theorem.

Theorem 1.4. *Let $n \in \mathbb{N}$, $0 < \alpha < \infty$, $1 < s \le n/\alpha$, and $1 < q, r < \infty$. Then, the vector valued inequality*

$$\left\|\left(\sum_j \left(\mathbf{M}^{\mathrm{loc}} f_j\right)^r\right)^{\frac{1}{r}}\right\|_{N_q^{\alpha,s}} \le C_{n,\alpha,s,q,r}\left\|\left(\sum_j |f_j|^r\right)^{\frac{1}{r}}\right\|_{N_q^{\alpha,s}} \tag{1.8}$$

holds for any sequence $\{f_j\}$ of functions such that $(\sum_j |f_j|^r)^{\frac{1}{r}} \in N_q^{\alpha,s}$. The vector valued inequality (1.8) fails for $r = 1$.

Note that the Fefferman–Stein type vector valued inequalities also hold for Morrey spaces and their preduals in place of $M_p^{\alpha,s}$ in (1.3) (see [5, Theorems 2.9 and 2.12]). Our method to prove Theorems 1.1 and 1.4 can also be applied to the Morrey spaces and their preduals (see Section 4 for details).

2. Proofs of the main results

Proposition 2.1. *Let $n \in \mathbb{N}$, $0 < \alpha < \infty$, $1 < s \le n/\alpha$, and $1 < q < \infty$. It holds that*

$$\|f\|_{L^1(\mathbb{R}^n)} \le C_{n,\alpha,s,q}\|f\|_{N_q^{\alpha,s}}$$

for any function $f \in N_q^{\alpha,s}$. *As a result,* (1.7) *fails by replacing* $\mathbf{M}^{\mathrm{loc}}$ *with* $\mathbf{M}$.

Proof. We first show that

$$\|f\|_{M_p^{\alpha,s}} \le \|f\|_{L^\infty(\mathbb{R}^n)}. \tag{2.1}$$

Indeed, by Young's inequality, one has

$$\|G_\alpha * \varphi\|_{L^s(\mathbb{R}^n)} \le \|G_\alpha\|_{L^1(\mathbb{R}^n)} \|\varphi\|_{L^s(\mathbb{R}^n)} = \|\varphi\|_{L^s(\mathbb{R}^n)},$$

which gives

$$|E| \le \mathrm{Cap}_{\alpha,s}(E), \quad E \subseteq \mathbb{R}^n, \tag{2.2}$$

and hence

$$\begin{aligned}
\|f\|_{M_p^{\alpha,s}} &= \sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)} \right)^{\frac{1}{p}} \\
&\le \|f\|_{L^\infty(\mathbb{R}^n)} \sup_K \left(\frac{\int_K 1 dx}{\mathrm{Cap}_{\alpha,s}(K)} \right)^{\frac{1}{p}} \\
&= \|f\|_{L^\infty(\mathbb{R}^n)} \sup_K \left(\frac{|K|}{\mathrm{Cap}_{\alpha,s}(K)} \right)^{\frac{1}{p}} \\
&\le \|f\|_{L^\infty(\mathbb{R}^n)},
\end{aligned}$$

which yields (2.1). Now, we use the fact that $N_q^{\alpha,s}$ is isomorphic to the Köthe dual $(M_p^{\alpha,s})'$ defined by

$$\|f\|_{(M_p^{\alpha,s})'} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| dx : g \in M_p^{\alpha,s},\ \|g\|_{M_p^{\alpha,s}} \le 1 \right\}, \quad q = \frac{p}{p-1}$$

(see [8, Theorem 1.7]). As a result, we obtain by (2.1) that

$$\begin{aligned}
\|f\|_{L^1(\mathbb{R}^n)} &= \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| dx : g \in L^\infty(\mathbb{R}^n),\ \|g\|_{L^\infty(\mathbb{R}^n)} \le 1 \right\} \\
&\le \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| dx : g \in M_p^{\alpha,s},\ \|g\|_{M_p^{\alpha,s}} \le 1 \right\} \\
&= \|f\|_{(M_p^{\alpha,s})'} \\
&\le C_{n,\alpha,s,q} \|f\|_{N_q^{\alpha,s}},
\end{aligned}$$

which yields the result. ∎

Readers may find that Verbitsky's proof of (1.2) in [6, Theorem, Section 2.6.3] is not complete, as the assumption $s < n/\alpha$ is made throughout the argument. We supply the proof which includes the case $s = n/\alpha$ in the following proposition.

Proposition 2.2. *Let $n \in \mathbb{N}$, $0 < \alpha < \infty$, $1 < s \leq n/\alpha$, and $1 < p < \infty$. It holds that*

$$\|\mathbf{M} f\|_{M_p^{\alpha,s}} \leq C_{n,\alpha,s,p}\|f\|_{M_p^{\alpha,s}}$$

for any function $f \in M_p^{\alpha,s}$.

Proof. Introduce

$$\widetilde{\mathbf{M}} f(x) = \sup_{r>1}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|dy, \quad x \in \mathbb{R}^n,$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. It is clear that $\mathbf{M} f \leq \mathbf{M}^{\mathrm{loc}} f + \widetilde{\mathbf{M}} f$. We first show that

$$\|\widetilde{\mathbf{M}} f\|_{M_p^{\alpha,s}} \leq C_{n,\alpha,s,p}\|f\|_{M_p^{\alpha,s}}. \tag{2.3}$$

Recall the estimate

$$C_{n,\alpha,s}^{-1}|B_r(x)| \leq \mathrm{Cap}_{\alpha,s}(B_r(x)) \leq C_{n,\alpha,s}|B_r(x)|, \quad r > 1, \quad x \in \mathbb{R}^n$$

(see [7, Proposition 3.1.4 (iii)]). Then, we have by Hölder's inequality that

$$\widetilde{M} f(x) \leq \sup_{r>1}\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|^p dy\right)^{\frac{1}{p}} \leq C_{n,\alpha,s,p}\|f\|_{M_p^{\alpha,s}}.$$

By using (2.2), one has

$$\|\widetilde{M} f\|_{M_p^{\alpha,s}} \leq C_{n,\alpha,s,p}\sup_K\left(\frac{|K|}{\mathrm{Cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}}\|f\|_{M_p^{\alpha,s}} = C_{n,\alpha,s,p}\|f\|_{M_p^{\alpha,s}},$$

which yields (2.3). It remains to show that

$$\|\mathbf{M}^{\mathrm{loc}} f\|_{M_p^{\alpha,s}} \leq C_{n,\alpha,s,p}\|f\|_{M_p^{\alpha,s}}. \tag{2.4}$$

To this end, let $K \subseteq \mathbb{R}^n$ be a compact set with $\mathrm{Cap}_{\alpha,s}(K) > 0$. Then, there exists a $V^K \geq 1$ on $K \setminus N$ for some set $N \subseteq \mathbb{R}^n$ with $\mathrm{Cap}_{\alpha,s}(N) = 0$ and

$$(V^K)^\delta \in A_1^{\mathrm{loc}}, \tag{2.5}$$

$$\left[(V^K)^\delta\right]_{A_1^{\mathrm{loc}}} \leq c(n,\alpha,s,\delta),$$

$$\left\|(V^K)^\delta\right\|_{L^1(\mathrm{Cap}_{\alpha,s})} \approx \mathrm{Cap}_{\alpha,s}(K), \tag{2.6}$$

where $\delta \in (1, n/(n-\alpha))$ for $s < 2$ and $\delta \in (s-1, n(s-1)/(n-\alpha s))$ for $s \geq 2$ (see [8, Lemma 3.2]). Note that in (2.5), a nonnegative weight ω is said to be in the A_1^{loc} class provided that

$$\mathbf{M}^{\text{loc}}\omega(x) \leq C\omega(x) \quad \text{a.e.} \tag{2.7}$$

for some constant $C > 0$. The infimum of all such constants is denoted by $[\omega]_{A_1^{\text{loc}}}$. Recall that if $\omega \in A_1^{\text{loc}}$, then

$$\int_{\mathbb{R}^n} \left(\mathbf{M}^{\text{loc}}\varphi(x)\right)^q \omega(x)dx \leq C(n,q,\mathbf{c}) \int_{\mathbb{R}^n} |\varphi(x)|^q \omega(x)dx$$

for some constant $C(n,q,\mathbf{c}) > 0$ depending only on $n, q, \mathbf{c}$ with $[\omega]_{A_1^{\text{loc}}} \leq \mathbf{c}$ (see [8, Lemma 8.2]). Further, note that $\text{Cap}_{\alpha,s}(N) = 0$ entails $|N| = 0$ by (2.2). Consequently, it follows that

$$\begin{aligned}
\int_K \left(\mathbf{M}^{\text{loc}} f(x)\right)^p dx &\leq \int_{\mathbb{R}^n} \left(\mathbf{M}^{\text{loc}} f(x)\right)^p (V^K)^\delta(x)dx \\
&\leq C_{n,\alpha,s} \int_{\mathbb{R}^n} |f(x)|^p (V^K)^\delta(x)dx \\
&= C_{n,\alpha,s} \int_0^\infty \int_{\{(V^K)^\delta > t\}} |f(x)|^p dxdt \\
&\leq C_{n,\alpha,s} \|f\|_{M_p^{\alpha,s}}^p \int_0^\infty \text{Cap}_{\alpha,s}\left(\{x \in \mathbb{R}^n : (V^K)^\delta(x) > t\}\right)dt \\
&= C_{n,\alpha,s} \|f\|_{M_p^{\alpha,s}}^p \left\|(V^K)^\delta\right\|_{L^1(\text{Cap}_{\alpha,s})} \\
&\leq C_{n,\alpha,s} \|f\|_{M_p^{\alpha,s}}^p \cdot \text{Cap}_{\alpha,s}(K).
\end{aligned}$$

Dividing both sides by $\text{Cap}_{\alpha,s}(K)$ and then taking supremum over all such compact sets K, one obtains the estimate (2.4) and the proof is now complete. ■

Before we proceed to the proof of Theorem 1.1, we recall some classic facts regarding weighted norm inequalities. For $1 < p < \infty$, we say that a nonnegative weight $\omega \in A_p$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C \tag{2.8}$$

for all balls B in $\mathbb{R}^n$. On the other hand, we say that $\omega \in A_1$ if

$$\mathbf{M}\omega(x) \leq C\omega(x) \quad \text{a.e.}$$

for some constant $C > 0$. The infima of all such constants $C > 0$ are denoted by $[\omega]_{A_p}$ for $1 \leq p < \infty$. We have $A_p \subseteq A_q$ for $1 \leq p \leq q < \infty$ with $[\omega]_{A_q} \leq [\omega]_{A_p}$. It is shown in [3, Theorem 3.1] that

$$\int_{\mathbb{R}^n} \Big(\sum_j (\mathbf{M} f_j(x))^r\Big)^{\frac{p}{r}} \omega(x)dx \leq \mathcal{N}([\omega]_{A_p}) \int_{\mathbb{R}^n} \Big(\sum_j |f_j(x)|^r\Big)^{\frac{p}{r}} \omega(x)dx \tag{2.9}$$

for some increasing function $\mathcal{N} : [0,\infty) \to [0,\infty)$, where $1 < p, r < \infty$ and $\omega \in A_p$.

On the other hand, the local version A_p^{loc} for $1 < p < \infty$ is defined to be the class consisting of nonnegative weights ω such that (2.8) is satisfied for all balls with radii less than 1. We have similarly that $A_p^{\mathrm{loc}} \subseteq A_q^{\mathrm{loc}}$ for $1 \leq p \leq q < \infty$ with $[\omega]_{A_q^{\mathrm{loc}}} \leq [\omega]_{A_p^{\mathrm{loc}}}$. Besides that, by [9, Lemma 2.11], one has

$$\int_{\mathbb{R}^n} \Big(\sum_j (\mathbf{M}^{\mathrm{loc}} f_j(x))^r\Big)^{\frac{p}{r}} \omega(x)dx \leq \mathcal{N}'([\omega]_{A_p^{\mathrm{loc}}}) \int_{\mathbb{R}^n} \Big(\sum_j |f_j(x)|^r\Big)^{\frac{p}{r}} \omega(x)dx \tag{2.10}$$

for some increasing function $\mathcal{N}' : [0,\infty) \to [0,\infty)$, where $1 < p, r < \infty$ and $\omega \in A_p^{\mathrm{loc}}$.

Proof of Theorem 1.1. Recall the characterization of $M_p^{\alpha,s}$ as

$$\|\varphi\|_{M_p^{\alpha,s}} \approx \sup_{\omega} \left(\int_{\mathbb{R}^n} |\varphi(x)|^p \omega(x)dx\right)^{\frac{1}{p}}, \tag{2.11}$$

where the supremum is taken over all nonnegative weights $\omega \in A_1^{\mathrm{loc}}$ with the properties $\|\omega\|_{L^1(\mathrm{Cap}_{\alpha,s})} \leq 1$ and $[\omega]_{A_1^{\mathrm{loc}}} \leq \mathbf{c}(n,\alpha,s)$ for some constant $\mathbf{c}(n,\alpha,s) > 0$ depending only on n, α, s (see [8, Theorem 1.2]). As a result, (1.4) follows by combining (2.11) and (2.10). To show that (1.4) does not hold if $\mathbf{M}^{\mathrm{loc}}$ is replaced by $\mathbf{M}$, we consider the following example in the one-dimensional case $\mathbb{R}$ (see also [10, Theorem 1, page 51]). Let $f_j = \chi_{(2^{j-1},2^j]}$ for $j = 1, 2, \ldots$ Then

$$\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}} = \chi_{(1,\infty)} \in L^\infty(\mathbb{R}).$$

Note that $\chi_{(1,\infty)} \in M_p^{\alpha,s}$ by (2.1). Since $\mathbf{M} f_j(x) \geq 1/8$ for all $|x| \leq 2^j$, we obtain

$$\sum_j (\mathbf{M} f_j(x))^r \geq \sum_{j:2^j \geq |x|} \frac{1}{8^r},$$

which yields

$$\sum_j (\mathbf{M} f_j(x))^r = \infty$$

everywhere $x \in \mathbb{R}$ and hence (1.4) fails for these f_j. ∎

Proof of Proposition 1.2. We consider the one-dimensional case $\mathbb{R}$. Let $N \in \mathbb{N}$, $N \geq 3$,

$$f_j = \chi_{\left[\frac{j-1}{N}, \frac{j}{N}\right)}, \quad j = 1, 2, \ldots, N,$$

and $f_j = 0$ for $j \geq N+1$. Fix $j = 1, 2, \ldots, N$ and $x \in [0,1]$. Assume that $x \leq (j-1)/N$. Let $\delta = j/n - x$. Clearly, $0 < \delta < 1$, and hence

$$\mathbf{M}^{\mathrm{loc}} f_j(x) \geq \frac{1}{|(x-\delta, x+\delta)|} \int_{x-\delta}^{x+\delta} f_j(y) dy = \frac{\frac{1}{N}}{2(\frac{j}{N} - x)}.$$

If $(j-1)/N < x < j/N$, then by choosing $0 < \delta < 1$ small enough such that $(x-\delta, x+\delta) \subseteq ((j-1)/N, j/N)$, one has $\mathbf{M}^{\mathrm{loc}} f_j(x) \geq 1$. Subsequently, if $x \geq j/N$, then by letting $\delta = x - (j-1)/N$, we have $0 < \delta < 1$ and hence

$$\mathbf{M}^{\mathrm{loc}} f_j(x) \geq \frac{1}{|(x-\delta, x+\delta)|} \int_{x-\delta}^{x+\delta} f_j(y) dy = \frac{\frac{1}{N}}{2(x - \frac{j-1}{N})}.$$

We conclude that for $x \in [0,1]$, it follows that

$$\mathbf{M}^{\mathrm{loc}} f_j(x) \geq \begin{cases} \frac{1}{2j}, & x \leq \frac{j-1}{N}, \\ \frac{1}{2(N-j+1)}, & x > \frac{j-1}{N}. \end{cases}$$

Consequently, for $x \in [0,1]$, we have

$$\begin{aligned} \sum_j \mathbf{M}^{\mathrm{loc}} f_j(x) &= \sum_{\substack{1 \leq j \leq N \\ x \leq \frac{j-1}{N}}} \mathbf{M}^{\mathrm{loc}} f_j(x) + \sum_{\substack{1 \leq j \leq N \\ x > \frac{j-1}{N}}} \mathbf{M}^{\mathrm{loc}} f_j(x) \\ &\geq \frac{1}{2} \sum_{\substack{1 \leq j \leq N \\ x \leq \frac{j-1}{N}}} \frac{1}{j} + \frac{1}{2} \sum_{\substack{1 \leq j \leq N \\ x > \frac{j-1}{N}}} \frac{1}{N-j+1} \\ &\geq \frac{1}{2} \sum_{j=2}^{N} \frac{1}{j}. \end{aligned}$$

Note that

$$\sum_j f_j = \chi_{[0,1)} \in M_p^{\alpha,s}$$

by (2.1). However,

$$\Big\|\sum_j \mathbf{M}^{\mathrm{loc}} f_j\Big\|_{M_p^{\alpha,s}}^p \geq \frac{1}{\mathrm{Cap}_{\alpha,s}([0,1])}\Big(\int_0^1 \sum_j \mathbf{M}^{\mathrm{loc}} f_j(x)\Big)^p dx \geq C\sum_{j=2}^N \frac{1}{j} \to \infty$$

as $N \to \infty$. Hence, (1.4) fails for $r = 1$. ∎

Proof of Theorem 1.3. Let $x_0 \in \mathbb{R}^n$, $R_0 > 0$, and $\omega \in A_1^{\mathrm{loc}}$. Then there exists $\overline{\omega} \in A_1$ such that $\overline{\omega} = \omega$ on $B_{R_0}(x_0)$ with $[\overline{\omega}]_{A_1} \leq C_{n,R_0}[\omega]_{A_1}$, where $C_{n,R_0} > 0$ is a constant depending only on n and R_0 but not on x_0 (see [8, Lemma 8.1]).

Note that $\overline{\omega} \in A_p$ with $[\overline{\omega}]_{A_p} \leq [\overline{\omega}]_{A_1}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n}\Big(\sum_j(\mathbf{M}f_j(x))^r\Big)^{\frac{p}{r}} \cdot \chi_{B_{R_0}(x_0)}(x)\omega(x)dx\\
&= \int_{\mathbb{R}^n}\Big(\sum_j(\mathbf{M}f_j(x))^r\Big)^{\frac{p}{r}}\overline{\omega}(x)dx\\
&\leq \mathcal{N}([\overline{\omega}]_{A_1})\int_{\mathbb{R}^n}\Big(\sum_j|f_j|^r\Big)^{\frac{p}{r}}\overline{\omega}(x)dx \qquad (2.12)\\
&= \mathcal{N}([\overline{\omega}]_{A_1})\int_{\mathbb{R}^n}\Big(\sum_j|f_j|^r\Big)^{\frac{p}{r}}\omega(x)dx \qquad (2.13)\\
&\leq \mathcal{N}(C_{n,R_0}[\omega]_{A_1})\int_{\mathbb{R}^n}\Big(\sum_j|f_j|^r\Big)^{\frac{p}{r}}\omega(x)dx
\end{aligned}$$

where we have used (2.9) in (2.12) and (2.13) follows by the fact that $(\sum_j |f_j|^r)^{\frac{p}{r}}$ is supported in $B_{R_0}(x_0)$. As a result, (1.5) follows by combining the above estimates with (2.11).

The example given in the proof of Proposition 1.2 also shows that (1.5) fails for $r = 1$.

The proof of (1.6) is postponed to Section 4 for technical reasons. ∎

Proof of Theorem 1.4. Note that

$$\|\varphi\|_{N_q^{\alpha,s}} \approx \inf_\omega\Big(\int_{\mathbb{R}^n}|\varphi(x)|^q\omega(x)^{1-q}dx\Big)^{\frac{1}{q}}, \qquad (2.14)$$

where the infimum is taken over all nonnegative weights $\omega \in A_1^{\mathrm{loc}}$ with

$$\|\omega\|_{L^1(\mathrm{Cap}_{\alpha,s})} \leq 1 \quad \text{and} \quad [\omega]_{A_1^{\mathrm{loc}}} \leq \mathbf{c}(n,\alpha,s)$$

for some constant $\mathbf{c}(n,\alpha,s) > 0$ depending only on n,α,s (see [8, page 6]). Now, observe that when $\omega \in A_1^{\rm loc} \subseteq A_p^{\rm loc}$, we have $\omega^{1-q} = \omega^{-1/(p-1)} \in A_q^{\rm loc}$, which yields (1.8) by combining (2.14) and (2.10).

The failure of (1.8) when $r = 1$ is exhibited by the same example in the proof of Proposition 1.2. To this end, we first show that $\chi_{[0,1]} \in N_q^{\alpha,s}$. Indeed, by letting $K = [0,1]$ and choosing $\left(V^K\right)^\delta$ as in (2.5), then

$$\begin{aligned}\|\chi_{[0,1]}\|_{N_q^{\alpha,s}}^q &\leq \int_0^1 \left(\frac{(V^K)^\delta}{\|(V^K)^\delta\|_{L^1(\mathrm{Cap}_{\alpha,s})}}\right)^{1-q} dx\\ &\leq \|(V^K)^\delta\|_{L^1(\mathrm{Cap}_{\alpha,s})}^{q-1}\\ &\leq C_{n,\alpha,s,q}\\ &< \infty,\end{aligned}$$

as claimed. Now, we see that for $N = 1,2,\ldots,$

$$\begin{aligned}C\sum_{j=2}^N \frac{1}{j}\|\chi_{[0,1]}\|_{N_q^{\alpha,s}} &\leq \Big\|\sum_j \mathbf{M}^{\rm loc} f_j\Big\|_{N_q^{\alpha,s}}\\ &\leq C_{n,\alpha,s,q}\Big\|\sum_j f_j\Big\|_{N_q^{\alpha,s}}\\ &= C_{n,\alpha,s,q}\|\chi_{[0,1)}\|_{N_q^{\alpha,s}}.\end{aligned}$$

Since $0 < \|\chi_{[0,1]}\|_{L^1(\mathbb{R}^n)} \leq C_{n,\alpha,s,q}\|\chi_{[0,1]}\|_{N_q^{\alpha,s}}$ by Proposition 2.1, taking $N \to \infty$ would yield a contradiction. ∎

3. Vector valued maximal inequalities in Sobolev multiplier spaces associated with Riesz capacities

Let $n \in \mathbb{N}$, $0 < \alpha < \infty$, and $1 < s < n/\alpha$. The Riesz capacities $\mathrm{cap}_{\alpha,s}(\cdot)$ are defined to be

$$\mathrm{cap}_{\alpha,s}(E) = \inf\left\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0,\ I_\alpha * \varphi \geq 1 \text{ on } E\right\},$$

where $E \subseteq \mathbb{R}^n$ is an arbitrary set, $I_\alpha(x) = |x|^{-(n-\alpha)}$, $x \in \mathbb{R}^n$ are the Riesz kernels. The (homogeneous) Sobolev multiplier spaces $\dot{M}_p^{\alpha,s}$ for $1 < p < \infty$ associated with Riesz capacities are defined to be the subspaces of $L_{\rm loc}^p(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{M}_p^{\alpha,s}} = \sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} < \infty$$

whenever $f \in \dot{M}_p^{\alpha,s}$. Here, the supremum is taken over all compact sets $K \subseteq \mathbb{R}^n$ with $\mathrm{cap}_{\alpha,s}(K) > 0$. Denote by $\dot{N}_q^{\alpha,s}$ the preduals of $\dot{M}_p^{\alpha,s}$, where $q = p/(p-1)$. We have similar characterizations as in (2.11) and (2.14):

$$\|\varphi\|_{\dot{M}_p^{\alpha,s}} \approx \sup_{\omega} \left(\int_{\mathbb{R}^n} |\varphi(x)|^p \omega(x) dx \right)^{\frac{1}{p}}, \tag{3.1}$$

where the supremum is taken over all nonnegative weights $\omega \in A_1$, with the properties $\|\omega\|_{L^1(\mathrm{cap}_{\alpha,s})} \leq 1$ and $[\omega]_{A_1} \leq \mathbf{c}(n, \alpha, s)$, and

$$\|\varphi\|_{\dot{N}_q^{\alpha,s}} \approx \inf_{\omega} \left(\int_{\mathbb{R}^n} |\varphi(x)|^q \omega(x)^{1-q} dx \right)^{\frac{1}{q}}, \tag{3.2}$$

where the infimum is taken over all nonnegative weights $\omega \in A_1$ with

$$\|\omega\|_{L^1(\mathrm{cap}_{\alpha,s})} \leq 1$$

and $[\omega]_{A_1} \leq \mathbf{c}'(n, \alpha, s)$. Here, the constants $\mathbf{c}(n, \alpha, s), \mathbf{c}'(n, \alpha, s) > 0$ depend only on n, α, s. As a result, combining (3.1) and (3.2) with (2.9), we have the vector valued maximal inequalities that

$$\left\| \left(\sum_j (\mathbf{M} f_j)^r \right)^{\frac{1}{r}} \right\|_X \leq C_{n,\alpha,s,p,r} \left\| \left(\sum_j |f_j|^r \right)^{\frac{1}{r}} \right\|_X \tag{3.3}$$

for $X = \dot{M}_p^{\alpha,s}$ and $\dot{N}_q^{\alpha,s}$. As in Proposition 1.2, the vector valued inequalities (3.3) fail for $r = 1$.

We remark that the example given in the proof of Theorem 1.1 does not counter the validity of (3.3) for $X = \dot{M}_p^{\alpha,s}$. In fact, $\chi_{(1,\infty)} \notin \dot{M}_p^{\alpha,s}$. To see this, for each $N = 1, 2, \ldots$, we compute that

$$\|\chi_{(1,\infty)}\|_{\dot{M}_p^{\alpha,s}}^p \geq \frac{\int_0^N \chi_{(1,\infty)}(x) dx}{\mathrm{cap}_{\alpha,s}([0, N])} \approx \frac{N-1}{N^{1-\alpha s}} \to \infty$$

as $N \to \infty$. Here, we have used the fact that $\mathrm{cap}_{\alpha,s}(B_r(x)) = r^{n-\alpha s} \cdot \mathrm{cap}_{\alpha,s}(B_1(0))$ (see [1, Theorem 5.1.2]).

It is worth mentioning that the embedding $L^\infty(\mathbb{R}^n) \hookrightarrow \dot{M}_p^{\alpha,s}$ fails, as already shown in the above example. Nevertheless, the following embedding

$$L^{\frac{np}{\alpha s},\infty}(\mathbb{R}^n) \hookrightarrow \dot{M}_p^{\alpha,s} \tag{3.4}$$

is valid. Indeed, by the Sobolev embedding theorem, one has

$$\|I_\alpha * \varphi\|_{L^{s^*}(\mathbb{R}^n)} \leq C_{n,\alpha,s} \|\varphi\|_{L^s(\mathbb{R}^n)}, \quad \frac{1}{s^*} = \frac{1}{s} - \frac{\alpha}{n},$$

which yields

$$|E|^{1-\frac{\alpha s}{n}} \leq C_{n,\alpha,s} \cdot \mathrm{cap}_{\alpha,s}(E), \quad E \subseteq \mathbb{R}^n.$$

As a result, the embedding (3.4) follows since

$$\begin{aligned}
\|f\|_{\dot{M}_p^{\alpha,s}} &= \sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{cap}_{\alpha,s}(K)} \right)^{\frac{1}{p}} \\
&\leq C_{n,\alpha,s,p} \sup_K \left(\frac{\int_K |f(x)|^p dx}{|K|^{1-\frac{\alpha s}{n}}} \right)^{\frac{1}{p}} \\
&= C_{n,\alpha,s,p} \sup_K |K|^{-\frac{1}{p}+\frac{\alpha s}{np}} \left(\int_K |f(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq C_{n,\alpha,s,p} \|f\|_{L^{\frac{np}{\alpha s},\infty}(\mathbb{R}^n)},
\end{aligned}$$

where we have used a standard fact that

$$\|f\|_{L^{q,\infty}(\mathbb{R}^n)} \approx \sup_{0<|E|<\infty} |E|^{-\frac{1}{r}+\frac{1}{q}} \left(\int_E |f(x)|^r dx \right)^{\frac{1}{r}},$$

here, $0 < q < \infty$ and $0 < r < q$.

4. Completion of the proof of Theorem 1.3

In this section, we prove the remaining estimate (1.6) in Theorem 1.3. In this case, the condition $\alpha s < n$ is assumed. We first observe that

$$\|\varphi\|_{M_p^{\alpha,s}} \approx \|\varphi\|_{\dot{M}_p^{\alpha,s}} \tag{4.1}$$

for any function φ with $\mathrm{supp}(\varphi) \subseteq B_{R_0}(x_0)$, where the implicit constants depend only on n, α, s, p, and R_0 but not on x_0. To show (4.1), recall that

$$\mathrm{cap}_{\alpha,s}(E) \leq C_{n,\alpha,s} \cdot \mathrm{Cap}_{\alpha,s}(E), \quad \alpha s < n, \tag{4.2}$$

where $E \subseteq \mathbb{R}^n$ is an arbitrary set (see [1, Proposition 5.1.4 (a)]). Then, it follows that

$$\begin{aligned}
\|\varphi\|_{M_p^{\alpha,s}} &= \sup_K \left(\frac{\int_K |\varphi(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)} \right)^{\frac{1}{p}} \\
&\leq C_{n,\alpha,s,p} \sup_K \left(\frac{\int_K |\varphi(x)|^p dx}{\mathrm{cap}_{\alpha,s}(K)} \right)^{\frac{1}{p}} \\
&= C_{n,\alpha,s,p} \|\varphi\|_{\dot{M}_p^{\alpha,s}}. \qquad (4.3)
\end{aligned}$$

We remark that the support condition of φ is irrelevant in showing (4.3). On the other hand, recall that

$$\|\varphi\|_{M_p^{\alpha,s}} = \sup_K \left(\frac{\int_K |\varphi(x)|^p dx}{\operatorname{Cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} \approx \sup_{K:\operatorname{diam}(K)\leq R_0} \left(\frac{\int_K |\varphi(x)|^p dx}{\operatorname{Cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}},$$

where the last supremum is taken over all compact sets $K \subseteq \mathbb{R}^n$ with $\operatorname{diam}(K) \leq R_0$ (see [7, Corollary 1.2.2]). As a consequence, we have

$$\begin{aligned}
\|\varphi\|_{\dot{M}_p^{\alpha,s}} &= \sup_K \left(\frac{\int_K |\varphi(x)|^p dx}{\operatorname{cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} \\
&= \sup_K \left(\frac{\int_{K\cap \overline{B}_{R_0}(x_0)} |\varphi(x)|^p dx}{\operatorname{cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} \\
&\leq \sup_K \left(\frac{\int_{K\cap \overline{B}_{R_0}(x_0)} |\varphi(x)|^p dx}{\operatorname{cap}_{\alpha,s}(K\cap \overline{B}_{R_0}(x_0))}\right)^{\frac{1}{p}} \\
&\leq \sup_{K:\operatorname{diam}(K)\leq R_0} \left(\frac{\int_K |\varphi(x)|^p dx}{\operatorname{Cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} \\
&\approx \|\varphi\|_{M_p^{\alpha,s}},
\end{aligned}$$

which yields (4.1).

Let $\{f_j\}$ be a sequence of functions such that $\operatorname{supp}(f_j) \subseteq B_{R_0}(x_0)$ for $j \in \mathbb{N}$. Note that $(\sum_j |f_j|^r)^{\frac{1}{r}}$ is supported in $B_{R_0}(x_0)$. Hence, (4.1) and (3.3) yield

$$\begin{aligned}
\left\|\left(\sum_j (\mathbf{M} f_j)^r\right)^{\frac{1}{r}}\right\|_{\dot{M}_p^{\alpha,s}} &\leq C_{n,\alpha,s,p,r}\left\|\left(\sum_j |f_j|^r\right)^{\frac{1}{r}}\right\|_{\dot{M}_p^{\alpha,s}} \\
&\leq C_{n,\alpha,s,p,r,R_0}\left\|\left(\sum_j |f_j|^r\right)^{\frac{1}{r}}\right\|_{M_p^{\alpha,s}}.
\end{aligned}$$

Inequality (1.6) follows by combining the above estimates with (4.3). The proof of Theorem 1.3 is now complete.

5. Vector valued maximal inequalities in Morrey spaces

The Morrey space $L^{p,\lambda}$ for $1 < p < \infty$, $0 < \lambda \leq n$ is defined to be the set of all locally p-integrable functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p,\lambda}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{\lambda-n}\left(\int_{B_r(x)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty.$$

Denote by $q = p/(p-1)$. The corresponding vector valued inequalities for $L^{p,\lambda}$ and their preduals $H^{q,\lambda}$ read as

$$\Big\| \Big(\sum_j (\mathbf{M} f_j)^r \Big)^{\frac{1}{r}} \Big\|_X \le C_{n,\lambda,p,r} \Big\| \Big(\sum_j |f_j|^r \Big)^{\frac{1}{r}} \Big\|_X \tag{5.1}$$

for $X = L^{p,\lambda}$ and $H^{q,\lambda}$. The proof of (5.1) can be obtained similarly to those of Theorems 1.1 and 1.4. Indeed, part of the results in [2] say that

(5.2)
$$\|f\|_{L^{p,\lambda}} \approx \sup\Big\{ \Big(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \Big)^{\frac{1}{p}} : \omega \in A_1,\ \|\omega\|_{L^1(\Lambda^{(\infty)}_{N-\lambda})} \le 1 \Big\},$$

(5.3)
$$\|f\|_{H^{p',\lambda}} \approx \inf\Big\{ \Big(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)^{1-q} dx \Big)^{\frac{1}{q}} : \omega \in A_1,\ \|\omega\|_{L^1(\Lambda^{(\infty)}_{N-\lambda})} \le 1 \Big\},$$

where $\Lambda^{(\infty)}_{N-\lambda}$ is the Hausdorff capacity. Combining (5.2) and (5.3) with (2.9) yields (5.1). This is a different approach than the proof of (5.1) in [5].

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