Groups Geom. Dyn. 19 (2025), 731–796
DOI 10.4171/GGD/810

Thurston's asymmetric metrics for Anosov representations

León Carvajales, Xian Dai, Beatrice Pozzetti, and Anna Wienhard

Abstract. We provide a good dynamical framework allowing to generalize Thurston's asymmetric metric and the associated Finsler norm from Teichmüller space to large classes of Anosov representations. In many cases, including the space of Hitchin representations, this gives a (possibly asymmetric) Finsler distance. In some cases, we explicitly compute the associated Finsler norm.

Contents

1. Introduction

Let S be a connected orientable surface without boundary, with finitely many punctures and negative Euler characteristic. The *Teichmüller space* $\mathrm{Teich}(S)$ of S is the space of isotopy classes of complete, finite area hyperbolic structures on S. For a pair of points $g_1, g_2 \in \mathrm{Teich}(S)$, Thurston [81] introduces the function

$$d_{\mathrm{Th}}(g_1, g_2) := \log \sup_c \left(\frac{L_{g_2}(c)}{L_{g_1}(c)} \right),$$

where the supremum is taken over all free isotopy classes c of closed curves in S and, for $g \in \mathrm{Teich}(S)$, the number $L_g(c)$ denotes the length of the unique geodesic in the class c,

Mathematics Subject Classification 2020: 53C15.
Keywords: Thurston's asymmetric metrics, Anosov representations, thermodynamic formalism.

with respect to the metric g. In [81, Theorem 3.1], Thurston shows that $d_{\mathrm{Th}}(\cdot,\cdot)$ defines an asymmetric distance on Teich(S) and investigates many properties of this metric. For instance, he shows (see [81, Theorem 8.5]) that $d_{\mathrm{Th}}(g_1, g_2)$ coincides with the least possible Lipschitz constant of homeomorphisms from (S, g_1) to (S, g_2) isotopic to id_S and constructs families of geodesic rays for this metric, called *stretch lines*.

Thurston also constructs a Finsler norm $\|\cdot\|_{\mathrm{Th}}$ on the tangent bundle of Teichmüller space: For $v \in T_g\mathrm{Teich}(S)$, he sets

$$\|v\|_{\mathrm{Th}} := \sup_c \frac{\mathrm{d}_g(L.(c))(v)}{L_g(c)}. \tag{1.1}$$

This is indeed a non-symmetric Finsler norm, namely it is non-negative, nondegenerate, $(\mathbb{R}_{\geq 0})$-homogeneous, and satisfies the triangle inequality. Moreover, Thurston shows that the path metric on Teich(S) induced by this Finsler norm coincides with $d_{\mathrm{Th}}(\cdot,\cdot)$.

Assume now that S is closed. Then Teich(S) identifies with a connected component $\mathfrak{T}(S)$ of the character variety

$$\mathfrak{X}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R})) := \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))/\!\!/\mathsf{PSL}(2,\mathbb{R}).$$

For a conjugacy class $[\gamma]$ in $\pi_1(S)$ and a point $\rho \in \mathfrak{T}(S)$, we set

$$L_\rho^{2\lambda_1}([\gamma]) := 2\lambda_1(\rho(\gamma)),$$

where $\lambda_1(\rho(\gamma))$ denotes the logarithm of the spectral radius of $\rho(\gamma)$. Identifying isotopy classes of closed curves in S with conjugacy classes in $\pi_1(S)$, one deduces from Thurston's result that

$$d_{\mathrm{Th}}^{2\lambda_1}(\rho_1, \rho_2) := \sup_{[\gamma]\in[\pi_1(S)]} \log\left(\frac{L_{\rho_2}^{2\lambda_1}([\gamma])}{L_{\rho_1}^{2\lambda_1}([\gamma])}\right) \tag{1.2}$$

defines an asymmetric distance on $\mathfrak{T}(S)$. Similarly, one gets an expression for the associated Finsler norm. The main goal of this note is to generalize this viewpoint, constructing asymmetric metrics and Finsler norms in other representation spaces that share many features with $\mathfrak{T}(S)$, namely spaces of *Anosov representations*, with a particular attention to *Hitchin*, *Benoist*, and *positive representations*.

1.1. Results

For a finitely generated group Γ and a semisimple Lie group G of non-compact type, we denote by $\mathfrak{X}(\Gamma, \mathsf{G})$ the character variety

$$\mathfrak{X}(\Gamma, \mathsf{G}) := \mathrm{Hom}(\Gamma, \mathsf{G})/\!\!/\mathsf{G}.$$

We furthermore denote by $\mathfrak{a}^+$ a chosen Weyl chamber of G and by $\lambda : \mathsf{G} \to \mathfrak{a}^+$ the Jordan projection. A functional $\varphi \in \mathfrak{a}^*$ is *positive on the limit cone* of a representation $\rho \in \mathfrak{X}(\Gamma, \mathsf{G})$

if for all $\gamma \in \Gamma$ of infinite order one has $\varphi(\lambda(\rho(\gamma))) \geq c\|\lambda(\rho(\gamma))\|$ for some $c > 0$ and some norm on $\mathfrak{a}$. With this at hand, for any functional $\varphi \in \mathfrak{a}^*$ positive on the limit cone of $\rho \in \mathfrak{X}(\Gamma, \mathsf{G})$, we can consider its *$\varphi$-marked length spectrum*

$$L_\rho^\varphi(\gamma) := \varphi(\lambda(\rho(\gamma)))$$

and its *φ-entropy*

$$h_\rho^\varphi := \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma] : L_\rho^\varphi(\gamma) \leq t\} \in [0, \infty].$$

If $\mathfrak{X} \subset \mathfrak{X}(\Gamma, \mathsf{G})$ is a subset, let $\varphi \in \mathfrak{a}^*$ be a functional positive on the limit cone of each representation $\rho \in \mathfrak{X}$. Naively, one would like to define $d_{\mathrm{Th}}^\varphi : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R} \cup \{\infty\}$ by

$$d_{\mathrm{Th}}^\varphi(\rho_1, \rho_2) := \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{L_{\rho_2}^\varphi(\gamma)}{L_{\rho_1}^\varphi(\gamma)}\right) \tag{1.3}$$

and prove that it defines an asymmetric metric for some specific choices of $\mathfrak{X}$. However, in this general setting, there could exist pairs of representations so that the φ-length spectrum of ρ_1 is uniformly larger than the φ-length spectrum of ρ_2: With the above definition, in that situation we would have $d_{\mathrm{Th}}^\varphi(\rho_1, \rho_2) < 0$ (see Remark 6.5 and references therein). To resolve this issue, we normalize the length ratio by the entropy:

$$d_{\mathrm{Th}}^\varphi(\rho_1, \rho_2) := \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h_{\rho_2}^\varphi}{h_{\rho_1}^\varphi} \frac{L_{\rho_2}^\varphi(\gamma)}{L_{\rho_1}^\varphi(\gamma)}\right)$$

(see Definition 6.1 for more details in the case when Γ has torsion). Observe that in the case when $\mathfrak{X}$ is the Teichmüller space, $h_\rho^{2\lambda_1} = 1$, and thus, this definition is compatible with the one given in equation (1.2).

By construction, d_{Th}^φ satisfies the triangular inequality. Our first result determines a setting in which such function is furthermore positive and separates points. For this, we consider the definition of the space of *Θ-Anosov representations*, an open subset of the character variety $\mathfrak{X}(\Gamma, \mathsf{G})$ depending on a subset Θ of the set of simple roots Π of G (we refer the reader to Section 4 for the precise definition). For any such set Θ, we denote by

$$\mathfrak{a}_\Theta := \bigcap_{\alpha\in\Pi\setminus\Theta} \ker \alpha$$

and by $\mathfrak{a}_\Theta^* < \mathfrak{a}^*$ the set functionals invariant under the unique projection $p_\Theta : \mathfrak{a} \to \mathfrak{a}_\Theta$ invariant under the subgroup W_Θ of the Weyl group of G fixing $\mathfrak{a}_\Theta$ pointwise.

Theorem 1.1 (See Theorems 6.2 and 6.8). *Assume that G is connected, real algebraic, simple, and center free. Assume furthermore that $\mathfrak{X} \subset \mathfrak{X}(\Gamma, \mathsf{G})$ consists only of Zariski-dense Θ-Anosov representations. Let $\varphi \in \mathfrak{a}_\Theta^*$ be positive on the limit cone of each representation in $\mathfrak{X}$, and suppose that an automorphism $\tau : \mathsf{G} \to \mathsf{G}$ leaving φ invariant is necessarily inner. Then $d_{\mathrm{Th}}^\varphi(\cdot, \cdot)$ defines a (possibly asymmetric) metric on $\mathfrak{X}$.*

The Thurston distance on the Teichmüller space of a closed surface is complete; however, in general the distance $d^{\varphi}_{\mathrm{Th}}$ might be incomplete also due to the entropy renormalization. This is, for example, the case for the Teichmüller space of surfaces with boundary of variable length. It would be interesting to investigate the relation between suitable metric completions and subsets of the length spectrum compactification, as introduced in [62].

Provided we have a good understanding of all possible Zariski closures in a given subset $\mathfrak{X} \subset \mathfrak{X}(\Gamma, \mathsf{G})$, we can weaken the Zariski density assumption. This is, for instance, the case for the set of *Benoist representations*. A Benoist representation is a representation $\rho : \Gamma \to \mathsf{PGL}(d+1, \mathbb{R})$ that preserves and acts cocompactly on a strictly convex domain $\Omega_\rho \subset \mathbb{P}(\mathbb{R}^d)$. We let $\mathrm{Ben}_d(\Gamma)$ be the space of conjugacy classes of Benoist representations, which by work of Koszul [53] and Benoist [11] is a union of connected components of the character variety $\mathfrak{X}(\Gamma, \mathsf{PGL}(d+1, \mathbb{R}))$. Benoist representations are Θ-Anosov for $\Theta = \{\alpha_1, \alpha_d\}$ (see [10] and [38, Proposition 6.1]). In particular, the logarithm of the spectral radius λ_1 and the *Hilbert length function* $\mathrm{H} := \lambda_1 - \lambda_{d+1}$ belong to $\mathfrak{a}^*_\Theta$. Here we recall that $\lambda_{d+1}(g)$ denotes the logarithm of the smallest eigenvalue of g.

Since Benoist computed the possible Zariski closures of a Benoist representation [8], the argument of Theorem 1.1 can be pushed further to show the following.

Theorem 1.2 (See Corollary 8.3 and Remark 8.4). *The following holds:*

(1) *The function* $d^{\lambda_1}_{\mathrm{Th}} : \mathrm{Ben}_d(\Gamma) \times \mathrm{Ben}_d(\Gamma) \to \mathbb{R}$ *given by*

$$d^{\lambda_1}_{\mathrm{Th}}(\rho, \widehat{\rho}) := \log \left(\sup_{[\gamma] \in [\Gamma]} \frac{h^{\lambda_1}_{\widehat{\rho}}}{h^{\lambda_1}_{\rho}} \frac{L^{\lambda_1}_{\widehat{\rho}}(\gamma)}{L^{\lambda_1}_{\rho}(\gamma)} \right)$$

defines a (possibly asymmetric) distance on $\mathrm{Ben}_d(\Gamma)$.

(2) *The function* $d^{\mathrm{H}}_{\mathrm{Th}} : \mathrm{Ben}_d(\Gamma) \times \mathrm{Ben}_d(\Gamma) \to \mathbb{R}$ *given by*

$$d^{\mathrm{H}}_{\mathrm{Th}}(\rho, \widehat{\rho}) := \log \left(\sup_{[\gamma] \in [\Gamma]} \frac{h^{\mathrm{H}}_{\widehat{\rho}}}{h^{\mathrm{H}}_{\rho}} \frac{L^{\mathrm{H}}_{\widehat{\rho}}(\gamma)}{L^{\mathrm{H}}_{\rho}(\gamma)} \right)$$

is non-negative, and one has

$$d^{\mathrm{H}}_{\mathrm{Th}}(\rho, \widehat{\rho}) = 0 \Leftrightarrow \rho = \widehat{\rho} \quad \text{or} \quad \widehat{\rho} = \rho^\star,$$

where $\rho^\star$ *is the contragredient of* ρ.

A similar result holds for a class of representations of fundamental groups of closed real hyperbolic manifolds into $\mathsf{PO}_0(2, q)$ called *AdS-quasi-Fuchsian*. These were introduced by Mess [60] and Barbot–Mérigot [4, 5] (see Corollary 8.5).

The renormalization by the entropy in equation (1.3) while necessary to ensure positivity might seem inconvenient: It may be difficult to obtain concrete control on the entropy, and thus, the relation between such distance and the best Lipschitz constant of associated equivariant maps is lost. There are, however, natural classes of representations on which the entropy of some explicit functionals in the Levi–Anosov subspace $\mathfrak{a}_\Theta^*$ is constant. For instance, this is the case for the *unstable Jacobian* $\mathrm{J}_{d-1} := d\lambda_1 + \lambda_{d+1}$ on Benoist components, thanks to work of Potrie–Sambarino [65, Corollary 1.7]. In Corollary 8.1, we define the corresponding metric. Another important example is the case of *Hitchin representations*, the representations in the connected component $\mathrm{Hit}(S, \mathsf{G})$ of $\mathfrak{X}(\pi_1(S), \mathsf{G})$, for a split real Lie group G and the fundamental group of a closed surface S, containing the composition of a lattice embedding $\pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$ and the principal embedding $\mathsf{PSL}(2, \mathbb{R}) \to \mathsf{G}$ [30, 54]. Hitchin representations are Anosov with respect to the minimal parabolic [30, 37], so that $\mathfrak{a}_\Theta^* = \mathfrak{a}^*$ and the entropy with respect to all simple roots is constant on $\mathrm{Hit}(S, \mathsf{G})$ and equal to 1, when G is classical [65, 67]. All possible Zariski closures of $\mathsf{PSL}(d, \mathbb{R})$-Hitchin representations have been determined by Guichard[1], and recently a written proof appeared in [73]. This result also covers $\mathsf{PSp}(2r, \mathbb{R})$ and $\mathsf{PSO}(p, p+1)$-Hitchin representations, but not the Hitchin component of $\mathsf{PSO}_0(p, p)$ (see Subsection 7.1 for details). As we explain in Subsection 7.1, Sambarino's approach also works in that case. We deduce the following.

Theorem 1.3 (See Corollary 7.3). *Let* G *be an adjoint, simple, real-split Lie group of classical type. Let* α *be any simple root of* G*, with the exception of the roots listed in Table 1. Then the function* $d_{\mathrm{Th}}^{\alpha} : \mathrm{Hit}(S, \mathsf{G}) \times \mathrm{Hit}(S, \mathsf{G}) \to \mathbb{R}$ *given by*

$$d_{\mathrm{Th}}^{\alpha}(\rho, \widehat{\rho}) := \log \left(\sup_{[\gamma] \in [\Gamma]} \frac{L_{\widehat{\rho}}^{\alpha}(\gamma)}{L_{\rho}^{\alpha}(\gamma)} \right)$$

defines an asymmetric distance on $\mathrm{Hit}(S, \mathsf{G})$.

Also in this case, even for the bad roots we can understand precisely when two representations have distance zero. See Subsection 8.3 for further families of representations for which we can generalize Theorem 1.3; this is notably the case for some

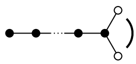
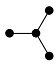
Type	Group	Diagram	Bad roots
A_{2n-1}	$\mathsf{PSL}_{2n}(\mathbb{R})$		$\{\alpha_n\}$
D_n	$\mathsf{PO}(n, n)\ \forall n \geq 5$		$\{\alpha_1, \ldots, \alpha_{n-2}\}$
	$\mathsf{PO}(4, 4)$		$\{\alpha_1, \ldots, \alpha_4\}$

Table 1. The roots marked in black are fixed by a nontrivial automorphism and are therefore not covered by Theorem 1.3.

[1] This is from unpublished work announced by Oliver Guichard.

connected components of Θ-*positive representations* of fundamental groups of surfaces in $\mathsf{PO}(p, p+1)$ [39], which are smooth and conjectured to only consist of Zariski-dense representations [26, Conjecture 1.7].

As a second theme in the paper, we give an explicit formula for the Finsler norm associated with the distance on the set $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ of Θ-Anosov representations. More specifically, we introduce a function $\|\cdot\|^{\varphi}_{\mathrm{Th}} : T\mathfrak{X}_\Theta(\Gamma, \mathsf{G}) \to \mathbb{R} \cup \{\pm\infty\}$ which is defined as follows. For a given tangent vector $v \in T_\rho\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$, we set

$$\|v\|^{\varphi}_{\mathrm{Th}} := \sup_{[\gamma]\in[\Gamma]} \frac{\mathrm{d}_\rho(h^{\varphi}_{\cdot})(v)L^{\varphi}_{\rho}(\gamma) + h^{\varphi}_{\rho}\mathrm{d}_\rho(L^{\varphi}_{\cdot}(\gamma))(v)}{h^{\varphi}_{\rho}L^{\varphi}_{\rho}(\gamma)}.$$

If $\rho \mapsto h^{\varphi}_{\rho}$ is constant, then this expression naturally generalizes Thurston's Finsler norm (1.1). We prove the following.

Proposition 1.4 (See Corollary 6.15). *Let $\{\rho_s\}_{s\in(-1,1)} \subset \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ be a real analytic family and set $\rho := \rho_0$ and $v := \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\rho_s$. Then $s \mapsto d^{\varphi}_{\mathrm{Th}}(\rho, \rho_s)$ is differentiable at $s = 0$ and*

$$\|v\|^{\varphi}_{\mathrm{Th}} = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} d^{\varphi}_{\mathrm{Th}}(\rho, \rho_s).$$

It is natural to ask whether $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ defines a Finsler norm. In this direction, we show the following.

Theorem 1.5 (See Corollary 6.16). *Let $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ be a point admitting an analytic neighborhood in $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$. Then the function $\|\cdot\|^{\varphi}_{\mathrm{Th}} : T_\rho\mathfrak{X}_\Theta(\Gamma, \mathsf{G}) \to \mathbb{R} \cup \{\pm\infty\}$ is real valued and non-negative. Furthermore, it is $(\mathbb{R}_{>0})$-homogeneous, satisfies the triangle inequality, and one has $\|v\|^{\varphi}_{\mathrm{Th}} = 0$ if and only if*

$$d_\rho(L^{\varphi}_{\cdot}(\gamma))(v) = -\frac{\mathrm{d}_\rho(h^{\varphi}_{\cdot})(v)}{h^{\varphi}_{\rho}}L^{\varphi}_{\rho}(\gamma) \tag{1.4}$$

for all $\gamma \in \Gamma$. In particular, if the function $\widehat{\rho} \mapsto h^{\varphi}_{\widehat{\rho}}$ is constant, then

$$\|v\|^{\varphi}_{\mathrm{Th}} = 0 \Leftrightarrow d_\rho(L^{\varphi}_{\cdot}(\gamma))(v) = 0$$

for all $\gamma \in \Gamma$.

Condition (1.4) has been studied by Bridgeman–Canary–Labourie–Sambarino [23,24] in some situations. By applying their results, we obtain the following.

Corollary 1.6 (See Corollaries 7.11 and 7.12). *The functions $\|\cdot\|^{\alpha_1}_{\mathrm{Th}}$ and $\|\cdot\|^{\lambda_1}_{\mathrm{Th}}$ define Finsler norms on* $\mathrm{Hit}_d(S) := \mathrm{Hit}(S, \mathsf{PSL}(d, \mathbb{R}))$.

We do not know, in this general setting, if the length metric induced by the Finsler norm $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ agrees with the distance $d^{\varphi}_{\mathrm{Th}}$: Indeed, it is not clear if the latter distance is geodesic.

Our final result is an application of Labourie–Wentworth's computation of the derivative of some length functions on $\mathrm{Hit}_d(S)$ along some special directions [55]. By the work of Hitchin [44], fixing a Riemann surface structure X_0 on S, we can parametrize $\mathrm{Hit}_d(S)$ by a vector space of holomorphic differentials (of different degrees) over X_0. Given a holomorphic differential q of degree k, we associate with a ray $t \mapsto tq$ for $t \geq 0$ a family $\{\rho_t\}_{t\geq 0}$ of Hitchin representations by the abovementioned Hitchin's parametrization. We denote by $v(q) \in T_{X_0}\mathrm{Hit}_d(S)$ its tangent direction at $t = 0$. The holomorphic differential q also defines a function $\mathrm{Re}(q) : T^1 X_0 \to \mathbb{R}$. Details for this construction will be given in Subsection 7.2.

Theorem 1.7 (See Proposition 7.13). *There exist constants C_1 and C_2, only depending on d and k, such that for every vector $v = v(q) \in T_{X_0}\mathrm{Hit}_d(S)$ as above, one has*

$$\|v(q)\|_{\mathrm{Th}}^{\lambda_1} = C_1 \sup_{[\gamma]\in[\Gamma]} \int \mathrm{Re}(q)\mathrm{d}\delta_\phi(a_{[\gamma]})$$

and

$$\|v(q)\|_{\mathrm{Th}}^{\alpha_1} = C_2 \sup_{[\gamma]\in[\Gamma]} \int \mathrm{Re}(q)\mathrm{d}\delta_\phi(a_{[\gamma]}),$$

where ϕ denotes the geodesic flow of X_0, $a_{[\gamma]} \subset T^1 X_0$ denotes the ϕ-periodic orbit corresponding to $[\gamma]$, and $\delta_\phi(a_{[\gamma]})$ denotes the ϕ-invariant Dirac probability measure supported on $a_{[\gamma]}$.

1.2. Outline of the proofs

The proofs of our main results follow closely the approach by Guillarmou–Knieper–Lefeuvre [40], which is based on work of Knieper [51] and Bridgeman–Canary–Labourie–Sambarino [23]. In [40], the authors work with the space $\mathfrak{M}$ of isometry classes of negatively curved, entropy one Riemannian metrics on a closed manifold M. For $g \in \mathfrak{M}$ and an isotopy class c of closed curves in M, one may define $L_g(c)$ as we did when g was a point in Teichmüller space. Guillarmou–Knieper–Lefeuvre define

$$d_{\mathrm{Th}}(g_1, g_2) := \log \sup_c \frac{L_{g_2}(c)}{L_{g_1}(c)},$$

where the supremum is taken over all isotopy classes c of closed curves in M. In [40, Proposition 5.4], the authors show

$$d_{\mathrm{Th}}(g_1, g_2) \geq 0 \tag{1.5}$$

for all $g_1, g_2 \in \mathfrak{M}$, and moreover,

$$d_{\mathrm{Th}}(g_1, g_2) = 0 \Leftrightarrow L_{g_1} = L_{g_2}. \tag{1.6}$$

Guillarmou–Lefeuvre's theorem about the local rigidity of the length spectrum [41, Theorem 1] (see also [40, Theorem 1.1]) gives that equation (1.6) is equivalent to $g_1 = g_2$, provided that these two metrics are sufficiently regular and close enough in some appropriate topology. Hence, $d_{\mathrm{Th}}(\cdot,\cdot)$ defines an asymmetric metric on a neighborhood of the diagonal of $\mathfrak{M}' \subset \mathfrak{M}$, where $\mathfrak{M}'$ is the subset of $\mathfrak{M}$ consisting of sufficiently regular metrics (see [40, 41] for details). Guillarmou–Knieper–Lefeuvre also construct an associated Finsler norm [40, Lemma 5.6].

Even though the local rigidity of the length spectrum is a geometric statement, the proofs of (1.5) and (1.6) can be abstracted to a more general dynamical framework inspired from [23, Section 3]. We develop this general dynamical framework in detail in Sections 2 and 3, as well as the specific statements needed for the construction of an asymmetric distance and a Finsler norm in that setting. As we explain, these general constructions can then be applied not only to the space $\mathfrak{M}$ as in Guillarmou–Knieper–Lefeuvre but also to other geometric settings, such as spaces of Anosov representations. We expect that this can be applicable in many more geometric contexts.

The general dynamical framework in Guillarmou–Knieper–Lefeuvre's setting arises as follows: Gromov [34] observed that the geodesic flows of any two $g_1, g_2 \in \mathfrak{M}$ are *orbit equivalent*. Roughly speaking, this means that the two flows have the same orbits, traveled at possibly different "speeds" (see Subsection 2.1 for details). The change of speed (or *reparametrization*) is encoded by a positive Hölder continuous function $r = r_{g_1,g_2}$ on the unit tangent bundle $X := T^1M$ of M. To be more precise, the function r_{g_1,g_2} is only well defined up to an equivalence relation, called *Livšic cohomology* (see Definition 2.2). Thus, we work in the general dynamical setting of studying the "geometry" of the space $\mathcal{L}_1(X)$ of Livšic cohomology classes of entropy one Hölder functions on X over the geodesic flow ϕ of g_1.

Since ϕ is an Anosov flow, one may study $\mathcal{L}_1(X)$ through the lens of *Thermodynamic formalism* (see Subsection 2.3). Crucial for us is the following rigidity result by Bridgeman–Canary–Labourie–Sambarino [23, Proposition 3.8] (see Proposition 2.18): There exists a distinguished ϕ-invariant probability measure $m^{\mathrm{BM}}(\phi)$ so that

$$\int r \, \mathrm{d}m^{\mathrm{BM}}(\phi) \geq 1 \tag{1.7}$$

and equality holds if and only if r is Livšic cohomologous to the constant function 1, namely the periods of periodic orbits of ϕ and the reparametrized flow by r coincide. Thus,

$$\sup_m \int r \, \mathrm{d}m \geq 1, \tag{1.8}$$

where the supremum is taken over all ϕ-invariant probability measures, and equality in the above formula holds if and only if r is Livšic cohomologous to 1. By Proposition 2.15, the quantity in (1.8) coincides with the supremum of ratios of periods of periodic orbits for ϕ and the reparametrized flow by r. These general dynamical considerations, when applied specifically to reparametrizing functions associated with $g_1, g_2 \in \mathfrak{M}$, readily imply (1.5) and (1.6).

Now as in [23] for their construction of a *pressure metric* (see Subsections 1.3 and 3.3 for a detailed comparison), the above general approach can also be applied to study spaces of Anosov representations. We use Sambarino's reparametrization theorem [72] (see Theorem 5.2) to map $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ to a space of Livšic cohomology classes of Hölder functions over *Gromov's geodesic flow* $\mathsf{U}\Gamma$ of Γ. More precisely, we associate with each Anosov representation ρ and each $\varphi \in \mathfrak{a}_\Theta^*$ a Hölder reparametrization of the geodesic flow $\mathsf{U}\Gamma$ encoding the φ-spectral data of ρ. This procedure is more involved than in the case of negatively curved metrics, not only because it depends on the additional choice of the functional φ but also because the entropy of φ is, in general, non-constant. While, when working with the space $\mathfrak{M}$ one can bypass this problem by normalizing the metric, this is not a natural procedure in our setting, this is why the extra normalization appears in the expression for $d_{\mathrm{Th}}^\varphi(\cdot,\cdot)$ (see Remark 2.16 for further comments on this point). Nevertheless, Bridgmeman–Canary–Labourie–Sambarino's rigidity statement (1.7) is adapted to the setting of arbitrary entropy and we deduce

$$d_{\mathrm{Th}}^\varphi(\rho_1, \rho_2) \geq 0 \tag{1.9}$$

for all $\rho_1, \rho_2 \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$, and moreover,

$$d_{\mathrm{Th}}^\varphi(\rho_1, \rho_2) = 0 \Leftrightarrow h_{\rho_1}^\varphi L_{\rho_1}^\varphi = h_{\rho_2}^\varphi L_{\rho_2}^\varphi, \tag{1.10}$$

which are the exact analogs of equations (1.5) and (1.6).

To finish the proof of Theorems 1.1, 1.2, and 1.3, we need to understand under which conditions one can guarantee *renormalized length spectrum rigidity*, that is, under which conditions the equality $h_{\rho_1}^\varphi L_{\rho_1}^\varphi = h_{\rho_2}^\varphi L_{\rho_2}^\varphi$ implies that ρ_1 and ρ_2 are conjugate. As in the case of negatively curved metrics, where length spectrum rigidity is only known to hold locally, this typically requires restriction to a subset of $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$. More precisely, we need to control the Zariski closure G_{ρ_i} of ρ_i, for $i = 1, 2$. Since central elements and compact factors are invisible to the Jordan projection, we must require that G_{ρ_i} is center free and without compact factors. Once this is assumed, and if we assume moreover that G_{ρ_i} is semisimple, renormalized length spectrum rigidity follows essentially from properties of Benoist's limit cone (see Theorem 6.8 and [23, Corollary 11.6]). In some special cases, such as Hitchin components and some components of Benoist and positive representations, these arguments can be pushed further to guarantee global rigidity (see Theorem 7.1 and Section 8).

We study the Finsler norm on $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ following the same approach, namely by finding a general dynamical construction inspired by [40] and then pulling back this construction to spaces of Anosov representations. Observe, however, that in this case we need a more complicated expression than what is available in [40] because we cannot assume that the entropy is constant.

We may summarize the above discussion by saying that the results of this paper are obtained by adapting the corresponding constructions in [40] to the context of Anosov representations: We can rely on the thermodynamic formalism, on which part of the constructions in [40] are based, using the work of Sambarino [72] and Bridgeman–Canary–Labourie–Sambarino [23], and the local rigidity statement needed in [40] is replaced here by rigidity statements for Anosov representations from [23]. One of the strong points of our approach is to find a suitable general setup where both contexts can be encompassed and which might prove useful for other geometric situations.

1.3. Other related work

In [23, 24], the authors construct $\mathtt{Out}(\Gamma)$-invariant analytic Riemannian metrics on $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$: They deduce from the aforementioned rigidity result that the Hessian of the renormalized intersection (1.7) is a semidefinite non-negative form, called the *pressure form*. This can be pulled back to spaces of Anosov representations, sometimes yielding a positive definite form [23, 24]. The construction of this paper is different: Instead of integrating with respect to a given measure and taking a second derivative, we integrate with respect to all invariant measures (see Subsection 3.3 for more detailed comparisons).

The rigidity result in equation (1.7) was previously known to hold in other settings. When restricted to geodesic flows of closed hyperbolic surfaces, this is a reinterpretation of Bonahon's intersection rigidity theorem [17, p. 156] (see Appendix A for more details). More generally, that same result was known to hold for pairs of convex co-compact, rank 1 representations ρ_1 and ρ_2 of a word hyperbolic group Γ (see Burger [25, p. 219]). Burger's results readily imply that

$$d_{\mathrm{Th}}(\rho_1, \rho_2) := \log \sup_{[\gamma]\in[\Gamma]} \frac{h_{\rho_2}}{h_{\rho_1}} \frac{L_{\rho_2}(\gamma)}{L_{\rho_1}(\gamma)}$$

defines an asymmetric distance on a subset of the space of conjugacy classes of convex cocompact representations $\Gamma \to \mathsf{G}$, where G has real rank 1 (note that in a rank 1 situation the choice of a functional φ is irrelevant). Burger also relates the number

$$\sup_{[\gamma]\in[\Gamma]} \frac{L_{\rho_2}(\gamma)}{L_{\rho_1}(\gamma)} \tag{1.11}$$

with one of the asymptotic slopes of the corresponding *Manhattan curve* (see [25, Theorem 1]). Guéritaud–Kassel [36, Proposition 1.13] extend Burger's asymmetric metric to some not necessarily convex cocompact representations into the isometry group of the real hyperbolic space. They also show that in some situations, the value (1.11) coincides with the best possible Lipschitz constant for maps between the two underlying real hyperbolic manifolds.

Our construction of the asymmetric metric is done on a very general dynamical setting and pulled back to Anosov representation spaces through Sambarino's reparametrization theorem. For reparametrizations of the geodesic flow of a closed surface, a construction with similar flavor was introduced by Tholozan [80, Theorem 1.31]. His construction leads to a symmetric distance, and it is described in terms of the projective geometry of some appropriate Banach space (see [80] and Remark 3.3 for further details). It would be intriguing to understand the relation between Tholozan's construction and the approach we carry out here.

1.4. Plan of the paper

In Section 2, we discuss the dynamical setup, and in Section 3, we construct the asymmetric metric and the corresponding Finsler norm in this general setting. In Section 4, we recall the definition and main examples of interest of Anosov representations. In Section 5, we recall Sambarino's reparametrization theorem. In Section 6, we pull back the construction of Section 3 to spaces of Anosov representations and also discuss the renormalized length spectrum rigidity in general. In Sections 7 and 8, we specify the discussion to Hitchin representations, as well as some components of Benoist and positive representations. In Appendix A, we discuss in detail the link between the rigidity statement (1.7) and Bonahon's intersection rigidity theorem.

2. Thermodynamic formalism

We begin by recalling some important terminology and results about the dynamics of topological flows on compact metric spaces. In Subsection 2.1, we recall the notions of Hölder orbit equivalence and Livšic cohomology. In Subsection 2.2, we recall the important concept of pressure and fix some terminology that will be used throughout the paper. In Subsection 2.3, we recall the notion of *Markov coding* of a topological flow and state the main consequences of admitting such a coding. We also recall the notion of *metric Anosov flows*, an important class of flows that admit Markov codings. Finally, in Subsection 2.4, we recall the notion of *renormalized intersection*, which is central in our study of the asymmetric metric. The exposition follows closely Bridgeman–Canary–Labourie–Sambarino [23, Section 3].

2.1. Topological flows, reparametrizations, and (orbit) equivalence

Let $\phi = (\phi_t : X \to X)$ be a Hölder continuous flow on a compact metric space X. In this paper, we always assume that ϕ is *topologically transitive*. This means that ϕ has a dense orbit.

The choice of a continuous function $r : X \to \mathbb{R}_{>0}$ induces a "reparametrization" ϕ^r of the flow ϕ. Informally, this is a flow with the same orbits as ϕ, but traveled at a different "speed." To define this notion properly, we first let $\kappa_r : X \times \mathbb{R} \to \mathbb{R}$ be given by

$$\kappa_r(x,t) := \int_0^t r(\phi_s(x))\mathrm{d}s.$$

The function $\kappa_r(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism for all $x \in X$ and therefore admits an (increasing) inverse $\alpha_r(x,\cdot) : \mathbb{R} \to \mathbb{R}$. That is, we have

$$\kappa_r(x, \alpha_r(x,t)) = \alpha_r(x, \kappa_r(x,t)) = t$$

for all $x \in X$ and $t \in \mathbb{R}$.

Definition 2.1. The *reparametrization* of ϕ by a continuous function $r : X \to \mathbb{R}_{>0}$ is the flow $\phi^r = (\phi^r_t : X \to X)$ defined by the formula

$$\phi^r_t(x) := \phi_{\alpha_r(x,t)}(x)$$

for all $x \in X$ and $t \in \mathbb{R}$. We say that ϕ^r is a *Hölder reparametrization* of ϕ if r is Hölder continuous. We let $\mathrm{HR}(\phi)$ be the set of Hölder reparametrizations of ϕ.

The reader may wonder why we choose the function α_r to reparametrize, instead of directly considering the function κ_r. One reason is the following. Let $\psi \in \mathrm{HR}(\phi)$ and denote by $r_{\phi,\psi}$ the corresponding reparametrizing function, that is, $\psi = \phi^{r_{\phi,\psi}}$. Denote by $\mathcal{O}$ the set of periodic orbits of ψ (note that this set is independent of the choice of ψ). Given $a \in \mathcal{O}$, we denote by $p_\psi(a)$ the period, according to the flow ψ, of the periodic orbit a. Then for every $x \in a$ one has the following equality:

$$\int_0^{p_\phi(a)} r_{\phi,\psi}(\phi_t(x))\mathrm{d}t = p_\psi(a).$$

Hence, by choosing the function $\alpha_{r_{\phi,\psi}}$ (instead of $\kappa_{r_{\phi,\psi}}$), we avoid a cumbersome formula involving the integral of $1/r_{\phi,\psi}$ when computing the periods of the new flow.

If we take another point $\widehat{\psi} \in \mathrm{HR}(\phi)$, then $\widehat{\psi}$ is a reparametrization of ψ, that is, one has $\widehat{\psi} = \psi^{r_{\psi,\widehat{\psi}}}$ for some positive continuous function $r_{\psi,\widehat{\psi}}$. In fact, an explicit computation shows

$$r_{\psi,\widehat{\psi}} = \frac{r_{\phi,\widehat{\psi}}}{r_{\phi,\psi}}. \tag{2.1}$$

As above, for every $a \in \mathcal{O}$ and every $x \in a$ one has

$$\int_0^{p_\psi(a)} r_{\psi,\widehat{\psi}}(\psi_t(x))\mathrm{d}t = p_{\widehat{\psi}}(a). \tag{2.2}$$

There are two notions of equivalence between topological flows that we now recall. A Hölder continuous flow $\phi' = (\phi'_t : X' \to X')$ on a compact metric space X' is said to be *(Hölder) conjugate* to ϕ if there is a (Hölder) homeomorphism $h : X \to X'$ satisfying

$$h \circ \phi_t = \phi'_t \circ h$$

for all $t \in \mathbb{R}$. A weaker notion is that of orbit equivalence: The flow $\phi' = (\phi'_t : X' \to X')$ is said to be *(Hölder) orbit equivalent* to ϕ if it is (Hölder) conjugate to a (Hölder) reparametrization of ϕ. One can see that every flow in the orbit equivalence class of ϕ is topologically transitive.

To single out elements in $\mathrm{HR}(\phi)$ which are conjugate to ϕ, one introduces *Livšic cohomology*. To motivate this notion, consider a Hölder continuous function $V : X \to \mathbb{R}$ of class C^1 along ϕ, and let

$$r(x) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} V(\phi_t(x)) \right) + 1.$$

If r is positive, then ϕ^r is conjugate to ϕ. Explicitly, if one defines $h(x) := \phi_{V(x)}(x)$, then

$$h \circ \phi_t^r = \phi_t \circ h$$

for all $t \in \mathbb{R}$.

Definition 2.2. Two Hölder continuous functions $f, g : X \to \mathbb{R}$ are said to be *Livšic cohomologous* (with respect to ϕ) if there is a Hölder continuous function $V : X \to \mathbb{R}$ of class C^1 along the direction of ϕ, so that for all $x \in X$ one has

$$f(x) - g(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} V(\phi_t(x)).$$

In that case we write $f \sim_\phi g$ and denote the Livšic cohomology class of f with respect to ϕ by $[f]_\phi$.

2.2. Invariant measures, entropy, and pressure

For $\psi \in \mathrm{HR}(\phi)$, we denote by $\mathcal{P}(\psi)$ the set of ψ-invariant probability measures on X. This is a convex compact metrizable space. We also let $\mathcal{E}(\psi) \subset \mathcal{P}(\psi)$ be the subset consisting of *ergodic* measures, that is, the subset of measures for which ψ-invariant measurable subsets have measure either equal to zero or one. The set $\mathcal{E}(\psi)$ is the set of extremal points of $\mathcal{P}(\psi)$.

By the Choquet representation theorem (see Walters [85, p. 153]), every element $m \in \mathcal{P}(\psi)$ admits an *ergodic decomposition*. This means that there exists a unique probability measure τ_m on $\mathcal{E}(\psi)$ such that

$$\int_X f(x)\mathrm{d}m(x) = \int_{\mathcal{E}(\psi)} \left(\int_X f(x)\mathrm{d}\mu(x) \right) \mathrm{d}\tau_m(\mu)$$

holds for every continuous function f on X.

The set of periodic orbits of ψ embeds into $\mathcal{P}(\psi)$ as follows: For $a \in \mathcal{O}$, recall that $p_\psi(a)$ is the period of the periodic orbit a according to the flow ψ. We denote by $\delta_\psi(a) \in \mathcal{P}(\psi)$ the *Dirac mass* supported on a, that is, the push-forward of the Lebesgue probability measure on $S^1 \cong [0,1]/\sim$ (where $0 \sim 1$) under the map

$$S^1 \to X : t \mapsto \psi_{p_\psi(a)t}(x),$$

where x is any point in a. Note that $\delta_\psi(a) \in \mathcal{E}(\psi)$. Using equation (2.2), we conclude that for every $\widehat{\psi} \in \mathrm{HR}(\phi)$ one has

$$p_{\widehat{\psi}}(a) = p_\psi(a) \int_X r_{\psi,\widehat{\psi}} \mathrm{d}\delta_\psi(a). \tag{2.3}$$

More generally, for $m \in \mathcal{P}(\psi)$, the map $m \mapsto \widehat{m}$ given by

$$\mathrm{d}\widehat{m} := \frac{r_{\psi,\widehat{\psi}}\mathrm{d}m}{\int r_{\psi,\widehat{\psi}}\mathrm{d}m} \tag{2.4}$$

defines an isomorphism $\mathcal{P}(\psi) \cong \mathcal{P}(\widehat{\psi})$.

We now recall the notion of *topological pressure*, which will be central for our purposes.

Definition 2.3. Let $f : X \to \mathbb{R}$ be a continuous function (or *potential*). The *topological pressure* (or *pressure*) of f is defined by

$$\mathbf{P}(\phi, f) := \sup_{m \in \mathcal{P}(\phi)} \left(h(\phi, m) + \int_X f \, \mathrm{d}m \right), \tag{2.5}$$

where $h(\phi, m)$ is the *metric entropy* of m.

The metric entropy (or *measure theoretic entropy*) $h(\phi, m)$ is defined using m-measurable partition of X and is a metric isomorphism invariant (see [85, Chapter 4]). When there is no risk of confusion, we will omit the flow ϕ in the notation and simply write $\mathbf{P}(f) = \mathbf{P}(\phi, f)$.

A special and important case is the pressure of the potential $f \equiv 0$, which is called the *topological entropy* of ϕ. It is denoted by $h_{\text{top}}(\phi)$, or simply by h_ϕ. The topological entropy is a topological invariant: Conjugate flows have the same topological entropy. In contrast, the topological entropy is not invariant under reparametrizations.

A measure $m \in \mathcal{P}(\phi)$ realizing the supremum in equation (2.5) is called an *equilibrium state* of f. An equilibrium state for $f \equiv 0$ is called a *measure of maximal entropy* of ϕ.

Livšic cohomologous functions share some common invariants defined in thermodynamic formalism.

Remark 2.4. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are Livšic cohomologous functions (w.r.t. ϕ), then $\mathbf{P}(\phi, f) = \mathbf{P}(\phi, g)$ and $m \in \mathcal{P}(\phi)$ is an equilibrium state for f if and only if it is an equilibrium state for g. Indeed, if $f \sim_\phi g$ and $m \in \mathcal{P}(\phi)$, then

$$\int_X f \, \mathrm{d}m = \int_X g \mathrm{d}m.$$

This is a consequence of ϕ-invariance of m and the mean value theorem for derivatives of real functions.

The following is well known and useful.

Proposition 2.5 (Bowen–Ruelle [21, Proposition 3.1], Sambarino [72, Lemma 2.4]). *Let $\phi = (\phi_t : X \to X)$ be a Hölder continuous flow on a compact metric space X and $r : X \to \mathbb{R}_{>0}$ be a Hölder continuous function. Then a real number h satisfies*

$$\boldsymbol{P}(\phi, -hr) = 0$$

if and only if $h = h_{\phi^r}$.

2.3. Symbolic coding and metric Anosov flows

We now specify an important class of topological flows for which pressure, equilibrium states, and Livšic cohomology behave particularly well. The property we are interested in is the existence of a *strong Markov coding* for the flow. Informally speaking, a Markov coding provides a way of modeling the flow by a suspension flow over a shift space. This allows us to obtain many properties about the dynamics of the flow, by studying the corresponding properties at the symbolic level. The reader can find a general introduction on how to model flows by Markov codings and suspension flows in Bowen [20] and Parry–Pollicott [63, Appendix III]. We give a cursory introduction of suspension flows and Markov partitions here.

Suppose (Σ, σ_A) is a two-sided shift of finite type. Given a "roof function" $r : \Sigma \to \mathbb{R}_{>0}$, the *suspension flow* of (Σ, σ_A) under r is the quotient space

$$\Sigma_r := \{(x,t) \in \Sigma \times \mathbb{R} : 0 \le t \le r(x)\}/(x, r(x)) \sim (\sigma_A(x), 0)$$

equipped with the natural flow $\sigma^r_{A,s}(x,t) := (x, t+s)$.

Definition 2.6. A *Markov coding* for the flow $\phi = (\phi_t : X \to X)$ is a 4-tuple $(\Sigma, \sigma_A, \pi, r)$ where (Σ, σ_A) is an irreducible two-sided subshift of finite type, the function $r : \Sigma \to \mathbb{R}_{>0}$ and the map $\pi : \Sigma_r \to X$ are continuous, and the following conditions hold:

- The map π is surjective and bounded-to-one.
- The map π is injective on a set of full measure (for any ergodic measure of full support) and on a dense residual set.
- For all $t \in \mathbb{R}$ one has $\pi \circ \sigma^r_{A,t} = \phi_t \circ \pi$.

If both π and r are Hölder continuous, we call the Markov coding a *strong Markov coding*.

The proof of the following proposition can be found in Sambarino [72, Lemma 2.9].

Proposition 2.7. *Let $\phi = (\phi_t : X \to X)$ be a topological flow admitting a strong Markov coding. Then every flow in the Hölder orbit equivalence class of ϕ admits a strong Markov coding.*

Thanks to the previous proposition, if ϕ admits a strong Markov coding, then every element $\psi \in \mathrm{HR}(\phi)$ also does. This has deep consequences for the dynamics of ψ that we will discuss in this section. However, before doing that we will discuss an important class of topological flows that admit Markov codings, namely *metric Anosov* flows. This class is important to us because, as proved by Bridgeman–Canary–Labourie–Sambarino [23, Sections 4 and 5], every Anosov representation induces a *geodesic flow* which is a topologically transitive and metric Anosov.

Among flows of class C^1 on compact manifolds, *Anosov* flows provide an important class exhibiting many interesting dynamical properties. They were introduced by Anosov [2] in his study of the geodesic flow of closed negatively curved manifolds. Anosov flows were generalized to *Axiom A* flows by Smale [77]; we do not give full definitions here and refer the reader to Smale's original paper. An example of an Axiom A flow which is not Anosov is the geodesic flow of a non-compact convex cocompact real hyperbolic manifold, the restriction of the flow to the set of vectors tangent to geodesics in the convex hull of the limit set shares many dynamical properties with Anosov flows, even though this set is not a manifold. In some contexts (and particularly in the setting we are focusing on), C^1-regularity is too much to expect; *metric Anosov* flows form a class that further generalize Axiom A flows to the topological setting and still share many desirable properties with them. They were introduced by Pollicott [64], who also showed that these flows admit a Markov coding, generalizing the corresponding results for Axiom A flows obtained previously by Bowen [20].

Let $\phi = (\phi_t : X \to X)$ be a continuous flow on a compact metric space X. For $\varepsilon > 0$, we define the ε-*local stable set* of x by

$$W^s_\varepsilon(x) := \{y \in X : d(\phi_t x, \phi_t y) \leq \varepsilon, \forall t \geq 0 \text{ and } d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\}$$

and the ε-*local unstable set* of x by

$$W^u_\varepsilon(x) := \{y \in X : d(\phi_{-t} x, \phi_{-t} y) \leq \varepsilon, \forall t \geq 0 \text{ and } d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to \infty\}.$$

Definition 2.8. A topological flow $\phi = (\phi_t : X \to X)$ is *metric Anosov* if the following conditions hold:

(1) There exist positive constants C, λ, ε such that

$$d(\phi_t(x), \phi_t(y)) \leq Ce^{-\lambda t} d(x, y) \quad \text{for all } y \in W^s_\varepsilon(x) \text{ and } t \geq 0,$$

and

$$d(\phi_{-t}(x), \phi_{-t}(y)) \leq Ce^{-\lambda t} d(x, y) \quad \text{for all } y \in W^u_\varepsilon(x) \text{ and } t \geq 0.$$

(2) There exists $\delta > 0$ and a continuous function v on the set

$$X_\delta := \{(x, y) \in X \times X : d(x, y) \leq \delta\}$$

such that for every $(x, y) \in X_\delta$, the number $v = v(x, y)$ is the unique value for which $W^u_\varepsilon(\phi_v x) \cap W^s_\varepsilon(y)$ is not empty consists of a single point, denoted by $\langle x, y \rangle$.

Theorem 2.9 (Pollicott [64]). *A topologically transitive metric Anosov flow on a compact metric space admits a Markov coding.*

For the rest of the section, we fix a topologically transitive flow $\phi = (\phi_t : X \to X)$ admitting a strong Markov coding. In this case, the entropy of ϕ agrees with the exponential growth rate of periodic orbits:

$$h_\phi = \lim_{t \to \infty} \frac{1}{t} \log \#\{a \in \mathcal{O} : p_\phi(a) \leq t\}. \tag{2.6}$$

Moreover, this number is positive and finite (see Bowen [19] and Pollicott [64]).

Another useful consequence of the existence of a Markov coding is the density of $\mathcal{O}$ in $\mathcal{E}(\phi)$. Combined with the ergodic decomposition (cf., Subsection 2.2), it provides a nice way of relating invariant measures and periodic orbits.

Theorem 2.10. *Let $\phi = (\phi_t : X \to X)$ be a topologically transitive flow admitting a strong Markov coding. Then for every measure $m \in \mathcal{E}(\phi)$, there is a sequence of periodic orbits $\{a_j\} \subset \mathcal{O}$ such that, as $j \to \infty$,*

$$\delta_\phi(a_j) \to m$$

in the weak-⋆ topology.

Proof. This is well known in hyperbolic dynamics (see, e.g., Sigmund [75, Theorem 1] when ϕ is Axiom A). We comment briefly on the ingredients of the proof, since we have not found an explicit reference in our specific setting.

By Pollicott [64, p.195], there is a σ_A-invariant ergodic measure μ on Σ so that $m = \pi_*(\widehat{\mu})$, where $\widehat{\mu}$ is the probability measure on Σ_r induced by the measure on $\Sigma \times \mathbb{R}$ given by

$$\frac{\mu \otimes \mathrm{d}t}{\int r \mathrm{d}\mu}.$$

Hence, it suffices to prove that μ can be approximated by periodic orbits of σ_A. This is a consequence of two dynamical properties of σ_A, called *expansiveness* and the *pseudo-orbit tracing property* (see, e.g., [42, Definition 3.2.11] and [84, Theorem 1]). Indeed, provided these properties, Sigmund's argument [76, Theorem 1] can be carried out in the present framework. ∎

With respect to equilibrium states, we have the following theorem.

Theorem 2.11 (Bowen–Ruelle [21], Pollicott [64], Parry–Pollicott [63, Proposition 3.6]). *Let $\phi = (\phi_t : X \to X)$ be a topologically transitive flow admitting a strong Markov coding. For every Hölder continuous function $f : X \to \mathbb{R}$, there exists a unique equilibrium state $m_f(\phi)$ for f with respect to ϕ. Furthermore, the equilibrium state is ergodic. Finally, if $g : X \to \mathbb{R}$ is Hölder continuous and $m_f(\phi) = m_g(\phi)$, then there exists a constant function c so that $f - g \sim_\phi c$.*

The equilibrium state for $f \equiv 0$ is called the *Bowen–Margulis measure* of ϕ and denoted by $m^{\mathrm{BM}}(\phi)$. For Anosov flows, the existence of this measure was proved by Margulis in his PhD Thesis [58]. Uniqueness was originally conjectured by Bowen [20] and this justifies the name. In a more geometric context, for example, for the geodesic flow of a convex cocompact real hyperbolic manifold, Sullivan [78] gave a description of this measure using Patterson–Sullivan theory. Because of this, the measure of maximal entropy in those contexts is sometimes called the *Bowen–Margulis–Sullivan measure*.

If $f \sim_\phi g$, then the integrals of f and g over every periodic orbit coincide. In the present setting, we also have a converse statement.

Theorem 2.12 (Livšic [56]). *Let $\phi = (\phi_t : X \to X)$ be a topologically transitive flow admitting a strong Markov coding. Suppose that f and g are two Hölder continuous functions such that for all $a \in \mathcal{O}$ and all $x \in a$ one has*

$$\int_0^{p_\phi(a)} f(\phi_t(x))\mathrm{d}t = \int_0^{p_\phi(a)} g(\phi_t(x))\mathrm{d}t.$$

Then $f \sim_\phi g$.

A proof of Livšic's Theorem 2.12 can be found in [83, Theorem 4.3]: Even though it is stated for C^1 hyperbolic flows, the proof only uses the existence of the Markov partition.

The final property of metric Anosov flows we will need is convexity of the pressure function and a characterization of its first derivative in terms of equilibrium states. Let S be a C^k (resp., smooth, analytic) manifold. A family of functions $\{f_s : X \to \mathbb{R}\}_{s \in S}$ is said to be a C^k *(resp., smooth, analytic) family*, if for all $x \in X$, the function $s \mapsto f_s(x)$ is C^k (resp., smooth, analytic).

Proposition 2.13 (Parry–Pollicott [63, Propositions 4.7, 4.10, and 4.12]). *Let $\phi = (\phi_t : X \to X)$ be a topologically transitive flow admitting a strong Markov coding. Then:*

(1) *For every pair of Hölder continuous functions $f, g : X \to \mathbb{R}$, the function*

$$s \mapsto \mathbf{P}(\phi, f + sg)$$

is convex. Furthermore, it is strictly convex if g is not Livšic cohomologous (w.r.t. ϕ) to a constant function.

(2) *Let $\{f_s\}_{s \in (-1,1)}$ be a C^k (resp., smooth, analytic) family of υ-Hölder continuous functions on X. Then $s \mapsto \mathbf{P}(\phi, f_s)$ is a C^k (resp, smooth, analytic) function, and*

$$\left.\frac{\mathrm{d}\mathbf{P}(\phi, f_s)}{\mathrm{d}s}\right|_{s=0} = \int_X \left(\left.\frac{\mathrm{d}f_s}{\mathrm{d}s}\right|_{s=0}\right)\mathrm{d}m_{f_0},$$

where $m_{f_0} = m_{f_0}(\phi)$ is the equilibrium state of f_0 (w.r.t. ϕ).

2.4. Intersection and renormalized intersection

Intersection and renormalized intersection provide a way of "measuring the difference" between two points in $\mathrm{HR}(\phi)$. The notion of intersection was introduced by Thurston in the context of Teichmüller space (see Wolpert [87]) and then reinterpreted by Bonahon [17] (see also Appendix A). Burger [25] generalized this notion to pairs of convex cocompact representations into Lie groups of real rank equal to 1, and noticed a rigid inequality for this number after renormalizing by entropy. Bridgeman–Canary–Labourie–Sambarino [23, Section 3.4] further generalized this (renormalized) intersection in the abstract dynamical setting we are focusing on. We will use these notions to study the asymmetric distance and Finsler norm in $\mathrm{HR}(\phi)$ in Section 3.

Definition 2.14. Let $\psi, \widehat{\psi} \in \mathrm{HR}(\phi)$. For $m \in \mathcal{P}(\psi)$, the *m-intersection number* between $\psi, \widehat{\psi} \in \mathrm{HR}(\phi)$ is defined by

$$\mathbf{I}_m(\psi, \widehat{\psi}) := \int_X r_{\psi,\widehat{\psi}} \mathrm{d}m,$$

where the positive continuous function $r_{\psi,\widehat{\psi}}$ is given by equation (2.1).

Recall that ϕ is a topologically transitive flow admitting a strong Markov coding. Intersection numbers and ratios of periods are linked as follows.

Proposition 2.15. *For every $\psi, \widehat{\psi} \in \mathrm{HR}$, the following equality holds:*

$$\sup_{a \in \mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} = \sup_{m \in \mathcal{P}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}).$$

Proof. The proof follows closely Guillarmou–Knieper–Lefeuvre [40, Lemma 4.10]. We include it for completeness.

First of all, we observe that

$$\sup_{m \in \mathcal{P}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}) = \sup_{m \in \mathcal{E}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}). \tag{2.7}$$

Indeed, let $m_0 \in \mathcal{P}(\psi)$ be such that

$$\sup_{m \in \mathcal{P}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}) = \mathbf{I}_{m_0}(\psi, \widehat{\psi}).$$

By ergodic decomposition (cf., Subsection 2.2), we have

$$\begin{aligned} \mathbf{I}_{m_0}(\psi, \widehat{\psi}) &= \int_{\mathcal{E}(\psi)} \left(\int_X r_{\psi,\widehat{\psi}}(x) \mathrm{d}\mu(x) \right) \mathrm{d}\tau_{m_0}(\mu) \\ &\leq \sup_{m \in \mathcal{E}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}) \times \int_{\mathcal{E}(\psi)} \mathrm{d}\tau_{m_0}(\mu) \\ &= \sup_{m \in \mathcal{E}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}). \end{aligned}$$

The reverse inequality being trivial, this proves equality (2.7).

We now prove

$$\sup_{a \in \mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} \leq \sup_{m \in \mathcal{E}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}).$$

To do that, take a sequence $a_j \in \mathcal{O}$ such that

$$\sup_{a \in \mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} = \lim_{j \to \infty} \frac{p_{\widehat{\psi}}(a_j)}{p_\psi(a_j)}.$$

Since $\mathcal{E}(\psi)$ is compact, we may assume $\delta_\psi(a_j) \to m$ for some $m \in \mathcal{E}(\psi)$. By equation (2.3), we have

$$\sup_{a\in\mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} = \lim_{j\to\infty} \int_X r_{\psi,\widehat{\psi}} \mathrm{d}\delta_\psi(a_j) = \int_X r_{\psi,\widehat{\psi}} \mathrm{d}m \leq \sup_{m\in\mathcal{E}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}).$$

To finish the proof, it remains to show

$$\sup_{a\in\mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} \geq \sup_{m\in\mathcal{E}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}).$$

By Theorem 2.10, given $m \in \mathcal{E}(\psi)$ we may find a sequence $a_j \in \mathcal{O}$ such that $\delta_\psi(a_j) \to m$. Proceeding as above, we have

$$\sup_{a\in\mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} \geq \lim_{j\to\infty} \int_X r_{\psi,\widehat{\psi}} \mathrm{d}\delta_\psi(a_j) = \int_X r_{\psi,\widehat{\psi}} \mathrm{d}m = \mathbf{I}_m(\psi, \widehat{\psi}).$$

The result follows taking supremum over all $m \in \mathcal{E}(\psi)$. ■

The supremum

$$\sup_{m\in\mathcal{P}(\psi)} \mathbf{I}_m(\psi, \widehat{\psi}) = \sup_{m\in\mathcal{P}(\psi)} \int r_{\psi,\widehat{\psi}} \mathrm{d}m$$

is a well-studied quantity in dynamics. Indeed, this number and the measure(s) attaining the sup is the subject of study of *ergodic optimization*. A general belief in this area is that "typically" among sufficiently regular functions, the maximizing measure is unique and supported on a periodic orbit. See Jenkinson [46] and references therein for a nice survey. However, for the geometric applications, we have in mind these types of generic results are not enough. In the specific case of reparametrizing functions arising from points in the Teichmüller space of a closed surface, Thurston gives a description of the measures realizing the sup above: These are always (partially) supported on a topological lamination on the surface, and this lamination is typically a simple closed geodesic (see [81, p.4 and Section 10] for details).

The function $m \mapsto \mathbf{I}_m(\psi, \widehat{\psi})$ is continuous with respect to the weak-$\star$ topology on $\mathcal{P}(\psi)$. Since $\mathcal{P}(\psi)$ is compact, Proposition 2.15 implies

$$\sup_{a\in\mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} < \infty. \tag{2.8}$$

Remark 2.16. Thanks to the above remark, one may try to use directly the log of the number in (2.8) to produce a metric on $\mathrm{HR}(\phi)$. However, the following problem arises. For a constant function $r = c > 1$, we have

$$\log\left(\sup_{a\in\mathcal{O}} \frac{p_\phi(a)}{p_{\phi^r}(a)}\right) = \log\left(\frac{1}{c}\right) < 0.$$

Hence, the quantity in equation (2.8) cannot define a distance in $\mathrm{HR}(\phi)$. This problem also arises in the geometric setting we will focus on (cf., Remark 6.5).

A way of resolving the above issue, natural from the viewpoint of dynamical systems, is to normalize by the entropy. Together with Proposition 2.15, this motivates the following definition.

Definition 2.17. Let $\psi, \widehat{\psi} \in \mathrm{HR}(\phi)$ and $m \in \mathcal{P}(\psi)$. The *m-renormalized intersection* between ψ and $\widehat{\psi}$ is

$$\mathbf{J}_m(\psi, \widehat{\psi}) := \frac{h_{\widehat{\psi}}}{h_\psi} \mathbf{I}_m(\psi, \widehat{\psi}).$$

Considering renormalized intersection fixes the above issue.

Proposition 2.18 (Bridgeman–Canary–Labourie–Sambarino [23, Proposition 3.8]). *For every* $\psi, \widehat{\psi} \in \mathrm{HR}$ *one has*

$$\mathbf{J}_{m^{\mathrm{BM}}(\psi)}(\psi, \widehat{\psi}) \geq 1.$$

Moreover, equality holds if and only if $(h_{\widehat{\psi}} r_{\phi,\widehat{\psi}}) \sim_\phi (h_\psi r_{\phi,\psi})$.

Proof. By equation (2.1), we have

$$\mathbf{J}_{m^{\mathrm{BM}}(\psi)}(\psi, \widehat{\psi}) = \frac{h_{\widehat{\psi}}}{h_\psi} \int \left(\frac{r_{\phi,\widehat{\psi}}}{r_{\phi,\psi}} \right) \mathrm{d}m^{\mathrm{BM}}(\psi).$$

Now the statement becomes precisely that of [23, Proposition 3.8]. ∎

3. Asymmetric metric and Finsler norm for flows

As always, we assume that ϕ is a topologically transitive flow admitting a strong Markov coding. We want to use the formula

$$\log \left(\sup_{a \in \mathcal{O}} \frac{h_{\widehat{\psi}}}{h_\psi} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} \right) = \log \left(\frac{h_{\widehat{\psi}}}{h_\psi} \sup_{a \in \mathcal{O}} \frac{p_{\widehat{\psi}}(a)}{p_\psi(a)} \right)$$

to define a distance on a suitable quotient of $\mathrm{HR}(\phi)$. We begin understanding which pairs are at distance zero:

Lemma 3.1. *For* ψ *and* $\widehat{\psi}$ *in* HR, *the following are equivalent:*

(1) *For every* $a \in \mathcal{O}$, $h_{\widehat{\psi}} p_{\widehat{\psi}}(a) = h_\psi p_\psi(a)$.

(2) $(h_{\widehat{\psi}} r_{\phi,\widehat{\psi}}) \sim_\phi (h_\psi r_{\phi,\psi})$.

(3) $r_{\psi,\widehat{\psi}} \sim_\psi h_\psi / h_{\widehat{\psi}}$.

(4) *There exists a constant function* c *so that* $r_{\psi,\widehat{\psi}} \sim_\psi c$.

Proof. Since ψ and $\widehat{\psi}$ are topologically transitive and admit a strong Markov coding (cf., Proposition 2.7), all results from Section 2 apply. In particular, the equivalence between (3) and (4) follows from equation (2.6).

The implications (2)⇒(1) and (3)⇒(1) are straightforward. The implications (1)⇒(2) and (1)⇒(3) hold thanks to Livšic's Theorem 2.12 (applied to ϕ and ψ, respectively). ■

We say that ψ and $\widehat{\psi}$ in $\mathrm{HR}(\phi)$ are *projectively equivalent* (and denote $\psi \sim \widehat{\psi}$) if any of the equivalent conditions of Lemma 3.1 hold. We denote by $\mathbb{P}\mathrm{HR}(\phi)$ the quotient space under this relation and denote by $[\psi] \in \mathbb{P}\mathrm{HR}(\phi)$ the equivalence class of ψ.

3.1. Asymmetric metric on $\mathbb{P}\mathrm{HR}(\phi)$

Define $d_{\mathrm{Th}} : \mathbb{P}\mathrm{HR}(\phi) \times \mathbb{P}\mathrm{HR}(\phi) \to \mathbb{R}$ by

$$d_{\mathrm{Th}}([\psi], [\widehat{\psi}]) := \log\left(\sup_{a\in\mathcal{O}} \frac{h_{\widehat{\psi}}}{h_\psi}\frac{p_{\widehat{\psi}}(a)}{p_\psi(a)}\right),$$

where ψ and $\widehat{\psi}$ are representatives of $[\psi]$ and $[\widehat{\psi}]$, respectively. Lemma 3.1 guarantees that d_{Th} is well defined, as it does not depend on the choice of these representatives.

Theorem 3.2. *The function d_{Th} defines a (possibly asymmetric) distance on $\mathbb{P}\mathrm{HR}(\phi)$.*

By "possibly asymmetric," we mean that there is no reason to expect that the equality $d_{\mathrm{Th}}([\psi], [\widehat{\psi}]) = d_{\mathrm{Th}}([\widehat{\psi}], [\psi])$ holds for all pairs $[\psi], [\widehat{\psi}] \in \mathbb{P}\mathrm{HR}(\phi)$. In fact, in some specific situations it is possible to show that $d_{\mathrm{Th}}(\cdot,\cdot)$ is indeed asymmetric (cf., Remark 7.10).

Proof. Let $[\psi], [\widehat{\psi}] \in \mathbb{P}\mathrm{HR}(\phi)$ and pick representatives $\psi, \widehat{\psi} \in \mathrm{HR}(\phi)$. By Proposition 2.15, we have

$$d_{\mathrm{Th}}([\psi], [\widehat{\psi}]) = \log\left(\sup_{m\in\mathcal{P}(\psi)} \mathbf{J}_m(\psi, \widehat{\psi})\right).$$

Proposition 2.18 implies

$$\sup_{m\in\mathcal{P}(\psi)} \mathbf{J}_m(\psi, \widehat{\psi}) \geq \mathbf{J}_{m^{\mathrm{BM}}(\psi)}(\psi, \widehat{\psi}) \geq 1,$$

and therefore $d_{\mathrm{Th}}([\psi], [\widehat{\psi}]) \geq 0$. Moreover, if $d_{\mathrm{Th}}([\psi], [\widehat{\psi}]) = 0$, then Proposition 2.18 implies $(h_{\widehat{\psi}} r_{\phi,\widehat{\psi}}) \sim_\phi (h_\psi r_{\phi,\psi})$, which by Lemma 3.1 means $[\psi] = [\widehat{\psi}]$. Since the triangle inequality for $d_{\mathrm{Th}}(\cdot,\cdot)$ is easily verified, the proof is complete. ■

Remark 3.3. When ϕ is a (not necessarily Hölder) continuous parametrization of the geodesic flow of a closed orientable surface of genus $g \geq 2$, Tholozan [80] defined a symmetric distance in $\mathbb{P}\mathrm{HR}(\phi)$ which has similar flavor to our $d_{\mathrm{Th}}(\cdot,\cdot)$. More precisely, he works in the space of (not necessarily Hölder) continuous reparametrizations of ϕ and considers an appropriate equivalence relation on this space, which restricts to $\sim$ in the Hölder setting. Tholozan proves that the quotient space under this equivalence relation sits as an open, weakly proper, convex domain in the projective space of some Banach space. Hence, it carries a natural *Hilbert metric* (see [80, Proposition 1.29] for details). In [80, Theorem 1.31], he gives an expression for this Hilbert metric which is a symmetrized version of $d_{\mathrm{Th}}(\cdot,\cdot)$.

3.2. Finsler norm

We now define a Finsler norm $\|\cdot\|_{\mathrm{Th}}$ on the "tangent space" $T_{[\psi]}\mathbb{P}\mathrm{HR}(\phi)$ of every $[\psi] \in \mathbb{P}\mathrm{HR}(\phi)$ and provide a link with the asymmetric distance $d_{\mathrm{Th}}(\cdot,\cdot)$ (Proposition 3.6). Recall that a *Finsler norm* on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ such that for all $v, w \in V$ and all $a \geq 0$ one has

- $\|v\| \geq 0$, with equality if and only if $v = 0$,
- $\|av\| = a\|v\|$, and
- $\|v + w\| \leq \|v\| + \|w\|$.

Before starting, we need to make sense of the "tangent space" $T_{[\psi]}\mathbb{P}\mathrm{HR}(\phi)$ (cf., also [23, Subsection 3.5.2]). To do this, we express our space of reparametrizations as a level set of the pressure function and apply Proposition 2.13 and the implicit function theorem in Banach spaces [45]. We need to be careful though, because the space of Hölder continuous functions on X is not closed in the topology of uniform convergence. To fix this issue, we will fix a Hölder exponent υ and work restricted to the space $\mathcal{H}^{\upsilon}(X)$ of υ-Hölder functions. In the geometric applications we have in mind, namely for spaces of Anosov representations, this is not a strong assumption as discussed in [23, Section 6] (see also Subsection 6.3).

Fix $\upsilon > 0$ and endow $\mathcal{H}^{\upsilon}(X)$ with the Banach norm

$$\|f\|_{\upsilon} := \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^{\upsilon}},$$

where $\|\cdot\|_{\infty}$ denotes the uniform norm. Let $\mathcal{B}^{\upsilon}(X) \subset \mathcal{H}^{\upsilon}(X)$ be the space of *ϕ-Livšic coboundaries*, that is, the set of υ-Hölder functions on X which are ϕ-Livšic cohomologus to zero. By Livšic's Theorem 2.12, $\mathcal{B}^{\upsilon}(X)$ is a closed (vector) subspace of $\mathcal{H}^{\upsilon}(X)$. We endow the quotient space $\mathcal{L}^{\upsilon}(X) := \mathcal{H}^{\upsilon}(X)/\mathcal{B}^{\upsilon}(X)$ of Livšic cohomology classes in $\mathcal{H}^{\upsilon}(X)$ with the norm

$$[f]_{\phi} \mapsto \inf_{u \in [f]_{\phi}} \|u\|_{\upsilon},$$

which by abuse of notations will also be denoted by $\|\cdot\|_{\upsilon}$. Note that $(\mathcal{H}^{\upsilon}(X), \|\cdot\|_{\upsilon})$ is a Banach space.

Let $\mathrm{HR}^{\upsilon}(\phi)$ be the set of reparametrizations $\psi \in \mathrm{HR}(\phi)$ so that $r_{\phi,\psi} \in \mathcal{H}^{\upsilon}(X)$ and $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ be its projection to $\mathbb{P}\mathrm{HR}(\phi)$. Let $[\psi] \in \mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ be any point and take a representative $\psi \in \mathrm{HR}^{\upsilon}(\phi)$ satisfying $h_{\psi} = 1$. By Proposition 2.5, we have

$$\mathbf{P}(\phi, -r_{\phi,\psi}) = 0.$$

Moreover, if $\widehat{\psi} \in [\psi]$ is another representative satisfying $h_{\widehat{\psi}} = 1$, Lemma 3.1 states that $r_{\phi,\widehat{\psi}} \sim_{\phi} r_{\phi,\psi}$. We then have an injective map from $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ to the space

$$\mathcal{P}^{\upsilon}(X) := \left\{[r]_{\phi} \in \mathcal{L}^{\upsilon}(X) : \mathbf{P}(\phi, -r) = 0\right\}.$$

Hence, $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ identifies with the open subset of $\mathscr{P}^{\upsilon}(X)$ consisting of Livšic cohomology classes of pressure zero, strictly positive, υ-Hölder continuous functions on X. In view of this discussion, throughout this section, all representatives ψ of points $[\psi]$ in $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ are assumed to satisfy $h_{\psi} = 1$.

From now on, we simply denote $[r]_{\phi}$ by $[r]$, omitting the underlying flow ϕ. By Proposition 2.13, for any positive $g \in \mathscr{H}^{\upsilon}(X)$ one has

$$\mathrm{d}_{[r]}\mathbf{P}(\phi, \cdot)([g]) > 0.$$

That same proposition and the implicit function theorem in Banach spaces imply that the tangent space to $\mathscr{P}^{\upsilon}(X)$ at $[r]$ is given by

$$T_{[r]}\mathscr{P}^{\upsilon}(X) = \left\{[g] \in \mathscr{L}^{\upsilon}(X) : \int_X g\mathrm{d}m_{-r} = 0\right\},$$

where $m_{-r} = m_{-r}(\phi)$ denotes the equilibrium state of $-r$ (w.r.t. ϕ). Since $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ sits as an open subset of $\mathscr{P}^{\upsilon}(X)$, it is natural to define the *tangent space* to $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ at $[\psi]$ by

$$T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi) := T_{[r_{\phi,\psi}]}\mathscr{P}^{\upsilon}(X).$$

We are now ready to define our Finsler norm.

Definition 3.4. Let $[g]$ be a vector in $T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$. We define

$$\|[g]\|_{\mathrm{Th}} := \sup_{m \in \mathcal{P}(\phi)} \frac{\int g\mathrm{d}m}{\int r_{\phi,\psi}\mathrm{d}m}.$$

Note that this is well defined, that is, it does not depend on the choice of the representatives g and $r_{\phi,\psi}$ in the respective ϕ-Livšic cohomology classes (cf., Remark 2.4). Furthermore, by equation (2.4), we have the following more succinct expression:

$$\|[g]\|_{\mathrm{Th}} = \sup_{m \in \mathcal{P}(\psi)} \int \left(\frac{g}{r_{\phi,\psi}}\right)\mathrm{d}m. \tag{3.1}$$

By definition of the tangent space, $\|[g]\|_{\mathrm{Th}} \geq 0$. Moreover, $(\mathbb{R}_{>0})$-homogeneity and the triangle inequality are easily verified. Hence, the following shows that $\|\cdot\|_{\mathrm{Th}}$ is a Finsler norm.

Lemma 3.5. *Let* $[g] \in T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ *be such that* $\|[g]\|_{\mathrm{Th}} = 0$. *Then* $[g] = 0$.

Proof. To prove the lemma, it suffices to show that g is Livšic cohomologous (w.r.t. ϕ) to a constant function c. Indeed, if this is the case, then by Remark 2.4, we have

$$c = \int c\mathrm{d}m_{-r_{\phi,\psi}} = \int g\mathrm{d}m_{-r_{\phi,\psi}} = 0.$$

Hence, $[g] = 0$ as desired.

Let us assume by contradiction that g is not Livšic cohomologous to a constant. By Proposition 2.13, the function $s \mapsto \mathbf{P}(\phi, -r_{\phi,\psi} + sg)$ is then strictly convex and

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathbf{P}(\phi, -r_{\phi,\psi} + sg) = \int g \mathrm{d}m_{-r_{\phi,\psi}} = 0.$$

Strict convexity implies then

$$\mathbf{P}(\phi, -r_{\phi,\psi} + g) > \mathbf{P}(\phi, -r_{\phi,\psi}) = 0.$$

On the other hand, we show that $\|[g]\|_{\mathrm{Th}} = 0$ implies $\mathbf{P}(\phi, -r_{\phi,\psi} + g) \leq 0$, giving the desired contradiction. Indeed, note that

$$\mathbf{P}(\phi, -r_{\phi,\psi} + g) \leq \sup_{m \in \mathcal{P}(\phi)} \left(h(\phi, m) - \int r_{\phi,\psi} \mathrm{d}m \right) + \sup_{m \in \mathcal{P}(\phi)} \int g \mathrm{d}m.$$

Since $\|[g]\|_{\mathrm{Th}} = 0$ and $r_{\phi,\psi}$ is positive, we have

$$\sup_{m \in \mathcal{P}(\phi)} \int g \mathrm{d}m \leq 0,$$

and therefore

$$\mathbf{P}(\phi, -r_{\phi,\psi} + g) \leq \sup_{m \in \mathcal{P}(\phi)} \left(h(\phi, m) - \int r_{\phi,\psi} \mathrm{d}m \right) = \mathbf{P}(\phi, -r_{\phi,\psi}) = 0. \qquad \blacksquare$$

We now link the Finsler norm $\|\cdot\|_{\mathrm{Th}}$ and the asymmetric distance $d_{\mathrm{Th}}(\cdot, \cdot)$. A path $\{[\psi^s]\}_{s \in (-1,1)} \subset \mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ is *analytic* (resp., C^k, *smooth*) if there is an analytic (resp., C^k, smooth) path $\{\widetilde{g}_s\}_{s \in (-1,1)} \subset \mathcal{H}^{\upsilon}(X)$ of strictly positive functions so that $\left[\phi^{\widetilde{g}_s}\right] = [\psi^s]$ for all $s \in (-1, 1)$.

Pick a path $\{[\psi^s]\}_{s \in (-1,1)} \subset \mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ of class C^1 and let $\{\widetilde{g}_s\}_{s \in (-1,1)} \subset \mathcal{H}^{\upsilon}(X)$ be as above. By Bridgeman–Canary–Labourie–Sambarino [23, Proposition 3.12], the function $s \mapsto h_{\phi^{\widetilde{g}_s}}$ is of class C^1. Hence, $s \mapsto g_s := h_{\phi^{\widetilde{g}_s}} \widetilde{g}_s$ is also C^1. Furthermore, we have

$$\left[\phi^{g_s}\right] = \left[\phi^{\widetilde{g}_s}\right] = [\psi^s]$$

for all s, and therefore we may choose $\psi^s = \phi^{g_s}$. By construction, we have $h_{\psi^s} = 1$, that is, $\mathbf{P}(\phi, -g_s) = 0$ for all $s \in (-1, 1)$ (Proposition 2.5). If we denote $\dot{g}_0 := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} g_s$, we have

$$[\dot{g}_0] = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} [g_s],$$

and Proposition 2.13 gives

$$0 = \int (-\dot{g}_0) \mathrm{d}m_{-g_0},$$

where $m_{-g_0} = m_{-g_0}(\phi)$ is the equilibrium state of $-g_0$ (w.r.t. ϕ). That is, setting $\psi := \psi^0$, we have $[\dot{g}_0] \in T_{[\psi]} \mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$.

Proposition 3.6. *With the notations above, the function* $s \mapsto d_{\mathrm{Th}}([\psi],[\psi^s])$ *is differentiable at* $s=0$. *Furthermore, one has*

$$\|[\dot{g}_0]\|_{\mathrm{Th}} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} d_{\mathrm{Th}}([\psi],[\psi^s]).$$

Proof. Compare Guillarmou–Knieper–Lefeuvre [40, Lemma 5.6]. Let

$$r_s := \frac{g_s}{r_{\phi,\psi}} = \frac{g_s}{g_0},$$

which is the reparametrizing function from ψ to ψ^s. Note that

$$\dot{r}_0 := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} r_s = \frac{\dot{g}_0}{r_{\phi,\psi}},$$

and by equation (3.1), we have

$$\|[\dot{g}_0]\|_{\mathrm{Th}} = \sup_{m\in\mathcal{P}(\psi)} \int \dot{r}_0 \mathrm{d}m. \tag{3.2}$$

On the other hand, let $u(s) := e^{d_{\mathrm{Th}}([\psi],[\psi^s])}$. Notice $h_{\psi^s} \equiv 1$. By Theorem 2.10, periodic orbit measures are dense in the space of invariant probability measures. We therefore have

$$u(s) = \sup_{m\in\mathcal{P}(\psi)} \int r_s \mathrm{d}m.$$

It suffices to show that u is differentiable at $s=0$ and $u'(0) = \|[\dot{g}_0]\|_{\mathrm{Th}}$. Since $r_0 \equiv 1$, we have

$$\frac{u(s)-u(0)}{s} = \frac{\sup_{m\in\mathcal{P}(\psi)} \int r_s \mathrm{d}m - \sup_{m\in\mathcal{P}(\psi)} \int 1 \mathrm{d}m}{s} = \sup_{m\in\mathcal{P}(\psi)} \int \left(\frac{r_s-1}{s}\right)\mathrm{d}m,$$

and thanks to equation (3.2), we need to show

$$\lim_{s\to 0}\left(\sup_{m\in\mathcal{P}(\psi)} \int \left(\frac{r_s-1}{s}\right)\mathrm{d}m\right) = \sup_{m\in\mathcal{P}(\psi)} \int \dot{r}_0 \mathrm{d}m.$$

Fix some $\varepsilon > 0$. The mean value theorem implies that $\frac{r_s-1}{s}$ converges uniformly to $\dot{r}_0$ as $s\to 0$. There exists then $\delta > 0$ so that, for all $0<|s|<\delta$ one has

$$\sup_{x\in X}\left|\frac{r_s(x)-1}{s} - \dot{r}_0(x)\right| < \varepsilon.$$

Fix any s so that $0<|s|<\delta$. For every $m\in\mathcal{P}(\psi)$, we have

$$\left|\int \frac{r_s-1}{s}\mathrm{d}m - \int \dot{r}_0 \mathrm{d}m\right| \leq \sup_{x\in X}\left|\frac{r_s(x)-1}{s} - \dot{r}_0(x)\right| < \varepsilon.$$

Therefore,

$$\int \dot{r}_0 \mathrm{d}m - \varepsilon < \int \frac{r_s-1}{s}\mathrm{d}m < \int \dot{r}_0 \mathrm{d}m + \varepsilon,$$

for all $m\in\mathcal{P}(\psi)$. Taking supremum over all $m\in\mathcal{P}(\psi)$, the result follows. ∎

Remark 3.7. We make some remarks about the Finster norm $\|\cdot\|_{\mathrm{Th}}$.

(1) Keeping the notations from above, Proposition 3.6 can be restated as

$$\left\|\left[\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} g_s\right]\right\|_{\mathrm{Th}} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\left(\sup_{m\in\mathcal{P}(\psi)} \mathbf{J}_m(\psi,\psi^s)\right).$$

We will come back to this equality in Subsection 3.3, comparing our viewpoint with previous work of Bridgeman–Canary–Labourie–Sambarino [23].

(2) Notice that although $\|\cdot\|_{\mathrm{Th}}$ is a Finsler norm induced from the asymmetric distance $d_{\mathrm{Th}}(\cdot,\cdot)$, it is not clear whether $d_{\mathrm{Th}}(\cdot,\cdot)$ is the length distance induced from $\|\cdot\|_{\mathrm{Th}}$. In the context of Teichmüller space (cf., Remark 7.10), Thurston [81] shows that $d_{\mathrm{Th}}(\cdot,\cdot)$ coincides with the length distance induced by the Finsler norm.

(3) The Finsler norm $\|\cdot\|_{\mathrm{Th}}$ is, in general, not induced by an inner product. Indeed, in some concrete examples (cf., Remark 7.10), one may find tangent vectors $[g]$ for which

$$\|[g]\|_{\mathrm{Th}} \neq \|-[g]\|_{\mathrm{Th}}.$$

3.3. Comparison with pressure norm

Thurston also introduced a Riemannian metric on the Teichmüller space of a closed surface S, which agrees with the Weil–Petersson metric (see Wolpert [87]). McMullen [59] reinterpreted this construction using thermodynamic formalism, and Bridgeman–Canary–Labourie–Sambarino [23] took inspiration from this to produce a Euclidean norm $\|\cdot\|_{\mathbf{P}}$ on $T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$. We now briefly recall the construction of [23] and point out the difference with our approach.

Let $[\psi]\in\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ and $[g]\in T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ be a tangent vector. Thanks to Proposition 2.13, one has $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\big|_{s=0}\mathbf{P}(-r_{\phi,\psi}+sg)\geq 0$. Hence, one may define

$$\|[g]\|_{\mathbf{P}} := \sqrt{\frac{\frac{\mathrm{d}^2}{\mathrm{d}s^2}\big|_{s=0}\mathbf{P}(-r_{\phi,\psi}+sg)}{\int r_{\phi,\psi}\,\mathrm{d}m_{-r_{\phi,\psi}}}}.$$

Work of Ruelle and Parry–Pollicott implies that $\|\cdot\|_{\mathbf{P}}$ is a norm[2] on $T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$, called the *pressure norm*. Moreover, this norm is induced from an inner product, and in fact, one has

$$\|[g]\|^2_{\mathbf{P}} = \frac{\lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T g(\phi_s(x))\mathrm{d}s\right)^2\mathrm{d}m_{-r_{\phi,\psi}}(x)}{\int r_{\phi,\psi}\,\mathrm{d}m_{-r_{\phi,\psi}}}.$$

See [23, Subsection 3.5.1] for details.

[2]In particular, one has to show that $\|[g]\|_{\mathbf{P}}=0$ if and only if $[g]=0$.

As noticed in [23, Subsection 3.5.2] the pressure norm is related to the $m^{\mathrm{BM}}(\psi)$-renormalized intersection. Indeed, consider the function $\mathbf{J}_{[\psi]}(\cdot)$ on $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ given by

$$\mathbf{J}_{[\psi]}([\widehat{\psi}]) := \mathbf{J}_{m^{\mathrm{BM}}(\psi)}(\psi, \widehat{\psi}),$$

where ψ (resp., $\widehat{\psi}$) is a representative of $[\psi]$ (resp., $[\widehat{\psi}]$). One may check that this is a well-defined function, as it does not depend on the choice of these representatives. Furthermore, by Proposition 2.18, this function has a minimum at $[\psi]$ and therefore its Hessian at $[\psi]$ defines a non-negative symmetric bilinear form on $T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$. In fact, if we let $\{g_s\}_{s\in(-1,1)}$ be a smooth path as in Proposition 3.6, then one has

$$\left\| \left[\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} g_s \right] \right\|_{\mathbf{P}}^2 = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} \mathbf{J}_{[\psi]}([\psi^s]).$$

See [23, Proposition 3.11] for details.

Hence, the second derivative of the $m^{\mathrm{BM}}(\psi)$-renormalized intersection defines an inner product on $T_{[\psi]}\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$. In contrast, our viewpoint is different: Rather than taking a second derivative of the renormalized intersection with respect to a given measure, we take the supremum of renormalized intersections over all measures and then take a first derivative (cf., Remark 3.7).

4. Anosov representations

Anosov representations were introduced by Labourie [54] for fundamental groups of negatively curved manifolds and then extended by Guichard–Wienhard [38] to general word hyperbolic groups. They provide a stable class of discrete representations with finite kernel into semisimple Lie groups, that share many features with holonomies of convex cocompact hyperbolic manifolds. We will briefly recall this notion in Subsection 4.2, after fixing some notations and terminology in Subsection 4.1. In Subsection 4.3, we discuss examples. For a more complete account on the state of the art of the field, see, for example, [49, 66, 86] and references therein.

4.1. Structure of semisimple Lie groups

Standard references for this part are the books of Knapp [50] and Helgason [43].

Let G be a connected real semisimple algebraic group of non-compact type with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G and τ be the corresponding Cartan involution of $\mathfrak{g}$. Let

$$\mathfrak{p} := \{v \in \mathfrak{g} : \tau v = -v\}.$$

We fix a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ and let M be the centralizer of $\mathfrak{a}$ in K.

A natural dynamical system one may look at when studying a discrete subgroup $\Delta < \mathsf{G}$, is the right action of $\mathfrak{a}$ on $\Delta\backslash\mathsf{G}/\mathsf{M}$. When G has real rank equal to 1, this action is conjugate to the action of the geodesic flow of the underlying negatively curved manifold. However, in general it may be hard to study the action $\mathfrak{a} \curvearrowright \Delta\backslash\mathsf{G}/\mathsf{M}$. In many situations (including the setting we are aiming for), it proves useful to consider a "more hyperbolic" dynamical system, namely the action of the center of the Levi group associated with a parallel set. We now fix the terminology needed to define this dynamical system.

Denote by Σ the set of *roots* of $\mathfrak{a}$ in $\mathfrak{g}$, that is, the set of functionals $\alpha \in \mathfrak{a}^* \setminus \{0\}$ for which the *root space*

$$\mathfrak{g}_\alpha := \{Y \in \mathfrak{g} : [X, Y] = \alpha(X)Y \text{ for all } X \in \mathfrak{a}\}$$

is non-zero. Fix a positive system $\Sigma^+ \subset \Sigma$ associated with a closed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. The set of simple roots for Σ^+ is denoted by Π.

Example 4.1. Suppose $\mathsf{G} = \mathsf{PSL}(V)$, where V is a real (resp., complex) vector space of dimension $d \geq 2$. The Lie algebra of G is the space of traceless linear operators in V. Hence, every element of $\mathfrak{g}$ acts on V. A maximal compact subgroup is the subgroup of orthogonal (resp., unitary) matrices with respect to an inner (resp., Hermitian inner) product o in V. A Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ is the subalgebra of matrices which are diagonal on a given projective basis $\mathcal{E}$ of V orthogonal with respect to o. The choice of a closed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ corresponds to the choice of a total order $\{\ell_1, \dots, \ell_d\}$ on $\mathcal{E}$. Explicitly, if $\lambda_j(X)$ denotes the eigenvalue of $X \in \mathfrak{a}$ on the eigenline ℓ_j, the Weyl chamber $\mathfrak{a}^+$ is given by the set of matrices $X \in \mathfrak{a}$ for which

$$\lambda_1(X) \geq \cdots \geq \lambda_d(X).$$

For $i \neq j$, we let $\alpha_{i,j}(X) := \lambda_i(X) - \lambda_j(X)$. Then

$$\Sigma = \{\alpha_{i,j} : i \neq j\} \quad \text{and} \quad \Sigma^+ = \{\alpha_{i,j} : i < j\}.$$

The set of simple roots is

$$\Pi = \{\alpha_{i,i+1} : i = 1, \dots, d-1\}.$$

Sometimes we will write the elements of Π simply by $\alpha_i := \alpha_{i,i+1}$.

Let W be the *Weyl group* of Σ. We realize it as

$$\mathsf{W} \cong \mathsf{N}_\mathsf{K}(\mathfrak{a})/\mathsf{M},$$

where $\mathsf{N}_\mathsf{K}(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in K. The group W acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$; thus, there exists a unique element $w_0 \in \mathsf{W}$ taking $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* associated with $\mathfrak{a}^+$ is $\iota := -w_0$.

We will furthermore need the structure of parabolic subgroups of G. Fix a non-empty subset $\Theta \subset \Pi$. Consider the subalgebras

$$\mathfrak{p}_\Theta := \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \bigoplus_{\alpha \in \langle \Pi - \Theta \rangle} \mathfrak{g}_{-\alpha}$$

and

$$\overline{\mathfrak{p}_\Theta} := \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha} \bigoplus_{\alpha\in\langle\Pi-\Theta\rangle} \mathfrak{g}_\alpha,$$

where $\langle\Pi - \Theta\rangle$ denotes the set of positive roots generated by roots in $\Pi - \Theta$. We let P_Θ and $\overline{\mathsf{P}}_\Theta$ be the corresponding subgroups of G. Every parabolic subgroup of G is conjugate to a unique P_Θ, for some $\Theta \subset \Pi$. Note that $\overline{\mathsf{P}}_\Theta$ is conjugate to $\mathsf{P}_{\iota(\Theta)}$, where

$$\iota(\Theta) := \{\alpha \circ \iota : \alpha \in \Theta\}.$$

The parabolic subgroup $\overline{\mathsf{P}}_\Theta$ is *opposite* to P_Θ.

Let

$$\mathcal{F}_\Theta := \mathsf{G}/\mathsf{P}_\Theta \quad \text{and} \quad \overline{\mathcal{F}}_\Theta := \mathsf{G}/\overline{\mathsf{P}}_\Theta$$

be the corresponding *flag manifolds* of G. Two flags $\xi \in \mathcal{F}_\Theta$ and $\overline{\xi} \in \overline{\mathcal{F}}_\Theta$ are *transverse* if $(\overline{\xi}, \xi)$ belongs to $\mathcal{F}_\Theta^{(2)}$, the unique open orbit of the action of G on $\overline{\mathcal{F}}_\Theta \times \mathcal{F}_\Theta$. We also let $\mathcal{F} := \mathcal{F}_\Pi$ and $\mathcal{F}^{(2)} := \mathcal{F}_\Pi^{(2)}$.

Example 4.2. Let G be as in Example 4.1. When $d = 2$, there is only one flag manifold, which identifies with $\mathbb{P}(\mathbb{R}^2)$ (the projective space of $\mathbb{R}^2$). When $d = 3$, there are three flag manifolds, namely $\mathbb{P}(\mathbb{R}^3)$, $\mathbb{P}((\mathbb{R}^3)^*)$, and

$$\mathcal{F} = \{(\xi^1, \xi^2) \in \mathbb{P}(\mathbb{R}^3) \times \mathbb{P}((\mathbb{R}^3)^*) : \xi^1 \subset \xi^2\},$$

where in the above formula, we have implicitly identified $\mathbb{P}((\mathbb{R}^3)^*)$ with the Grassmannian of two-dimensional subspaces of $\mathbb{R}^3$.

More generally, for arbitrary $d \geq 2$ the choice of Θ is equivalent to the choice of a subset $\{1 \leq i_1 < \cdots < i_p \leq d - 1\}$, for some $1 \leq p \leq d - 1$. Then $\mathcal{F}_\Theta$ identifies with the space of *partial flags* indexed by Θ, that is, the space of sequences ξ of the form $(\xi^{i_1} \subset \cdots \subset \xi^{i_p})$, where ξ^{i_j} is a linear subspace of V of dimension i_j, for all $j = 1, \ldots, p$. Furthermore, one has $\iota(\Theta) = \{1 \leq d - i_p < \cdots < d - i_1 \leq d - 1\}$. A flag $\overline{\xi} \in \overline{\mathcal{F}}_\Theta$ is transverse to $\xi \in \mathcal{F}_\Theta$ if and only if for all $j = 1, \ldots, p$, the sum $\overline{\xi}^{d-i_j} + \xi^{i_j}$ is direct.

A point in $(\overline{\xi}, \xi) \in \mathcal{F}_\Theta^{(2)}$ determines a *parallel set* of the Riemannian symmetric space X_G of G. It is the union of all parametrized flat subspaces f of X_G so that the flag associated with $f(\mathfrak{a}^+)$ (resp., $f(-\mathfrak{a}^+)$) belongs to the fiber over ξ (resp., $\overline{\xi}$), for the fibration $\mathcal{F} \to \mathcal{F}_\Theta$ (resp., $\mathcal{F} \to \overline{\mathcal{F}}_\Theta$). When the real rank of G is equal to 1, this is just a geodesic of X_G. When $\Theta = \Pi$, it is a maximal flat subspace of X_G. Any parallel set is identified with the Riemannian symmetric space of the Levi subgroup $L_\Theta = \mathsf{P}_\Theta \cap \overline{\mathsf{P}}_\Theta$, a reductive subgroup of G.

Let

$$\mathfrak{a}_\Theta := \bigcap_{\alpha\in\Pi-\Theta} \ker\alpha$$

be the Lie algebra of the center of $L_\Theta = \mathsf{P}_\Theta \cap \overline{\mathsf{P}}_\Theta$ (in particular, $\mathfrak{a}_\Pi = \mathfrak{a}$). There is a unique projection $p_\Theta : \mathfrak{a} \to \mathfrak{a}_\Theta$ invariant under the group

$$\mathsf{W}_\Theta := \{w \in \mathsf{W} : w|_{\mathfrak{a}_\Theta} = \mathrm{id}_{\mathfrak{a}_\Theta}\}.$$

The dual space $\mathfrak{a}_\Theta^*$ identifies naturally with $\{\varphi \in \mathfrak{a}^* : \varphi \circ p_\Theta = \varphi\}$. We will use this identification throughout the paper.

Consider the space $\mathcal{F}_\Theta^{(2)} \times \mathfrak{a}_\Theta$, endowed with the action of $\mathfrak{a}_\Theta$ by translations on the last coordinate. This action commutes with a natural action of G that we now describe, and the quotient dynamics is the "more hyperbolic" dynamical system we have referred to at the beginning of this subsection.

Let N be the *unipotent radical* of $\mathsf{P} = \mathsf{P}_\Pi$, that is, the connected subgroup of G associated with the Lie algebra $\sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. The *Iwasawa decomposition* is

$$\mathsf{G} = \mathsf{K} \exp(\mathfrak{a}) \mathsf{N}.$$

In particular, $\mathcal{F} \cong \mathsf{K}/\mathsf{M}$ and for $\xi \in \mathcal{F}$. We may find $k \in \mathsf{K}$ such that $k\mathsf{M} = \xi$. Quint [70] defines a map $\sigma : \mathsf{G} \times \mathcal{F} \to \mathfrak{a}$ by the formula

$$gk = l \exp(\sigma(g, k\mathsf{M}))n,$$

where $n \in \mathsf{N}$ and $l \in \mathsf{K}$. Quint [70, Lemme 6.11] also shows that $p_\Theta \circ \sigma : \mathsf{G} \times \mathcal{F} \to \mathfrak{a}_\Theta$ factors through a map $\sigma_\Theta : \mathsf{G} \times \mathcal{F}_\Theta \to \mathfrak{a}_\Theta$. For every $g, h \in \mathsf{G}$ and $\xi \in \mathcal{F}_\Theta$ one has

$$\sigma_\Theta(gh, \xi) = \sigma_\Theta(g, h \cdot \xi) + \sigma_\Theta(h, \xi).$$

The map σ_Θ is called the Θ-*Busemann–Iwasawa cocycle* of G. Observe that the action of $\mathfrak{a}_\Theta$ on $\mathcal{F}_\Theta^{(2)} \times \mathfrak{a}_\Theta$ commutes with the action of G given by

$$g \cdot (\overline{\xi}, \xi, X) := (g \cdot \overline{\xi}, g \cdot \xi, X - \sigma_\Theta(g, \xi)).$$

Remark 4.3. The Busemann–Iwasawa cocycle of G is a vector–valued version of the *Busemann function* of the Riemannian symmetric space X_G of G. Indeed, when G has real rank equal to 1, then $\mathcal{F}$ identifies with the visual boundary ∂X_G of X_G. Let $o \in X_\mathsf{G}$ be the point fixed by K. After identifying $\mathfrak{a}$ with $\mathbb{R}$ suitably, one has

$$\sigma(g, \xi) = b_\xi(o, g^{-1} \cdot o),$$

where $b.(\cdot, \cdot) : \partial X_\mathsf{G} \times X_\mathsf{G} \times X_\mathsf{G} \to \mathbb{R}$ is the Busemann function. A similar interpretation holds in higher rank (cf., [70, Lemme 6.6]).

In Section 5, we will consider a flow space which is even better behaved than the action of $\mathfrak{a}_\Theta$ associated with a parallel set. It will be induced by the choice of a functional in $\mathfrak{a}_\Theta^*$. Natural generators of $\mathfrak{a}_\Theta^*$ are the *fundamental weights* associated with Θ, whose definition we now recall.

Denote by $(\cdot,\cdot)$ the inner product on $\mathfrak{a}^*$ dual to the Killing form of $\mathfrak{g}$. For $\varphi, \psi \in \mathfrak{a}^*$ set

$$\langle \varphi, \psi \rangle := 2\frac{(\varphi, \psi)}{(\psi, \psi)}.$$

Given $\alpha \in \Pi$, the corresponding *fundamental weight* is the functional $\omega_\alpha \in \mathfrak{a}^*$ defined by the formulas $\langle \omega_\alpha, \beta \rangle = \delta_{\alpha\beta}$ for $\beta \in \Pi$. One has

$$\omega_\alpha \circ p_\Theta = \omega_\alpha \tag{4.1}$$

for all $\alpha \in \Theta$ (cf., Quint [69, Lemme II.2.1]). In particular, we have $\omega_\alpha \in \mathfrak{a}_\Theta^*$.

Fundamental weights are related to a special set of linear representations of G introduced by Tits [82]. If $\Lambda : \mathsf{G} \to \mathsf{PGL}(V)$ is an irreducible representation, a functional $\chi \in \mathfrak{a}^*$ is a *weight* of Λ if the *weight space*

$$V_\chi := \{v \in V : \Lambda(\exp(X)) \cdot v = e^{\chi(X)} v, \text{ for all } X \in \mathfrak{a}\}$$

is non-zero. Tits [82] shows that there exists a unique weight χ_Λ which is maximal with respect to the order given by $\chi \geq \chi'$ if $\chi - \chi'$ is a linear combination of simple roots with non-negative coefficients. The functional χ_Λ is called the *highest weight* of Λ, and the representation is *proximal* if the associated weight space V_{χ_Λ} is one-dimensional. The next proposition is useful.

Proposition 4.4 (Tits [82]). *For every $\alpha \in \Pi$, there exists a finite-dimensional real vector space V_α and a proximal irreducible representation $\Lambda_\alpha : \mathsf{G} \to \mathsf{PGL}(V_\alpha)$ such that the highest weight $\chi_\alpha = \chi_{\Lambda_\alpha}$ is of the form $k_\alpha \omega_\alpha$, for some integer $k_\alpha \geq 1$.*

We fix from now on a set of representations $\{\Lambda_\alpha\}_{\alpha \in \Pi}$ as in Proposition 4.4. Observe that for all $\alpha \in \Theta$, we have

$$\chi_\alpha \circ p_\Theta = \chi_\alpha, \tag{4.2}$$

and therefore χ_α belongs to $\mathfrak{a}_\Theta^*$.

We conclude recalling the definitions of Cartan and Jordan projections of G for later use. The *Cartan projection* of $g \in \mathsf{G}$ is the unique element $\mu(g) \in \mathfrak{a}^+$ satisfying

$$g \in \mathsf{K} \exp(\mu(g)) \mathsf{K}.$$

The *Jordan projection* of g is defined by

$$\lambda(g) := \lim_{n \to \infty} \frac{\mu(g^n)}{n}.$$

One may show that for all $\alpha \in \Pi$ and all $g \in \mathsf{G}$ one has

$$\lambda_1(\Lambda_\alpha(g)) = \chi_\alpha(\lambda(g)) = k_\alpha \omega_\alpha(\lambda(g)), \tag{4.3}$$

where $\lambda_1(\Lambda_\alpha(g))$ denotes the logarithm of the modulus of the highest eigenvalue of $\Lambda_\alpha(g)$.

We denote

$$\mu_\Theta := p_\Theta \circ \mu \quad \text{and} \quad \lambda_\Theta := p_\Theta \circ \lambda.$$

4.2. Anosov representations and their length functions

We now define Anosov representations and their corresponding length functions and entropies. The definition that we present here is not the original definition, but an equivalent one established in [15, 35, 47].

Let Γ be a finitely generated group and $|\cdot|$ be the word length associated with a finite generating set (that we fix from now on).

Definition 4.5. Let $\Theta \subset \Pi$ be a non-empty set. A representation $\rho : \Gamma \to \mathsf{G}$ is P_Θ-*Anosov* (or Θ-*Anosov*) if there exist positive constants C and c such that for all $\alpha \in \Theta$ one has

$$\alpha(\mu(\rho(\gamma))) \geq C|\gamma| - c.$$

When $\Theta = \Pi$ and G is split, ρ is sometimes called *Borel–Anosov*. When $\mathsf{G} = \mathsf{PSL}(V)$ with V as in Example 4.1, $\{\alpha_1\}$-Anosov representations are also called *projective Anosov*.

An immediate consequence of Definition 4.5 is that Anosov representations are quasi-isometric embeddings from Γ to G. In particular, they are discrete and have finite kernels. A deeper consequence is a theorem by Kapovich–Leeb–Porti [48, Theorem 1.4] (see also [15, Section 3]): If $\rho : \Gamma \to \mathsf{G}$ is Θ-Anosov, then Γ is word hyperbolic. Throughout the paper, we shall assume that Γ is nonelementary and denote by $\partial\Gamma$ its Gromov boundary. We also let $\partial^{(2)}\Gamma$ be the space of ordered pairs of different points in $\partial\Gamma$. Every infinite order element $\gamma \in \Gamma$ has a unique attracting (resp., repelling) fixed point in $\partial\Gamma$, denoted by γ_+ (resp., γ_-). We let $\Gamma_{\mathrm{H}} \subset \Gamma$ be the subset consisting of infinite order elements. The conjugacy class of $\gamma \in \Gamma$ is denoted by $[\gamma]$, and the set of conjugacy classes of elements of Γ (resp., Γ_{H}) will be denoted by $[\Gamma]$ (resp., $[\Gamma_{\mathrm{H}}]$).

A central feature of Θ-Anosov representations is that they admit *limit maps*. By definition, these are Hölder continuous, ρ-equivariant, dynamics preserving maps

$$\xi_\rho : \partial\Gamma \to \mathcal{F}_\Theta \quad \text{and} \quad \overline{\xi}_\rho : \partial\Gamma \to \overline{\mathcal{F}}_\Theta,$$

which are moreover *transverse*, that is, for every $x \neq y$ in $\partial\Gamma$ one has

$$(\overline{\xi}_\rho(x), \xi_\rho(y)) \in \mathcal{F}^{(2)}_\Theta.$$

The limit maps exist and are unique (see [15, 35, 47] for details).

Example 4.6. Let G be as in Example 4.1 and $\Theta = \{1 \leq i_1 < \cdots < i_p \leq d-1\}$ for some $1 \leq p \leq d-1$ (cf., Example 4.2). For $j = 1, \ldots, p$, we let

$$\xi^{i_j}_\rho : \partial\Gamma \to \mathbb{G}_{i_j}(V)$$

be the i_j-coordinate of ξ_ρ into the Grassmannian $\mathbb{G}_{i_j}(V)$ of i_j-dimensional subspaces of V.

The set of Θ-Anosov representations from Γ to G is an open subset of the space of all representations $\Gamma \to \mathsf{G}$. This is a consequence of the original definition [38, 54]. Indeed, the original definition requires a priori the word hyperbolicity of Γ and the existence of the limit maps, with them one constructs a flow space which, by definition, satisfies certain form of uniform hyperbolicity. General results in hyperbolic dynamics give that this is an open condition.

Projective Anosov representations are very general.

Proposition 4.7 (Guichard–Wienhard [38, Proposition 4.3]). *Let $\rho : \Gamma \to \mathsf{G}$ be Θ-Anosov. Then for every $\alpha \in \Theta$, the representation $\Lambda_\alpha \circ \rho : \Gamma \to \mathsf{PGL}(V_\alpha)$ is projective Anosov.*

We denote by $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ the space of conjugacy classes of P_Θ-Anosov representations from Γ to G. Length functions and entropies are important invariants to study this space. By work of Sambarino [71, 72] that we will recall in Section 5, they provide a way of associating with each $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ certain flow space as in Sections 2 and 3, and therefore one may use the thermodynamic formalism to study $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$. To define length functions and entropies properly, we need to recall the definition of a fundamental object, introduced by Benoist [7] for general discrete subgroups of G.

Definition 4.8. The Θ-*limit cone* of $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ is the smallest closed cone $\mathcal{L}_\rho^\Theta \subset \mathfrak{a}_\Theta^+$ containing the set $\{\lambda_\Theta(\rho(\gamma)) : \gamma \in \Gamma\}$. The *limit cone* $\mathcal{L}_\rho$ of ρ is the Π-limit cone.

In the above definition, we abuse notations, because ρ is a conjugacy class of representations. However, it is clear that the Θ-limit cone is independent of the choice of a representative in this conjugacy class.

Under the assumption that ρ is Zariski dense, Benoist [7] showed that $\mathcal{L}_\rho$ is a convex cone with non-empty interior.[3] Since p_Θ is a surjective linear map, the same properties hold for the Θ-limit cone.

Let

$$(\mathcal{L}_\rho^\Theta)^* := \{\varphi \in \mathfrak{a}_\Theta^* : \varphi|_{\mathcal{L}_\rho^\Theta} \geq 0\}$$

be the *dual cone*. We denote by $\operatorname{int}((\mathcal{L}_\rho^\Theta)^*)$ the interior of $(\mathcal{L}_\rho^\Theta)^*$, that is, the set of functionals in $\mathfrak{a}_\Theta^*$ which are positive on $\mathcal{L}_\rho^\Theta \setminus \{0\}$.

Fix a functional

$$\varphi \in \bigcap_{\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})} \operatorname{int}((\mathcal{L}_\rho^\Theta)^*).$$

The above intersection is non-empty. For example, it contains λ_1 and more generally ω_α for all $\alpha \in \Pi$.

[3] In fact, Benoist shows this result for any Zariski-dense discrete subgroup of G.

Definition 4.9. The *φ-marked length spectrum* (or simply *φ-length spectrum*) of $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ is the function $L^\varphi_\rho : \Gamma \to \mathbb{R}_{\geq 0}$ given by

$$L^\varphi_\rho(\gamma) := \varphi(\lambda_\Theta(\rho(\gamma))).$$

Observe that for a Θ-Anosov representation ρ, $L^\varphi_\rho(\gamma) > 0$ if and only if $\gamma \in \Gamma_{\mathrm{H}}$ (i.e., if it has infinite order). Furthermore, the φ-length spectrum is invariant under conjugation in Γ and therefore descends to a function $[\Gamma] \to \mathbb{R}_{\geq 0}$. We will often abuse notations and denote this function by L^φ_ρ as well.

Definition 4.10. The *φ-entropy* of ρ is defined by

$$h^\varphi_\rho := \limsup_{t \to \infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma] : L^\varphi_\rho(\gamma) \leq t\} \in [0, \infty].$$

The φ-entropy of ρ was introduced by Sambarino [71,72], who showed that this quantity is defined by a true limit, is positive, finite, and coincides with the topological entropy of a suitable flow associated with ρ and φ. We will briefly recall these results and facts in Section 5.

Example 4.11. Here is a concrete set of length spectra that will be of interest (the corresponding entropies are named accordingly). Let $\mathsf{G} = \mathsf{PSL}(V)$ with V as in Example 4.1:

- If $\rho : \Gamma \to \mathsf{G}$ is Θ-Anosov and $\alpha_i \in \Theta$ belongs to $\mathfrak{a}^*_\Theta$ (this is always the case if $\Theta = \Pi$), then $L^{\alpha_i}_\rho$ is called the ith-*simple root length spectrum* of ρ.
- If $\rho : \Gamma \to \mathsf{G}$ is projective Anosov, then $L^{\alpha_{1,d}}_\rho$ is called the *Hilbert length spectrum* of ρ. We denote it by L^{H}_ρ.
- If $\rho : \Gamma \to \mathsf{G}$ is projective Anosov, then $L^{\lambda_1}_\rho$ is called the *spectral radius length spectrum* of ρ.

4.3. Examples of Anosov representations

Schottky-type constructions as in Benoist [6] provide basic examples of Θ-Anosov representations of free groups. In this subsection, we give a list of other examples that will be of interest to us.

Example 4.12 (Teichmüller space). Let S be a connected, closed, orientable surface of genus ≥ 2 and $\Gamma = \pi_1(S)$ be its fundamental group (in short, Γ is a *surface group*). The *Teichmüller space* of S is the space of isotopy classes of Riemannian metrics on S of constant curvature equal to -1. Throughout the paper, we identify this space with a connected component $\mathfrak{T}(S)$ of the space of $\mathsf{PSL}(2, \mathbb{R})$-conjugacy classes of faithful and discrete representations $\Gamma \to \mathsf{PSL}(2, \mathbb{R})$. By the Švarc–Milnor Lemma (see [31, Proposition 19 of Chapter 3]), representations in $\mathfrak{T}(S)$ are Anosov.

Example 4.13 (Hitchin representations). An important class of Anosov representations is given by Hitchin representations. For every split real Lie group G, we denote by $\tau : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{G}$ the *principal embedding* [52], which is well defined up to conjugation. In the case of $\mathsf{G} = \mathsf{PSL}(d,\mathbb{R})$, τ gives the unique irreducible linear representation of $\mathsf{PSL}(2,\mathbb{R})$. It was proven by Labourie [54] and Fock–Goncharov [30] that, given the holonomy $\rho_h : \Gamma \to \mathsf{PSL}(2,\mathbb{R})$ of any chosen hyperbolization h of S, the entire connected component of $\tau \circ \rho_h$ consists of Borel–Anosov representations. This component is usually referred to as the *Hitchin component*. An element in it is called a (conjugacy class of) *Hitchin representation*. We will denote by $\mathrm{Hit}_d(S)$ (resp., $\mathrm{Hit}(S,\mathsf{G})$) the Hitchin component of Γ in $\mathsf{PSL}(d,\mathbb{R})$ (resp., in G). Any Hitchin representation is Borel–Anosov, that is, it is Anosov with respect to any subset of Π. It was proven in [65, 67] that the entropy of each simple root is constant and equal to 1 on each Hitchin component, when G is not of exceptional type.

Example 4.14 (Θ-positive representations). A general framework encompassing all cases of connected components of character varieties of fundamental groups of surfaces only consisting of Anosov representations was proposed by Guichard–Wienhard [39] (see also [37]). They introduce the class of Θ-positive representations, which includes, apart from Hitchin components, *maximal representations* in Hermitian Lie groups, as well as the connected components of representation in the $\mathsf{PO}_0(p,q)$-character variety and some components in the character varieties of the four exceptional Lie groups with restricted root system of type F_4. While Hitchin representations are Borel–Anosov, the other representations are, in general, only Anosov with respect to a proper subset $\Theta < \Pi$, which consists of a single root in the case of maximal representations and has $p-1$ elements in the case of $\mathsf{PO}_0(p,q)$-positive representations. It was proven in [68] that for maximal and Θ-positive representations in $\mathsf{PO}_0(p,q)$, the entropy with respect to any root in Θ is equal to 1.

Example 4.15 (Hyperconvex representations). Another important class of Anosov representations are $(1,1,p)$-*hyperconvex representations* studied in [67]. These are representations $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{R})$ that are $\{\alpha_1,\alpha_p\}$-Anosov and satisfy the additional transversality property that for all triples of pairwise distinct points $x,y,z \in \partial\Gamma$, the sum $\xi^1_\rho(x) + \xi^1_\rho(y) + \xi^{d-p}_\rho(z)$ is direct. If Γ is a cocompact lattice in $\mathsf{PO}(1,p)$, so that $\partial\Gamma = \mathbb{S}^{p-1}$, it follows from [67] that $\xi^1_\rho(\partial\Gamma)$ is a C^1-submanifold of $\mathbb{P}(\mathbb{R}^d)$. Furthermore, it was proven in [68] that for these representations, which sometimes admit nontrivial deformations, the entropy for the functional $p\omega_{\alpha_1} - \omega_{\alpha_p}$ is constant and equal to 1. Important examples of this class are the groups Γ dividing a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$ studied by Benoist [9–12]. These are $(1,1,d-1)$-hyperconvex and were already studied by Potrie–Sambarino [65].

Example 4.16 (AdS-quasi-Fuchsian representations). Let $q \geq 2$ and Γ be the fundamental group of a closed q-dimensional manifold. A representation $\rho : \Gamma \to \mathsf{PO}(2,q)$ is said to be *AdS-quasi-Fuchsian* if it is faithful, discrete, and preserves an acausal topological $(q-1)$-sphere on the boundary of the anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^{1,q}$. Recall that $\mathbb{A}\mathrm{d}\mathbb{S}^{1,q}$ is defined as the set of negative lines for the underlying quadratic form $\langle\cdot,\cdot\rangle_{2,q}$, and its boundary is the space $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{1,q}$ of isotropic lines. A subset of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{1,q}$ is said to be *acausal* if it lifts to a cone in $\mathbb{R}^{2+q}\setminus\{0\}$ in which all $\langle\cdot,\cdot\rangle_{2,q}$-products of noncollinear vectors are negative. The fundamental example of an AdS-quasi-Fuchsian representation is given by *AdS-Fuchsian* representations, that is, representations of the form

$$\Gamma \to \mathsf{PO}(1,q) \to \mathsf{PO}(2,q),$$

where the first map is the holonomy of a closed real hyperbolic manifold and the second arrow is the standard embedding stabilizing a negative line in $\mathbb{R}^{2+q}$.

AdS-quasi-Fuchsian representations were introduced in seminal work by Mess [60] for $q=2$ and then generalized by Barbot–Mérigot [5] and Barbot [4] for $q>2$. They are $\{\alpha_1\}$-Anosov representations, where α_1 is the simple root in $\mathsf{PO}(2,q)$ corresponding to the stabilizer of an isotropic line (see [5]). Furthermore, the space of AdS-quasi-Fuchsian representations is a union of connected components of the representation space (see [4]). AdS-quasi-Fuchsian representations were generalized to $\mathbb{H}^{p-1,q}$-*convex-cocompact* representations by Danciger–Guéritaud–Kassel [29].

5. Flows associated with Anosov representations

We now recall Sambarino's reparametrization theorem [71,72]. This result associates with each $\rho\in\mathfrak{X}_\Theta(\Gamma,\mathsf{G})$ and each $\varphi\in\mathrm{int}((\mathcal{L}_\rho^\Theta)^*)$ a topological flow on a compact space, recording the data of the φ-length spectrum of ρ and admitting a strong Markov coding. Through the thermodynamic formalism, this provides a powerful tool to study the representation ρ and the space $\mathfrak{X}_\Theta(\Gamma,\mathsf{G})$ of P_Θ-Anosov representations.

Sambarino deals originally with Anosov representations of the fundamental group of a closed negatively curved manifold. In that case he uses the geodesic flow of the manifold (which is Anosov) as a "reference" flow, and from ρ and φ builds a Hölder reparametrization of that flow encoding the periods $L_\rho^\varphi(\gamma)=\varphi(\lambda_\Theta(\rho(\gamma)))$. In the present framework, we are dealing with more general word hyperbolic groups. Nevertheless, his result is known to still hold: One may replace the reference geodesic flow of the manifold by the *Gromov–Mineyev geodesic flow* of Γ. This is a topologically transitive Hölder continuous flow on a compact metric space $\mathrm{U}\Gamma$, well defined up to Hölder orbit equivalence. It was introduced by Gromov [33] (see also Mineyev [61] for details). To define this flow space, one considers a proper and cocompact action of Γ on $\partial^{(2)}\Gamma\times\mathbb{R}$, extending the natural action of Γ on $\partial^{(2)}\Gamma$. The space $\partial^{(2)}\Gamma\times\mathbb{R}$ equipped with this action will be denoted by $\widetilde{\mathrm{U}\Gamma}$, and we refer to this action as the Γ-*action* on $\partial^{(2)}\Gamma\times\mathbb{R}$. In the sequel, we will consider many different actions of Γ on $\partial^{(2)}\Gamma\times\mathbb{R}$, depending on various choices, and this justifies this specific terminology and notation.

The Γ-action commutes with the $\mathbb{R}$-action given by

$$t : (x, y, s) \mapsto (x, y, s + t).$$

We let $\phi = (\phi_t : \mathrm{U}\Gamma \to \mathrm{U}\Gamma)$ be the quotient *Gromov–Mineyev geodesic* flow. Central in all what follows is a result by Bridgeman–Canary–Labourie–Sambarino [23, Sections 4 and 5], stating that in the present setting ϕ is metric Anosov, and one has the following (see also [27]).

Theorem 5.1 ([19, 20, 23, 64]). *Let Γ be a word hyperbolic group admitting an Anosov representation. Then ϕ admits a strong Markov coding.*

5.1. The reparametrization theorem

Provided Theorem 5.1, Sambarino's reparametrization theorem carries on to this more general setting, as summarized in detail in [74]. More precisely, Sambarino shows that to define a Hölder reparametrization of ϕ, it suffices to consider a *Hölder cocycle* over Γ with non-negative *periods* and finite *entropy*. We do not give full definitions here and refer the reader to [74, Sections 3.1 and 3.2] for details, but let us now recall how this construction works specifically for the φ-*Busemann–Iwasawa cocycle* of ρ (also called the φ-*refraction cocycle* of ρ in [74, Definition 3.5.1]).

Let $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ and consider the pullback $\beta^\rho_\Theta : \Gamma \times \partial\Gamma \to \mathfrak{a}_\Theta$ of the Busemann–Iwasawa cocycle of G through the representation ρ, that is,

$$\beta^\rho_\Theta(\gamma, x) := \sigma_\Theta(\rho(\gamma), \xi_\rho(x)).$$

The group Γ acts on $\partial^{(2)}\Gamma \times \mathbb{R}$ by

$$\gamma \cdot (x, y, s) := (\gamma \cdot x, \gamma \cdot y, s - \varphi \circ \beta^\rho_\Theta(\gamma, y)).$$

The space $\partial^{(2)}\Gamma \times \mathbb{R}$ equipped with this action will be denoted by $\widetilde{\mathrm{U}\Gamma}^{\rho,\varphi}$, and we refer to this action as the (ρ, φ)-*refraction action* (or simply the (ρ, φ)-*action*). We let $\mathrm{U}\Gamma^{\rho,\varphi}$ be the quotient space. The (ρ, φ)-action commutes with the $\mathbb{R}$-action given by

$$t : (x, y, s) \mapsto (x, y, s - t).$$

We let $\phi^{\rho,\varphi} = (\phi^{\rho,\varphi}_t : \mathrm{U}\Gamma^{\rho,\varphi} \to \mathrm{U}\Gamma^{\rho,\varphi})$ be the quotient flow, called the (ρ, φ)-*refraction* flow. As shown by Sambarino, to prove that $\phi^{\rho,\varphi}$ is Hölder orbit equivalent to ϕ one needs to analyze the *periods* and *entropy* of the (ρ, φ)-refraction cocycle. Let us now recall these notions.

For every $\gamma \in \Gamma_\mathrm{H}$ one has $\beta^\rho_\Theta(\gamma, \gamma_+) = \lambda_\Theta(\rho\gamma)$ (cf., [72, Lemma 7.5]). In particular, the *period* $\varphi(\beta^\rho_\Theta(\gamma, \gamma_+)) = L^\varphi_\rho(\gamma)$ of $\gamma \in \Gamma_\mathrm{H}$ is positive. In [74, Section 3.2], the *entropy* of $\varphi \circ \beta^\rho_\Theta$ is defined by

$$\limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma_\mathrm{H}] : \varphi(\beta^\rho_\Theta(\gamma, \gamma_+)) \leq t\} \in [0, \infty].$$

Note that the definition of this entropy differs from the φ-entropy of ρ by the fact that here we are only considering conjugacy classes of infinite order elements in Γ, while for h_ρ^φ we also allow conjugacy classes represented by finite order elements. However, the two numbers coincide: A theorem by Bogopolskii–Gerasimov [16] (see also Brady [22]) states that there exists a positive K_Γ such that every finite subgroup of Γ has at most K_Γ elements. In particular, there are only finitely many conjugacy classes of finite order elements in Γ and therefore

$$h_\rho^\varphi = \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma_{\mathrm{H}}] : \varphi(\beta_\Theta^\rho(\gamma,\gamma_+)) \leq t\} \in [0,\infty]. \tag{5.1}$$

Moreover, the φ-entropy is positive and finite. Indeed, let $\alpha \in \Theta$ and consider the function $\mathbb{P}(\mathcal{L}_\rho^\Theta) \to \mathbb{R}_{>0}$ given by

$$\mathbb{R}v \mapsto \frac{\varphi(v)}{\chi_\alpha(v)},$$

where $v \neq 0$ is any vector representing the line $\mathbb{R}v$. Since $\mathbb{P}(\mathcal{L}_\rho^\Theta)$ is compact, we find a constant $c > 1$ so that

$$c^{-1} \leq \frac{L_\rho^\varphi(\gamma)}{\chi_\alpha(\lambda(\rho(\gamma)))} \leq c$$

for all $\gamma \in \Gamma_{\mathrm{H}}$. Applying equation (4.3), we conclude

$$c^{-1} \leq \frac{L_\rho^\varphi(\gamma)}{\lambda_1(\Lambda_\alpha(\rho(\gamma)))} \leq c$$

for all $\gamma \in \Gamma_{\mathrm{H}}$. Thanks to Proposition 4.7, to show $0 < h_\rho^\varphi < \infty$ it suffices to show that the spectral radius entropy of a projective Anosov representation is positive and finite. On the one hand, finiteness follows by an easy geometric argument (see [74, Lemma 5.1.2]). Positiveness though follows from dynamical reasons: The spectral radius entropy coincides with the topological entropy of the *geodesic flow* of ρ, introduced in [23, Section 4]. Since the latter flow is metric Anosov, we know by Subsection 2.3 that its topological entropy is positive (see [74, Theorem 5.1.3] for details).

We have checked the hypothesis on periods and entropy needed to have Sambarino's reparametrization theorem.

Theorem 5.2 (See [74, Corollary 5.3.3]). *Let* $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ *and* $\varphi \in \mathrm{int}((\mathcal{L}_\rho^\Theta)^*)$. *Then there exists an equivariant Hölder homeomorphism*

$$\widetilde{\nu}^{\rho,\varphi} : \widetilde{\mathrm{U}\Gamma} \to \widetilde{\mathrm{U}\Gamma}^{\rho,\varphi},$$

such that for all $(x,y) \in \partial^{(2)}\Gamma$, *there exists an increasing homeomorphism* $\widetilde{h}_{(x,y)}^{\rho,\varphi} : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\widetilde{\nu}^{\rho,\varphi}(x,y,s) = (x,y,\widetilde{h}_{(x,y)}^{\rho,\varphi}(s)) \tag{5.2}$$

for all $s \in \mathbb{R}$. *In particular, the* (ρ,φ)*-refraction action is proper and cocompact. Moreover, if we let* $\nu^{\rho,\varphi} : \mathrm{U}\Gamma \to \mathrm{U}\Gamma^{\rho,\varphi}$ *be the map induced by* $\widetilde{\nu}^{\rho,\varphi}$, *then the flow*

$$(\nu^{\rho,\varphi})^{-1} \circ \phi^{\rho,\varphi} \circ \nu^{\rho,\varphi}$$

is a Hölder reparametrization of ϕ.

Define $\mathtt{R}_\varphi : \mathfrak{X}_\Theta(\Gamma, \mathsf{G}) \to \mathbb{P}\mathrm{HR}(\phi)$ by

$$\mathtt{R}_\varphi(\rho) := [(\nu^{\rho,\varphi})^{-1} \circ \phi^{\rho,\varphi} \circ \nu^{\rho,\varphi}].$$

The map $\mathtt{R}_\varphi$ is well defined because the map $\nu^{\rho,\varphi}$, while not canonical, is well defined up to Livšic equivalence. We will use $\mathtt{R}_\varphi$ together with the work in Sections 2 and 3 to define and study an asymmetric metric on a suitable quotient of $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$: $\mathtt{R}_\varphi$ might not be injective. To this aim, we will relate, in Section 6, the φ-length spectrum (resp., φ-entropy) of ρ with the periods of periodic orbits (resp., topological entropy) of $\phi^{\rho,\varphi}$. We conclude this section discussing the equality:

$$h_\rho^\varphi = h_{\mathrm{top}}(\phi^{\rho,\varphi}).$$

When Γ is torsion free, this follows directly from [74, Theorem 3.2.2]; we include in the next subsection a proof allowing for finite order elements in Γ.

5.2. Strongly primitive elements, periodic orbits, and entropy

The *axis* of an element $\gamma \in \Gamma_{\mathrm{H}}$ is $A_\gamma := (\gamma_-, \gamma_+) \times \mathbb{R} \subset \partial^{(2)}\Gamma \times \mathbb{R}$. The element γ acts via (ρ, φ) on A_γ as translation by $-\varphi(\lambda_\Theta(\rho(\gamma))) = -L_\rho^\varphi(\gamma)$. The axis A_γ descends to a periodic orbit $a_\rho^\varphi(\gamma) = a_\rho^\varphi([\gamma])$ of $\phi^{\rho,\varphi}$: Conjugate elements in Γ determine the same periodic orbit. We let $\mathcal{O}^{\rho,\varphi}$ be the set of periodic orbits of $\phi^{\rho,\varphi}$. The period $p_{\phi^{\rho,\varphi}}(a_\rho^\varphi(\gamma))$ of $a_\rho^\varphi(\gamma)$ divides the number $L_\rho^\varphi(\gamma)$, and we say that γ is *strongly primitive* (w.r.t. the pair (ρ, φ)) if this period is precisely $L_\rho^\varphi(\gamma)$. Denote by $\Gamma_{\mathrm{SP}} \subset \Gamma_{\mathrm{H}}$ the set of strongly primitive elements. A priori, this set depends on the (ρ, φ)-action. However, we will show in Lemma 6.3 that this is not the case.

Remark 5.3. When Γ is torsion free, strongly primitive elements coincide with *primitive* elements of Γ, that is, elements that cannot be written as a power of another element. In that case, there is a one-to-one correspondence between periodic orbits of $\phi^{\rho,\varphi}$ and conjugacy classes of primitive elements in Γ. However, if Γ contains finite order elements, this correspondence no longer holds (see, e.g., Blayac [14, Section 3.4] for a detailed discussion).

The discussion above yields a well-defined map

$$[\Gamma_{\mathrm{H}}] \to \mathcal{O}^{\rho,\varphi} \times (\mathbb{Z}_{>0}) : [\gamma] \mapsto (a_\rho^\varphi(\gamma), n_\rho^\varphi(\gamma)), \tag{5.3}$$

where $n_\rho^\varphi(\gamma) = n_\rho^\varphi([\gamma])$ is determined by the equality

$$L_\rho^\varphi(\gamma) = n_\rho^\varphi(\gamma) p_{\phi^{\rho,\varphi}}(a_\rho^\varphi(\gamma)).$$

To prove the equality $h_\rho^\varphi = h_{\mathrm{top}}(\phi^{\rho,\varphi})$, we first show the following technical lemma (recall that $K_\Gamma > 0$ is the constant given by Bogopolskii–Gerasimov's Theorem [16]).

Lemma 5.4. *The fibers of the map* (5.3) *have at most* K_Γ *elements.*

Proof. Take $(a,n) \in \mathcal{O}^{\rho,\varphi} \times (\mathbb{Z}_{>0})$ and fix $\gamma_0 \in \Gamma_{\mathrm{SP}}$ such that $a^{\varphi}_{\rho}(\gamma_0) = a$. Let $H(\gamma_0)$ be the set of elements in Γ_{H} that act trivially on A_{γ_0}. Since the (ρ,φ)-action is proper, the subgroup $H(\gamma_0)$ is finite and therefore $\#H(\gamma_0) \leq K_\Gamma$. We conclude observing that the fiber over (a,n) is contained in

$$\{[\gamma_0^n \eta] : \eta \in H(\gamma_0)\}. \qquad \blacksquare$$

Corollary 5.5. *Let* $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ *and* $\varphi \in \operatorname{int}((\mathcal{L}^{\Theta}_{\rho})^*)$. *Then the* φ*-entropy of* ρ *coincides with the topological entropy of the refraction flow* $\phi^{\rho,\varphi}$.

Proof. The inequality $h_{\mathrm{top}}(\phi^{\varphi}_{\rho}) \leq h^{\varphi}_{\rho}$ is easily seen. To show the reverse inequality, recall from equation (5.1) that

$$h^{\varphi}_{\rho} = \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma_{\mathrm{H}}] : L^{\varphi}_{\rho}(\gamma) \leq t\}.$$

Lemma 5.4 implies then

$$h^{\varphi}_{\rho} \leq \limsup_{t\to\infty} \frac{1}{t} \log \#\{(a,n) \in \mathcal{O}^{\rho,\varphi} \times (\mathbb{Z}_{>0}) : n p_{\phi^{\rho,\varphi}}(a) \leq t\}.$$

If we let

$$k := \min_{a \in \mathcal{O}^{\rho,\varphi}} p_{\phi^{\rho,\varphi}}(a) > 0,$$

we have

$$\#\{(a,n) \in \mathcal{O}^{\rho,\varphi} \times (\mathbb{Z}_{>0}) : n p_{\phi^{\rho,\varphi}}(a) \leq t\} \leq \frac{t}{k} \times \#\{a \in \mathcal{O}^{\rho,\varphi} : p_{\phi^{\rho,\varphi}}(a) \leq t\}.$$

Equation (2.6) implies the desired inequality. $\blacksquare$

6. Thurston's metric and Finsler norm for Anosov representations

Fix a functional

$$\varphi \in \bigcap_{\rho \in \mathfrak{X}_\Theta(\Gamma,\mathsf{G})} \operatorname{int}((\mathcal{L}^{\Theta}_{\rho})^*).$$

Recall from Section 5 that this induces a map

$$\mathrm{R}_\varphi : \mathfrak{X}_\Theta(\Gamma, \mathsf{G}) \to \mathbb{P}\mathrm{HR}(\phi),$$

where ϕ is a Hölder parametrization of the Gromov–Mineyev geodesic flow of Γ. In view of the contents of Section 3 (and thanks to Theorem 5.1), it is natural to try to "pull back" the asymmetric metric on $\mathbb{P}\mathrm{HR}(\phi)$ to $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ under this map. This motivates the following definition.

Definition 6.1. Define $d^{\varphi}_{\mathrm{Th}} : \mathfrak{X}_{\Theta}(\Gamma, \mathsf{G}) \times \mathfrak{X}_{\Theta}(\Gamma, \mathsf{G}) \to \mathbb{R} \cup \{\infty\}$ by[4]

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) := \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h^{\varphi}_{\widehat{\rho}}}{h^{\varphi}_{\rho}} \frac{L^{\varphi}_{\widehat{\rho}}(\gamma)}{L^{\varphi}_{\rho}(\gamma)} \right).$$

The main theorem of this section is the following.

Theorem 6.2. *The function $d^{\varphi}_{\mathrm{Th}}(\cdot,\cdot)$ is real valued, non-negative, and satisfies the triangle inequality. Furthermore,*

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) = 0 \Leftrightarrow h^{\varphi}_{\rho} L^{\varphi}_{\rho} = h^{\varphi}_{\widehat{\rho}} L^{\varphi}_{\widehat{\rho}}.$$

We deduce Theorem 6.2 from Theorem 3.2: In Corollary 6.4, we show that for all $\rho, \widehat{\rho} \in \mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$,

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) = d_{\mathrm{Th}}(\mathtt{R}_{\varphi}(\rho), \mathtt{R}_{\varphi}(\widehat{\rho})),$$

and in Corollary 6.6, we prove that $\mathtt{R}_{\varphi}(\rho) = \mathtt{R}_{\varphi}(\widehat{\rho})$ if and only if $h^{\varphi}_{\rho} L^{\varphi}_{\rho} = h^{\varphi}_{\widehat{\rho}} L^{\varphi}_{\widehat{\rho}}$. Both Corollaries 6.4 and 6.6 are straightforward when Γ is torsion free (see Remark 5.3). We explain the details in Subsection 6.1 allowing for finite order elements in Γ. In Subsection 6.2, we discuss general conditions that guarantee renormalized length spectrum rigidity. As a consequence, we will have an asymmetric metric defined in interesting subsets of $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$ (under some assumptions on G). More examples will be discussed in Sections 7 and 8. In Subsection 6.3, we use the map $\mathtt{R}_{\varphi}$ to pull back the Finsler norm of $\mathbb{P}\mathrm{HR}(\phi)$ to $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$.

6.1. Proof of Theorem 6.2

Let $\rho \in \mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$. Recall from Subsection 5.2 that $\gamma \in \Gamma_{\mathrm{H}}$ is strongly primitive (w.r.t. (ρ, φ)) if the (ρ, φ)-action of γ on the axis A_{γ} is a translation by the period of the corresponding periodic orbit of $\phi^{\rho,\varphi}$. The following technical lemma implies in particular that this notion is independent of ρ (recall the notation introduced in equation (5.3)). We note that this holds in the more general setting of Hölder reparametrizations of the Gromov geodesic flow (see also Remark 6.7).

Lemma 6.3. *Let ρ and $\widehat{\rho}$ in $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$, then for every $\gamma \in \Gamma_{\mathrm{H}}$ one has*

$$n^{\varphi}_{\rho}(\gamma) = n^{\varphi}_{\widehat{\rho}}(\gamma).$$

In particular, γ is strongly primitive for the (ρ, φ)-action if and only if it is strongly primitive for the $(\widehat{\rho}, \varphi)$-action.

[4]When $\gamma \notin \Gamma_{\mathrm{H}}$ one has $L^{\varphi}_{\rho}(\gamma) = 0 = L^{\varphi}_{\widehat{\rho}}(\gamma)$. In the above definition, it is understood that in that case we set

$$\frac{L^{\varphi}_{\widehat{\rho}}(\gamma)}{L^{\varphi}_{\rho}(\gamma)} = 0.$$

Proof. To ease notations, we let $n := n^{\varphi}_{\rho}(\gamma)$ and $\hat{n} := n^{\varphi}_{\hat{\rho}}(\gamma)$. Suppose by contradiction that $n \neq \hat{n}$, say $n < \hat{n}$.

Let $a = a^{\varphi}_{\rho}(\gamma)$ (resp., $\hat{a} = a^{\varphi}_{\hat{\rho}}(\gamma)$) be the periodic orbit of $\phi^{\rho,\varphi}$ (resp., $\phi^{\hat{\rho},\varphi}$) associated with $[\gamma]$. Fix a strongly primitive γ_0 (resp., $\hat{\gamma}_0$) representing a (resp., $\hat{a}$) for the (ρ, φ)-action (resp., $(\hat{\rho}, \varphi)$-action). By definition of n and $\hat{n}$, we have

$$L^{\varphi}_{\rho}(\gamma) = nL^{\varphi}_{\rho}(\gamma_0) \text{ and } L^{\varphi}_{\hat{\rho}}(\gamma) = \hat{n}L^{\varphi}_{\hat{\rho}}(\hat{\gamma}_0). \tag{6.1}$$

We may assume furthermore that $(\gamma_0)_{\pm} = (\hat{\gamma}_0)_{\pm}$.

On the other hand, by Theorem 5.2, there exists an equivariant Hölder homeomorphism

$$\nu : \widetilde{\mathrm{U}\Gamma}^{\rho,\varphi} \to \widetilde{\mathrm{U}\Gamma}^{\hat{\rho},\varphi},$$

such that for all $(x, y) \in \partial^{(2)}\Gamma$ there exists an increasing homeomorphism $h_{(x,y)} : \mathbb{R} \to \mathbb{R}$ satisfying

$$\nu(x, y, s) = (x, y, h_{(x,y)}(s)).$$

Hence, for all $\eta \in \Gamma$ and all $(x, y, s) \in \widetilde{\mathrm{U}\Gamma}^{\rho,\varphi}$ one has

$$h_{(\eta\cdot x,\eta\cdot y)}(s - \varphi \circ \beta^{\rho}_{\Theta}(\eta, y)) = h_{(x,y)}(s) - \varphi \circ \beta^{\hat{\rho}}_{\Theta}(\eta, y).$$

In particular, equation (6.1) gives

$$h_{((\gamma_0)_-,(\gamma_0)_+)}(s - nL^{\varphi}_{\rho}(\gamma_0)) = h_{((\gamma_0)_-,(\gamma_0)_+)}(s - L^{\varphi}_{\rho}(\gamma)) = h_{((\gamma_0)_-,(\gamma_0)_+)}(s) - L^{\varphi}_{\hat{\rho}}(\gamma),$$

and therefore

$$h_{((\gamma_0)_-,(\gamma_0)_+)}(s - nL^{\varphi}_{\rho}(\gamma_0)) = h_{((\gamma_0)_-,(\gamma_0)_+)}(s) - \hat{n}L^{\varphi}_{\hat{\rho}}(\hat{\gamma}_0).$$

Hence,

$$h_{((\gamma_0)_-,(\gamma_0)_+)}(s - nL^{\varphi}_{\rho}(\gamma_0)) = h_{((\gamma_0)_-,(\gamma_0)_+)}(s) - L^{\varphi}_{\hat{\rho}}(\hat{\gamma}_0^{\hat{n}}) = h_{((\gamma_0)_-,(\gamma_0)_+)}(s - L^{\varphi}_{\rho}(\hat{\gamma}_0^{\hat{n}})).$$

We then conclude

$$h_{((\gamma_0)_-,(\gamma_0)_+)}(s - nL^{\varphi}_{\rho}(\gamma_0)) = h_{((\gamma_0)_-,(\gamma_0)_+)}(s - \hat{n}L^{\varphi}_{\rho}(\hat{\gamma}_0)).$$

This implies

$$nL^{\varphi}_{\rho}(\gamma_0) = \hat{n}L^{\varphi}_{\rho}(\hat{\gamma}_0) > nL^{\varphi}_{\rho}(\hat{\gamma}_0).$$

This is a contradiction because γ_0 was assumed to be strongly primitive for the (ρ, φ)-action. ∎

Corollary 6.4. *For every ρ and $\hat{\rho}$ in $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$ one has*

$$d^{\varphi}_{\mathrm{Th}}(\rho, \hat{\rho}) = d_{\mathrm{Th}}(\mathrm{R}_{\varphi}(\rho), \mathrm{R}_{\varphi}(\hat{\rho})).$$

Proof. By Corollary 5.5, we have

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h_{\mathrm{top}}(\phi^{\widehat{\rho},\varphi})}{h_{\mathrm{top}}(\phi^{\rho,\varphi})} \frac{L^{\varphi}_{\widehat{\rho}}(\gamma)}{L^{\varphi}_{\rho}(\gamma)} \right).$$

Equation (5.3) gives then

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h_{\mathrm{top}}(\phi^{\widehat{\rho},\varphi})}{h_{\mathrm{top}}(\phi^{\rho,\varphi})} \frac{n^{\varphi}_{\widehat{\rho}}(\gamma)}{n^{\varphi}_{\rho}(\gamma)} \frac{p_{\phi^{\widehat{\rho},\varphi}}(a^{\varphi}_{\widehat{\rho}}(\gamma))}{p_{\phi^{\rho,\varphi}}(a^{\varphi}_{\rho}(\gamma))} \right).$$

By Lemma 6.3, we have

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h_{\mathrm{top}}(\phi^{\widehat{\rho},\varphi})}{h_{\mathrm{top}}(\phi^{\rho,\varphi})} \frac{p_{\phi^{\widehat{\rho},\varphi}}(a^{\varphi}_{\widehat{\rho}}(\gamma))}{p_{\phi^{\rho,\varphi}}(a^{\varphi}_{\rho}(\gamma))} \right).$$

This finishes the proof. ∎

Remark 6.5. There are geometric settings in which the renormalization by entropy in the definition of the asymmetric metric is essential (see also Section 2.4). For instance, Tholozan [79, Theorem B] shows that there exist pairs ρ and j in $\mathrm{Hit}_3(S)$ for which there is a $c > 1$ so that

$$L^{\mathrm{H}}_{\rho}(\gamma) \geq c L^{\mathrm{H}}_{j}(\gamma) \tag{6.2}$$

for all $\gamma \in \pi_1(S)$ (recall the notation introduced in Example 4.11). Hence,

$$\log\left(\sup_{[\gamma]\in[\pi_1(S)]} \frac{L^{\mathrm{H}}_{j}(\gamma)}{L^{\mathrm{H}}_{\rho}(\gamma)} \right) \leq \log\left(\frac{1}{c}\right) < 0.$$

On the other hand, some length functions on some spaces of Anosov representations have constant entropies (cf., Subsection 4.3). In these situations, renormalizing by entropy is not needed.

We now compute the set of points which are identified under the map $\mathtt{R}_{\varphi}$, finishing the proof of Theorem 6.2.

Corollary 6.6. *Let ρ and $\widehat{\rho}$ be two points in $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$. Then*

$$\mathtt{R}_{\varphi}(\rho) = \mathtt{R}_{\varphi}(\widehat{\rho}) \Leftrightarrow h^{\varphi}_{\rho} L^{\varphi}_{\rho} = h^{\varphi}_{\widehat{\rho}} L^{\varphi}_{\widehat{\rho}}.$$

Proof. By definition of $\mathbb{P}\mathrm{HR}(\phi)$ and Corollary 5.5, we have

$$\mathtt{R}_{\varphi}(\rho) = \mathtt{R}_{\varphi}(\widehat{\rho}) \Leftrightarrow h^{\varphi}_{\rho} p_{\phi^{\rho,\varphi}}(a^{\varphi}_{\rho}(\gamma)) = h^{\varphi}_{\widehat{\rho}} p_{\phi^{\widehat{\rho},\varphi}}(a^{\varphi}_{\widehat{\rho}}(\gamma))$$

for all $\gamma \in \Gamma_{\mathrm{H}}$. Thanks to Lemma 6.3, this is equivalent to

$$h^{\varphi}_{\rho} n^{\varphi}_{\rho}(\gamma) p_{\phi^{\rho,\varphi}}(a^{\varphi}_{\rho}(\gamma)) = h^{\varphi}_{\widehat{\rho}} n^{\varphi}_{\widehat{\rho}}(\gamma) p_{\phi^{\widehat{\rho},\varphi}}(a^{\varphi}_{\widehat{\rho}}(\gamma))$$

for all $\gamma \in \Gamma_{\mathrm{H}}$. Since for all $\gamma \in \Gamma_{\mathrm{H}}$ we have

$$n_\rho^\varphi(\gamma) p_{\phi^{\rho,\varphi}}(a_\rho^\varphi(\gamma)) = L_\rho^\varphi(\gamma) \quad \text{and} \quad n_{\widehat{\rho}}^\varphi(\gamma) p_{\phi^{\widehat{\rho},\varphi}}(a_{\widehat{\rho}}^\varphi(\gamma)) = L_{\widehat{\rho}}^\varphi(\gamma),$$

the proof is finished. ∎

To finish this subsection, we record the following technical remark for future use.

Remark 6.7. One may define the notion of strongly primitive elements for the action $\Gamma \curvearrowright \widetilde{\mathrm{U}\Gamma}$, in a way analogous to the definition for the action $\Gamma \curvearrowright \widetilde{\mathrm{U}\Gamma}^{\rho,\varphi}$. As in Lemma 6.3, one shows that γ is strongly primitive for $\Gamma \curvearrowright \widetilde{\mathrm{U}\Gamma}$ if and only if it is strongly primitive for the (ρ,φ)-action, for some (any) $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$.

On the other hand, if we let $\mathcal{O}$ be the set of periodic orbits of ϕ, we may take for each $a \in \mathcal{O}$ a strongly primitive representative $\gamma_a \in \Gamma_{\mathrm{SP}}$. We see that

$$a \mapsto [A_{\gamma_a}]$$

defines a one-to-one correspondence between $\mathcal{O}$ and $\mathcal{O}^{\rho,\varphi}$ for all $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$, where $[A_{\gamma_a}]$ is the image of the axis A_{γ_a} under the quotient map $\widetilde{\mathrm{U}\Gamma}^{\rho,\varphi} \to \mathrm{U}\Gamma^{\rho,\varphi}$. A set $\{\gamma_a\}_{a\in\mathcal{O}}$ of strongly primitive elements representing each periodic orbit will be fixed from now on.

6.2. Renormalized length spectrum rigidity

Recall that G is a connected semisimple real algebraic group of non-compact type. In this subsection, we discuss necessary conditions that two Θ-Anosov representations with the same renormalized length spectra must satisfy.

For a Lie group G_1, we denote by $(\mathsf{G}_1)_0$ the connected component, in the Hausdorff topology, containing the identity. If $\sigma : \mathsf{G}_1 \to \mathsf{G}_2$ is a Lie group isomorphism, we denote, with a slight abuse of notation, by $\sigma : \mathfrak{a}_{\mathsf{G}_1}^+ \to \mathfrak{a}_{\mathsf{G}_2}^+$ the induced linear isomorphism between Weyl chambers. Furthermore, if $\mathsf{G}_1 < \mathsf{G}$ is a Lie group inclusion, we denote by $\pi_{\mathsf{G}_1} : \mathfrak{a}_{\mathsf{G}_1}^+ \to \mathfrak{a}_{\mathsf{G}}^+$ the induced piecewise linear map.

We will need the following fairly general classical rigidity result, which is an application of Benoist [7, Theorem 1]. See, for instance, [23, Corollary 11.6], Burger [25], and Dal'bo–Kim [28].

Theorem 6.8. *Let ρ and $\widehat{\rho}$ be two Θ-Anosov representations into G. Denote by G_ρ (resp., $\mathsf{G}_{\widehat{\rho}}$) the Zariski closure of $\rho(\Gamma)$ (resp., $\widehat{\rho}(\Gamma)$). Assume that G_ρ and $\mathsf{G}_{\widehat{\rho}}$ are simple, real algebraic, and center free. Assume furthermore $\rho(\Gamma) \subset (\mathsf{G}_\rho)_0$ and $\widehat{\rho}(\Gamma) \subset (\mathsf{G}_{\widehat{\rho}})_0$. Then if the equality $h_\rho^\varphi L_\rho^\varphi = h_{\widehat{\rho}}^\varphi L_{\widehat{\rho}}^\varphi$ holds, there exists an isomorphism $\sigma : (\mathsf{G}_\rho)_0 \to (\mathsf{G}_{\widehat{\rho}})_0$ such that $\sigma \circ \rho = \widehat{\rho}$. It furthermore holds $\varphi \circ \pi_{\mathsf{G}_{\widehat{\rho}}} \circ \sigma = \varphi \circ \pi_{\mathsf{G}_\rho}$.*

Denote by $\mathfrak{X}_\Theta^{\mathrm{Z}}(\Gamma, \mathsf{G}) \subset \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ the subset consisting of Zariski-dense representations.

Corollary 6.9. *Assume that* G *is simple, center free, and for every non-inner automorphism* σ *of* G *one has* $\varphi \circ \sigma \neq \varphi$. *Then* $d^{\varphi}_{\mathrm{Th}}(\cdot,\cdot)$ *defines a (possibly asymmetric) metric on* $\mathfrak{X}^{Z}_{\Theta}(\Gamma, \mathsf{G})$.

Remark 6.10. The group G needs to be center free in Theorem 6.8 and Corollary 6.9: the Jordan and Cartan projections of G factor through the adjoint form of G, thus any two representations differing by a central character will have the same renormalized length spectrum, and thus distance zero.

6.3. Finsler norm for Anosov representations

Bridgeman–Canary–Labourie–Sambarino [23, 24] used the map $\mathtt{R}_{\varphi}$ to pull back the pressure norm on $\mathbb{P}\mathrm{HR}^{\upsilon}(\phi)$ to produce a pressure metric on $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$ (for some choices of φ). We now imitate this procedure working with the Finsler norm defined in Subsection 3.2.

A family of representations $\{\rho_z : \Gamma \to \mathsf{G}\}_{z \in D}$ parametrized by a real analytic disk D is *real analytic* if for all $\gamma \in \Gamma$, the map $z \mapsto \rho_z(\gamma)$ is real analytic. We fix a real analytic neighborhood of $\rho \in \mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$ and a real analytic family $\{\rho_z\}_{z \in D} \subset \mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$, parametrized by some real analytic disk D around 0, so that $\rho_0 = \rho$ and $\bigcup_{z \in D} \rho_z$ coincides with this neighborhood. By abuse of notation, we will sometimes identify the neighborhood with D itself.

Definition 6.11. Given a tangent vector $v \in T_{\rho}\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$, we set

$$\|v\|^{\varphi}_{\mathrm{Th}} := \sup_{[\gamma] \in [\Gamma_{\mathrm{H}}]} \frac{\mathrm{d}_{\rho}(h^{\varphi}_{\cdot})(v) L^{\varphi}_{\rho}(\gamma) + h^{\varphi}_{\rho} \mathrm{d}_{\rho}(L^{\varphi}_{\cdot}(\gamma))(v)}{h^{\varphi}_{\rho} L^{\varphi}_{\rho}(\gamma)},$$

where $\mathrm{d}_{\rho}(h^{\varphi}_{\cdot})$ (resp., $\mathrm{d}_{\rho}(L^{\varphi}_{\cdot}(\gamma))$) is the derivative of $\widehat{\rho} \mapsto h^{\varphi}_{\widehat{\rho}}$ (resp., $\widehat{\rho} \mapsto L^{\varphi}_{\widehat{\rho}}(\gamma)$) at ρ. In particular, if $\widehat{\rho} \mapsto h^{\varphi}_{\widehat{\rho}}$ is constant one has

$$\|v\|^{\varphi}_{\mathrm{Th}} = \sup_{[\gamma] \in [\Gamma_{\mathrm{H}}]} \frac{\mathrm{d}_{\rho}(L^{\varphi}_{\cdot}(\gamma))(v)}{L^{\varphi}_{\rho}(\gamma)}. \tag{6.3}$$

Remark 6.12. We make two remarks about the φ-entropy and the Finster norm $\|\cdot\|^{\varphi}_{\mathrm{Th}}$.

(1) Recall that by [23, Section 8], entropy varies in an analytic way over $\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$. In particular, $h^{\varphi}_{\cdot}$ is differentiable.

(2) Equation (6.3) generalizes Thurston's Finsler norm on Teichmüller space [81, p. 20].

We want conditions guaranteeing that $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ defines a Finsler norm on $T_{\rho}\mathfrak{X}_{\Theta}(\Gamma, \mathsf{G})$; a priori it is not even clear that $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ is real valued and non-negative. To link $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ and the Finsler norm of Subsection 3.2, we need the following proposition. We fixed a set of strongly primitive elements $\{\gamma_a\}$ representing each periodic orbit $a \in \mathcal{O}$ in Remark 6.7.

Proposition 6.13 ([23, Proposition 6.2], [24, Proposition 6.1]). *Let $\{\rho_z\}_{z\in D}$ be a real analytic family of Θ-Anosov representations. Then up to restricting D to a smaller disk around 0, there exists $\upsilon > 0$ and a real analytic family $\{\widetilde{g}_z : \mathrm{U}\Gamma \to \mathbb{R}_{>0}\}_{z\in D} \subset \mathcal{H}^\upsilon(\mathrm{U}\Gamma)$ so that for all $z \in D$, all $a \in \mathcal{O}$, and all $x \in a$ one has*

$$\int_0^{p_\phi(a)} \widetilde{g}_z(\phi_s(x))\mathrm{d}s = L^\varphi_{\rho_z}(\gamma_a).$$

In particular, the map $D \to \mathbb{P}\mathrm{HR}^\upsilon(\phi)$ given by $z \mapsto \mathrm{R}_\varphi(\rho_z) = \left[\phi^{\widetilde{g}_z}\right]$ is real analytic.

Proof. The argument follows [24, Proposition 6.1]. Since $\{\omega_\alpha\}_{\alpha\in\Theta}$ span $\mathfrak{a}^*_\Theta$, there exist real numbers a_α so that $\varphi = \sum_{\alpha\in\Theta} a_\alpha\omega_\alpha$. The result for projective Anosov representations and the spectral radius length function is given by [23, Proposition 6.2]; thus, the proof of [24, Proposition 6.1] applies (cf., Proposition 4.7 and equation (4.3)). ∎

Fix a real analytic family $\{\widetilde{g}_z\}$ as in Proposition 6.13. By [23, Proposition 3.12], the function $z \mapsto h_{\phi^{\widetilde{g}_z}}$ is real analytic. By Corollary 5.5, we get that $z \mapsto h^\varphi_{\rho_z}$ is real analytic, as claimed in Remark 6.12.

Proposition 6.13 bridges between $\|\cdot\|^\varphi_{\mathrm{Th}}$ and the Finsler norm on $\mathbb{P}\mathrm{HR}^\upsilon(\phi)$, as we now explain. First, observe that in Definition 6.11, it suffices to consider only strongly primitive elements when taking the sup, that is,

$$\|v\|^\varphi_{\mathrm{Th}} = \sup_{[\gamma]\in[\Gamma_{\mathrm{SP}}]} \frac{\mathrm{d}_\rho(h^\varphi_\cdot)(v)L^\varphi_\rho(\gamma) + h^\varphi_\rho \mathrm{d}_\rho(L^\varphi_\cdot(\gamma))(v)}{h^\varphi_\rho L^\varphi_\rho(\gamma)}.$$

Indeed, the function $\widehat{\rho} \mapsto n^\varphi_{\widehat{\rho}}(\gamma)$ is constant for all $\gamma \in \Gamma_{\mathrm{H}}$ (Lemma 6.3), and Remark 6.7 gives

$$\|v\|^\varphi_{\mathrm{Th}} = \sup_{a\in\mathcal{O}} \frac{\mathrm{d}_\rho(h^\varphi_\cdot)(v)L^\varphi_\rho(\gamma_a) + h^\varphi_\rho \mathrm{d}_\rho(L^\varphi_\cdot(\gamma_a))(v)}{h^\varphi_\rho L^\varphi_\rho(\gamma_a)}. \tag{6.4}$$

Recalling the notations from Subsection 3.2, we have the following.

Lemma 6.14. *Let $\{\rho_z\}_{z\in D} \subset \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ be a real analytic family parametrizing an open neighborhood around $\rho = \rho_0$. Fix an analytic path $z : (-1, 1) \to D$ so that $z(0) = 0$ and set $\rho_s := \rho_{z(s)}$ and $v := \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\rho_s$. Let also $h_s := h^\varphi_{\rho_s}$ and $g_s := h_s\widetilde{g}_{z(s)}$. Then*

$$\|v\|^\varphi_{\mathrm{Th}} = \|[\dot{g}_0]\|_{\mathrm{Th}}.$$

In the above statement, by construction, the Livšic cohomology class $[\dot{g}_0] = [\dot{g}_0]_\phi$ belongs to the tangent space $T_{[\phi^{g_0}]}\mathbb{P}\mathrm{HR}^\upsilon(\phi)$.

Proof. Combining equations (6.4) and (2.3) and Proposition 6.13, we have

$$\|v\|^\varphi_{\mathrm{Th}} = \sup_{a\in\mathcal{O}} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \frac{h_s L^\varphi_{\rho_s}(\gamma_a)}{h^\varphi_\rho L^\varphi_\rho(\gamma_a)} = \sup_{a\in\mathcal{O}} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \frac{h_s \int \widetilde{g}_s \mathrm{d}\delta_\phi(a)}{h_0 \int \widetilde{g}_0 \mathrm{d}\delta_\phi(a)}.$$

Hence,

$$\|v\|^{\varphi}_{\mathrm{Th}} = \sup_{a\in\mathcal{O}} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \frac{\int g_s \mathrm{d}\delta_\phi(a)}{\int g_0 \mathrm{d}\delta_\phi(a)} = \sup_{a\in\mathcal{O}} \frac{\int \dot{g}_0 \mathrm{d}\delta_\phi(a)}{\int g_0 \mathrm{d}\delta_\phi(a)}.$$

By Theorem 2.10, we get

$$\|v\|^{\varphi}_{\mathrm{Th}} = \sup_{m\in\mathcal{P}(\phi)} \frac{\int \dot{g}_0 \mathrm{d}m}{\int g_0 \mathrm{d}m}.$$

This finishes the proof. ∎

From Propositions 3.6 and 6.13 and Corollary 6.4, we obtain the following.

Corollary 6.15. *Keep the notations from Lemma 6.14. Then $s \mapsto d^{\varphi}_{\mathrm{Th}}(\rho, \rho_s)$ is differentiable at $s = 0$ and*

$$\|v\|^{\varphi}_{\mathrm{Th}} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} d^{\varphi}_{\mathrm{Th}}(\rho, \rho_s).$$

We now turn to the study of conditions guaranteeing that $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ defines a Finsler norm.

Corollary 6.16. *Let $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$ be a point admitting an analytic neighborhood in $\mathfrak{X}_\Theta(\Gamma, \mathsf{G})$. Then function $\|\cdot\|^{\varphi}_{\mathrm{Th}} : T_\rho\mathfrak{X}_\Theta(\Gamma, \mathsf{G}) \to \mathbb{R} \cup \{\pm\infty\}$ is real valued and non-negative. Furthermore, it is $(\mathbb{R}_{>0})$-homogeneous, satisfies the triangle inequality, and $\|v\|^{\varphi}_{\mathrm{Th}} = 0$ if and only if*

$$d_\rho(L^{\varphi}_{\cdot}(\gamma))(v) = -\frac{\mathrm{d}_\rho(h^{\varphi}_{\cdot})(v)}{h^{\varphi}_\rho} L^{\varphi}_\rho(\gamma) \tag{6.5}$$

for all $\gamma \in \Gamma_{\mathrm{H}}$. In particular, if the function $\rho \mapsto h^{\varphi}_\rho$ is constant, then

$$\|v\|^{\varphi}_{\mathrm{Th}} = 0 \Leftrightarrow d_\rho(L^{\varphi}_{\cdot}(\gamma))(v) = 0$$

for all $\gamma \in \Gamma_{\mathrm{H}}$.

Proof. By Lemmas 3.5 and 6.14, the function $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ is real valued, non-negative, $(\mathbb{R}_{>0})$-homogeneous and satisfies the triangle inequality. Furthermore, keeping the notation from Lemma 6.14, if $\|v\|^{\varphi}_{\mathrm{Th}} = 0$ then $\dot{g}_0 \sim_\phi 0$, and this condition is equivalent to

$$0 = \int_0^{p_\phi(a)} \dot{g}_0(\phi_t(x))\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_0^{p_\phi(a)} g_s(\phi_t(x))\mathrm{d}t$$

for all $a \in \mathcal{O}$ and $x \in a$. Hence,

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} h_s \int_0^{p_\phi(a)} \widetilde{g}_s(\phi_t(x))\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} h_s L^{\varphi}_{\rho_s}(\gamma_a).$$

Thus,

$$d_\rho(L^{\varphi}_{\cdot}(\gamma_a))(v) = -\frac{\mathrm{d}_\rho(h^{\varphi}_{\cdot})(v)}{h^{\varphi}_\rho} L^{\varphi}_\rho(\gamma_a)$$

for all $a \in \mathcal{O}$. Now by Lemma 6.3, for every $\gamma \in \Gamma_{\mathrm{H}}$ there is some $n \geq 1$ and $a \in \mathcal{O}$ so that $L^{\varphi}_\rho(\gamma) = nL^{\varphi}_\rho(\gamma_a)$ for all $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G})$. This finishes the proof. ∎

In view of Corollary 6.16, to show that $\|\cdot\|^{\varphi}_{\mathrm{Th}}$ is a Finsler norm, one needs to guarantee that condition (6.5) implies $v = 0$. These type of questions have been addressed by Bridgeman–Canary–Labourie–Sambarino [23, 24] in some situations. Rather than discussing these results here, we will recall them in the next sections, when needed.

7. Hitchin representations

In this section, we focus on Hitchin representations. The Zariski closures of $\mathsf{PSL}(d,\mathbb{R})$-Hitchin representations have been classified by Guichard. Hence, the results of the previous section apply nicely in this setting giving global rigidity results and leading to asymmetric distances in the whole component. This is explained in detail in Subsection 7.1, where we also treat the case of $\mathsf{PSO}_0(p,p)$, the remaining classical case not covered by Guichard's classification, using recent results by Sambarino [73]. In Subsection 7.2, we discuss Finsler norms associated with some special length functionals in the $\mathsf{PSL}(d,\mathbb{R})$-Hitchin component, showing that they are nondegenerate (this will be a consequence of Corollary 6.16 and results in [23, 24]).

Throughout this section, we let S be a closed oriented surface of genus $g \geq 2$ and denote by $\Gamma = \pi_1(S)$ its fundamental group. We also let G be an adjoint, connected, simple real-split Lie group. Apart from exceptional cases, G is one of the following:

$$\mathsf{PSL}(d,\mathbb{R}), \quad \mathsf{PSp}(2r,\mathbb{R}), \quad \mathsf{SO}_0(p,p+1), \quad \text{or} \quad \mathsf{PSO}_0(q,q),$$

for $q > 2$. Hitchin representations are Π-Anosov (cf., Example 4.13). We denote by $\mathrm{Hit}(S,\mathsf{G})$ the Hitchin component into G, when $\mathsf{G} = \mathsf{PSL}(d,\mathbb{R})$ we also use the special notation $\mathrm{Hit}_d(S)$.

7.1. Length spectrum rigidity

For $\rho \in \mathrm{Hit}(S,\mathsf{G})$ denote $\mathcal{L}^*_\rho := (\mathcal{L}^{\Pi}_\rho)^*$ and consider $\varphi \in \bigcap_{\rho\in\mathrm{Hit}(S,\mathsf{G})} \mathrm{int}(\mathcal{L}^*_\rho) \subset \mathfrak{a}^*_\Pi = \mathfrak{a}^*$.

The main goal of this section is to prove the following.

Theorem 7.1. *Let G be an adjoint, simple, real-split Lie group of classical type. In the case $\mathsf{G} = \mathsf{PSO}_0(p,p)$, assume furthermore $p \neq 4$. Let $\varphi \in \bigcap_{\rho\in\mathrm{Hit}(S,\mathsf{G})} \mathrm{int}(\mathcal{L}^*_\rho)$ be so that $\varphi \circ \sigma \neq \varphi$ for every non-inner automorphism σ of G. If $\rho, \widehat{\rho} \in \mathrm{Hit}(S,\mathsf{G})$ satisfy $h^\varphi_\rho L^\varphi_\rho = h^\varphi_{\widehat{\rho}} L^\varphi_{\widehat{\rho}}$, then $\rho = \widehat{\rho}$.*

Before going into the proof of Theorem 7.1, we make few remarks and establish the main corollaries of interest.

Remark 7.2. We make two remarks for the groups $\mathsf{G} = \mathsf{PSL}(d,\mathbb{R})$ and $\mathsf{G} = \mathsf{PSO}_0(4,4)$.

- When $\mathsf{G} = \mathsf{PSL}(d,\mathbb{R})$, Bridgeman–Canary–Labourie–Sambarino [23, Corollary 11.8] proved Theorem 7.1 for the spectral radius length function $\varphi = \lambda_1$. The proof of Theorem 7.1 follows the same approach.

- We aim to define a simple root asymmetric metric on $\mathrm{Hit}(S, \mathsf{G})$ (Corollary 7.3). As every simple root of $\mathsf{PSO}_0(4,4)$ is fixed by a non-inner automorphism, the function

$$d^{\alpha}_{\mathrm{Th}} : \mathrm{Hit}(S, \mathsf{PSO}_0(4,4)) \times \mathrm{Hit}(S, \mathsf{PSO}_0(4,4)) \to \mathbb{R}$$

  does not separate points for any simple root α. This is the main reason why we exclude the case $\mathsf{G} = \mathsf{PSO}_0(4,4)$ in the statement of Theorem 7.1.

We have the following two consequences of Theorem 7.1.

Corollary 7.3. *Let* G *be an adjoint, simple, real-split Lie group of classical type. Let* α *be any simple root of* G*, with the exception of the roots listed in Table 1. Then the function* $d^{\alpha}_{\mathrm{Th}} : \mathrm{Hit}(S, \mathsf{G}) \times \mathrm{Hit}(S, \mathsf{G}) \to \mathbb{R}$ *given by*

$$d^{\alpha}_{\mathrm{Th}}(\rho, \widehat{\rho}) := \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{L^{\alpha}_{\widehat{\rho}}(\gamma)}{L^{\alpha}_{\rho}(\gamma)} \right)$$

defines an asymmetric distance on $\mathrm{Hit}(S, \mathsf{G})$.

Proof. By Potrie–Sambarino [65, Theorem B] and Pozzetti–Sambarino–Wienhard [67, Theorem 9.9], we have $h^{\alpha}_{\rho} = 1$ for all $\rho \in \mathrm{Hit}(S, \mathsf{G})$. Since roots as in the statement are not fixed by non-inner automorphisms of G, then by Theorems 6.2 and 7.1, the function d^{α}_{Th} defines a possibly asymmetric metric.

It remains to show that d^{α}_{Th} is indeed asymmetric. But Thurston [81, p. 5] exhibits examples of points $\rho, \widehat{\rho} \in \mathrm{Teich}(S)$ for which the distance from ρ to $\widehat{\rho}$ is different from the distance from $\widehat{\rho}$ to ρ. Since $\mathrm{Hit}(S, \mathsf{G})$ contains a copy of $\mathrm{Teich}(S)$, the claim follows. ■

Corollary 7.4. *Let* $\mathsf{G} = \mathsf{PSL}(d, \mathbb{R})$ *and* $\varphi = \lambda_1$ *be the spectral radius length function. Then the function* $d^{\lambda_1}_{\mathrm{Th}} : \mathrm{Hit}_d(S) \times \mathrm{Hit}_d(S) \to \mathbb{R}$ *given by*

$$d^{\lambda_1}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h^{\lambda_1}_{\widehat{\rho}}}{h^{\lambda_1}_{\rho}} \frac{L^{\lambda_1}_{\widehat{\rho}}(\gamma)}{L^{\lambda_1}_{\rho}(\gamma)} \right)$$

defines an asymmetric distance on $\mathrm{Hit}_d(S)$.

Proof. The action on α of the unique non-inner automorphism of $\mathsf{PSL}(d, \mathbb{R})$ coincides with the opposition involution ι. When $d > 2$ note that $\lambda_1 \neq \lambda_1 \circ \iota$; hence, in this case the result follows from Theorems 6.2 and 7.1. If $d = 2$, the result follows from Theorem 6.2 and the length spectrum rigidity for hyperbolic surfaces. ■

We now turn to the proof of Theorem 7.1. In view of the natural inclusions

$$\mathrm{Hit}(S, \mathsf{PSp}(2r, \mathbb{R})) \subset \mathrm{Hit}_{2r}(S) \quad \text{and} \quad \mathrm{Hit}(S, \mathsf{SO}_0(p, p+1)) \subset \mathrm{Hit}_{2p+1}(S),$$

we may assume that G is either $\mathsf{PSL}(d, \mathbb{R})$ or $\mathsf{PSO}_0(p, p)$. We will focus on the case $\mathsf{G} = \mathsf{PSO}_0(p, p)$, the argument for $\mathsf{G} = \mathsf{PSL}(d, \mathbb{R})$ is similar (and further, in that case the reader can also compare with [23, Corollary 11.8]).

The main step in the proof is to carefully analyze the possible Zariski closures of $\mathsf{PSO}_0(p, p)$-Hitchin representations and show that they satisfy the hypotheses of Theorem 6.8. This is achieved in Corollaries 7.8 and 7.9, as an application of recent work by Sambarino [73].

Let then $p > 2$ and consider a principal embedding $\tau : \mathsf{PSL}(2, \mathbb{R}) \to \mathsf{PSO}_0(p, p)$. Then τ factors as

$$\tau : \mathsf{PSL}(2, \mathbb{R}) \to \mathsf{SO}_0(p, p-1) \to \mathsf{PSO}_0(p, p),$$

where the first map is the irreducible representation into $\mathsf{SL}(2p-1, \mathbb{R})$ and the second is induced by the standard embedding stabilizing a non-isotropic line $\ell_\tau \subset \mathbb{R}^{2p}$. We let π_τ be the complementary $(p, p-1)$-hyperplane. Note that τ lifts to a principal embedding $\widehat{\tau} : \mathsf{PSL}(2, \mathbb{R}) \to \mathsf{SO}_0(p, p)$. A *Fuchsian representation* is a Hitchin representation into $\mathsf{PSO}_0(p, p)$ (resp., $\mathsf{SO}_0(p, p-1)$) whose image is contained in a conjugate of $\tau(\mathsf{PSL}(2, \mathbb{R}))$ (resp., $\widehat{\tau}(\mathsf{PSL}(2, \mathbb{R}))$). The following is well known (see, e.g., [73, p. 25] for a proof).

Lemma 7.5. *Let $\rho \in \mathrm{Hit}(S, \mathsf{PSO}_0(p, p))$. Then there exists a representation $\widehat{\rho} : \Gamma \to \mathsf{SO}_0(p, p)$ lifting ρ that may be deformed to a Fuchsian representation.*

Here is another useful lemma.

Lemma 7.6. *Let $\widehat{\rho} : \Gamma \to \mathsf{SO}_0(p, p)$ be a Hitchin representation. Then the Zariski closure $\mathsf{G}_{\widehat{\rho}}$ of $\widehat{\rho}$ is reductive.*

Proof. Suppose by contradiction that $\mathsf{G}_{\widehat{\rho}}$ is not reductive. Then $\mathsf{G}_{\widehat{\rho}}$ is contained in a proper parabolic subgroup of $\mathsf{SO}_0(p, p)$ [18]. That is, $\widehat{\rho}(\Gamma)$ stabilizes a totally isotropic subspace W of $\mathbb{R}^{2p}$.

Now the proof reduces to Sambarino [73, Theorem B]. Indeed, let $\mathfrak{g}_{\widehat{\rho}}^{ss}$ be the semisimple part of the Lie algebra $\mathfrak{g}_{\widehat{\rho}}$ of $\mathsf{G}_{\widehat{\rho}}$. By assumption, we have $\mathfrak{g}_{\widehat{\rho}}^{ss} \neq \mathfrak{g}_{\widehat{\rho}}$ and so [73, Theorem B] implies that $\mathfrak{g}_{\widehat{\rho}}^{ss}$ is either a principal $\mathfrak{sl}_2(\mathbb{R})$ or isomorphic to a copy of $\mathfrak{so}(p, p-1)$, stabilizing a non-isotropic line ℓ and a complementary hyperplane π. In either case, $\mathfrak{g}_{\widehat{\rho}}^{ss}$ acts irreducibly on a hyperplane π of signature $(p, p-1)$. But then $\ell \oplus W$ intersects π nontrivially and is $\mathfrak{g}_{\widehat{\rho}}^{ss}$-invariant, a contradiction. ∎

For a Hitchin representation $\widehat{\rho} : \Gamma \to \mathsf{SO}_0(p, p)$, let $\mathfrak{g}_{\widehat{\rho}}^{ss}$ be the semisimple part of the Lie algebra $\mathfrak{g}_{\widehat{\rho}}$ of $\mathsf{G}_{\widehat{\rho}}$. By Sambarino [73, Theorem A], if $p \neq 4$ then $\mathfrak{g}_{\widehat{\rho}}^{ss}$ is either $\mathfrak{so}(p, p)$, a principal $\mathfrak{sl}_2$, or the image of the standard embedding $\mathfrak{so}(p, p-1) \to \mathfrak{so}(p, p)$. In each case $\mathfrak{g}_{\widehat{\rho}}^{ss}$ contains, up to conjugation, the Lie subalgebra $\mathrm{d}\widehat{\tau}(\mathfrak{sl}_2)$.

Lemma 7.7. *Let $\widehat{\rho} : \Gamma \to \mathsf{SO}_0(p, p)$ be a Hitchin representation. Suppose that $g \in \mathsf{G}_{\widehat{\rho}}$ satisfies $ghg^{-1} = \pm h$ for all $h \in \mathsf{G}_{\widehat{\rho}}$. Then $g \in \{\mathrm{id}, -\mathrm{id}\}$.*

Proof. Let $g \in \mathsf{G}_{\hat\rho}$ be as in the statement. Since $\hat\tau(\mathsf{PSL}(2,\mathbb{R})) \subset (\mathsf{G}_{\hat\rho})_0$, then g centralizes (up to a sign) the principal $\mathsf{PSL}(2,\mathbb{R})$, which factors through $\mathsf{SO}_0(p,p-1)$.

Now if $h \in \mathsf{PSL}(2,\mathbb{R})$ is a hyperbolic element with eigenvalues $\pm\lambda$ (well defined up to ± 1), then $\hat\tau(h)$ acting on π_τ is diagonalizable with eigenvalues

$$\lambda^{2(p-1)},\dots,\lambda^2,1,\lambda^{-2},\dots,\lambda^{-2(p-1)}.$$

Note that these are positive independently on whether we choose λ or $-\lambda$ for the eigenvalues of h; hence, to fix ideas we will assume $\lambda > 1$. In particular, all the eigenvalues of $\hat\tau(h)$ are positive. We let π_h be the two-dimensional plane spanned by ℓ_τ and the eigenline in π_τ of eigenvalue 1, which we denote by ℓ_h^1. That is, π_h is the eigenspace of $\hat\tau(h)$ associated with the eigenvalue 1. We also let ℓ_h^i be the eigenline of eigenvalue $i = \lambda^{2(p-1)},\dots,\lambda^2,\lambda^{-2},\dots,\lambda^{-2(p-1)}$.

Note that $g\hat\tau(h)g^{-1} = \hat\tau(h)$. Otherwise, we would have $g\hat\tau(h)g^{-1} = -\hat\tau(h)$ and for $v \in \ell_h^i$ we have

$$g\cdot v = \frac{1}{\lambda^i} g\hat\tau(h)\cdot v = -\frac{1}{\lambda^i}\hat\tau(h)g\cdot v.$$

We would then find a negative eigenvalue of $\hat\tau(h)$, a contradiction. We conclude that $g\hat\tau(h)g^{-1} = \hat\tau(h)$ as claimed.

It follows that g preserves ℓ_h^i for all i, and also preserves π_h. We claim that g preserves ℓ_τ. Indeed, note that there is some $m \in \mathsf{PSL}(2,\mathbb{R})$ so that $\hat\tau(m)\cdot\ell_h^1 \neq \ell_h^1$, as the action of $\hat\tau(\mathsf{PSL}(2,\mathbb{R}))$ on π_τ is irreducible. Furthermore, $\hat\tau(m)\cdot\ell_h^1$ is different from ℓ_h^i, as all these lines are isotropic, while $\hat\tau(m)\cdot\ell_h^1$ is not. By what we just proved, g preserves $\pi_{mhm^{-1}}$ and therefore preserves $\pi_{mhm^{-1}} \cap \pi_h = \ell_\tau$. Hence, $g\cdot\ell_\tau = \ell_\tau$ and therefore $g\cdot\ell_h^1 = \ell_h^1$ for every hyperbolic $h \in \mathsf{PSL}(2,\mathbb{R})$.

We conclude that for every hyperbolic $h \in \mathsf{PSL}(2,\mathbb{R})$, the element g preserves the projective basis

$$\mathcal{B}_h := \{\ell_h^{2(p-1)},\dots,\ell_h^2,\ell_h^1,\ell_\tau,\ell_h^{-2},\dots,\ell_h^{-2(p-1)}\}.$$

Fix such an h. Let $m \in \mathsf{PSL}(2,\mathbb{R})$ be so that $\hat\tau(m)\cdot\ell_h^1 \notin \mathcal{B}_h$. Then g preserves the elements of the basis $\mathcal{B}_{mhm^{-1}}$ as well and therefore preserves $2p+1$ lines in general position in $\mathbb{R}^{2p}$. It follows that $g = \mu\mathrm{id}$ for some $\mu \in \mathbb{R}$. Since $g \in \mathsf{SO}_0(p,p)$, then $\mu = \pm 1$. ∎

Corollary 7.8. *Assume $p \neq 4$ and let $\rho \in \mathrm{Hit}(S,\mathsf{PSO}_0(p,p))$. Then the Zariski closure G_ρ of ρ is simple and center free, and with Lie algebra $\mathfrak{so}(p,p)$, $\mathfrak{so}(p,p-1)$, or a principal $\mathfrak{sl}_2$.*

Proof. Let $\hat\rho$ be a lift of ρ. Then $\mathsf{G}_\rho = \mathsf{G}_{\hat\rho}/\{\pm\mathrm{id}\}$ and by Lemmas 7.6 and 7.7, G_ρ is reductive and center free. In particular, it is semisimple and by Sambarino [73, Theorem A], the result follows. ∎

The proof of the following well-known fact can be found in [73, Corollary 6.2] for $\mathsf{PSL}(d,\mathbb{R})$-Hitchin representations, but the proof applies in our setting.

Corollary 7.9. *Let* $\rho \in \mathrm{Hit}(S, \mathsf{PSO}_0(p,p))$. *Then* $\rho(\Gamma) \subset (\mathsf{G}_\rho)_0$.

We have now completed the analysis of the possible Zariski closures of $\mathsf{PSO}_0(p,p)$-Hitchin representations, and we can prove Theorem 7.1.

Proof of Theorem 7.1. By Corollaries 7.8 and 7.9 and Theorem 6.8, there exists an isomorphism $\sigma : (\mathsf{G}_\rho)_0 \to (\mathsf{G}_{\widehat{\rho}})_0$ so that $\sigma \circ \rho = \widehat{\rho}$. In particular, $(\mathsf{G}_\rho)_0 \cong (\mathsf{G}_{\widehat{\rho}})_0$ and we have three possibilities. If $(\mathsf{G}_\rho)_0$ is a principal $\mathsf{PSL}(2,\mathbb{R})$, then the result follows from length spectrum rigidity in Teichmüller space. If $(\mathsf{G}_\rho)_0 \cong \mathsf{PSO}_0(p,p-1)$, then the corresponding Dynkin diagram is of type B_{p-1} and therefore admits no nontrivial automorphism. Hence, in that case σ is inner as desired.

Finally, assume $(\mathsf{G}_\rho)_0 = \mathsf{PSO}_0(p,p)$ and suppose by contradiction that $\rho \neq \widehat{\rho}$. Hence, σ is a noninternal automorphism. But on the other hand, by Theorem 6.8, we have $\varphi \circ \sigma = \varphi$, contradicting our hypothesis. ∎

Remark 7.10. A natural length function on $\mathrm{Hit}_d(S)$, specially relevant in the case $d = 3$, is the Hilbert length (cf., Example 4.11). However, the Hilbert length is not rigid, as the *contragredient* representation $\rho^\star(\gamma) := {}^t\rho(\gamma)^{-1}$ of ρ satisfies $h^{\mathrm{H}}_\rho L^{\mathrm{H}}_\rho = h^{\mathrm{H}}_{\rho^\star} L^{\mathrm{H}}_{\rho^\star}$, but in general, one has $\rho^\star \neq \rho$. Hence, $d^{\mathrm{H}}_{\mathrm{Th}}(\cdot,\cdot)$ does not separate points of $\mathrm{Hit}_d(S)$. It follows from the proof of Theorem 7.1 that this is the only possible situation where two different $\mathsf{PSL}(d,\mathbb{R})$-Hitchin representations can have the same Hilbert length spectra. Similar comments apply to the simple roots listed in Table 1.

7.2. Simple root and spectral radius Finsler norms

We now restrict to $\mathsf{G} = \mathsf{PSL}(d,\mathbb{R})$. We list some useful consequences of Corollary 6.16 and [23, 24]. For the first simple root, we have the following.

Corollary 7.11. *Let* $\varphi = \alpha_1 \in \Pi$ *be the first simple root. The function on* $T\mathrm{Hit}_d(S)$

$$\|v\|^{\alpha_1}_{\mathrm{Th}} = \sup_{[\gamma]\in[\Gamma]} \frac{\mathrm{d}_\rho(L^{\alpha_1}_{\cdot}(\gamma))(v)}{L^{\alpha_1}_\rho(\gamma)}$$

defines a Finsler norm on $\mathrm{Hit}_d(S)$.

Proof. By Potrie–Sambarino [65, Theorem B], we have $h^{\alpha_1}_\rho = 1$ for all $\rho \in \mathrm{Hit}_d(S)$. Hence, thanks to Corollary 6.16, we only have to show that $\|v\|^{\alpha_1}_{\mathrm{Th}} = 0$ implies $v = 0$. But this follows from Corollary 6.16 and [24, Theorem 1.7]: The set $\{\mathrm{d}_\rho(L^{\alpha_1}_{\cdot}(\gamma))\}_{\gamma\in\Gamma}$ generates the cotangent space $T^*_\rho\mathrm{Hit}_d(S)$. ∎

When $d = 2j > 2$, it is shown in [24, Proposition 8.1] that the middle root pressure quadratic form is degenerate along representations that factor through $\mathsf{PSp}(2j,\mathbb{R})$. The proof shows that $\|\cdot\|^{\alpha_j}_{\mathrm{Th}}$ is degenerate as well.

With the same argument as in Corollary 7.11 (but applying [23, Lemma 9.8 and Proposition 10.1] instead of [24, Theorem 1.7]), we obtain the following.

Corollary 7.12. *Let $\varphi = \lambda_1$ be the spectral radius length function. Then the function* $\|\cdot\|^{\lambda_1}_{\mathrm{Th}} : T\mathrm{Hit}_d(S) \to \mathbb{R}_{\geq 0}$*, taking* $v \in T_\rho \mathrm{Hit}_d(S)$ *to*

$$\|v\|^{\lambda_1}_{\mathrm{Th}} = \sup_{[\gamma]\in[\Gamma]} \frac{\mathrm{d}_\rho(h^{\lambda_1}_{\cdot})(v) L^{\lambda_1}_\rho(\gamma) + h^{\lambda_1}_\rho \mathrm{d}_\rho(L^{\lambda_1}_{\cdot}(\gamma))(v)}{h^{\lambda_1}_\rho L^{\lambda_1}_\rho(\gamma)}$$

defines a Finsler norm on $\mathrm{Hit}_d(S)$.

We finish this subsection with a comment on Labourie and Wentworth work [55], which explicitly compute the derivative of the spectral radius and simple root length functions at points of the Fuchsian locus $\mathrm{Teich}(S) \subset \mathrm{Hit}_d(S)$, along some special directions. More explicitly, fixing a Riemann surface structure X_0 on S, the canonical line bundle K associated with X_0 is the $(1,0)$-part of the complexified cotangent bundle $T^*X_0^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} T^*X_0$. A *holomorphic k-differential* is a holomorphic section of the bundle K^k, where the power k is taken with respect to tensor operation. In local holomorphic coordinates $z = x + iy$, a holomorphic k-differential can be written as

$$q_k = q_k(z) \underbrace{dz \otimes \cdots \otimes dz}_{k \text{ times}} = q_k(z) dz^k,$$

with $q_k(z)$ holomorphic. Hitchin's seminal work [44] parametrizes $\mathrm{Hit}_d(S)$ by the space of holomorphic differentials over X_0. More precisely, there exists a homeomorphism

$$\mathrm{Hit}_d(S) \cong \bigoplus_{k=2}^{d} H^0(X_0, K^k),$$

where $H^0(X_0, K^k)$ denotes the space of holomorphic k-differentials over X_0. Given a holomorphic k-differential $q_k \in H^0(X_0, K^k)$, one may consider a natural family of Hitchin representations $\{\rho_t\}_{t\geq 0}$, corresponding to $\{tq_k\}_{t\geq 0} \subset H^0(X_0, K^k)$ under this parametrization, with ρ_0 corresponding to the point X_0 in the Teichmüller space $\mathrm{Teich}(S)$. Infinitesimally, this gives a vector space isomorphism:

$$T_{\rho_0}\mathrm{Hit}_d(S) \cong \bigoplus_{k=2}^{d} H^0(X_0, K^k).$$

Given a family of Hitchin representations $\{\rho_t\}_{t\geq 0}$ as above, we denote by $v = v(q_k) := \frac{d}{dt}\big|_{t=0}\rho_t \in T_{X_0}\mathrm{Hit}_d(S)$ the corresponding tangent vector. The computation of the derivatives $\mathrm{d}_{\rho_0}(L^{\lambda_j}_{\cdot}(\gamma))(v)$, for $1 \leq j \leq d$, has been carried out by Labourie–Wentworth [55, Theorem 4.0.2], using the above identification and information of $H^0(X_0, K^k)$. To be more precise, define the function $\mathrm{Re}\, q_k : T^1X_0 \to \mathbb{R}$ as the real part of the holomorphic differential q_k evaluated on unit tangent vectors. More precisely,

$$\mathrm{Re}\, q_k(x) := \mathrm{Re}\left(q_k|_p(w, w, \ldots, w)\right)$$

for $x = (p, w) \in T^1 X_0$.

Let ϕ be the geodesic flow on $T^1 X_0$. For $\gamma \in \Gamma$, let $l_{\rho_0}(\gamma) := \frac{2}{d-1} L^{\lambda_1}_{\rho_0}(\gamma)$ be the hyperbolic length of the closed geodesic on X_0 corresponding to the free homotopy class $[\gamma]$.

Proposition 7.13. *There exist constants C_1 and C_2, only depending on d and k, such that for any vector $v = v(q_k) \in T_{X_0}\mathrm{Hit}_d(S)$ as above,*

$$\|v(q_k)\|^{\lambda_1}_{\mathrm{Th}} = C_1 \sup_{[\gamma]\in[\Gamma]} \frac{1}{l_{\rho_0}(\gamma)} \int_0^{l_{\rho_0}(\gamma)} \operatorname{Re} q_k(\phi_s(x))\mathrm{d}s$$

and

$$\|v(q_k)\|^{\alpha_1}_{\mathrm{Th}} = C_2 \sup_{[\gamma]\in[\Gamma]} \frac{1}{l_{\rho_0}(\gamma)} \int_0^{l_{\rho_0}(\gamma)} \operatorname{Re} q_k(\phi_s(x))\mathrm{d}s,$$

where $x = x_\gamma$ is any point on $T^1 X_0$ that lies in the periodic orbit corresponding to γ.

Proof. The proof is a simple combination of Definition 6.11 together with [55, Theorem 4.0.2 and Corollary 4.0.5]. One also needs the fact that $h^{\lambda_1}_\rho \leq 1$ with equality precisely when ρ is Fuchsian, and $h^{\alpha_1}_{\cdot} \equiv 1$ (by [65, Theorem B]). ∎

8. Other examples

As discussed in Subsection 1.2, we need two ingredients to gain a good understanding of the asymmetric metric $d^{\varphi}_{\mathrm{Th}}(\cdot,\cdot)$:

- A reparametrization of the geodesic flow of Γ with periods given by the functional φ: This is needed to show that $d^{\varphi}_{\mathrm{Th}}(\cdot,\cdot)$ is non-negative, degenerating if and only if the renormalized length spectra coincide. Sambarino provides such a reparametrization whenever $\varphi \in \mathrm{int}((\mathcal{L}^{\Theta}_{\rho})^*)$ and Θ is the set of Anosov roots (see Section 5).
- A good understanding of the Zariski closure and its outer automorphism group for representations belonging to a given class of interests: This is necessary to obtain renormalized length spectrum rigidity.

Furthermore, on subsets of representations for which the entropy of some functional is constant, one can avoid the renormalization by entropy.

We discuss here further classes in which simultaneous knowledge of some of these aspects can be achieved.

8.1. Benoist representations

Let Γ be a torsion-free word hyperbolic group. A *Benoist representation* is a faithful and discrete representation $\rho : \Gamma \to \mathsf{PSL}(d+1, \mathbb{R})$ dividing an open, strictly convex set $\Omega_\rho \subset \mathbb{RP}^d$ (recall Example 4.15). We denote by $\mathrm{Ben}_d(\Gamma) \subset \mathfrak{X}(\Gamma, \mathsf{PSL}(d+1, \mathbb{R}))$ the space of conjugacy classes of Benoist representations. Koszul [53] showed that $\mathrm{Ben}_d(\Gamma)$ is an open subset of the character variety, and Benoist [11] showed it is closed. Hence, $\mathrm{Ben}_d(\Gamma)$ is a union of connected components of $\mathfrak{X}(\Gamma, \mathsf{PSL}(d+1, \mathbb{R}))$.

As Benoist representations are Θ-Anosov for $\Theta = \{\alpha_1, \alpha_d\}$, both the unstable Jacobian $\mathrm{J}_{d-1} := d\omega_1 - \omega_d = d\lambda_1 + \lambda_{d+1}$ and $\mathrm{H} := \lambda_1 - \lambda_{d+1}$ belong to $\mathrm{int}((\mathcal{L}_\rho^\Theta)^*)$ for every $\rho \in \mathrm{Ben}_d(\Gamma)$. We focus here on these two functionals since it was proven in [65, Corollary 7.1] that J_{d-1} has constant entropy, and the Hilbert length function has particular geometric significance as $L_\rho^{\mathrm{H}}(\gamma)$ coincides with the length of the unique Hilbert geodesic in $\rho(\Gamma)\backslash\Omega_\rho$ in the isotopy class corresponding to $[\gamma]$.

Corollary 8.1. *The function* $d_{\mathrm{Th}}^{\mathrm{J}_{d-1}} : \mathrm{Ben}_d(\Gamma) \times \mathrm{Ben}_d(\Gamma) \to \mathbb{R}$ *given by*

$$d_{\mathrm{Th}}^{\mathrm{J}_{d-1}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{L_{\widehat{\rho}}^{\mathrm{J}_{d-1}}(\gamma)}{L_{\rho}^{\mathrm{J}_{d-1}}(\gamma)}\right)$$

defines a (possibly asymmetric) distance on $\mathrm{Ben}_d(\Gamma)$.

Proof. Benoist [8, Théorème 3.6] showed that if $\rho \in \mathrm{Ben}_d(\Gamma)$ is not Zariski dense, then $\rho(\Gamma) \subset \mathsf{PSO}(d, 1)$. Hence, by Theorems 6.2 and 6.8, if $d_{\mathrm{Th}}(\rho, \widehat{\rho}) = 0$ then there exists an isomorphism $\sigma : (\mathsf{G}_\rho)_0 \to (\mathsf{G}_{\widehat{\rho}})_0$ so that $\sigma \circ \rho = \widehat{\rho}$. If $(\mathsf{G}_\rho)_0 \cong (\mathsf{G}_{\widehat{\rho}})_0 \cong \mathsf{PSO}_0(1, d)$, then the equality $\rho = \widehat{\rho}$ follows from length spectrum rigidity in Teichmüller space (when $d = 2$) or by Mostow rigidity (when $d > 2$).

On the other hand, if $(\mathsf{G}_\rho)_0 \cong (\mathsf{G}_{\widehat{\rho}})_0 \cong \mathsf{PSL}(d+1, \mathbb{R})$ and σ is non-inner, it acts non-trivially on the Dynkin diagram of type A_d; hence, its action on $\mathtt{a}$ coincides with the opposition involution ι. Since J_{d-1} is not ι-invariant and has constant entropy by [65, Corollary 7.1], Corollary 6.9 finishes the proof. ∎

Remark 8.2. The same applies for all $(1, 1, p)$-hyperconvex representations $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ of hyperbolic groups having as boundary a $(p-1)$-dimensional sphere (see Example 4.15): It follows from [67, Proposition 7.4] that their projective limit set is a C^1-sphere and from [68, Theorem A] that then the entropy of the unstable Jacobian $\mathrm{J}_{p-1} := p\omega_1 - \omega_p$ is constant and equal to 1. If we then denote by $\mathrm{Hyp}^{\mathrm{Z}}(\Gamma)$ the open subset of the character variety consisting of Zariski-dense $(1, 1, p)$-hyperconvex representations, the function

$$d_{\mathrm{Th}}^{\mathrm{J}_{p-1}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{L_{\widehat{\rho}}^{\mathrm{J}_{p-1}}(\gamma)}{L_{\rho}^{\mathrm{J}_{p-1}}(\gamma)}\right)$$

defines a (possibly asymmetric) distance on $\mathrm{Hyp}^{\mathrm{Z}}(\Gamma)$.

With the same proof as in Corollary 8.1, we get the following result.

Corollary 8.3. *The function* $d^{\mathrm{H}}_{\mathrm{Th}} : \mathrm{Ben}_d(\Gamma) \times \mathrm{Ben}_d(\Gamma) \to \mathbb{R}$ *given by*

$$d^{\mathrm{H}}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h^{\mathrm{H}}_{\widehat{\rho}}}{h^{\mathrm{H}}_{\rho}} \frac{L^{\mathrm{H}}_{\widehat{\rho}}(\gamma)}{L^{\mathrm{H}}_{\rho}(\gamma)} \right)$$

is real valued, non-negative and $d^{\mathrm{H}}_{\mathrm{Th}}(\rho, \widehat{\rho}) = 0$ *if and only if* $\rho = \widehat{\rho}$ *or* $\rho = \widehat{\rho}^{\star}$, *where* $\rho^{\star}(\gamma) := {}^{t}\rho(\gamma)^{-1}$ *for all* $\gamma \in \Gamma$.

Remark 8.4. The Hilbert length function H is the only element in $\mathrm{int}((\mathcal{L}^{\Theta}_{\rho})^{*})$ which is fixed by the opposition involution, and the unstable Jacobian J_{d-1} and its image $\mathrm{J}_{d-1} \circ \iota = -d\lambda_{d+1} - \lambda_1$ are the only elements in $\mathrm{int}((\mathcal{L}^{\Theta}_{\rho})^{*})$ that have constant entropy on the whole $\mathrm{Ben}_d(\Gamma)$. In particular, for all other functionals $\varphi \in \mathrm{int}((\mathcal{L}^{\Theta}_{\rho})^{*})$, such as, for example, the spectral radius λ_1,

$$d^{\varphi}_{\mathrm{Th}}(\rho, \widehat{\rho}) := \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h^{\varphi}_{\widehat{\rho}}}{h^{\varphi}_{\rho}} \frac{L^{\varphi}_{\widehat{\rho}}(\gamma)}{L^{\varphi}_{\rho}(\gamma)} \right)$$

defines a (possibly asymmetric) distance on $\mathrm{Ben}_d(\Gamma)$. In all these cases, the renormalization by entropy is, however, necessary.

8.2. AdS-quasi-Fuchsian representations

Let $q \geq 2$ and Γ be the fundamental group of a closed real hyperbolic q-dimensional manifold. Denote by $\mathrm{QF}_q(\Gamma)$ the space of AdS-quasi-Fuchsian representations $\Gamma \to \mathsf{PO}_0(2, q)$, which is a union of connected components of the character variety (recall Example 4.16). Since representations in $\mathrm{QF}_q(\Gamma)$ are Anosov with respect to the space of isotropic lines, the Hilbert length functional $\mathrm{H} = \omega_1 - \omega_{q+1}$ belongs to the Anosov–Levi space $\mathfrak{a}^{*}_{\Theta}$. This functional is a multiple of the spectral radius functional on $\mathsf{PO}_0(2, q)$.

Corollary 8.5. *If* $q > 2$, *the function* $d^{\mathrm{H}}_{\mathrm{Th}} : \mathrm{QF}_q(\Gamma) \times \mathrm{QF}_q(\Gamma) \to \mathbb{R}$ *given by*

$$d^{\mathrm{H}}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{h^{\mathrm{H}}_{\widehat{\rho}}}{h^{\mathrm{H}}_{\rho}} \frac{L^{\mathrm{H}}_{\widehat{\rho}}(\gamma)}{L^{\mathrm{H}}_{\rho}(\gamma)} \right)$$

defines a (possibly asymmetric) distance on $\mathrm{QF}_q(\Gamma)$.

Proof. For $q > 2$, the group $\mathsf{PO}_0(2, q)$ is simple, and the associated root system is of type B_2. In particular, it has no nontrivial automorphisms and therefore an automorphism of $\mathsf{PO}_0(2, q)$ is necessarily inner. Corollary 6.9 then proves the result when restricting to Zariski-dense AdS-quasi-Fuchsian representations.

Furthermore, Glorieux–Monclair [32, Proposition 1.4] computed the possible Zariski closures of an AdS-quasi-Fuchsian representation: If ρ is not Zariski dense, then it is AdS-Fuchsian. This means that ρ preserves a totally geodesic copy of $\mathbb{H}^q$ inside the anti-de Sitter space and acts cocompactly on it (cf., [29, Remark 1.13]). Therefore, $\rho(\Gamma) \subset \mathsf{PO}(1, q) \subset \mathsf{PO}_0(2, q)$. Hence, the length spectrum rigidity of closed real hyperbolic manifolds finishes the proof. ∎

In the special case $q = 2$, the function $d^{\mathrm{H}}_{\mathrm{Th}}$ does not separate points. Indeed, $\mathsf{PSO}_0(2,2) \cong \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R})$ and every representation of the form

$$\rho = (\rho^{\mathrm{L}}, \rho^{\mathrm{R}}) : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R}),$$

where ρ^{ε} is a point in Teichmüller space for $\varepsilon \in \{\mathrm{L},\mathrm{R}\}$, is AdS-quasi-Fuchsian. However, the representation $\widehat{\rho} := (\rho^{\mathrm{R}}, \rho^{\mathrm{L}})$ has the same Hilbert length spectrum as ρ, but $\rho \neq \widehat{\rho}$ (unless $\rho^{\mathrm{L}} = \rho^{\mathrm{R}}$).

Remark 8.6. Since AdS-quasi-Fuchsian representations have Lipschitz limit set, it follows again from [68, Theorem A] that the entropy of the unstable Jacobian $\mathrm{J}_{q-1} := q\omega_1 - \omega_q$ is constant and equal to 1 on $\mathrm{QF}_q(\Gamma)$. In particular, the function $d^{\mathrm{J}_{q-1}}_{\mathrm{Th}} : \mathrm{QF}_q(\Gamma) \times \mathrm{QF}_q(\Gamma) \to \mathbb{R}$ given by

$$d^{\mathrm{J}_{q-1}}_{\mathrm{Th}}(\rho, \widehat{\rho}) := \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{L^{\mathrm{J}_{q-1}}_{\widehat{\rho}}(\gamma)}{L^{\mathrm{J}_{q-1}}_{\rho}(\gamma)}\right)$$

is non-negative.

However, in this case the unstable Jacobian does not belong to the Levi–Anosov subspace. As a result, it is not clear whether a metric Anosov flow with periods J_{q-1} exists, allowing us to apply the thermodynamic formalism which is at the basis of this work. Thus, we do not know if the condition $d^{\mathrm{J}_{q-1}}_{\mathrm{Th}}(\rho, \widehat{\rho}) = 0$ leads to an equality between length spectra that allows to conclude that $d^{\mathrm{J}_{q-1}}_{\mathrm{Th}}$ separates points.

8.3. Zariski-dense Θ-positive representations in $\mathsf{PO}_0(p, p+1)$

Let $2 \leq p \leq q$. Let $\Gamma = \pi_1(S)$ be a surface group and $\mathrm{Pos}_{p,q}(S)$ be the space of Θ-positive representations $\Gamma \to \mathsf{PO}_0(p,q)$ (cf., Example 4.14).

Corollary 8.7. *For $2 < p \leq q$ and $j = 1, \ldots, p-2$, let α_j be the corresponding simple root of $\mathsf{PO}_0(p,q)$. Let $\mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma) \subset \mathrm{Pos}_{p,q}(\Gamma)$ be the subset consisting of Zariski-dense representations. Then the function*

$$d^{\alpha_j}_{\mathrm{Th}} : \mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma) \times \mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma) \to \mathbb{R}$$

given by

$$d^{\alpha_j}_{\mathrm{Th}}(\rho, \widehat{\rho}) = \log\left(\sup_{[\gamma]\in[\Gamma]} \frac{L^{\alpha_j}_{\widehat{\rho}}(\gamma)}{L^{\alpha_j}_{\rho}(\gamma)}\right)$$

defines a (possibly asymmetric) distance on $\mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma)$.

Proof. As $\mathsf{PO}_0(p,q)$ Θ-positive representations are Θ-Anosov for $\Theta = \{\alpha_1, \ldots, \alpha_{p-1}\}$ (see [13,37]), we have $\alpha_j \in \mathrm{int}((\mathcal{L}^{\Theta}_{\rho})^*)$ for every $\rho \in \mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma)$. Furthermore, α_j-entropy is constant on the space of $\mathsf{PO}_0(p,q)$-positive representations [68, Corollary 1.7]. Thus, to finish the proof, it only remains to show that α_j-length spectrum rigidity holds on $\mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma)$.

Since $\mathsf{PO}_0(p,q)$ is simple and center free, Theorem 6.8 guarantees that two representations in $\mathrm{Pos}^{\mathrm{Z}}_{p,q}(\Gamma)$ having the same renormalized length spectra differ by an automorphism of $\mathsf{PO}_0(p,q)$. Since the Dynkin diagram associated with the root system of $\mathsf{PO}_0(p,q)$ is of type B_p and admits no nontrivial automorphism, the outer automorphism group of $\mathsf{PO}_0(p,q)$ is trivial and this finishes the proof. ∎

Remark 8.8. The space $\mathrm{Pos}_{2,3}(\Gamma)$ contains connected components only consisting of Zariski-dense representations [1, Theorem 4.40]. More generally, for all $p > 2$ the space $\mathrm{Pos}_{p,p+1}(\Gamma)$ contains smooth connected components. It is conjectured that these consist only of Zariski-dense representations as well (see [26, Conjecture 1.7]), if the conjecture were true, the functions in Corollary 8.7 would define metrics on these connected components.

On the other hand, it follows from the classification in [3] that for $q \geq p$ all connected components of $\mathrm{Pos}_{p,q}(S)$, with the exception of the Hitchin component if $p = q$, contain representations with compact centralizer.

A. Geodesic currents

Bridgeman–Canary–Labourie–Sambarino [24, p. 60] remarked that the renormalized intersection number of Subsection 2.4 can be linked to *Bonahon's intersection number*, in the specific case of geodesic flows associated with points in the Teichmüller space of a surface. We explain this in more detail for the reader's convenience.

Let S be a connected closed orientable surface of genus bigger than one and $\widetilde{S}$ be its universal cover. Let Γ be the fundamental group of S. A *(complete) geodesic* of $\widetilde{S}$ is an element of $\partial^{(2)}\Gamma$. A *geodesic current* is a Borel, locally finite, Γ-invariant measure on the space of geodesics of $\widetilde{S}$, which is also invariant under the map $(x,y) \mapsto (y,x)$. We let $\mathcal{C}(S)$ be the space of geodesic currents in S. An important example of a geodesic current is given by isotopy classes of closed curves in S: Every such class α defines an element $\delta_\alpha \in \mathcal{C}(S)$ by representing α as a conjugacy class c_α in Γ and then considering the sum of Dirac masses supported on the axes of elements in c_α. Another interesting example is given by *measured geodesic laminations* on S (cf., Bonahon [17, p. 153]).

Bonahon [17] defined a continuous, bilinear, symmetric pairing

$$i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}_{\geq 0},$$

called the *intersection number* between geodesic currents. This terminology is motivated by the following property: If α and β are isotopy classes of closed curves in S, then one has

$$i(\delta_\alpha, \delta_\beta) = \inf_{\alpha' \in \alpha, \beta' \in \beta} \#(\alpha' \cap \beta').$$

Furthermore, Bonahon defines an embedding

$$\mathtt{L} : \mathrm{Teich}(S) \hookrightarrow \mathcal{C}(S)$$

from the Teichmüller space Teich(S) into the space of geodesic currents that can be described as follows. Since every point $\rho \in \mathfrak{T}(S)$ is Anosov, we have an equivariant limit map $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^2)$ and we may pull back the Haar measure on $\mathbb{P}(\mathbb{R}^2) \times \mathbb{P}(\mathbb{R}^2) \setminus \{(\eta,\eta) : \eta \in \mathbb{P}(\mathbb{R}^2)\}$ under this map. We obtain an element $\mathtt{L}_\rho \in \mathcal{C}(S)$ which is called the *Liouville current* of ρ. Furthermore, the Haar measure on $\mathbb{P}(\mathbb{R}^2) \times \mathbb{P}(\mathbb{R}^2) \setminus \{(\eta,\eta) : \eta \in \mathbb{P}(\mathbb{R}^2)\}$ can be normalized so that for every isotopy class of closed curves α in S

$$i(\mathtt{L}_\rho, \delta_\alpha) = L_\rho(\alpha), \tag{A.1}$$

where $L_\rho(\alpha)$ is the length of the unique closed geodesic (for the metric ρ) in the isotopy class α (cf., [17, Proposition 14]).

The embedding $\mathtt{L} : \mathrm{Teich}(S) \to \mathcal{C}(S)$ allows us to relate renormalized intersection and Bonahon's intersection. Indeed, pick a base point $\rho_0 \in \mathrm{Teich}(S)$ and denote by S_{ρ_0} the underlying hyperbolic surface. The associated geodesic flow $\phi = \phi_{\rho_0}$ is a topologically transitive Anosov flow and admits a strong Markov coding (cf., Theorem 2.9). Furthermore, the choice of ρ_0 induces a homeomorphism between $\mathcal{C}(S)$ and the space $\mathcal{P}(\phi)$. Indeed, the Busemann–Iwasawa cocycle of ρ_0 induces an identification between the unit tangent bundle of the Riemannian universal cover of S_{ρ_0} with

$$\partial^{(2)}\Gamma \times \mathbb{R},$$

in such a way that the action of the (lifted) geodesic flow is given by translation in the $\mathbb{R}$-coordinate. The identification $\mathcal{C}(S) \cong \mathcal{P}(\phi)$ is defined by associating with a geodesic current ν the probability measure m_ν homothetic to the quotient measure of $\nu \otimes \mathrm{d}t$.

The geodesic flow $\psi = \psi_\rho$ corresponding to another point $\rho \in \mathrm{Teich}(S)$ is Hölder orbit equivalent to $\phi = \phi_{\rho_0}$, and therefore we may think ψ as an element of[5] HR(ϕ).

Lemma A.1. *Let ρ_0 and ρ be two points in* Teich(S) *and take $\nu \in \mathcal{C}(S)$. Then*

$$\mathbf{I}_{m_\nu}(\phi,\psi) = \mathbf{J}_{m_\nu}(\phi,\psi) = \frac{i(\nu, \mathtt{L}_\rho)}{i(\nu, \mathtt{L}_{\rho_0})}.$$

Proof. The function $\mathbf{J}_\cdot(\phi,\psi)$ is continuous on $\mathcal{P}(\phi)$. Similarly, $i(\cdot, \mathtt{L}_{\rho_0})$ and $i(\cdot, \mathtt{L}_\rho)$ are continuous on $\mathcal{C}(S)$. Since $\nu \mapsto m_\nu$ is a homeomorphism and multicurves are dense in $\mathcal{C}(S)$ (see Bonahon [17, Proposition 2]), it suffices to prove the statement for $\nu = \delta_\alpha$, where α is any isotopy class of closed curves in S.

[5]Formally, there is no canonical way of identifying ψ with a specific reparametrization of ϕ, but just to a Livšic cohomology class (cf., Livšic's Theorem 2.12). For simplicity, we will ignore this detail in this discussion and think that the choice of ρ induces a specific element $\psi \in \mathrm{HR}(\phi)$. As it will become clear, the discussion is independent of this arbitrary choice.

Assume then $\nu = \delta_\alpha$. By equation (A.1), we have

$$i(\nu, \mathrm{L}_{\rho_0}) = L_{\rho_0}(\alpha) \quad \text{and} \quad i(\nu, \mathrm{L}_\rho) = L_\rho(\alpha).$$

On the other hand, it is well known that $h_\phi = h_\psi = 1$ (cf., Manning [57]), hence $\mathbf{J}_{m_\nu}(\phi, \psi) = \mathbf{I}_{m_\nu}(\phi, \psi)$. Also, α defines a periodic orbit $a_\alpha \in \mathcal{O}$ satisfying $p_\phi(a_\alpha) = L_{\rho_0}(\alpha)$ and $p_\psi(a_\alpha) = L_\rho(\alpha)$. Since $m_{\delta_\alpha} = \delta_\phi(a_\alpha)$, equation (2.3) completes the proof. ∎

One can check that $m^{\mathrm{BM}}(\phi) = m_{\mathrm{L}_{\rho_0}}$. Hence, combining Lemma A.1 and Bonahon [17, Proposition 15], we have

$$\mathbf{J}_{m^{\mathrm{BM}}(\phi)}(\phi, \psi) = \frac{i(\mathrm{L}_{\rho_0}, \mathrm{L}_\rho)}{i(\mathrm{L}_{\rho_0}, \mathrm{L}_{\rho_0})} = \frac{i(\mathrm{L}_{\rho_0}, \mathrm{L}_\rho)}{\pi^2 |\chi(S)|}. \tag{A.2}$$

As an interesting consequence, one gets

$$\mathbf{J}_{m^{\mathrm{BM}}(\phi)}(\phi, \psi) = \mathbf{J}_{m^{\mathrm{BM}}(\psi)}(\psi, \phi)$$

for all $\rho_0, \rho \in \mathfrak{T}(S)$. However, if one considers the supremum of all renormalized intersections (rather than just the Bowen–Margulis-renormalized intersection), this symmetry no longer holds: Combine Theorem 3.2 with Thurston's example [81, p. 5].

Another interesting consequence of equation (A.2) is that it recovers a result by Bonahon [17, p. 156]. Indeed, combining that equation with Proposition 2.18 and length spectrum rigidity on Teich(S), one has

$$i(\mathrm{L}_{\rho_0}, \mathrm{L}_\rho) \geq \pi^2 |\chi(S)|$$

for all $\rho_0, \rho \in \mathfrak{T}(S)$, with equality if and only if $\rho = \rho_0$.

Acknowledgments. We are grateful to Gerhard Knieper, Rafael Potrie, and Andrés Sambarino for several helpful discussions and comments.

Funding. This investigation is part of the project B5 in the CRC/TRR 191 funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 281071066 – TRR 191. L.C., B.P., and A.W. acknowledge funding by the DFG through the project 338644254 (SPP2026). L.C. acknowledges funding by Agencia Nacional de Investigación e Innovación - FCE_3_2020_1_162840. A.W. is supported by the European Research Council under ERC-Advanced Grant 101018839. This work is supported by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy EXC-2181/1 - 390900948 (the Heidelberg STRUCTURES Cluster of Excellence).

References

[1] D. Alessandrini and B. Collier, The geometry of maximal components of the $PSp(4, \mathbb{R})$ character variety. *Geom. Topol.* **23** (2019), no. 3, 1251–1337 Zbl 1418.22007 MR 3956893

[2] D. V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.* **90** (1967), pp. 209 MR 0224110

[3] M. Aparicio-Arroyo, S. Bradlow, B. Collier, O. García-Prada, P. B. Gothen, and A. Oliveira, $\mathrm{SO}(p,q)$-Higgs bundles and higher Teichmüller components. *Invent. Math.* **218** (2019), no. 1, 197–299 Zbl 1473.14017 MR 3994589

[4] T. Barbot, Deformations of Fuchsian AdS representations are quasi-Fuchsian. *J. Differential Geom.* **101** (2015), no. 1, 1–46 Zbl 1327.53089 MR 3356068

[5] T. Barbot and Q. Mérigot, Anosov AdS representations are quasi-Fuchsian. *Groups Geom. Dyn.* **6** (2012), no. 3, 441–483 Zbl 1333.53106 MR 2961282

[6] Y. Benoist, Actions propres sur les espaces homogènes réductifs. *Ann. of Math. (2)* **144** (1996), no. 2, 315–347 Zbl 0868.22013 MR 1418901

[7] Y. Benoist, Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.* **7** (1997), no. 1, 1–47 Zbl 0947.22003 MR 1437472

[8] Y. Benoist, Automorphismes des cônes convexes. *Invent. Math.* **141** (2000), no. 1, 149–193 Zbl 0957.22008 MR 1767272

[9] Y. Benoist, Convexes divisibles. II. *Duke Math. J.* **120** (2003), no. 1, 97–120 Zbl 1037.22022 MR 2010735

[10] Y. Benoist, Convexes divisibles. I. In *Algebraic groups and arithmetic*, pp. 339–374, Tata Institute of Fundamental Research, Mumbai, 2004 Zbl 1084.37026 MR 2094116

[11] Y. Benoist, Convexes divisibles. III. *Ann. Sci. École Norm. Sup. (4)* **38** (2005), no. 5, 793–832 Zbl 1085.22006 MR 2195260

[12] Y. Benoist, Convexes divisibles. IV. *Invent. Math.* **164** (2006), no. 2, 249–278 Zbl 1107.22006 MR 2218481

[13] J. Beyrer and B. Pozzetti, Positive surface group representations in PO(p,q). 2021, arXiv:2106.14725v1

[14] P.-L. Blayac, *Aspects dynamiques des structures projectives convexes*. Ph.D. thesis, Université Paris-Saclay, 2021

[15] J. Bochi, R. Potrie, and A. Sambarino, Anosov representations and dominated splittings. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 11, 3343–3414 Zbl 1429.22011 MR 4012341

[16] O. V. Bogopol'skiĭ and V. N. Gerasimov, Finite subgroups of hyperbolic groups. *Algebra i Logika* **34** (1995), no. 6, 619–622, 728 Zbl 0901.20022 MR 1400705

[17] F. Bonahon, The geometry of Teichmüller space via geodesic currents. *Invent. Math.* **92** (1988), no. 1, 139–162 Zbl 0653.32022 MR 0931208

[18] A. Borel and J. Tits, Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I. *Invent. Math.* **12** (1971), 95–104 Zbl 0238.20055 MR 0294349

[19] R. Bowen, Periodic orbits for hyperbolic flows. *Amer. J. Math.* **94** (1972), 1–30 Zbl 0254.58005 MR 0298700

[20] R. Bowen, Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973), 429–460 Zbl 0282.58009 MR 0339281

[21] R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), no. 3, 181–202 Zbl 0311.58010 MR 0380889

[22] N. Brady, Finite subgroups of hyperbolic groups. *Internat. J. Algebra Comput.* **10** (2000), no. 4, 399–405 Zbl 1010.20030 MR 1776048

[23] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino, The pressure metric for Anosov representations. *Geom. Funct. Anal.* **25** (2015), no. 4, 1089–1179 Zbl 1360.37078 MR 3385630

[24] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino, Simple root flows for Hitchin representations. *Geom. Dedicata* **192** (2018), 57–86 Zbl 1383.53038 MR 3749423

[25] M. Burger, Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2. *Internat. Math. Res. Notices* **7** (1993), 217–225 Zbl 0829.57023 MR 1230298

[26] B. Collier, $SO(n,n+1)$-surface group representations and Higgs bundles. *Ann. Sci. Éc. Norm. Supér. (4)* **53** (2020), no. 6, 1561–1616 Zbl 1466.53031 MR 4203035

[27] D. Constantine, J.-F. Lafont, and D. J. Thompson, Strong symbolic dynamics for geodesic flows on CAT(−1) spaces and other metric Anosov flows. *J. Éc. polytech. Math.* **7** (2020), 201–231 Zbl 1436.37046 MR 4054334

[28] F. Dal'Bo and I. Kim, A criterion of conjugacy for Zariski dense subgroups. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), no. 8, 647–650 Zbl 0953.22013 MR 1763904

[29] J. Danciger, F. Guéritaud, and F. Kassel, Convex cocompactness in pseudo-Riemannian hyperbolic spaces. *Geom. Dedicata* **192** (2018), 87–126 Zbl 1428.53078 MR 3749424

[30] V. Fock and A. Goncharov, Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.* (2006), no. 103, 1–211 Zbl 1099.14025 MR 2233852

[31] É. Ghys and P. de la Harpe, Espaces métriques hyperboliques. In *Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988)*. Progr. Math. 83, pp. 27–45, Birkhäuser, Boston, MA, 1990 MR 1086650

[32] O. Glorieux and D. Monclair, Regularity of limit sets of AdS quasi-Fuchsian groups. [v1] 2018, [v2] 2023, arXiv:1809.10639v2

[33] M. Gromov, Hyperbolic groups. In *Essays in group theory*. Math. Sci. Res. Inst. Publ. 8, pp. 75–263, Springer, New York, 1987 Zbl 0634.20015 MR 0919829

[34] M. Gromov, Three remarks on geodesic dynamics and fundamental group. *Enseign. Math. (2)* **46** (2000), no. 3-4, 391–402 Zbl 1002.53028 MR 1805410

[35] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, Anosov representations and proper actions. *Geom. Topol.* **21** (2017), no. 1, 485–584 Zbl 1373.37095 MR 3608719

[36] F. Guéritaud and F. Kassel, Maximally stretched laminations on geometrically finite hyperbolic manifolds. *Geom. Topol.* **21** (2017), no. 2, 693–840 Zbl 1472.30017 MR 3626591

[37] O. Guichard and A. Wienhard, Anosov representations: domains of discontinuity and applications. *Invent. Math.* **190** (2012), no. 2, 357–438 Zbl 1270.20049 MR 2981818

[38] O. Guichard, F. Labourie, and A. Wienhard. Positivity and representations of surface groups. 2021, arXiv:2106.14584v3

[39] O. Guichard and A. Wienhard, Positivity and higher Teichmüller theory. In *European Congress of Mathematics*, pp. 289–310, European Mathematical Society (EMS), Zürich, 2018 Zbl 1404.22034 MR 3887772

[40] C. Guillarmou, G. Knieper, and T. Lefeuvre, Geodesic stretch, pressure metric and marked length spectrum rigidity. *Ergodic Theory Dynam. Systems* **42** (2022), no. 3, 974–1022 Zbl 1493.37036 MR 4374964

[41] C. Guillarmou and T. Lefeuvre, The marked length spectrum of Anosov manifolds. *Ann. of Math. (2)* **190** (2019), no. 1, 321–344 Zbl 1506.53054 MR 3990606

[42] B. Hasselblatt, Anatole Katok. *Ergodic Theory Dynam. Systems* **42** (2022), no. 2, 321–388 Zbl 1482.01023 MR 4362895

[43] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. Pure Appl. Math. 80, Academic Press, New York-London, 1978 Zbl 0451.53038 MR 0514561

[44] N. J. Hitchin, Lie groups and Teichmüller space. *Topology* **31** (1992), no. 3, 449–473 Zbl 0769.32008 MR 1174252

[45] L. Hörmander, *Implicit function theorems*. Stanford Univ., 1977

[46] O. Jenkinson, Ergodic optimization in dynamical systems. *Ergodic Theory Dynam. Systems* **39** (2019), no. 10, 2593–2618 Zbl 1435.37009 MR 4000508
[47] M. Kapovich, B. Leeb, and J. Porti, Anosov subgroups: dynamical and geometric characterizations. *Eur. J. Math.* **3** (2017), no. 4, 808–898 Zbl 1403.22013 MR 3736790
[48] M. Kapovich, B. Leeb, and J. Porti, A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings. *Geom. Topol.* **22** (2018), no. 7, 3827–3923 Zbl 1454.53048 MR 3890767
[49] F. Kassel, Geometric structures and representations of discrete groups. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 1115–1151, World Scientific, Hackensack, NJ, 2018 Zbl 1447.57028 MR 3966802
[50] A. W. Knapp, *Lie groups beyond an introduction*. Progr. Math. 140, Birkhäuser, Boston, MA, 1996 Zbl 0862.22006 MR 1399083
[51] G. Knieper, Volume growth, entropy and the geodesic stretch. *Math. Res. Lett.* **2** (1995), no. 1, 39–58 Zbl 0846.53028 MR 1312976
[52] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.* **81** (1959), 973–1032 Zbl 0099.25603 MR 0114875
[53] J.-L. Koszul, Déformations de connexions localement plates. *Ann. Inst. Fourier (Grenoble)* **18** (1968), 103–114 Zbl 0167.50103 MR 0239529
[54] F. Labourie, Anosov flows, surface groups and curves in projective space. *Invent. Math.* **165** (2006), no. 1, 51–114 Zbl 1103.32007 MR 2221137
[55] F. Labourie and R. Wentworth, Variations along the Fuchsian locus. *Ann. Sci. Éc. Norm. Supér. (4)* **51** (2018), no. 2, 487–547 Zbl 1404.37036 MR 3798306
[56] A. N. Livšic, Cohomology of dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1296–1320 Zbl 0273.58013 MR 0334287
[57] A. Manning, Topological entropy for geodesic flows. *Ann. of Math. (2)* **110** (1979), no. 3, 567–573 Zbl 0426.58016 MR 0554385
[58] G. A. Margulis, Certain applications of ergodic theory to the investigation of manifolds of negative curvature. *Funkcional. Anal. i Priložen.* **3** (1969), no. 4, 89–90 Zbl 0207.20305 MR 0257933
[59] C. T. McMullen, Thermodynamics, dimension and the Weil-Petersson metric. *Invent. Math.* **173** (2008), no. 2, 365–425 Zbl 1156.30035 MR 2415311
[60] G. Mess, Lorentz spacetimes of constant curvature. *Geom. Dedicata* **126** (2007), 3–45 Zbl 1206.83117 MR 2328921
[61] I. Mineyev, Flows and joins of metric spaces. *Geom. Topol.* **9** (2005), 403–482 Zbl 1137.37314 MR 2140987
[62] A. Parreau, Compactification d'espaces de représentations de groupes de type fini. *Math. Z.* **272** (2012), no. 1-2, 51–86 Zbl 1322.22022 MR 2968214
[63] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque* (1990), no. 187-188, 268 Zbl 0726.58003 MR 1085356
[64] M. Pollicott, Symbolic dynamics for Smale flows. *Amer. J. Math.* **109** (1987), no. 1, 183–200 Zbl 0628.58042 MR 0878205
[65] R. Potrie and A. Sambarino, Eigenvalues and entropy of a Hitchin representation. *Invent. Math.* **209** (2017), no. 3, 885–925 Zbl 1380.30032 MR 3681396
[66] M. B. Pozzetti, Higher rank Teichmüller theories. In *Séminaire Bourbaki*, Exp. No. 1159, pp. 327–354, 2020 Zbl 1470.32034 MR 4224639
[67] M. B. Pozzetti, A. Sambarino, and A. Wienhard, Conformality for a robust class of non-conformal attractors. *J. Reine Angew. Math.* **774** (2021), 1–51 Zbl 1483.37037 MR 4250471

[68] M. B. Pozzetti, A. Sambarino, and A. Wienhard, Anosov representations with Lipschitz limit set. *Geom. Topol.* **27** (2023), no. 8, 3303–3360 Zbl 07777462 MR 4668098

[69] J.-F. Quint, Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.* **77** (2002), no. 3, 563–608 Zbl 1010.22018 MR 1933790

[70] J.-F. Quint, Mesures de Patterson-Sullivan en rang supérieur. *Geom. Funct. Anal.* **12** (2002), no. 4, 776–809 Zbl 1169.22300 MR 1935549

[71] A. Sambarino, Hyperconvex representations and exponential growth. *Ergodic Theory Dynam. Systems* **34** (2014), no. 3, 986–1010 Zbl 1308.37014 MR 3199802

[72] A. Sambarino, Quantitative properties of convex representations. *Comment. Math. Helv.* **89** (2014), no. 2, 443–488 Zbl 1295.22016 MR 3229035

[73] A. Sambarino, Infinitesimal Zariski closures of positive representations. [v1] 2020, [v2] 2023, arXiv:2012.10276v2, to appear in *J. Differential Geom.*

[74] A. Sambarino, A report on an ergodic dichotomy. *Ergodic Theory Dynam. Systems* **44** (2024), no. 1, 236–289 Zbl 07779085 MR 4676211

[75] K. Sigmund, On the space of invariant measures for hyperbolic flows. *Amer. J. Math.* **94** (1972), 31–37 Zbl 0242.28014 MR 0302866

[76] K. Sigmund, On dynamical systems with the specification property. *Trans. Amer. Math. Soc.* **190** (1974), 285–299 Zbl 0286.28010 MR 0352411

[77] S. Smale, Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817 Zbl 0202.55202 MR 0228014

[78] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.* (1979), no. 50, 171–202 Zbl 0439.30034 MR 0556586

[79] N. Tholozan, Volume entropy of Hilbert metrics and length spectrum of Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$. *Duke Math. J.* **166** (2017), no. 7, 1377–1403 Zbl 1375.53022 MR 3649358

[80] N. Tholozan, Teichmüller geometry in the highest Teichmüller space. 2019, https://www.math.ens.psl.eu/~tholozan/Annexes/CocyclesReparametrizations2.pdf, visited on 8 June 2024

[81] W. P. Thurston, Minimal stretch maps between hyperbolic surfaces. In *Collected works of William P. Thurston with commentary*. Vol. I. Foliations, surfaces and differential geometry, pp. 533–585, American Mathematical Society, Providence, RI, 2022 MR 4554454

[82] J. Tits, Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.* **247** (1971), 196–220 Zbl 0227.20015 MR 0277536

[83] C. P. Walkden, Livšic theorems for hyperbolic flows. *Trans. Amer. Math. Soc.* **352** (2000), no. 3, 1299–1313 Zbl 0995.37016 MR 1637106

[84] P. Walters, On the pseudo-orbit tracing property and its relationship to stability. In *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*. Lecture Notes in Math. 668, pp. 231–244, Springer, Berlin-New York, 1978 Zbl 0403.58019 MR 0518563

[85] P. Walters, *An introduction to ergodic theory*. Grad. Texts in Math. 79, Springer, New York-Berlin, 1982 Zbl 0475.28009 MR 0648108

[86] A. Wienhard, An invitation to higher Teichmüller theory. In *Proceedings of the international congress of mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 1013–1039, World Scientific, Hackensack, NJ, 2018 Zbl 1447.32023 MR 3966798

[87] S. A. Wolpert, Thurston's Riemannian metric for Teichmüller space. *J. Differential Geom.* **23** (1986), no. 2, 143–174 Zbl 0592.53037 MR 0845703

Received 21 October 2022.

León Carvajales
Facultad de Ciencias Económicas y de Administración, Instituto de Estadística, Universidad de la República, Gonzalo Ramirez 1926, 11200 Montevideo, Uruguay; leon.carvajales@fcea.edu.uy

Xian Dai
Fakultät für Mathematik, Ruhr-Universität Bochum, IB, Universitätsstraße 150, 44801 Bochum, Germany; xian.dai@ruhr-uni-bochum.de

Beatrice Pozzetti
Mathematisches Institut, Ruprecht-Karls Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany; pozzetti@mathi.uni-heidelberg.de

Anna Wienhard
Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103 Leipzig, Germany; wienhard@mis.mpg.de