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Boundary maps and reducibility for cocycles into the isometries of CAT(0)-spaces

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Abstract. Let Γ be a discrete countable group acting isometrically on a measurable field $\mathbf{X}$ of CAT(0)-spaces of finite telescopic dimension over some ergodic standard Borel probability Γ-space (Ω, μ). If $\mathbf{X}$ does not admit any invariant Euclidean subfield, we prove that the measurable field $\widehat{\mathbf{X}}$ extended to a Γ-boundary admits an invariant section. In the case of constant fields, this shows the existence of Furstenberg maps for measurable cocycles, extending results by Bader, Duchesne and Lécureux. When $\Gamma < \mathrm{PU}(n,1)$ is a torsion-free lattice and the CAT(0)-space is $\mathcal{X}(p,\infty)$, we show that a maximal cocycle $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p,\infty)$ with a suitable boundary map is finitely reducible. As a consequence, we prove an infinite-dimensional rigidity phenomenon for maximal cocycles in $\mathrm{PU}(1,\infty)$.

1. Introduction

The concept of boundary maps for representations in algebraic groups was first introduced by Furstenberg [18, 19] and represent a powerful tool in the investigation of rigidity phenomena. Examples of results involving such maps are Mostow rigidity [29] and Margulis superrigidity [23]. Boundary maps gained even more importance in the context of bounded cohomology, where their application to the work by Burger and Monod [12] has generated a prolific literature [8, 9, 11, 21, 30].

More recently, several authors focused their attention on actions on CAT(κ)-spaces and the associated boundary maps. For instance, CAT(-1)-spaces have been studied by Burger and Mozes [13] and by Monod and Shalom [26]. Duchesne focused first on actions on the Hermitian symmetric space $\mathcal{X}(p,\infty)$ [15] and then, together with Bader and Lécureux, on a general CAT(0)-space $\mathcal{X}$ of finite telescopic dimension [5]. There the authors proved the existence of a boundary map $B \to \partial\mathcal{X}$ whenever the action on $\mathcal{X}$ is not elementary. In this setting, B denotes a Γ-boundary in the sense of Bader and Furman [6]. Such boundary can be seen as an extension both of the Furstenberg–Poisson boundary [18] and of the strong boundary in the sense of Burger–Monod [12]. In the general setting studied in [5], we lose the structure of spherical building of $\partial\mathcal{X}(p,\infty)$

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exploited in [15], but we can still rely on the rich structure of CAT(0)-spaces. For instance, a useful tool is the Euclidean de Rham decomposition (see [7]). Additionally, when the telescopic dimension is finite, Caprace and Lytchak [14] proved that a filtering family of closed convex subspaces of a CAT(0)-space $\mathcal{X}$ has a point fixed by $\mathrm{Isom}(\mathcal{X})$ in the bordification $\overline{\mathcal{X}}$. Both in [15] and in [5], the arguments rely on the notion of *measurable fields of* CAT(0)*-spaces* that were first introduced by Anderegg and Henry [3] and then developed by Duchesne [15].

Following the line of some recent works about measurable cocycles by the authors and Moraschini [27, 28, 31–34], in the first part of this paper we prove a generalization of [5, Theorem 1.1] to measurable fields. For all the definitions concerning measurable fields, we refer to Section 2.2.

Theorem 1. *Let Γ be a discrete countable group, (Ω, μ) be an ergodic standard Borel probability Γ-space and B a Γ-boundary. Consider a measurable field $\mathbf{X}$ of complete separable* CAT(0)*-spaces of finite telescopic dimension endowed with an isometric Γ-action. If $\mathbf{X}$ does not admit any invariant Euclidean subfield, then there exists an invariant section of the boundary field $\partial\widehat{\mathbf{X}}$, where $\widehat{\mathbf{X}}$ is the extension of $\mathbf{X}$ to the boundary B.*

An immediate consequence of the previous theorem is the existence of invariant sections for the boundary of constant fields and hence the existence of boundary maps for measurable cocycles.

Proposition 2. *Let Γ be a discrete countable group, (Ω, μ) be an ergodic standard Borel probability space and B a Γ-boundary. Consider a complete separable* CAT(0)*-space $\mathcal{X}$ of finite telescopic dimension and a measurable cocycle $\sigma : \Gamma \times \Omega \to \mathrm{Isom}(\mathcal{X})$. If $\mathcal{X}$ does not admit any Euclidean subfield on Ω which is σ-invariant, then there exists a boundary map $\phi : B \times \Omega \to \partial\mathcal{X}$ for σ.*

The proof of Theorem 1 is based on the arguments used in [5, Theorem 1], where the crucial point is the measurable version of the Adams–Ballmann theorem [15, Theorem 1.8]. Thanks to [15, Proposition 8.11], we can work with minimal invariant subfields. Moreover, by applying the measurable Euclidean de Rham decomposition [15, Propositions 9.2 and 9.3], we reduce ourselves to the particular case when there exists an invariant minimal family of closed convex spaces with trivial Euclidean factors.

Starting from Theorem 1, we investigate the case of the constant field $\mathcal{X} = \mathcal{X}(p, \infty)$. Recently, Duchesne, Lécureux and Pozzetti [16] proved that any maximal representation $\rho : \Gamma \to \mathrm{PU}(p, \infty)$ of a lattice $\Gamma < \mathrm{PU}(n, 1)$, with $n \geq 2$, preserves a finite-dimensional totally geodesic Hermitian symmetric space $\mathcal{Y} \subset \mathcal{X}(p, \infty)$. Moreover, under the additional hypothesis of Zariski density, they ruled out the existence of any such representation for any $p \geq 1$.

Motivated by such results, we will focus our attention on measurable cocycles $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p, \infty)$, where Γ is a complex hyperbolic lattice in $\mathrm{PU}(n, 1)$ and (Ω, μ) is an ergodic standard Borel probability Γ-space. We actually need to assume something more, namely the existence of a boundary map $\partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \mathcal{I}_p$, where $\mathcal{I}_p$ denotes the set of p-isotropic subspaces of $\mathcal{H} = \mathbb{C}^{p,\infty}$. The existence of those maps will be discussed in the last section.

The most important difference with the finite-dimensional case [30, 31] is that the group $\mathrm{PU}(p,\infty)$ is not algebraic in the usual meaning. The absence of such structure motivates the notions of *algebraic* and of *standard algebraic subgroups* given by Duchesne, Lécureux and Pozzetti [16]. In this way, they were able to define the notion of Zariski density inside $\mathrm{PU}(p,\infty)$.

The lack of an algebraic structure can be overcome, for instance, when σ is cohomologous to a cocycle whose image is contained in a *finite-dimensional* algebraic subgroup. We call such cocycles *finitely reducible*. Using the machinery of numerical invariants and maximality developed by Moraschini and the second author [27, 28], we get a statement similar to [16, Theorem 6.7] for cocycles.

Theorem 3. *Let $\Gamma < \mathrm{PU}(n,1)$ be a complex hyperbolic lattice with $n \geq 1$ and let (Ω,μ) be an ergodic standard Borel probability Γ-space. Consider a measurable cocycle $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p,\infty)$ with $p \geq 1$ and suppose that there exists a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \mathcal{I}_p$. If σ is maximal, then it is finitely reducible.*

The structure of the proof is the following. We first refine [16, Proposition 6.2], namely we show that any slice of the boundary map has image essentially contained in a unique copy of $\partial\mathcal{X}(p,q)$ embedded in $\partial\mathcal{X}(p,\infty)$ for some $p \leq q \leq np$. Since such construction varies measurably, ergodicity implies that q does not depend on the slice. Using the transitive action of $\mathrm{PU}(p,\infty)$ (Lemma 3.2), we twist the cocycle and the boundary map in such a way to find a cohomologous cocycle σ^f and a boundary map ϕ^f with image of the latter essentially contained in some embedding of $\partial\mathcal{X}(p,q)$ in $\partial\mathcal{X}(p,\infty)$, so that finite reducibility follows.

It seems natural to ask whether Theorem 1 provides a suitable boundary map in the context of Theorem 3. We notice that by our first result we have an equivariant map $\partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \mathcal{I}_k$ for some $k \leq p$. In particular, for cocycles $\sigma : \Gamma \times \Omega \to \mathrm{PU}(1,\infty)$, since maximality implies non-elementarity, Theorem 1 provides a boundary map $\partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \partial\mathbb{H}^\infty_{\mathbb{C}}$ and, by applying Theorem 3 and [31, Theorem 2], we get the following version of Mostow rigidity for infinite-dimensional cocycles.

Theorem 4. *Let $\Gamma < \mathrm{PU}(n,1)$ be a complex hyperbolic lattice with $n \geq 1$ and let (Ω,μ) be an ergodic standard Borel probability Γ-space. Any maximal cocycle $\sigma : \Gamma \times \Omega \to \mathrm{PU}(1,\infty)$ is cohomologous to a cocycle preserving a copy of $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{H}^\infty_{\mathbb{C}}$ and acting on it via the standard lattice embedding.*

Plan of the paper. The paper is divided into three sections. Section 2 focuses on the existence of invariant boundary sections and boundary maps. After a brief introduction of basics about CAT(0)-spaces (Section 2.1), we define measurable fields of metric and CAT(0)-spaces. We recall the measurable Euclidean de Rham decomposition and the measurable version of the Adams–Ballmann theorem (Section 2.2). Then we define boundaries and we prove Theorem 1 (Section 2.3). We conclude the section recalling the notion of boundary maps and proving Proposition 2.

Section 3 is devoted to reducibility of cocycles into the isometries of $\mathcal{X}(p,\infty)$. We first recall some notions about bounded cohomology (Section 3.1). Then we introduce Hermitian symmetric spaces, and we characterize embeddings of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$ (Section 3.2). In this context, we define the *Toledo invariant* associated to a measurable cocycle, passing through the definition of *Bergman class* and the machinery developed by [27] about numerical invariants of measurable cocycles (Sections 3.3 and 3.4). Then we move to the notion of *algebraic* and *finite-dimensional algebraic subgroups* of $\mathrm{GL}(\mathcal{H})$ (Section 3.5), and we finally provide the proof of Theorem 3 (Section 3.6).

We conclude with Section 4, where we prove Theorem 4.

2. Existence of invariant boundary sections and boundary maps

This section is devoted to prove the existence of boundary maps for measurable cocycles in the isometries of a complete separable CAT(0)-space. After a brief introduction about CAT(0)-spaces, measurable fields and boundaries in the sense of Bader and Furman [6], we give the proof of Theorem 1. Then we move to measurable cocycles and boundary maps to prove Proposition 2.

2.1. CAT(0)-spaces

A metric space $(\mathcal{X}, d)$ is a CAT(0)-*space* if it is geodesic and for every triple of distinct points $x, y, z \in \mathcal{X}$, given a point m in the geodesic segment between y and z, the following inequality holds:

$$d(x,m)^2 \leq \frac{1}{2}(d(x,y)^2 + d(x,z)^2) - \frac{1}{4}d(y,z)^2.$$

For the purposes of this paper, CAT(0)-spaces will always be assumed to be *complete* and *separable*.

Since embedded flats into CAT(0)-spaces play an important role in the study of their geometry, we recall the following decomposition into Euclidean and non-Euclidean factors. Precisely, the *Euclidean de Rham decomposition* of a CAT(0)-space $\mathcal{X}$ is its canonical isometric splitting into a Hilbert space H and a factor Z which cannot be further decomposed as a product with non-trivial Euclidean factor [7, Theorem 6.15]. Moreover, for every point $x \in \mathcal{X}$, the space H (respectively Z) identifies with a unique closed convex subspace of $\mathcal{X}$ containing x.

Given a subset $\mathcal{Y} \subset \mathcal{X}$ of a metric space, its *diameter* is defined as

$$\mathrm{diam}(\mathcal{Y}) := \sup_{x,y \in \mathcal{Y}} d(x,y),$$

and $\mathcal{Y}$ is said to be *bounded* if it has finite diameter. A convex bounded set $\mathcal{Y}$ has some preferred points called *circumcenters*, which are the centers of balls of minimal radius containing $\mathcal{Y}$. Notice that, without the assumption of convexity, one can still give the notion of circumcenter, but such points may not belong to $\mathcal{Y}$. In the case of CAT(0)-spaces, it turns out that every bounded subset has a unique circumcenter, which we call *center*. An equivalent definition can be given in terms of actions of isometries. Precisely, the center of a bounded subset $\mathcal{Y} \subset \mathcal{X}$ of a CAT(0)-space is the unique point fixed by any isometry stabilizing $\mathcal{Y}$.

Before introducing the notion of telescopic dimension, we need the one of *geometric dimension*. This concept was first introduced by Kleiner [22] in terms of the space of directions at each point and then has been reformulated by Caprace and Lytchak [14, Theorem 1.3] in the following way. If $\mathcal{X}$ is a CAT(0)-space, then its geometric dimension is $\leq n$ if for each subset $\mathcal{Y}$ of finite diameter the following inequality holds:

$$\mathrm{rad}(\mathcal{Y}) \leq \sqrt{\frac{n}{2(n+1)}}\,\mathrm{diam}(\mathcal{Y}).$$

The number $\mathrm{rad}(\mathcal{Y})$ is the *circumradius* of $\mathcal{Y}$, namely the infimum of all $r > 0$ such that $\mathcal{Y}$ is contained in some closed ball of radius r. The result by Caprace and Lytchak leads to a characterization of telescopic dimension, originally given by Kleiner [22], that we assume here as a definition (refer to [14] for more details).

Definition 2.1. A CAT(0)-space $\mathcal{X}$ has *telescopic dimension* $\leq n$ if for any $\delta > 0$ there exists some constant $D > 0$ such that for every bounded set $\mathcal{Y}$ of diameter $> D$, we have

$$\mathrm{rad}(\mathcal{Y}) \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}}\right)\mathrm{diam}(\mathcal{Y}).$$

As we will recall in Section 3.2, the Hermitian symmetric space $\mathcal{X}(p,\infty)$ is a CAT(0)-space of telescopic dimension p [15, Corollary 1.4]. This implies that the visual boundary $\partial\mathcal{X}(p,\infty)$ has geometric dimension $p-1$ [14, Proposition 2.1].

For a complete CAT(0)-space $\mathcal{X}$ with finite telescopic dimension, Caprace and Lytchak proved that every filtering family of closed convex subspaces of $\mathcal{X}$ either intersects at $\mathcal{X}$ or at $\partial\mathcal{X}$ [14, Theorem 1.1]. Notice that this is equivalent to quasi-compactness of the bordification $\overline{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$ endowed with the topology defined by Monod [25, Section 3.7]. The following technical result is an example of application of [14, Theorem 1.1], and it turned out to be useful in the proof of [5, Theorem 1.1] and [16, Theorem 1.7]. It will be exploited to prove Theorem 1.

Proposition 2.2 ([5, Proposition 2.1]). *Let E be a Euclidean space and $f : E \to \mathbb{R}$ be a convex function. If we denote by $m = \inf\{f(x) \mid x \in E\}$, then:*

(i) *If m is not attained, then $\bigcap_{\varepsilon>0} \partial E_\varepsilon \neq \emptyset$, where $E_\varepsilon := f^{-1}((m, m+\varepsilon))$ is not empty and has a center.*

If m is attained, we denote by $E_m = f^{-1}(m)$ and by $E_m = F \times T$ its Euclidean de Rham decomposition. Then one of the following holds:

(ii) *E_m is bounded and thus it has a center.*

(iii) *T is bounded and $\partial E_m = \partial F$ is a sphere.*

(iv) *T is not bounded and $\partial T \subset \partial E$ has radius less than $\frac{\pi}{2}$.*

Notice that, as mentioned in point (iii), boundaries of flats are Euclidean spheres that can also be interpreted as CAT(1)-spaces. In particular, boundaries of maximal flats are subcomplexes, called apartments, of the building structure of the visual boundary. We refer to [1] for the general theory of such building. We only point out that the existence of circumcenters for bounded subsets [7, Proposition 2.7] holds also in this case. More precisely, every subset of radius at most $\frac{\pi}{2}$ in a sphere has a center, and this property will be used in the proof of Theorem 1.

2.2. Measurable fields and the Adams–Ballmann dichotomy

In this section, we introduce measurable fields of metric spaces together with some results that we will exploit in the next section to prove the existence of boundary maps. We refer to Anderegg and Henry [3] for the general theory of measurable fields and to Duchesne [15] for the measurable version of both the Euclidean de Rham decomposition and the Adams–Ballmann dichotomy.

Definition 2.3. Given a standard probability space (Ω, μ), a *measurable field of metric spaces* on Ω is a collection of metric spaces $\mathbf{X} = \{X_\omega\}_{\omega \in \Omega}$ together with a countable family $\mathcal{F} \subset \prod_{\omega \in \Omega} X_\omega$ such that

- for all $x, y \in \mathcal{F}$, the map $\omega \mapsto d_\omega(x_\omega, y_\omega)$ is measurable;
- for almost every $\omega \in \Omega$, the set $\{f_\omega \mid f \in \mathcal{F}\}$ is dense in X_ω.

A *section* of $\mathbf{X}$ is an element $x \in \prod_{\omega \in \Omega} X_\omega$ such that for every $y \in \mathcal{F}$, the map $\omega \mapsto d_\omega(x_\omega, y_\omega)$ is measurable.

If the X_ω's are CAT(0)-spaces, then we call $\mathbf{X}$ a measurable field of CAT(0)-spaces. A *subfield* $\mathbf{Y}$ of $\mathbf{X}$ is a collection of non-empty closed convex subsets $Y_\omega \subset X_\omega$ such that for every section Ω of $\mathbf{X}$, the map $\omega \mapsto d_\omega(x_\omega, Y_\omega)$ is measurable.

If G is a locally compact group acting on a standard probability space (Ω, μ) by preserving the measure class, we say that Ω is a *Lebesgue* G-space. A G*-action* on $\mathbf{X}$ is a collection $\{\sigma(g, \omega)\}_{g \in G, \omega \in \Omega}$, where

- for every $g \in G$ and almost every $\omega \in \Omega$, we have $\sigma(g, \omega) \in \mathrm{Isom}(X_\omega, X_{g\omega})$;
- for every $g, h \in G$ and almost every $\omega \in \Omega$, the following equality holds:

$$\sigma(gh, \omega) = \sigma(g, h\omega)\sigma(h, \omega); \tag{1}$$

- for every $x, y \in \mathcal{F}$, the map $(g, \omega) \mapsto d(x_\omega, \sigma(g, g^{-1}\omega) y_{g^{-1}\omega})$ is measurable.

Example 2.4. Given a standard Borel probability space (Ω, μ) and a complete separable metric space $\mathcal{X}$, we can build a measurable field $\mathbf{X}$ by setting $X_\omega := \mathcal{X}$ and taking as fundamental family the collection $\{f^x\}_{x \in \mathcal{X}_0}$, where $\mathcal{X}_0 \subset \mathcal{X}$ is a countable dense subset and $f^x_\omega := x$ for every $\omega \in \Omega$.

Given a locally compact group G such that (Ω, μ) is a Lebesgue G-space, a G-action $\{\sigma(g,\omega)\}_{g \in G, \omega \in \Omega}$ on $\mathbf{X}$ boils down to a map $\sigma : G \times \Omega \to \mathrm{Isom}(\mathcal{X})$ satisfying equation (1) and such that the function

$$(g, \omega) \mapsto d(x, \sigma(g, g^{-1}\omega)y)$$

is measurable for all $x, y \in \mathcal{X}_0$.

Example 2.5. Let G be a locally compact group and consider a Lebesgue G-space (Ω, μ). Given another Lebesgue G-space (Θ, ν), the product space $(\Omega \times \Theta, \mu \otimes \nu)$ is again a Lebesgue G-space with respect to the diagonal action. Consider a measurable field $\mathbf{X} = \{\mathcal{X}_\omega\}_{\omega \in \Omega}$ on Ω of CAT(0)-spaces of finite telescopic dimension with an isometric G-action $\{\sigma(g,\omega)\}_{g \in G, \omega \in \Omega}$. We can consider the *extension* $\widehat{\mathbf{X}}$ of the field $\mathbf{X}$ to the space Θ in the following way:

- We set $\widehat{\mathcal{X}}_{(\omega,\theta)} := \mathcal{X}_\omega$ for $(\omega, \theta) \in \Omega \times \Theta$.
- Any section x_ω of $\mathbf{X}$ can be changed into a section of $\widehat{\mathbf{X}}$ by setting $\hat{x}_{(\omega,\theta)} := x_\omega$.
- We define an isometric G-action on $\widehat{\mathbf{X}}$ by setting $\{\hat{\sigma}(g, \omega, \theta) := \sigma(g, \omega)\}$ for all $g \in G, \omega \in \Omega, \theta \in \Theta$.

A G-action $\{\sigma(g,\omega)\}_{g \in G, \omega \in \Omega}$ on a measurable field $\mathbf{X}$ of CAT(0)-spaces induces a natural G-action on both sections and subfields. Similarly, we have an induced action on the boundary field $\partial\mathbf{X}$ of $\mathbf{X}$. The latter is obtained by collecting the boundaries of each $\mathcal{X}_\omega$ and by controlling measurability via the Busemann functions. The G-action on $\mathbf{X}$ induces an action on the sections of $\partial\mathbf{X}$ given by $(g\xi)_\omega = (\sigma(g, g^{-1}\omega)\xi_{g^{-1}\omega})$. Furthermore, a subfield $\mathbf{Y} \subset \mathbf{X}$ is *minimal* if it is invariant under the G-action and it does not contain a proper invariant subfield.

As proved by Caprace and Lytchak [14, Proposition 1.8], any isometric action of a locally compact group on a complete CAT(0)-space of finite telescopic dimension either has a fixed point in the boundary or admits an invariant non-empty closed convex subset which is minimal with respect to inclusion. This allows us to reduce the investigation of existence of boundary maps to minimal actions (see [15, Theorem 1.7] and [5, Theorem 1.1]). The following result can be seen as the generalization of [14, Proposition 1.8] to measurable fields.

Proposition 2.6 ([15, Proposition 8.11]). *Let G be a locally compact group and let (Ω, μ) be a Lebesgue G-space. Suppose that $\mathbf{X}$ is a measurable field on Ω of* CAT(0)*-spaces of finite telescopic dimension, G acts on $\mathbf{X}$ and Ω is ergodic. Then either there exists a minimal invariant subfield of $\mathbf{X}$ or there exists an invariant section of $\partial\mathbf{X}$.*

A second construction that we will use is the extension of the Euclidean de Rham decomposition for measurable fields of CAT(0)-spaces.

Proposition 2.7 ([15, Proposition 9.2]). *Let G be a locally compact group and let (Ω, μ) be a Lebesgue G-space. Let x be a section of a measurable field $\mathbf{X}$ on Ω of* CAT(0)*-spaces of finite telescopic dimension. Suppose G acts minimally on $\mathbf{X}$ via $\sigma = \{\sigma(g,\omega)\}_{g\in G,\omega\in\Omega}$ and assume that the action Ω is ergodic. There exist $n \in \mathbb{N}$ and two subfields $\mathbf{E}$ and $\mathbf{Y}$ of $\mathbf{X}$ containing x such that $\mathbf{X} = \mathbf{E} \times \mathbf{Y}$ and $E_\omega \cong \mathbb{R}^n$ for almost every $\omega \in \Omega$. Moreover, $\mathbf{E}$ is maximal for those properties.*

If y is another section of $\mathbf{X}$ and $\mathbf{X} = \mathbf{E}' \times \mathbf{Y}'$ is another such decomposition associated to y, then for almost every $\omega \in \Omega$, the projections $\pi_{E_\omega|E'_\omega}$ and $\pi_{Y_\omega|Y'_\omega}$ are isometries. As a consequence, the G-action on $\mathbf{X}$ splits as

$$\sigma(g,\omega) = \sigma_{\mathbf{E}}(g,\omega) \times \sigma_{\mathbf{Y}}(g,\omega),$$

where $\{\sigma_{\mathbf{E}}(g,\omega)\}_{g\in G,\omega\in\Omega}$ and $\{\sigma_{\mathbf{Y}}(g,\omega)\}_{g\in G,\omega\in\Omega}$ are, respectively, actions on $\mathbf{E}$ and $\mathbf{Y}$.

The last preliminary result that we recall is the measurable version of the Adams–Ballmann dichotomy [2]. In order to state it, we need to recall the definition of *amenable space* due to Zimmer [36]. Given a locally compact second countable group G, a Lebesgue G-space (Ω, μ) is *G-amenable* if for every G-action on a separable measurable field $\mathbf{E}$ of Banach spaces over Ω and every G-invariant subfield $\mathbf{K}$ of weakly compact subsets of the unit balls of $\mathbf{E}^*$, there exists an invariant section of $\mathbf{K}$.

We conclude the section with the following.

Theorem 2.8 ([15, Theorem 1.8]). *Let G be a locally compact second countable group and (Ω, μ) be a Lebesgue G-space which is ergodic and amenable. Let $\mathbf{X}$ be a measurable field on Ω of complete* CAT(0)*-spaces of finite telescopic dimension. If G acts on $\mathbf{X}$, then either there is an invariant section of the boundary field $\partial\mathbf{X}$ or there exists an invariant Euclidean subfield of $\mathbf{X}$.*

2.3. Existence of invariant sections for extended fields

In this section, we prove Theorem 1. We need first to recall the definition of boundary in the sense of Bader and Furman [6].

Definition 2.9. Let G be a locally compact second countable group. A *fiberwise isometric G-action* on a measurable map $p : M \to T$ between standard Borel spaces is a G-invariant Borel map $d : M \times_p M \to \mathbb{R}_{\geq 0}$ such that any fiber $p^{-1}(t) \subset M$ endowed with the induced metric $d_{|p^{-1}(t)\times p^{-1}(t)}$ is a separable metric space on which G acts in a compatible way, namely

$$d(gx, gy) = d(x, y)$$

for every $g \in G$ and every $x, y \in M$ with $p(x) = p(y)$.

A map $q : X \to Y$ between Lebesgue G-spaces is *relatively metrically ergodic* if for any fiberwise isometric G-action on a measurable map $p : M \to T$ between standard Borel spaces and measurable G-equivariant maps $f : X \to M$ and $h : Y \to T$, there exists a measurable G-equivariant map $\psi : Y \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ \downarrow q & \nearrow \psi & \downarrow p \\ Y & \xrightarrow{h} & T \end{array}$$

Definition 2.10 ([6, Definition 2.3]). Let Γ be a locally compact and second countable group. A Γ-*boundary* is an amenable Lebesgue Γ-space (B, ν) such that the projections $\pi_1 : B \times B \to B$ and $\pi_2 : B \times B \to B$ on the first and second factors, respectively, are relatively metrically ergodic.

Remark 2.11. The notion of Γ-boundary contains other versions of boundaries.

(i) The Furstenberg–Poisson boundary of a locally compact and second countable group [18] turns out to be a boundary in the sense of Definition 2.10 [6, Theorem 2.7].

(ii) Suppose that $\Gamma < H$ is a lattice into a connected semisimple Lie group of non-compact type. Given a minimal parabolic subgroup $P < H$, the quotient H/P is the Furstenberg–Poisson boundary for Γ and hence a Γ-boundary by [6, Theorem 2.3].

(iii) In general, a Γ-boundary is a strong Γ-boundary in the sense of Burger and Monod [12] and [6, Remark 2.4].

Since the arguments in the proof of Theorem 1 strongly rely on the objects introduced in Section 2.2, we recall the following result, due to Duchesne, Lécureux and Pozzetti.

Lemma 2.12 ([16, Lemma 4.11]). *Let Γ be a countable group and let $\mathbf{X}$ be a measurable field over a Lebesgue Γ-space (Ω, μ). Then there exists a full-measure subset $\Omega_0 \subset \Omega$, a standard Borel structure on $X := \bigsqcup_{\omega \in \Omega_0} X_\omega$ and a Borel map $p : X \to \Omega_0$ that admits a fiberwise isometric Γ-action. Moreover, $p^{-1}(\omega)$ is X_ω with the metric d_ω.*

Proof of Theorem 1. Without loss of generality, we can suppose that the Γ-action on the measurable field $\mathbf{X} := \{\mathcal{X}_\omega\}_{\omega \in \Omega}$ is minimal. In fact, by Proposition 2.6, either we have a minimal subfield $\mathbf{Y} \subset \mathbf{X}$ on Ω or there exists an invariant section of $\partial \mathbf{X}$. In the second case, the same section can be viewed as a section of the boundary field $\partial \widehat{\mathbf{X}}$ of the extension of $\mathbf{X}$ on B, and hence we would conclude.

According to Proposition 2.7, we consider the Euclidean de Rham decomposition $\mathbf{Y} = \mathbf{F} \times \mathbf{Z}$ and we denote by $\sigma_{\mathbf{Z}}$ and $\sigma_{\mathbf{Y}}$ the Γ-actions induced, respectively, on $\mathbf{Z}$ and $\mathbf{Y}$. We claim that the Γ-action $\sigma_{\mathbf{Z}}$ on $\mathbf{Z}$ is minimal. By contradiction, assume that it is not. Thus, by Proposition 2.6, there exists a minimal invariant subfield $\mathbf{W} \subset \mathbf{Z}$ whose product with $\mathbf{F}$ is a strict subfield of $\mathbf{F} \times \mathbf{Z} = \mathbf{Y}$, contradicting the minimality of $\mathbf{Y}$. Moreover, any $\sigma_{\mathbf{Z}}$-invariant Euclidean subfield of $\mathbf{Z}$ would produce an invariant Euclidean subfield of $\mathbf{X}$, and this would contradict the hypothesis. Notice also that the boundary ∂Z_ω is contained in ∂Y_ω and a fortiori in $\partial \mathcal{X}_\omega$.

Now we have a measurable field $\mathbf{Z}$ on Ω which is minimal, and it does not admit any invariant Euclidean subfield. Following Example 2.5, we consider the extension $\widehat{\mathbf{Z}}$ of the field $\mathbf{Z}$ to B. By [26, Proposition 2.4], the spaces $B \times \Omega$ and $B \times B \times \Omega$ are ergodic Γ-spaces. By [37, Proposition 4.3.4], $B \times \Omega$ is also Γ-amenable. In this context, we apply [15, Theorem 1.8] to $\widehat{\mathbf{Z}}$ and we have two possible cases: either there exists a section of $\partial \widehat{\mathbf{Z}}$ or there exists an invariant Euclidean subfield $\mathbf{E} \subset \widehat{\mathbf{Z}}$.

We consider the distance map

$$d : B \times B \times \Omega \to \mathbb{R}, \quad (\xi_1, \xi_2, \omega) \mapsto d(E_{\xi_1,\omega}, E_{\xi_2,\omega}) := \inf_{y \in E_{\xi_1,\omega}} d(y, E_{\xi_2,\omega}),$$

where $\mathbf{E} = \{E_{\xi,\omega}\}_{(\xi,\omega) \in B \times \Omega}$. Following [5], we have four possible cases, and by ergodicity one of them must happen almost surely. For the same reason, the distance map is essentially equal to some value, say d_0, for almost every $\omega \in \Omega$ and $\xi_1, \xi_2 \in B$.

Case (i). Suppose that d_0 is not attained for almost every $\omega \in \Omega$ and $\xi_1, \xi_2 \in B$. Hence, for almost every $\omega \in \Omega$ and $\xi_1 \in B$, we can define the subspaces

$$E^n_{\xi_1,\xi_2,\omega} := \left\{ y \in E_{\xi_1,\omega} \mid d(y, E_{\xi_2,\omega}) < d_0 + \frac{1}{n} \right\}$$

which are nested subspaces of $E_{\xi_1,\omega}$. By [15, Proposition 8.10], we have a σ-equivariant map

$$\psi : B \times B \times \Omega \to \partial \mathbf{E}.$$

To ensure the correct application of [15, Proposition 8.10], we are considering the extended field $\{E'_{\xi_1,\xi_2,\omega}\}_{(\xi_1,\xi_2,\omega) \in B \times B \times \Omega}$ such that $E'_{\xi_1,\xi_2,\omega} = E_{\xi_1,\omega}$ for every $\omega \in \Omega$ and $\xi_1, \xi_2 \in B$. It follows directly from Lemma 2.12 that the projection p of (a full-measure subset of) $\partial \mathbf{E}$ on $B \times \Omega$ has a Γ-fiberwise isometric action. The metric structure on each fiber is the same as the one given in [5]. Thus, we can apply relative metric ergodicity to the following diagram:

$$\begin{array}{ccc} B \times B & \xrightarrow{\ \Psi\ } & \mathrm{L}^0(\Omega, \partial \mathbf{E}) \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle p_\Omega} \\ B & \xrightarrow{\ j\ } & \mathrm{L}^0(\Omega, B \times \Omega) \end{array}$$

Here $\mathrm{L}^0(\Omega, \cdot)$ denotes the space of measurable sections identified μ-almost everywhere with the standard measurable structure coming from the topology of convergence in measure [35, Section 4.4] and [17, Notation 2.4], Ψ is defined by $\Psi(\xi_1, \xi_2)(x) := \psi(x, \xi_1, \xi_2)$, j is given by $j(\xi)(x) := (\xi, \omega)$, π_1 is the projection on the first factor and p_Ω is defined as $p_\Omega(f)(x) := p(f(\omega))$. The function p_Ω can be equipped with a fiberwise isometric Γ-action obtained by the one on p by integrating along Ω the functions in each fiber (see [31, Theorem 1]).

By relative metric ergodicity, we have a lifting $B \to \mathrm{L}^0(\Omega, \partial\mathbf{E})$, thus Ψ does not depend on the second factor. Hence, we have a σ-invariant map $B \times \Omega \to \partial\mathbf{E} \subset \partial\widehat{\mathbf{Z}}$, whose existence is ruled out by the dichotomy of Theorem 2.8.

We can suppose that the distance $d_{\xi_1,\xi_2,\omega}$ is attained almost surely, and we define the non-empty subsets

$$W_{\xi_1,\xi_2,\omega} := \{w \in E_{\xi_1,\omega} \mid d(w, E_{\xi_2,\omega}) = d_0\} \subset E_{\xi_1,\omega}.$$

Case (ii). If the $W_{\xi_1,\xi_2,\omega}$'s are bounded, we can associate to any such subset its circumcenter $c_{\xi_1,\xi_2,\omega}$. The map

$$\psi : B \times B \times \Omega \to \mathbf{E}, \quad \psi(\xi_1, \xi_2, \omega) := c_{\xi_1,\xi_2,\omega}$$

is σ-equivariant, and by applying twice the relative metric ergodicity, we obtain a map $\psi : \Omega \to \mathbf{E}$ such that

$$\psi(\gamma\omega) = \sigma(\gamma, \omega)\psi(\omega).$$

Since points are 0-dimensional flats, this contradicts the hypothesis on $\mathbf{X}$.

Thus, the $W_{\xi_1,\xi_2,\omega}$'s are not bounded, and we can consider their Euclidean de Rham decomposition

$$W_{\xi_1,\xi_2,\omega} = F_{\xi_1,\xi_2,\omega} \times T_{\xi_1,\xi_2,\omega},$$

where the $F_{\xi_1,\xi_2,\omega}$'s are maximal Euclidean factors.

Case (iii). If $T_{\xi_1,\xi_2,\omega}$ is not bounded, as in Case (i) we realize a map

$$\psi : B \times B \times \Omega \to \partial\mathbf{T}, \quad \psi(\xi_1, \xi_2, \omega) := c_{\xi_1,\xi_2,\omega},$$

where $c_{\xi_1,\xi_2,\omega}$ is the center of $\partial T_{\xi_1,\xi_2,\omega}$ and $\mathbf{T}$ denotes the measurable field given by the $T_{\xi_1,\xi_2,\omega}$'s. Notice that $c_{\xi_1,\xi_2,\omega}$ can be defined, thanks to Proposition 2.2 (iv). Using the same arguments of Case (i), we get a contradiction.

Case (iv). Finally, if the $T_{\xi_1,\xi_2,\omega}$'s are bounded, we consider a subfield $\mathbf{E}'$ of $\mathbf{E}$ whose sheets are defined as follows:

$$E'_{\xi_1,\xi_2,\omega} := F_{\xi_1,\xi_2,\omega} \times \{t_{\xi_1,\xi_2,\omega}\}$$

for every $\omega \in \Omega$ and $\xi_1, \xi_2 \in B$, where $t_{\xi_1,\xi_2,\omega}$ is the circumcenter of $T_{\xi_1,\xi_2,\omega}$. The same argument used in [5] shows that in fact $E'_{\xi_1,\xi_2,\omega} = E_{\xi_1,\omega}$ for almost every $\omega \in \Omega$ and $\xi_1, \xi_2 \in B$. Moreover, $E_{\xi,\omega}$ and $E_{\xi',\omega}$ are *parallel* for almost every $\omega \in \Omega$ and almost every $\xi, \xi' \in B$. Recall that two Euclidean subspaces are parallel if the restriction on the first one of the distance to the second one is constant and vice versa (in this context, the sandwich lemma [7, Exercise II.2.12 (2)] guarantees that their convex hull splits isometrically as $\mathbb{R}^n \times [0, d_0]$, for some n). We use the notation

$$E_{\xi,\omega} /\!/ E_{\xi',\omega}.$$

By Fubini's theorem, there exists an element $\xi_0 \in B$ and a full-measure subset $\Delta \subset B \times \Omega$ such that

$$E_{\xi,\omega} \mathbin{/\!/} E_{\xi_0,\omega}$$

for every $(\xi, \omega) \in \Delta$. We denote by $\Delta^\Gamma = \bigcap_{\gamma\in\Gamma} \gamma\Delta$ which is still of full measure since Γ is countable. We consider the set

$$C_\omega := \text{ convex hull } (\{E_{\xi,\omega}\}_{(\xi,\omega)\in\Delta^\Gamma})$$

that can be decomposed into Euclidean de Rham factors $E_\omega \times T_\omega$ such that

$$E_\omega \mathbin{/\!/} E_{\xi,\omega} \mathbin{/\!/} E_{\xi_0,\omega} \tag{2}$$

for every $(\xi, \omega) \in \Delta^\Gamma$. Moreover, for almost every $\omega \in \Omega$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}\sigma_{\mathbf{Z}}(\gamma,\omega)C_\omega &= \text{convex hull } (\sigma_{\mathbf{Z}}(\gamma,\omega)E_{\xi,\omega})_{(\xi,\omega)\in\Delta^\Gamma} \\ &= \text{convex hull } (E_{\gamma\xi,\gamma\omega})_{(\xi,\omega)\in\Delta^\Gamma} \\ &= \text{convex hull } (E_{\xi,\gamma\omega})_{(\xi,\gamma\omega)\in\Delta^\Gamma} = C_{\gamma\omega},\end{aligned}$$

where to pass from the first line to the second one we used the fact that **E** is a subfield of **Z** and to pass from the second line to the third one we exploited the action on Δ^Γ. Now, by the minimality of **Z** we must have $C_\omega = Z_\omega$ for almost every $\omega \in \Omega$, and since Z_ω has trivial Euclidean factor, by equation (2) we have

$$\dim(E_{\xi,\omega}) = 0$$

for every $(\xi, \omega) \in \Delta^\Gamma$. Hence, we have a section $B \times \Omega \to \mathbf{Z}$ and, by the same argument used in Case (ii), we have a contradiction. ■

2.4. Measurable cocycles and boundary maps

In this section, we finally prove Proposition 2. We will first need a short introduction to measurable cocycles. We refer to [27,28,31] for further details. We will assume that G is a locally compact and second countable group endowed with its Haar measurable structure, H is a topological group endowed with its Borel σ-algebra and (Ω, μ) be a standard Borel probability space endowed with a measure-preserving G-action.

Definition 2.13. A *measurable cocycle* is a measurable function $\sigma : G \times \Omega \to H$ such that

$$\sigma(g_1g_2, \omega) = \sigma(g_1, g_2\omega)\sigma(g_2, \omega)$$

holds for almost every $g_1, g_2 \in G$ and for almost every $\omega \in \Omega$.

For the reader who is confident with measured groupoid theory, measurable cocycles are almost representations of the measured groupoid $G \times \Omega$ with values in H. The notion of homotopic representations can be rephrased as follows.

Definition 2.14. Let $\sigma_1, \sigma_2 : G \times \Omega \to H$ be two measurable cocycles, let $f : \Omega \to H$ be a measurable map and denote by σ_1^f the cocycle defined as

$$\sigma_1^f(g, \omega) := f(g\omega)^{-1}\sigma_1(g, \omega) f(\omega)$$

for every $g \in G$ and almost every $\omega \in \Omega$. The cocycle σ_1^f is the *f-twisted cocycle associated to* σ_1. We say that σ_1 is *cohomologous* to σ_2 if there exists a measurable map f such that $\sigma_2 = \sigma_1^f$.

Example 2.15. Even if there are plenty of examples of measurable cocycles in different areas, we recall the following ones, since they play a predominant role in our results.

(i) For any standard Borel probability G-space, a homomorphism $\rho : G \to H$ between a locally compact and second countable group G and a topological group H defines a cocycle as follows:

$$\sigma_\rho : G \times \Omega \to H, \quad \sigma_\rho(g, \omega) := \rho(g).$$

Conjugated representations give cohomologous cocycles in the sense of Definition 2.14.

(ii) Let $\mathcal{X}$ be a CAT(0)-space of finite telescopic dimension. Fix a discrete countable group Γ and a standard Borel probability Γ-space. We can consider the constant field $\mathbf{X} = \{\mathcal{X}\}_{\omega \in \Omega}$ on Ω. Following Example 2.4, we know that an isometric Γ-action boils down to a function $\sigma : \Gamma \times \Omega \to \mathrm{Isom}(\mathcal{X})$ which satisfies equation (1) and is measurable with respect to the topology on $\mathrm{Isom}(\mathcal{X})$ induced by the family of pseudometrics $(g, h) \mapsto (gx, hx)$, where $x \in \mathcal{X}_0$ varies in a countable dense subset of $\mathcal{X}$.
Vice versa, any cocycle $\sigma : \Gamma \times \Omega \to \mathrm{Isom}(\mathcal{X})$ which is measurable with respect to the Borel structure induced by the previous pseudometrics on $\mathrm{Isom}(\mathcal{X})$ gives rise to an isometric Γ-action on the constant field $\mathbf{X}$.

Now we are ready to give the definition of a boundary map.

Definition 2.16. Let Γ be a discrete countable group, (Ω, μ) be a standard Borel probability Γ-space, H a topological group and Y a measurable H-space. Consider a measurable cocycle $\sigma : \Gamma \times \Omega \to H$. For any Γ-boundary B, a *boundary map* is a measurable map

$$\phi : B \times \Omega \to Y$$

which is *σ-equivariant*, namely

$$\phi(\gamma b, \gamma\omega) = \sigma(\gamma, \omega)\phi(b, \omega)$$

for every $\gamma \in \Gamma$ and almost every $b \in B, \omega \in \Omega$.

We are finally ready to prove the existence of boundary maps.

Proof of Proposition 2. We consider the constant field $\mathbf{X} := \{\mathcal{X}\}_{\omega\in\Omega}$ given by Example 2.4. By Example 2.15 (ii), the measurable cocycle $\sigma : \Gamma \times \Omega \to \mathrm{Isom}(\mathcal{X})$ induces an isometric G-action on $\mathbf{X}$. Since we assumed by hypothesis that there are no invariant Euclidean subfield of $\mathbf{X}$, we can apply Theorem 1 to obtain an invariant section of the boundary field $\partial\widehat{\mathbf{X}}$, where $\widehat{\mathbf{X}}$ is the extension of $\mathbf{X}$ to B. By [5, Lemma 3.10], this is equivalent to a boundary map $\phi : B \times \Omega \to \partial\mathcal{X}$. ■

Remark 2.17. Fix positive integers n and $p \le q$. In the setting of Proposition 2, suppose that Γ is a complex hyperbolic lattice in $\mathrm{PU}(n,1)$ and $\mathcal{X}(p,q)$ denotes the Hermitian symmetric space associated to the group $\mathrm{SU}(p,q)$ (see Section 3.2). By Proposition 2, we have a boundary map $\phi : \partial\mathbb{H}^n \times \Omega \to \partial\mathcal{X}(p,q)$. By the ergodicity of $B \times \Omega$, we have that ϕ takes values in the set of isotropic k-dimensional subspace in the boundary $\partial\mathcal{X}(p,q)$, for some $k \le p$. To see this, for each pair $(\xi,\omega) \in \partial\mathbb{H}^n_{\mathbb{C}} \times \Omega$ one can take the smallest cell in the spherical building of $\partial\mathcal{X}(p,q)$ which contains $\phi(\xi,\omega)$, that corresponds to a totally isotropic flag of $\mathbb{C}^{p,q}$ (see [15]). By ergodicity, the type of this flag must be the same for almost every pair in $\partial\mathbb{H}^n_{\mathbb{C}} \times \Omega$, and by taking the maximal isotropic spaces of any flag, we get the desired map. If we assume that σ is Zariski dense, the same argument in [16, Theorem 1.7] shows that $k = p$, namely the target is the Shilov boundary of $\mathcal{X}(p,q)$.

Now, since Zariski density implies non-elementarity, this gives an alternative proof of [31, Theorem 1].

3. Finite reducibility

In the second part of this paper, we study cocycle actions on the Hermitian symmetric space $\mathcal{X}(p,\infty)$. We first give a brief overview about bounded cohomology in order to define the *Toledo invariant* and *maximal cocycles*. Then, we recall basic facts about $\mathcal{X}(p,\infty)$, and we parametrize the space of embeddings of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$. Later, we characterize *finite algebraic subgroups*, and we use this notion to define *finite reducibility*. Finally, we prove Theorem 3.

3.1. Bounded cohomology

Let G be a locally compact and second countable group and let E be a Banach G-module (namely a Banach space together with an action of G by isometries). Continuous bounded cohomology is usually defined via the complex of continuous bounded functions on G. Burger–Monod [12] showed that more generally we can consider the cohomology of any strong resolution of E by relatively injective G-modules. More precisely, we have that the *continuous bounded cohomology* of G with coefficients in E is the cohomology of the

G-invariant vectors of any such resolution $(E^\bullet, \delta^\bullet)$, namely

$$\mathrm{H}^k_{\mathrm{cb}}(G; E) \cong \mathrm{H}^k((E^\bullet)^G, \delta^\bullet).$$

If E is the dual of some Banach G-module endowed with the weak-* topology and assuming that G is a semisimple Lie group of non-compact type, we can define the cochain complex of essentially bounded weak-* measurable functions on the Furstenberg boundary $B(G)$, denoted by $(\mathrm{L}^\infty_{\mathrm{w}*}(B(G)^{\bullet+1}; E), \delta^\bullet)$, where $\delta^\bullet$ is the standard homogeneous coboundary operator. Since the previous complex can be completed to a strong resolution of E by relatively injective modules, we have an isomorphism

$$\mathrm{H}^k_{\mathrm{cb}}(G; E) \cong \mathrm{H}^k(\mathrm{L}^\infty_{\mathrm{w}*}(B(G)^{\bullet+1}; E)^G, \delta^\bullet) \tag{3}$$

for any $k \geq 0$. By [12, Corollary 1.5.3], the isomorphism is actually isometric, that is, it preserves the natural seminormed structures on those spaces.

If we consider the complex of bounded weak-* measurable functions on a measurable G-space X, denoted by $(\mathcal{B}^\infty_{\mathrm{w}*}(X^{\bullet+1}; E), \delta^\bullet)$, we obtain only a strong resolution of E. Nevertheless, Burger and Iozzi [10] showed that there exists a canonical non-trivial map

$$\mathrm{c}^k : \mathrm{H}^k(\mathcal{B}^\infty(X^{\bullet+1}; E)^G, \delta^\bullet) \to \mathrm{H}^k_{\mathrm{cb}}(G; E) \tag{4}$$

for every $k \geq 0$.

Let $\Gamma < G$ be a lattice. As in the case of representations, given a measurable cocycle $\sigma : \Gamma \times \Omega \to H$, there exists a natural notion of pullback in bounded cohomology. More precisely, for any Banach H-module E, the map

$$\begin{aligned} &\mathrm{C}^\bullet_{\mathrm{b}}(\sigma) : \mathrm{C}^\bullet_{\mathrm{cb}}(H; E)^H \to \mathrm{C}^\bullet_{\mathrm{cb}}(\Gamma; E)^\Gamma, \\ &\mathrm{C}^\bullet_{\mathrm{b}}(\sigma)(\psi)(\gamma_0, \ldots, \gamma_\bullet) := \int_\Omega \psi(\sigma(\gamma_0^{-1}, \omega)^{-1}, \ldots, \sigma(\gamma_\bullet^{-1}, \omega)^{-1}) d\mu(\omega), \end{aligned} \tag{5}$$

is a well-defined cochain map [33, Lemma 2.7], inducing a map at the level of cohomology groups

$$\mathrm{H}^k_{\mathrm{b}}(\sigma) : \mathrm{H}^k_{\mathrm{cb}}(H; E) \to \mathrm{H}^k_{\mathrm{b}}(\Gamma; E), \quad \mathrm{H}^k_{\mathrm{b}}(\sigma)([\psi]) := [\mathrm{C}^k_{\mathrm{b}}(\sigma)(\psi)]$$

for every $k \geq 0$. If σ additionally admits a boundary map $\phi : B(G) \times \Omega \to Y$, one can define

$$\begin{aligned} &\mathrm{C}^\bullet(\Phi^\Omega) : \mathcal{B}^\infty(Y^{\bullet+1}; E)^H \to \mathrm{L}^\infty_{\mathrm{w}*}(B(G)^{\bullet+1}; E)^\Gamma, \\ &\mathrm{C}^\bullet(\Phi^\Omega)(\psi)(\xi_0, \ldots, \xi_\bullet) := \int_\Omega \psi(\phi(\xi_0, \omega), \ldots, \phi(\xi_\bullet, x)) d\mu(\omega). \end{aligned} \tag{6}$$

As shown by the second author and Moraschini [27, 28], the above map is a cochain map which does not increase the norm and it induces well-defined maps in cohomology

$$\mathrm{H}^k(\Phi^\Omega) : \mathrm{H}^k(\mathcal{B}^\infty(Y^{\bullet+1}; E)^H) \to \mathrm{H}^k_{\mathrm{b}}(\Gamma; E), \quad \mathrm{H}^k(\Phi^\Omega)([\psi]) := [\mathrm{C}^k(\Phi^\Omega)(\psi)]$$

for every $k \geq 0$.

Thanks to [33, Lemma 2.10], one can check that the class $\mathrm{H}^k_{\mathrm{b}}(\sigma)([\psi])$ admits as a natural representative the cocycle $\mathrm{C}^k(\Phi^\Omega)(\psi)$.

3.2. The symmetric space $\mathcal{X}(p,q)$

Let $(p,q) \in \mathbb{N} \times \mathbb{N} \cup \{\infty\}$ with $p \leq q$ and consider a $(p+q)$-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$. Let $\{e_i\}_{i=1}^{p+q}$ be a Hilbert basis for $\mathcal{H}$. We denote by $L(\mathcal{H})$ the set of $\mathbb{C}$-linear bounded operators with respect to the operator norm and by $\mathrm{GL}(\mathcal{H})$ the group of bounded invertible $\mathbb{C}$-linear operators of $\mathcal{H}$ with bounded inverse.

We define the Hermitian form Q of signature (p,q) as follows:

$$Q(x) = \sum_{i=1}^{p} x_i \overline{x}_i - \sum_{i=p+1}^{p+q} x_i \overline{x}_i,$$

where $x = \sum_{i=1}^{p+q} x_i e_i$ for all $x \in \mathcal{H}$. We denote with $\mathrm{U}(p,q)$ the subgroup of $\mathrm{GL}(\mathcal{H})$ of isometries with respect to Q, that means linear maps $h : \mathcal{H} \to \mathcal{H}$ such that $Q(v,w) = Q(h(v),h(w))$ for all $v, w \in \mathcal{H}$. If we define the space

$$\mathcal{X}(p,q) := \{V < \mathcal{H} \mid \dim V = p, Q_{|V} > 0\},$$

then by Witt's theorem, the group $\mathrm{U}(p,q)$ acts transitively on it [4, Theorem 3.9]. Moreover, the stabilizer of $V_0 := \mathrm{Span}\{e_1,\dots,e_p\}$ is the product $\mathrm{U}(p) \times \mathrm{U}(q)$, where $\mathrm{U}(m)$ is the orthogonal group of the Hilbert space of dimension m, for any $m \in \mathbb{N} \cup \{\infty\}$. Hence, we can identify $\mathcal{X}(p,q)$ with the quotient

$$\mathrm{U}(p,q)/\mathrm{U}(p) \times \mathrm{U}(q),$$

and one can show that it has a structure of simply connected non-positively curved Riemannian symmetric space with real rank p [15].

Homotheties act trivially on $\mathcal{X}(p,q)$, so we have an isometric action by isometries of the quotient

$$\mathrm{PU}(p,q) := \mathrm{U}(p,q)/\{\lambda\,\mathrm{Id}, |\lambda| = 1\}$$

on $\mathcal{X}(p,q)$.

We define the boundary $\partial\mathcal{X}(p,q)$ as the set of subspaces of $\mathcal{H}$ on which the restriction of Q is identically zero (i.e., *totally isotropic subspaces*). In $\partial\mathcal{X}(p,q)$, for each $1 \leq k \leq p$, we denote as $\mathcal{I}_k(p,q)$ (or simply $\mathcal{I}_k$) the set of totally isotropic subspaces of dimension k. In particular, we will be interested in the set $\mathcal{I}_p$ of maximal totally isotropic subspaces. Finally, two points in $\mathcal{I}_p(p,q)$ defined by totally isotropic subspaces V_1 and V_2 are said to be *opposite* if $V_1 \cap V_2 = 0$.

We denote by $Q_{p,q}$ the Hermitian form of signature (p,q) with $p \leq q < +\infty$. Let $\{E_i\}_{i=1}^{p+q}$ and $(e_i)_{i\in\mathbb{N}}$ be two bases of $\mathbb{C}^{p,q}$ and of an infinite-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$, respectively.

Definition 3.1. An *embedding* of $\mathcal{X}(p,q)$ into $\mathcal{X}(p,\infty)$ is a linear map $\iota : \mathbb{C}^{p,q} \to \mathcal{H}$ that preserves the Hermitian forms $Q_{p,q}$ and $Q_{p,\infty}$, namely $Q_{p,\infty}(\iota(x),\iota(y)) = Q_{p,q}(x,y)$ for every $x, y \in \mathbb{C}^{p,q}$. The group $\mathrm{U}(p,q)$ of linear bounded transformations preserving $Q_{p,q}$ embeds in $\mathrm{U}(p,\infty)$ in the following way: the action on $\iota(\mathbb{C}^{p,q})$ is the one of $\mathrm{U}(p,q)$ and is trivial on the orthogonal complement of $\iota(\mathbb{C}^{p,q})$.

Among all embeddings of $\mathcal{X}(p,q)$ in $\mathcal{X}(p,\infty)$, we consider the *standard embedding* defined by the map $\iota_0 : \mathbb{C}^{p,q} \to \mathcal{H}$ defined as $\iota_0(E_i) = e_i$ for $i = 1, \dots, p+q$. In this special case, the space $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$ can be identified with the set

$$\mathcal{V}_0 = \{V < \operatorname{Span}\{e_1, \dots, e_{p+q}\} \mid \dim V = p, Q_{p,\infty|V} > 0\}$$

and the group $\mathrm{U}(p,q)$ is identified with elements g in $\mathrm{U}(p,\infty)$ such that

$$g(e_i) = \sum_{j \in \mathbb{N}} a_{ij} e_j,$$

where for either i or j bigger than $p+q$, then $a_{ij} = \delta_{ij}$, and the matrix $A = (a_{ij})_{i,j=1}^{p+q}$ represents an element in $\mathrm{U}(p,q)$, namely it satisfies

$$A^* \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix} A = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}.$$

The notion of embedding given in Definition 3.1 corresponds to the one of standard embedding given in [16]. This choice is motivated by the fact that here we need to distinguish the particular objects described above, whose role among all embeddings is clarified by the following.

Lemma 3.2. *Any embedding of $\mathcal{X}(p,q)$ of $\mathcal{X}(p,\infty)$ can be obtained by composition of an element $g \in \mathrm{U}(p,\infty)$ with the standard embedding.*

Proof. Let $\iota : \mathbb{C}^{p,q} \to \mathcal{H}$ be an isometric linear map. For each e_i, we set $u_i := \iota(e_i)$ and

$$U_\iota := \operatorname{Span}\{u_1, \dots, u_{p+q}\}.$$

There is a natural identification of $\mathcal{X}(p,q)$ with the subspace of $\mathcal{X}(p,\infty)$ defined by

$$\mathcal{V}_\iota = \{V < U_\iota \mid \dim V = p, Q_{p,\infty|V} > 0\}.$$

If we denote with U_0 the subspace of $\mathcal{H}$ spanned by the first $p+q$ vectors of the basis $(e_i)_{i\in\mathbb{N}}$, we can define an isometric linear map $h : U_0 \to U_\iota$ on the basis as follows:

$$h(e_i) = u_i,$$

and then extend it by linearity. Since h preserves the Hermitian form Q, by Witt's theorem it extends to an isometry of $\mathcal{H}$ with respect to Q, namely to an element $g \in \mathrm{U}(p,\infty)$. The thesis follows noticing that the isometric linear map $g \circ \iota$ actually gives the standard embedding. ∎

Remark 3.3. As a subspace of the Grassmannian $\mathrm{Gr}(p+q,\mathcal{H})$, the set of embedding of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$ naturally inherits the topology induced by principal angles, that in this case coincides with the Wijsman topology (see [16]). Since by Lemma 3.2 the group $\mathrm{U}(p,\infty)$ acts transitively on the set of all such embeddings, we have an identification with the $\mathrm{PU}(p,\infty)$-orbit of the standard embedding in $\mathrm{Gr}(p+q,\mathcal{H})$. Moreover, this can be identified with the quotient $\mathrm{PU}(p,\infty)/\operatorname{Stab}_{\mathrm{PU}(p,\infty)} V_0$, where V_0 is the image of the standard embedding.

3.3. The Kähler class and the Bergman cocycle

A crucial difference between the finite case and the infinite one in the context of symmetric spaces is that $\mathrm{PU}(p,q)$ is locally compact for $q < \infty$, whereas $\mathrm{PU}(p,\infty)$ is not. To overcome this problem, we will deal with the bounded cohomology groups $\mathrm{H}^\bullet_{\mathrm{b}}(\mathrm{PU}(p,\infty);\mathbb{R})$, namely its continuous bounded cohomology if we endow $\mathrm{PU}(p,\infty)$ with the discrete topology.

Since $\mathcal{X}(p,\infty)$ is a Hermitian symmetric space, there exists a Kähler form ω, that is, a $\mathrm{PU}(p,\infty)$-invariant closed 2-form on $\mathcal{X}(p,\infty)$. Using such an invariant form, we can define

$$\omega_x : \mathrm{PU}(p,\infty)^3 \to \mathbb{R}, \quad \omega_x(g_0,g_1,g_2) = \frac{1}{\pi}\int_{\Delta(g_0x,g_1x,g_2x)} \omega,$$

where x is a point in $\mathcal{X}(p,\infty)$ and $\Delta(g_0x,g_1x,g_2x)$ is a triangle in $\mathcal{X}(p,\infty)$ with vertices g_0x, g_1x, g_2x and geodesic edges. The map ω_x defines a strict $\mathrm{PU}(p,\infty)$-invariant cocycle, and by [16, Lemma 5.3] different choices of the basepoint Ω lead to cohomologous cocycles. In this way, we obtain a well-defined cohomology class $k^{\mathrm{b}}_{\mathrm{PU}(p,\infty)} \in \mathrm{H}^2_{\mathrm{b}}(\mathrm{PU}(p,\infty);\mathbb{R})$, called *bounded Kähler class* of $\mathrm{PU}(p,\infty)$. Now, taking the Gromov norm $\|\cdot\|_\infty$, it follows from the definition that

$$\|k^{\mathrm{b}}_{\mathrm{PU}(p,\infty)}\|_\infty = \mathrm{rk}\,\mathcal{X}(p,\infty) = p. \tag{7}$$

We will need to define the Bergman class extending the one given in finite case, namely to construct a cocycle on the boundary $\mathcal{I}_p$.

Given any three maximal totally isotropic subspaces $V_0, V_1, V_2 \in \mathcal{I}_p$, since they are contained in a finite-dimensional subspace of dimension at most $3p$, we can use the definition of the Bergman cocycle associated to $\mathrm{SU}(p,2p)$ to get a strict $\mathrm{PU}(p,\infty)$-invariant cocycle

$$\beta : \mathcal{I}_p^3 \to [-p,p].$$

We recall that the maximal value is taken on triples of pairwise opposite totally isotropic subspaces which lie in a $2p$-dimensional subspace [30, Proposition 2.1]. Now, given a point $V \in \mathcal{I}_p$, the cocycle C_V defined as

$$C_V(g_0,g_1,g_2) = \beta(g_0V, g_1V, g_2V)$$

still represents the bounded Kähler class $k^{\mathrm{b}}_{\mathrm{PU}(p,\infty)} \in \mathrm{H}^2_{\mathrm{b}}(\mathrm{PU}(p,\infty);\mathbb{R})$ [16, Lemma 5.4].

3.4. The Toledo invariant

Let $\Gamma < \mathrm{PU}(n,1)$ be a complex hyperbolic lattice, (Ω,μ) be a standard Borel probability Γ-space and let $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p,\infty)$ be a measurable cocycle. Following [31], we define the transfer map

$$\mathrm{T}^2_{\mathrm{b}} : \mathrm{H}^2_{\mathrm{b}}(\Gamma;\mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(n,1);\mathbb{R})$$

as the map induced in cohomology by the function

$$\widehat{\mathrm{T}}^2_{\mathrm{b}} : \mathrm{L}^\infty((\partial\mathbb{H}^n_{\mathbb{C}})^3;\mathbb{R})^\Gamma \to \mathrm{L}^\infty((\partial\mathbb{H}^n_{\mathbb{C}})^3;\mathbb{R})^{\mathrm{PU}(n,1)},$$
$$\widehat{\mathrm{T}}^2_{\mathrm{b}}(c)(\xi_0,\xi_1,\xi_2) = \int_{\Gamma\backslash \mathrm{PU}(n,1)} c(\overline{g}\xi_0,\overline{g}\xi_1,\overline{g}\xi_2)d\mu_{\Gamma\backslash \mathrm{PU}(n,1)}(\overline{g}).$$

Considering $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(n,1);\mathbb{R}) = \mathbb{R}k^{\mathrm{b}}_{\mathrm{PU}(n,1)}$, where $k^{\mathrm{b}}_{\mathrm{PU}(n,1)}$ is the bounded Kähler class of $\mathrm{PU}(n,1)$, by composing with the pullback of equation (5), we get

$$\mathrm{T}^2_{\mathrm{b}} \circ \mathrm{H}^2_{\mathrm{b}}(\sigma)(k^{\mathrm{b}}_{\mathrm{PU}(p,\infty)}) = \mathrm{t}_\sigma k^{\mathrm{b}}_{\mathrm{PU}(n,1)} \tag{8}$$

for some real number t_σ.

Definition 3.4. Given a lattice $\Gamma < \mathrm{PU}(n,1)$ and a standard Borel Γ-space (Ω,μ), let $\sigma : \Gamma\times\Omega \to \mathrm{PU}(p,\infty)$ be a measurable cocycle. The number t_σ is the *Toledo invariant* associated to σ.

Since both $\mathrm{T}^2_{\mathrm{b}}$ and $\mathrm{H}^2_{\mathrm{b}}(\sigma)$ are norm non-increasing, by equation (7) we have $|\mathrm{t}_\sigma| \leq p$ and the following.

Definition 3.5. Given $\Gamma < \mathrm{PU}(n,1)$ and a standard Borel Γ-space (Ω,μ), a measurable cocycle $\sigma : \Gamma\times\Omega \to \mathrm{PU}(p,\infty)$ is *maximal* if its Toledo invariant is equal to p.

If σ admits a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \mathcal{I}_p$, exploiting the version of the pullback given by equation (6), we get a map

$$\mathrm{T}^2_{\mathrm{b}} \circ \mathrm{H}^2(\Phi^\Omega) : \mathrm{H}^2((\mathcal{B}^\infty(\mathcal{I}_p)^{\bullet+1};\mathbb{R})^{\mathrm{PU}(p,\infty)}) \to \mathrm{H}^2_{\mathrm{b}}(\mathrm{PU}(n,1);\mathbb{R}).$$

In this way, we can rewrite equation (8) as follows

$$\mathrm{T}^2_{\mathrm{b}} \circ \mathrm{H}^2_{\mathrm{b}}(\Phi^\Omega)([\beta]) = t_\sigma k^{\mathrm{b}}_{\mathrm{PU}(n,1)}.$$

Writing down the above equation in terms of cochains, we get the formula

$$\int_{\Gamma\backslash \mathrm{PU}(n,1)} \int_\Omega \beta(\phi(\overline{g}\xi_0,\omega),\phi(\overline{g}\xi_1,\omega),\phi(\overline{g}\xi_2,\omega))d\mu(\omega)d\mu_{\Gamma\backslash \mathrm{PU}(n,1)}(\overline{g})$$
$$= \mathrm{t}_\sigma \cdot c_n(\xi_0,\xi_1,\xi_2) \tag{9}$$

that holds for every triple of distinct points (ξ_0,ξ_1,ξ_2) in $\partial\mathbb{H}^n_{\mathbb{C}}$ [30,31]. Here c_n is Cartan's angular invariant that represents the bounded Kähler class of $\mathrm{PU}(n,1)$ [20].

3.5. Algebraic subgroups of GL($\mathcal{H}$)

We first introduce the notion of polynomial map.

Definition 3.6. A map $f : L(\mathcal{H}) \to \mathbb{R}$ is a *polynomial map* if it is a finite sum of maps $f_1, \dots, f_k$, where for each $i = 1, \dots, k$, there exists an n_i-linear map $h_i \in L^{n_i}(L(\mathcal{H}), \mathbb{R})$ such that $f_i(g) = h_i(g, \dots, g)$ for every $g \in L(\mathcal{H})$. The *degree* of f is the maximum of the n_i's.

Now, in parallel to the finite-dimensional case, we define an algebraic subgroup as the set of the zero locus of some family of polynomial maps.

Definition 3.7. A subgroup G of $\mathrm{GL}(\mathcal{H})$ is *algebraic* if there exists a positive integer n and family $\mathcal{P}$ of polynomial maps of degrees at most n such that

$$G = \{g \in \mathrm{GL}(\mathcal{H}) \mid P(g, g^{-1}) = 0,\ \forall P \in \mathcal{P}\}.$$

A *strict algebraic subgroup* is a proper algebraic subgroup of $\mathrm{GL}(\mathcal{H})$.

To define a linear algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$, we consider polynomial equations in matrix coefficients. The generalization to infinite dimension of this notion is the content of the following definition.

Definition 3.8 ([16, Definition 3.4]). Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and choose an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. A homogeneous polynomial map $P : L(\mathcal{H}) \times L(\mathcal{H}) \to \mathbb{R}$ is *standard* of degree d if there exist two naturals ℓ, m such that $\ell + m = d$ and a family of real coefficients $(\lambda_i)_{i\in\mathbb{N}^{2\ell}}$ and $(\mu_j)_{j\in\mathbb{N}^{2m}}$ such that for any $(M, N) \in L(\mathcal{H}) \times L(\mathcal{H})$, we have that P can be expressed as the absolute convergent series

$$P(M, N) = \sum_{i\in\mathbb{N}^{2\ell}, j\in\mathbb{N}^{2m}} \lambda_i \mu_j P_i(M) P_j(N),$$

where $P_i(M) = \prod_{k=0}^{\ell-1} \langle M e_{i_{2k}}, e_{i_{2k+1}} \rangle$ and $P_j(N) = \prod_{k=0}^{m-1} \langle N e_{j_{2k}}, e_{j_{2k+1}} \rangle$.

A *standard polynomial map* is a finite sum of standard homogeneous polynomial maps.

An algebraic subgroup of $L(\mathcal{H})$ is *standard* if it is defined by a family of standard polynomial maps.

Hence, we have the following interesting property that shows how proper standard algebraic subgroups are closely related to finite-dimensional subspace of $\mathcal{H}$.

Lemma 3.9 ([16, Lemma 3.6]). *If H is a strict standard algebraic group, then there exists a finite-dimensional subspace E of $\mathcal{H}$ such that the group $H_E := \{g \in H \mid g(E) = E, g_{|E^\perp} = \mathrm{id}\}$ is a strict algebraic subgroup of* $\mathrm{GL}(E)$.

We call the subspace E *support* of the strict algebraic subgroup H and the group H_E the E-*part* of H. We are now ready to give the following.

Definition 3.10. A *finite-dimensional algebraic subgroup* is a standard algebraic subgroup of $\mathrm{GL}(\mathcal{H})$ of the form H_E.

Hence, it follows by Lemma 3.9 a characterization of finite-dimensional algebraic subgroups.

Lemma 3.11. *If E is a finite-dimensional subspace of $\mathcal{H}$ and H is a subgroup of* $\mathrm{GL}(\mathcal{H})$ *contained in* $\mathrm{GL}(E)$*, then H is algebraic in* $\mathrm{GL}(E)$ *if and only if it is a finite-dimensional algebraic subgroup of* $\mathrm{GL}(\mathcal{H})$.

Proof. If H is a finite-dimensional algebraic subgroup of $\mathrm{GL}(\mathcal{H})$, then $H = H_E$, and by Lemma 3.9 it is algebraic in $\mathrm{GL}(E)$. Conversely, if H is algebraic in $\mathrm{GL}(E)$, it is also an algebraic subgroup in $\mathrm{GL}(\mathcal{H})$. Moreover, any polynomial which defines H on $\mathrm{GL}(E)$ can be turned into a polynomial on the entries of the matrices. Hence, the same polynomials, seen as standard polynomial maps in the sense of Definition 3.8, define a standard algebraic subgroup in $\mathrm{GL}(\mathcal{H})$. Since it fixes $E^{\perp}$, then it coincides with its E-part and we are done. ∎

The group $\mathrm{U}(p,\infty)$ is algebraic subgroup of $\mathrm{GL}(\mathcal{H})$. Indeed, suppose $V_0 := \mathrm{Span}\{e_1,\dots,e_p\}$, we have that

$$\mathrm{U}(p,\infty) = \{g \in \mathrm{GL}(\mathcal{H}) \mid g^* \,\mathrm{Id}_{p,\infty}\, g = \mathrm{Id}_{p,\infty}\},$$

where $\mathrm{Id}_{p,\infty}$ is the linear map $\mathrm{Id}_{V_0} \oplus -\mathrm{Id}_{V_0^{\perp}}$. Since the map $(A,B) \mapsto A^* \,\mathrm{Id}_{p,\infty}\, B - \mathrm{Id}_{p,\infty}$ is bilinear on $L(\mathcal{H}) \times L(\mathcal{H})$, then $\mathrm{U}(p,\infty)$ is algebraic in $\mathrm{GL}(\mathcal{H}_{\mathbb{R}})$ and hence in $\mathrm{GL}(\mathcal{H})$ (see [16] for more details). By Proposition 3.11, we can say immediately that the groups $\mathrm{U}(p,q)$ with $q < \infty$, seen as subgroups of $\mathrm{U}(p,\infty)$ inside $\mathrm{GL}(\mathcal{H})$, are actually finite algebraic since they stabilize the embedding of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$.

Since we work with the quotients $\mathrm{PU}(p,q)$ instead of the groups $\mathrm{U}(p,q)$, we call *finite algebraic* a subgroup of $\mathrm{PU}(p,\infty)$ if its preimage under the projection $\mathrm{U}(p,\infty) \to \mathrm{PU}(p,\infty)$ is finite algebraic in $\mathrm{GL}(\mathcal{H})$ in the sense of Definition 3.10.

3.6. Proof of finite reducibility

Given a measurable cocycle $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p,\infty)$, one can ask when its image is contained in some suitable subgroup of $\mathrm{PU}(p,\infty)$.

Definition 3.12. A cocycle $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p,\infty)$ is *finitely reducible* if it admits a cohomologous cocycle with image contained in a finite-dimensional algebraic subgroup of $\mathrm{PU}(p,\infty)$.

Before proving the main theorem, we recall that a *p-chain* is a copy of $\mathcal{I}_p(p,p)$ in $\mathcal{I}_p(p,\infty)$ determined by an embedding of $\mathcal{X}(p,p)$ in $\mathcal{X}(p,\infty)$.

Definition 3.13. A measurable map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{I}_p$ *almost surely maps chains to chains* if for almost every chain $\mathcal{C} \subset \partial\mathbb{H}^n_{\mathbb{C}}$ there is a p-chain $\mathcal{T} \subset \mathcal{I}_p$ such that for almost every point $\xi \in \mathcal{C}$, $\phi(\xi) \in \mathcal{T}$.

An equivalent condition [30, Lemma 4.2] to the one above is to check that for almost every pair $(x, y) \in \partial\mathbb{H}^n_{\mathbb{C}} \times \partial\mathbb{H}^n_{\mathbb{C}}$, the points $\phi(\xi_0)$ and $\phi(\xi_1)$ are opposite and, for almost every $z \in \mathcal{C}_{\xi_0,\xi_1}$, the subspace $\phi(z)$ is contained in $\langle \phi(\xi_0), \phi(\xi_1)\rangle$. Before passing to the proof of Theorem 3, we need the following result about maps that almost surely maps chains to chains, which is a slight refinement of [16, Proposition 6.2]. Since there is a natural embedding $\partial\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}^n\mathbb{C}$, we can say that a set of $k \leq n+1$ points in $\partial\mathbb{H}^n_{\mathbb{C}}$ is *generic* if, for every $1 < h \leq k$, any subset of h points does not span an $(h-2)$-dimensional subspace.

Lemma 3.14. *Let $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{I}_p$ be a measurable map that almost surely maps chains to chains. Then there exists a unique minimal totally geodesic embedded copy of $\mathcal{X}(p,q) \subset \mathcal{X}(p,\infty)$ that contains the image of almost every $(n+1)$-tuple of generic points in $\partial\mathbb{H}^n_{\mathbb{C}}$. Moreover, $p \leq q \leq np$.*

Proof. We argue by induction on n. The case $n = 1$ is clear, since there is only one chain $\mathcal{C}$ in $\partial\mathbb{H}^1_{\mathbb{C}}$, and for almost every $\eta_1, \eta_2 \in \mathcal{C}$ the subspace $\langle \phi(\eta_1), \phi(\eta_2)\rangle \subset \mathcal{H}$ defines a copy of $\mathcal{X}(p,p) \subset \mathcal{X}(p,\infty)$. The fact that ϕ almost surely maps chains to chains implies that for almost every ξ in $\partial\mathbb{H}^n_{\mathbb{C}}$ we have $\phi(\xi) < \langle \phi(\eta_1), \phi(\eta_2)\rangle$.

Assume that the statement holds for $n-1$. Thanks to the construction in [16, Section 7.1], we can define a full-measure subset $\mathcal{G}$ of the set of $(n+1)$-tuple of points in general position of $\partial\mathbb{H}^n_{\mathbb{C}}$ such that for every $(\xi_0, \ldots, \xi_n) \in \mathcal{G}$, the following conditions hold:

- $\phi_{|\langle \xi_0,\ldots,\xi_{n-1}\rangle}$ almost surely maps chains to chains;
- for almost every $\eta \in \langle \xi_0, \ldots, \xi_{n-1}\rangle$, then $\langle \phi(\eta), \phi(\xi_{n-1})\rangle$ is a $2p$-dimensional subspace on which the restriction of Q has signature (p,p);
- for almost every $\eta \in \langle \xi_{n-1}, \xi_n\rangle$, then $\langle \phi(\eta), \phi(\xi_{n-1})\rangle$ is a $2p$-dimensional subspace on which the restriction of Q has signature (p,p);
- for almost every $\eta_1 \in \langle \xi_{n-1}, \xi_n\rangle$, $\eta_2 \in \langle \xi_0, \ldots, \xi_{n-1}\rangle$, the space $\langle \phi(\eta_1), \phi(\eta_2)\rangle$ has dimension $2p$ and the restriction of Q has signature (p,p).

As proved in [16, Proposition 6.2], for almost every $(\xi_0, \ldots, \xi_n) \in \mathcal{G}$, the space

$$V_{\xi_0,\ldots,\xi_n} := \langle \phi(\xi_0), \ldots, \phi(\xi_n)\rangle$$

contains $\phi(\eta)$ for almost every $\eta \in \partial\mathbb{H}^n_{\mathbb{C}}$. Furthermore, the restriction of Q to $V_{\xi_0,\ldots,\xi_n}$ is non-degenerate of signature (p,q) with $p \leq q \leq np$.

We now prove that almost every pair of tuples $((\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n)) \in \mathcal{G}^2$ gives the same subspace. We first note that since $V_{\xi_0,\ldots,\xi_n}$ contains the image of almost every point in $\partial\mathbb{H}^n_{\mathbb{C}}$, it clearly contains $\phi(\eta_0), \ldots, \phi(\eta_n)$, and hence $\langle \phi(\eta_0), \ldots, \phi(\eta_n)\rangle$, for almost every $(\eta_0, \ldots, \eta_n) \in \mathcal{G}$. Hence, there exists a full-measure subset $\mathcal{Q} \subset \mathcal{G} \times \mathcal{G}$ such that

$$V_{\xi_0,\ldots,\xi_n} < V_{\eta_0,\ldots,\eta_n}$$

for almost every $((\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n)) \in \mathcal{Q}$. By taking the measure-preserving idempotent function of $\mathcal{G} \times \mathcal{G}$ which swaps the tuple, one gets a second full-measure subset $\overline{\mathcal{Q}}$. Hence, the intersection $\mathcal{Q} \cap \overline{\mathcal{Q}}$ is a full-measure subset of $\mathcal{G} \times \mathcal{G}$ of pairs $(\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n)$ such that

$$V_{\xi_0,\ldots,\xi_n} = V_{\eta_0,\ldots,\eta_n},$$

which implies the uniqueness.

A similar argument can be used to prove minimality, namely that every linear subspace $W < \mathcal{H}$ containing the image of a full-measure subset of $\partial \mathbb{H}^n_{\mathbb{C}}$ must contain the spaces constructed above. ∎

Remark 3.15. It seems natural to investigate the effective dimension of the copy of $\partial \mathcal{X}(p, q)$ which contains the essential image of ϕ provided by Lemma 3.14. For instance, given a chain-preserving map $\psi : \partial \mathbb{H}^n_{\mathbb{C}} \to \partial \mathbb{H}^p_{\mathbb{C}}$, Burger and Iozzi [11] proved the following dichotomy: if the image of almost every triple (ξ_0, ξ_1, ξ_2) of generic points is generic as well, then ψ coincides almost everywhere with the map induced on boundaries by an isometric holomorphic embedding $\mathbb{H}^n_{\mathbb{C}} \to \mathbb{H}^p_{\mathbb{C}}$. If not, then the image is essentially contained into a chain in $\partial \mathbb{H}^p$.

In our more general context, we do not know if such a dichotomy holds. However, in our setting, the two cases described above can be interpreted as the limit cases as follows. In fact, if $\phi : \partial \mathbb{H}^n_{\mathbb{C}} \to \mathcal{I}_p$ as in Lemma 3.14 sends almost every $(n+1)$-tuple of generic points of $\partial \mathbb{H}^n_{\mathbb{C}}$ to $(n+1)$ generic points of $\mathcal{I}_p$, then we have that the essential image of ϕ is contained in $\partial \mathcal{X}(p, np)$. On the other hand, by the same argument used in [11, Proposition 2.2], if there is a positive measure subset of triple in $(\partial \mathbb{H}^n_{\mathbb{C}})^3$ not on a chain whose image lies on a chain, then the image of ϕ is essentially contained into one copy of $\partial \mathcal{X}(p, p)$. We point out that these two cases do not produce a dichotomy but a characterization of the cases when $q = p$ and $q = np$ in the notation of Lemma 3.14.

Now we are ready to give the proof of the main result of this part.

Proof of Theorem 3. By equation (9) and using [16, Corollary 6.1], it follows that almost every slice ϕ_ω almost maps chains to chains. Hence, by Lemma 3.14, for almost every $\omega \in \Omega$, there exists a unique minimal totally geodesic embedding $\mathcal{X}_\omega(p, q_\omega) \subset \mathcal{X}(p, \infty)$ such that $\mathrm{Ess\,Im}(\phi_\omega) \subset \partial \mathcal{X}_\omega(p, q_\omega)$ for some $p \le q_\omega \le np$. Notice the $\mathcal{X}_\omega$ depends only on the measure class of ϕ_ω. By [24, Chapter VII, Lemma 1.3], the function

$$\Phi : \Omega \to \mathrm{L}^0(\Omega, \mathcal{I}_p), \quad \Phi(\omega) := \phi_\omega$$

is measurable, where $\mathrm{L}^0(\Omega, \mathcal{I}_p)$ has again the measurable structure coming from the topology of convergence in measure (the distance on $\mathcal{I}_p$ is that of Remark 3.3).

According to Fubini's theorem, there exists $\xi_0, \dots, \xi_n \in \partial\mathbb{H}^n_{\mathbb{C}}$ so that $\mathcal{X}_\omega(p, q_\omega) := \langle \phi_\omega(\xi_0), \dots, \phi_\omega(\xi_n)\rangle$ for almost every $\omega \in \Omega$. Since ϕ_ω depends measurably on ω, the same holds for $\mathcal{X}(p, q_\omega)$.

Moreover, the equivariance of ϕ implies that

$$\sigma(\gamma, \omega)\mathcal{X}_\omega(p, q_\omega) = \mathcal{X}_{\gamma\omega}(p, q_{\gamma\omega})$$

for almost every $\gamma \in \Gamma$ and $\omega \in \Omega$. By the ergodicity of Ω, the number q_ω is essentially constant, namely $q_\omega = q$ for almost every $\omega \in \Omega$. If we denote by ι_ω the isometric linear map that induces the embedding $\mathcal{X}_\omega(p, q) \subset \mathcal{X}(p, \infty)$, the uniqueness of $\mathcal{X}_\omega(p, q)$, together with the σ-equivariance of ϕ, implies that the map

$$\Omega \to \mathrm{PU}(p, \infty)/\operatorname{Stab}_{\mathrm{PU}(p,\infty)}(V_0), \quad \omega \mapsto \mathcal{X}_\omega(p, q) \tag{10}$$

is measurable (with respect to the measurable structure discussed in Remark 3.3) and σ-equivariant. Here $\operatorname{Stab}_{\mathrm{PU}(p,\infty)} V_0$ is the subgroup of $\mathrm{PU}(p, \infty)$ preserving the subspace V_0. Now, thanks to the differentiable structure of the group $\mathrm{PU}(p, \infty)$, we can compose the function in equation (10) with a measurable section

$$\mathrm{PU}(p, \infty)/\operatorname{Stab}_{\mathrm{PU}(p,\infty)}(V_0) \to \mathrm{PU}(p, \infty)$$

in order to obtain a measurable map

$$f : \Omega \to \mathrm{PU}(p, \infty), \quad f(\omega) = g_\omega^{-1}.$$

By construction, $f(\omega)$ sends $\mathcal{X}_\omega(p, q)$ to the standard embedded copy $\mathcal{X}(p, q) \subset \mathcal{X}(p, \infty)$.

We consider the twisted cocycle $\sigma^f : \Gamma \times \Omega \to \mathrm{PU}(p, \infty)$ defined as

$$\sigma^f(\gamma, \omega) := f(\gamma\omega)^{-1}\sigma(\gamma, \omega) f(\omega)$$

and the associated twisted boundary map $\phi^f : \partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \mathcal{I}_p$ defined by

$$\phi^f(\xi, \omega) := f(\omega)^{-1}\phi(\xi, \omega).$$

Now, by definition of f, for almost every $\omega \in \Omega$, the image of almost every slice ϕ_ω is contained in the boundary of a fixed $\mathcal{X}(p, q)$. For almost every $\omega \in \Omega$, denote by E_ω the full-measure set of points ξ in $\partial\mathbb{H}^n_{\mathbb{C}}$ such that $\phi^f_\omega(\xi) \in \partial\mathcal{X}(p, q)$. Consider now the set $E = \bigcup_{\omega\in\Omega} E_\omega \times \{\omega\}$ (that is of full measure in $\partial\mathbb{H}^n_{\mathbb{C}} \times \Omega$, by Fubini's theorem) and the diagonal action of Γ given by

$$\gamma(\xi, \omega) = (\gamma\xi, \gamma\omega).$$

Since Γ is countable, we find an invariant full-measure subset $\overline{E}$ such that $\phi^f(\overline{E}) \subset \partial\mathcal{X}(p, q)$. More precisely, we set

$$\overline{E} = \bigcap_{\gamma\in\Gamma} \gamma E,$$

where γ acts diagonally. Being the countable intersection of full-measure sets, it is clear that $\overline{E}$ has full measure. Now, since the image of a full-measure set under ϕ^f is contained in the boundary of the embedded $\mathcal{X}(p,q)$, it follows that the image of the twisted cocycle σ^f is contained in $\mathrm{Stab}_{\mathrm{PU}(p,\infty)} V_0$, which is finite algebraic as desired. ■

Remark 3.16. The descending chain condition that holds for Noetherian spaces (as algebraic groups are) allows us to define the algebraic hull for cocycles into algebraic groups. This cannot be adapted for $\mathrm{PU}(p,\infty)$, namely there exists no well-defined minimal strict algebraic group containing the image of a twisted cocycle. Nevertheless, by Theorem 3, any maximal cocycles have a representative in its cohomology class whose image is contained into the embedding of $\mathrm{PU}(p,q)$ in $\mathrm{PU}(p,\infty)$, which is algebraic. For such particular measurable cocycles, our result recovers an algebraic flavor.

4. Consequences of finite reducibility

The aim of this last section is to link Theorems 1 and 3. We consider the setting of Theorem 3, namely Γ is a complex hyperbolic lattice, (Ω,μ) is an ergodic standard Borel probability Γ-space and $\sigma : \Gamma \times \Omega \to \mathrm{PU}(p,\infty)$ is a maximal cocycle. If we assume that σ is non-elementary, Theorem 1 provides a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \partial\mathcal{X}(p,\infty)$. Moreover, by Remark 2.17 such a map takes values into $\mathcal{I}_k(p,\infty)$ for some $k \leq p$. Unfortunately, this is not sufficient to prove reducibility as in Theorem 3, since such k might be strictly less than p.

However, for cocycles $\sigma : \Gamma \times \Omega \to \mathrm{PU}(1,\infty)$, one can exploit the geometry of $\mathcal{X}(1,\infty) = \mathbb{H}^\infty_{\mathbb{C}}$ and of its boundary to prove Theorem 4.

Proof of Theorem 4. We first show that maximal cocycles cannot be non-elementary. In fact, by ergodicity, a σ-equivariant family of flats can be made of points or lines. In both cases, one can twist σ into a cocycle whose image is contained either in the stabilizer of a point or a geodesic, which are both amenable. Since amenable groups have trivial bounded cohomology, we have a contradiction to maximality.

Since σ is not elementary, Theorem 1 provides a boundary map $\partial\mathbb{H}^n_{\mathbb{C}} \times \Omega \to \partial\mathbb{H}^\infty_{\mathbb{C}}$ and then we can apply Theorem 3. Hence, we have that σ is cohomologous to a cocycle $\widetilde{\sigma}$ whose image is contained in the stabilizer of an embedded copy of $\mathbb{H}^n_{\mathbb{C}}$ in $\mathbb{H}^\infty_{\mathbb{C}}$. The stabilizer $\mathrm{Stab}_{\mathrm{PU}(1,\infty)}(\mathbb{H}^n_{\mathbb{C}})$ is an almost direct product with one factor isomorphic to $\mathrm{PU}(n,1)$. By composing with the projection on such factor, we get a maximal cocycle. Hence, we can apply [28, Theorem 1.5] and we are done. ■

Remark 4.1. In the general setting of Theorem 3, as pointed out in Remark 2.17, Theorem 3 provides a boundary map into some $\mathcal{I}_k(p,\infty)$. In [16], the authors exploited Proposition 2.2 to rule out the case $k < p$ for Zariski dense representations. In an attempt to adapt such an argument in the context of cocycles, we got stuck in the final part. Precisely, following the proof of Theorem 1.7 of [16], one can construct a σ-equivariant family $\{W_\omega\}_{\omega\in\Omega}$ of non-trivial subspaces of $\Lambda^d\mathcal{H}$ for some d. Since the stabilizers of such spaces are standard algebraic subgroups, it would be enough to twist the cocycle in order to get a cocycle with image contained in one of these stabilizers. However, the action of $\mathrm{PU}(p,\infty)$ on the subspaces (a priori of infinite dimension) of $\Lambda^d\mathcal{H}$ seems quite mysterious to us. Even before, one should clarify the measurable structures involved. To conclude as in the proof of Theorem 3 or [31, Theorem 2], one should identify the $\mathrm{PU}(p,\infty)$-orbit of some W_ω with the quotient $\mathrm{PU}(p,\infty)/\operatorname{Stab}_{\mathrm{PU}(p,\infty)} W_\omega$, for instance by proving that the action is smooth, which is also not clear to us.

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References

[1] P. Abramenko and K. S. Brown, *Buildings: theory and applications*. Grad. Texts in Math. 248, Springer, New York, 2008 Zbl 1214.20033 MR 2439729

[2] S. Adams and W. Ballmann, Amenable isometry groups of Hadamard spaces. *Math. Ann.* **312** (1998), no. 1, 183–196 Zbl 0913.53012 MR 1645958

[3] M. Anderegg and P. Henry, Actions of amenable equivalence relations on CAT(0) fields. *Ergodic Theory Dynam. Systems* **34** (2014), no. 1, 21–54 Zbl 1330.37004 MR 3163023

[4] E. Artin, *Geometric algebra*. Wiley Classics Lib., John Wiley & Sons, New York, 2011 Zbl 0642.51001 MR 1009557

[5] U. Bader, B. Duchesne, and J. Lécureux, Furstenberg maps for CAT(0) targets of finite telescopic dimension. *Ergodic Theory Dynam. Systems* **36** (2016), no. 6, 1723–1742 Zbl 1378.37055 MR 3530464

[6] U. Bader and A. Furman, Boundaries, rigidity of representations, and Lyapunov exponents. In S. Y. Jang, Y. R. Kim, D.-W. Lee, and I. Yie (eds.), *Proceedings of the International Congress of Mathematicians – Seoul 2014*, pp. 71–96, Invited lectures III, Kyung Moon Sa, Seoul, Korea, 2014 Zbl 1378.37004 MR 3729019

[7] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren Math. Wiss. 319, Springer Berlin, Heidelberg, 2013 Zbl 0988.53001 MR 1744486

[8] M. Bucher, M. Burger, and A. Iozzi, A dual interpretation of the Gromov–Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices. In M. A. Picardello (ed.), *Trends in harmonic analysis*, pp. 47–76, Springer INdAM Ser. 3, Springer, Milano, 2013 Zbl 1268.53056 MR 3026348

[9] M. Bucher, M. Burger, and A. Iozzi, The bounded Borel class and 3-manifold groups. *Duke Math. J.* **167** (2018), no. 17, 3129–3169 Zbl 1417.22009 MR 3874650

[10] M. Burger and A. Iozzi, Boundary maps in bounded cohomology. *Geom. Funct. Anal.* **12** (2002), no. 2, 281–292 (Appendix to: Continuous bounded cohomology and applications to rigidity theory by: M. Burger and N. Monod) Zbl 1006.22011 MR 1911668

[11] M. Burger and A. Iozzi, A measurable Cartan theorem and applications to deformation rigidity in complex hyperbolic geometry. *Pure Appl. Math. Q.* **4** (2008), no. 1, 181–202 Zbl 1145.32013 MR 2406001

[12] M. Burger and N. Monod, Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.* **12** (2002), no. 2, 219–280 Zbl 1006.22010 MR 1911660

[13] M. Burger and S. Mozes, CAT(−1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.* **9** (1996), no. 1, 57–93 Zbl 0847.22004 MR 1325797

[14] P.-E. Caprace and A. Lytchak, At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.* **346** (2010), no. 1, 1–21 Zbl 1184.53038 MR 2558883

[15] B. Duchesne, Infinite-dimensional nonpositively curved symmetric spaces of finite rank. *Int. Math. Res. Not. IMRN* **2013** (2013), no. 7, 1578–1627 Zbl 1315.53054 MR 3044451

[16] B. Duchesne, J. Lécureux, and M. B. Pozzetti, Boundary maps and maximal representations on infinite-dimensional Hermitian symmetric spaces. *Ergodic Theory Dynam. Systems* **43** (2023), no. 1, 140–189 Zbl 1544.22012 MR 4518493

[17] D. Fisher, D. W. Morris, and K. Whyte, Nonergodic actions, cocycles and superrigidity. *New York J. Math.* **10** (2004), 249–269 Zbl 1074.37001 MR 2114789

[18] H. Furstenberg, A Poisson formula for semi-simple Lie groups. *Ann. of Math. (2)* **77** (1963), no. 2, 335–386 Zbl 0192.12704 MR 0146298

[19] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces. In C. C. Moore (ed.), *Harmonic analysis on homogeneous spaces*, pp. 193–229, Proc. Sympos. Pure Math. 26, American Mathematical Society, Providence, RI, 1973 Zbl 0289.22011 MR 0352328

[20] W. M. Goldman, *Complex hyperbolic geometry*. Oxford Math. Monogr., Clarendon Press, Oxford, 1999 Zbl 0939.32024 MR 1695450

[21] A. Iozzi, Bounded cohomology, boundary maps, and rigidity of representations into $\mathrm{Homeo}_+(S^1)$ and $\mathrm{SU}(1,n)$. In M. Burger and A. Iozzi (eds.), *Rigidity in dynamics and geometry*, pp. 237–260, Springer, Berlin, Heidelberg, 2002 Zbl 1012.22023 MR 1919404

[22] B. Kleiner, The local structure of length spaces with curvature bounded above. *Math. Z.* **231** (1999), no. 3, 409–456 Zbl 0940.53024 MR 1704987

[23] G. A. Margulis, Discrete groups of motions of manifolds of nonpositive curvature. In *Proceedings of the International Congress of Mathematicians, 2, Vancouver, B.C., 1974*, pp. 21–34, Canadian Mathematical Congress, Montreal, QC, 1975 Zbl 0367.57012 MR 0492072

[24] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*. Ergeb. Math. Grenzgeb. (3) 17, Springer Berlin, Heidelberg, 1991 Zbl 0732.22008 MR 1090825

[25] N. Monod, Superrigidity for irreducible lattices and geometric splitting. *J. Amer. Math. Soc.* **19** (2006), no. 4, 781–814 Zbl 1105.22006 MR 2219304

[26] N. Monod and Y. Shalom, Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differential Geom.* **67** (2004), no. 3, 395–455 Zbl 1127.53035 MR 2153026

[27] M. Moraschini and A. Savini, A Matsumoto–Mostow result for Zimmer's cocycles of hyperbolic lattices. *Transform. Groups* **27** (2022), no. 4, 1337–1392 Zbl 1515.22006 MR 4507989

[28] M. Moraschini and A. Savini, Multiplicative constants and maximal measurable cocycles in bounded cohomology. *Ergodic Theory Dynam. Systems* **42** (2022), no. 11, 3490–3525 Zbl 1523.57041 MR 4492631

[29] G. D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *Publ. Math. Inst. Hautes Études Sci.* **34** (1968), no. 1, 53–104 Zbl 0189.09402 MR 0236383

[30] M. B. Pozzetti, Maximal representations of complex hyperbolic lattices into $\mathrm{SU}(m,n)$. *Geom. Funct. Anal.* **25** (2015), no. 4, 1290–1332 Zbl 1325.22007 MR 3385634

[31] F. Sarti and A. Savini, Superrigidity of maximal measurable cocycles of complex hyperbolic lattices. *Math. Z.* **300** (2022), no. 1, 421–443 Zbl 1497.22010 MR 4359531

[32] F. Sarti and A. Savini, Parametrized Kähler class and Zariski dense orbital 1-cohomology. *Math. Res. Lett.* **30** (2023), no. 6, 1895–1929 Zbl 07946711 MR 4779157

[33] A. Savini, Algebraic hull of maximal measurable cocycles of surface groups into Hermitian Lie groups. *Geom. Dedicata* **213** (2021), no. 1, 375–400 Zbl 1497.22011 MR 4278334

[34] A. Savini, Borel invariant for measurable cocycles of 3-manifold groups. *J. Topol. Anal.* **16** (2024), no. 3, 385–408 Zbl 1547.57041 MR 4770560

[35] R. L. Wheeden and A. Zygmund, *Measure and integral: an introduction to real analysis*. CRC Press, New York-Basel, 1977 Zbl 1326.26007 MR 0492146

[36] R. J. Zimmer, Amenable ergodic group actions and an application to Poisson boundaries of random walks. *J. Funct. Anal.* **27** (1978), no. 3, 350–372 Zbl 0391.28011 MR 0473096

[37] R. J. Zimmer, *Ergodic theory and semisimple groups*. Monogr. Math. 81, Birkhäuser Boston, MA, 1984 Zbl 0571.58015 MR 0776417

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