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First-order sentences in random groups II: ∀∃-sentences

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Abstract. We prove that a random group, in Gromov's density model with $d < 1/16$, satisfies with overwhelming probability a universal-existential first-order sentence σ (in the language of groups) if and only if σ is true in a nonabelian free group. It is remarked that one can also obtain the optimal result, that is, replace $d < 1/16$ by $d < 1/2$.

1. Introduction

In this paper, we continue our work that connects random groups with the first-order theory of nonabelian free groups (see [6]). We generalize our previous result, that a random group (of density $d < 1/16$) satisfies with overwhelming probability a universal sentence in the language of groups if and only if the sentence is satisfied in a nonabelian free group, to ∀∃-sentences. Our main result is the following theorem.

Theorem 1. *Let σ be a ∀∃ first-order sentence in the language of groups. Let $0 \leq d < 1/16$ be a real number. Then a random group of density d satisfies, with overwhelming probability, the sentence σ if and only if a nonabelian free group satisfies σ.*

We will make heavy use of the machinery developed for answering Tarski's question and in particular *formal solutions, towers, closures of towers* (see [3, 11]), and the *process of validating a* ∀∃*-sentence* (see [4, 12]).

2. Preliminaries

2.1. The density model

Recall Gromov's density model of randomness.

Definition 2.1 (Gromov's density model). Let $\mathbb{F}_n := \langle e_1, \ldots, e_n \rangle$ be a free group of rank n. Let S_ℓ be the set of reduced words on $e_1, \ldots, e_n$ of length ℓ.

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Let $0 \leq d \leq 1$. Then a random set of relators of density d at length ℓ is a subset of S_ℓ that consists of $(2n-1)^{d\ell}$-many elements picked randomly (uniformly and independently) among all elements of S_ℓ.

A group $G := \langle e_1, \ldots, e_n \mid \mathcal{R} \rangle$ is called random of density d at length ℓ if $\mathcal{R}$ is a random set of relators of density d at length ℓ.

A random group of density d satisfies some property (of presentations) P, with overwhelming probability, if the probability of occurrence of P tends to 1 as ℓ goes to infinity.

We note in passing that at density 0, we have one relator of length ℓ.

Heuristically, one can understand this as follows: the ratio of groups in $(2n-1)^{d\ell}$ relators all of length ℓ that satisfy property P over all such groups is a number, say p, that for "interesting properties" will depend on ℓ, that is, $p := p(\ell)$. If $p(\ell)$ goes to 1 as ℓ goes to ∞, then we say that, with overwhelming probability, a random group has property P.

We work in an expansion of the language of groups $\mathcal{L}$ where we add n constant symbols $c_1, \ldots, c_n$ that will be interpreted as $e_1, \ldots, e_n$ in $\mathbb{F}_n$ and as the images of those generators under the canonical maps in a random group Γ_ℓ.

Definition 2.2. Let Γ_ℓ be a random group of density d at length ℓ. Let b_ℓ be an element in Γ_ℓ. Then b_ℓ is a constant if there exists an element $b \in \mathbb{F}_n$ such that b_ℓ is the image of b, under the canonical quotient map, for all $\ell \in \mathbb{N}$.

Hence, under the above definition, a sentence with constants in $\mathbb{F}_n$ makes sense in a random group and likewise a sentence with constants in a random group makes sense in $\mathbb{F}_n$.

We will need the following results from [6].

Theorem 2.3 ([6, Theorem 7.19]). *Let $d < 1/16$. Let σ be a universal sentence in the language $\mathcal{L} \cup \{c_1, \ldots, c_n\}$. Then σ is almost surely true in a random group (of density d) with constants interpreted as $\pi(e_1), \ldots, \pi(e_n)$ if and only if it is true in $\mathbb{F}_n$.*

Theorem 2.3 was obtained as a corollary of the following.

Theorem 2.4 ([6, Proposition 7.18]). *Let $d < 1/16$. Let $V(\bar{x}, e_1, \ldots, e_n) = 1$ be a system of equations over $\mathbb{F}_n$. Suppose Γ_ℓ is a random group of density d at length ℓ and $\pi : \mathbb{F}_n \to \Gamma_\ell$ the natural quotient map.*

Then, every solution $\bar{b}_\ell$ of $V(\bar{x}) = 1$ in Γ_ℓ is the image of a solution $\bar{c}_\ell$ of $V(\bar{x}) = 1$ in $\mathbb{F}_n$, under the canonical quotient maps, that is, $\pi(\bar{c}_\ell) = \bar{b}_\ell$, with probability approaching 1 as ℓ goes to infinity.

To understand Theorem 2.4 in the light of Definition 2.1, one considers the following property P: for a fixed $V(\bar{x}, e_1, \ldots, e_n) = 1$, every solution of $V(\bar{x}, e_1, \ldots, e_n) = 1$ is the pre-image of a solution of $V(\bar{x}, e_1, \ldots, e_n) = 1$ in $\mathbb{F}$, under the canonical quotient map.

Alternatively, under the same interpretation, one can think that the above theorem says that either $\overline{b}_\ell$ is the image (under the canonical map) of a solution of $V(\overline{x}, e_1, \dots, e_n) = 1$ in $\mathbb{F}_n$ or the probability that $V(\overline{b}_\ell, e_1, \dots, e_n) = 1$ tends to 0 as ℓ goes to ∞.

2.2. Boolean combinations of universal-existential axioms

The following lemma is due to Malcev. Its proof uses the fact that if (a, b, c) is a solution of $x^2y^2z^2 = 1$ in a free group, then $[a, b] = [a, c] = [b, c] = 1$ [7].

Lemma 2.5. *Let $\mathbb{F} := \langle e_1, e_2, \dots \rangle$ be a nonabelian free group. Then, conjunctions of equations are equivalent (over constants), in $\mathbb{F}$, to one equation:*

$$\mathbb{F} \models \forall x, y((x = 1 \wedge y = 1) \leftrightarrow (x^2 e_1)^2 e_1^{-2} = ((y e_2)^2 e_2^{-2})^2).$$

For disjunctions, we get the following.

Lemma 2.6. *Let $\mathbb{F} := \langle e_1, e_2, \dots \rangle$ be a nonabelian free group. Then, a disjunction of equations is equivalent (over constants), in $\mathbb{F}$, to four conjunctions of equations:*

$$\mathbb{F} \models \forall x, y\Bigg((x = 1 \vee y = 1) \leftrightarrow \bigwedge_{\substack{a \in \{e_1, e_1^{-1}\} \\ b \in \{e_2, e_2^{-1}\}}} [x^a, y^b] = 1\Bigg).$$

In particular, one easily obtains the following corollary.

Corollary 2.7. *A disjunction of conjunctions (or conjunction of disjunctions) of equations with constants in $\mathbb{F}$ is equivalent, in $\mathbb{F}$, to one equation.*

Since the above corollary can be expressed by a universal formula, we also get the following corollary.

Corollary 2.8. *Let $d < 1/16$ and Γ be a random group at density d. A disjunction of conjunctions (or conjunction of disjunctions) of equations with constants in Γ is equivalent in Γ to the same one equation as in $\mathbb{F}$.*

Lemma 2.9 (Cf. [3, Lemma 6]). *Let τ be a ∀∃ first-order sentence in the language of groups. Then, τ is equivalent in $\mathbb{F}_n$ to a sentence ζ of the form*

$$\forall \overline{x} \exists \overline{y} (\sigma(\overline{x}, \overline{y}, \overline{a}) = 1 \wedge \psi(\overline{x}, \overline{y}, \overline{a}) \neq 1),$$

where $\sigma(\overline{x}, \overline{y}, \overline{a}) = 1$ is an equation and $\psi(\overline{x}, \overline{y}, \overline{a}) \neq 1$ is an inequation, both over constants from $\mathbb{F}_n$.

Moreover, if $d < 1/16$, then τ is almost surely true in a random group of density d if and only if the sentence ζ is.

Proof. Every ∀∃-sentence in the language of groups is (logically) equivalent to a formula in prenex (disjunctive) normal form:

$$\forall \bar{x} \exists \bar{y} \Big(\bigvee_{i=1}^{m} (\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) \neq 1) \Big).$$

In any nontrivial group, the quantifier-free part, $\bigvee_{i=1}^{m} \big(\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) \neq 1\big)$, of the above sentence is equivalent to

$$\exists \bar{z}_1, \ldots, \bar{z}_m \Big(\Big(\bigwedge_{i=1}^{m} \bar{z}_i \neq 1 \Big) \wedge \bigvee_{i=1}^{m} (\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) = \bar{z}_i) \Big).$$

By Corollary 2.7, the disjunction of conjunctions of equations

$$\bigvee_{i=1}^{m} \big(\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) = \bar{z}_i\big)$$

is equivalent to one equation $\sigma(\bar{x}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_m, \bar{a}) = 1$ over constants in $\mathbb{F}_n$. Similarly, the conjunction $\bigwedge_{i=1}^{m} \bar{z}_i \neq 1$ is equivalent to a single inequation $\psi(\bar{z}_1, \ldots, \bar{z}_m, \bar{a}) \neq 1$ over constants in $\mathbb{F}_n$. Hence, we can take for ζ the following sentence:

$$\forall \bar{x} \exists \bar{y} \exists z_1, \ldots, z_m \big(\psi(\bar{z}_1, \ldots, \bar{z}_m, \bar{a}) \neq 1 \wedge \sigma(\bar{x}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_m, \bar{a}) = 1\big).$$

For a random group Γ of density $d < 1/16$, we argue as follows. In this case, the sentence $\forall \bar{x} \exists \bar{y} (\bigvee_{i=1}^{m} (\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) \neq 1))$ is almost surely true in Γ if and only if $\forall \bar{x} \exists \bar{y} \exists \bar{z}_1, \ldots, \bar{z}_m ((\bigwedge_{i=1}^{m} \bar{z}_i \neq 1) \wedge \bigvee_{i=1}^{m} (\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) = \bar{z}_i))$ is almost surely true in Γ. In addition,

$$\mathbb{F}_n \models \forall \bar{x} \forall \bar{y} \forall \bar{z} \Big(\Big[\Big(\bigwedge_{i=1}^{m} \bar{z}_i \neq 1 \Big) \wedge \bigvee_{i=1}^{m} (\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) = \bar{z}_i) \Big]$$
$$\leftrightarrow (\psi(\bar{z}_1, \ldots, \bar{z}_m, \bar{a}) \neq 1 \wedge \sigma(\bar{x}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_m, \bar{a}) = 1) \Big).$$

By Theorem 2.3, the above sentence is almost surely true in Γ. In particular, this implies that $\forall \bar{x} \exists \bar{y} \exists \bar{z}_1, \ldots, \bar{z}_m ((\bigwedge_{i=1}^{m} z_i \neq 1) \wedge \bigvee_{i=1}^{m} (\Sigma_i(\bar{x}, \bar{y}) = 1 \wedge \Psi_i(\bar{x}, \bar{y}) = \bar{z}_i))$ is almost surely true in Γ if and only if $\forall \bar{x} \exists \bar{y} \exists \bar{z}_1, \ldots, \bar{z}_m (\psi(\bar{z}_1, \ldots, \bar{z}_m, \bar{a}) \neq 1 \wedge \sigma(\bar{x}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_m, \bar{a}) = 1)$ is almost surely true in Γ. ■

2.3. Limit groups, towers, and resolutions

For the convenience of the reader, we define some basic notions and constructions that were used in the solution of Tarski's problem (see also [14]).

A *limit group* L is a finitely generated group for which there exists a sequence of morphisms, $(h_n)_{n<\omega} : L \to \mathbb{F}$, such that for every nontrivial $g \in L$, $h_n(g) \neq 1$ for all but finitely many n. A *restricted limit group* R is a finitely generated group with a distinguished nonabelian free subgroup $\mathbb{F}$, for which there exists a sequence of morphisms $(h_n)_{n<\omega} : R \to \mathbb{F}$, such that $h_n \restriction \mathbb{F} = \mathrm{Id}$, for $n < \omega$, and for every nontrivial $g \in R$, $h_n(g) \neq 1$ for all but finitely many n. When we consider morphisms, $h : R \to G$, from a restricted limit group, we tacitly assume that they are $\mathbb{F}$-morphisms, that is, G contains a copy of $\mathbb{F}$ and $h \restriction \mathbb{F}$ is the identity. Note that a restricted limit group is, in particular, a limit group.

We also define a special subclass of limit groups, namely groups that have the structure of a *tower*.

A tower is built recursively by adding *floors* to a given ground floor that consists of a nonabelian free group. There are two types of floors: *surface floors* and *abelian floors*. The corresponding notion in the work of Kharlampovich–Myasnikov is the notion of an *NTQ group*, that is, the coordinate group of a nondegenerate triangular quasiquadratic system of equations (see [2, Definition 9]).

Towers can be thought of as groups equipped with construction instructions. The instructions consist of the nonabelian free group of the ground floor, the additional floors, and finally the way and order each floor is added to the already constructed tower. Towers also admit closures. The closure of a tower is obtained by "augmenting" the abelian floors of the original tower in a way that the original floor sits as a finite index subgroup in the augmented floor. It is still a tower with the same number, type, and order of floors; moreover, it contains the original tower as a subgroup. When no abelian floors take part in the construction of a tower, then we call it hyperbolic and it, of course, coincides with any of its closures.

We assume some familiarity with Bass–Serre theory [13]. We start by defining the notion of a surface floor.

Definition 2.10 (Surface-type vertex). A vertex v of a graph of groups Γ is called a *surface-type vertex* if the following conditions hold:

- The group G_v carried by v is the fundamental group of a compact surface Σ (usually with boundary), with Euler characteristic $\chi(\Sigma) < 0$.
- Incident edge groups are maximal boundary subgroups of $\pi_1(\Sigma)$, and this induces a bijection between the set of incident edges and the set of boundary components of Σ.

Definition 2.11 (Exceptional surfaces). Four hyperbolic surfaces with $\chi(\Sigma) = -1$ are considered exceptional because their mapping class group is "too small" (they do not carry pseudo-Anosov diffeomorphisms): the thrice-punctured sphere, the twice-punctured projective plane, the once-punctured Klein bottle, and the closed non-orientable surface of genus 3.

Definition 2.12 (Centered splitting). A centered splitting of G is a graph of groups decomposition $G = \pi_1(\Gamma)$ such that the vertices of Γ are $v, v_1, \ldots, v_m$, with $m \geq 1$, where v is surface type and every edge joins v to some v_i (see Figure 1).

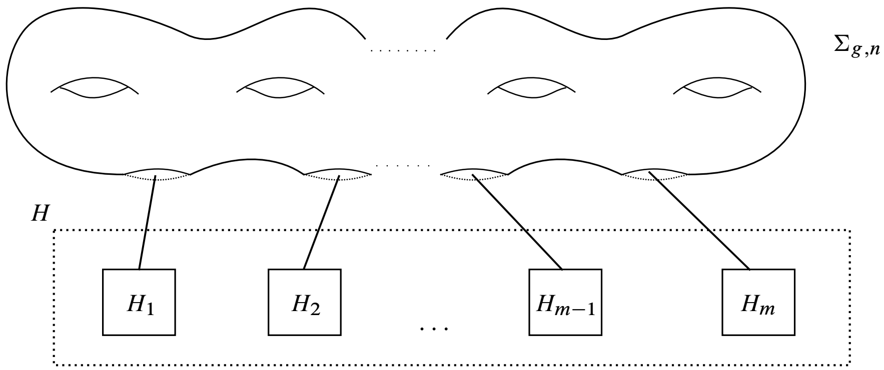

Figure 1. A centered splitting.

The vertex v is called the *central vertex* of Γ. The vertices $v_1, \ldots, v_n$ are the *bottom vertices*, and we denote by H_i the *bottom group* carried by v_i. The base of Γ is the abstract free product $H = H_1 * \cdots * H_m$.

Definition 2.13 (Surface floor). Let G be a group and H be a nonabelian subgroup of G. Then G has the structure of a surface floor over H, if the following conditions hold:

- The group G admits a simple non-exceptional centered splitting with bottom group H.
- There exists a retraction $r : G \to H$ that sends the group carried by the central vertex of Γ to a nonabelian image.

An abelian floor is defined in a similar way.

Definition 2.14. Let G be a group and H be a subgroup of G. Then G has the structure of an abelian floor over H, if G admits a splitting as an amalgamated free product $H *_E (E \oplus \mathbb{Z}^m)$, where (the image of) E is a maximal abelian subgroup of H and $\mathbb{Z}^m$ is a free abelian group of rank m (see Figure 2).

We call the image of E in H the peg of the abelian floor.

We can now define towers.

Definition 2.15. A group G has the structure of a tower (of height m) over a non-abelian subgroup $\mathbb{F}$ if there exists a (possibly trivial) free group $\mathbb{F}_n$ and a sequence

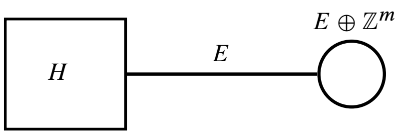

Figure 2. An abelian floor.

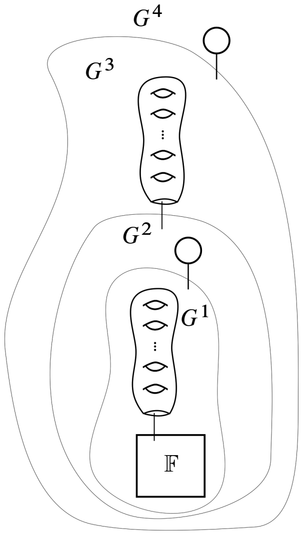

Figure 3. A tower of height 4.

$G = G^m > G^{m-1} > \cdots > G^0 = \mathbb{F} * \mathbb{F}_n$ such that for each i, $0 \leq i < m$, G^{i+1} is either a surface floor or an abelian floor over G^i (see Figure 3).

In [1] (or rather in [15]), it was proved that the class of limit groups coincides with the class of constructive limit groups [1, Definition 1.14]. Thus, one can easily deduce the following fact.

Fact 2.16. If G has the structure of a tower, then G is a limit group.

We next define the notion of a closure of a tower.

Definition 2.17 (Abelian closure). Let $\mathbb{Z} \oplus \mathbb{Z}^m := \langle z, z_1, z_2, \ldots, z_m \rangle$ and $\mathbb{Z} \oplus A^m := \langle z, a_1, a_2, \ldots, a_m \rangle$ be free abelian groups of rank $m+1$. Then an embedding $f : \mathbb{Z} \oplus \mathbb{Z}^m \to \mathbb{Z} \oplus A^m$ such that $f(z) = z$ is called a closure embedding.

Equivalently, we can see that $f : \mathbb{Z} \oplus \mathbb{Z}^m \to \mathbb{Z} \oplus A^m$ is a closure embedding if and only if $f(z) = z$ and $f(\mathbb{Z} \oplus \mathbb{Z}^m)$ has finite index in $\mathbb{Z} \oplus A^m$. We call the latter group the *group closure of* $\mathbb{Z} \oplus \mathbb{Z}^m$ with respect to f.

A closure of a tower is obtained by "augmenting" the free abelian vertex group of each abelian floor by a group closure with respect to some closure embedding. It is still a tower with the same number of floors.

Definition 2.18 (Tower closure). Let $\mathcal{T}(G, \mathbb{F})$ be a tower of height m and $\{E_i \oplus \mathbb{Z}^{m_i}\}_{i \leq k}$ be the collection of the free abelian vertex groups of its abelian floors. Let, for each $i \leq k$, $f_i : E_i \oplus \mathbb{Z}^{m_i} \to E_i \oplus A^{m_i}$ be a closure embedding. Then the closure of $\mathcal{T}(G, \mathbb{F})$ with respect to $\{f_i\}_{i\leq k}$, denoted $\mathrm{Cl}(\mathcal{T}(G,\mathbb{F}))$, is a tower of height m, defined recursively as follows:

- The ground floor of $\mathrm{Cl}(\mathcal{T})$ is identical to the ground floor of $\mathcal{T}$, that is, $\mathrm{Cl}(G^0) := \mathbb{F} * \mathbb{F}_n$.
- Suppose the tower $\mathrm{Cl}(\mathcal{T})_i$ up to the i-th floor $(\hat{\mathcal{G}}(\mathrm{Cl}(G^{i+1}), \mathrm{Cl}(G^i)), \hat{r}_i)$ has been constructed and has the following properties:
  - (1) The order and the type of floors of $\mathrm{Cl}(\mathcal{T})_i$ are identical to those of $\mathcal{T}_i$.
  - (2) $\mathrm{Cl}(G^{i+1}) \supseteq G^{i+1}$.
  - (3) The pegs of abelian floors of $\mathrm{Cl}(\mathcal{T})_i$ as subgroups of $\mathrm{Cl}(G^i)$ are identical with the pegs of the corresponding abelian floors of $\mathcal{T}_i$ as subgroups of G^i.
  - (4) The surface-type vertex groups of the surface floors of $\mathrm{Cl}(T)_i$ are identical to the surface-type vertex groups of the corresponding surface floors of $\mathcal{T}_i$ and the boundary components are attached through the amalgamation embeddings to the same elements as the boundary components of the corresponding surface floors of $\mathcal{T}_i$.
- We construct the $i+1$-th floor of $\mathrm{Cl}(\mathcal{T})$:
  - (1) If the $i+1$-th floor of $\mathcal{T}$ is a surface floor, then the $i+1$-th floor of $\mathrm{Cl}(\mathcal{T})$ is identical except that it is added on top of $\mathrm{Cl}(G^{i+1})$ which contains G^{i+1}. The retraction $\hat{r}_{i+1}$ coincides with r_{i+1} on the surface-type vertex and the Bass–Serre elements and is the identity on $\mathrm{Cl}(G^{i+1})$. Points (1)–(4) in the recursive assumption hold by construction.
  - (2) If the $i+1$-th floor of $\mathcal{T}$ is an abelian floor, with free abelian vertex group $E_j \oplus \mathbb{Z}^{m_j}$ for some $j \leq k$, then the $i+1$-th floor of $\mathrm{Cl}(\mathcal{T})$ is an abelian floor with free abelian vertex group, $E_j \oplus A^{m_j}$, the group closure of $E_j \oplus \mathbb{Z}^{m_j}$ with respect to f_j, and it is attached (in $\mathrm{Cl}(G^{i+1})$) to the same cyclic subgroup that $E_j \oplus \mathbb{Z}^{m_j}$ was attached (in G^{i+1}). This latter group remains maximal abelian in $\mathrm{Cl}(G^i)$, since the only maximal abelian groups in G^i that are not maximal abelian in $\mathrm{Cl}(G_i)$ are free abelian of higher rank. Again points (1)–(4) in the recursive assumption hold by construction.

The group, $\mathrm{Cl}(G)$, that corresponds to a closure of a tower $\mathcal{T}(G,\mathbb{F})$ contains G as a subgroup. A *covering set of closures* of $\mathcal{T}$ is a finite set of towers, $\mathcal{T}_1, \ldots, \mathcal{T}_k$, which are closures of $\mathcal{T}$ and for any morphism $h: G \to \mathbb{F}$ which is the identity on $\mathbb{F}$, there exists $i \leq k$ such that h extends to the group that corresponds to $\mathcal{T}_i$. Equivalently, for each abelian floor, $\mathbb{Z} \oplus \mathbb{Z}^m$, of $\mathcal{T}$ the union of cosets, $\bar{\mu}_i + \Delta^i$, that each corresponds to the group closure of $\mathbb{Z} \oplus \mathbb{Z}^m$ in $\mathcal{T}_i$, cover Z^m.

Towers are constructed from *well-structured resolutions* of (restricted) limit groups (see [11, Definition 1.12]). As a matter of fact in the aforementioned paper, they are called "completions of well-structured resolutions". It will be helpful for the sequel to explain the notion of a resolution of a (restricted) limit group. Toward this end, we will record some background results on limit groups.

We now define the modular automorphism group of a one-ended limit group. We quickly note that when an automorphism of a vertex group (of a graph of groups) restricts to conjugation on its edge groups, then there is a natural way to extend it to an automorphism of the whole group. We call such an extension the *standard extension*.

Definition 2.19. Let L be a one-ended limit group and Λ an abelian graph of groups decomposition of L. Then the group of modular automorphisms with respect to Λ, $\mathrm{Mod}_\Lambda(L) \leq \mathrm{Aut}(L)$, is generated by the following automorphisms:

- Inner automorphisms of L.
- Dehn twists along edges by elements that centralize an edge group.
- Standard extensions of automorphisms of QH subgroups – these automorphisms are obtained by surface homeomorphisms fixing all the boundary components.
- Standard extensions of automorphisms of abelian vertex groups that are the identity on their peripheral subgroups.

The modular automorphism group, $\mathrm{Mod}(L)$, of a one-ended limit group L is the group of automorphisms generated by $\mathrm{Mod}_\Lambda(L)$ for every abelian graph of groups decomposition, Λ, of L. In the case of a restricted limit group, we impose the extra condition that the modular automorphisms fix pointwise the distinguished free subgroup.

Fact 2.20. Let $L_0 \twoheadrightarrow L_1 \twoheadrightarrow \cdots \twoheadrightarrow L_n \twoheadrightarrow \cdots$ be a chain of proper quotients of limit groups. Then it terminates after finitely many steps.

Fact 2.21. Let L be a (restricted) freely indecomposable limit group (respectively indecomposable with respect to $\mathbb{F}$). Then there exist finitely many proper quotients $M_1, \ldots, M_k$ of L such that any ($\mathbb{F}$)-morphism factors through one of them after pre-composing by a modular automorphism.

We are now able to define a diagram that encodes all possible morphisms from a limit group to a free group. Let L be a limit group, and $L_1 * L_2 * \cdots * L_q * \mathbb{F}_m$ its Grushko decomposition. For each non-free (freely indecomposable) factor, we draw arrows from L_i toward $M_{i1}, \ldots, M_{ik_i}$, the proper quotients given by Fact 2.21. We iterate the procedure for each M_{ij}. By Fact 2.20, the procedure stops in finitely many steps. Likewise, we can construct a diagram that encodes all $\mathbb{F}$-morphisms from a restricted limit group. We call such diagram a *(restricted) Makanin–Razborov diagram* (see [11, p. 67]). As a matter of fact, such a diagram is canonical.

Definition 2.22 (Resolution). Let L be a (restricted) limit group and $\mathcal{MR}(L)$ its (restricted) Makanin–Razborov diagram. Then the subdiagram that consists of one arrow for each Grushko factor at each stage of the iterative procedure is called a resolution (see Figure 4).

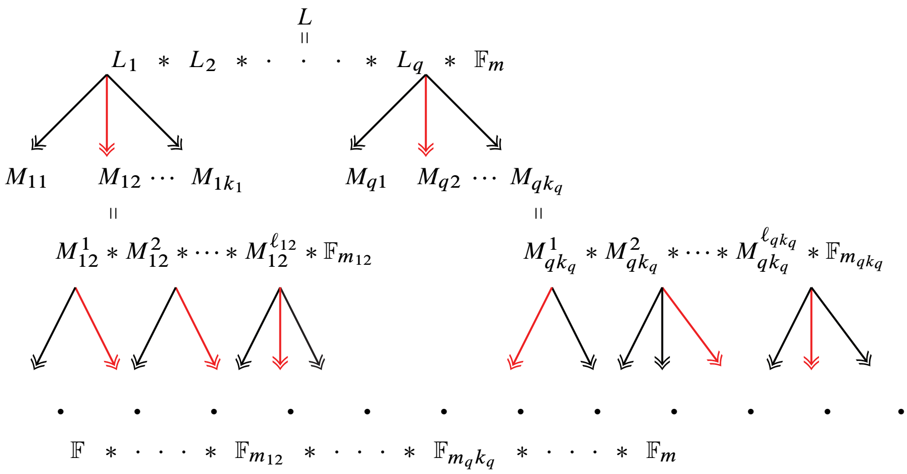

Figure 4. A Makanin–Razborov diagram for L and a resolution in red.

Well-structured resolutions, that are used to construct towers, are resolutions satisfying some extra conditions. The main properties of well-structured resolutions are as follows: every subgroup generated by a rigid subgroup and its edge groups is mapped monomorphically to the next level, and images of different factors in the Grushko decomposition are maximally separated on the next level and separated from the subsurfaces mapped to free groups. As defining them properly is technical and beyond the purpose of this paper, we refer the reader to [11, Definition 1.11]. In [4], the analogue of a resolution is called a fundamental sequence and there are restrictions on fundamental sequences that make them similar to well-structured resolutions [4, Sections 7.3, 7.8, 7.9].

2.4. Validation of a ∀∃-sentence in nonabelian free groups

Let $\forall\bar{x}\exists\bar{y}(\Sigma(\bar{x},\bar{y},\bar{a}) = 1 \wedge \Psi(\bar{x},\bar{y},\bar{a}) \neq 1)$, where $\Sigma(\bar{x},\bar{y},\bar{a}) = 1$ is a conjunction of equations and $\Psi(\bar{x},\bar{y},\bar{a}) \neq 1$ a conjunction of inequations, be a true sentence in a nonabelian free group $\mathbb{F} := \langle\bar{a}\rangle$. The idea, for validating the above sentence, is to find witnesses for the existentially quantified variables $\bar{y}$ in terms of the universally quantified variables $\bar{x}$ and the constants $\bar{a}$ as words in $\langle\bar{x},\bar{a}\rangle$. Indeed, the first step of the validating process is based on the following theorem [3] and [11, Theorem 1.2].

Theorem 2.23. *Let* $\mathbb{F} \models \forall\bar{x}\exists\bar{y}(\Sigma(\bar{x},\bar{y},\bar{a}) = 1 \wedge \Psi(\bar{x},\bar{y},\bar{a}) \neq 1)$. *Then, there exists a tuple of words* $\bar{w}(\bar{x},\bar{a})$ *in the free group* $\langle\bar{x},\bar{a}\rangle$, *such that* $\Sigma(\bar{x},\bar{w}(\bar{x},\bar{a}),\bar{a})$ *is trivial in* $\langle\bar{x},\bar{a}\rangle$ *and, moreover,* $\mathbb{F} \models \exists\bar{x}\Psi(\bar{x},\bar{w}(\bar{x},\bar{a}),\bar{a}) \neq 1$.

In the special case where no inequations exist, Theorem 2.23 is known as *Merzlyakov's theorem* and leads to the equality of the positive theories of nonabelian free groups. We think of $\bar{w}(\bar{x},\bar{a})$ as validating the sentence in a particular subset of $\mathbb{F}^{|\bar{x}|}$. What is left to do is find validating witnesses for the complement of this subset. The subset of $\mathbb{F}^{|\bar{x}|}$ for which the formal solution *does not* work is first order definable by the union of the following "varieties" $\psi_1(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1, \ldots, \psi_k(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1$, where each $\psi_i(\bar{x}, \bar{y}, \bar{a})$, for $i \leq k$, is a word in $\Psi(\bar{x}, \bar{y}, \bar{a})$. One can further split each variety $\psi_i(\bar{x}, \bar{w}(\bar{x}, \bar{a}))$ in finitely many irreducible varieties, that is, systems of equations $\Sigma_{i1}(\bar{x}, \bar{a}) = 1, \ldots, \Sigma_{im_i}(\bar{x}, \bar{a}) = 1$ for $i \leq k$, such that $L_{ij} := \langle \bar{x}, \bar{a} \mid \Sigma_{ij}(\bar{x}, \bar{a}) \rangle$, for $i \leq k$ and $j \leq m_i$, is a (restricted) limit group.

The iterative step of the process uses a further generalization of Merzlyakov's theorem that we record next (see [3] and [11, Theorem 1.18]). For convenience of notation, we denote by $G(\bar{x})$ a group G with generating set $\bar{x}$.

Theorem 2.24. *Let $L(\bar{x}, \bar{a}) := \langle \bar{x}, \bar{a} \mid R(\bar{x}, \bar{a}) \rangle$ be a restricted limit group, and $T(\bar{x}, \bar{z}, \bar{a})$ a tower constructed from a well-structured resolution of $L(\bar{x}, \bar{a})$.*

Suppose

$$\mathbb{F} \models \forall \bar{x}(R(\bar{x}, \bar{a}) = 1 \rightarrow \exists \bar{y}(\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1 \wedge \Psi(\bar{x}, \bar{y}, \bar{a}) \neq 1)).$$

Then there exist, $C_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, C_q(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ a covering closure of $T(\bar{x}, \bar{z}, \bar{a})$ and "formal solutions", $\bar{w}_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, \bar{w}_q(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ with the following properties:

- *For each $1 \leq i \leq q$, the words $\Sigma(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a})$ are trivial in $C_i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$.*
- *For each $1 \leq i \leq q$, there exists a morphism $h_i : C_i \rightarrow \mathbb{F}$, which is the identity on $\mathbb{F}$ and such that $\Psi(h_i(\bar{x}), h_i(\bar{w}_i), \bar{a}) \neq 1$.*

In principle, formal solutions do not exist in arbitrary limit groups, but only in limit groups that admit a special structure – the structure of a tower. A tower is constructed as a nested graph of groups based on a free group and "gluing" at each step, either a surface with boundaries (along the boundaries) or a free abelian group along a direct factor. The covering closure of a tower is a finite set of towers where the free abelian "floors" are augmented to finite index supergroups (see [11, Definition 1.15]). It covers the original tower in the sense that every morphism from the original tower to a free group extends (as a closure contains the tower as a subgroup) to a morphism from some closure. The precise construction is of no practical importance for this paper, but we note that as a consequence, the (new) subset of $\mathbb{F}^{|x|}$ for which the (new) formal solution *does not work* is not a union of varieties anymore but is *contained* in a union of Diophantine sets. This subtlety makes the proof that the process terminates quite involved. In any case, the important fact for us is that after repeatedly using the above theorem, we will eventually cover all of $\mathbb{F}^{|x|}$ with finitely many subsets for which some formal solution works.

We will give some more details on the procedure avoiding the technical results that imply its termination after finitely many steps. It will be important to carefully collect the "bad" tuples in a set in such a way that the procedure terminates. This set will occasionally be larger than needed, that is, it will also contain tuples for which a formal solution already works, but this is unavoidable, under the existing methods, if one wants to guarantee the termination. We next explain how this works. For presentational purposes, we will first record a special case, called *the minimal rank case*.

The minimal rank case. The assumption in this case is that all (restricted) limit groups L_{ij}, for $i \leq k$ and $j \leq m_i$, that collect the "bad" tuples in the first step of the procedure *do not admit* a surjection (that is the identity on $\mathbb{F}$) to a free product $\mathbb{F} * \mathbb{F}_n$, for some nontrivial free group $\mathbb{F}_n$. This simplified version of the iterative procedure is presented, for example, in [12, Section 1]. This assumption considerably simplifies the technicalities in the construction of the towers on each consecutive step and guarantees that their complexity decreases.

(1) In the first step of the procedure, we now apply Theorem 2.23 to the sentence $\forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1 \wedge \Psi(\bar{x}, \bar{y}, \bar{a}) \neq 1)$ and obtain a formal solution $\bar{w}(\bar{x}, \bar{a})$ that *does not work* in the union of varieties $\psi_1(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1, \dots, \psi_k(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1$.

(2) The latter union of varieties can be further decomposed as $\mathrm{Hom}_{\mathbb{F}}(L_{ij}, \mathbb{F})$, for $i \leq k$ and $j \leq m_i$, where L_{ij} is a restricted limit group, and $\mathrm{Hom}_{\mathbb{F}}(L_{ij}, \mathbb{F})$ is the set of restricted homomorphisms from L_{ij} to $\mathbb{F}$. Equivalently, this can be seen as a decomposition of each variety to its irreducible components.

(3) In the iterative step of the procedure, we work with each L_{ij} in parallel. Thus, we can fix $L := L_{ij}$ for some $i \leq k$ and $j \leq m_i$. To each (restricted) limit group L, we can assign finitely many towers (based on well-structured resolutions of L), $T_1, \dots, T_\ell$, such that any (restricted) morphism $h : L \to \mathbb{F}$ factors through T_i, for some $i \leq \ell$. We will, again, work in parallel with each T_i. Thus, we can fix $T := T_i$, for some $i \leq \ell$.

(4) We apply Theorem 2.24 for the couple L and T and obtain finitely many closures $C_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \dots, C_q(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ and the corresponding formal solutions $\bar{w}_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \dots, \bar{w}_q(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ for each closure, that each *does not work* in the definable set

$$\exists \bar{z}, \bar{s} (\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1) \\ \wedge \forall \bar{z}, \bar{s} \big(\psi_1(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1 \vee \dots \vee \psi_k(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1\big).$$

(5) We do not continue with the previous definable set but rather with the larger set defined by

$$\exists \bar{z}, \bar{s} \Big(\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) \\ = 1 \wedge (\psi_1(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1 \vee \dots \vee \psi_k(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1) \Big).$$

(6) We work with each closure in parallel. Thus, for some $i \leq q$, we fix $C(\bar{x}, \bar{z}, \bar{s}, \bar{a}) := C_i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$. We consider the set of (restricted) morphisms, $\mathrm{Hom}_{\mathbb{F}}(C, \mathbb{F})$, whose images satisfy $\psi_1(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1 \vee \dots \vee \psi_k(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1$ in $\mathbb{F}$. There exist finitely many (restricted) limit groups, $QL_1, \dots, QL_m$, which are quotients of C and such that a (restricted) morphism $h : C \to \mathbb{F}$ satisfies the previous condition if and only if it factors through one of these quotients.

(7) We repeat the procedure for each QL_i and towers based on resolutions constructed as in [12, Proposition 1.13]; by [12, Theorem 1.18], the procedure terminates after finitely many steps.

We next record the general case.

The general case. In the general case, not only the resolutions (used to construct towers) are modified but also the family of morphisms that descend to the next step of the procedure. Instead of well-structured resolutions, we restrict to a special subclass called *well-separated resolutions* (see [12, Definition 2.2]). One of the properties of well-separated resolutions is that they can be used to endow each surface that corresponds to a QH subgroup of every JSJ decomposition along the resolution with a family of (two-sided, disjoint, non-null-homotopic, non-parallel) simple closed curves. This family of curves induces a splitting, as a graph of groups, of the fundamental group of the surface by cutting it along them.

Definition 2.25. Let Σ be a compact surface. Given a homomorphism $h : \pi_1(\Sigma) \to H$ a family of pinched curves is a collection $\mathcal{C}$ of disjoint, non-parallel, two-sided simple closed curves, none of which is null-homotopic, such that the fundamental group of each curve is contained in $\ker h$ (the curves may be parallel to a boundary component).

The map ϕ is *non-pinching* if there is no pinched curve: ϕ is injective in restriction to the fundamental group of any simple closed curve which is not null-homotopic.

Let $\eta_{i+1} : L_i \to L_{i+1}$ be part of a well-separated resolution and $\pi_1(\Sigma) := Q$ is a QH subgroup in a JSJ decomposition of a freely indecomposable free factor of L_i. Then, a family of pinched curves on Σ may be assigned to η_{i+1} and the free decomposition of L_{i+1}. This family will be used later in order to define a special class of morphisms called *taut with respect to the given resolution* (see [12, Definition 2.4]).

Remark 2.26. When a morphism $h : L \to \mathbb{F}$ factors through a resolution, $L := L_0 \to L_1 \to L_2 \to \cdots \to L_p$, then for each $i \leq p$, there exists a section with respect to the quotient map $\eta_i : L_{i-1} \to L_i$, for $0 < i \leq p$, that is, a morphism, $h_i : L_i \to \mathbb{F}$, such that $h = h_i \circ \eta_i \circ \alpha_{i-1} \circ \cdots \circ \eta_1 \circ \alpha_0$, where α_i is in $\mathrm{Mod}(L_i)$.

For economy of notation when we say that h (does not) kill an element of L_i, we mean that h_i (does not) kill it.

Definition 2.27 (Taut morphism). Let h be a morphism that factors through a well-separated resolution

$$L := L_0 \to L_1 \to L_2 \to \cdots \to L_p.$$

Then h is taut with respect to the given resolution if the following conditions hold:

- Every abelian edge group in each abelian decomposition at every level of the resolution is not killed by h.

- Every QH subgroup in each abelian decomposition at every level of the resolution does not have an abelian image under h.
- For any vertex group of the graph of groups decomposition of a QH subgroup of any JSJ decomposition in any level of the resolution, obtained by the family of pinched curves, the morphism, $h \circ \alpha$, that is, h pre-composed by any modular automorphism of the QH subgroup, restricted to this vertex group is non-pinching.

A (restricted) limit group L admits a finite family of well-separated resolutions, called the *taut Makanin–Razborov diagram of* L, such that any (restricted) morphism from L to $\mathbb{F}$ factors through a well-separated resolution, and, moreover, it is compatible with its structure (see [12, Proposition 2.5]).

We next refine the class of taut morphisms by those that are in the *shortest form*.

Definition 2.28 (Shortest form morphism). Let Res be a well-separated resolution of a limit group $L := L_0$:

$$L_0 \to L_1 \to L_2 \to \cdots \to L_p.$$

For each $i \leq p$, let $\mathrm{Comp}(L_i)$ be the completion up to the step i (see [11, Definition 1.12]), and consider the multi-graded Makanin–Razborov diagram (see [10, Section 12]) of $\mathrm{Comp}(L_{i+1})$ with respect to the images of the rigid subgroups and edge groups of the JSJ decompositions of each freely indecomposable free factor of L_i (see Figure 5).

For each morphism $h : L \to \mathbb{F}$ that factors through Res, there exists a sequence of morphisms $h_0, h_1, \ldots, h_p$ from $\mathrm{Comp}(L_i)$ to $\mathbb{F}$ which are defined as sections with respect to the canonical epimorphisms $\pi_i : \mathrm{Comp}(L) \to \mathrm{Comp}(L_i)$, that is, $h = h_i \circ \pi_i$, $1 \leq i \leq p$.

We say that a taut morphism, $h : L \to \mathbb{F}$, with respect to the given resolution, is in the shortest form with respect to Res, if every h_i, $1 \leq i \leq p$, as above is shortest with respect to the modular automorphism of the solid (or rigid) limit group it factors out from, in the multi-graded Makanin–Razborov diagram it factors through.

Note that any morphism of a given (restricted) limit group can be extended to a (taut) shortest form morphism that factors through a completed resolution in the taut Makanin–Razborov diagram.

Having defined taut shortest form morphisms, we are now ready to briefly describe the process of the validation of a $\forall\exists$ first-order sentence which is true in a nonabelian free group.

We perform the same steps (1)–(3) that were described for the minimal rank case. In item (3), we only consider taut well-structured resolutions.

In items (5) and (6), we now require the values for the variables $\bar{z}$, $\bar{s}$ to be images of (taut) shortest form morphisms for the fixed resolution we work with.

Item (6) differs in a yet another way; it may happen that the natural image of L_i in QL_i is a proper quotient of L_i. In this case, we go back to item (3) and work with this proper quotient instead of L_i. Otherwise, we go to item (7).

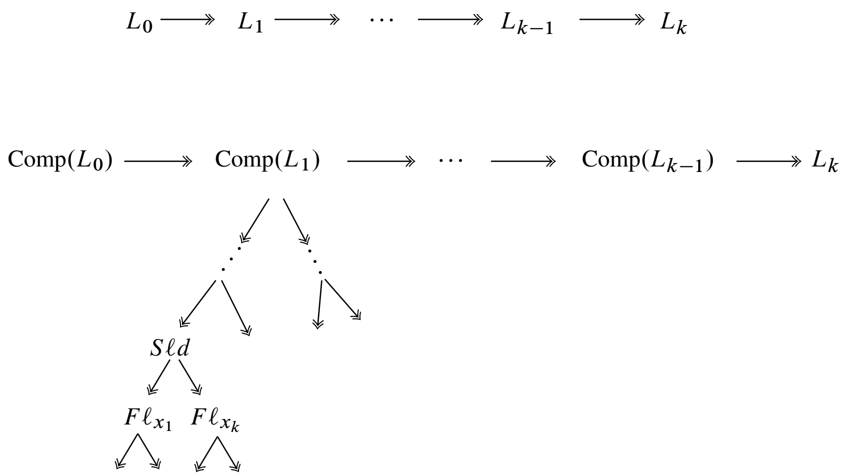

Figure 5. A resolution and a multi-graded Makanin–Razborov diagram of a completed limit group.

For item (7), the construction of towers and well-separated resolutions for each QL_i is much more complicated than in the minimal rank case. It is described, for example, in [12, Section 4]. Here is where we essentially use that specializations of $\overline{z}$, $\overline{s}$ are shortest form specializations (see [12, Proposition 4.3]). For the purposes of this paper, it is enough to know that this process terminates after finitely many steps (see [5, Theorem 3.5] and [12, Theorem 4.12]); the precise construction is not essential.

3. Main theorem

We are now ready to prove the following theorem.

Theorem 3.1. *Let $d < 1/16$. Then any ∀∃-sentence (with coefficients) that is true in a nonabelian free group $\mathbb{F}$ is true with overwhelming probability in a random group of density d.*

Proof. Allowing constants, by Lemma 2.9, a ∀∃-sentence τ is equivalent in $\mathbb{F}$ to a sentence of the form $\forall\overline{x}\exists\overline{y}(\sigma(\overline{x},\overline{y},\overline{a}) = 1 \wedge \psi(\overline{x},\overline{y},\overline{a}) \neq 1)$, where $\sigma(\overline{x},\overline{y},\overline{a})$ and $\psi(\overline{x},\overline{y},\overline{a})$ are single words in $\langle\overline{x},\overline{y},\overline{a}\rangle$. Suppose τ is true in $\mathbb{F}$. We claim that the iterative procedure presented in Section 2.4 is also valid for a random group (of density d). Hence, the formal solutions produced along the procedure will validate the sentence with overwhelming probability in the corresponding definable sets in a random group.

By Theorem 2.23, there exists a formal solution $\overline{w}(\overline{x},\overline{a})$ such that $\sigma(\overline{x},\overline{w}(\overline{x},\overline{a}),\overline{a})$ is trivial in $\langle\overline{x},\overline{a}\rangle$ and $\overline{w}(\overline{x},\overline{a})$ witnesses that the inequation also holds in $\mathbb{F}$ for tuples in $V^c := \mathbb{F}^{|\overline{x}|} \setminus V$, where V is the solution set of $\psi(\overline{x},\overline{w}(\overline{x},\overline{a}),\overline{a}) = 1$ in $\mathbb{F}$, which is a proper subset of $\mathbb{F}^{|\overline{x}|}$. We claim that for any tuple $\overline{b}_\ell$ in a random group Γ_ℓ with the property that all of its pre-images are in V^c, then with overwhelming probability $\sigma(\bar{b}_\ell, \bar{w}(\bar{b}_\ell, \bar{a}), \bar{a}) = 1 \wedge \psi(\bar{b}_\ell, \bar{w}(\bar{b}_\ell, \bar{a}), \bar{a}) \neq 1$. Indeed, obviously, since $\sigma(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a})$ is trivial in $\langle \bar{x}, \bar{a} \rangle$, we get $\sigma(\bar{b}_\ell, \bar{w}(\bar{b}_\ell, \bar{a}), \bar{a}) = 1$ for any $\bar{b}_\ell$ in a random group Γ_ℓ (and actually in any group). For the inequation, by our hypothesis, we have $\psi(\bar{c}_\ell, \bar{w}(\bar{c}_\ell, \bar{a}), \bar{a}) \neq 1$ in $\mathbb{F}$ for any pre-image $\bar{c}_\ell$ of $\bar{b}_\ell$. Hence, by Theorem 2.4, either the probability that $\psi(\bar{b}_\ell, \bar{w}(\bar{b}_\ell, \bar{a}), \bar{a}) = 1$ goes to 0 as ℓ goes to ∞ and, consequently, the probability that $\psi(\bar{b}_\ell, \bar{w}(\bar{b}_\ell, \bar{a}), \bar{a}) \neq 1$ goes to 1, or $\bar{b}_\ell$ is the image of a tuple $\bar{c}_\ell$ that satisfies $\psi(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1$ in $\mathbb{F}$, which is a contradiction to the choice of $\bar{b}_\ell$. Therefore, only the first alternative holds and the result follows.

We continue with the tuples in a random group whose pre-images not all belong to V^c.

Recall that the variety V can be further seen as the union of irreducible varieties, that is, varieties that correspond to limit groups. We continue with one such (restricted) limit group, say $L(\bar{x}, \bar{a})$, and we apply Theorem 2.24 for $L(\bar{x}, \bar{a})$ and a tower $T(\bar{x}, \bar{z}, \bar{a})$ in order to obtain finitely many formal solutions, $\bar{w}_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, \bar{w}_q(\bar{x}, \bar{z}, \bar{s}, \bar{a})$, and a set of covering closures, $C_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, C_q(\bar{x}, \bar{z}, \bar{s}, \bar{a})$, of T, so that $\sigma(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a})$ is trivial in C_i for any $i \leq q$, that is, for any (restricted) morphism $h : C_i \to \mathbb{F}$, h kills $\sigma(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a})$. Moreover, for each $1 \leq i \leq q$, there exists a (restricted) morphism $h_i : C_i \to \mathbb{F}$ such that $\psi(h_i(\bar{x}), h_i(\bar{w}_i), \bar{a}) \neq 1$ in $\mathbb{F}$.

At this point, we split the proof into two cases. For clarity, we will first prove the result in the minimal rank case. Recall that in this case, the termination of the validating process is easier and the choices for the stratification of $\mathbb{F}^{|\bar{x}|}$ are less complicated.

The minimal rank case: For each $i \leq q$, we consider the subset $D^i_{L,T}$ of $\mathbb{F}^{|x|}$ that contains all tuples with the following properties: they extend to (restricted) morphisms from C_i to $\mathbb{F}$ and any such extension *does not* satisfy $\psi(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}), \bar{a}) = 1$ in $\mathbb{F}$.

$$\begin{aligned} D^i_{L,T}(\bar{x}) := \exists \bar{z}, \bar{s}(\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1) \\ \wedge \forall \bar{z}, \bar{s}(\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1 \to \psi(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}), \bar{a}) \neq 1). \end{aligned}$$

We note in passing that the solution set of the above formula in $\mathbb{F}$ is a proper subset (in principle) of the "good tuples", that is, the tuples for which the formal solution $\bar{w}_i$ validates the sentence at this stage.

By [12, Proposition 1.2], it follows that there exists a finite number of proper quotients $Q^i_1(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, Q^i_n(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ of C_i such that $\bar{c}$ belongs to $D^i_{L,T}$ if and only if it extends to a (restricted) morphism from C_i to $\mathbb{F}$ and no such (restricted) extension factors through one of the quotients $Q^i_1, \ldots, Q^i_n$.

For any fixed ℓ, we take a tuple $\bar{b}_\ell$ in a random group Γ_ℓ such that it has a pre-image $\bar{c}_\ell$ that extends to satisfy $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$ in $\mathbb{F}$ and any such pre-image (that extends to satisfy $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$) belongs to $D^i_{L,T}$.

We will show that for such $\bar{b}_\ell$, we can find a tuple $\bar{d}_\ell$ so that $\sigma(\bar{b}_\ell, \bar{d}_\ell, \bar{a}) = 1 \wedge \psi(\bar{b}_\ell, \bar{d}_\ell, \bar{a}) \neq 1$ with probability tending to 1 as ℓ goes to ∞.

Denote the formula that collects the tuples corresponding to $\bigcup_{j=1}^n \mathrm{Hom}_{\mathbb{F}}(Q^i_j, \mathbb{F})$ by $\Phi_i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$. The solution set of Φ_i in $\mathbb{F}$ is a union of varieties and, since we work over parameters, it is equivalent to a variety that, by abusing notation, we denote by $\Phi_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$. Note that, since $\bar{b}_\ell$ has a pre-image that extends to satisfy $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$, $\bar{b}_\ell$ itself extends to satisfy $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$, say by $\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell$. To see this, just consider the projection of the solution of $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$ in $\mathbb{F}$ to Γ_ℓ. By Theorem 2.4, for the triple (of tuples) $\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell$, either the probability that $\Phi_i(\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell, \bar{a}) = 1$ tends to 0, as ℓ goes to ∞, or each such triple is the image of a solution $\bar{c}_\ell, \bar{c}^1_\ell, \bar{c}^2_\ell$ of the same formula in $\mathbb{F}$.

- In the first alternative, equivalently, $\Phi_i(\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell, \bar{a}) \neq 1$ with overwhelming probability, and we are done, since

$$\mathbb{F} \models \forall \bar{x}, \bar{z}, \bar{s}(\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1 \to (\Phi_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1 \leftrightarrow \psi(\bar{x}, \bar{w}_i(\bar{x}, z, \bar{s}, \bar{a}), \bar{a}) = 1)).$$

  Thus, since $\Sigma_{C_i}(\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell, \bar{a}) = 1$, we also get that $\psi(\bar{b}_\ell, w_i(\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell, \bar{a}), \bar{a}) \neq 1$ with overwhelming probability.
- In the second alternative, that is, when the tuple $\bar{b}_\ell, \bar{b}^1_\ell, \bar{b}^2_\ell$ is the image of a solution $\bar{c}_\ell, \bar{c}^1_\ell, \bar{c}^2_\ell$ of the same formula in $\mathbb{F}$, by the choice of $\bar{b}_\ell$ and since a fortiori $\bar{c}_\ell, \bar{c}^1_\ell, \bar{c}^2_\ell$ satisfies $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$ in $\mathbb{F}$ (as the solution set of $\Phi_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$ is a subset of the solution set of $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$), we get a contradiction.

We recall that $\psi(\bar{x}, w(\bar{x}, \bar{a}), \bar{a}) = 1$ is the union of finitely many sets of the form $\mathrm{Hom}_{\mathbb{F}}(L_i(\bar{x}, \bar{a}), \mathbb{F})$, for $i \leq p$, and for each $i \leq p$, there exist finitely many towers $T_{i,j}(\bar{x}, \bar{z}, \bar{a})$, for $j \leq m_i$, such that any (restricted) morphism from L_i to $\mathbb{F}$ extends to a morphism from $T_{i,j}$ to $\mathbb{F}$, for some $j \leq m_i$. Moreover, for each $T_{i,j}$, for $i \leq p$, and $j \leq m_i$, there exist finitely many towers $C_{i,j,r}$, for $r \leq q_{i,j}$, that form a covering closure of $T_{i,j}$.

At the end of this second step, we have validated all tuples $\bar{b}_\ell$ in a random group Γ_ℓ that have a pre-image that extends to some $C_{i,j,r}$, for $i \leq p$, $j \leq m_i$, and $r \leq q_{i,j}$, and any such pre-image belongs to $D^r_{L_i, T_{i,j}}$.

The validation process in a nonabelian free group continues with each $QL^{i,j,r}_k$, which is a quotient of $C_{i,j,r}$ that collects "bad tuples", in parallel. For each such quotient, there exist finitely many towers (of lower complexity than $C_{i,j,r}$) and for each such tower a set of towers that form a covering closure and bear formal solutions. At this step, we validate all $\bar{b}_\ell$ that extend to one of these closures and any of such extensions does not satisfy $\psi(\bar{x}, \bar{y}, \bar{a}) = 1$ with the new formal solution corresponding to the particular closure at the place of $\bar{y}$. This process produces a stratification of $\mathbb{F}^{|\bar{x}|}$ by finitely many subsets $\mathrm{Str}_1, \mathrm{Str}_2, \ldots, \mathrm{Str}_\alpha$, such that Str_i collects the tuples that work at step i.

We finally fix a ∀∃-sentence, say τ, which is true in $\mathbb{F}$. For $\ell \in \mathbb{N}$, consider a group Γ_ℓ that does not satisfy τ. Suppose $\bar{b}_\ell$ witnesses it. As ℓ goes to ∞, the witnesses $\bar{b}_\ell$ will have pre-images that vary through the finitely many strata, but, for any stratum, the ratio of the number of groups for which the sentence will be false over the number of all groups goes to 0; hence, since there are only finitely many strata, τ is true in a random group with overwhelming probability.

The general case. In this case, we start with the restricted limit group $L(\overline{x}, \overline{a})$ and a tower $T(\overline{x}, \overline{z}, \overline{a})$ that corresponds to a well-separated resolution of L. We apply Theorem 2.24 for L and T in order to obtain finitely many formal solutions $\overline{w}_i(\overline{x}, \overline{z}, \overline{s}, \overline{a})$, for $i \leq q$, and a set of covering closures $C_i(\overline{x}, \overline{z}, \overline{s}, \overline{a})$, for $i \leq q$, of T, so that $\sigma(\overline{x}, \overline{w}_i(\overline{x}, \overline{z}, \overline{s}, \overline{a}), \overline{a})$ is trivial in C_i, for any $i \leq q$, and, moreover, there exists a restricted morphism $h_i : C_i \to \mathbb{F}$ with $\psi(h_i(\overline{x}), h_i(\overline{w}_i), \overline{a}) \neq 1$.

We fix some C_i, for $i \leq q$. Let $S_1, \ldots, S_k$ be a collection of solid (or rigid) limit groups that appear in the multi-graded Makanin–Razborov diagrams of different levels of its resolution (see Definition 2.28).

Recall that any morphism $h : L \to \mathbb{F}$ extends to some C_i as a (taut) shortest form with respect to some collection of solid limit groups.

We consider the subset $C^i_{S_1,\ldots,S_k}$ of $\mathbb{F}^{|\overline{x}|}$ that contains all tuples with the following properties:

- They extend to taut morphisms that factor through the fixed C_i, and factor out, as solid (or rigid) morphisms, through the fixed S_i, for $i \leq k$.
- No shortest form, with respect to $\text{Mod}(S_1), \ldots, \text{Mod}(S_k)$, such extension satisfies $\psi(\overline{x}, \overline{w}_i(\overline{x}, \overline{z}, \overline{s}, \overline{a}), \overline{a}) = 1$.

We consider tuples $\overline{b}_\ell$ in the random group Γ_ℓ with the following properties:

(1) The tuple $\overline{b}_\ell$ has a pre-image $\overline{c}_\ell$, that admits a (taut) shortest form extension, with respect to $\text{Mod}(S_1), \ldots, \text{Mod}(S_k)$, factoring through C_i.

(2) Any such pre-image of $\overline{b}_\ell$ belongs to $C^i_{S_1,\ldots,S_k}$.

(3) No pre-image of $\overline{b}_\ell$ extends to factor through some well-separated resolution that imposes a more refined pinching than (the well-separated resolution that corresponds to) C_i.

The last point guarantees that any pre-image of $\overline{b}_\ell$ that extends to a morphism from C_i is taut with respect to the fixed resolution.

We will show that such $\overline{b}_\ell$ can be extended to $\overline{b}_\ell, \overline{b}^1_\ell, \overline{b}^2_\ell$ in such a way that $\sigma(\overline{b}_\ell, \overline{w}_i(\overline{b}_\ell, \overline{b}^1_\ell, \overline{b}^2_\ell), \overline{a}) = 1$ and $\psi(\overline{b}_\ell, \overline{w}_i(\overline{b}_\ell, \overline{b}^1_\ell, \overline{b}^2_\ell), \overline{a}) \neq 1$ with overwhelming probability.

Indeed, since $\overline{b}_\ell$ has a pre-image $\overline{c}_\ell$ that extends to a morphism that factors through C_i and does not satisfy $\psi(\overline{x}, \overline{w}_i(\overline{x}, \overline{z}, \overline{s}, \overline{a}), \overline{a}) = 1$, we consider the image, say $\overline{b}_\ell, \overline{b}^1_\ell, \overline{b}^2_\ell$, of that triple (of tuples). It is obvious that it satisfies $\Sigma_{C_i}(\overline{x}, \overline{z}, \overline{s}, \overline{a}) = 1$ with overwhelming probability. We next consider the conjunction of $\psi(\overline{x}, \overline{w}_i(\overline{x}, \alpha(\overline{z}), \alpha(\overline{s}), \overline{a}), \overline{a}) = 1$ for α in $\text{Mod}(S_1) \cdot \ldots \cdot \text{Mod}(S_k)$. Since $\mathbb{F}$ is equationally Noetherian, the above (infinite) system of equations is equivalent to a finite subsystem. Denote this finite subsystem by $\Phi^{\text{Mod}}_i(\overline{x}, \overline{z}, \overline{s}, \overline{a}) = 1$; we further add to $\Phi^{\text{Mod}}_i(\overline{x}, \overline{z}, \overline{s}, \overline{a}) = 1$ the equations $\Sigma_{C_i}(\overline{x}, \overline{z}, \overline{s}, \overline{a}) = 1$ and the equations for the fixed solid (or rigid) limit groups

$\Sigma_{S_1}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1, \ldots, \Sigma_{S_k}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$. We still denote this system of equations by $\Phi_i^{\mathrm{Mod}}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$.

By Theorem 2.4, either the probability that $\Phi_i^{\mathrm{Mod}}(\bar{b}_\ell, \bar{b}_\ell^1, \bar{b}_\ell^2, \bar{a}) = 1$ is zero or $\bar{b}_\ell, \bar{b}_\ell^1, \bar{b}_\ell^2$ is the image of $\bar{c}_\ell, \bar{c}_\ell^1, \bar{c}_\ell^2$ such that $\Phi_i^{\mathrm{Mod}}(\bar{c}, \bar{c}_1, \bar{c}_2, \bar{a}) = 1$ in $\mathbb{F}$.

We first consider the case where the probability that $\Phi_i^{\mathrm{Mod}}(\bar{b}_\ell, \bar{b}_\ell^1, \bar{b}_\ell^2, \bar{a}) = 1$ is zero, then, equivalently, $\Phi_i^{\mathrm{Mod}}(\bar{b}_\ell, \bar{b}_\ell^1, \bar{b}_\ell^2, \bar{a}) \neq 1$ with overwhelming probability. Thus, since $\bar{b}_\ell, \bar{b}_\ell^1, \bar{b}_\ell^2$ satisfies $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$, $\Sigma_{S_1}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1, \ldots, \Sigma_{S_k}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$, we get that $\psi(\bar{b}_\ell, \bar{w}_i(\bar{b}_\ell, \alpha(\bar{b}_\ell^1), \alpha(\bar{b}_\ell^2), \bar{a}), \bar{a}) \neq 1$ for some $\alpha \in \mathrm{Mod}(S_1) \cdot \ldots \cdot \mathrm{Mod}(S_k)$ with overwhelming probability and $\bar{w}_i(\bar{b}_\ell, \alpha(\bar{b}_\ell^1), \alpha(\bar{b}_\ell^2), \bar{a})$ validates the sentence for $\bar{b}_\ell$.

We now consider the case that the tuple $\bar{b}_\ell, \bar{b}_\ell^1, \bar{b}_\ell^2$ is the image of a solution $\bar{c}, \bar{c}_1, \bar{c}_2$ such that $\Phi_i^{\mathrm{Mod}}(\bar{c}, \bar{c}_1, \bar{c}_2) = 1$ in $\mathbb{F}$, then, $\bar{c}, \bar{c}_1, \bar{c}_2$ satisfies $\Sigma_{C_i}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$, and $\Sigma_{S_1}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1, \ldots, \Sigma_{S_k}(\bar{x}, \bar{z}, \bar{s}, \bar{a}) = 1$ in addition, we may change $\bar{c}_1$, $\bar{c}_2$ using an automorphism α to make them the shortest form, given that they satisfy $\Phi_i(\bar{x}, \bar{w}_i(\bar{x}, \alpha(\bar{z}), \alpha(\bar{s}), \bar{a}), \bar{a}) = 1$ for any α in $\mathrm{Mod}(S_1) \cdot \ldots \cdot \mathrm{Mod}(S_n)$. Therefore, we assume that $\bar{c}, \bar{c}_1, \bar{c}_2$ is the shortest form. Hence, this solution, by the choice of $\bar{b}_\ell$, must belong to $C^i_{S_1,\ldots,S_k}$ and, in particular, it factors out as solid morphisms from $S_1, \ldots, S_k$ and does not satisfy $\psi(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1$, a contradiction.

Those tuples $\bar{b}$ that have pre-images that extend to $\bar{c}_1, \bar{c}_2$ satisfying equations of $S_1, \ldots, S_k$ that are not in $C_{S_1,\ldots,S_k}$ either correspond to morphisms that factor out from a different set of solid (or rigid) limit groups, $S_{i_1}, \ldots, S_{i_k}$, in the multi-graded Makanin–Razborov diagrams of the different levels of the completed resolution, or have pre-images $\bar{c}$ that extend to $\bar{c}_1, \bar{c}_2$ which are the shortest form with respect to $\mathrm{Mod}(S_1) \cdots \mathrm{Mod}(S_n)$ and also satisfy $\psi(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1$. In the first case, we tackle the finitely many cases in parallel at the same step. In the second case, we define quotients of each C_i with which we will proceed to the next step and will repeat similar constructions.

Let $Q_1^i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, Q_{p_i}^i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ be the quotient groups of $C_i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ obtained by all (taut) shortest form morphisms, with respect to $\mathrm{Mod}(S_1), \ldots, \mathrm{Mod}(S_k)$, that factor through the fixed resolution and in addition satisfy $\psi(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \bar{a}) = 1$. We assume that the natural image of $L(\bar{x}, \bar{a})$ in $Q_j^i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$, for $j \leq p_i$, is not a proper quotient of L; otherwise, we repeat this step with the image of L. Each C_i and every set of solid (or rigid) limit groups in the multi-graded Makanin–Razborov diagrams from the different levels of the completed resolution are considered independently. At this step, we verified the sentence simultaneously for all tuples $\bar{b}_\ell$ satisfying the three conditions for some $i \leq q$ and some set, $S_1, \ldots, S_k$, of solid (or rigid) limit groups.

Remaining tuples $\bar{b}_\ell$ either have pre-images $\bar{c}_\ell$ that extend to $\bar{c}_1, \bar{c}_2$ that factor through one of $Q_1^i(\bar{x}, \bar{z}, \bar{s}, \bar{a}), \ldots, Q_k^i(\bar{x}, \bar{z}, \bar{s}, \bar{a})$ or have pre-images $\bar{c}_\ell$ that extend to $\bar{c}_3, \bar{c}_4$ that factor through another C_j that imposes more refined pinching than C_i. In other words, these $\bar{b}_\ell$ have pre-images $\bar{c}_\ell$ that belong to a different stratum in the process for $\mathbb{F}$.

We now give the final argument of the proof. Suppose τ is a ∀∃-sentence, which is true for a nonabelian free group. The main point is that the validating process decomposes $\mathbb{F}^{|\bar{x}|}$ in finitely many different strata. For $\ell \in \mathbb{N}$, consider a group Γ_ℓ that does not satisfy τ. Suppose $\bar{b}_\ell$ witnesses it. As ℓ goes to ∞, the witnesses $\bar{b}_\ell$ will have pre-images that vary through the finitely many strata, but, for any stratum, the ratio of the number of groups for which the sentence will be false over the number of all groups goes to 0; hence, since there are only finitely many strata, τ is true in a random group with overwhelming probability. ∎

Theorem 3.2. *Let* $d < 1/16$. *Every* $\exists\forall$*-sentence that is true in a nonabelian free group is true in the random group.*

Proof. Let $Q(\bar{x}, \bar{y})$ be a quantifier-free formula such that

$$\mathbb{F} \models \exists\bar{x}\forall\bar{y}Q(\bar{x}, \bar{y})$$

for a nonabelian free group $\mathbb{F}$. By [9], the sentence is true in the nonabelian free group $\mathbb{F}_n$ of rank n. Hence, for some $\bar{b}$ in $\mathbb{F}_n$, we have $\forall\bar{y}Q(\bar{b}, \bar{y})$ is true in $\mathbb{F}_n$. Thus, by Theorem 2.3, it is true with overwhelming probability in a random group of density d. Therefore, $\exists\bar{x}\forall\bar{y}Q(\bar{x}, \bar{y})$ is true with overwhelming probability in a random group of density d. ∎

The last two theorems imply Theorem 1.

Remark 3.3. In Theorem 1, one can obtain the optimal result, that is, replace $d < 1/16$ by $d < 1/2$, if, instead of Theorems 2.3 and 2.4, they use the same statements for $d < 1/2$ proven in the recent paper [8, Theorem 1.2]. That paper follows the same strategy as [6] and improves the density parameter by using a couple of clever tricks from hyperbolic geometry.

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