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Torsion homology growth and cheap rebuilding of inner-amenable groups

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Abstract. We prove that virtually torsion-free, residually finite groups that are inner-amenable and non-amenable have the cheap 1-rebuilding property, a notion recently introduced by Abért, Bergeron, Frączyk and Gaboriau. As a consequence, the first ℓ^2-Betti number with arbitrary field coefficients and log-torsion in degree 1 vanish for these groups. This extends results previously known for amenable groups to inner-amenable groups. We use a structure theorem of Tucker-Drob for inner-amenable groups showing the existence of a chain of q-normal subgroups.

1. Introduction

In 1994, Lück proved the approximation theorem about ℓ^2-Betti numbers. Its group-theoretic version asserts that for a residually finite group Γ with finite type model for $E\Gamma$, and any residual chain $(\Gamma_i)_{i\in\mathbb{N}}$ of Γ (i.e., a chain of nested, normal, finite-index subgroups in G whose intersection is trivial), the following holds: for all $n \in \mathbb{N}$, we have ([18, Theorem 0.1], [15, Theorem 5.3])

$$b_n^{(2)}(\Gamma) = \lim_{i\to\infty} \frac{b_n(\Gamma_i)}{[\Gamma : \Gamma_i]}.$$

Here, $b_n(\cdot)$ denotes the (ordinary) n-th Betti number and $b_n^{(2)}(\cdot)$ the ℓ^2-Betti number as defined by Atiyah [2] originally for spaces (for an introduction, see the book by Kammeyer [15]).

We obtain a different viewpoint by taking the right-hand side of this equality as a definition. The main advantage is that we can replace the n-th ordinary Betti number by different homology-related invariants; e.g., we can consider the invariants

$$\limsup_{i\to\infty} \frac{b_n(\Gamma_i, \mathbb{K})}{[\Gamma : \Gamma_i]} \quad \text{and} \quad \limsup_{i\to\infty} \frac{\log |H_n(\Gamma_i, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_i]},$$

where $\mathbb{K}$ is any field and tors denotes the torsion subgroup of an abelian group. We call the resulting invariant the *gradient* of the invariant that we insert instead of ordinary Betti numbers. For these gradient invariants, a priori, we do not know whether the lim sup is actually a proper limit or if its value depends on the chosen residual chain. The Betti

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number gradients do however depend on the field: Avramidi, Okun and Schreve exhibited an example of a right-angled Artin group where the $\mathbb{Q}$-Betti number gradients (i.e., the ℓ^2-Betti numbers) and the $\mathbb{F}_2$-Betti number gradients do *not* coincide [3, Corollary 2].

An efficient way to show the vanishing of Betti number gradients for all fields and the vanishing of log-torsion gradients is via the cheap α-rebuilding property, recently introduced by Abért, Bergeron, Frączyk and Gaboriau [1]. Roughly, for a fixed $\alpha \in \mathbb{N}$, a group Γ has the cheap α-rebuilding property if for all Farber sequence $(\Gamma_i)_{i\in\mathbb{N}}$, the following holds in a uniform way: Because the Γ_i are finite-index subgroups, we obtain a tower of finite degree coverings $B\Gamma_i \to B\Gamma$. If i is large enough, we can find a model of $B\Gamma_i$ (i.e., a CW-complex of the homotopy type of $B\Gamma_i$) with few cells up to dimension α, maintaining tame norms on the cellular boundary operators and homotopies. (We will give a more precise definition of the cheap α-rebuilding property in Appendix B.)

In this article, we will prove that certain inner-amenable groups have the cheap 1-rebuilding property. Inner-amenable groups are defined as follows and generalise the notion of amenability of groups.

Definition 1.1 (Inner amenability, [23, Definition 0.7]). A group Γ is *inner-amenable* if the conjugation action of Γ on itself admits an atomless invariant mean, i.e., if there is a finitely additive probability measure $\mu\colon \mathcal{P}(\Gamma) \to [0,1]$ such that for all subsets $A \subset \Gamma$ and $g \in \Gamma$, we have

$$\mu(g \cdot A \cdot g^{-1}) = \mu(A)$$

and additionally $\mu(\{x\}) = 0$ for all $x \in \Gamma$.

We collect examples of such groups in Section 2. Our results follow from a structure result of Tucker-Drob [23] for this class of groups (Theorem 3.3) about the existence of q-normal subgroups, suggesting a strategy for proving properties for inner-amenable groups (Theorem 3.5).

Main results

Recall that a group is *virtually torsion-free* if there exists a torsion-free subgroup of finite index.

Theorem 1.2 (Theorem 6.4). *Let Γ be a finitely generated, virtually torsion-free, residually finite group that is inner-amenable and non-amenable. Then, Γ has the cheap 1-rebuilding property.*

In particular, we have the following corollary.

Corollary 1.3. *Let Γ be a finitely* presented*, virtually torsion-free, residually finite, inner-amenable group. Then, for every Farber sequence $(\Gamma_i)_{i\in\mathbb{N}}$ and every coefficient field $\mathbb{K}$, we have*

$$\lim_{i\to\infty} \frac{\dim_K H_1(\Gamma_i, \mathbb{K})}{[\Gamma : \Gamma_i]} = 0 \quad \text{and} \quad \lim_{i\to\infty} \frac{\log \left|H_1(\Gamma_i, \mathbb{Z})_{\mathrm{tors}}\right|}{[\Gamma : \Gamma_i]} = 0.$$

Proof. If Γ is amenable, the claims are already known: The second claim about torsion growth was proved by Kar, Kropholler and Nikolov [16, Theorem 1]. To derive the first claim, we can apply [19, Remark 1.3] and we obtain that it suffices to prove the claim for $K = \mathbb{Q}$. By Lück's approximation theorem [18], it suffices to show that ℓ^2-Betti numbers of amenable groups vanish (in degree 1) [7, Theorem 0.2].

If Γ is non-amenable, by Theorem 1.2, Γ has the cheap 1-rebuilding property. Because Γ is finitely presented, it is of type F_2. Thus, the corollary follows from the work of Abért, Bergeron, Frączyk and Gaboriau [1, Theorem 10.20]. ■

Note that we prove a stronger assertion for inner-amenable groups that are non-amenable. It is unknown if all amenable groups have the cheap 1-rebuilding property [1, Question 10.21].

By work of Chifan, Sinclair and Udrea, using ergodic-theoretic methods, it was already known that for all countable inner-amenable groups, the first ℓ^2-Betti number vanishes [8, Corollary D]. Later, Tucker-Drob proved that the *cost* of these groups is equal to 1 [23, Theorem 5], thus implying the same result. In this article, we obtain a more restrictive result, as we require the groups in question to be virtually torsion-free and residually finite. However, we can present a topological proof without using ergodic theoretic methods.

We point out that the proof of the amenable case in Corollary 1.3 only uses results that hold in all degrees (instead of just degree 1). It is thus natural to ask the following question.

Question 1.4. Let $\alpha \in \mathbb{N}$ and let Γ be a virtually torsion-free, residually finite, inner-amenable group of type F_α. Does Γ have the cheap α-rebuilding property?

If the answer should be negative, at least the analogue of Corollary 1.3 could be true.

Question 1.5. If, additionally, Γ is of type $F_{\alpha+1}$, does the following hold? For every Farber sequence $(\Gamma_i)_{i\in\mathbb{N}}$ and every coefficient field $\mathbb{K}$, we have

$$\lim_{i\to\infty} \frac{\dim_K H_n(\Gamma_i, \mathbb{K})}{[\Gamma : \Gamma_i]} = 0 \quad \text{and} \quad \lim_{i\to\infty} \frac{\log \left|H_n(\Gamma_i, \mathbb{Z})_{\mathrm{tors}}\right|}{[\Gamma : \Gamma_i]} = 0.$$

For right-angled Artin groups, we have a positive answer.

Proposition 1.6 (Corollary 7.3). *Let Γ be a right-angled Artin group. If Γ is inner-amenable, then Γ has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$. In particular, the conclusions of Question 1.5 hold for Γ.*

Organisation of this article

In Section 2, we collect many examples for inner-amenable groups.

Theorem 1.2 is proved in Section 6, using a strategy by transport through q-normality outlined in Section 3. For this, we need to construct actions out of q-normality (Section 4). We explain how to obtain such an action using a more general principle in Section 5.

We will prove Proposition 1.6 in Section 7, where we also give an outlook on why these results might not generalise to higher degrees.

In Appendix B, we recall the definition of the cheap α-rebuilding property.

2. Inner-amenable groups

Inner-amenable groups were originally defined by Effros [11], who showed that property Gamma implies inner amenability. Effros' question whether the converse holds was answered negatively by Vaes [24].

We define inner-amenable groups by the existence of an *atomless, conjugation-invariant* mean (see Definition 1.1). Note that this is the special case $H = G$ of Tucker-Drob's relative definition of inner amenability [23, p. 5].

Remark 2.1. The restriction to *atomless* means (instead of just means μ that satisfy $\mu(\{e\}) = 0$, as originally demanded by Effros) has the consequence that fewer groups are inner-amenable (e.g., finite groups are not inner-amenable in this sense), but also non-ICC groups [22, Definition 1.2] are not automatically inner-amenable. This is the case, e.g., in Stalder's article [22, below Definition 1.2]. For ICC groups (i.e., groups where every non-trivial conjugation class is infinite), the two notions coincide.

Similar to amenability, there is a characterisation in terms of Følner sequences.

Lemma 2.2 (Inner-Følner sequence [4, Théorème (F)]). *A countable group Γ is inner-amenable if and only if it admits an* inner-Følner sequence*, i.e., a sequence $(F_n)_{n\in\mathbb{N}}$ of finite, nonempty subsets of Γ with $\lim_{n\to\infty} |F_n| = \infty$ such that for all $\gamma \in \Gamma$,*

$$\lim_{n\to\infty} \frac{\left|(\gamma \cdot F_n \cdot \gamma^{-1}) \triangle F_n\right|}{|F_n|} = 0.$$

Note that the condition $\mu(\{e\}) = 0$ originally translates to the condition $e \notin F_n$. In our context, the atomlessness of the mean translates to the property that $|F_n| \to \infty$.

Example 2.3. We collect examples of inner-amenable groups given in the literature.

(1) *Infinite* amenable groups are inner-amenable (because we can choose bi-invariant means).

(2) Products $A \times \Gamma$, where A is inner-amenable, and Γ is any group, are inner-amenable [4, Corollaire 2 (iii)].

(3) Extensions $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$, where Γ' is inner-amenable and Γ'' is amenable [4, Corollaire 2 (iv)].

(4) The direct limits of inner-amenable groups are inner-amenable [4, Corollaire 2 (v)].

(5) Let $\Lambda \subseteq \Gamma$ be a finite-index subgroup. Then, Γ is inner-amenable if and only if Λ is [10, Proposition 2.7].

(6) All Baumslag-Solitar groups $\mathrm{BS}(m, n)$ (where $m, n \neq 0$) are inner-amenable [22, Exemple 3.2].

(7) If $H = \Lambda$ is abelian, then every HNN-extension $\mathrm{HNN}(\Lambda, H, K, \phi)$ is inner-amenable [22, Exemple 3.3].

(8) There is a criterion for the inner amenability of non-ascending HNN extensions [10, Theorem 1.2]. Two specific instances of this phenomenon are given by examples of Kida and Ozawa, where the associated subgroups are cyclic [17, Theorems 1.1 and 1.4].

(9) There is a criterion for the inner amenability of wreath products [10, Theorem 1.5].

(10) Groups that are (JS-)stable, McDuff or have property Gamma, are inner-amenable [9, Figure 1]. Sufficient conditions for stability can be found in the article by Tucker-Drob [23, Theorem 18, Corollary 19].

(11) Thompson's group F is inner-amenable [14]. In fact, it is even stable [23, Corollary 21].

(12) Thompson's groups T and V are *not* inner-amenable [13, Theorem 4.4].

(13) Nonabelian free groups are *not* inner-amenable [4, Corollaire 3 (iii)].

(14) Discrete ICC groups having property (T) are *not* inner-amenable [4, Corollaire 3 (i)].

Another large class of examples was pointed out by Francesco Fournier-Facio.

Example 2.4. Let Γ be a countable group with commuting conjugates; i.e., for every finitely generated subgroup $H \leq \Gamma$, there is $f \in \Gamma$ such that H commutes with fHf^{-1}. Then, Γ is inner-amenable.

Proof. We will show that Γ admits an inner-Følner sequence (Lemma 2.2). Because Γ is countable, let $\Gamma = \{\gamma_0, \gamma_1, \ldots\}$. Since Γ has commuting conjugates, for all $n \in \mathbb{N}$, there is $f_n \in \Gamma$ such that $\{\gamma_0, \ldots, \gamma_n\}$ commutes with $\{f_n \gamma_0 f_n^{-1}, \ldots, f_n \gamma_n f_n^{-1}\}$. In particular, this implies that $F_n := \{f_n \gamma_0 f_n^{-1}, \ldots, f_n \gamma_n f_n^{-1}\}$ defines an inner-Følner sequence. ■

Together with Example 2.3 (3), we obtain that the following groups are inner-amenable.

Example 2.5. Let Γ be a countable group with commuting conjugates, or a group extension of the form

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow K \longrightarrow 1,$$

where H is countable with commuting conjugates and K is amenable. Then, Γ is inner-amenable.

In particular, this includes countable groups of piecewise linear or piecewise projective homomorphisms of the real line [12, Proof of Theorem 1.3] such as Thompson's group F. More examples of this type can be found in the work of Fournier-Facio and Lodha [12].

A natural question to ask is whether the result of Fournier-Facio and Lodha that second bounded cohomology vanishes for group extensions as in Example 2.5 [12, Theorem 1.2] extends to all inner-amenable groups. Already the group $\mathbb{Z} \times F_2$ shows that this is *not* the case, as the second bounded cohomology $H_b^2(\mathbb{Z} \times F_2)$ retracts onto $H_b^2(F_2) \ncong 0$.

3. Transport through q-normality

In a recent article, Tucker-Drob proved a structure theorem for inner-amenable groups, which suggests a strategy for proving results about inner-amenable groups. The theorem guarantees the existence of q-normal subgroups. Before stating the strategy (Theorem 3.5), we recall the definition of q-normality, which was originally introduced by Popa [21, Definition 2.3].

Definition 3.1 (q-normal, q^*-normal [23, p. 2]). A subgroup $H \subset \Gamma$ is *q-normal* (resp. q^*-normal) if there is a generating set S of Γ, such that for all $s \in S$, the subgroup $sHs^{-1} \cap H$ is infinite (resp. non-amenable).

In this case, we write $H \leq_q \Gamma$ (resp. $H \leq_{q^*} \Gamma$).

Example 3.2. Infinite, normal subgroups are q-normal. Indeed, if $H \trianglelefteq \Gamma$, then we have $sHs^{-1} \cap H = H$ for all $s \in \Gamma$, so we can take any generating set as witness.

The converse is not true. For example, in the *Lamplighter group* $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$, the subgroup $\bigoplus_{\mathbb{N}} \mathbb{Z}$, indexed only over the natural numbers, is q-normal but not normal.

We can now state a variant of Tucker-Drob's structure theorem.

Theorem 3.3 (The structure of inner-amenable groups). *Let Γ be a finitely generated, virtually torsion-free, inner-amenable and non-amenable group. Then, there exist $g \in \Gamma \backslash \{e\}$ and finitely generated non-amenable subgroups $L \subseteq K \subseteq \Gamma$ such that*

$$\mathbb{Z} \cong \langle g \rangle \leq Z(L) \trianglelefteq L \leq_q K \leq_q \Gamma.$$

Here, $Z(L)$ denotes the centre of L.

Proof. We show how to deduce this result from the work of Tucker-Drob [23] (where we always choose $H = \Gamma$ and $\mathcal{F} = \{\Gamma\}$). The main effort goes into proving that a chain of length 3 suffices and that L and K can be assumed to be finitely generated. Moreover, we have to show that we can choose g to be of infinite order.

Let μ be an atomless, conjugation-invariant mean on Γ. Let $\Gamma' \subseteq \Gamma$ be a finite-index, torsion-free subgroup. We can assume $\mu(\Gamma') = 1$ [10, Proposition 2.3]. By a strengthening of a classical theorem, called Rosenblatt's theorem [13, Proposition 4.2], the following holds: The centraliser $C_\Gamma(g)$ is non-amenable for μ-almost all $g \in \Gamma$ [23, Lemma 4.2]. In particular, we find such an element $g \in \Gamma' \backslash \{e\}$. Because Γ' is torsion-free, we have $\langle g \rangle \cong \mathbb{Z}$. Since $C_\Gamma(g)$ is non-amenable, there is a finitely generated subgroup $L \subset C_\Gamma(g)$ that is non-amenable. We can suppose that $g \in L$. In particular, $\langle g \rangle$ is central in L.

Since L is a non-amenable subgroup of an inner-amenable group, we have that [23, Theorem 4.3 (i)]

$$L \leq_{q^*} \langle L \cup S_L \rangle \leq_q \Gamma, \quad \text{where } S_L := \big\{ g \in \Gamma \mid L \cap C_\Gamma(g) \text{ is non-amenable} \big\}.$$

Note that, in particular, q^*-normality implies q-normality. We will show how to pick a suitable finitely generated subgroup $K \subset \langle L \cup S_L \rangle$: Since $\langle L \cup S_L \rangle \leq_q \Gamma$ and Γ is finitely generated, we can pick a finite generating set $g_1, \dots, g_n$ of Γ such that for all $i \in \{1, \dots, n\}$, we have that

$$g_i \cdot \langle L \cup S_L \rangle \cdot g_i^{-1} \cap \langle L \cup S_L \rangle$$

is infinite. In particular, its intersection with the finite-index subgroup Γ' is still infinite and we can pick a non-trivial element in this intersection and write

$$g_i \cdot w_i \cdot g_i^{-1} = w_i', \tag{1}$$

where w_i and w_i' are words in $L \cup S_L \cup S_L^{-1}$. Let $\widetilde{S}_L$ be the (finite) set of letters in S_L that occur in any of the words $w_1, w_1', \dots, w_n, w_n'$. We set $K := \langle L \cup \widetilde{S}_L \rangle$. As L is finitely generated and $\widetilde{S}_L$ is finite, K is finitely generated. The relation in equation (1) witnesses that for all $i \in \{1, \dots, n\}$, the intersection $g_i \cdot K \cdot g_i^{-1} \cap K$ is non-trivial. Because the words w_i' were chosen in Γ', which is torsion-free, this intersection is infinite. This shows that $K \leq_q \Gamma$. Moreover, the generating set $L \cup \widetilde{S}_L$ witnesses that $L \leq_q K$. ∎

We illustrate this result with an example of a non-trivial chain of q-normal subgroups.

Example 3.4. Consider the inner-amenable and non-amenable group

$$\Gamma := F_2 \times F_2 \times \mathbb{Z} = \big\langle a, b, c, d, e \mid [a, c], [a, d], [b, c], [b, d], [a, e], [b, e], [c, e], [d, e] \big\rangle.$$

Let $g := a$ and $L := \langle a, c, d \rangle$, $K := \langle a, b^2, c, d \rangle$. Then,

$$\mathbb{Z} \cong \langle g \rangle \leq Z(L) \trianglelefteq L \leq_q K \leq_q \Gamma.$$

Note that L is *not* normal in K and K is *not* normal in Γ.

Theorem 3.3 suggests the following strategy for proving results about virtually torsion-free inner-amenable groups, which Tucker-Drob used to prove that inner-amenable groups are cheap and of fixed price [23, Theorem 5].

Theorem 3.5. *Let $\mathcal{C}$ be a class of finitely generated, torsion-free groups that is closed under taking finitely generated subgroups and contains the integers $\mathbb{Z}$. Let P be an (isomorphism) invariant defined for groups in $\mathcal{C}$. Suppose that the following two conditions hold:*

(1) *The integers $\mathbb{Z}$ satisfy P.*

(2) *Let $L, G \in \mathcal{C}$ such that $L \leq_q G$ (see Definition 3.1). Then, if L satisfies P, then so does G.*

Then, P holds for all groups in $\mathcal{C}$ that are inner-amenable but non-amenable .

Proof. Let Γ be a virtually torsion-free, finitely generated group in $\mathcal{C}$ that is inner-amenable and non-amenable. By Theorem 3.3, there exist $g \in \Gamma \backslash \{e\}$ and finitely generated subgroups $L \subseteq K \subseteq \Gamma$ such that

$$\mathbb{Z} \cong \langle g \rangle \leq Z(L) \trianglelefteq L \leq_q K \leq_q \Gamma.$$

Note that $\langle g \rangle \cong \mathbb{Z}$ satisfies P by assumption (1). Moreover, $\langle g \rangle$ is central in L, thus normal in L. In particular, it is also q-normal in L (Example 3.2). Thus, we have that $\langle g \rangle \leq_q L \leq_q K \leq_q \Gamma$. As L and K are finitely generated, $L, K \in \mathcal{C}$. Hence, we obtain that Γ satisfies P by applying assumption (2) three times. ■

4. An action induced by q-normality

In this section, we will explain how to construct an action out of q-normality. If $L \leq_q G$ is a q-normal subgroup, we want to construct an action of G on a graph whose vertex stabilisers are (isomorphic to) L.

Recall that $G/L := \{gL \mid g \in G\}$ is a transitive G-set, and an action $G \curvearrowright G/L$ is given by the left-translation of the cosets.

Lemma 4.1 (Blow up-action out of q-normality). *Let G be a finitely generated group and let $L \leq_q G$ be a q-normal subgroup. There is a cocompact action $G \curvearrowright (G/L, E)$ on a connected graph whose edge stabilisers are all infinite.*

Proof. We choose a generating set S of G as in Definition 3.1. Because G is finitely generated, we can assume S to be finite. We define

$$E := \big\{\{gL, gsL\} \mid g \in G,\ s \in S\big\}.$$

Then, E is closed under G-orbits (where the action is by left-translation) and by construction, the action is cocompact (the quotient has one 0-cell and $|S|$ many 1-cells). Moreover, because S is a generating set of G, the graph $(G/L, E)$ is connected. Finally, fix an edge $\{gL, gsL\}$. Its stabiliser contains

$$gLg^{-1} \cap (gs)L(gs)^{-1} = g \cdot (L \cap sLs^{-1}) \cdot g^{-1},$$

which is infinite by choice of S. ■

5. A different approach to actions induced by q-normality

In this section, we present an alternative proof of Lemma 4.1. We show how to get an appropriate action out of a "blow up" constructing, combining two actions. This exhibits an interesting technique and shows a more conceptual way how to obtain the desired action. We postpone all technical proofs to Appendix A.

If we have a q-normal subgroup, we can define the following action on a graph.

Lemma 5.1 (Action out of q-normality). *Let $L \leq_q G$ be a q-normal subgroup (see Definition 3.1). Then, we define a graph $\Omega := (V, E)$ where*

$$V = \{gLg^{-1} \mid g \in G\},$$

$$E = \left\{\{g_1Lg_1^{-1}, g_2Lg_2^{-1}\} \;\middle|\; \begin{array}{l} g_1, g_2 \in G,\ g_1Lg_1^{-1} \cap g_2Lg_2^{-1} \textit{ is infinite,} \\ g_1Lg_1^{-1} \neq g_2Lg_2^{-1} \end{array}\right\}.$$

Then, G acts on Ω by conjugation. The vertex set contains exactly one orbit. The stabiliser of these vertices is given by the normaliser $N_G(L)$ of L in G.

Moreover, the graph Ω is connected and for each $\{g_1Lg_1^{-1}, g_2Lg_2^{-1}\} \in E$, we have by construction that $g_1Lg_1^{-1} \cap g_2Lg_2^{-1}$ is infinite.

The vertex set consists of exactly one G-orbit, and the stabilisers of the vertices are conjugates of the normaliser $N_G(L)$ of L. Thus, as a G-set, V is isomorphic to $G/N_G(L)$ (where G acts by left-translation). Thus, we will view Ω as a graph with G-action and vertex set $G/N_G(L)$ in the following.

On the other hand, the normaliser $N_G(L)$ acts on the classifying space of the quotient $N_G(L)/L$.

Lemma 5.2 (Action on the quotient). *Let $L \trianglelefteq H$ be an infinite, normal subgroup. Then, H acts on $E(H/L)^{(1)}$, i.e., the 1-skeleton of a classifying space of the quotient H/L. Explicit constructions show that we can assume the 0-skeleton to be given by H/L. Moreover, $E(H/L)^{(1)}$ is connected and stabilisers are given by L, which is infinite by assumption.*

We can use these two actions to obtain the statement in Lemma 4.1 using the following method.

Lemma 5.3 (Blowing up actions). *Let G be a group and let $L \subseteq H \subseteq G$ be subgroups. Let $H \curvearrowright (H/L, E_{H/L})$ and $G \curvearrowright (G/H, E_{G/H})$ be actions on connected graphs, where the action on the vertex sets is given by left-translation. We suppose the following:*

- *In $(H/L, E_{H/L})$, all edge stabilisers are infinite.*
- *In $E_{G/H}$, for each G-orbit, we pick a representative $f \in E_{G/H}$ incident to eH. We denote by F the set containing the representatives we pick. For each $f \in F$, we choose $g(f) \in G$ such that $f = \{eH, g(f)H\}$ and we demand that $L \cap g(f)Lg(f)^{-1}$ is infinite.*

Then, there is an action $G \curvearrowright \Omega := (G/L, E)$, given by left-translation on the vertex set G/L, where Ω is a connected graph and each stabiliser of an edge is infinite.

Concretely, we note that G/L is isomorphic (as a G-set) to $G \times_H (H/L)$ via the isomorphism

$$\begin{aligned} G/L &\longrightarrow G \times_H H/L \\ gL &\longmapsto \big[(g, eL)\big] \end{aligned}$$

and we can thus define E as follows:

$$\begin{aligned} E := \big\{\{\big[(g, h_1L)\big], \big[(g, h_2L)\big]\} \mid g \in G,\ \{h_1L, h_2L\} \in E_{H/L}\big\} \\ \cup \big\{\{\big[(g, eL)\big], \big[\big(g \cdot g(f), eL\big)\big]\} \mid g \in G,\ f \in F\big\}. \end{aligned}$$

To explain the notation, we would like to recall the following. If $H \curvearrowright X$ is an action of H, we can induce an action of G as follows: We consider the quotient

$$G \times_H X := (G \times X)/\big(\forall_{g \in G, x \in X, h \in H}\ (gh, x) \sim (g, hx)\big),$$

which inherits a G-action by left-translation on the first component. We call this construction the *induction* of $H \curvearrowright X$ by G.

We would like to point out that the resulting action might not be cocompact. Because G is finitely generated, we can fix this issue easily. The idea was pointed out by Gaboriau.

Lemma 5.4 (Cocompactness). *Let G be finitely generated, $L \subseteq G$ be a subgroup and let $G \curvearrowright (G/L, E)$ be an action on a connected graph induced by left-translation on G/L. Then, there is a G-invariant subset $E' \subset E$ such that the action $G \curvearrowright (G/L, E')$ is cocompact and $(G/L, E')$ is connected.*

It is worth mentioning that in Lemma 5.3, we could also be more indifferent about the choice of edges of the seond type; more precisely, we could define the edge set E to be

$$\begin{aligned} E := &\left\{\left\{\left[(g, h_1 L)\right], \left[(g, h_2 L)\right]\right\} \mid g \in G,\ \{h_1 L, h_2 L\} \in E_{H/L}\right\} \\ &\cup \left\{\left\{\left[(g_1, eL)\right], \left[(g_2, eL)\right]\right\} \mid g_1, g_2 \in G,\ \{g_1 H, g_2 H\} \in E_{G/H}\right\}. \end{aligned}$$

However, with the choice as in Lemma 5.3, the resulting graph coincides with the following construction.

Remark 5.5. The graph $\Omega_{G/H} := (G/H, E_{G/H})$ is a G-CW complex of dimension 1, i.e., a pushout of the following type:

$$\begin{array}{ccc} \coprod_{I_1} G/H \times S^0 & \longrightarrow & G/H \\ \big\downarrow & & \big\downarrow \\ \coprod_{I_1} G/H \times D^1 & \longrightarrow & \Omega_{G/H} \end{array}$$

for some index set I_1. The complex Ω is then obtained by replacing G/H with the induction of

$$\Omega_{H/L} := (H/L, E_{H/L});$$

i.e., Ω is (the 1-skeleton of) a pushout as follows:

$$\begin{array}{ccc} \coprod_{I_1} (G \times_H \Omega_{H/L}) \times S^0 & \longrightarrow & G \times_H \Omega_{H/L} \\ \big\downarrow & & \big\downarrow \\ \coprod_{I_1} (G \times_H \Omega_{H/L}) \times D^1 & \longrightarrow & \tilde{\Omega}. \end{array}$$

In Lemma 5.3, we just give an explicit description of the resulting edge set.

This construction was inspired by a similar construction to build classifying spaces: If $N \trianglelefteq G$ is a normal subgroup, and the G-CW complex $E(G/N)$ is given, we can blow up by $G \times_N EN$ (where EN is a model for the classifying space of N) to obtain a model of EG. An instance of this method in a slightly different setting is elaborated in an article by Lück and Weiermann [20, Proof of Proposition 5.1].

6. Application to the cheap 1-rebuilding property

In this section, we will prove Theorem 1.2. For completeness, we provide the definition of the cheap 1-rebuilding property, which is a property of groups, in Appendix B. However, for the following proof, it suffices to understand the following two properties of this notion.

Proposition 6.1 ([1, Lemma 10.10]). *The group $\mathbb{Z}$ has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.*

Proposition 6.2 ([1, Example 10.12]). *Let Γ be a residually finite group that acts cocompactly on a graph Ω such that*

- *vertex stabilisers have the cheap 1-rebuilding property, and*
- *edge stabilisers are infinite.*

Then, Γ has the cheap 1-rebuilding property.

We can now prove the following generalisation of an inheritance property for normal subgroups [1, Corollary 10.13 (2)].

Lemma 6.3 (Transport through q-normality). *Let Γ be a finitely generated, residually finite group and $L \leq_q \Gamma$ a q-normal subgroup. If L has the cheap 1-rebuilding property, then so does Γ.*

Proof. It suffices to show the hypotheses of Proposition 6.2. By Lemma 4.1, Γ acts cocompactly on a connected graph $\Omega := (\Gamma/L, E)$ such that all edge stabilisers are infinite. All vertex stabilisers are conjugates of L and thus by assumption have the cheap 1-rebuilding property. ∎

Theorem 6.4. *Let Γ be a finitely generated, virtually torsion-free, residually finite group that is inner-amenable and non-amenable. Then, Γ has the cheap 1-rebuilding property.*

Proof. We apply Theorem 3.5 to the class of finitely generated, virtually torsion-free, residually finite groups and the property P “has the cheap 1-rebuilding property”. It suffices therefore to show that the two hypotheses are satisfied.

The group $\mathbb{Z}$ has the cheap 1-rebuilding property by Proposition 6.1. The second condition is satisfied by Lemma 6.3. ∎

Remark 6.5. Lemma 6.3 shows the cheap 1-rebuilding property for a larger class of groups. If Γ is a finitely generated, residually finite group and there exist finitely generated subgroups $G_0, \ldots, G_n$ such that

$$\mathbb{Z} \cong G_0 \leq_q \cdots \leq_q G_n = \Gamma,$$

then Γ has the cheap 1-rebuilding property.

Question 6.6. Which groups can be obtained via a sequence of q-normal subgroups as in the above remark?

This class contains all virtually torsion-free, inner-amenable and non-amenable groups by Theorem 3.3. This inclusion is strict, as e.g. Example 7.1 shows. Moreover, the following groups satisfy this condition, recovering a result of Abért, Bergeron, Frączyk and Gaboriau [1, Proposition 10.15] where an additional hypothesis of finite generation for some normalisers was necessary.

Corollary 6.7. *Let Γ be a residually finite group that is chain-commuting; i.e., there is a finite generating set $\{\gamma_1, \dots, \gamma_m\}$ of elements of infinite order such that for all $i \in \{1, \dots, m-1\}$, we have $[\gamma_i, \gamma_{i+1}] = e$. Then, Γ has the cheap 1-rebuilding property.*

Proof. We have that

$$\mathbb{Z} \cong \langle \gamma_1 \rangle \leq_q \langle \gamma_1, \gamma_2 \rangle \leq_q \cdots \leq_q \langle \gamma_1, \dots, \gamma_m \rangle = \Gamma.$$

Here, q-normality is witnessed by the given generating sets and the fact that because γ_i and γ_{i+1} commute, we have

$$\mathbb{Z} \cong \langle \gamma_i \rangle \subseteq \gamma_{i+1} \cdot \langle \gamma_1, \dots, \gamma_i \rangle \cdot \gamma_{i+1}^{-1} \cap \langle \gamma_1, \dots, \gamma_i \rangle.$$

Thus, Proposition 6.1 and repeated application of Lemma 6.3 imply that Γ has the cheap 1-rebuilding property. ∎

Examples of chain-commuting groups are right-angled Artin groups with connected nerve. As Damien Gaboriau pointed out to us, in fact all Artin groups with connected nerve are chain-commuting. They thus have the cheap 1-rebuilding property, provided that they are residually finite, recovering a special case of a result by Abért, Bergeron, Frączyk and Gaboriau [1, Theorem 10.17].

Corollary 6.8. *Let Γ be an Artin group with connected nerve. Then, Γ is chain-commuting. In particular, if Γ is residually finite, it has the cheap 1-rebuilding property.*

Proof. Since the nerve of Γ is connected, we can find a finite sequence of standard generators of Γ such that for all two subsequent generators s, t, the corresponding vertices in the nerve v_s, v_t are connected by an edge. Recall that this means that the Artin group $\langle s, t \rangle_\Gamma$ generated by s and t is spherical. This implies that the centre of $\langle s, t \rangle_\Gamma$ is infinite cyclic [5, Satz 7.2]. In particular, we can choose a generator $\gamma_{s,t}$ in this centre. Then, the sequence $(s, \gamma_{s,t}, t)$ is chain-commuting. Choosing a generator of the centre for every two subsequent standard generators, we obtain a set of generators of Γ that is chain-commuting.

We obtain that Γ has the cheap 1-rebuilding property by Corollary 6.7. ∎

7. Generalisations and outlook

In Question 1.4, we asked if the result of Theorem 1.2 generalises to higher degrees. We do not expect this to be the case. If it does, it would require new insights about inner-amenable groups. By Tucker-Drob's strategy (Theorem 3.5), a natural approach would be to prove an analogue of Lemma 6.3 in higher degrees. However, already in degree 2, this fails.

Example 7.1. We have that $\mathbb{Z} \trianglelefteq \mathbb{Z} \times F_2 \leq_q F_2 \times F_2$, and $\mathbb{Z}$ as well as $\mathbb{Z} \times F_2$ have the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$ [1, Lemma 10.10, Corollary 10.13 (2)]. The group $F_2 \times F_2$ does *not* have the cheap 2-rebuilding property because its second ℓ^2-Betti number is positive [1, Theorem 10.20]. Note that $F_2 \times F_2$ is *not* inner-amenable [4, Théorème 5] (alternatively, by Theorem 7.2).

However, the desired generalisation holds for right-angled Artin groups. For an introduction to these groups, we refer to a survey by Charney [6]. The following is known about the inner amenability of right-angled Artin groups.

Theorem 7.2 ([10, Corollary 4.21]). *A right-angled Artin group is inner-amenable if and only if it splits as a direct product with* $\mathbb{Z}$.

We obtain the following corollary.

Corollary 7.3. *Let* Γ *be a right-angled Artin group. If* Γ *is inner-amenable, then* Γ *has the cheap* α*-rebuilding property for all* $\alpha \in \mathbb{N}$. *In particular, also the conclusions of Question 1.5 hold for* Γ.

Proof. As Γ is inner-amenable, it splits as a direct product with $\mathbb{Z}$ by Theorem 7.2. We can then conclude by two lemmas of Abért, Bergeron, Frączyk and Gaboriau [1, Corollary 10.13 (2) and Lemma 10.10]. ∎

A. Proofs of Section 5

We present the missing proofs of Section 5.

Proof of Lemma 5.1. The only non-obvious claim is the connectedness of the graph. It suffices to show that for $g \in G$, there is a path from eLe^{-1} to gLg^{-1}. Because of q-normality (Definition 3.1), there is a generating set S of G such that for all $s \in S$, the set $sLs^{-1} \cap L$ is infinite. We pick $s_1, \dots, s_n \in S$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ such that $g = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$. Using that cardinalities are conjugation-invariant, i.e., if $sLs^{-1} \cap L$ is infinite, then also for all $g' \in G$, we have that $g'sLs^{-1}g'^{-1} \cap g'Lg'^{-1}$ is infinite, and we obtain that

$$eLe^{-1}, s_1^{\varepsilon_1} L s_1^{-\varepsilon_1}, \dots, s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} L s_n^{-\varepsilon_n} \cdots s_1^{-\varepsilon_1} = gLg^{-1}$$

is a path in Ω. ∎

Proof of Lemma 5.3. By construction, we obtain an action $G \curvearrowright \Omega := (G \times_H H/L, E)$ by left-translation on the vertex set. This graph is connected: Let $[(g, eL)] \in G \times_H H/L$. Note that for all $f \in F$, we obtain the edges $\{[(g, eL)], [(g \cdot g(f), eL)]\}$ for all $g \in G$. In particular, since $(G/H, E_{G/H})$ is connected, we obtain a path from $[(g, eL)]$ to $[(h, eL)]$ for some $h \in H$. Now, because $(H/L, E_{H/L})$ is connected, we can find a path from $[(h, eL)] = [(e, hL)]$ to $[(e, eL)]$. Concatenating these two paths shows that the graph Ω is connected.

It remains to prove that edge stabilisers are infinite:

- Let $\{[(g, h_1 L)], [(g, h_2 L)]\} \in E$ with $g \in G$, $\{h_1 L, h_2 L\} \in E_{H/L}$. Because edge stabilisers in $(H/L, E_{H/L})$ are infinite, there are infinitely many $h \in H$ that fix $\{h_1 L, h_2 L\}$. For all these $h \in H$, its conjugate $g \cdot h \cdot g^{-1}$ fixes $\{[(g, h_1 L)], [(g, h_2 L)]\}$. Thus, the stabiliser of this edge is infinite.
- Let $g \in G$, $f \in F$ and consider the edge $\{[(g, eL)], [(g \cdot g(f), eL)]\}$. This edge is fixed by $g \cdot (L \cap g(f) \cdot L \cdot g(f)^{-1}) \cdot g^{-1}$ which, by the second hypothesis, is an infinite set. ∎

Proof of Lemma 5.4. Because G is finitely generated, we can pick a finite generating set $g_1, \ldots, g_n$ of G. Because $(G/L, E)$ is connected, we can pick paths from eL to $g_i L$ for all $i \in \{1, \ldots, n\}$. We then define E' to be the union of the G-orbits of all edges occuring in one of these paths. ∎

B. Cheap α-rebuilding property

For completeness, we include the definition of the cheap α-rebuilding property, quality of rebuildings and Farber neighbourhoods. We refer to the work of Abért, Bergeron, Frączyk and Gaboriau [1] for more details and examples.

Definition B.1 (Rebuilding [1, Definition 1]). Let $\alpha \in \mathbb{N}$ and let Y be a CW-complex with finite α-skeleton. An α-*rebuilding* of Y is a tuple $(Y, Y', \mathbf{g}, \mathbf{h}, \mathbf{P})$, consisting of the following data:

(1) Y' is a CW-complex with finite α-skeleton,

(2) $\mathbf{g}\colon Y^{(\alpha)} \to Y'^{(\alpha)}$ and $\mathbf{h}\colon Y'^{(\alpha)} \to Y^{(\alpha)}$ are cellular maps that are homotopy inverse to each other up to dimension $\alpha - 1$, i.e., $\mathbf{h} \circ \mathbf{g}_{\restriction Y^{(\alpha-1)}} \simeq \mathrm{id}_{\restriction Y^{(\alpha-1)}}$ within $Y^{(\alpha)}$ and $\mathbf{g} \circ \mathbf{h}_{\restriction Y'^{(\alpha-1)}} \simeq \mathrm{id}_{\restriction Y'^{(\alpha-1)}}$ within $Y'^{(\alpha)}$, and

(3) a cellular homotopy $\mathbf{P}\colon [0,1] \times Y^{(\alpha-1)} \to Y^{(\alpha)}$ between the identity and $\mathbf{h} \circ \mathbf{g}$, i.e., $\mathbf{P}(0, \cdot) = \mathrm{id}_{\restriction Y^{(\alpha-1)}}$ and $\mathbf{P}(1, \cdot) = \mathbf{h} \circ \mathbf{g}_{\restriction Y^{(\alpha-1)}}$.

Definition B.2 (Quality of a rebuilding [1, Definition 2]). Given real numbers $T, \kappa \geq 1$, we say that an α-rebuilding $(Y, Y', \mathbf{g}, \mathbf{h}, \mathbf{P})$ is of *quality* (T, κ) if we have for all $j \leq \alpha$

$$|X'^{(j)}| \leq \kappa T^{-1} |X^{(j)}|, \qquad \text{(cells bound)}$$

$$\log \|g_j\|, \log \|h_j\|, \log \|\rho_{j-1}\|, \log \|\partial'_j\| \leq \kappa(1 + \log T), \qquad \text{(norms bound)}$$

where $|\cdot|$ denotes the number of cells and ∂' is the cellular boundary map on Y', g and h are the chain maps respectively associated with $\mathbf{g}$ and $\mathbf{h}$, $\rho\colon C_\bullet(Y) \to C_{\bullet+1}(Y)$ is the chain homotopy induced by $\mathbf{P}$ in the cellular chain complexes:

$$\begin{array}{ccccccccc}
C_\alpha(Y) & \xrightarrow{\partial_\alpha} & \cdots & & \cdots \longrightarrow & C_1(Y) & \xrightarrow{\partial_1} & C_0(Y) \\
g_\alpha \downarrow\uparrow h_\alpha & \swarrow \rho_{\alpha-1} & & & \swarrow \rho_1 & g_1 \downarrow\uparrow h_1 & \swarrow \rho_0 & g_0 \downarrow\uparrow h_0 \\
C_\alpha(Y') & \xrightarrow{\partial'_\alpha} & \cdots & & \cdots \longrightarrow & C_1(Y') & \xrightarrow{\partial'_1} & C_0(Y'),
\end{array}$$

and the norms $\|\cdot\|$ are the canonical ℓ^2-norms on the cellular chain complexes given by the basis of open cells.

Definition B.3 ([1, Section 10.1]). Let Γ be a countable group and let $\mathrm{Sub}_\Gamma^{\mathrm{fi}}$ denote the space of finite-index subgroups of Γ with the topology induced from the topology of pointwise convergence on $\{0,1\}^\Gamma$. For $\gamma \in \Gamma$, we consider the following function:

$$\mathrm{fx}_{\Gamma,\gamma} : \mathrm{Sub}_\Gamma^{\mathrm{fi}} \longrightarrow [0,1], \quad \Gamma' \longmapsto \frac{|\{g\Gamma' \mid \gamma g\Gamma' = g\Gamma'\}|}{[\Gamma : \Gamma']}.$$

We also recall the definition of a Farber sequence.

Definition B.4 (Farber sequence [1, Definition 10.1]). A sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subgroups of Γ is a *Farber sequence* if it consists of finite-index subgroups and for every $\gamma \in \Gamma\setminus\{e\}$, we have $\lim_{n\to\infty} \mathrm{fx}_{\Gamma,\gamma}(\Gamma_n) = 0$.

Note that Farber sequences exist if and only if Γ is residually finite. The most common example of Farber sequences is *residual chains*, i.e., nested sequences of finite-index normal subgroups whose intersection is trivial.

Definition B.5 (Farber neighbourhood [1, Definition 10.2]). Let Γ be a residually finite group. An open subset $U \subseteq \mathrm{Sub}_\Gamma^{\mathrm{fi}}$ is a Γ-*Farber neighbourhood* if it is invariant by the conjugation action of Γ on $\mathrm{Sub}_\Gamma^{\mathrm{fi}}$ and every Farber sequence in $\mathrm{Sub}_\Gamma^{\mathrm{fi}}$ eventually belongs to U.

Finally, we can define the cheap α-rebuilding property.

Definition B.6 (Cheap α-rebuilding property [1, Definition 10.5]). Let Γ be a countable group and $\alpha \in \mathbb{N}$. Then, Γ has the *cheap α-rebuilding property* if it is residually finite and there is a $K(\Gamma,1)$-space X with finite α-skeleton and a constant $\kappa_X \geq 1$ such that the following holds: For every real number $T \geq 1$, there exists a Farber neighbourhood $U = U(X,T) \subset \mathrm{Sub}_\Gamma^{\mathrm{fi}}$ such that for every finite covering $Y \to X$ with $\pi_1(Y) \in U$, there is an α-rebuilding (Y, Y') of quality (T, κ_X).

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