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Compatible pants decomposition for $SL_2(\mathbb{C})$ representations of surface groups

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Abstract. For any irreducible representation of a surface group into $SL_2(\mathbb{C})$, we show that there exists a pants decomposition where the restriction to any pair of pants is irreducible and where no curve of the decomposition is sent to a trace ± 2 element. We prove a similar property for SO_3-representations. We also investigate the type of pants decomposition that can occur in this setting for a given representation. This result was announced by Detcherry and Santharoubane (2022), motivated by the study of the Azumaya locus of the skein algebra of surfaces at roots of unity.

1. Introduction

Let Σ be a compact connected oriented surface (without boundary) of genus at least two. Let $\rho : \pi_1(\Sigma) \to G = SL_2(\mathbb{C})$ or SO_3 be a group homomorphism. We start with the following definition.

Definition 1.1. A pants decomposition $\mathcal{P}$ of Σ is called compatible with ρ if for any curve $c \in \mathcal{P}$, the elements $\pm\rho(c)$ are not unipotent and for any pants P in $\mathcal{P}$, the restriction $\rho|_{\pi_1(P)}$ is irreducible.

The purpose of this paper is to prove that any irreducible representation of $\pi_1(\Sigma)$ into $SL_2(\mathbb{C})$ or SO_3 admits a compatible pants decomposition. Remark that in the case of a $SL_2(\mathbb{C})$ representation, the compatibility condition can be translated into a condition on the traces of the curves of the pants decomposition. The first condition is equivalent to $\mathrm{Tr}(\rho(c)) \neq \pm 2$, and the second is $x^2 + y^2 + z^2 - xyz - 4 \neq 0$, where x, y, z are the traces of boundary curves of a pair of pants in the decomposition. One type of pants decomposition important for us is the sausage type which is a pants decomposition in the same orbit, under the action of the mapping class group of Σ, as the one shown in Figure 1.

Theorem 1.2. *Let $\rho : \pi_1(\Sigma) \to G = SL_2(\mathbb{C})$ or SO_3 be an irreducible representation. Then there is a compatible pants decomposition of Σ for ρ. Moreover, if ρ is a representation whose image is not conjugated to the quaternion $Q_8 \subset SU_2$, then there is a compatible pants decomposition for ρ of sausage type.*

The second part of this theorem was announced in [4, Theorem 1.5] without a proof. The sausage type pants decomposition is very important in [4] and Theorem 1.2 is key

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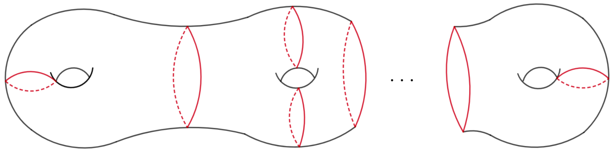

Figure 1. A sausage type pants decomposition of Σ.

result to understand the Azumaya locus of the skein algebra of Σ at roots of unity. Notice that the existence of compatible pants decomposition for non-elementary representations in $SL_2(\mathbb{C})$ is a key step used by Gallo–Kapovich–Marden in [7] to prove that holonomies of $\mathbb{C}P^1$-structures are Zariski dense in the $SL_2(\mathbb{C})$-character variety of a given surface. It is quite intriguing that the exact same condition appeared in [4] in the context of quantum topology.

A result of Baba [2] shows that the pants decomposition compatible with a non-elementary representation ρ arising from [7] enables us to construct explicitly all the projective structures with holonomy ρ. Our result for representations in SO_3 might also be used to describe the branched spherical structures with given holonomy.

The question of finding compatible pants decomposition also emerged in the first author's thesis, motivated by Witten's asymptotic expansion conjecture. This conjecture expresses the asymptotics of Witten–Reshetikhin–Turaev invariants of a 3-manifold M as a sum of contributions associated with SU_2 representations of $\pi_1(M)$ and involving Chern–Simons invariants and Reidemeister torsions. The first author conjectures that the geometric quantization techniques from [3] may be used to estimate the contribution of representations that admit a compatible pants decomposition.

The proof of Theorem 1.2 is split in several steps. In Section 2, we deal with representations which are non-elementary, representations with dense images in SU_2 and representations with images in a non-compact dihedral group. For the non-elementary case, the existence of compatible pants decomposition is already proved in [7], and we adapt these techniques to get the sausage type pants decomposition. For the case of a representation with dense image in SU_2, Theorem 1.2 is a direct application of Previte–Xia's result (see [12]) that proved that any such representation has a dense orbit in the SU_2 character variety under the action of the mapping class group. The last case is dealt by hand. In Section 3, we treat the remaining cases, namely, representations with finite images. Such representations are classified; the proof is done by studying the orbits under the mapping class group and building explicit compatible pants decomposition for each orbit.

2. Reduction to representations with finite image

In this section, we reduce the proof of Theorem 1.2 to the case of representations ρ with finite images. More precisely, we show Theorem 1.2 assuming that Proposition 3.1 holds. This proposition will be proven in Section 3.

2.1. Mapping class group action

Let us begin with some observations on the mapping class group action on sets of representations that will be used throughout the rest of the paper. Recall that the mapping class group of Σ is defined by

$$\mathrm{Mod}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0^+(\Sigma).$$

The theorem of Dehn, Nielsen and Baer, see for example [6, Chapter 8], states that its natural action on the fundamental group induces a group isomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Out}^+(\Sigma)$ on the index two subgroup $\mathrm{Out}^+(\Sigma)$ of $\mathrm{Out}(\Sigma) = \mathrm{Aut}(\Sigma)/\mathrm{Inn}(\Sigma)$ induced by the automorphisms preserving orientation. Therefore, $\mathrm{Mod}(\Sigma)$ acts by precomposition as $\mathrm{Out}^+(\pi_1(\Sigma))$ on the space of conjugacy classes of representations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, for any group G.

A key observation to prove Theorem 1.2 is that this action preserves the set of representations admitting a compatible pants decomposition. If $\rho\colon \pi_1(\Sigma) \to G$ is a representation, we denote by $[\rho]$ its conjugacy class.

Lemma 2.1. *Suppose that ρ_∞ admits a compatible pants decomposition $\mathcal{P}$. If the conjugacy class $[\rho_\infty]$ of ρ_∞ is in the closure of* $\mathrm{Mod}(\Sigma) \cdot [\rho]$*, then ρ admits a compatible pants decomposition of the type of $\mathcal{P}$.*

Proof. For a given pants decomposition P of Σ, let us denote by $\mathcal{C}(\mathrm{P})$ the set of conjugacy classes of representations

$$\pi_1(\Sigma) \to G$$

compatible with P. It follows from the definition of compatibility that these sets are open. Therefore, there exists a representation in $(\mathrm{Mod}(\Sigma) \cdot [\rho]) \cap \mathcal{C}(\mathcal{P})$. Hence, there exists $f \in \mathrm{Mod}(\Sigma)$ such that $f \cdot [\rho] \in \mathcal{C}(\mathcal{P})$ and therefore $[\rho] \in \mathcal{C}(f^{-1} \cdot \mathcal{P})$. ∎

We will thus prove Theorem 1.2 by studying the orbits of representations $\pi_1(\Sigma) \to SL_2(\mathbb{C})$. We will use different methods depending on the image of the representation we wish to study.

2.2. Non-elementary case

Let us begin with the case where ρ is a non-elementary representation; i.e., the action of its image on the Riemann sphere $\mathbb{CP}^1$ by Möbius transformations has no finite orbit. We can in that case adapt the strategy of Gallo, Kapovich and Marden in [7, Part A] to find a compatible pants decomposition.

Proposition 2.2. *Let $\rho\colon \pi_1(\Sigma) \to PSL_2(\mathbb{C})$ be a non-elementary representation. For any trivalent graph Γ with $3g-3$ edges that has at least one one-edge loop, there is a pants decomposition of Σ, with associated graph Γ which is compatible with ρ.*

Proof. Let us begin by recalling the main steps of the construction by Gallo, Kapovich and Marden in [7] of a Schottky pants decomposition for ρ: a pants decomposition such that

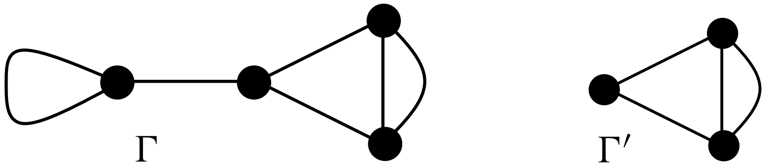

Figure 2. Example of a trivalent graph Γ and Γ'.

the restriction of ρ to each pair of pants is an isomorphism onto a Schottky group. Note that this construction of Gallo, Kapovich and Marden works for every non-elementary representation $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ except in genus $g = 2$ for the pentagon representations. We suppose for now that ρ is not a pentagon representation.

The first step is to find special handle in Σ. That is a handle $\mathcal{H}$ whose fundamental group $\pi_1(\mathcal{H})$ is sent by ρ onto a non-elementary subgroup of $\mathrm{SL}_2(\mathbb{C})$. This handle $\mathcal{H}$ allows us to find $g-1$ disjoint simple curves away from it, which are sent by ρ to loxodromic elements. Cutting the surface along those curves leads to a genus one surface with $2g-2$ boundary components. Choosing any two of these components, we find a curve separating them from the rest of the surface, and such that the restriction of ρ to the pair of pants they bound is an isomorphism onto a Schottky group. Cutting along this curve takes off a pair of pants and gives a genus one surface with $2g-3$ boundary components. We repeat the same procedure until we get a genus one surface with two boundary components where a special cutting process is applied to get a Schottky pants decomposition. This process finds a curve bounding the two boundary components and cuts the handle. We now show that choosing wisely the curves at each step allows us to create a decomposition with any trivalent graph with $3g-3$ edges and at least one one-edge loop. We thus are reduced to the following combinatorial lemma.

Lemma 2.3. *Any trivalent graph* Γ *with* $3g-3$ *edges that has at least one one-edge loop can be created by this procedure.*

Proof. We start from a genus one surface Σ with $2g-2$ boundary components that come from cutting $g-1$ curves from a closed surface. Let us label the boundary components by integers $1 \leqslant k \leqslant g-1$ with the same label if they come from cutting the same curve. Pick a one-edge loop e. Let us consider the graph Γ' that is the graph Γ with e and all the edges connected to it removed, as for example in Figure 2.

Let us cut Γ' along $g-1$ edges that do not disconnect it. We obtain a new graph Γ' with $2g-2$ boundary components that we label with integers $1 \leqslant k \leqslant g-1$. We require that two boundaries have the same label if they come from the same edge; see the left side of Figure 3.

The graph Γ' has $2g-2$ boundary components and $2g-3$ vertices. Therefore, one of the vertices has two boundary components; let us choose such a vertex f. Denote by x and y the labels of the boundary components next to f. Let us cut the last edge joining f to the rest of the graph and remove f from Γ'. We give a new label z to the resulting

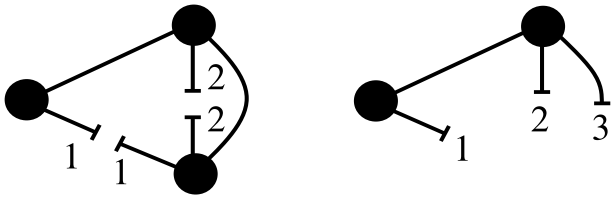

Figure 3. The graph Γ' in two of the steps.

boundary component; see the right side of Figure 3. In the surface Σ, we pick a curve that bounds the curves labeled by x and y and take off the pair of pants it defines. We label by z the new boundary curve.

Repeat this procedure until Γ' has only one vertex, with two boundary components. Note that at each step, the number of vertices decreases by one, and so does the number of boundary components of Γ'. We are left with Σ of genus one surface with two boundary components. We then apply the special cutting procedure to finish creating the pants decomposition. This pants decomposition is isomorphic to Γ by construction. ■

We now deal with the special case in genus two where ρ is a pentagon representation. By [10, Proposition 5.1], we can make the mapping class group act so that the sausage type pants decomposition of Figure 6 satisfies the following:

- One of the pair of pants is sent by ρ isomorphically onto a Schottky group.
- The other pair of pants is sent by ρ to a non-elementary group and each of its curves is sent by ρ to a loxodromic element.

This decomposition is thus compatible with ρ. ■

We have proven Proposition 2.2 for representation with values in $PSL_2(\mathbb{C})$, which is stronger than the same result for representation in $SL_2(\mathbb{C})$. In particular, the pentagon representations are not representation $\pi_1(\Sigma) \to SL_2(\mathbb{C})$: they do not lift to $SL_2(\mathbb{C})$.

2.3. Elementary case

We now turn to the case where $\rho\colon \pi_1(\Sigma) \to PSL_2(\mathbb{C})$ is elementary: the action of its image on $\mathbb{CP}^1$ has finite orbits. It is known, see for example [13, Chapter 5], that ρ falls in one of the following three categories:

(1) ρ is affine: it has a conjugate into the upper trianglar matrices.

(2) ρ is spherical: it has a conjugate into the group $PSU_2 = SO_3$ that preserves the round metric of $\mathbb{CP}^1 = \mathbb{S}^2$.

(3) ρ is dihedral: it has a conjugate into the group D of matrices that are either diagonal or that have their two diagonal entries vanishing.

Observe that affine representations are reducible. We will therefore only consider representations that are either spherical or dihedral. We begin with the case where ρ is spherical.

2.3.1. Spherical case. We now show Theorem 1.2 for representation with infinite image in $\mathrm{SO}_3 = \mathrm{PSU}_2$.

Proposition 2.4. *Let $\rho\colon \pi_1(\Sigma) \to \mathrm{PSU}_2$ be a representation with infinite non-abelian image. There is a pants decomposition of Σ of sausage type which is compatible with ρ.*

Let us begin by recalling the description of the closed subgroups of SU_2. Let us denote by K the subgroup of SU_2 generated by the diagonal matrices and the matrices with both diagonal entries vanishing. In other words, $K = \mathrm{SU}_2 \cap D$.

Lemma 2.5. *Let H be a subgroup SU_2. One of the following holds:*

(1) *H is finite.*

(2) *H has a conjugate into the diagonal matrices.*

(3) *H has a conjugate dense in K.*

(4) *H is dense in SU_2.*

Proof. Replacing H by its closure if necessary, we may assume that H is closed and is a Lie subgroup of SU_2. Let us recall that the Lie algebra $\mathfrak{su}(2)$ is isomorphic to $\mathbb{R}^3$ endowed with the cross-product. This Lie algebra does not admit any two-dimensional subalgebra. Hence, the Lie algebra $\mathfrak{h}$ of H must be of dimension 0, 1 or 3.

- If $\dim \mathfrak{h} = 0$, then H is a discrete group, and since SU_2 is compact, H is finite.
- If $\dim \mathfrak{h} = 3$, then H is the whole connected group SU_2.
- If $\dim \mathfrak{h} = 1$, then let us conjugate H such that its connected component containing the identity is
$$\mathrm{Diag} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$
Let $h \in H$. The group $h \,\mathrm{Diag}\, h^{-1}$ is connected and contains the identity and hence is included in H. Therefore, h must send the eigenvectors of matrices of Diag to themselves. Let us denote by (e_1, e_2) the canonical basis of $\mathbb{C}^2$. The vector he_1 is an eigenvector of a matrix in Diag and thus is in either $\mathbb{R}e_1$ or $\mathbb{R}e_2$. Hence, h is either diagonal or has the form $\left(\begin{smallmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{smallmatrix}\right)$. ∎

It follows that the infinite subgroups of $\mathrm{SO}_3(\mathbb{R}) = \mathrm{PSU}_2$ are, after conjugation, either abelian, dense in $\mathrm{O}_2(\mathbb{R})$ or dense in $\mathrm{SO}_3(\mathbb{R})$. The abelian representations are reducible. Therefore, we will only consider the two other cases. To prove Proposition 2.4, it suffices to show that in the closure of the orbit of such a ρ, there exists a representation with finite non-abelian image. Indeed by Lemma 2.1 and Proposition 3.1, the representation ρ then admits a compatible pants decomposition of sausage type. Proposition 2.4 follows from the following lemma.

Lemma 2.6. *Every representation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSU}_2$ with infinite non-abelian image admits a representation with finite non-abelian image in the closure of its $\mathrm{Mod}(\Sigma)$-orbit.*

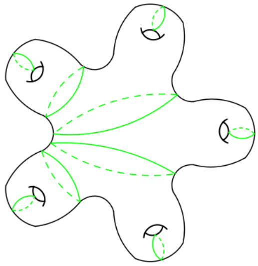

Figure 4. Part of a sausage type pants decomposition.

We will actually show that for representations in SO_3 with dense image, we can obtain any trivalent graph Γ.

Lemma 2.7. *For every trivalant graph Γ with $3g-3$ edges and every representation $\rho\colon \pi_1(\Sigma) \to SO_3$ with dense image, there exists a compatible pants decomposition whose graph is Γ.*

Proof. The mapping class group orbit of ρ is dense in its connected component of

$$\mathrm{Hom}\big(\pi_1(\Sigma), SO_3(\mathbb{R})\big)/SO_3(\mathbb{R}).$$

This is a consequence of the main result of [12]; see also [9, Lemma 4.4]. Let us recall that this connected component is the set of all representations $\pi_1(\Sigma) \to PSU_2 = SO_3(\mathbb{R})$ that lift to SU_2 if ρ does (resp. that do not lift if ρ does not). Each of these components admits at least one representation with a compatible pants decomposition of graph Γ; see [8] and [11, Section 6]. The result follows from Lemma 2.1. ∎

We now prove Lemma 2.6 for the representation $\pi_1(\Sigma) \to SO_3$ whose image is not dense in SO_3.

Proof. We are left with the case where ρ has its image dense in $O_2(\mathbb{R})$. Its $\mathrm{Mod}(\Sigma)$-orbit is dense in the representations $\pi_1(\Sigma) \to O_2(\mathbb{R})$ that are not abelian and lift to SU_2 if ρ does (resp. do not lift if ρ does not); see [9, Proposition 4.2]. These sets contain representations with finite non-abelian image. For example, we can take a representation with image as a finite dihedral group. ∎

2.3.2. Dihedral case. We now turn to the case where ρ is dihedral, that is, when it has image in D. The group D is naturally isomorphic to $\mathbb{C}^\star \rtimes \mathbb{Z}_2$. Let us denote by $p\colon \mathbb{C}^\star \rtimes \mathbb{Z}_2 \to \mathbb{Z}_2$ the projection on the second factor.

Proposition 2.8. *Let $\rho\colon \pi_1(\Sigma) \to H$ be an irreducible representation with infinite non-abelian image. There exists a decomposition of sausage type which is compatible with ρ.*

Fix $a_1, \dots, b_g$ a standard generating set for $\pi_1(\Sigma)$. Let us observe that the curves freely homotopic to a_i for $1 \leqslant i \leqslant g$ and the ones freely homotopic to $\prod_{i=k}^{g}[a_i, b_i]$ can be completed in a sausage type pants decomposition; see Figure 4.

Proof. We have proven this result when ρ takes its values in SU_2. Therefore, we now assume that ρ does not admit a conjugate in SU_2.

It suffices to prove that we can assume that the two following conditions are met, after precomposing ρ with an automorphism of $\pi_1(\Sigma)$:

(1) $p \circ \rho(a_i) = 1$ for all $1 \leqslant i \leqslant g$;

(2) $\rho\big(\prod_{i=k}^{g}[a_i, b_i]\big)$ is loxodromic for $1 < k \leqslant g$.

Indeed we have seen that the curve freely homotopic to these loops forms part of a pants decomposition of sausage type. If ρ satisfies these two conditions, then the restriction of ρ to each pair of pants is not reducible: the curve homotopic to $\prod_{i=k}^{g}[a_i, b_i]$ preserves only the lines generated by the vectors of the standard basis of $\mathbb{C}^2$, while the one homotopic to a_i does not preserve any of them. Moreover, two of the curves of each pair of pants have zero entries on the diagonal and hence are not central. The third one is non-central as well because it is loxodromic.

We now prove that we may assume that those two conditions are met, with an action of the mapping class group.

The mapping class group acts transitively on the set of epimorphisms $\pi_1(\Sigma) \to \mathbb{Z}_2$; therefore as before we may assume that $p \circ \rho(\gamma) = 0$ for all $\gamma \in \{a_1, b_1, \dots, b_g\} \setminus \{b_1\}$ and $p \circ \rho(\gamma) = 1$ for $\gamma = b_1$. As before we can conjute ρ by a diagonal matrix so that

$$\rho(b_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrices $\rho(a_1)$ and $\rho(b_1)$ must commute. Hence,

$$\rho(a_1) \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Note that the restriction $\rho_{|\Sigma_{g-1}}$ of ρ to the subsurface obtained by removing the handle containing a_1 and b_1 takes its values in $\ker p \simeq \mathbb{C}^\star$. Let $f = \log|\rho_{\Sigma_{g-1}}| \colon \pi_1(\Sigma) \to \mathbb{R}$. This homomorphism is not trivial since ρ does not take its values in SU_2. We may act by the mapping class group of Σ_{g-1} to make $f(a_g)$ non-zero: we may exchange the handles so that either $f(a_g)$ or $f(b_g)$ does not vanish. Then applying a Dehn twist along b_g if necessary, we may assume that $f(a_g) \neq 0$. Applying a power Dehn twist along a_g, we may assume that $f(b_g) + \sum_{i=1}^{g-1} f(b_i) > 1$.

Let $\gamma = b_1 \dots b_{g-1} b_g$. Let us apply a Dehn twist along a curve freely homotopic to γ. It leaves unchanged each b_i and changes each a_i so that $p \circ \rho(a_i) = 1$. Moreover,

$$\prod_{i=k}^{g} [\rho(a_i), \rho(b_i)] = \prod_{i=k}^{g} \left[\begin{pmatrix} 0 & \mu_i \\ -\mu_i^{-1} & 0 \end{pmatrix}, \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \right] = \prod_{i=k}^{g} \left[\begin{pmatrix} \lambda_i^{-2} & 0 \\ 0 & \lambda_i^{2} \end{pmatrix} \right].$$

Therefore, we have

$$\rho\left(\prod_{i=k}^{g} [a_i, b_i] \right) = \begin{pmatrix} \prod_{i \geqslant k} \lambda_i^{-2} & 0 \\ 0 & \prod_{i \geqslant k} \lambda_i^{2} \end{pmatrix}$$

that is loxodromic for every $1 < k \leqslant g$. ■

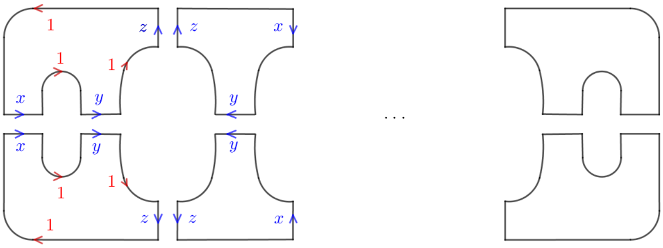

Figure 5. The top half of a cellular decomposition of Σ_g associated with the square chain dual graph, and an associated cocycle c. The cocycle c has value 1 on all edges of the lower copy of the dual graph.

3. Representations onto finite subgroups of SU_2 or SO_3

We recall that pants decomposition of sausage was introduced in Figure 1. We will also consider pants decomposition of *square type*, to be the ones whose dual graph is represented in Figure 5. This section is devoted to the proof of the following proposition.

Proposition 3.1. *Let Σ be a closed compact oriented surface.*

- *Let G be a nonabelian finite subgroup of* SO_3 *and $\rho : \pi_1(\Sigma) \to G$ an epimorphism. Then there is a pants decomposition of Σ of sausage type, and there is one of square type, which are compatible with ρ.*
- *Let $\rho : \pi_1(\Sigma) \to Q_8$ be an epimorphism. Then there is a pants decomposition of Σ of square type, which is compatible with ρ.*

To exhibit such appropriate epimorphisms into finite groups, we will define them using cocycles on a cellular decomposition of the surface, which we define in Section 3.1.

3.1. Representations as holonomy of cocycles

For X a CW-complex, we write $\mathcal{C}^i(X)$ for the set of its oriented i-dimensional cells.

Definition 3.2. Let X be a CW-complex and G a group. A G-cocycle on X is a map $c : \mathcal{C}^1(X) \to G$ that satisfies the following properties:

- For any oriented edge $e \in \mathcal{C}^1(X)$, we have $c(\bar{e}) = c(e)^{-1}$ where $\bar{e}$ is the edge e with opposite orientation.
- For any 2-cell $w \in \mathcal{C}^2(X)$ with boundary $e_1 e_2 \dots e_k$, we have

$$c(e_1)c(e_2)\dots c(e_k) = 1_G.$$

For G a group, G-cocycles on a CW complex X correspond to the "local" version of representations of $\pi_1(X)$ into G. Indeed, one recovers representations of $\pi_1(X)$ taking the holonomy.

Lemma 3.3. *For c a G-cocycle on a CW-complex X, for $x_0 \in \mathcal{C}^0(X)$ and $\gamma = e_1 \dots e_n$ a loop on the 1-skeleton of X based at x_0, let us write*

$$\mathrm{hol}_c(\gamma) = c(e_1)\dots c(e_n).$$

Then, the following hold:

- *For c a G-cocycle, the map $\gamma \mapsto \mathrm{hol}_c(\gamma)$ is a morphism $\pi_1(X, x_0) \to G$.*
- *For any morphism $\varphi : \pi_1(X, x_0) \to G$, there is a G-cocycle on X such that $\varphi = \mathrm{hol}_c$.*

Proof. Any loop in X can be homotoped to lie in the 1-skeleton of X, and any homotopy between loops in the 1-skeleton can be homotoped to lie in the 2-skeleton. Moreover, the second condition of Definition 3.2 ensures that the holonomy of a loop depends only on its homotopy class.

For the second point, let us consider a covering tree T in the 1-skeleton and let $\widetilde{X} = X/\langle T\rangle$ be the CW-complex obtained by contracting T to a point. Then since $\widetilde{X}$ has only one 0-cell, a choice of G-cocycle on $\widetilde{X}$ is equivalent to a choice of representation

$$\rho : \pi_1(\widetilde{X}) \to G.$$

Note that $\widetilde{X}$ has the same π_1 as X, and extending a G-cocycle on $\widetilde{X}$ by setting $c(e) = 1_G$ for any $e \in T$, one gets a G-cocycle whose holonomy is ρ. ∎

G-cocycles with the same holonomy can be related thanks to the following proposition.

Lemma 3.4. *Let X be a CW-complex, G a group, $x_0 \in \mathcal{C}^0(X)$ and let c and c' be two G-cocycle with the same holonomy. Then there exists a map $d : C^0(X) \to G$ such that $d(x_0) = 1_G$ and for any oriented edge e with $\partial e = x \cup \bar{y}$, we have*

$$c'(e) = d(x)c(e)d(y)^{-1}.$$

Proof. Choose a maximal covering tree out of the 1-skeleton of X. We can pick the value of d one vertex of $C^0(X)$ at a time, so that the relation $c'(e) = d(x)c(e)d(y)^{-1}$ is satisfied for any edge e belonging to T. One can further assume that $d(x_0) = 1_G$, by picking x_0 as the first vertex.

Now define c'' by $c''(e) = d(x)c(e)d(y)^{-1}$ for any edge e of $C^1(X)$. It is clear that c'' is also a 1-cocycle, which has the same holonomy as c or c', and c'' coincides with c' on T. We claim that it coincides with c' on the remaining edges. Indeed, for any edge e not in T, there is a loop on $C^1(X)$ based at x_0 whose only edge not in T is e. Then the fact that c' and c'' have the same holonomy and coincide on edges in T implies that they coincide on e too. ∎

Now let $X = \Sigma_g$ with some fixed cellular decomposition. We will need a criterion for when a representation $\rho = \mathrm{hol}_c$ of $\pi_1(\Sigma_g)$ into SO_3 lifts to a representation $\widetilde{\rho} : \pi_1(X) \to \mathrm{SU}_2$. Since any representation of $\pi_1(\Sigma_g)$ into SU_2 is represented by a SU_2-cocycle, ρ will admit a lift if and only if c lifts to a SU_2-cocycle $\tilde{c}$.

For any oriented edge e of the 1-skeleton of Σ_g, choose a lift $\tilde{c}(e) \in SU_2$ of $c(e) \in SO_3$. The lifts can be chosen so that $\tilde{c}(\bar{e}) = \tilde{c}(e)^{-1}$ are unique up to multiplication by $\pm I_2$. Let us fix a choice of lift of the SO_3-cocycle c to a map $\tilde{c} : \mathcal{C}^1(\Sigma_g) \to SU_2$, and an orientation of Σ_g. For $f \in \mathcal{C}^{2,+}(\Sigma_g)$, the set of oriented 2-cells whose orientations agree with that of Σ_g, let $\partial f = e_1 \dots e_k$ be the oriented boundary. We define

$$\varepsilon(c) = \prod_{f \in \mathcal{C}^{2,+}(\Sigma_g)} \mathrm{hol}_{\tilde{c}}(\partial f) \in SU_2.$$

Note that since c is a SO_3-cocycle, we have that $\mathrm{hol}_c(\partial f) = 1_{SO_3}$ for any $f \in \mathcal{C}^{2,+}(\Sigma_g)$; hence, $\varepsilon(c) \in \{\pm I_2\}$. Moreover, since each edge appears twice in the product, and the choices of lifts $\tilde{c}(e)$ are unique up to $\pm I_2$, the quantity $\varepsilon(c)$ is independent of the choice of lift $\tilde{c}$.

Lemma 3.5. *Let $\rho = \mathrm{hol}_c$ be a representation of $\pi_1(\Sigma_g, x_0)$ into SO_3. Then ρ lifts to a SU_2 representation of $\pi_1(\Sigma_g, x_0)$ if and only if $\varepsilon(c) = I_2$.*

Proof. Note that if c lifts to a SU_2 cocycle $\tilde{c}$, then $\varepsilon(c) = I_2$; since $\varepsilon(c)$ does not depend on the choice of a lift $\tilde{c} : \mathcal{C}^1(\Sigma_g) \to SU_2$, we must have $\varepsilon(c) = I_2$. Moreover, we claim that if the representation $\rho = \mathrm{hol}_c$ lifts, then so does the cocycle c. Indeed, the lift $\widetilde{\rho}$ will be the holonomy of a SU_2 cocycle c'. We may not have that $r(c') = c$; however, $r(c')$ and c have the same holonomy, and hence they are related by a map $d : \mathcal{C}^0(\Sigma_g) \to SO_3$, so that $r(c')(e) = d(a)c(e)d(b)^{-1}$ if $\partial e = a \cup \bar{b}$. Since there is no obstruction to lift the map d to SU_2, after correcting c' by a lift d' of d, we get a lift of the cocycle c to SU_2. ■

3.2. Compatible pants for representations with finite image

We now want to produce for an irreducible representation of $\pi_1(\Sigma_g)$ in SO_3 or SU_2 with finite image a compatible pants decomposition. We note that for the groups SO_3 and SU_2 being irreducible is equivalent to having nonabelian image. Let us write $r : SU_2 \to SO_3 \simeq SU_2/\{\pm I_2\}$ for the projection map. When $\rho : \pi_1(\Sigma_g) \to SU_2$ is irreducible, it can happen that the composition $r \circ \rho$ is also irreducible; in that case, a compatible pants decomposition for $r \circ \rho$ will also be one for ρ. This will allow us to focus on the case of representations into SO_3, with the exception of representations with image the quaternion group Q_8, as the following lemma shows.

Lemma 3.6. *Let G be a nonabelian finite subgroup of SU_2 such that $r(G) < SO_3$ is abelian. Then G is conjugated in SU_2 to Q_8.*

Proof. First we claim that $-I_2$ must be in G; otherwise G and $r(G)$ would be isomorphic. Then $r(G) \simeq G/\{\pm I_2\}$ is the quotient of G by a subgroup of its center. The subgroup $r(G) < SO_3$ being abelian, it is either cyclic or (up to conjugation) the subgroup $D < SO_3$ of diagonal matrices with ± 1 diagonal coefficients and determinant 1. However, if $r(G)$ was cyclic, this would imply that G itself is abelian. Hence, we are in the second case which proves the claim since $r^{-1}(D) = Q_8$. ■

The next lemma is standard; a proof may be for example found in [1].

Lemma 3.7. *Nonabelian finite subgroups of* SO_3 *are isomorphic to either a dihedral group* D_n *with* $n \geq 3$*, to the symmetric group* S_4*, or the alternating group* A_4 *or* A_5*.*

Proposition 3.8 ([9, Proposition 1.8]). *Let* $g \geq 2$*, and let* $G = D_n$ *with* $n \geq 3$ *or* S_4*,* A_4 *or* A_5*. For* $\varepsilon \in \{+,-\}$*, let* $\mathrm{Hom}^{s,\varepsilon}(\pi_1(\Sigma_g), G)$ *be the set surjective morphisms* $\pi_1(\Sigma_g) \to G$ *that lift (resp. do not lift) to* SU_2 *if* $\varepsilon = +$ *(resp.* $\varepsilon = -$*). Then* $\mathrm{Mod}(\Sigma_g)$ *acts transitively on* $\mathrm{Hom}^{s,\varepsilon}(\pi_1(\Sigma_g), G)$*.*

We note that for $G = D_n$ with n odd, any surjective morphism lifts to SU_2. We need to complement this proposition with the case of surjective morphisms onto the quaternion group.

Proposition 3.9. *Let* $g \geq 2$*. The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *acts transitively on* $\mathrm{Hom}^s(\pi_1(\Sigma_g), Q_8)$*.*

Proof. Let $a_1, b_1, \ldots, a_g, b_g$ be standard generators of $\pi_1(\Sigma_g)$. We recall that the quaternion group has generators and relations:

$$Q_8 = \big\langle I, J, K, -1 \mid I^2 = J^2 = K^2 = -1,\ (-1)^2 = 1,\ IJ = K, KI = J,\ JK = I \big\rangle.$$

Following the strategy of the proof of Proposition 3.8, we will deduce the proposition from the fact that $\mathrm{Mod}(\Sigma_g)$ acts transitively on surjective morphisms onto a cyclic group [5] (see also [9, Theorem 3.3] for a short proof). Let $\rho : \pi_1(\Sigma_g) \to Q_8$ be a surjective morphism. Then the quotient morphism onto $Q_8/\{J\} \simeq \mathbb{Z}/2\mathbb{Z}$ is surjective. Up to mapping class group action, we can assume that $\rho(a_1) \in I\langle J\rangle$ and that $\rho(b_1), \rho(a_i), \rho(b_i) \in \{J\}$ for $i \geq 2$. We claim that actually $\rho(b_1) = \pm 1$; indeed, otherwise, one would have $[\rho(a_1), \rho(b_1)] = -1$, while $[\rho(a_i), \rho(b_i)] = 1$ for $i \geq 2$. Therefore, ρ would not satisfy the surface group relation.

Moreover, up to applying $t_{a_1}^2$ where t_{a_1} is the Dehn twist along a_1, one may assume that $\rho(b_1) = 1$ (as $t_{a_1}^2(b_1) = b_1 a_1^2$).

Now since $\rho(a_1) \in I\langle J\rangle$ $\rho(b_1) = 1$ and ρ is surjective onto Q_8, one must have that $\rho|_{\langle a_2, b_2, \ldots, a_g, b_g\rangle}$ is surjective onto $\langle J\rangle \simeq \mathbb{Z}/4\mathbb{Z}$. Without loss of generality, assume that a_2 is mapped to a generator of $\langle J\rangle$. The loop $b_1 a_2$ is represented by a simple closed curve disjoint from b_1, and one has $t_{b_1 a_2}^k(a_1) = a_1(b_1 a_2)^k$. Hence, up to applying a power of this Dehn twist, one may assume that $\rho(a_1) = I$. Finally, up to mapping class group action of the subsurface with fundamental group $\langle a_2, b_2, \ldots, a_g, b_g\rangle$, one may assume that $\rho(a_1) = I$, $\rho(b_1) = 1$, $\rho(a_2) = J$, $\rho(b_2) = 1$ and $\rho(a_i) = \rho(b_i) = 1$ for $i \geq 3$. ∎

We recall that the orbits of the action of $\mathrm{Mod}(\Sigma_g)$ on pants decomposition of Σ_g correspond to the isomorphism classes of the dual trivalent graphs of pants decompositions. Thanks to Propositions 3.8 and 3.9, to show that every surjective morphism of $\pi_1(\Sigma_g)$ onto a finite nonabelian group G of SU_2 or SO_3 admits a compatible pants decomposition

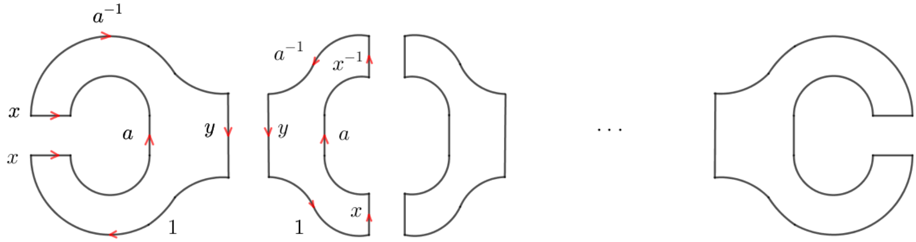

Figure 6. The top half of a cellular decomposition of Σ_g associated with the sausage dual graph, and an associated cocycle c. The cocycle c has value 1 on all remaining edges of the lower copy of the dual graph.

of some fixed type, we only need to exhibit one such surjective morphism which is compatible to a fixed pants decomposition of the same type. In addition, for $G = D_n$ with n even, S_4, A_4 or A_5, we need to find one such surjective morphism that lifts to SU_2 and one that does not.

We will exhibit such surjective morphism for two types of pants decomposition: one with a sausage dual graph, and one with "the square chain" dual graph. They are represented in Figures 5 and 6. Note that the pants in the pants decomposition can be further cut by three arcs into a pair of hexagons, so that Σ_g is obtained by gluing two copies of (a banded version of) its dual graph, and so that we get a cellular decomposition of Σ_g, with 2-cells that are hexagons. This is represented in Figures 5 and 6.

We define a G-cocycle for these cellular decompositions by specifying the holonomy of each arc, and checking that the holonomy around each hexagon is 1.

Proposition 3.10. *Let Σ_g be a compact oriented genus $g \geq 2$ surface with a cellular decomposition as in Figure 5. Let $G = D_n$, $n \geq 3$ or A_4, S_4, A_5 or Q_8. Then, the following hold:*

- *There exist $x, y, z \in G$ non-central elements such that $xyz = 1_G$, and $G = \langle x, y \rangle$.*
- *If $x, y, z \in G$ satisfies those conditions, then the holonomy $\rho = \mathrm{hol}_c$ of the cocycle c described in Figure 5 is compatible with the pants decomposition shown in the same figure.*
- *If moreover $G \neq Q_8$, then ρ may be interpreted as a representation $\rho : \pi_1(\Sigma_g) \to SO_3$, and as such, it lifts to SU_2.*

Proof. It follows from the definition of c that the value of its holonomy $\rho = \mathrm{hol}_c$ on any curve of the pants decomposition is x, y or z, up to conjugation and possibly inversion. Hence, any curve of the pants decomposition is mapped to a non-central element. Moreover, the restriction of ρ to any pants of the decomposition is conjugated to the map $F_2 \to G$ which maps the two generators of the free group F_2 to x and y. This and the hypothesis $G = \langle x, y \rangle$ imply that the restriction of ρ to each pair of pants of the decomposition is surjective onto G, hence non-abelian. This of course also implies that ρ itself is surjective.

If G is a subgroup of SO_3, then the SO_3-cocycle c may be lifted to SU_2. Indeed, the top hexagons of the decomposition are all identical; to lift c it suffices to choose lifts $\tilde{x}, \tilde{y}$ and $\tilde{z}$ so that $\tilde{x}\tilde{y}\tilde{z} = 1_{SU_2}$. On the edges that belong to the bottom hexagons, we simply lift $I_3 \in SO_3$ to $I_2 \in SU_2$. This defines indeed a SU_2 cocycle $\tilde{c}$ that lifts c.

Finally, we prove that in each case $G = D_n, n \geq 3$ or A_4, S_4, A_5 or Q_8, a triple $x, y, z \in G$ of non-central elements such that $xyz = 1_G$ and $G = \langle x, y\rangle$ exists.

- For $G = D_n = \langle r, s \mid s^2 = 1,\ rs = sr^{-1}\rangle$, one can take $x = s$, $y = r$ and $z = sr$. These elements are non-central when $n \geq 3$.
- For $G = A_4$, one can take $x = (12)(34)$, $y = (123)$ and $z = (234)$. Then x and y do not commute (in particular, x, y and z are non-central), and one can see that $\langle x, y\rangle$ contains both the subgroup of all double transpositions and a 3-cycle; hence, its order is divisible by 3 and 4; hence, $\langle x, y\rangle = A_4$.
- For $G = S_4$, one can take $x = (12)$, $y = (1234)$ and $z = (324)$. Again, x, y and z are non-central and $S_4 = \langle x, y\rangle$ since it contains all transpositions $(i\ i+1) = y^{i-1}xy^{1-i}$ with $1 \leq i \leq 4$.
- For $G = A_5$, one can take $x = (12)(34)$, $y = (12345)$, $z = (254)$. These elements are obviously non-central. We claim that $A_5 = \langle x, y\rangle$. Indeed, since x, y, z have orders $2, 5$ and 3, the subgroup $\langle x, y\rangle$ has order a multiple of 30, and index at most 2. But A_5 is simple, and a subgroup of index 2 is always normal, since the index must be 1.
- For $G = Q_8 = \langle I, J, K, -1 \mid I^2 = J^2 = K^2 = -1,\ (-1)^2 = 1,\ IJ = K\rangle$, we can take $x = I$, $y = J$ and $z = -K$, as these elements are non-central and Q_8 is generated by I and J. ■

Proposition 3.10, together with Propositions 3.8 and 3.9, shows that any finite non-abelian image SO_3 representation of $\pi_1(\Sigma_g)$ that lift to SU_2 has a compatible pants decomposition. It remains to treat the case of representations that do not lift to SU_2. We will use the following lemma.

Lemma 3.11. *Let $x, y \in SO_3$ be commuting order 2 elements such that $x \neq y^{\pm 1}$, and let $\tilde{x}, \tilde{y}$ be any lifts of x, y in SU_2. Then $[\tilde{x}, \tilde{y}] = -I_2$.*

Proof. Elements of order 2 in SO_3 lift to order 4 elements in SU_2 since the only order 2 elements in SU_2 are $\pm I_2$. Moreover, two order 4 elements in SU_2 must be co-diagonalizable with eigenvalues $\pm i$; hence, they must be equal or inverse of one another. Since $x \neq y^{\pm 1}$, one also has that $\tilde{x} \neq \tilde{y}^{-1}$. So $\tilde{x}$ and $\tilde{y}$ are non-commuting, and therefore $[\tilde{x}, \tilde{y}] = -I_2$ since their projections to SO_3 are commuting. ■

Proposition 3.12. *For $G = D_n$, $n \geq 3$ even, or A_4, S_4 or A_5, there is a surjective representation $\rho : \pi_1(\Sigma_g) \to SO_3$ with image isomorphic to G, compatible with a square type pants decomposition and such that ρ does not lift to SU_2.*

Proof. Let c be the cocycle defined in the proof of Proposition 3.10. We will modify c on the edges of a single hexagon of the cellular decomposition of Figure 5 and obtain another cocycle c' whose holonomy is still surjective onto G and compatible with the pants decomposition, but so that $\varepsilon(c') = -I_2$.

By Proposition 3.5, this shows that $\rho' = \mathrm{hol}_{c'}$ does not lift. Take a top hexagon of the cellular decomposition, and replace the holonomy of edges: $1, x, 1, y, 1, z$ by a, x, a^{-1}, $y, 1, z$ where $a \neq x^{\pm 1}$ has order 2 and commutes with x. The holonomy of this hexagon is still 1. The corresponding bottom hexagon also still has holonomy 1, so this defines a G-cocycle c'. We have not changed the holonomy of the pants decomposition curves, so they are still non-central, the restriction to any pair of pants is still non-abelian since y and z did not commute, and ρ' is still surjective since its restriction to at least 1 pair of pants of the decomposition coincides with the restriction of ρ, which was surjective.

Notice that in each case covered in the proof of Proposition 3.10, the element x was chosen of order 2. Lemma 3.11 implies that $\varepsilon(c) = -\varepsilon(c')$; hence, ρ' does not lift to SU_2.

To conclude, it remains to show in each case that there is a choice of an element a of order 2 such that $a \neq x^{\pm 1}$ and $[a, x] = 1$. We list again the cases:

- If $G = D_n$, $n \geq 3$ even, the element $a = r^{n/2}$ commutes with $x = s$.
- If $G = A_4$, the element $a = (13)(24)$ commutes with $x = (12)(34)$.
- If $G = S_4$, the element $a = (34)$ commutes with $x = (12)$.
- If $G = A_5$, the element $a = (13)(24)$ commutes with $x = (12)(34)$. ■

Proposition 3.13. *Let Σ_g be a compact oriented genus $g \geq 2$ surface with a cellular decomposition as in Figure 6. Let $G = D_n$, $n \geq 3$ or A_4, S_4 or A_5. Then, the following hold:*

- *There exists $x, y, a \in G$, with x, y non-commuting elements and such that $[x, a]y = 1_G$, and $G = \langle x, a\rangle$.*
- *If $x, y, a \in G$ satisfies those conditions, then the holonomy $\rho = \mathrm{hol}_c$ of the cocycle c described in Figure 6 is compatible with the pants decomposition shown in the same figure.*
- *The representation ρ interpreted as a representation $\rho : \pi_1(\Sigma_g) \to SO_3$ lifts to SU_2.*

Proof. We again start by proving the last two points, which are easier: notice that the holonomy of the top hexagons of the cellular decomposition are all $[x, a]y$, while the holonomy of the bottom hexagons are all $aa^{-1} = 1_G$. Thus, c defines a G-cocycle if and only if $[x, a]y = 1_G$. The representation $\rho = \mathrm{hol}_c$ is surjective since $G = \langle a, x\rangle$ and the image of ρ contains a and x, and it is irreducible in restriction to each pair of pants since x and y do not commute.

For the lifting property, notice that one can choose lifts $\tilde{x}, \tilde{a}, \tilde{y} \in SU_2$ of $x, y, a \in SO_3$ so that $[\tilde{x}, \tilde{a}]\tilde{y} = I_2$ since $\tilde{y}$ appears with odd power in the product $[\tilde{x}, \tilde{a}]\tilde{y}$. Then using the same lifts for each hexagon, it is clear that we get SU_2-cocycle $\tilde{c}$ that is a lift of c.

We now produce elements x, a, y for each choice of finite group G listed in the proposition:

- If $G = D_n$, $n \geq 3$ odd, take $x = s$, $a = r^{(n+1)/2}$ and $y = [a, x] = r^{n+1} = r$. Then x and y are non-commuting and generate G (thus a and x also generate G).

- If $G = D_n, n \geq 3$ even, take $x = s$, $a = r$ and $y = [a, x] = r^2$. Since $n \geq 3$, x and y are non-commuting and moreover $G = \langle a, x \rangle$.
- If $G = A_4$, take $x = (123)$, $a = (234)$ and $y = [a, x] = (14)(23)$. The elements x and y are non-commuting. We claim that $\langle a, x \rangle = G$: indeed, $\langle a, x \rangle$ contains the 3-cycle x and contains a double transposition y. It actually contains the subgroup generated by double transpositions since it also contains $xyx^{-1} = (13)(24)$. Thus, its order is divisible both by 3 and 4; therefore, it is A_4.
- If $G = S_4$, take $x = (1234)$, $a = (23)$ and $y = [a, x] = (234)$. It is again clear that x and y are non-commuting. The subgroup $\langle a, x \rangle$ contains a 3 and a 4 cycle; hence, its index is 1 or 2. But the only index 2 subgroup of any S_n is A_n, and $\langle a, x \rangle$ is not included in A_4 since $x \notin A_4$, so we must have $S_4 = \langle x, a \rangle$.
- If $G = A_5$, take $x = (13)(24)$, $a = (345)$ and $y = [a, x] = (13452)$. Then x and y are non-commuting, and $\langle a, x \rangle$ has order divisible by 2, 3 and 5. Thus, it has index at most 2 in A_5, and therefore by simplicity of A_5 one has $A_5 = \langle a, x \rangle$. ■

Again we complement the previous proposition with a proposition which deals with non-abelian finite image SO_3 representations that do not lift to SU_2.

Proposition 3.14. *For $G = D_n$, $n \geq 3$ even, or A_4, S_4 or A_5, there is a surjective representation $\rho : \pi_1(\Sigma_g) \to SO_3$ with image isomorphic to G, compatible with a sausage type pants decomposition and such that ρ does not lift to* SU_2.

Proof. We follow the same strategy as for Proposition 3.12, modifying the cocycle c of Proposition 3.13 on the edges of a single top hexagon. This time the top hexagon will be the leftmost hexagon in Figure 6. The new holonomy will still be surjective as the restriction to the one-holed torus on the right side of the decomposition will not have changed, and furthermore it will be clear that it is still compatible with the pants decomposition. The sequence of holonomy of edges for the distinguished top hexagon was initially: $x, a, x^{-1}, a^{-1}, y, 1$, and we change it to the following:

- if $G = D_n$, $n \geq 3$ even, to $x, ab, x^{-1}, b^{-1}a^{-1}, y, 1$ where $b = r^{n/2}$ commutes with $x = s$;
- if $G = A_4$, to $x, a, x^{-1}, a^{-1}b, y, b^{-1}$ where $b = (12)(34)$ commutes with $y = (14)(23)$;
- if $G = S_4$, to $xb, a, b^{-1}x^{-1}, a^{-1}, y, 1$ where $b = (14)$ commutes with $a = (23)$;
- if $G = A_5$, to $x, ab, x^{-1}, b^{-1}a^{-1}, y, 1$ where $b = (12)(34)$ commutes with $x = (13)(24)$.

Lemma 3.11 then implies in each case that $\varepsilon(c') = -\varepsilon(c)$, and hence the new holonomy does not lift. ■

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