Groups Geom. Dyn. 19 (2025), 1041–1080
DOI 10.4171/GGD/830

A law of iterated logarithm on lamplighter diagonal products

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Abstract. We prove a law of iterated logarithm for random walks on a family of diagonal products constructed by Brieussel and Zheng (2021). This provides a wide variety of new examples of law of iterated logarithm behaviors for random walks on groups. In particular, it follows that for any $\frac{1}{2} \leq \beta \leq 1$ there is a group G and random walk W_n on G with $\mathbb{E}|W_n| \simeq n^\beta$ such that

$$0 < \limsup \frac{|W_n|}{n^\beta (\log\log n)^{1-\beta}} < \infty \quad \text{and} \quad 0 < \liminf \frac{|W_n|(\log\log n)^{1-\beta}}{n^\beta} < \infty.$$

1. Introduction

Let G be some finitely generated group, with a finite symmetric generating set S. Let W_n be a random walk on G, with finitely supported symmetric step measure such that the support of W_1 generates G. A central object of study in the theory of random walks on groups is the distribution of the distance $|W_n|$ of the random walk from its origin point, and its connection to other geometric and algebraic properties of G. One is interested in understanding both the behavior of $|W_n|$ for specific families of groups and the set of possible behaviors of the distance function for random walks on groups in general (often referred to as the "inverse problem"). Usually, fine understanding of the underlying metric structure of G is lacking, and beyond some examples of polynomial growth groups and some non-amenable groups, most works focused on understanding the expected distance on some families of groups, with few works also looking at some moderate and large deviation regimes (see Subsections 1.1 and 1.2).

The main contribution of this paper is proving a law of iterated logarithm type result for random walks on a family of diagonal products studied in [3]. These groups were used to capture a variety of behaviors of random walks on groups, in particular in terms of the expected distance. Our result provides a wide variety of law of iterated logarithm type behaviors for random walks on groups.

Mathematics Subject Classification 2020: 05C81 (primary); 20E22, 20F69, 60B15, 60F15 (secondary).
Keywords: random walks on groups, law of iterated logarithm, diagonal products, excursions.

1.1. The expected distance $\mathbb{E}|W_n|$

Throughout this paper, we will only consider symmetric finitely supported random walks W_n on a group G, whose support generates the group G. Much advancement has been done in the last years regarding the expected value $\mathbb{E}|W_n|$, also known as the *speed* or *rate of escape* of the random walk. It is trivial to see that the expected distance is sub-additive, and in particular that $\mathbb{E}|W_n| \lesssim n$ (where $\lesssim, \simeq$ denote (in)equalities up to an absolute constant that may depend on the choice of group and random walk measure but not on n). By [20], for any random walk W_n on an infinite finitely generated group, one has $\mathbb{E}|W_n| \gtrsim \sqrt{n}$. For many "small" groups, such as virtually nilpotent groups, the random walk is always diffusive, that is, $\mathbb{E}|W_n| \simeq \sqrt{n}$. In contrast, for non-Liouville measures, the entropy of the random walk is $H(W_n) \simeq n$ by works of Avez, Derriennic and Kaĭmanovich–Vershik [2, 7, 15], which in turn is equivalent to $\mathbb{E}|W_n| \simeq n$ by Guivarc'h, Varopoulos and Karlsson–Ledrappier [10, 16, 26]. For a long time all known examples of random walks on groups exhibited one of the two extreme behaviors above. We note that some of the above works extend beyond the case of finitely supported symmetric measure considered in this paper.

This leads to the following natural question, attributed to Vershik.

Question 1.1. What functions can be realized as the speed function of a random walk on some group?

The first "non-classical" examples toward this question were given by Erschler [9], who proved that if a random walk on a group G satisfies $\mathbb{E}|W_n| \simeq n^{\alpha}$, then the random walk on $G \wr \mathbb{Z}$ satisfies $\mathbb{E}|W_n| \simeq n^{\frac{1+\alpha}{2}}$. Starting with $G = \mathbb{Z}$ and using this result recursively, one can realize all of the functions $f(n) = n^{1-2^{-k}}$ as speed functions of iterated wreath product.

The next step was taken by Amir and Virág [1], who constructed groups where $W_n \simeq n^{\beta}$ for $\frac{3}{4} \leq \beta < 1$ (and more generally with $\mathbb{E}|W_n| \simeq f(n)$ for any "nice enough" function f within that range). Their construction uses permutational wreath product over the natural action of the mother groups $\mathcal{M}_m$ on the boundary of their tree with $\mathbb{Z}$-valued lamps.

In [3], Brieussel and Zheng constructed random walks on groups that, up to some regularity condition, capture the whole range of possible behaviors of the expected distance.

Theorem 1.2 ([3, Theorem 1.1]). *Let $f\colon [1,\infty) \to [1,\infty)$ be a continuous function with $f(1) = 1$ such that $\frac{f(x)}{\sqrt{x}}$ and $\frac{x}{f(x)}$ are non-decreasing. Then there exist a group Δ and a random walk W_n on Δ such that $\mathbb{E}|W_n| \simeq f(n)$.*

The examples constructed and analyzed in Brieussel and Zheng also gave new behaviors of other geometric quantities such as the entropy of the random walk, the return probabilities and the isoperimetric profile.

We remark that, in general, it is not known whether the speed of a (symmetric, finitely supported) random walk on a group depends on the choice of the step distribution (up to the equivalence $\simeq$). However, we believe that for the examples discussed in this article, the speed and the law of iterated logarithm do not depend on the choice. We did not pursue this direction.

1.2. The law of iterated logarithm

Since the possible behavior of the expected distance is well understood in many cases, one may try and study further questions regarding the behavior of the distance function of the random walk W_n, for example, moderate and large deviations of the distance function. One such direction is to study the times in which the random walk is atypically far from its origin. This can be formulated by the law of iterated logarithm.

The classical law of iterated logarithm for random walks on $\mathbb{Z}$, first established by Khintchine [18] and Kolmogoroff [19], states that any random walk W_n with zero mean and unit variance satisfies

$$\limsup_{n\to\infty} \frac{|W_n|}{\sqrt{2n\log\log n}} = 1$$

almost surely.

Unlike the expected distance which was well studied, there are much fewer examples where the law of iterated logarithm of random walks on groups is understood. There are several ways to phrase a law of iterated logarithm for a random walk on a general group. Perhaps the strongest one is finding a function $g(n)$ such that

$$\limsup_{n\to\infty} \frac{|W_n|}{g(n)} = 1$$

almost surely. However, finding such a function requires a tight estimation for the distance $|W_n|$, which is in many cases hard to achieve.

We say that $g(n)$ is an *upper scaling function* for W_n, if

$$0 < \limsup_{n\to\infty} \frac{|W_n|}{g(n)} < \infty$$

almost surely, and that $h(n)$ is a *lower scaling function* for W_n, if

$$0 < \liminf_{n\to\infty} \frac{|W_n|}{h(n)} < \infty$$

almost surely. Scaling functions can be thought of as a way of measuring the rate with which $|W_n|$ goes to infinity.

These definitions can also be rephrased in terms of inner and outer radii. We say that a function $R(n)$ is an *outer radius* for W_n if $|W_n| \geq R(n)$ finitely often with probability 1, and a function $r(n)$ is an *inner radius* for W_n if $|W_n| \leq r(n)$ finitely often with probability 1. It follows that $g(n)$ is an upper scaling function if $Cg(n)$ is an outer radius for some constant $C > 0$, but $cg(n)$ is not an outer radius for some other constant $c > 0$ (and one can similarly rephrase lower scaling functions in terms of inner radii).

Let us briefly review previous works in this direction. The classical results for $\mathbb{Z}$ and $\mathbb{Z}^d$ can be found, for example, in [8], where Dvoretzky and Erdös give a characterization for the inner and outer radii of a simple random walk on $\mathbb{Z}^d$. Hebisch and Saloff-Coste generalize this theorem for arbitrary groups with polynomial volume growth of order $d \geq 3$ (see [11, Theorem 9.2]).

In [21], Revelle studies groups of the form $G \wr \mathbb{Z}$, where the random walk on G has α-tight degree of escape (which is a form of control over the tail of the distance function), and proves that $G \wr \mathbb{Z}$ has $\alpha' = \frac{1+\alpha}{2}$-tight degree of escape; he also shows that $n^{\alpha'}(\log\log n)^{1-\alpha'}$ is an upper scaling function for $G \wr \mathbb{Z}$, whereas $n^{\alpha'}/(\log\log n)^{1-\alpha'}$ is a lower scaling function for this group. Revelle studies in addition several Baumslag–Solitar groups, proving they have an inner radius of order $\sqrt{n/\log\log n}$ and an outer radius of order $\sqrt{n \log\log n}$.

Finally, in [24] Thompson proves laws of iterated logarithm for certain polycyclic and metabelian groups. He shows that these groups are all diffusive, have $\frac{1}{2}$-tight degree of escape and have an upper scaling function $g(n) = \sqrt{n \log\log n}$.

Note that in all previously known cases, there was either a very detailed understanding of the metric properties of the groups and the behavior of random walks on them (e.g., for polynomial growth groups), or some tight form of control over the tail behavior of the walk is assumed (as in the results of Revelle [21] and Thompson [24]). Such understanding and control is generally lacking for most groups, and is the main reason why there are only few families of groups for which laws of iterated logarithm (or other moderate and large deviation estimates) are known.

There is no example of a Liouville group where it is known that no form of the law of iterated logarithm holds, and we expect a law of iterated logarithm to hold, for instance, for the Amir–Virág construction mentioned above. However, proving such a law would require better understanding of the metric structure of automata groups and of rare events concerning their natural actions than is currently available.

1.3. Main results

In this article, we consider the groups constructed in [3]. These groups are diagonal products of lamplighter groups with finite lamp groups. We focus on the case where the lamp groups are expanders, and prove a law of iterated logarithm on these groups. Our main theorem is the following.

Theorem 1.3. *Let $f\colon [1,\infty) \to [1,\infty)$ be a continuous function such that $f(1) = 1$ and such that $\frac{x}{f(x)}$ and $\frac{f(x)}{\sqrt{x}}$ are non-decreasing. Let Δ be the group from [3] (in the expander case) for which the speed function is equivalent to $f(n)$. Write W_n for the random walk on Δ with the appropriate generators.*

(i) *If* $\frac{f(x)}{\sqrt{x}\log\log\log x}$ *is non-decreasing, then*

$$0 < \limsup_{n\to\infty} \frac{|W_n|}{\log\log n \; f\left(\frac{n}{\log\log n}\right)} < \infty$$

almost surely.

(ii) *If* $\frac{f(x)}{\sqrt{x}(\log\log x)^{1+\varepsilon}}$ *is non-decreasing for some* $\varepsilon > 0$, *then*

$$0 < \liminf_{n\to\infty} \frac{|W_n|}{\frac{1}{\log\log n} f(n\log\log n)} < \infty$$

almost surely.

Example 1.4. Let $\frac{1}{2} < \alpha < 1$, and suppose that the speed function we choose is $f(n) = n^\alpha$. Then the theorem states that

$$\mathrm{P}\left(0 < \limsup_{n\to\infty} \frac{|W_n|}{n^\alpha(\log\log n)^{1-\alpha}} < \infty\right) = 1$$

and

$$\mathrm{P}\left(0 < \liminf_{n\to\infty} \frac{|W_n|(\log\log n)^{1-\alpha}}{n^\alpha} < \infty\right) = 1.$$

Our proof draws ideas from the works of Revelle [21], Brieussel and Zheng [3] and classical proofs of the law of iterated logarithm, together with a careful analysis of random walk excursions and the dependencies between the number of excursions of different lengths at different positions and times. Let us describe briefly the main sources of difficulty when trying to prove a law of iterated logarithm (or other tail estimates), compared to estimating the expected distance $\mathbb{E}|W_n|$ (see also the proof sketch in Section 3).

First, expectation is additive, regardless of the dependency structure between different parts, which allows for the simplification of many of the estimates. For instance, if one wants (as in the analysis of Brieussel and Zheng) to estimate the expected total number of length k excursions completed by some random walk on $\mathbb{Z}$, it is enough to estimate the probability that the random walk completes a length k excursion from a given point x at time n, and then sum up these probabilities. However, to understand the tail behavior of the total number of length k excursions completed by the walk, one must understand the dependency between completing excursions at different times and positions.

Second, it is harder to handle rare events than typical events. The expected distance is governed by the typical behavior of the random walk, while the law of iterated logarithm, and in greater generality the tail behavior of $|W_n|$, is governed by rare events (but not those that are too rare). The groups we analyze are constructed as diagonal products of lamplighter groups over $\mathbb{Z}$ (see Section 2). To get distance estimates on the diagonal product, one must use distance estimates on the different layers. Typical behavior happens in all layers at once. However, rare events do not a priori happen simultaneously in all layers;

thus, one must analyze the dependency between the different layers more carefully. A key step in the proof is showing that these rare events boil down to rare events of the projection of the random walk to $\mathbb{Z}$, and then bounding the contribution of the different layers under these events.

Revelle's approach [21] plays a key role in analyzing these rare events, but new ideas are required to deal with the difficulties arising from the Brieussel–Zheng construction. Revelle studies lamplighter groups over $\mathbb{Z}$, while the Brieussel–Zheng construction is an infinite diagonal product of such lamplighters. Furthermore, the local times used by Revelle in his analysis are here replaced by the number of long excursions, and so we need to estimate the number of long excursions starting from any point and of many scales together. For this reason our analysis requires a more refined understanding of excursions especially at extremal times of the random walk.

Let us also remark that one could try to construct examples of groups satisfying a law of iterated logarithm as in the example above by considering the wreath products $G \wr \mathbb{Z}$, where G is taken to be a group with speed n^{β}, and then using Revelle's results [21]. However, there are two caveats to this approach. First, the wreath product $G \wr \mathbb{Z}$ will have a rate of escape $n^{(\beta+1)/2}$; thus, it could only provide examples with $\alpha \geq \frac{3}{4}$. Second, in order to apply Revelle's theorems, one must know that the random walk on G satisfies a *tight degree of escape* which amounts to proving tail bounds on the distance of the random walk on G, which requires the same kind of analysis done in our paper.

Remark 1.5. Throughout this article, we did not optimize the constants. Any unnumbered constant is some universal constant, but its value may change between claims. However, numbered constants keep their values for the rest of the paper.

2. Realizing speed functions with diagonal products

In this section, we describe the groups from [3], and give an outline of the technique used in this article to estimate the speed of the random walk on these groups, proving Theorem 1.2. These groups are diagonal products of a sequence of lamplighter groups with finite lamp groups, where the lamp groups are chosen to be expanders or diffusive groups. In this article, we focus on the expander case, although we first give the general description for the groups.

2.1. Diagonal product

Definition 2.1. Let $X = \{x_1, \dots, x_{|X|}\}$ be a set, and let $\{\Delta_s\}_{s\geq 0}$ be a sequence of groups. Suppose that each Δ_s is generated by a set $X(s)$ which we identify as a copy of X in Δ_s, that is, $X(s) = \{x_1(s), \dots, x_{|X|}(s)\}$. The *diagonal product of* $\{\Delta_s\}_{s\geq 0}$ *with respect to* $\{X(s)\}_{s\geq 0}$, which will be denoted Δ, is the subgroup of the direct product $\prod_{s\geq 0} \Delta_s$ generated by the diagonal elements $(x_i(s))_{s\geq 0}$.

Alternatively, we can construct Δ as follows: Let $F(X)$ be the free group generated by X, and let $\pi_s\colon F(X) \to \Delta_s$ denote the natural projections. Then $\Delta = F(X)/\bigcap_{s\geq 0} \ker \pi_s$.

Note that Δ has a natural generating set, given by the diagonal elements.

Remark 2.2. If X has a natural algebraic structure, for example a union of several groups, it is natural to require that the above construction will be compatible with that structure. This can be done by requiring that the identifications of $X(s)$ with X respect that structure.

2.2. The lamp groups

Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$ be two groups. Let $\{\Gamma_s\}_{s\geq 0}$ be a sequence of groups, where each Γ_s is generated by a set of the form $A(s) \cup B(s)$, where $A(s)$ and $B(s)$ are subgroups of Γ_s isomorphic to A and B, respectively.

Fix a sequence of strictly increasing integers $\{k_s\}$. For each s let $\Delta_s = \Gamma_s \wr \mathbb{Z}$, with a generating set given by $\tau(s) = (e, +1)$, $\alpha_i(s) = (a_i(s)\delta_0, 0)$ for all $1 \leq i \leq |A|$, and $\beta_j(s) = (b_j(s)\delta_{k_s}, 0)$ for all $1 \leq j \leq |B|$.

Finally, we take Δ to be the diagonal product of the groups Δ_s with respect to the above generating sets, marked with the generating set $\mathcal{T} = (\tau, \alpha_1, \ldots, \alpha_{|A|}, \beta_1, \ldots, \beta_{|B|})$. If U_α and U_β are the uniform measures on the subgroups $\mathcal{A} = \{\alpha_1, \ldots, \alpha_{|A|}\}$ and $\mathcal{B} = \{\beta_1, \ldots, \beta_{|B|}\}$, and μ is the uniform measure on $\{\tau, \tau^{-1}\}$, we use the "switch-walk-switch" measure for our random walk on Δ, that is

$$q = (U_\alpha * U_\beta) * \mu * (U_\alpha * U_\beta).$$

Denote by W_n the random walk on Δ with step distribution q. By the choice of q, we may write $W_n = (f_s^{W_n}, S_n)$, where S_n is a simple random walk on $\mathbb{Z}$, and $f_s^{W_n}$ denotes the lamp configuration at the layer Δ_s at time n.

Following [3], we make the following assumptions on the groups Γ_s.

Assumption 2.3. We assume the following about the groups $\{\Gamma_s\}$ and the sequences $\{k_s\}$ and $\{l_s\}$:

(i) $k_0 = 0$ and $\Gamma_0 = A(0) \times B(0) \cong A \times B$.

(ii) Let $[A(s), B(s)]^{\Gamma_s}$ denote the normal closure of the subgroup generated by commutators $[a_i(s), b_j(s)]$. Then we assume

$$\Gamma_s/[A(s), B(s)]^{\Gamma_s} \cong A(s) \times B(s) \cong A \times B.$$

(iii) Letting $l_s = \operatorname{diam}(\Gamma_s)$, we assume that k_s and l_s grow at least exponentially.

In [3], the authors treat two cases of families of groups Γ_s: one of them is when the random walks on Γ_s are diffusive, and the other one is when the groups $\{\Gamma_s\}$ satisfy the following linear speed assumption.

Definition 2.4 ([3, Definition 3.1]). Let $\{\Gamma_s\}$ be a sequence of finite groups where each Γ_s is marked with a generating set $A(s) \cup B(s)$. Let $\eta_s = U_{A(s)} * U_{B(s)} * U_{A(s)}$, where $U_{A(s)}, U_{B(s)}$ are uniform distribution on $A(s), B(s)$. We say $\{\Gamma_s\}$ satisfies the (σ, T_s)-*linear speed assumption* if in each Γ_s,

$$L_{\eta_s}(t) = \mathbb{E}|X_t^{(s)}|_{\Gamma_s} \geq \sigma t \quad \text{for all } t \leq T_s,$$

where $X_t^{(s)}$ has distribution η_s^{*t}.

Note that for the definition to be meaningful, we must have $\sigma \leq 1$.

In this paper, we focus only on the case where the groups satisfy the linear speed assumption. Such groups can be constructed, for instance, using Lafforgue super expanders or with lamplighter groups over a d-dimensional infinite dihedral group (see [3, Examples 3.2 and 3.3]).

2.3. Excursions and the speed in a single layer

In a single layer $\Delta_s = \Gamma_s \wr \mathbb{Z}$, the lamp value at $x \in \mathbb{Z}$ can only be changed when the random walk reaches x or $x - k_s$. As $A(s)$ is a subgroup and the measure chosen on it is uniform, a product of random elements from $A(s)$ is again a uniform element from $A(s)$ (and the same holds for $B(s)$). Therefore, conditioning on the trajectory of the simple random walk S_t, the distribution of the lamp value at $x \in \mathbb{Z}$ depends only on the number of k_s-excursions.

Definition 2.5. Let S_t be a random walk on $\mathbb{Z}$, and let $k > 0$. We say that the random walk begins a k-*excursion* in time j, if it visits $S_j - k$ before its next visit in S_j. We denote by $T(k, x, n)$ the number of k-excursions the random walk S_t performs before time n starting at x.

To estimate the speed of the random walk on Δ_s, Brieussel and Zheng give bounds on the speed in terms of the number of k_s-excursions and the diameter $l_s = \operatorname{diam}(\Gamma_s)$. By dividing the range into intervals of length k_s and working in each one separately one after the other, one can show that for some universal constant $C > 0$,

$$|W_n|_{\Delta_s} \leq \frac{C k_s}{2} \sum_{j \in \mathbb{Z}} T\left(\frac{k_s}{2}, \frac{k_s}{2} j, n\right) + C|\text{range}(S_n)|, \tag{1}$$

which has an expected value of $\frac{C'n}{k_s} + C'\sqrt{n} \lesssim \frac{n}{k_s}$ if $k_s \leq \sqrt{n}$. Also, since $l_s = \operatorname{diam}(\Gamma_s)$, we also have the bound

$$|W_n|_{\Delta_s} \leq |\text{range}(S_n)| l_s,$$

which has an expected value $\lesssim \sqrt{n}\, l_s$. Together we get

$$\mathbb{E}|W_n|_{\Delta_s} \lesssim \sqrt{n}\, l_s + \frac{n}{k_s}.$$

Brieussel and Zheng show that this also gives a matching lower bound because of the linear speed assumption, so one gets the asymptotic behavior of the speed of the random walk on Δ_s.

2.4. Estimating the speed of the random walk

The random walk on Δ can be thought of as parallel random walks on each Δ_s, so in order to estimate the speed in Δ one needs to use the estimates for the speed in each layer Δ_s. In [3], Brieussel and Zheng show that there is some universal constant $C > 0$ such that

$$|W_n|_{\Delta_s} \leq |W_n|_{\Delta} \leq C \sum_{s:k_s \leq |\mathrm{range}(S_n)|} |W_n|_{\Delta_s}. \tag{2}$$

As the speed in each layer Δ_s is of order $\sqrt{n}\, l_s + \frac{n}{k_s}$ and the sequences $\{k_s\}$ and l_s grow at least exponentially, it follows that for each n there is a layer $s_1(n) = \max\{s \mid k_s l_s \leq \sqrt{n}\}$ that determines the speed of the random walk, that is, $\mathbb{E}[|W_n|_{\Delta}]$ is equivalent to $\mathbb{E}[|W_n|_{\Delta_{s_1(n)}}]$ up to universal constants. This proves the following.

Proposition 2.6 ([3, Proposition 3.6]).

$$\mathbb{E}[|W_n|_{\Delta}] \simeq \mathbb{E}[|W_n|_{\Delta_{s_1(n)}}] \simeq \sqrt{n}\, l_{s_1(n)} + \frac{n}{k_{s_1(n)+1}}.$$

This proposition allows one to realize various functions as the speed functions of such groups. To do so, given a function $f\colon \mathbb{N} \to \mathbb{N}$ one must choose appropriate sequences $\{k_s\}$ and $\{l_s\}$. This is done in the following way.

Proposition 2.7 ([3, Corollary B.3]). *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that $\frac{f(n)}{\sqrt{n}}$ and $\frac{n}{f(n)}$ are non-decreasing, and let $m_0 > 1$. Then one can find sequences $\{k_s\}$ and $\{l_s\}$ of positive integers such that $k_{s+1} \geq m_0 k_s$, $l_{s+1} \geq m_0 l_s$, and also such that*

$$f(n) \simeq \bar{f}(n),$$

where

$$\bar{f}(x) = \sqrt{x}\, l_s + \frac{x}{k_{s+1}} \quad \text{if } (k_s l_s)^2 \leq x < (k_{s+1} l_{s+1})^2.$$

This concludes the proof of Theorem 1.2.

3. Sketch of the proof

As mentioned in the introduction, in order to show that

$$0 < \limsup_{n\to\infty} \frac{|W_n|_{\Delta}}{a_n} < \infty$$

almost surely, we need to prove that there are constants $C, c > 0$ such that Ca_n is an outer radius but ca_n is not an outer radius. For the analogous claim

$$0 < \liminf_{n\to\infty} \frac{|W_n|_\Delta}{b_n} < \infty,$$

we need to prove that there are constants $C', c' > 0$ such that $c'b_n$ is an inner radius for $|W_n|_\Delta$, but $C'b_n$ is not an inner radius.

Our overall framework (as in [3]) is to first study the distance of the random walk on a single layer Δ_s, and then deduce a bound for the distance of the random walk on Δ. For a single layer, Brieussel and Zheng provide an upper bound for the distance of the random walk in terms of k-excursions (1), and a lower bound on the speed of the random walk

$$\mathbb{E}[|W_n|_{\Delta_s}] \geq c\,\mathbb{E}\Bigg[\sum_{x\in\mathbb{Z}} \min\{T(k_s, x, n), l_s\}\Bigg]. \tag{3}$$

To estimate the fluctuations of the distance of the random walk, we use the same upper bound (1), but one needs to prove a version of (3) that holds for the distance almost surely and not only for expectation, that is, for large enough n

$$|W_n|_{\Delta_s} \geq c \sum_{x\in\mathbb{Z}} \min\{T(k_s, x, n), l_s\}. \tag{4}$$

Proving (4) requires modifying ideas from Revelle [21] (with the role of local times being replaced by the number of long excursions), together with concentration bounds on the number of long excursions. This, in turn, requires understanding the tail behavior of $T(k, n) = k \sum_{x\in\mathbb{Z}} T(k, kx, n)$ and of $\sum_{x\in\mathbb{Z}} \min\{T(k, x, n), l\}$ for all k, l. Finer understanding of these tail behaviors is also required to get a good lower bound on the RHS of (4). This part takes the bulk of the work, and the dependencies between excursions in the same layer and between different layers come into play.

The above steps provide upper and lower bounds on the distance in a single layer in terms of k_s, l_s, which we then add up by (2) to bound the distance of the random walk on Δ.

Because of the law of iterated logarithm on $\mathbb{Z}$, we know that there are times in which the range is small or large compared to its expected value. We show that in these times, the distance reaches its lim inf and lim sup values, respectively. As in the case of the speed of the random walk, we prove that there is one layer Δ_s that is equivalent to the distance in Δ; however, since we consider the times in which the range is extremal, this layer is different from the critical layer used to estimate the speed of the random walk.

The article is constructed as follows. In Section 4, we associate another random walk to S_t, so that the numbers of k-excursions of S_t at all points correspond to the local times of the new random walk. To do so, we use the results in Appendix A, which study the induced random walk of S_t on $k\mathbb{Z}$. In Section 5, we study the tail behavior of $T(k, n)$, showing that under appropriate assumptions it is highly concentrated around its mean $\frac{n}{2k}$.

In Sections 6 and 7, we study the tail behavior of $\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\}$, for the two critical layers. These results rely on the tail behavior of the maximal local time of a random walk, which we study in Appendix B.

We then return to the random walk on Δ. In Section 8, we prove the lower bound for the distance of the random walk in terms of the k-excursions. Section 9 is devoted to deduce the bounds for the distance in a single layer, and in Section 10 we use the bounds for a single layer to bound the distance of the random walk on Δ. Finally, in Section 11, we express our results in terms of the speed function, and conclude the proof of Theorem 1.3.

Remark 3.1. In Theorem 1.3, we demand that the random walks are slightly super-diffusive. These conditions stem from the fact that the more diffusive the random walk is, the harder it is to get strong concentration inequalities for the number of k-excursions.

For the lim sup statement (Theorem 1.3 (i)), this manifests in not having a precise enough bound for the contribution of the layers near the lim sup critical layer. It may be that by having a better understanding of the joint behavior of the distance in those layers, one could improve the bounds.

For the lim inf statement (Theorem 1.3 (ii)), the difficulty arises from the sequence of parameters k_s, l_s used in the approximation of the speed function. When f is almost diffusive, these sequences may grow extremely fast. One can still write an explicit formula for the lim inf in terms of k_s, l_s (see Proposition 10.2); however, this may not coincide with the expression in the theorem.

4. The k-induced random walk

Let S_t be a simple random walk on $\mathbb{Z}$. Recall that $T(k,x,n)$ denotes the number of k-excursions starting from x that are completed until time n. To study the distribution of $T(k,x,n)$ for each k,x,n, we associate a new simple random walk on $\mathbb{Z}$ with S_t, so that k-excursions in S_t can be approximated by the local times of the new random walk.

Definition 4.1. Let S_t be a simple random walk on $\mathbb{Z}$, and let $k \geq 0$. The *k-induced random walk of S_t* is the simple random walk $Y_n^{(k)}$ on $\mathbb{Z}$ defined in the following manner. We first set $n_0^{(k)} = 0$, and for any $j \geq 1$ define inductively

$$n_j^{(k)} = \min\Big\{t > n_{j-1}^{(k)} \;\Big|\; \big|S_t - S_{n_{j-1}^{(k)}}\big| = k\Big\}.$$

The random walk $Y_t^{(k)}$ is then given by $Y_t^{(k)} = S_{n_t^{(k)}}/k$. We also write $N_k(n) = \max\{j \mid n_j^{(k)} \leq n\}$ for the number of steps $Y_n^{(k)}$ made until time n.

It is easy to see that $Y_n^{(k)}$ is indeed a simple random walk on $\mathbb{Z}$. In addition, as the expected time to reach $\pm k$ starting from 0 is k^2, we have $\mathbb{E}[N_k(n)] = \frac{n}{k^2}$.

Proposition 4.2. *There are universal constants $d_1, c, C > 0$ such that for all n and for all $k \le \frac{d_1\sqrt{n}}{\sqrt{\log\log n}}$,*

$$\mathrm{P}\left(N_k(n) \notin \left[\frac{cn}{k^2}, \frac{Cn}{k^2}\right]\right) \le \frac{2}{\log^3 n}.$$

Proof. By choosing appropriate $c, c' > 0$, Appendix A shows that

$$\mathrm{P}\left(N_k(n) \notin \left[\frac{cn}{k^2}, \frac{Cn}{k^2}\right]\right) \le 2\exp\left(-\frac{n}{20k^2}\right).$$

Note that if $k \le \frac{\sqrt{n}}{8\sqrt{\log\log n}}$, we have $\frac{n}{k^2} \ge 64\log\log n$, so

$$\exp\left(-\frac{n}{20k^2}\right) \le \exp\left(-3\log\log n\right) = \frac{1}{\log^3 n}$$

completing the proof. ∎

We turn to give bounds on the k-excursions of S_t by means of the induced random walks. Denote the local time of $Y_n^{(k)}$ at x by

$$L^{(k)}(x,n) = |\{0 \le t \le n \mid Y_t^{(k)} = x\}|,$$

and let

$$\ell^{(k)}(x,n) = |\{0 \le t < n \mid Y_t^{(k)} = x, Y_{t+1}^{(k)} = x-1\}|.$$

Proposition 4.3. *For any $x \in \mathbb{Z}$,*

$$\ell^{(2k)}\left(\left\lceil \frac{x}{2k}\right\rceil, n\right) - 1 \le T(k,x,n) \le L^{(k/2)}\left(\left\lfloor \frac{x}{k/2}\right\rfloor, n\right).$$

We remark that when $\frac{k}{2}$ is not an integer, it can be replaced with $\lfloor\frac{k}{2}\rfloor$ and the proposition still holds. However, we treat $\frac{k}{2}$ as an integer for convenience.

Proof. For the upper bound, we write $x = \frac{k}{2}j + r$ with $0 \le r < \frac{k}{2}$. Any k-excursion starting at x must include at least one $\frac{k}{2}$-excursion starting at $\frac{k}{2}j$, so $T(k,x,n) \le T\left(\frac{k}{2}, \frac{k}{2}j, n\right) \le L^{(k/2)}(j,n)$.

For the lower bound, write $x = 2kj' + r'$ with $0 \le r' < 2k$. A similar argument shows that $T(2k, 2k(j'+1), n) \le T(k,x,n)$. We note that $\ell^{(2k)}(2k(j'+1), n)$ counts the number of $2k$-excursions starting at $2k(j'+1)$ which are completed before time n, with the possibility that we do not finish the last $2k$-excursion before time n. This proves the lower bound. ∎

5. The total number of k-excursions

We turn to give upper and lower bounds for

$$T(k,n) = \sum_{x\in\mathbb{Z}} kT(k,kx,n).$$

Note that this also provides bounds on the total number of k-excursions, by the following proposition.

Proposition 5.1. *For any n,k,*

$$\frac{1}{2}T(2k,n) \le \sum_{x\in\mathbb{Z}} T(k,x,n) \le T\Big(\frac{k}{2},n\Big).$$

Proof. For the lower bound, note that any $2k$-excursion starting at $2kx$ must contain at least one k-excursion from each of $2kx-k,\dots,2kx$. Therefore,

$$T(2k,2kx,n) \le \frac{1}{k}\sum_{j=0}^{k-1} T(k,2kx-j,n),$$

and we have

$$T(2k,n) = 2k\sum_{x\in\mathbb{Z}} T(2k,2kx,n) \le 2\sum_{x\in\mathbb{Z}} T(k,x,n).$$

For the upper bound, notice that any k-excursion from $\frac{k}{2}x+j$ for some $0\le j<\frac{k}{2}$ contains at least one $\frac{k}{2}$-excursion from $\frac{k}{2}x$. Therefore,

$$\sum_{x\in\mathbb{Z}} T(k,x,n) = \sum_{x\in\mathbb{Z}}\sum_{j=0}^{k/2-1} T\Big(k,\frac{k}{2}x+j,n\Big) \le \frac{k}{2}\sum_{x\in\mathbb{Z}} T\Big(\frac{k}{2},\frac{k}{2}x,n\Big) = T\Big(\frac{k}{2},n\Big). \qquad \blacksquare$$

Before studying the asymptotic behavior of $T(k,n)$, we prove an (almost) monotonicity result for $T(k,n)$.

Lemma 5.2. *Suppose $2k\le k'$. Then $T(k',n)\le 3T(k,n)$.*

Proof. For any $x\in\mathbb{Z}$, each k'-excursion from $k'x$ contains at least $\lfloor\frac{k'}{k}\rfloor-1$ many k-excursions from points $k\mathbb{Z}$. Therefore,

$$\begin{aligned} T'(k',n) &= k'\sum_{x\in\mathbb{Z}} T(k',k'x,n) \le k'\frac{1}{\lfloor\frac{k'}{k}\rfloor-1}\sum_{x\in\mathbb{Z}} T(k,kx,n) \\ &= \frac{\frac{k'}{k}}{\lfloor\frac{k'}{k}\rfloor-1} T(k,n) \le 3T(k,n). \end{aligned} \qquad \blacksquare$$

For specific values of n and k of order at most $\frac{\sqrt{n}}{\sqrt{\log\log n}}$, we use the k-induced random walks defined in Section 4 to get a concentration result.

Lemma 5.3. *There are universal constants $d_2, c, C > 0$ such that the following holds: For large enough n, if $k \leq \frac{d_2\sqrt{n}}{\sqrt{\log\log n}}$,*

$$\mathrm{P}\left(T(k,n) \notin \left[\frac{cn}{k}, \frac{Cn}{k}\right]\right) \leq \frac{6}{\log^3 n}.$$

Proof. Using Proposition 4.2, let $d_1, c, C > 0$ be the constants such that

$$\mathrm{P}\left(N_k(n) \notin \left[\frac{cn}{k^2}, \frac{Cn}{k^2}\right]\right) \leq \frac{2}{\log^3 n}$$

for all $k \leq \frac{d_1\sqrt{n}}{\sqrt{\log\log n}}$. Then for any $k \leq \frac{d_1\sqrt{n}}{\sqrt{\log\log n}}$, by Proposition 4.3,

$$\begin{aligned}\mathrm{P}\left(T(k,n) > \frac{4Cn}{k}\right) &\leq \mathrm{P}\left(k\sum_{x\in\mathbb{Z}} L^{(k/2)}(x,n) > \frac{4Cn}{k}\right)\\ &= \mathrm{P}\left(N_{k/2}(n) > \frac{Cn}{(k/2)^2}\right) \leq \frac{2}{\log^3 n}.\end{aligned}$$

For the lower bound, we use again Proposition 4.3:

$$\begin{aligned}&\mathrm{P}\left(T(k,n) < \frac{k}{8}N_{2k}(n) \;\middle|\; N_{2k}(n)\right)\\ &\quad\leq \mathrm{P}\left(k\sum_{x\in\mathbb{Z}}(\ell^{(2k)}(x,n)-1) < \frac{k}{8}N_{2k}(n) \;\middle|\; N_{2k}(n)\right)\\ &\quad\leq \mathrm{P}\left(\sum_{x\in\mathbb{Z}}(\ell^{(2k)}(x,n)-1) < \frac{1}{8}N_{2k}(n) \;\middle|\; N_{2k}(n)\right).\end{aligned}$$

Let $Z = |\{x : L^{(2k)}(x,n) > 0\}|$. This is the range of a simple random walk on $\mathbb{Z}$ after $N_{2k}(n)$ steps. We thus have

$$\mathrm{P}\left(Z \geq \frac{1}{8}N_{2k}(n) \;\middle|\; N_{2k}(n)\right) \leq \exp\left(-\frac{(N_{2k}(n)/8)^2}{2N_{2k}(n)}\right) = \exp\left(-\frac{N_{2k}(n)}{128}\right),$$

so

$$\begin{aligned}&\mathrm{P}\left(T(k,n) < \frac{k}{8}N_{2k}(n) \;\middle|\; N_{2k}(n)\right)\\ &\quad\leq \mathrm{P}\left(\sum_{x\in Z}\ell^{(2k)}(x,n) < \frac{1}{8}N_{2k}(n) + Z \;\middle|\; N_{2k}(n)\right)\\ &\quad\leq \mathrm{P}\left(\sum_{x\in Z}\ell^{(2k)}(x,n) < \frac{1}{4}N_{2k}(n) \;\middle|\; N_{2k}(n)\right) + \exp\left(-\frac{N_{2k}(n)}{128}\right).\end{aligned}$$

Conditioning on $N_{2k}(n)$, $\sum_{x\in Z}\ell^{(2k)}(x,n)$ is a binomial random variable, with parameters $N_{2k}(n)$ and $\frac{1}{2}$. Therefore, by Chernoff's inequality,

$$\mathrm{P}\Big(\sum_{x\in Z}\ell^{(2k)}(x,n)<\frac{1}{4}N_{2k}(n)\;\Big|\;N_{2k}(n)\Big)\le\exp\Big(-\frac{1}{16}N_{2k}(n)\Big).$$

We therefore have

$$\mathrm{P}\Big(T(k,n)<\frac{k}{8}N_{2k}(n)\;\Big|\;N_{2k}(n)\Big)\le 2\exp\Big(-\frac{1}{128}N_{2k}(n)\Big),$$

so

$$\begin{aligned}\mathrm{P}\Big(T(k,n)<\frac{cn}{16k}\Big)&\le\mathrm{P}\Big(N_{2k}(n)<\frac{cn}{2k^2}\Big)+\mathrm{P}\Big(T(k,n)<\frac{cn}{16k}\;\Big|\;N_{2k}(n)\ge\frac{cn}{2k^2}\Big)\\&\le\frac{2}{\log^3 n}+\mathrm{P}\Big(T(k,n)<\frac{k}{8}N_{2k}(n)\;\Big|\;N_{2k}(n)\ge\frac{cn}{2k^2}\Big)\\&\le\frac{2}{\log^3 n}+2\exp\Big(-\frac{cn}{256k^2}\Big).\end{aligned}$$

Let $d_2>0$ be a constant such that $d_2^2\le\sqrt{c/768}$. For any $k\le\frac{d_2\sqrt{n}}{\sqrt{\log\log n}}$,

$$\frac{n}{k^2}\ge\frac{1}{d_2^2}\log\log n\ge\frac{768}{c}\log\log n,$$

and thus

$$\mathrm{P}\Big(T(k,n)<\frac{cn}{16k}\Big)\le\frac{4}{\log^3 n}$$

concluding the proof. ∎

We are now ready to get a concentration result for all values of n,k simultaneously.

Proposition 5.4. *There are universal constants $d_2,c_1,C_1>0$ such that the following holds almost surely: For all but finitely many n and for all $k\le\frac{d_2\sqrt{n}}{2\sqrt{\log\log n}}$,*

$$\frac{c_1 n}{k}\le T(k,n)\le\frac{C_1 n}{k}.$$

Proof. We choose an exponential sequence of times $t_m=2^m$, and for each m an exponential sequence $k_{m,i}=2^i$ defined for all i such that $k_{m,i}\le\frac{d_2\sqrt{t_m}}{\sqrt{\log\log t_m}}$. Note that for each m there are at most $K\log t_m$ such indices for some constant K.

By Lemma 5.3, there are constants $c,C>0$ such that for each m,i,

$$\mathrm{P}\Big(T(k_{m,i},t_m)\notin\Big[\frac{ct_m}{k_{m,i}},\frac{Ct_m}{k_{m,i}}\Big]\Big)\le\frac{6}{\log^3 t_m}.$$

Taking a union bound over i, we have

$$\mathrm{P}\left(\exists i : T(k_{m,i}, t_m) \notin \left[\frac{ct_m}{k_{m,i}}, \frac{Ct_m}{k_{m,i}}\right]\right) \leq \frac{6K}{\log^2 t_m} = \frac{6K}{m^2 \log^2 2}.$$

As the latter expression is summable over m, by the Borel–Cantelli lemma we have that for all but finitely many m, for all i

$$\frac{ct_m}{k_{m,i}} \leq T(k_{m,i}, t_m) \leq \frac{Ct_m}{k_{m,i}}. \tag{5}$$

We extend this result to general n, k in two steps.

First, let m be such that (5) holds for all i, and let $k \leq \frac{d_2\sqrt{t_m}}{2\sqrt{\log\log t_m}}$. Take i such that $k_{m,i} \leq k < k_{m,i+1}$. By Lemma 5.2,

$$T(k, t_m) \leq 3T(k_{m,i-1}, t_m) \leq \frac{3Ct_m}{k_{m,i-1}} \leq \frac{6Ct_m}{k}$$

and

$$T(k, t_m) \geq \frac{1}{3}T(k_{m,i+1}, t_m) \geq \frac{ct_m}{3k_{m,i+1}} \geq \frac{ct_m}{6k}.$$

Finally, take any n, and let m be such that $t_m \leq n < t_{m+1}$. Suppose n is large enough so that (5) holds. Then,

$$T(k, n) \leq T(k, t_{m+1}) \leq \frac{6Ct_{m+1}}{k} \leq \frac{12Cn}{k}$$

and

$$T(k, n) \geq T(k, t_m) \geq \frac{ct_m}{6k} \geq \frac{cn}{12k}$$

as required. ∎

6. The truncated sum of k-excursions, lower layer

We turn to study the sum $\sum_{x\in\mathbb{Z}} \min\{T(k, x, n), l\}$ for $k, l > 0$. Our two cases of interest are when $kl \sim \sqrt{n \log\log n}$ (which will turn out to be the lim inf layer) and when $kl \sim \frac{\sqrt{n}}{\sqrt{\log\log n}}$ (which will be the lim sup layer). We begin with the lim inf layer; for this we study the maximal number of k-excursions from a given point, that is, $\max_{x\in\mathbb{Z}} T(k, x, n)$.

6.1. The maximal number of k-excursions

Similarly to the way we estimated $T(k, n)$, we get tail bounds on the maximal number of k-excursions in two steps. We first use the induced random walks to get a tail bound for given values of n and k, and then use exponential scales to combine these bounds into a bound for the maximal number of k-excursions.

Lemma 6.1. *There is a universal constant $c > 0$ such that for all large enough n and for all $k \le \frac{d_1\sqrt{n}}{\sqrt{\log\log n}}$,*

$$\mathrm{P}\left(\max_{x\in\mathbb{Z}} T(k,x,n) \ge \frac{c\sqrt{n\log\log n}}{k}\right) \le \frac{3}{\log^3 n}.$$

Proof. We use the induced random walks from Section 4. By Proposition 4.3,

$$\mathrm{P}\left(\max_{x\in\mathbb{Z}} T(k,x,n) \ge \frac{c\sqrt{n\log\log n}}{k}\right) \le \mathrm{P}\left(\max_{x\in\mathbb{Z}} L^{(k/2)}(x,n) \ge \frac{c\sqrt{n\log\log n}}{k}\right).$$

From Appendix B,

$$\mathrm{P}\left(\max_{x\in\mathbb{Z}} L(x,n) \ge mn^{1/2}\right) \le Cm^2\exp(-C'm^2)$$

for large enough n, m. Let $c' > 0$ be a constant such that for $k \le \frac{d_1\sqrt{n}}{\sqrt{\log\log n}}$

$$\mathrm{P}\left(N_{k/2}(n) \ge \frac{c'n}{k^2}\right) \le \frac{2}{\log^3 n}.$$

Using Proposition 4.2,

$$\begin{aligned}
&\mathrm{P}\left(\max_{x\in\mathbb{Z}} L^{(k/2)}(x,n) \ge \frac{c\sqrt{n\log\log n}}{k}\right)\\
&\quad\le \mathrm{P}\left(N_{k/2}(n) \ge \frac{c'n}{k^2}\right) + \mathrm{P}\left(\max_{x\in\mathbb{Z}} L\left(x,\frac{c'n}{k^2}\right) \ge \frac{c\sqrt{n\log\log n}}{k}\right)\\
&\quad\le \frac{2}{\log^3 n} + \frac{c^2}{c'}\log\log n\exp\left(-\frac{c^2}{c'}\log\log n\right).
\end{aligned}$$

Finally, choosing $c = 2\sqrt{c'}$, we have

$$\frac{c^2}{c'}\log\log n\exp\left(-\frac{c^2}{c'}\log\log n\right) = \frac{2\log\log n}{\log^4 n} \le \frac{1}{\log^3 n}$$

for large enough n, as required. ∎

Proposition 6.2. *There are universal constants $d_1, C > 0$ such that the following holds almost surely: For all but finitely many n and for all $k \le \frac{d_1\sqrt{n}}{2\sqrt{\log\log n}}$,*

$$\max_{x\in\mathbb{Z}} T(k,x,n) \le \frac{C\sqrt{n\log\log n}}{k}.$$

Proof. We choose an exponential sequence of times $t_m = 2^m$, and for each m an exponential sequence $k_{m,i} = 2^i$ defined for all i such that $k_{m,i} \le \frac{\sqrt{t_m}}{2\sqrt{\log\log t_m}}$. Note that for each m there are at most $\log t_m$ such indices.

By Lemma 6.1, there is a constant $c > 0$ such that for all m, i we have

$$\mathrm{P}\left(\max_{x\in\mathbb{Z}} T(k_{m,i}, x, t_m) \geq \frac{c\sqrt{t_m}\log\log t_m}{k_{m,i}}\right) \leq \frac{3}{\log^3 t_m}.$$

Taking a union bound,

$$\mathrm{P}\left(\exists i : \max_{x\in\mathbb{Z}} T(k_{m,i}, x, t_m) \geq \frac{c\sqrt{t_m}\log\log t_m}{k_{m,i}}\right) \leq \frac{3}{\log^2 t_m} = \frac{3}{m^2\log^2 2}.$$

The latter expression is summable over m, so by Borel–Cantelli lemma we have that for all but finitely many m, for all i we have

$$\max_{x\in\mathbb{Z}} T(k_{m,i}, x, t_m) \leq \frac{c\sqrt{t_m}\log\log t_m}{k_{m,i}}. \tag{6}$$

Let n, k be such that $k \leq \frac{d_1\sqrt{n}}{2\sqrt{\log\log n}}$. Take m such that $t_m \leq n < t_{m+1}$, and suppose n is large enough so that (6) holds. Take i such that $k_{m,i} \leq k < k_{m,i+1}$. Then

$$\begin{aligned}\max_{x\in\mathbb{Z}} T(k, x, n) &\leq \max_{x\in\mathbb{Z}} T(k_{m,i}, x, t_{m+1})\\ &\leq \frac{c\sqrt{t_{m+1}}\log\log t_{m+1}}{k_{m,i}}\\ &\leq \frac{4c\sqrt{n}\log\log n}{k}\end{aligned}$$

as required. ∎

6.2. The truncated sum

We are now ready to get an asymptotic bound for the truncated sum of k-excursions.

Proposition 6.3. *There are universal constants $d_2, c_2 > 0$ such that the following holds almost surely: For all but finitely many n, for all $k \leq \frac{d_2\sqrt{n}}{2\sqrt{\log\log n}}$ and for all l,*

$$\sum_{x\in\mathbb{Z}} \min\{T(k, x, n), l\} \geq c_2 \min\left\{\frac{n}{k}, \frac{\sqrt{n}}{\sqrt{\log\log n}} l\right\}.$$

Proof. We first assume that $l \geq \frac{\sqrt{n}\log\log n}{k}$. By Proposition 6.2, there is a constant $C > 0$ such that for large enough n we have

$$\max_{x\in\mathbb{Z}} T(k, x, n) \leq \frac{C\sqrt{n}\log\log n}{k} \leq Cl,$$

so by Proposition 5.4

$$\sum_{x\in\mathbb{Z}} \min\{T(k, x, n), l\} \geq \frac{1}{C}\sum_{x\in\mathbb{Z}} T(k, x, n) \geq \frac{1}{2C} T(2k, n) \geq \frac{c'n}{k}$$

for large enough n and a universal constant c'.

Now suppose $l < \frac{\sqrt{n \log\log n}}{k}$. Let

$$A_1 = \{x \in \mathbb{Z} \mid T(k,x,n) < l\}, \quad A_2 = \{x \in \mathbb{Z} \mid T(k,x,n) \geq l\}.$$

Note that by Propositions 5.1 and 5.4, for large enough n we have

$$\begin{aligned}\frac{c_1 n}{4k} \leq \frac{1}{2} T(2k,n) &\leq \sum_{x\in\mathbb{Z}} T(k,x,n) \\ &= \sum_{x \in A_1} T(k,x,n) + \sum_{x\in A_2} T(k,x,n).\end{aligned}$$

We split into two cases:

(i) If $\sum_{x\in A_1} T(k,x,n) \geq \frac{c_1 n}{8k}$, then clearly

$$\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} \geq \sum_{x\in A_1} T(k,x,n) \geq \frac{c_1 n}{8k} \geq \frac{c_1\sqrt{n}}{8\sqrt{\log\log n}} l.$$

(ii) Otherwise, we must have $\sum_{x\in A_2} T(k,x,n) \geq \frac{c_1 n}{8k}$. By Proposition 6.2,

$$\frac{c_1 n}{8k} \leq \sum_{x\in A_2} T(k,x,n) \leq \frac{c\sqrt{n\log\log n}}{k} \cdot |A_2|,$$

so $|A_2| \geq \frac{c_1\sqrt{n}}{8c\sqrt{\log\log n}}$, showing

$$\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} \geq l \cdot |A_2| \geq \frac{c_1\sqrt{n}}{8c\sqrt{\log\log n}} l.$$

The proposition is thus proved. ∎

7. The truncated sum of k-excursions, upper layer

For the remaining case, we switch strategy. We first need the exact distribution of $T(k,0,n)$, which can be found, for example, in [5]. The main idea is the following: We use the reflection principle successively, reflecting the random walk whenever we first visit any jk for any $j < 0$. In this manner, the number of k-excursions starting from 0 has the same distribution (up to a factor of $2k$) as the minimum (or maximum) of the random walk. This proves the following.

Lemma 7.1 ([5]). *For any $a \in \mathbb{N}$,*

$$\begin{aligned}\mathrm{P}(T(k,0,n) \geq a) &= \mathrm{P}\Big(\min_{0\leq t\leq n} S_t \leq -2ka\Big) \\ &= \mathrm{P}\Big(\max_{0\leq t\leq n} S_t \geq 2ka\Big).\end{aligned}$$

Lemma 7.2. *There are universal constants $c, c' > 0$ such that for large enough n and for all $k \leq \frac{c\sqrt{n}}{\sqrt{\log\log n}}$,*

$$\mathrm{P}\left(T(k,0,n) \leq \frac{c\sqrt{n}}{k\sqrt{\log\log n}}\right) \leq \frac{c'}{\log^3 n}.$$

Proof. By the reflection principle (see Lemma 7.1),

$$\begin{aligned}\mathrm{P}\left(T(k,0,n) \leq \frac{c\sqrt{n}}{k\sqrt{\log\log n}}\right) &\leq \mathrm{P}\left(T(k,0,n) \leq \left\lceil \frac{c\sqrt{n}}{k\sqrt{\log\log n}} \right\rceil\right) \\ &= \mathrm{P}\left(\max_{0\leq t\leq n} S_t \leq 2k\left\lceil \frac{c\sqrt{n}}{k\sqrt{\log\log n}} \right\rceil\right) \\ &\leq \mathrm{P}\left(\max_{0\leq t\leq n} S_t \leq \frac{2c\sqrt{n}}{\sqrt{\log\log n}} + 2k\right) \\ &\leq \mathrm{P}\left(\max_{0\leq t\leq n} S_t \leq \frac{4c\sqrt{n}}{\sqrt{\log\log n}}\right),\end{aligned}$$

where the last inequality follows from $k \leq \frac{c\sqrt{n}}{\sqrt{\log\log n}}$. To bound the latter probability, we use [12, Lemma 2]. Following their proof, one can choose $c = \frac{\pi}{4}$ and get

$$\mathrm{P}\left(\max_{0\leq t\leq n} S_t \leq \frac{\pi\sqrt{n}}{4\sqrt{\log\log n}}\right) \leq \frac{c'}{\log^3 n}$$

for large enough n, concluding our proof. ■

Proposition 7.3. *There exists a universal constant $c > 0$ such that the following holds almost surely: For all but finitely many values of n and for all $k \leq \frac{c\sqrt{n}}{2\sqrt{\log\log n}}$,*

$$\sum_{x\in\mathbb{Z}} \min\left\{T(k,x,n), \frac{c\sqrt{n}}{k\sqrt{\log\log n}}\right\} \geq \frac{c\sqrt{n}}{k\sqrt{\log\log n}} \cdot |\mathrm{range}(S_{n/4})|.$$

Proof. For convenience, write $l = \frac{c\sqrt{n}}{k\sqrt{\log\log n}}$. We first note that

$$\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} \geq \sum_{x\in\mathrm{range}(S_{n/2})} \min\{T(k,x,n), l\}.$$

Writing

$$Z_{n,k} = \sum_{x\in\mathrm{range}(S_{n/2})} \max\{l - T(k,x,n), 0\},$$

we have

$$\sum_{x\in\mathrm{range}(S_{n/2})} \min\{T(k,x,n), l\} = l \cdot |\mathrm{range}(S_{n/2})| - Z_{n,k}, \tag{7}$$

so we turn to upper bound $Z_{n,k}$.

Note that for any $x \in \mathrm{range}(S_{n/2})$, by Lemma 7.2

$$\mathrm{P}(T(k,x,n) \le l) \le \mathrm{P}(T(k,0,n/2) \le l) \le \frac{c'}{\log^3 n}$$

for large enough n. Therefore, for any $x \in \mathrm{range}(S_{n/2})$,

$$\mathbb{E}[\max\{l - T(k,x,n), 0\}] \le l \cdot \mathrm{P}(T(k,x,n) \le l) \le \frac{c'}{\log^3 n} l,$$

and thus

$$\mathbb{E}[Z_{n,k}] \le \frac{c'}{\log^3 n} l \cdot \mathbb{E}[\mathrm{range}(S_{n/2})] \le \frac{c'' \sqrt{n}}{\log^3 n} l.$$

By Markov's inequality,

$$\mathrm{P}\left(Z_{n,k} \ge \frac{\sqrt{n}}{4\sqrt{\log\log n}} l\right) \le \frac{\frac{c''\sqrt{n}}{\log^3 n} l}{\frac{\sqrt{n}}{4\sqrt{\log\log n}} l} \le \frac{4c''}{\log^{2.5} n}$$

for large enough n.

Let $t_m = 2^m$ and let $k_{m,i} = \frac{c\sqrt{t_m}}{2^i \sqrt{\log\log t_m}}$ (so $l_{m,i} = \frac{c\sqrt{t_m}}{k_{m,i}\sqrt{\log\log t_m}} = 2^i$) for all i such that $k_{m,i} \ge 1$. As there are at most $\log t_m$ such indices, we have

$$\mathrm{P}\left(\exists i : Z_{t_m,k_{m,i}} \ge \frac{\sqrt{t_m}}{4\sqrt{\log\log t_m}} l_{m,i}\right) \le \frac{4c''}{\log^{1.5} t_m} = \frac{4c''}{m^{1.5}\log^{1.5} 2}.$$

The latter expression is summable over m, so by Borel–Cantelli lemma we have that for all but finitely many m and for all i as above,

$$Z_{t_m,k_{m,i}} \le \frac{\sqrt{t_m}}{4\sqrt{\log\log t_m}} l_{m,i}.$$

From the lim inf law of iterated logarithm (see [4, 12]), we know that for large enough m, $|\mathrm{range}(S_{t_m/2})| > \frac{\sqrt{t_m}}{2\sqrt{\log\log t_m}}$, and thus (7) yields

$$\sum_{x\in\mathbb{Z}} \min\{T(k_{m,i}, x, t_m), l_{m,i}\} \ge \frac{l_{m,i}}{4} \cdot |\mathrm{range}(S_{t_m/2})| \tag{8}$$

for all but finitely many m and for all i as above.

Now, let n, let $k \le \frac{c\sqrt{n}}{2\sqrt{\log\log n}}$, and write $l = \frac{c\sqrt{n}}{k\sqrt{\log\log n}}$. Let m such that $t_m \le n < t_{m+1}$, and let i such that $k_{m,i} \le k < k_{m,i+1}$. Assume n is large enough so that (8) holds. Then

$$\begin{aligned}\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} &\ge \sum_{x\in\mathbb{Z}} \min\{T(k_{m,i+1}, x, t_m), l_{m,i+1}\} \\ &\ge \frac{l_{m,i+1}}{4} \cdot |\mathrm{range}(S_{t_m/2})| \\ &\ge \frac{l}{8} \cdot |\mathrm{range}(S_{n/4})| \\ &= \frac{c\sqrt{n}}{8k\sqrt{\log\log n}} \cdot |\mathrm{range}(S_{n/4})|.\end{aligned}$$

■

Corollary 7.4. *There are universal constants $d_3, c_3 > 0$ such that the following holds almost surely: For infinitely many n and for all k, l with $k \leq \frac{d_3\sqrt{n}}{2\sqrt{\log\log n}}$,*

$$\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} \geq c_3 \min\left\{\frac{n}{k}, \sqrt{n\log\log n}\, l\right\}.$$

Proof. By the usual law of iterated logarithm, there are infinitely many n such that $|\mathrm{range}(S_{n/4})| \geq \frac{1}{4}\sqrt{n\log\log n}$. Using Proposition 7.3, we have two cases:

(i) If $l \geq \frac{\sqrt{n}}{k\sqrt{\log\log n}}$, then

$$\begin{aligned}\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} &\geq \sum_{x\in\mathbb{Z}} \min\left\{T(k,x,n), \frac{\sqrt{n}}{k\sqrt{\log\log n}}\right\}\\ &\geq \frac{c\sqrt{n}}{4k\sqrt{\log\log n}}\sqrt{n\log\log n} = \frac{cn}{4k}.\end{aligned}$$

(ii) If $l < \frac{\sqrt{n}}{k\sqrt{\log\log n}}$, let $k' = \frac{\sqrt{n}}{l\sqrt{\log\log n}} > k$. Then

$$\begin{aligned}\sum_{x\in\mathbb{Z}} \min\{T(k,x,n), l\} &\geq \sum_{x\in\mathbb{Z}} \min\{T(k',x,n), l\}\\ &\geq \frac{c\sqrt{n}}{4k'\sqrt{\log\log n}}\sqrt{n\log\log n}\\ &= \frac{c}{4}\sqrt{n\log\log n}\, l.\end{aligned}$$

This concludes the proof. ■

8. Bounds on the distance in Δ

We return to the construction of [3]. Recall that our group Δ is a diagonal product of lamplighter groups $\Delta_s = \Gamma_s \wr \mathbb{Z}$, where each Γ_s is generated by a set of the form $A(s) \cup B(s)$, and the diagonal product is taken with respect to the generating set $\alpha_i(s) = (a_i(s)\delta_0, 0)$, $\beta_i(s) = (b_i(s)\delta_{k_s}, 0)$ and $\tau(s) = (e, +1)$.

As explained in Subsection 2.3, for any given layer s, the number of Γ_s-steps the random walk makes at x in Δ_s depends only on whether we reached x and on the number of k_s-excursions from x. In this section, we provide bounds for the distance of a simple random walk on Δ and on Δ_s in terms of the k_s-excursions.

For the upper bound on the distance, we use the bound found in [3, Proposition 2.14 and proof of Lemma 3.4].

Proposition 8.1. *For all $s \geq 0$,*

$$|W_n|_{\Delta_s} \leq 11 \min\left\{\sum_{j\in\mathbb{Z}} k_s T\left(\frac{k_s}{2}, j\frac{k_s}{2}, n\right) + |\mathrm{range}(S_n)|, |\mathrm{range}(S_n)|\, l_s\right\}.$$

In addition, writing $s_0(n) = \max\{s \geq 0 \mid k_s \leq |\mathrm{range}(S_n)|\}$, *we have*

$$|W_n|_\Delta \leq 500 \sum_{s \leq s_0(n)} |W_n|_{\Delta_s}.$$

We now turn to the lower bound. For all $s \geq 0$, we have

$$|W_n|_\Delta \geq |W_n|_{\Delta_s} \geq \sum_{x \in \mathbb{Z}} |f_n(x)|_{\Gamma_s}.$$

To get a lower bound for the RHS, we use the following lemma.

Lemma 8.2. *Let* $\{\Gamma_s\}$ *be a sequence of finite groups, let* $l_s = \mathrm{diam}(\Gamma_s)$, *and suppose that* $\{\Gamma_s\}$ *satisfies* $(\sigma, c_0 l_s)$*-linear speed assumption for some constants* $0 < \sigma, c_0 < 1$ *with* $\sigma c_0 l_s \geq 4$. *Then for all* $t \geq 0$,

$$\mathrm{P}\left(\left|X_t^{(s)}\right|_{\Gamma_s} \geq \frac{\sigma}{8} \min\{t, c_0 l_s\}\right) \geq \frac{\sigma c_0}{8}.$$

Proof. Assume first that $t \leq c_0 l_s$. Let $p = \mathrm{P}(|X_t^{(s)}|_{\Gamma_s} \geq \frac{\sigma}{2} t)$. Then

$$\sigma t \leq \mathbb{E}|X_t^{(s)}|_{\Gamma_s} \leq (1-p)\frac{\sigma}{2} t + pt \leq \frac{\sigma}{2} t + pt,$$

so $p \geq \frac{\sigma}{2}$, and the assertion follows.

Now, let $t \geq c_0 l_s$. By [20, Lemma 4.1], we have

$$\mathbb{E}|X_t^{(s)}|_{\Gamma_s} \geq \frac{1}{2}\mathbb{E}\left|X_{c_0 l_s}^{(s)}\right|_{\Gamma_s} - 1 \geq \frac{\sigma}{2} c_0 l_s - 1.$$

If $\sigma c_0 l_s \geq 4$, we therefore have

$$\mathbb{E}|X_t^{(s)}| \geq \frac{\sigma c_0 l_s}{4}.$$

Let $p = \mathrm{P}\left(|X_t^{(s)}| \geq \frac{\sigma c_0 l_s}{8}\right)$. Then

$$\frac{\sigma c_0 l_s}{4} \leq \mathbb{E}|X_t^{(s)}| \leq (1-p)\frac{\sigma c_0 l_s}{8} + p l_s \leq \frac{\sigma c_0 l_s}{8} + p l_s.$$

We therefore have $p \geq \frac{\sigma c_0}{8}$, as required. ■

We are now ready to prove our lower bound on the metric in a single layer.

Proposition 8.3. *Let* $\{\Gamma_s\}$ *be a sequence of finite groups, let* $l_s = \mathrm{diam}(\Gamma_s)$, *and suppose that* $\{\Gamma_s\}$ *satisfies* $(\sigma, c_0 l_s)$*-linear speed assumption for some constants* $\sigma, c_0 > 0$. *Also suppose there is* $m_0 > 1$ *such that*

$$k_{s+1} > 2k_s \quad \text{and} \quad l_{s+1} \geq m_0 l_s$$

for all s. *Then, almost surely, for large enough* n, *for all* s *such that* $k_s \leq \frac{d_2\sqrt{n}}{\sqrt{\log\log n}}$ *we have*

$$|W_n|_{\Delta_s} \geq \frac{\sigma c_0}{16} \sum_{x \in \mathbb{Z}} \min\{T(k_s, x, n), c_0 l_s\}.$$

Proof. We use a similar approach to Revelle [21]. It is trivial that for all n, s,

$$|W_n|_{\Delta_s} \geq \frac{1}{2} \sum_{x\in\mathbb{Z}} |f_n(x)|_{\Gamma_s}.$$

Let $I_{x,n}$ denote the indicator of whether the random walk on $\mathbb{Z}$ visited x before time n. Conditioning on $T(k_s, x, n)$ and $I_{x,n}$ for all x, the random variables $|f_n(x)|_{\Gamma_s}$ are independent.

Let $\theta_x = \frac{\sigma}{8}\min\{T(k_s, x, n), c_0 l_s\}$, and write $Z_x = |f_n(x)|_{\Gamma_s}\theta_x^{-1}$. By Lemma 8.2, $\mathrm{P}(Z_x \geq 1) \geq \frac{\sigma c_0}{8}$, so we may apply [21, Lemma 2] to get

$$\begin{aligned}
&\mathrm{P}\Bigg(\sum_{x\in\mathbb{Z}} |f_n(x)|_{\Gamma_s} < \frac{\sigma c_0}{16}\sum_{x\in\mathbb{Z}}\theta_x \;\Bigg|\; T(k_s, x, n), I_{x,n}\Bigg) \\
&\quad = \mathrm{P}\Bigg(\sum_{x\in\mathbb{Z}} \frac{|f_n(x)|_{\Gamma_s}}{\theta_x}\frac{\theta_x}{\max_x \theta_x} < \frac{\sigma c_0}{16}\sum_{x\in\mathbb{Z}}\frac{\theta_x}{\max_x\theta_x} \;\Bigg|\; T(k_s, x, n), I_{x,n}\Bigg) \\
&\quad \leq \exp\Bigg(-\frac{\sigma c_0}{96\max_x\theta_x}\sum_{x\in\mathbb{Z}}\theta_x \;\Bigg|\; T(k_s, x, n), I_{x,n}\Bigg).
\end{aligned} \tag{9}$$

Let $d_2, c, c_2 > 0$ be constants such that Propositions 6.2 and 6.3 hold. For each m, let A_m denote the event that for all $n \geq m$ and for all s with $k_s \leq \frac{d_2\sqrt{n}}{2\sqrt{\log\log n}}$,

$$\max_{x\in\mathbb{Z}}\theta_x \leq \min\left\{\frac{c\sqrt{n\log\log n}}{k_s}, l_s\right\}$$

and

$$\sum_{x\in\mathbb{Z}}\theta_x \geq c_2\min\left\{\frac{n}{k_s}, \frac{\sqrt{n}}{\sqrt{\log\log n}}l_s\right\} \geq \frac{c'\sqrt{n}}{\sqrt{\log\log n}}\max_{x\in\mathbb{Z}}\theta_x.$$

We have $\mathrm{P}\big(\bigcup_{m=1}^{\infty} A_m\big) = 1$. Also, let B_n denote the event that for all s with $k_s \leq \frac{d_2\sqrt{n}}{2\sqrt{\log\log n}}$,

$$\sum_{x\in\mathbb{Z}}|f_n(x)|_{\Gamma_s} < \frac{\sigma}{4}\sum_{x\in\mathbb{Z}}\theta_x.$$

Conditioning on A_m, (9) shows that

$$\mathrm{P}\Bigg(\sum_{x\in\mathbb{Z}}|f_n(x)|_{\Gamma_s} < \frac{\sigma}{4}\sum_{x\in\mathbb{Z}}\theta_x \;\Bigg|\; A_m\Bigg) \leq \exp\left(-\frac{c'\sqrt{n}}{\sqrt{\log\log n}}\right)$$

for all $n \geq m$. As $\{k_s\}$ increases exponentially, we may take a union bound to see

$$\mathrm{P}(B_n \mid A_m) \leq c''\log n\exp\left(-\frac{c'\sqrt{n}}{\sqrt{\log\log n}}\right)$$

for all $n \geq m$ and a constant c''. The latter expression is summable, so the Borel–Cantelli lemma shows that for each m, conditioning on A_m we have that $\mathrm{P}(\liminf B_n^c \mid A_m) = 1$. But as $\mathrm{P}\big(\bigcup_{m=1}^{\infty} A_m\big) = 1$, we must have $\mathrm{P}(\liminf B_n^c) = 1$, as required. ∎

9. Bounds on the distance in a single layer

We use the results obtained in the previous section to bound the distance of the random walk in each layer separately. As explained before, for a given s, the number of steps that the random walk on $\Delta_s = \Gamma_s \wr \mathbb{Z}$ makes in the copy of the lamp group Γ_s at any given point $x \in \mathbb{Z}$ until time n is $T(k_s, x, n)$, and the distance in each copy of Γ_s is also bounded from above by l_s.

When s is small, we expect to reach saturation in Γ_s relatively fast, so the effective bound should be l_s. However, when s is big, the number of excursions can be small compared to l_s, so the bound should depend on $T(k_s, x, n)$. We are therefore looking for the layers which will separate these two cases.

To do this, let d_1, d_2, d_3 be the constants from Propositions 5.4 and 6.3 and Corollary 7.4. Take a constant $r \leq \frac{1}{2}\min\{d_1, d_2, d_3\}$, and consider the following layers:

$$\begin{aligned}
s_0'(n) &= \max\Big\{s \geq 0 \;\Big|\; k_s \leq \frac{r\sqrt{n}}{\sqrt{\log\log n}}\Big\},\\
s_0''(n) &= \max\Big\{s \geq 0 \;\Big|\; k_s \leq \frac{2\sqrt{n}}{\sqrt{\log\log n}}\Big\},\\
s_2(n) &= \max\Big\{s \geq 0 \;\Big|\; k_s l_s \leq \frac{r\sqrt{n}}{\sqrt{\log\log n}}\Big\},\\
s_3(n) &= \max\Big\{s \geq 0 \;\Big|\; k_s l_s \leq \sqrt{n\log\log n}\Big\},\\
\widetilde{s}_3(n) &= \min\{s_0'(n), s_3(n)\}.
\end{aligned}$$

We will prove that the dominating layer for the lim sup is $s_2(n)$, whereas the dominating layer for the lim inf is $\widetilde{s}_3(n)$.

9.1. Upper bounds

Proposition 9.1. *There is a universal constant $C > 0$ such that almost surely, for large enough n and all $s \geq 0$,*

$$|W_n|_{\Delta_s} \leq \begin{cases} C\sqrt{n\log\log n}\, l_s, & s \leq s_2(n),\\ \dfrac{Cn}{k_s}, & s_2(n) < s \leq s_0'(n),\\ C\sqrt{n\log\log n}, & s > s_0'(n). \end{cases}$$

Proof. By Proposition 8.1, we have

$$|W_n|_{\Delta_s} \leq 11\min\Big\{T\Big(\frac{k_s}{2}, n\Big) + |\mathrm{range}(S_n)|, |\mathrm{range}(S_n)|\, l_s\Big\}$$

for all $s \geq 0$. Also, by the classical law of iterated logarithm, for large enough n we have

$$|\mathrm{range}(S_n)| \leq 2\sqrt{n\log\log n}. \tag{10}$$

The case $s \le s_2(n)$ follows from $|W_n|_{\Delta_s} \le 11 \cdot |\mathrm{range}(S_n)|\, l_s$ and (10).

We move to the case where $s_2(n) < s \le s_0'(n)$. By Proposition 5.4, almost surely for large enough n and for all $s \le s_0'(n)$,

$$T\Big(\frac{k_s}{2}, n\Big) \le \frac{C_1 n}{k_s/2} = \frac{2C_1 n}{k_s}.$$

Also, by (10), for large enough n we have

$$|\mathrm{range}(S_n)| \le 2\sqrt{n \log\log n} \le \frac{2rn}{k_s},$$

proving the second case.

For the case where $s > s_0'(n)$, as $k_s > \frac{r\sqrt{n}}{\sqrt{\log\log n}}$, we may apply Lemma 5.2 and Proposition 5.4:

$$T(k_s, n) \le 3T\Big(\frac{r\sqrt{n}}{2\sqrt{\log\log n}}, n\Big) \le \frac{3C_1 n}{\frac{r\sqrt{n}}{2\sqrt{\log\log n}}} = \frac{6C_1}{r}\sqrt{n \log\log n}.$$

Now the bound on $|W_n|_{\Delta_s}$ follows from (10). ∎

Proposition 9.2. *There is a universal constant $C > 0$ such that almost surely, for all large enough n for which $|\mathrm{range}(S_n)| \le \frac{2\sqrt{n}}{\sqrt{\log\log n}}$ and all $s \ge 0$,*

$$|W_n|_{\Delta_s} \le \begin{cases} \dfrac{C\sqrt{n}}{\sqrt{\log\log n}}\, l_s, & s \le \tilde{s}_3(n), \\ \dfrac{Cn}{k_s}, & \tilde{s}_3(n) < s \le s_0''(n). \end{cases}$$

Proof. By Proposition 8.1, we have

$$|W_n|_{\Delta_s} \le 11 \min\Big\{T\Big(\frac{k_s}{2}, n\Big) + |\mathrm{range}(S_n)|, |\mathrm{range}(S_n)|\, l_s\Big\}$$

for all $s \ge 0$. Also, by our assumption on n, $|\mathrm{range}(S_n)| \le \frac{2\sqrt{n}}{\sqrt{\log\log n}}$, so the case $s \le \tilde{s}_3(n)$ follows automatically.

We move to the case where $\tilde{s}_3(n) < s \le s_0'(n)$. By Proposition 5.4, almost surely for large enough n and for all $s \le s_0'(n)$,

$$T\Big(\frac{k_s}{2}, n\Big) \le \frac{C_1 n}{k_s/2} = \frac{2C_1 n}{k_s}.$$

In addition,

$$|\mathrm{range}(S_n)| \le \frac{2\sqrt{n}}{\sqrt{\log\log n}} \le \frac{2rn}{k_s},$$

proving part of the second case.

For the case where $s_0'(n) < s \le s_0''(n)$, as $\frac{r\sqrt{n}}{\sqrt{\log\log n}} < k_s \le \frac{2\sqrt{n}}{\sqrt{\log\log n}}$, we may apply Lemma 5.2 and Proposition 5.4:

$$T(k_s, n) \le 3T\left(\frac{r\sqrt{n}}{2\sqrt{\log\log n}}, n\right) \le \frac{C_1 n}{\frac{r\sqrt{n}}{2\sqrt{\log\log n}}} = \frac{2C_1}{r}\sqrt{n\log\log n} < \frac{2rn}{k_s}.$$

The conclusion follows. ∎

9.2. Lower bounds

Proposition 9.3. *There is a universal constant $c > 0$ such that almost surely, for all but finitely many n,*

$$|W_n|_{\Delta_s} \ge \begin{cases} \frac{c\sqrt{n}}{\sqrt{\log\log n}} l_s, & s \le \tilde{s}_3(n), \\ \frac{cn}{k_s}, & \tilde{s}_3(n) < s \le s_0'(n). \end{cases}$$

Proof. Using Proposition 8.3, for large enough n we have

$$|W_n|_{\Delta_s} \ge c \sum_{x \in \mathbb{Z}} \min\{T(k_s, x, n), l_s\}$$

for all $s \le s_0'(n)$. The result follows from Proposition 6.3. ∎

Proposition 9.4. *There is a universal constant $c > 0$ such that almost surely, for infinitely many n,*

$$|W_n|_{\Delta_s} \ge \begin{cases} c\sqrt{n\log\log n}\, l_s, & s \le s_2(n), \\ \frac{cn}{k_s}, & s_2(n) < s \le s_0'(n). \end{cases}$$

Proof. Using Proposition 8.3, for large enough n we have

$$|W_n|_{\Delta_s} \ge c \sum_{x \in \mathbb{Z}} \min\{T(k_s, x, n), l_s\}$$

for all $s \le s_0'(n)$. The result follows from Corollary 7.4. ∎

10. Bounds on the total distance

We now use the bounds on each layer achieved in the previous section to find a bound on the distance of the random walk on Δ in terms of the sequences $\{k_s\}$ and $\{l_s\}$. Recall the following bounds from [3]:

$$|W_n|_\Delta \le C \sum_{s \le s_0(n)} |W_n|_{\Delta_s}, \tag{11}$$

where $s_0(n) = \min\{s \geq 0 \mid k_s \leq |\mathrm{range}(S_n)|\}$, and

$$|W_n|_\Delta \geq |W_n|_{\Delta_s} \tag{12}$$

for any $s \geq 0$.

For the lim sup, recall the critical layer

$$s_2(n) = \max\left\{s \geq 0 \,\middle|\, k_s l_s \leq \frac{r\sqrt{n}}{\sqrt{\log\log n}}\right\},$$

and that

$$s_0'(n) = \max\left\{s \geq 0 \,\middle|\, k_s \leq \frac{r\sqrt{n}}{\sqrt{\log\log n}}\right\},$$

where $r > 0$ is some universal constant.

Proposition 10.1. *Suppose that $\{\Gamma_s\}$ satisfy the $(\sigma, c_0 l_s)$-linear speed assumption and* $\mathrm{diam}(\Gamma_s) \leq C_0 l_s$. *Suppose there exists a constant $m_0 > 1$ such that*

$$k_{s+1} > 2k_s, \quad l_{s+1} \geq m_0 l_s$$

for all s. Let

$$g(n) = \frac{n}{k_{s_2(n)+1}} + \sqrt{n \log\log n}\, l_{s_2(n)}.$$

Then almost surely,

$$0 < \limsup_{n\to\infty} \frac{|W_n|_\Delta}{g(n)} \quad \text{and} \quad \limsup_{n\to\infty} \frac{|W_n|_\Delta}{g(n) + \sqrt{n\log\log n}\,\log\log\log n} < \infty.$$

Proof. For the lower bound, we use (12) and Proposition 9.4, showing that for infinitely many n we have

$$\begin{aligned} |W_n|_\Delta &\geq \frac{1}{2}\left(|W_n|_{\Delta_{s_2(n)}} + |W_n|_{\Delta_{s_2(n)+1}}\right) \\ &\geq \frac{c}{2}\left(\sqrt{n\log\log n}\, l_{s_2(n)} + \frac{n}{k_{s_2(n)+1}}\right) = \frac{c}{2} g(n), \end{aligned}$$

so $\limsup_{n\to\infty} \frac{|W_n|_\Delta}{g(n)} > 0$ almost surely.

For the upper bound, we use (11). We divide the interval $0 \leq s \leq s_0(n)$ into three parts, and use Proposition 9.1 to analyze the contribution of each of them:

- For $0 \leq s \leq s_2(n)$,

$$\begin{aligned} \sum_{s \leq s_2(n)} |W_n|_{\Delta_s} &\leq \sum_{s\leq s_2(n)} C_1 \sqrt{n\log\log n}\, l_s \\ &\leq \frac{C_1}{1 - 1/m_0} \sqrt{n\log\log n}\, l_{s_2(n)} \end{aligned}$$

  where the last inequality follows from $l_{s+1} \geq m_0 l_s$.

- For $s_2(n)+1 \le s \le s_0'(n)$,

$$\sum_{s_2(n)+1\le s\le s_0'(n)} |W_n|_{\Delta_s} \le \sum_{s_2(n)+1\le s\le s_0'(n)} \frac{C_1 n}{k_s} \le \frac{2C_1 n}{k_{s_2(n)+1}},$$

  where the last inequality follows from $k_{s+1} > 2k_s$.
- For $s_0'(n)+1 \le s \le s_0(n)$, recall that for large enough n we have $|\mathrm{range}(S_n)| \le 2\sqrt{n\log\log n}$. As $k_{s+1} > 2k_s$, the number of layers $s_0'(n)+1 \le s \le s_0(n)$ is at most $c\log\log\log n$ for some constant $c>0$, so

$$\sum_{s_0'(n)+1\le s\le s_0(n)} |W_n|_{\Delta_s} \le C_1\sqrt{n\log\log n}\log\log\log n.$$

Summing the above yields

$$|W_n|_\Delta \le C g(n) + C\sqrt{n\log\log n}\log\log\log n,$$

proving the second lim sup is finite. ■

For the lim inf, recall that

$$s_0''(n) = \max\left\{ s\ge 0 \;\middle|\; k_s \le \frac{2\sqrt{n}}{\sqrt{\log\log n}}\right\},$$
$$s_3(n) = \max\left\{ s\ge 0 \;\middle|\; k_s l_s \le \sqrt{n\log\log n}\right\},$$

and that $\tilde{s}_3(n) = \min\{s_0'(n), s_3(n)\}$.

Proposition 10.2. *Suppose that $\{\Gamma_s\}$ satisfy the $(\sigma, c_0 l_s)$-linear speed assumption and* $\mathrm{diam}(\Gamma_s) \le C_0 l_s$. *Suppose there exists a constant $m_0 > 1$ such that*

$$k_{s+1} > 2k_s, \quad l_{s+1} \ge m_0 l_s$$

for all s. Let

$$h(n) = \begin{cases} \dfrac{n}{k_{s_3(n)+1}} + \dfrac{\sqrt{n}}{\sqrt{\log\log n}} l_{s_3(n)}, & s_3(n) < s_0'(n), \\ \dfrac{\sqrt{n}}{\sqrt{\log\log n}} l_{s_0'(n)}, & \text{otherwise.} \end{cases}$$

Then almost surely

$$0 < \liminf_{n\to\infty} \frac{|W_n|_\Delta}{h(n)} < \infty.$$

Proof. For the upper bound we use again (11). Take n such that $|\mathrm{range}(S_n)| \le \frac{2\sqrt{n}}{\sqrt{\log\log n}}$, and assume n is large enough such that Proposition 9.2 holds. We divide the interval $0 \le s \le s_0''(n)$ into several parts and analyze the contribution of each of them:

- For $0 \le s \le \tilde{s}_3(n)$,

$$\sum_{s \le \tilde{s}_3(n)} |W_n|_{\Delta_s} \le \sum_{s \le \tilde{s}_3(n)} \frac{C\sqrt{n}}{\sqrt{\log\log n}} l_s \le \frac{C}{1-1/m_0} \frac{\sqrt{n}}{\sqrt{\log\log n}} l_{\tilde{s}_3(n)},$$

  where the last inequality follows from $l_{s+1} \ge m_0 l_s$.
- For $\tilde{s}_3(n) < s \le s_0''(n)$, we have

$$\sum_{\tilde{s}_3(n)+1 \le s \le s_0''(n)} |W_n|_{\Delta_s} \le \sum_{\tilde{s}_3(n)+1 \le s \le s_0''(n)} \frac{Cn}{k_s} \le \frac{2Cn}{k_{\tilde{s}_3(n)+1}},$$

  where the last inequality follows from $k_{s+1} > 2k_s$.

Summing the above, we have

$$|W_n|_\Delta \le \begin{cases} \frac{C\sqrt{n}}{\sqrt{\log\log n}} l_{s_3(n)} + \frac{Cn}{k_{s_3(n)+1}}, & s_3(n) < s_0'(n), \\ \frac{C\sqrt{n}}{\sqrt{\log\log n}} l_{s_0'(n)} + \frac{Cn}{k_{s_0'(n)}}, & \text{otherwise.} \end{cases}$$

In the second case, $s_0'(n) \le s_3(n)$, so $k_{s_0'(n)} l_{s_0'(n)} \le \sqrt{n \log\log n}$, and thus the latter term is inessential. We therefore get $|W_n|_\Delta \le 2Ch(n)$ for each such n, proving the above lim inf is finite.

To prove that the lim inf is positive, we use (12). Take n large enough so that Proposition 9.3 holds. If $s_3(n) < s_0'(n)$, we have

$$\begin{aligned} |W_n|_\Delta &\ge \frac{1}{2}\Big(|W_n|_{\Delta_{s_3(n)}} + |W_n|_{\Delta_{s_3(n)+1}}\Big) \\ &\ge \frac{c}{2}\left(\frac{\sqrt{n}}{\sqrt{\log\log n}} l_{s_3(n)} + \frac{n}{k_{s_3(n)+1}}\right) = \frac{c}{2} h(n). \end{aligned}$$

Finally, if $s_0'(n) \le s_3(n)$, we have

$$|W_n|_\Delta \ge |W_n|_{\Delta_{s_0'(n)}} \ge \frac{c\sqrt{n}}{\sqrt{\log\log n}} l_{s_0'(n)} = c\, h(n).$$

This concludes the proof. ∎

11. Proof of the main theorem

We return to the idea of [3] for approximating a given function. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\frac{f(n)}{\sqrt{n}}$ and $\frac{n}{f(n)}$ are non-decreasing. Choose sequences $\{k_s\}$ and $\{l_s\}$ as in Proposition 2.7, and let Δ be the corresponding diagonal product. Denote by W_n the random walk on Δ with respect to the "switch-walk-switch" measure.

Theorem 11.1. *If* $\frac{f(n)}{\sqrt{n}\log\log\log n}$ *is non-decreasing, then almost surely*

$$0 < \limsup_{n\to\infty} \frac{|W_n|_\Delta}{\log\log n \cdot f(\frac{n}{\log\log n})} < \infty.$$

Proof. We first note that, by Proposition 2.7,

$$f\Big(\frac{rn}{\log\log n}\Big) \simeq \sqrt{\frac{rn}{\log\log n}}\, l_{s_2(n)} + \frac{\frac{rn}{\log\log n}}{k_{s_2(n)+1}}$$
$$\simeq \frac{1}{\log\log n}\bigg(\sqrt{n\log\log n}\, l_{s_2(n)} + \frac{n}{k_{s_2(n)+1}}\bigg).$$

Therefore, Proposition 10.1 shows that almost surely,

$$0 < \limsup_{n\to\infty} \frac{|W_n|_\Delta}{\log\log n\Big(f\big(\frac{n}{\log\log n}\big) + \frac{\sqrt{n}}{\sqrt{\log\log n}}\log\log\log n\Big)} < \infty.$$

As $\frac{f(n)}{\sqrt{n}\log\log\log n}$ is non-decreasing,

$$\frac{\sqrt{n}}{\sqrt{\log\log n}}\log\log\log n \lesssim f\Big(\frac{n}{\log\log n}\Big),$$

and the conclusion follows. ∎

Theorem 11.2. *If* $\frac{f(n)}{\sqrt{n}(\log\log n)^{1+\varepsilon}}$ *is non-decreasing for some* $\varepsilon > 0$*, and the sequences* $\{k_s\}$ *and* $\{l_s\}$ *are chosen so that* $\log\log k_s \le l_s$ *(as in Lemma C.1), then almost surely*

$$0 < \liminf_{n\to\infty} \frac{|W_n|_\Delta}{\frac{1}{\log\log n}\cdot f(n\log\log n)} < \infty.$$

Proof. We first note that, by Proposition 2.7,

$$f(c_1 n\log\log n) \simeq \sqrt{c_1 n\log\log n}\, l_{s_3(n)} + \frac{c_1 n\log\log n}{k_{s_3(n)+1}}$$
$$\simeq \log\log n\bigg(\frac{\sqrt{n}}{\sqrt{\log\log n}}\, l_{s_3(n)} + \frac{n}{k_{s_3(n)+1}}\bigg).$$

Also, the assumption that $\log\log k_s \le l_s$ is non-decreasing shows that $s_3(n) \le s_0'(n)$ for large enough n. Therefore, Proposition 10.2 shows that almost surely

$$0 < \liminf_{n\to\infty} \frac{|W_n|_\Delta}{\frac{1}{\log\log n} f(n\log\log n)} < \infty.$$ ∎

A. k-jumps of a simple random walk on $\mathbb{Z}$

Let S_t be a simple random walk on $\mathbb{Z}$, and fix $k > 0$. Recall from Definition 4.1 the sequence $\{n_j^{(k)}\}_{j=0}^{\infty}$: We set $n_0^{(k)} = 0$, and

$$n_j^{(k)} = \inf\{t > n_{j-1}^{(k)} \mid S_t \in \{S_{n_{j-1}^{(k)}} \pm k\}\}.$$

Recall also $N_k(n) = \max\{j \mid n_j^{(k)} \leq n\}$. Let $X_j = n_j^{(k)} - n_{j-1}^{(k)}$. Our goal is to estimate $N_k(n)$.

Proposition A.1. *Let* $0 < r < \frac{1}{2}$. *There exists* $c > 0$ *such that for any* $n, k \geq 1$,

$$\mathrm{P}\left(N_k(n) > \frac{cn}{k^2}\right) \leq \exp\left(-\frac{rn}{k^2}\right).$$

Proof. For $k = 1$, we may choose $c = 1$ and the inequality will hold; so we assume $k > 1$. Note that

$$\begin{aligned}
\mathrm{P}\left(X_j > k^2\right) &= \mathrm{P}\left(\max_{n_{j-1} \leq t \leq n_{j-1}+k^2} |S_t - S_{n_{j-1}}| < k\right) \\
&= \mathrm{P}\left(\max_{0 \leq t \leq k^2} |S_t| < k\right) = 1 - \mathrm{P}\left(\max_{0 \leq t \leq k^2} |S_t| \geq k\right) \\
&\geq 1 - \frac{\mathbb{E}|S_{k^2}|}{k},
\end{aligned}$$

where the last step follows from Doob's maximal inequality. Since $\mathbb{E}|S_n| \sim \sqrt{\frac{2}{\pi}n}$, there is a constant $p_0 > 0$ such that $\mathrm{P}(X_j > k^2) \geq p_0$ for all k. As the random variables $X_1, X_2, \ldots,$ are i.i.d., for all $c > 1$ we have

$$\begin{aligned}
\mathrm{P}\left(N_k(n) > \frac{cn}{p_0 k^2}\right) &= \mathrm{P}\left(\sum_{j=1}^{cn/p_0k^2} X_j \leq n\right) \\
&\leq \mathrm{P}\left(\sum_{j=1}^{cn/p_0k^2} k^2 \cdot 1_{\{X_j > k^2\}} \leq n\right) \\
&\leq \mathrm{P}\left(\mathrm{Bin}\left(\frac{cn}{p_0k^2}, p_0\right) \leq \frac{n}{k^2}\right).
\end{aligned}$$

Chernoff's inequality shows that

$$\mathrm{P}\left(\mathrm{Bin}\left(\frac{cn}{p_0k^2}, p_0\right) \leq \frac{n}{k^2}\right) \leq \exp\left(-\left(1 - \frac{1}{c}\right)^2 \frac{n}{2k^2}\right).$$

Now, let $0 < r < 1$. Choosing a large enough c so that $r \leq (1 - \frac{1}{c})^2$, we have

$$\mathrm{P}\left(N_k(n) > \frac{cn}{p_0k^2}\right) \leq \exp\left(-\frac{rn}{k^2}\right)$$

as required. ∎

Proposition A.2. *Let* $0 < r < \frac{1}{10}$. *Then there exists* $c' > 0$ *such that for all* $n, k \geq 1$,

$$\mathrm{P}\left(N_k(n) < \frac{c'n}{k^2}\right) \leq \exp\left(-\frac{rn}{k^2}\right).$$

Proof. First, for $k = 1$ we may take $c' = \frac{1}{2}$, and the desired probability is 0; so we assume $k > 1$. Note that

$$\mathrm{P}\left(X_i \geq k^2\right) \leq \mathrm{P}\left(|S_{k^2}| \leq k\right),$$

and the latter converges from below to $\mathrm{P}(-1 \leq \mathrm{N}(0,1) \leq 1) \approx 0.683$. One can check that by taking $p_1 = 0.9$ we have $\mathrm{P}(X_i \geq k^2) \leq p_1$ for all $k \geq 2$.

Our next step is to show by induction that $\frac{1}{k^2} X_j$ is stochastically dominated by a geometric random variable $G_j \sim \mathrm{Geo}(1 - p_1)$. To this end, we use that for simple random walk $\{Y_t\}$ on the interval $[-k, k]$, the hitting time of $\{-k, k\}$ when starting at j is stochastically dominated by the same hitting time when starting at 0. To see this, simply let the walker starting at 0 walk until the first time it hits $\pm j$. The remaining time until the walker hits $\pm k$ has the same distribution as for a walker starting at j.

Using this fact, we deduce

$$\begin{aligned}
&\mathrm{P}\left(X_j \geq ik^2 \mid (i-1)k^2\right) \\
&\quad = \sum_{-k<a<k} \mathrm{P}(X_j \geq ik^2 \mid X_j \geq (i-1)k^2, S_{(i-1)k^2} = a)\, \mathrm{P}(S_{(i-1)k^2} = a) \\
&\quad \leq \mathrm{P}(X_j \geq ik^2 \mid X_j \geq (i-1)k^2, S_{(i-1)k^2} = 0) = \mathrm{P}(X_j \geq k^2),
\end{aligned}$$

so for any $m \geq 1$,

$$\begin{aligned}
\mathrm{P}(X_j \geq mk^2) &= \prod_{i=1}^{m} \mathrm{P}(X_j \geq ik^2 \mid X_j \geq (i-1)k^2) \\
&\leq \prod_{i=1}^{m} \mathrm{P}(X_j \geq k^2) = \mathrm{P}(X_j \geq k^2)^m.
\end{aligned}$$

This proves that $\frac{1}{k^2} X_j$ is stochastically dominated by G_j. Therefore, for any $c' \leq 1 - p_1$,

$$\mathrm{P}\left(N_k(n) < \frac{c'n}{k^2}\right) = \mathrm{P}\left(\sum_{j=1}^{c'n/k^2} X_j > n\right) \leq \mathrm{P}\left(\sum_{j=1}^{c'n/k^2} G_j > \frac{n}{k^2}\right).$$

By [14, Theorem 2.1],

$$\mathrm{P}\left(\sum_{j=1}^{c'n/k^2} G_j > \frac{n}{k^2}\right) \leq \exp\left(-\frac{c'n}{k^2}(\lambda - 1 - \ln\lambda)\right)$$

for $\lambda = \frac{1-p_1}{c'}$.

Finally, let $0 < r < \frac{1}{10}$. As $c'(\lambda - 1 - \ln\lambda) = 1 - p_1 - c' - c' \ln\frac{1-p_1}{c'} \to 1 - p_1$ as $c' \to 0$, by choosing small enough c' we have $c'(\lambda - 1 - \ln\lambda) > r$, and thus

$$\mathrm{P}\left(N_k(n) > \frac{c'n}{k^2}\right) \leq \exp\left(-\frac{rn}{k^2}\right). \qquad \blacksquare$$

B. On the maximal local time of a simple random walk

For a simple random walk S_t on $\mathbb{Z}$, recall that

$$L(x,n) = |\{0 \le k \le n \mid S_k = x\}|$$

denotes the local time of S_t at x until time n. We are interested in the tail behavior of $L(n) = \max_{x\in\mathbb{Z}} L(x,n)$. The main aim of this section is to prove an upper tail bound on the maximal local time of a simple random walk on $\mathbb{Z}$ (Corollary B.4). This seems like a classical result; however, we could not find the exact statement we needed in the literature, and therefore provide the statement with proof and background for completeness.

For the local time of a simple random walk S_t on $\mathbb{Z}$ at a given point, we have the following bound.

Lemma B.1. *For any $n \ge 1$, any $x \in \mathbb{Z}$ and any $u \ge 1$,*

$$\mathrm{P}(L(x,n) \ge u) \le \exp\left(-\frac{u^2}{2n}\right).$$

Proof. First, for any $x \in \mathbb{Z}$ we have

$$\mathrm{P}(L(x,n) \ge u) \le \mathrm{P}(L(0,n) \ge u),$$

so we may assume that $x = 0$. A standard use of the reflection principle shows that

$$\mathrm{P}(L(0,n) \ge u) = \mathrm{P}\left(\max_{0\le t\le n-u} S_t \ge u\right) \le \mathrm{P}\left(\max_{0\le t\le n} S_t \ge u\right)$$

(see, for instance, [23, Theorem 9.3]). Then, by the maximal Azuma–Hoeffding inequality,

$$\mathrm{P}\left(\max_{0\le t\le n} S_t \ge u\right) \le \exp\left(-\frac{u^2}{2n}\right)$$

as required. ∎

B.1. Brownian motion and the Skorokhod embedding

Let B_t be a Brownian motion on $\mathbb{R}$ starting at 0. Define a sequence of stopping times by $\tau_0 = 0$ and, inductively,

$$\tau_k = \inf\{t > \tau_{k-1} \mid |B_t - B_{\tau_{k-1}}| = 1\}.$$

Then $S_k = B_{\tau_k}$ is a simple random walk on $\mathbb{Z}$, and $\tau_k - \tau_{k-1}$ are i.i.d.r.v. with $\mathbb{E}[\tau_k - \tau_{k-1}] = 1$ and $\sigma^2 = \mathbb{E}[(\tau_k - \tau_{k-1})^2] < \infty$. We therefore have

$$\mathrm{P}(|\tau_n - n| \ge \sqrt{n}u) \le 2\exp\left(-\frac{u^2}{2\sigma^2}\right). \tag{13}$$

B.2. Brownian local time

Given a Brownian motion B_t on $\mathbb{R}$, a theorem of Trotter [25] shows that almost surely there exists a function $\eta(x,t)$, jointly continuous in x and t, such that

$$\eta(x,t) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^t 1_{(-\infty,x)}(B_s)\,\mathrm{d}s,$$

where $\eta(x,t)$ is called the *local time of B_t at x*. The distribution of $\eta(0,t)$ is well known: For all $u > 0$,

$$\mathrm{P}\left(\frac{\eta(0,t)}{\sqrt{t}} \le u\right) = 2\Phi(u) - 1 \tag{14}$$

(see, for example, [17]).

Let $\eta(t) = \sup_{x\in\mathbb{R}} \eta(x,t)$. By the scaling property of Brownian motion, $\frac{\eta(t)}{\sqrt{t}}$ has the same distribution as $\eta(1)$, which, by [17], satisfies

$$\mathrm{P}(\eta(t) \ge \sqrt{t}u) = \mathrm{P}(\eta(1) \ge u) \le \exp\left(-\frac{u^2}{4}\right) \tag{15}$$

for large enough u.

B.3. Strong approximation for the local time

We give a strong approximation theorem for the maximal local time of a random walk and a Brownian motion, using the Skorokhod embedding. This is similar to other results (such as in [6, 13, 22]), but with a more quantitative flavor.

We use the notations of [22]. Fix $x \in \mathbb{Z}$. Define

$$\nu_1 = \inf\{k \ge 0 \mid B_{\tau_k} = S_k = x\}$$

and, inductively,

$$\nu_j = \inf\{k > \nu_{j-1} \mid B_{\tau_k} = S_k = x\}.$$

For each j, let

$$a_j(x) = \eta(x, \tau_{\nu_j+1}) - \eta(x, \tau_{\nu_j-1}).$$

From [22], $a_1(x), a_2(x), \ldots$, are i.i.d.r.v. with

$$\mathrm{P}(a_j(x) \ge u) = \mathrm{P}(\eta(0,\tau_1) \ge u) \le C \exp\left(-\frac{\pi}{4}u\right) \tag{16}$$

(for some universal constant $C > 0$), and $\mathbb{E}[a_j(x)] = 1$. In addition, the random matrices $L = (L(x,n))_{x\in\mathbb{Z},n\ge 0}$ and $A = (a_j(x))_{x\in\mathbb{Z},j\ge 1}$ are independent.

Lemma B.2. *There is a universal constant $c_1 > 0$ such that for any $x \in \mathbb{Z}$ and $u \ge 1$,*

$$\mathrm{P}(|a_1(x) + \cdots + a_{L(x,n)}(x) - L(x,n)| \ge \sqrt{n}u) \le 3\exp\left(-\frac{n^{1/3}u^{4/3}}{2}\right)$$

Proof. By the above, we have

$$\begin{aligned}
&\mathrm{P}(|a_1(x)+\cdots+a_{L(x,n)}(x)-L(x,n)|\geq\sqrt{n}u)\\
&\quad=\sum_{k=0}^{n}\mathrm{P}(|a_1(x)+\cdots+a_k(x)-k|\geq\sqrt{n}u)\,\mathrm{P}(L(x,n)=k)\\
&\quad=\sum_{k=0}^{n}\mathrm{P}\left(\left|\frac{a_1(x)+\cdots+a_k(x)-k}{\sqrt{k}}\right|\geq\frac{\sqrt{n}u}{\sqrt{k}}\right)\mathrm{P}(L(x,n)=k)\\
&\quad\leq\sum_{k=0}^{(nu)^{2/3}}\mathrm{P}\left(\left|\frac{a_1(x)+\cdots+a_k(x)-k}{\sqrt{k}}\right|\geq\frac{\sqrt{n}u}{\sqrt{k}}\right)\mathrm{P}(L(x,n)=k)\\
&\qquad+\mathrm{P}\left(L(x,n)>(nu)^{2/3}\right)\\
&\quad\leq\sum_{k=0}^{(nu)^{2/3}}\mathrm{P}\left(\left|\frac{a_1(x)+\cdots+a_k(x)-k}{\sqrt{k}}\right|\geq n^{1/6}u^{2/3}\right)\mathrm{P}(L(x,n)=k)\\
&\qquad+\exp\left(-\frac{n^{1/3}u^{4/3}}{2}\right)\\
&\quad\leq 2\exp\left(-\frac{n^{1/3}u^{4/3}}{2}\right)+\exp\left(-\frac{n^{1/3}u^{4/3}}{2}\right)\leq 3\exp\left(-\frac{n^{1/3}u^{4/3}}{2}\right).
\end{aligned}$$

Note that we used Lemma B.1 during the proof to show that

$$\mathrm{P}(L(x,n)>(nu)^{2/3})\leq\exp\left(-\frac{n^{1/3}u^{4/3}}{2}\right).\qquad\blacksquare$$

Lemma B.3. *There is a universal constant $C_1>0$ such that for any $x\in\mathbb{Z}$ and $u\geq 1$,*

$$\mathrm{P}(|a_1(x)+\cdots+a_{L(x,n)}(x)-\eta(x,\tau_n)|\geq\sqrt{n}u)\leq C_1\exp\left(-\frac{\pi\sqrt{n}u}{4}\right).$$

Proof. By [22, equation (2.6)],

$$|a_1(x)+\cdots+a_{L(x,n)}(x)-\eta(x,\tau_n)|\leq\gamma+a_{L(x,n)}(x)$$

for some constant $\gamma\geq 0$. Therefore, by (16),

$$\begin{aligned}
&\mathrm{P}(|a_1(x)+\cdots+a_{L(x,n)}(x)-\eta(x,\tau_n)|\geq\sqrt{n}u)\\
&\qquad\leq\mathrm{P}(\gamma+a_{L(x,n)}(x)\geq\sqrt{n}u)\leq C\exp\left(-\frac{\pi}{4}(\sqrt{n}u-\gamma)\right)
\end{aligned}$$

as required. $\blacksquare$

Corollary B.4. *There are universal constants $C,C'>0$ such that for all $n,u\geq 1$,*

$$\mathrm{P}(L(n)\geq\sqrt{n}u)\leq Cu^2\exp(-C'u^2).$$

Proof. If $u \geq \sqrt{n}$, the claim is trivial. So we assume $u < \sqrt{n}$. By combining Lemmas B.2 and B.3, for any $x \in \mathbb{Z}$ we have

$$\begin{aligned} &\mathrm{P}(|L(x,n) - \eta(x,\tau_n)| \geq 2\sqrt{n}u) \\ &\quad\leq 3\exp\Big(-\frac{n^{1/3}u^{4/3}}{2}\Big) + C_1\exp\Big(-\frac{\pi\sqrt{n}u}{4}\Big). \end{aligned}$$

Taking a union bound, we have

$$\begin{aligned} &\mathrm{P}\Big(\sup_{|x|\leq n} |L(x,n) - \eta(x,\tau_n)| \geq 2\sqrt{n}u\Big) \\ &\quad\leq Cn\Big(3\exp\Big(-\frac{n^{1/3}u^{4/3}}{2}\Big) + C_1\exp\Big(-\frac{\pi\sqrt{n}u}{4}\Big)\Big). \end{aligned}$$

The function $n \mapsto n\exp(-n^\alpha u^\beta)$ for fixed $u, \alpha, \beta \geq 1$ is decreasing for $n \geq 1$, so we may use $n > u^2$ and get

$$\mathrm{P}\Big(\sup_{|x|\leq n} |L(x,n) - \eta(x,\tau_n)| \geq 2\sqrt{n}u\Big) \leq Cu^2\exp\Big(-\frac{u^2}{2}\Big)$$

for an appropriate constant $C > 0$. The claim now follows from (13) and (15). ∎

C. Approximation of functions

In [3, Appendix B], Brieussel and Zheng show how to approximate a function $f\colon[1,\infty)\to[1,\infty)$ such that $\frac{f(x)}{\sqrt{x}}$ and $\frac{x}{f(x)}$ are non-decreasing by a function of the form

$$\bar{f}(x) = \frac{x}{k_{s+1}} + \sqrt{x}\,l_s, \quad (k_s l_s)^2 \leq x < (k_{s+1}l_{s+1})^2$$

for appropriate sequences of nonnegative integers $\{k_s\}$ and $\{l_s\}$. Here, we prove the following lemma.

Lemma C.1. *Let $f\colon[1,\infty)\to[1,\infty)$ be a continuous function such that $f(1) = 1$ and $\frac{x}{f(x)}$ and $\frac{f(x)}{\sqrt{x}(\log\log x)^{1+\varepsilon}}$ are non-decreasing for some $\varepsilon > 0$, and let $m_0 > 1$. Then one can find sequences $\{k_s\}$ and $\{l_s\}$, possibly finite with last value infinity in one of them, such that:*

- *$k_{s+1} \geq m_0 k_s$ and $l_{s+1} \geq m_0 l_s$ for all s;*
- *$f(x) \simeq_{2m_0} \bar{f}(x)$; and*
- *$\log\log k_s \leq l_s$ for large enough s.*

Proof. Writing $g(x) = \frac{f(x^2)}{x}$, we have $\frac{g(x)}{(\log\log x)^{1+\varepsilon}}$ and $\frac{x}{g(x)}$ are non-decreasing. We can now use [3, Lemma B.1] to find sequences $\{k_s\}$ and $\{l_s\}$ so that the first two conditions of the lemma are satisfied.

To prove the last assertion, we recall the construction of $\{k_s\}$ and $\{l_s\}$. They are defined inductively, according to the following procedure: Define first $k_0 = l_0 = 1$. Assuming that k_s and l_s were defined and that $g(k_s l_s) = l_s$, we have $l_s \le g(x) \le \frac{x}{k_s}$ for all $x \ge k_s l_s$. Take the minimal $y \ge m_0^2 k_s l_s$ such that $m_0 l_s \le g(y) \le \frac{y}{m_0 k_s}$. If such y exists, we have two cases:

(i) $g(y) = \frac{y}{m_0 k_s}$ – in which case we take $k_{s+1} = m_0 k_s$ and $l_{s+1} = \frac{y}{m_0 k_s} \ge m_0 l_s$;

(ii) $g(y) = m_0 l_s$ – in which case we take $l_{s+1} = m_0 l_s$ and $k_{s+1} = \frac{y}{m_0 l_s} \ge m_0 k_s$.

If such y does not exist, the assumption on $\frac{g(x)}{(\log\log x)^{1+\varepsilon}}$ shows that we have $g(x) \ge \frac{x}{m_0 k_s}$ for all $x \ge k_s l_s$, in which case we take $k_{s+1} = m_0 k_s$ and $l_{s+1} = \infty$.

To ensure the last condition, note that by our assumption on g, for any $x > 1$ we have

$$1 \le \frac{g(x \exp\exp x)}{(\log\log(x \exp\exp x))^{1+\varepsilon}} = \frac{g(x \exp\exp x)}{x} \cdot \frac{x}{(x + \log(1 + \frac{\log x}{\exp x}))^{1+\varepsilon}}.$$

The right term tends to 0 as x tends to ∞, hence for large enough x we have $g(x \exp\exp x) \ge x$. In particular, for large enough s we have

$$g(m_0 l_s \exp(\exp(m_0 l_s))) \ge m_0 l_s. \tag{17}$$

Now, if (17) holds for some s, then in the second case we have $y \le m_0 l_s \exp\exp(m_0 l_s)$, so $k_{s+1} \le \exp\exp(m_0 l_s) = \exp\exp(l_{s+1})$ as required. ∎

Acknowledgments. We thank the anonymous referee for their helpful comments.

Funding. The authors were supported by Israeli Science Foundation grant #957/20. The second author was also supported by the Bar-Ilan President's Doctoral Fellowships of Excellence.

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Received 15 December 2022.

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