Publ. RIMS Kyoto Univ. **61** (2025), 447–486
DOI 10.4171/PRIMS/61-3-5

Special K-Stability and Positivity of CM Line Bundles

by

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Abstract

We show that the CM line bundle on a proper family parametrizing specially K-stable varieties with maximal variation is big and nef. As an application, we show projectivity of any proper subspace of the coarse moduli space of uniformly adiabatically K-stable klt-trivial fibrations over curves constructed in Hashizume and Hattori (Geom. Topol. 29 (2025), 1619–1691).

Mathematics Subject Classification 2020: 14J10 (primary); 14J17, 14J27, 14J40 (secondary).
Keywords: klt-trivial fibration, adiabatically K-stable, coarse moduli space.

§1. Introduction

§1.1. Positivity of CM line bundles

K-stability was first introduced by Tian [Tia97] for Fano manifolds and reformulated by Donaldson [Don02] for other polarized varieties. This notion is defined in a purely algebro-geometric way and is considered to be closely related to the existence of constant scalar curvature Kähler (for short cscK) metrics as the Yau–Tian–Donaldson conjecture predicts. On the other hand, K-stability is closely related to birational geometry and moduli problems. For log Fano pairs, K-stability is completely detected by the δ-invariant, which was first introduced by Fujita–Odaka [FO18] and Blum–Jonsson [BJ20] and the moduli scheme parametrizing K-polystable log Fano pairs is constructed (see [X25]). On the other hand, Hashizume and the author [HH25] construct a moduli space parametrizing uniformly adiabatically K-stable klt-trivial fibrations over curves and show that this moduli space

Communicated by Y. Namikawa. Received November 12, 2023. Revised May 20, 2024.

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is a kind of K-moduli space if we choose some polarizations. Here, uniform adiabatic K-stability is first introduced in the author's works [Hat24b, Hat25] as the K-stability of a fiber space when its polarization is very close to an ample line bundle on the base.

Odaka ([Oda10, Conj. 5.2]) conjectured the K-moduli conjecture 10 years ago; it predicts that there will exist a quasi-projective moduli scheme parametrizing all K-polystable varieties with some fixed numerical data. This conjecture is shown for the Fano case (cf. [XZ20, LXZ22, X25]). He further conjectured that we can choose a $\mathbb{Q}$-ample line bundle on the moduli space to be the CM line bundle, which was introduced by Paul–Tian [PT09, PT06]. Indeed, the CM line bundle of a KSBA moduli space (see [Ko23] for details) is ample by [PX17]. Recently, it was shown that a K-moduli space of log Fano pairs with fixed dimension and volume is a projective scheme ([CP21, CP23, Pos22, XZ20]) with the CM line bundle ample. On the other hand, Fujiki–Schumacher [FS90] showed that compact subspaces of moduli spaces parametrizing some cscK manifolds are projective by using the generalized Weil–Petersson metric, which is closely related to the CM line bundle. Their result is enhanced by Dervan–Naumann [DN25] for any moduli space parametrizing all cscK manifolds. Recently, Ortu [Ort23] has constructed moduli spaces parametrizing manifolds with optimal symplectic connections (cf. [DS21]). Ortu also shows that any compact subspace of her moduli spaces is projective.

§1.2. Main results

In this paper, we settle Odaka's conjecture on positivity of the CM line bundle for special K-stability.

Theorem 1.1. *Let $\pi\colon (X,\Delta,L) \to S$ be a polarized log $\mathbb{Q}$-Gorenstein family, where S is projective and $(X_{\bar{s}},\Delta_{\bar{s}})$ is klt for any geometric point $\bar{s}\in S$. Suppose that π has maximal variation along any curve (cf. Definition 2.20). If (X_s,Δ_s,L_s) is specially* K*-stable for any closed point $s\in S$, then the CM-line bundle $\lambda_{\mathrm{CM},\pi}$ is ample.*

Special K-stability was first introduced by [Hat24a]. We note that K-stability of $\mathbb{Q}$-Fano varieties, log Calabi–Yau varieties, and varieties with the ample canonical divisors are equivalent to special K-stability of them and special K-stability is compatible with the theory of filtrations as Ding stability of log Fano pairs (see [Fuj19] and [Li17]). On the other hand, special K-stability ensures the existence of cscK metrics by the result of Zhang [Zha24].

We recall the following fact shown in [HH25].

Theorem 1.2 (For details, see Theorem 3.11)**.** *We fix $d\in\mathbb{Z}_{>0}$, $u\in\mathbb{Q}_{>0}$, $v\in\mathbb{Q}_{>0}$. Then there exists a separated Deligne–Mumford moduli stack $\mathscr{M}_{d,v,u,r}$ of finite type* *over* $\mathbb{C}$ *with a coarse moduli space* $M_{d,v,u,r}$ *parametrizing uniformly adiabatically* K*-stable klt-trivial fibrations* $f\colon (X,0,A)\to\mathbb{P}^1$ *such that*

(1) $\dim X = d$,
(2) *for any general fiber* F, $F\cdot A^{d-1}=v$, *and*
(3) $K_X\sim_{\mathbb{Q}} -uf^*\mathcal{O}(1)$.

Furthermore, there exists $w\in\mathbb{Q}_{>0}$ *such that for any uniformly adiabatically* K*-stable klt-trivial fibration* $f\colon (X,0,A)\to\mathbb{P}^1$ *as above, if* $\mathrm{vol}(A)=w$, *then* (X,A) *is specially* K*-stable.*

For the case when u is nonpositive in Theorem 1.2, $\mathscr{M}_{d,v,u,r}$ parametrizes polarized klt minimal models. In this case, the quasi-projectivity of the open locus of $M_{d,v,u,r}$ parameterizing varieties with only canonical singularities has been solved by [Vie95] We also remark that $M_{d,v,u,r}$ is not proper in general. Indeed, the moduli space of uniformly adiabatically K-stable rational Weierstrass fibrations with a fixed section is contained in the projective GIT moduli scheme of Miranda [Mi81] as an open subset, but they do not coincide.

By Theorem 1.1, we show that proper subspaces of the coarse moduli spaces constructed by [HH25] are all projective.

Corollary 1.3. *Any proper subspace* B *of* $M_{d,v,u,r}$ *is projective.*

§1.3. Outline of the proof

We briefly explain the idea of the proof of Theorem 1.1 here. Special K-stability of a polarized klt pair (X,Δ,L) consists of the following two properties.

(i) $H\coloneqq \delta(X,\Delta,L)L+K_X+\Delta$ is ample, and
(ii) uniform J^H-stability of (X,L).

Consider a polarized log family $\pi\colon (X,\Delta,L)\to S$ of relative dimension n and an ample line bundle H on X. Recall that the log CM line bundle $\lambda_{\mathrm{CM},\pi}$ is defined as

$$\pi_*\Big(-\frac{n(K_{X_s}+\Delta_s)\cdot L_s^{n-1}}{L_s^n}L^{n+1}+(n+1)L^n\cdot(K_{X/S}+\Delta)\Big)$$

(see Definition 3.2). We define the following variant of the CM line bundle

$$\lambda_{\mathrm{J},\pi,H}\coloneqq \pi_*\Big(-\frac{nH_s\cdot L_s^{n-1}}{L_s^n}L^{n+1}+(n+1)L^n\cdot H\Big).$$

We call this the J^H-line bundle (cf. Definition 3.4). The following is a key observation in this paper.

Proposition 1.4. *Let* $\pi\colon (X, L) \to S$ *be a polarized family with an ample* $\mathbb{Q}$*-line bundle* H*, where* S *is projective and every geometric fiber of* π *is normal. If* (X_s, L_s) *is* J^{H_s}*-semistable for any closed point* $s \in S$*, then* $\lambda_{\mathrm{J},\pi,H}$ *is ample.*

This phenomenon is first observed by Murakami [Mur23, Lem. 2.7] for the case when π is smooth and all fibers are uniformly J-stable. In this paper, we show that the same phenomenon occurs also when π is nonsmooth but flat. This is the first ingredient to show Theorem 1.1. Furthermore, we remark that although we need some assumptions on the variation on the family to deduce the positivity of the CM line bundle, we need no such assumption to deduce Proposition 1.4. Indeed, Proposition 1.4 also holds for trivial test configurations as we will see in Example 5.2. On the other hand, we show that $\pi_*(H^{n+1})$ is ample by using a similar technique to obtain the ampleness of the CM line bundle of a family of K-stable log Fano pairs with maximal variation as follows.

Theorem 1.5. *Let* $\pi\colon (X, \Delta, L) \to S$ *be a polarized log* $\mathbb{Q}$*-Gorenstein family of relative dimension* n*, where* S *is projective and* $(X_{\bar{s}}, \Delta_{\bar{s}})$ *is klt for any geometric point* $\bar{s} \in S$*. Suppose that* π *has maximal variation along any curve and* $\pi_* L^{n+1} \equiv 0$*. If there exists* $\lambda \in \mathbb{Q}_{>0}$ *such that* $\lambda < \delta(X_s, \Delta_s, L_s)$ *for any closed point* $s \in S$ *and* $K_{X/S} + \Delta + \lambda L$ *is* π*-ample, then the* $\mathbb{Q}$*-line bundle* $\pi_*(K_{X/S} + \Delta + \lambda L)^{n+1}$ *is ample.*

This is the second ingredient. With these ingredients, we obtain Theorem 1.1. We give the proofs of these ingredients in Section 5.

§2. Preliminaries

We work over the field of complex numbers $\mathbb{C}$.

Notation and conventions

(i) If we say that X is a scheme, then we assume X to be of finite type over $\mathbb{C}$. If X is further separated, irreducible, and reduced, then we say that X is a variety. For any point $x \in X$, let $\kappa(x)$ denote the residue field of the local ring $\mathcal{O}_{X,x}$. That is, if we set $\mathfrak{m}_x$ as the maximal ideal of $\mathcal{O}_{X,x}$, $\kappa(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$.

(ii) Let X be a scheme. We denote

$$X(S) \coloneqq \mathrm{Hom}(S, X)$$

and call this the set of all S-valued points of X. If $S = \operatorname{Spec} \Omega$, where Ω is an algebraically closed field, then we call elements of $X(S)$ geometric points of X. If the image of $\operatorname{Spec} \Omega \to X$ is $x \in X$, we denote this by $\bar{x} \in X$.

(iii) Let X be a scheme of finite type over $\mathbb{C}$, and U an open subset. We say that U is *big* if $\operatorname{codim}_X(X \setminus U) \geq 2$.

(iv) For any coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$, $\mathscr{F}(d)$ denotes $\mathscr{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)$ for any $d \in \mathbb{Z}$.

(v) We say that a proper morphism of schemes $f\colon X \to Y$ is a *contraction* if $f_*\mathcal{O}_X \cong \mathcal{O}_Y$.

(vi) Let X be a scheme of finite type over $\mathbb{C}$ with a ($\mathbb{Q}$-)line bundle H. Let $|H|$ (resp. $|H|_{\mathbb{Q}}$) denote the set of all effective divisors linearly equivalent (resp. effective $\mathbb{Q}$-divisors $\mathbb{Q}$-linearly equivalent) to H.

(vii) Let X be a scheme of finite type over $\mathbb{C}$. We say that a property $\mathcal{P}$ *holds for any very general closed point* $x \in X$ if there exist countably many closed subvarieties $V_i \subsetneq X$ such that $\mathcal{P}$ holds for any closed point $x \in X \setminus \bigcup_{i=1}^{\infty} V_i$.

(viii) Let $f\colon X \to S$ be a proper morphism of normal varieties. Let $g\colon T \to S$ be a morphism from a normal variety. Then we set $X_T := X \times_S T$ and $f_T\colon X_T \to T$ as the morphism induced by f and g. Let $h\colon X_T \to X$ be the canonical morphism. If L is a $\mathbb{Q}$-line bundle on X, we set $L_T := h^*L$. For any $s \in S$, we denote $X_s = X_T$ and $L_s = L_T$, where $T = \operatorname{Spec} \kappa(s)$. Suppose that f is flat, all geometric fibers are connected and normal; there exists a $\mathbb{Q}$-divisor Δ such that $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and Δ does not contain any fiber of X over S. Then we set Δ_T as follows. Take an open subset $U \subset X$ such that $\operatorname{codim}_{X_s}(X_s \setminus U) \geq 2$ for any $s \in S$ and $f|_U$ is smooth. Then we see that $\Delta|_U$ is $\mathbb{Q}$-Cartier and can consider $h|^*_{h^{-1}(U)}\Delta|_U$. Let Δ_T be the closure of $h|^*_{h^{-1}(U)}\Delta|_U$. We note that then

$$K_{X_T/T} + \Delta_T = h^*(K_{X/S} + \Delta).$$

(ix) Let D be a $\mathbb{Q}$-Weil divisor on a projective normal variety X. We say that D is big if there exist an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-Weil divisor E such that $D = A + E$. If $D + A'$ is big for any ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A', then we say that D is *pseudo-effective*.

(x) Let X be a projective normal variety. We say that C is a *movable curve* of X if there exist a projective birational morphism $\mu\colon X' \to X$ and very ample hypersurfaces $H_1, \ldots, H_{\dim X - 1}$ such that $C = H_1 \cap \cdots \cap H_{\dim X - 1}$. Let L be a line bundle on X. We denote $L \cdot C := \mu^*L \cdot C$ for simplicity. By [BDPP13], $L \cdot C \geq 0$ for any movable curve C if and only if L is pseudo-effective.

(xi) Let $f\colon X \to S$ be a proper morphism such that any geometric fiber is normal and connected. Then we consider the following functor. For any morphism of schemes $T \to S$, we attain the following set:

$$\mathfrak{Pic}_{X/S}(T) := \{L \mid L \text{ is a line bundle on } X_T\} / \sim_T,$$

where $L_1 \sim_T L_2$ if and only if $L_1 \otimes f_T^* B \sim L_2$ for some line bundle B on T. Then we have the relative Picard scheme $\mathrm{Pic}_{X/S}$, which represents the étale sheafification of the above functor. If $S = \operatorname{Spec} \mathbb{C}$, then we simply denote $\mathrm{Pic}(X) := \mathrm{Pic}_{X/S}$. Furthermore, $\mathrm{Pic}^0(X)$ denotes the identity component of $\mathrm{Pic}(X)$ and parametrizes all line bundles algebraically equivalent to $\mathcal{O}_X$. See [F+05, §9] for details. Let $[L]$ denote an element of $\mathrm{Pic}(X)$ whose representative is a line bundle L.

(xii) Let E be a locally free sheaf on a smooth projective variety S. We say that E is *nef* (resp. *ample*) if $\mathcal{O}_{\mathbb{P}_S(E)}(1)$ is nef (resp. ample). Also, E is called *weakly positive* if for any $a \in \mathbb{Z}_{>0}$ and ample line bundle A, the stalk of $\mathrm{Sym}^{ab} E \otimes \mathcal{O}_S(bA)$ at the generic point of S is generated by $H^0(S, \mathrm{Sym}^{ab} E \otimes \mathcal{O}_S(bA))$ for some $b \in \mathbb{Z}_{>0}$. If S is a curve, then the nefness of S is equivalent to the weak positivity.

(xiii) Let X be a proper normal variety and $\pi \colon Y \to X$ be a resolution of singularities of X. Then $\mathrm{Alb}(Y)$ denotes the Albanese variety of Y. Let $\beta \colon Y \to \mathrm{Alb}(Y)$ be a canonical morphism. Then it is well known that there exists a canonical morphism $\alpha \colon X \to \mathrm{Alb}(Y)$ such that $\beta = \alpha \circ \pi$. Thus, we denote $\mathrm{Alb}(Y)$ by $\mathrm{Alb}(X)$ and call this the *Albanese variety* of X. We also call α an *Albanese morphism.*

(xiv) Let X be a Noetherian scheme. Let $\mathscr{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say that $\mathscr{F}^\bullet$ is a *perfect complex* if there exist a family of open subsets $\{U_i\}_{i=1}^r$ and bounded complexes $\mathscr{G}_i^\bullet$ of finite free $\mathcal{O}_{U_i}$-modules with a quasi-isomorphism $\mathscr{G}_i^\bullet \to \mathscr{F}^\bullet|_{U_i}$ for each i. On the other hand, let E be a coherent sheaf on X. Then E is called an $\mathcal{O}_X$-module *of finite Tor-dimension* if for any $x \in X$, the stalk E_x admits a resolution of finitely generated free $\mathcal{O}_{X,x}$-modules of finite length ([MFK94, p. 111]).

§2.1. K-stability

We first recall the fundamental concepts of birational geometry and K-stability.

Definition 2.1. Let X be a quasi-projective normal variety. Suppose that B is a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then we call (X, B) a *sublog pair*. If B is further effective, then we say that (X, B) is a *log pair*. For any prime divisor E over X, choose a proper birational morphism $\pi \colon Y \to X$ such that E is defined on Y. Then we set the log discrepancy of (X, B) with respect to E as

$$A_{(X,B)}(E) = \mathrm{ord}_E(K_Y - \pi^*(K_X + B)) + 1.$$

The above value is independent of the choice of π. We say that (X, B) is *subklt* (resp. *sublc*) if $A_{(X,B)}(E) > 0$ (resp. ≥ 0) for any prime divisor E over X. If E is further effective, then we say that (X, B) is *klt* (resp. *lc*).

For any coherent ideal $\mathfrak{a}$ of X and rational number $r > 0$, we consider the pair $(X, B + r\mathfrak{a})$. We call $r\mathfrak{a}$ a $\mathbb{Q}$-ideal. We define the *log discrepancy* of $(X, B + r\mathfrak{a})$ as follows. Take E a prime divisor over X and $\pi\colon Y \to X$ such that π is a log resolution of (X, B) and $\mathfrak{a}$, i.e. $\mathrm{Ex}(\pi) + \pi_*^{-1}B + \pi^{-1}\mathfrak{a}$ is a simple normal crossing and Y is a smooth variety proper over X such that E is a prime divisor defined on Y. Here, we note that $\pi^{-1}\mathfrak{a}$ is a Cartier divisor. Then

$$A_{(X,B+r\mathfrak{a})}(E) = \mathrm{ord}_E(K_Y - \pi^*(K_X + B) - r\pi^{-1}\mathfrak{a}) + 1.$$

We say that $(X, B + r\mathfrak{a})$ is *subklt* (resp. *sublc*) if $A_{(X,B+r\mathfrak{a})}(E) > 0$ (resp. ≥ 0) for any prime divisor E over X.

Let (X, B) be a log pair. We set

$$\mathrm{Aut}(X, B) := \big\{\sigma \in \mathrm{Aut}\, X \;\big|\; \sigma_* B = B\big\}.$$

It is well known that $\mathrm{Aut}(X, B)$ is a group scheme and let $\mathrm{Aut}_0(X, B)$ be the identity component of $\mathrm{Aut}(X, B)$.

Definition 2.2 (Log canonical threshold)**.** Let (X, B) be a log subpair and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Take an arbitrary $\mathbb{Q}$-ideal $\mathfrak{a}$ on X. Then we define the *log canonical threshold* for (X, B) with respect to D as

$$\mathrm{lct}(X, B; D) = \sup\big\{t \in \mathbb{Q} \;\big|\; (X, B + tD) \text{ is sublc}\big\}.$$

On the other hand, we set the log canonical threshold of (X, B) with respect to $\mathfrak{a}$ as

$$\mathrm{lct}(X, B; \mathfrak{a}) = \sup\big\{t \in \mathbb{Q} \;\big|\; (X, B + t\mathfrak{a}) \text{ is sublc}\big\}.$$

Definition 2.3. We say that $\mathfrak{a}_\bullet$ is a graded sequence of nonzero ideals if there exists a sequence of ideals $\{\mathfrak{a}_m\}_{m\in\mathbb{Z}_{\geq 0}}$ satisfying

$$\mathfrak{a}_m \cdot \mathfrak{a}_n \subset \mathfrak{a}_{n+m}$$

for any $n, m \in \mathbb{Z}_{>0}$. For any prime divisor E over X, we set (cf. [JM12, Lem. 2.3])

$$\mathrm{ord}_E(\mathfrak{a}_\bullet) := \lim_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m} = \inf_{m\geq 0} \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m}.$$

By [JM12, Cor. 2.16], we can set

$$\mathrm{lct}(X,B;\mathfrak{a}_\bullet) \coloneqq \lim_{m\to\infty} m\cdot \mathrm{lct}(X,B;\mathfrak{a}_m) = \inf_E \frac{A_{(X,B)}(E)}{\mathrm{ord}_E(\mathfrak{a}_\bullet)}.$$

Further, (X,B,L) is called a *polarized log pair* if (X,B) is a log pair and L is an ample $\mathbb{Q}$-line bundle.

Definition 2.4. Let (X,B,L) be a polarized klt pair and take $r_0 \in \mathbb{Z}_{>0}$ such that r_0L is Cartier. For any $m\in\mathbb{Z}_{>0}$, we call D an mr_0*-basis-type* divisor of L if there exists a basis $\{D_i\}_{i=1}^{h^0(X,\mathcal{O}_X(mr_0L))}$ of $H^0(X,\mathcal{O}_X(mr_0L))$ such that

$$D = \frac{1}{mr_0h^0(X,\mathcal{O}_X(mr_0L))}\sum_{i=1}^{h^0(X,\mathcal{O}_X(mr_0L))} D_i.$$

Let $|L|_{mr_0\text{-basis}}$ denote the set of all mr_0-basis-type divisors of L. We set

$$\delta_{r_0m}(X,B,L) \coloneqq \inf_{D\in|L|_{mr_0\text{-basis}}} \mathrm{lct}(X,B;D)$$

and $\delta(X,B,L) \coloneqq \limsup_{m\to\infty}\delta_{r_0m}(X,B,L)$. We call $\delta(X,B,L)$ the δ*-invariant* and know by [BJ20, Thm. A] that $\delta(X,B,L) = \lim_{m\to\infty}\delta_{r_0m}(X,B,L)$.

Let (X,B,L) be a polarized klt pair as above. Let $|L|_{\mathbb{Q}}$ denote the set of all effective $\mathbb{Q}$-divisors $\mathbb{Q}$-linearly equivalent to L. Suppose that r_0L is Cartier for some $r_0\in\mathbb{Z}_{>0}$. We set for any prime divisor E over X,

$$T_L(E) = \sup_{D\in|L|_{\mathbb{Q}}} \mathrm{ord}_E(D), \quad S_L(E) = \lim_{m\to\infty}\sup_{D\in|L|_{m\text{-basis}}} \mathrm{ord}_E(D).$$

Indeed, we see that the above limit exists (cf. [BJ20]). We set the α*-invariant* as

$$\alpha(X,B,L) \coloneqq \inf_E \frac{A_{(X,B)}(E)}{T_L(E)}.$$

On the other hand, we see that

$$\delta(X,B,L) = \inf_E \frac{A_{(X,B)}(E)}{S_L(E)}.$$

Definition 2.5 (K-stability)**.** Let (X,B,L) be a polarized log pair of dimension n. A *normal semiample test configuration* $(\mathcal{X},\mathcal{L})$ for (X,L) is defined as follows:

(1) $\mathcal{X}$ is a normal variety with a $\mathbb{G}_m$-action and $\mathcal{L}$ is a semiample $\mathbb{G}_m$-linearized $\mathbb{Q}$-line bundle on $\mathcal{X}$.

(2) There exists a proper surjective and $\mathbb{G}_m$-equivariant morphism $\pi\colon \mathcal{X}\to\mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication.

(3) $(\pi^{-1}(1), \mathcal{L}|_{\pi^{-1}(1)}) \cong (X, L)$.

Let $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$ denote a semiample test configuration $(X \times \mathbb{A}^1, L \times \mathbb{A}^1)$ with the trivial $\mathbb{G}_m$-action on the first component X. It is well known that there exists another semiample test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ and two $\mathbb{G}_m$-equivariant birational morphisms $\sigma\colon \mathcal{Y} \to \mathcal{X}$ and $\rho\colon \mathcal{Y} \to X_{\mathbb{A}^1}$ such that $\mathcal{L}_{\mathcal{Y}} = \sigma^*\mathcal{L}$ and their restrictions to $(\pi \circ \sigma)^{-1}(1)$ are nothing but the identity morphisms of X. Then consider the $\mathbb{G}_m$-equivariant canonical compactification $(\overline{\mathcal{Y}}, \overline{\mathcal{L}_{\mathcal{Y}}}) \to \mathbb{P}^1$ of $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}}) \to \mathbb{A}^1$ such that the fiber over $\infty \in \mathbb{P}^1$ coincides with (X, L) with the trivial $\mathbb{G}_m$-action. Let H be an arbitrary $\mathbb{R}$-line bundle over X and let $\mathcal{B}_{\mathcal{Y}}$ be the Zariski-closure of $B \times \mathbb{G}_m$ in $\overline{\mathcal{Y}}$. Then we set the non-Archimedean Mabuchi functional and the non-Archimedean J^H-functional as

$$M_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = (K_{\overline{\mathcal{Y}}/\mathbb{P}^1} + \mathcal{B}_{\mathcal{Y}} - \mathcal{Y}_0 + \mathcal{Y}_{0,\mathrm{red}}) \cdot \overline{\mathcal{L}_{\mathcal{Y}}}^n - \frac{n(K_X + B) \cdot L^{n-1}}{(n+1)L^n} \overline{\mathcal{L}_{\mathcal{Y}}}^{n+1},$$

$$\mathcal{J}^{H,\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \overline{\rho^* H_{\mathbb{A}^1}} \cdot \overline{\mathcal{L}_{\mathcal{Y}}}^n - \frac{nH \cdot L^{n-1}}{(n+1)L^n} \overline{\mathcal{L}_{\mathcal{Y}}}^{n+1}.$$

It is well known that $M_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ and $\mathcal{J}^{H,\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ do not depend on the choice of $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ (cf. [BHJ17, §7] and [Hat21]).

We say that (X, B, L) is *uniformly* K-*stable* (resp. (X, L) is *uniformly* J^H-*stable*) if there exists $\varepsilon > 0$ such that

$$M_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \ \ (\text{resp. } \mathcal{J}^{H,\mathrm{NA}}(\mathcal{X}, \mathcal{L})) \ \ \geq \varepsilon \mathcal{J}^{L,\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any normal semiample test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L). We say that (X, B, L) is K-*semistable* (resp. (X, L) is J^H-*semistable*) if

$$M_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \ \ (\text{resp. } \mathcal{J}^{H,\mathrm{NA}}(\mathcal{X}, \mathcal{L})) \ \ \geq 0$$

for any normal semiample test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L). It is well known that $\mathcal{J}^{L,\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ (cf. [BHJ17, Prop. 7.8]) and hence uniform K-stability implies K-semistability.

For J^H-semistability in the case when H is ample, there exists a useful criterion. We prepare the following notion.

Definition 2.6. Let (X, L) be a polarized normal variety of dimension n with an $\mathbb{R}$-line bundle H. We say that (X, L) is J^H-*nef* if for any subvariety $V \subset X$ of dimension p,

$$\left(n\frac{H \cdot L^{n-1}}{L^n} L^p - pH \cdot L^{p-1}\right) \cdot V \geq 0.$$

We also say that (X, L) is *uniformly* J^H-*positive* if there exists $\varepsilon > 0$ such that (X, L) is $\mathrm{J}^{H-\varepsilon L}$-nef.

Theorem 2.7 ([Hat25, Thm. 3.2])**.** *Let (X, L) be a polarized normal variety with a nef $\mathbb{R}$-line bundle H. Then (X, L) is J^H-nef if and only if (X, L) is J^H-semistable.*

If H is further ample, then (X, L) is uniformly J^H-positive if and only if (X, L) is uniformly J^H-stable.

By using J-stability and the δ-invariant, we can set the following key notion.

Definition 2.8 (Special K-stability, [Hat24a, Def. 3.10])**.** Let (X, B, L) be a polarized klt pair. If (X, B, L) is uniformly $\mathrm{J}^{\delta(X,B,L)L+K_X+B}$-stable and $\delta(X, B, L)L + K_X + B$ is ample, then we say that (X, B, L) is *specially* K-*stable.* If (X, B, L) is $\mathrm{J}^{\delta(X,B,L)L+K_X+B}$-semistable and $\delta(X, B, L)L + K_X + B$ is nef, then we say that (X, B, L) is *specially* K-*semistable.*

By [Hat24a, Cor. 3.21], specially K-stable log pairs are uniformly K-stable.

§2.2. Filtered linear series and good filtrations

We collect some fundamental concepts of filtrations.

Definition 2.9. Let $R = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} R_m$ be a finitely generated graded $\mathbb{C}$-algebra such that $R_0 = \mathbb{C}$. Then F is called a (*decreasing, left-continuous, and multiplicative*) *filtration* of R if $F^\lambda R_m \subset R_m$ is a vector subspace for any $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$ and the following hold:

- $F^\lambda R_m \subset F^{\lambda'} R_m$ for any $\lambda > \lambda'$ and $F^\lambda R_m = \bigcap_{\lambda>\lambda'} F^{\lambda'} R_m$,
- $F^\lambda R_m \cdot F^{\lambda'} R_{m'} \subset F^{\lambda+\lambda'} R_{m+m'}$, and
- $F^0 R_0 = R_0$.

We say that this filtration is *linearly bounded* if there exist a positive real number C and $m_0 \in \mathbb{Z}_{>0}$ such that $F^\lambda R_m = 0$ (resp. $= R_m$) for any $m \geq m_0$ and $\lambda \geq Cm$ (resp. $\leq -Cm$).

Let X be a normal projective variety and L an ample line bundle on X. Let F be a linearly bounded multiplicative filtration on $R = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mL))$. Then we define a graded subalgebra $FR^{(\lambda)} := \bigoplus_{m\geq 0} F^{m\lambda} H^0(X, \mathcal{O}_X(mL)) \subset R$. We set

$$\lambda_{\max}(F) := \limsup_{k\to\infty} \frac{\sup\{t \in \mathbb{R} \mid F^t R_k \neq 0\}}{k}$$

and

$$\lambda_{\min}(F) := \inf\{\lambda \in \mathbb{R} \mid \mathrm{vol}(FR^{(\lambda)}) > 0\}.$$

We set the *weight* $w_F(m)$ of R_m with respect to F as

$$w_F(m) = \sum_{\lambda\in\mathbb{R}} \dim\left(F^\lambda R_m / \bigcup_{\lambda<\lambda'} F^{\lambda'} R_m\right)$$

and we call w_F the *weight function* of F. Set

$$S_m(F) := \frac{w_F(m)}{mh^0(X, \mathcal{O}_X(mL))}$$

and

$$S(F) := \lambda_{\min}(F) + \frac{1}{(L^n)} \int_{\lambda_{\min}(F)}^{\lambda_{\max}(F)} \mathrm{vol}(FR^{(\lambda)})\, d\lambda.$$

It is well known that $S(F) = \lim_{m\to\infty} S_m(F)$ (cf. [BJ20, Cor. 2.12]).

Two fundamental examples follow.

Example 2.10. Let X be a proper variety with an ample line bundle L on X. Let $R := \bigoplus H^0(X, \mathcal{O}_X(mL))$ and D be a closed subvariety of X. Let F be a linearly bounded multiplicative filtration on R. Then we set a filtration $F|_D$ on $\bigoplus H^0(D, \mathcal{O}_D(mL|_D))$ as

$$F|_D^{\lambda} H^0(D, \mathcal{O}_D(mL|_D)) := \mathrm{Image}\big(F^{\lambda} H^0(X, \mathcal{O}_X(mL)) \to H^0(D, \mathcal{O}_D(mL|_D))\big)$$

for any $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$. Then we call $F|_D$ the *restricted filtration* on D. We can check that $F|_D$ is multiplicative and linearly bounded.

Example 2.11. Let F be a linearly bounded multiplicative filtration of $R = \bigoplus_{m\geq 0} R_m$. If we set for any $\lambda \in \mathbb{R}$,

$$F_{\mathbb{Z}}^{\lambda} R_m := F^{\lceil \lambda \rceil} R_m,$$

then we see that $F_{\mathbb{Z}}$ is also a linearly bounded multiplicative filtration.

Definition 2.12 (Good filtrations, cf. [Hat24a, Def. 2.12])**.** Let X be a normal projective variety of dimension n and L an ample line bundle on X. Let F be a linearly bounded multiplicative filtration on $R = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mL))$. If there exist $a_0, a_1 \in \mathbb{R}$ and $C \in \mathbb{R}_{>0}$ such that

$$|w_F(m) - a_0 m^{n+1} - a_1 m^n| < Cm^{n-2}$$

for any $m \in \mathbb{Z}_{>0}$, then we say that F is a *good filtration*.

We note that if $F = F_{\mathbb{Z}}$ and $\bigoplus F^{\lambda} R_m$ is a finitely generated bigraded algebra, then F is good.

The following notion was first introduced by [XZ20].

Definition 2.13. Let (X, B, L) be a polarized klt pair. Take $r \in \mathbb{Z}_{>0}$ such that rL is Cartier and a linearly bounded filtration F on $R := \bigoplus_{m\geq 0} R_m$, where $R_m = H^0(X, \mathcal{O}_X(mrL))$. Now we set the base ideal of $F^{\lambda} R_m$ as

$$I_{m,\lambda}(F) := \mathrm{Image}(F^{\lambda} R_m \otimes \mathcal{O}_X(-mrL) \to \mathcal{O}_X)$$

for $\lambda \in \mathbb{R}$. Set $I_\bullet^{(\lambda)}(F) \coloneqq \{I_{m,m\lambda}(F)\}_{m\geq 0}$ as a graded sequence of ideals. For any $\delta > 0$, we set the *δ-lc slope* of F as

$$\mu_\delta(F) \coloneqq \sup\bigl\{\lambda \in \mathbb{R} \bigm| \mathrm{lct}(X, B; I_\bullet^{(\lambda)}(F)) \geq \tfrac{\delta}{r}\bigr\}.$$

Furthermore, we set

$$\beta_\delta(F) \coloneqq \frac{\mu_\delta(F) - S(F)}{r}.$$

We remark that the following holds.

Theorem 2.14 ([XZ20, Prop. 4.5]). *Let (X, B, L) be a polarized klt pair. Take $r \in \mathbb{Z}_{>0}$ such that rL is Cartier and set $R \coloneqq R(X, rL)$. Then*

$$\delta(X, \Delta, L) = \sup\bigl\{\delta > 0 \bigm| \beta_\delta(F) \geq 0 \text{ for any linearly bounded filtration } F \text{ on } R\bigr\}.$$

We remark that Xu and Zhuang showed Theorem 2.14 only for log Fano pairs, but their proof also works for Theorem 2.14 in the same way. We also note the following useful lemma, which also holds for general polarized klt pairs.

Lemma 2.15 ([XZ20, Lem. 4.13]). *Let (X, B, L) be a polarized klt pair. Take $r \in \mathbb{Z}_{>0}$ such that rL is Cartier and a linearly bounded filtration F on $R \coloneqq \bigoplus_{m\geq 0} R_m$, where $R_m = H^0(X, \mathcal{O}_X(mrL))$. For any real numbers $s, \varepsilon \in (0, 1)$, it holds that*

$$\mu_{1+(1-\varepsilon)s}(F) \geq s \cdot \mu_{\varepsilon^{-1}}(F) + (1 - s)\mu_1(F).$$

Definition 2.16 (Donaldson–Futaki invariant and J-functional for filtrations). Let (X, L) be a polarized variety of dimension n with L a line bundle and let F be a multiplicative filtration on R, where $R = \bigoplus_{m\geq 0} R_m$ and $R_m = H^0(X, \mathcal{O}_X(mL))$. Let $w_F(m)$ be the weight function of F. Then we see that (cf. [Hat24a, eq. (2)])

$$\lim_{m\to\infty} \frac{w_F(m)}{m^{n+1}}$$

exists and write this as b_0. On the other hand, it is well known that $\chi(X, \mathcal{O}_X(mL))$ is a polynomial of degree n. We denote this by $a_0 m^n + a_1 m^{n-1} + O(m^{n-2})$.

Take H an ample $\mathbb{Q}$-line bundle on X. Take a sufficiently divisible $r \in \mathbb{Z}_{>0}$ such that rH is very ample. Let $D \in |rH|$ be a very general member and let $F|_D$ be the restriction (cf. Example 2.10). Let $\tilde{a}_0 \coloneqq \frac{H\cdot L^{n-1}}{(n-1)!}$ and $\tilde{b}_0 \coloneqq \lim_{m\to\infty} \frac{w_{F|_D}(m)}{rm^n}$. By [Hat24a, Lem. 2.20], we see that the value

$$\frac{\tilde{b}_0 a_0 - \tilde{a}_0 b_0}{a_0^2}$$

is independent of the choice of r and very general D. We denote this by $\mathcal{J}^{H,\mathrm{NA}}(F)$ and call it the J^H-functional of F. For an arbitrary $\mathbb{Q}$-line bundle T on X, there exist two ample line bundles H_1 and H_2 such that $T = H_1 - H_2$. It is easy to see that $\mathcal{J}^{H,\mathrm{NA}}(F)$ is linear with respect to H. We set $\mathcal{J}^{T,\mathrm{NA}}(F) := \mathcal{J}^{H_1,\mathrm{NA}}(F) - \mathcal{J}^{H_2,\mathrm{NA}}(F)$. Then $\mathcal{J}^{T,\mathrm{NA}}(F)$ is independent of the choice of H_1 and H_2.

If F is a good filtration and $w_F(m) = b_0 m^{n+1} + b_1 m^n + O(m^{n-1})$, then we set the Donaldson–Futaki invariant of F as

$$\mathrm{DF}(F) = 2\frac{b_0 a_1 - a_0 b_1}{a_0^2}.$$

Remark 2.17. We do not know much about the relationship between this Donaldson–Futaki invariant and the Futaki invariant introduced by Székelyhidi [Szé15]. For details, we refer to [Hat24a, Rem. 2.20].

Proposition 2.18 ([Hat24a, Lem. 2.20])**.** *Let (X, L) be a polarized variety with L a line bundle and let H be a nef $\mathbb{Q}$-line bundle. If (X, L) is J^H-semistable, then*

$$\mathcal{J}^{H,\mathrm{NA}}(F) \geq 0$$

for any multiplicative linearly bounded filtration F on $\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mL))$.

Proof. Let F be a multiplicative linearly bounded filtration on $\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mL))$. By the fact that $\mathcal{J}^{H,\mathrm{NA}}(F)$ is linear with respect to H, we may assume that H is very ample and take a very general member $D \in |H|$. By [Hat24a, Lem. 2.20], we see that

$$\mathcal{J}^{H,\mathrm{NA}}(F_{\mathbb{Z}}) \geq 0.$$

By [BJ20, Cor. 2.12], $S(F) = S(F_{\mathbb{Z}})$. On the other hand, $F_{\mathbb{Z}}|_D = (F|_D)_{\mathbb{Z}}$. Since

$$\mathcal{J}^{H,\mathrm{NA}}(F) = n\Big(S(F|_D) - \frac{H \cdot L^{n-1}}{L^n} S(F)\Big),$$

we have $\mathcal{J}^{H,\mathrm{NA}}(F_{\mathbb{Z}}) = \mathcal{J}^{H,\mathrm{NA}}(F)$, which completes the proof. □

§2.3. Polarized log family

In this subsection, we discuss the following concept.

Definition 2.19. Let $\pi\colon X \to S$ be a proper flat morphism of normal varieties such that $\pi_*\mathcal{O}_X \cong \mathcal{O}_S$, Δ an effective $\mathbb{Q}$-Weil divisor on X, and L a π-ample $\mathbb{Q}$-line bundle on X. We say that $\pi\colon (X, \Delta, L) \to S$ is a *polarized log family* if any fiber X_s over $s \in S$ is normal and no irreducible component of Δ contains some fiber X_s. If $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier, then we say that $\pi\colon (X, \Delta, L) \to S$ is $\mathbb{Q}$-*Gorenstein*. We say that π is of relative dimension n if $\dim X_s = n$ for general $s \in S$.

A polarized log $\mathbb{Q}$-Gorenstein family $\pi\colon (X, \Delta, L) \to S$ satisfies the condition of (viii) in "Notation and conventions". Hence, for any morphism $g\colon T \to S$ from a normal variety, we can set Δ_T as (viii). To state Theorem 1.1, we prepare the following notion.

Definition 2.20 (Maximal variation)**.** Let $\pi\colon (X,\Delta,L)\to S$ be a polarized log family. Suppose that for any irreducible curve $C\subset S$ containing a general point of S and two general distinct closed points $p,q\in C$, (X_p,Δ_p) and (X_q,Δ_q) are not isomorphic. Then we say that π has *maximal variation*.

On the other hand, if the restriction $\pi|_{\pi^{-1}(C)}$ has maximal variation for any irreducible curve $C\subset S$, we say that π has *maximal variation along any curve.*

Next we show that if some fiber of a polarized log $\mathbb{Q}$-Gorenstein family is specially K-stable, then so are very general fibers. We make use of this assertion to show Theorem 1.1. To prove this, we first show the following on J-semistability.

Proposition 2.21. *Let $\pi\colon (X,\Delta,L)\to S$ be a polarized log family and let H be a $\mathbb{Q}$-line bundle on X. Suppose that there exists a closed point $s_0\in S$ such that H_{s_0} is nef and (X_{s_0},L_{s_0}) is $\mathrm{J}^{H_{s_0}}$-semistable. Then for any very general point $s\in S$, (X_s,L_s) is J^{H_s}-semistable.*

Proof. It is well known that if H_{s_0} is nef, then so is H_s for any very general point $s\in S$. On the other hand, we deal with J^{H_s}-nefness. Consider the Hilbert scheme $\mathrm{Hilb}_{X/S}$. Recall that for any S-scheme T, the set of T-valued points of $\mathrm{Hilb}_{X/S}$ is the set of closed subschemes of $X\times_S T$ flat over T. It is well known that $\mathrm{Hilb}_{X/S}$ has countably many connected components and each of them is proper over S. Here, we claim that for a very general point $s\in S$ and any p-dimensional closed subvariety $V\subset X_s$,

$$\left(n\frac{H_s\cdot L_s^{n-1}}{L_s^n}L_s-pH_s\right)\cdot L_s^{p-1}\cdot V\geq 0.$$

Indeed, for any closed subvariety $V\subset X_s$ of dimension p, there exists a connected component $\mathcal{H}\subset\mathrm{Hilb}_{X/S}$ containing the point corresponding to V. Since s is very general, $\mathcal{H}\to S$ is surjective. Then

$$\left(n\frac{H_s\cdot L_s^{n-1}}{L_s^n}L_s-pH_s\right)\cdot L_s^{p-1}\cdot V=\left(n\frac{H_{s_0}\cdot L_{s_0}^{n-1}}{L_{s_0}^n}L_{s_0}-pH_{s_0}\right)\cdot L_{s_0}^{p-1}\cdot V'\geq 0$$

by the $\mathrm{J}^{H_{s_0}}$-nefness of (X_{s_0},L_{s_0}), where $V'\subset X_{s_0}$ is a p-dimensional closed subscheme whose corresponding point in $\mathrm{Hilb}_{X/S}$ is contained in $\mathcal{H}$. This means that for any very general closed point $s\in S$, (X_s,L_s) is J^{H_s}-nef and H_s is nef. By Theorem 2.7, we obtain the assertion. □

Corollary 2.22. *Let $\pi\colon (X,\Delta,L)\to S$ be a polarized log $\mathbb{Q}$-Gorenstein family. Suppose that there exists a closed point $s_0\in S$ such that $(X_{s_0},\Delta_{s_0},L_{s_0})$ is specially*

K-*stable. Then there exist positive rational numbers* λ *and* ε *such that for any very general point* $s \in S$, $\delta(X_s, \Delta_s, L_s) \geq \lambda + \varepsilon$ *and* (X_s, L_s) *is* $\mathrm{J}^{K_{X_s}+\Delta_s+\lambda L_s}$*-semistable. In particular,* (X_s, Δ_s, L_s) *is specially* K*-stable for any very general closed point* $s \in S$.

Proof. We know that the correspondence

$$S \ni s \mapsto \delta(X_{\bar{s}}, \Delta_{\bar{s}}, L_{\bar{s}})$$

is lower-semicontinuous by [BL22, Thm. 6.6]. Thus, for any sufficiently small $\varepsilon \in \mathbb{Q}_{>0}$, there exists a nonempty open subset $U \subset S$ such that $\delta(X_{\bar{s}}, \Delta_{\bar{s}}, L_{\bar{s}}) \geq \delta(X_{s_0}, \Delta_{s_0}, L_{s_0}) - \varepsilon$ for any geometric point $\bar{s} \in U$. Choose ε small enough such that there exists $\lambda \in \mathbb{Q}_{>0}$ such that (X_{s_0}, L_{s_0}) is uniformly $\mathrm{J}^{K_{X_{s_0}}+\Delta_{s_0}+\lambda L_{s_0}}$-stable and $\lambda + 2\varepsilon \leq \delta(X_{s_0}, \Delta_{s_0}, L_{s_0})$. Then we see that $\delta(X_{\bar{s}}, \Delta_{\bar{s}}, L_{\bar{s}}) \geq \lambda + \varepsilon$ for any geometric point $\bar{s} \in U$. On the other hand, (X_s, L_s) is $\mathrm{J}^{K_{X_s}+\Delta_s+\lambda L_s}$-semistable by Proposition 2.21 for any very general closed point $s \in S$. □

§3. CM line bundle

In this section, we discuss the CM line bundle.

§3.1. CM line bundle and J-line bundle

First, we explain how to define the log CM line bundle for a polarized log $\mathbb{Q}$-Gorenstein family.

Definition 3.1. Let S be a Noetherian scheme and let E be a vector bundle over S. Let $X \subset \mathbb{P}_S(E)$ be a closed subscheme such that $\mathcal{O}_X$ is a perfect complex as an $\mathcal{O}_{\mathbb{P}_S(E)}$-module. Suppose that the generic fiber of the canonical morphism $\pi\colon X \to S$ is of dimension n. We say that $\mathcal{O}_X$ satisfies the condition $Q_{(r)}$ if the following hold:

(1) for each point $s \in S$ of depth 0,

$$\dim((\mathrm{Supp}(\mathcal{O}_X))_s) \leq r,$$

(2) for each point $s \in S$ of depth 1,

$$\dim((\mathrm{Supp}(\mathcal{O}_X))_s) \leq r + 1.$$

Let $L := \mathcal{O}_{\mathbb{P}_S(E)}(1)|_X$. Assume now that $\mathcal{O}_X$ satisfies the condition $Q_{(r)}$. For any sufficiently large $m \in \mathbb{Z}_{>0}$, consider the *Knudsen–Mumford expansion* [KM76, Thm. 4] (cf. [MFK94, Lem. 5.8])

$$\det(\pi_* \mathcal{O}_X(mL)) \cong \bigotimes_{i=0}^{n+1} \mathcal{M}_i^{\otimes \binom{m}{i}},$$

where $\mathcal{M}_i$ is a uniquely determined line bundle on S for $i = 0, \dots, n+1$. It is well known that the Knudsen–Mumford expansion is compatible with base changes. More precisely, for any morphism $g\colon T \to S$, consider the Knudsen–Mumford expansion

$$\det(\pi_{T*}\mathcal{O}_{X_T}(mL_T)) \cong \bigotimes_{i=0}^{n+1} \mathcal{N}_i^{\otimes\binom{m}{i}},$$

where π_T, X_T, and L_T are the base changes by $T \to S$. Then $\mathcal{N}_i = g^*\mathcal{M}_i$ for $0 \le i \le n+1$ (cf. [CP21, Lem. 3.5]).

Definition 3.2 (CM line bundle)**.** Let $\pi\colon (X, \Delta, L) \to S$ be a polarized log family of relative dimension n. Take $r \in \mathbb{Z}_{>0}$ such that rL is a π-very ample line bundle. Then $X \subset \mathbb{P}_S(\pi_*\mathcal{O}_X(mrL))$. We set the *log CM line bundle* of π as

$$\lambda_{\mathrm{CM},\pi} \coloneqq \pi_*(\mu_L L^{n+1} + (n+1)L^n \cdot (K_{X/S} + \Delta)),$$

where $\mu_L \coloneqq n\frac{-(K_{X_t}+\Delta_t)\cdot L_t^{n-1}}{L_t^n}$ for general closed point $t \in S$, and $\pi_*(L^n \cdot D)$ is defined to be a $\mathbb{Q}$-divisor unique up to $\mathbb{Q}$-linear equivalence on S for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X as follows. Suppose that mL is relatively very ample over S. Take a line bundle M on S such that $N \coloneqq mL + \pi^*M$ is very ample. By the Bertini theorem and [Har77, II, Exer. 8.2], we see that there exist a positive integer l and $D_1, \dots, D_n \in |lN|$ such that $Y \coloneqq D_1 \cap D_2 \cap \cdots \cap D_n$ is normal, irreducible, and finite over S. We may further assume that $Y \not\subset \operatorname{Supp} D$ and then we can define $D \cap Y$ as a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then we set a $\mathbb{Q}$-Weil divisor

$$\pi_*(L^n \cdot D) \coloneqq \frac{1}{(ml)^n}\pi_*(D \cap Y) - nm(D_t \cdot L_t^{n-1})M.$$

Then we see the following.

Proposition 3.3. *Let $\pi\colon (X, \Delta, L) \to S$ be a log $\mathbb{Q}$-Gorenstein polarized family with a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X as above. Then $\pi_*(L^n \cdot D)$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S uniquely determined up to $\mathbb{Q}$-linear equivalence independent of the choices of m, l, and $D_1, \dots, D_n$. In particular, $\lambda_{\mathrm{CM},\pi}$ is a well-defined $\mathbb{Q}$-line bundle on S and for any morphism $g\colon T \to S$ from a normal variety, it holds that $\lambda_{\mathrm{CM},\pi_T} = g^*(\lambda_{\mathrm{CM},\pi})$.*

Proof. We note that for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors E_1 and E_2 on X, we have

$$\pi_*(Y \cap (E_1 + E_2)) \sim_{\mathbb{Q}} \pi_*(Y \cap E_1) + \pi_*(Y \cap E_2)$$

and if $E_1 \sim_{\mathbb{Q}} E_2$, then $\pi_*(Y \cap E_1) \sim_{\mathbb{Q}} \pi_*(Y \cap E_2)$. Thus, we may replace D with a very ample line bundle on X by the linearity of $\pi_*(Y \cap D)$ with respect to D and by decomposing $mD = A_1 - A_2$, where A_1 and A_2 are very ample and

$m \in \mathbb{Z}_{>0}$ is sufficiently large. We may further assume that D is an effective normal Cartier divisor on X such that every fiber of $\pi|_D\colon D \to S$ is equidimensional and of dimension $n-1$ by the Bertini theorem and [Har77, II, Exer. 8.2]. Then we see that $\mathcal{O}_D$ satisfies the assumption of [KM76, Thm. 4]. Indeed, it suffices to show that $\mathcal{O}_D$ is a perfect complex of $\mathbb{P}_S(E)$, where $E = \pi_*(\mathcal{O}_X(rL))$, since the condition $Q_{(r)}$ is satisfied (see [KM76, p. 50] and [CP21, Lem. A.1]). For this, it is enough to show that $\mathcal{O}_D$ is an $\mathcal{O}_{\mathbb{P}_S(E)}$-module of finite Tor-dimension (cf. [MFK94, p. 111]). By the fact that D is a Cartier divisor of X and [MFK94, Lem. 5.8], we know that $\mathcal{O}_D$ is also of finite Tor-dimension. Thus, we may apply [KM76, Thm. 4] for $\pi|_D$ and there exists the Knudsen–Mumford expansion

$$\det((\pi|_D)_*\mathcal{O}_D(mrL|_D)) \cong \bigotimes_{i=0}^{n} \mathcal{N}_i^{\otimes\binom{m}{i}}.$$

We assert that $\pi_*(D\cdot L^n) = (\pi|_D)_*(L|_D^n) = \frac{1}{r^n}\mathcal{N}_n$. Indeed, take a big open subset $S^\circ \subset S$ such that S° is smooth and D is flat over S°. Over S°, we have

$$\pi_*(D\cdot L^n)|_{S^\circ} = (\pi|_D)_*(L|_D^n)|_{S^\circ} = \frac{1}{r^n}\mathcal{N}_n|_{S^\circ}$$

by [CP21, Lems A.1, A.2]. Hence, $\pi_*(D\cdot L^n) \sim_{\mathbb{Q}} \frac{1}{r^n}\mathcal{N}_n$ is $\mathbb{Q}$-Cartier and for any morphism $g\colon T \to S$ from a normal variety, we see that $g^*(\pi_*(L^n\cdot D)) = \pi_{T*}(L_T^n\cdot D_T)$. It follows from this that $\lambda_{\mathrm{CM},\pi_T} = g^*(\lambda_{\mathrm{CM},\pi})$. We complete the proof. □

Definition 3.4 (J-line bundle)**.** Let $\pi\colon (X,\Delta,L) \to S$ be a polarized log pair with an $\mathbb{R}$-line bundle H on X. We set the J^H*-line bundle* with respect to H as

$$\lambda_{\mathrm{J},\pi,H} \coloneqq \pi_*\Big((n+1)L^n\cdot H - n\frac{H_t\cdot L_t^{n-1}}{L_t^n}L^{n+1}\Big),$$

where $t \in S$ is a general closed point. As Proposition 3.3, we have the following.

Proposition 3.5. *Let $\pi\colon (X,\Delta,L) \to S$ be a polarized log pair with an $\mathbb{R}$-line bundle H on X. Then, $g^*(\lambda_{\mathrm{J},\pi,H}) = \lambda_{\mathrm{J},\pi_T,H_T}$ for any morphism $g\colon T \to S$ from a normal variety.*

If $H = K_{X/S} + \Delta$ for a $\mathbb{Q}$-Gorenstein family, then we obtain that $\lambda_{\mathrm{J},\pi,H} = \lambda_{\mathrm{CM},\pi}$.

Definition 3.6 (CM degree and J-degree)**.** Let $\pi\colon (X,\Delta,L) \to C$ be a polarized log family of relative dimension n with C a proper smooth curve. Let t be a closed point of C and $v \coloneqq (L_t^n)$. Then we set the CM degree as

$$\mathrm{CM}((X,\Delta,L)/C) \coloneqq \frac{1}{(n+1)v}\deg_C \lambda_{\mathrm{CM},\pi}|_C.$$

On the other hand, let H be an $\mathbb{R}$-line bundle on X. We set the J^H-degree as

$$\mathcal{J}^H((X,L)/C) := \frac{1}{(n+1)v}\deg_C \lambda_{\mathrm{J},\pi,H}|_C.$$

We note that

$$\mathrm{CM}((X,\Delta,L)/C) = v^{-1}\Big((K_{X/S}+\Delta)\cdot L^n - \frac{n(K_{X_t}+\Delta_t)\cdot L_t^{n-1}}{(n+1)L_t^n}L^{n+1}\Big),$$
$$\mathcal{J}^H((X,L)/C) = v^{-1}\Big(H\cdot L^n - \frac{nH_t\cdot L_t^{n-1}}{(n+1)L_t^n}L^{n+1}\Big).$$

Next, we explain a relationship between the CM degree and the Harder–Narasimhan filtration.

Definition 3.7 (Harder–Narasimhan filtration)**.** Let C be a proper smooth curve. For any locally free sheaf E on C, we set the *slope* of E as

$$\mu(E) := \frac{\deg_C E}{\operatorname{rank} E}.$$

We say that E is *semistable* if $\mu(E) \geq \mu(F)$ for any nonzero subsheaf $F \subsetneq E$. It is well known that there exists the unique sequence (cf. [HL10, Thm. 1.3.4])

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = E$$

such that E_i/E_{i-1} is a semistable locally free sheaf and $\mu_i := \mu(E_i/E_{i-1})$ satisfies that $\mu_i > \mu_{i+1}$ for $1 \leq i \leq k$. We denote $\mu_{\min} := \mu_k$ and call this the *minimal slope* of E. For any $\lambda \in \mathbb{R}$, we set $\mathscr{F}^\lambda_{\mathrm{HN}}E$ as the union of subsheaves of minimal slope at least λ. We call $\mathscr{F}_{\mathrm{HN}}$ the *Harder–Narasimhan filtration* of E.

Definition 3.8. Let $\pi\colon (X,\Delta,L) \to C$ be a polarized log family of relative dimension n, where C is a proper smooth curve. Take $r \in \mathbb{Z}_{>0}$ such that rL is Cartier. Take $s \in C$ such that X_s is normal. Let $\mathcal{R}_m := \pi_*\mathcal{O}_X(mrL)$ and $R_m := H^0(X_s, \mathcal{O}_{X_s}(mrL_s))$ for any $m \in \mathbb{Z}_{\geq 0}$. We set the Harder–Narasimhan filtration $\mathscr{F}_{\mathrm{HN}}$ on $\mathcal{R}_m$ as Definition 3.7. We define a filtration F_{HN} as

$$F^\lambda_{\mathrm{HN}}R_m := \mathrm{Image}(\mathscr{F}^\lambda_{\mathrm{HN}}\mathcal{R}_m \subset \mathcal{R}_m \to R_m).$$

It is known by [XZ20, Lem.-Def. 2.26] that the filtration $\mathscr{F}_{\mathrm{HN}}$ is linearly bounded and multiplicative and so is F_{HN}. We call F_{HN} the *induced* filtration of $R = \bigoplus_{m\geq 0} R_m$ (see [XZ20, §2.8]).

The following is important to calculate the CM degree or the J-degree.

Proposition 3.9. *Let $\pi\colon (X,\Delta,L) \to C$ be a polarized log pair of relative dimension n, where C is a proper smooth curve. Let $s \in C$ be a closed point. Take*

$r \in \mathbb{Z}_{>0}$ such that rL is Cartier and define the induced filtration F_{HN} on $R = \bigoplus_{m\geq 0} H^0(X_s, \mathcal{O}_{X_s}(mrL_s))$. Then F_{HN} is a good filtration and

$$S(F_{\mathrm{HN}}) = r\frac{L^{n+1}}{(n+1)L_s^n}.$$

Proof. By definition (cf. [XZ20, Prop. 4.6]), we see that

$$S_m(F_{\mathrm{HN}}) = \frac{\deg_C \pi_*\mathcal{O}_X(mrL)}{mh^0(X_s, \mathcal{O}_{X_s}(mrL_s))}.$$

Furthermore, let $g(C)$ be the genus of C. By the Leray spectral sequence and the Serre vanishing theorem, we have that

$$\chi(X, \mathcal{O}_X(mrL)) = \chi(C, \pi_*\mathcal{O}_X(mrL))$$

for any sufficiently large $m > 0$. Then, by the Riemann–Roch theorem on locally free sheaves on C, we have

(1) $$\deg_C \pi_*\mathcal{O}_X(mrL) = h^0(X_s, \mathcal{O}_{X_s}(mrL_s))(g(C)-1) + \chi(X, \mathcal{O}_X(mrL)).$$

By this, we see that F_{HN} is a good filtration. Note that

$$h^0(X_s, \mathcal{O}_{X_s}(mrL_s)) = \frac{(mr)^n}{n!}L_s^n + O(m^{n-1}),$$
$$\chi(X, \mathcal{O}_X(mrL)) = \frac{(mr)^{n+1}}{(n+1)!}L^{n+1} + O(m^n).$$

Thus we have the second assertion by $\lim_{m\to\infty} S_m(F_{\mathrm{HN}}) = S(F_{\mathrm{HN}})$. □

As [Hat24a, Cor. 3.9], we obtain that the CM degree is nonnegative when a fiber is smooth and admits a unique cscK metric.

Corollary 3.10. *We keep the notation as above. Suppose further that $\Delta = 0$ and X_s is a smooth variety with a cscK metric in $\mathrm{c}_1(L_s)$ and $\mathrm{Aut}(X_s, L_s)$ discrete for some $s \in C$. Then $\mathrm{CM}((X, 0, L)/C) \geq 0$.*

Proof. Let $\chi(X, \mathcal{O}_X(mrL)) = b_0m^{n+1} + b_1m^n + O(m^{n-1})$ and $h^0(X_s, \mathcal{O}_{X_s}(mrL_s)) = a_0m^n + a_1m^{n-1} + O(m^{n-2})$. We see by (1) that

$$\deg_C \pi_*\mathcal{O}_X(mrL) = b_0m^{n+1} + (b_1 + a_0(g(C)-1))m^n + O(m^{n-1}).$$

Thus, we see that $\mathrm{CM}((X, 0, L)/C) = \mathrm{DF}(F_{\mathrm{HN}})$. On the other hand, we see that (X_s, L_s) is asymptotically Chow stable by [Don01]. Thus, we see as [Hat24a, Thm. 2.18] that

$$\mathrm{DF}(F_{\mathrm{HN}}) \geq 0.$$

We complete the proof. □

§3.2. Moduli of uniformly adiabatically K-stable klt-trivial fibrations over curves and the CM line bundle

Let $f\colon X \to C$ be a morphism of normal varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_C$ and suppose that C is a proper smooth curve. We say that $f\colon (X,\Delta) \to C$ is a *klt-trivial fibration* over a curve if $K_X+\Delta \sim_{\mathbb{Q},C} 0$ and (X,Δ) is klt. Then we set the *discriminant divisor*

$$B \coloneqq \sum_{P\in C} (1-\mathrm{lct}(X,\Delta;f^*P))P.$$

Let M be a $\mathbb{Q}$-divisor on C such that

$$K_X+\Delta \sim_{\mathbb{Q}} f^*(K_C+M+B).$$

We call M the *moduli divisor*. Take an f-ample $\mathbb{Q}$-Cartier Weil divisor A. We say that $f\colon (X,\Delta,A) \to C$ is *uniformly adiabatically* K-*stable* if there exist positive constants ε_0 and δ such that

$$M_\Delta^{\mathrm{NA}}(\mathcal{X},\mathcal{M}) \geq \delta \mathcal{J}^{\varepsilon A+L,\mathrm{NA}}(\mathcal{X},\mathcal{M})$$

for any $\varepsilon \in (0,\varepsilon_0)$ and normal semiample test configuration $(\mathcal{X},\mathcal{M})$ for $(X,\varepsilon A+L)$, where L is a fiber of f. It is known by [Hat25, Thm. 1.1] that f is uniformly adiabatically K-stable if and only if one of the following holds.

- $-K_X-\Delta$ is nef but not numerically trivial and $\delta(C,B,-K_C-M-B)>1$, or
- $K_X+\Delta$ is nef.

Let $\mathfrak{Z}_{d,v,u}$ be the following set of isomorphism classes of uniformly adiabatically K-stable klt-trivial fibrations for $d\in\mathbb{Z}_{>0}$, $u\in\mathbb{Q}_{>0}$, and $v\in\mathbb{Q}_{>0}$:

$$\left\{ f\colon (X,0,A)\to\mathbb{P}^1 \;\middle|\; \begin{array}{l} \text{(i) } f \text{ is a uniformly adiabatically K-stable klt-trivial fibration} \\ \quad\ \text{over } \mathbb{P}^1 \text{ with } \dim X = d, \\ \text{(ii) } A \text{ is an } f\text{-ample line bundle such that } K_X\cdot A^{d-1} = -uv, \\ \text{(iii) } K_X \sim_{\mathbb{Q}} -uf^*\mathcal{O}_{\mathbb{P}^1}(1) \end{array} \right\}.$$

Theorem 3.11 (Cf. [HH25, Thm. 1.2])**.** *We fix $d\in\mathbb{Z}_{>0}$, $u\in\mathbb{Q}_{>0}$, $v\in\mathbb{Q}_{>0}$. Then we have the following for some $r\in\mathbb{Z}_{>0}$. For any locally Noetherian scheme*

S over $\mathbb{C}$, we attain a groupoid $\mathscr{M}_{d,v,u,r}(S)$ whose objects are

$$\left\{ \begin{array}{c|l} \begin{array}{ccc} (\mathcal{X},\mathscr{A}) & \xrightarrow{f} & \mathcal{C} \\ & {\scriptstyle \pi_{\mathcal{X}}}\searrow \quad \swarrow & \\ & S & \end{array} & \begin{array}{l} \text{(i) } \pi_{\mathcal{X}} \text{ is a flat projective morphism and } \mathcal{X} \text{ is a scheme,} \\ \text{(ii) } \mathscr{A} \in \mathrm{Pic}_{\mathcal{X}/S}(S) \text{ such that } \mathscr{A} \text{ is } f\text{-ample,} \\ \text{(iii) } \omega^{[r]}_{\mathcal{X}/S} \text{ exists as a line bundle, whose restriction to any geometric} \\ \quad \text{fiber } \mathcal{X}_{\bar{s}} \text{ over } \bar{s} \in S \text{ coincides with } \mathcal{O}_{\mathcal{X}_{\bar{s}}}(rK_{\mathcal{X}_{\bar{s}}}), \\ \text{(iv) } \pi_{\mathcal{X}*}\omega^{[-lr]}_{\mathcal{X}/S} \text{ is locally free and it generates } H^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(-lrK_{\mathcal{X}_s})), \\ \quad \text{for any point } s \in S \text{ and any } l \in \mathbb{Z}_{>0}, \\ \text{(v) } f \text{ is the canonical } S\text{-morphism to } \mathcal{C} \coloneqq \mathbf{Proj}_S(\bigoplus \pi_{\mathcal{X}*}\omega^{[-lr]}_{\mathcal{X}/S}) \\ \quad \text{and } (\mathcal{X}_{\bar{s}}, \mathscr{A}_{\bar{s}}) \to \mathcal{C}_{\bar{s}} \in \mathfrak{Z}_{d,v,u} \text{ for any geometric point } \bar{s} \in S \end{array} \end{array} \right\}.$$

and isomorphisms are S-isomorphisms $\alpha\colon (\mathcal{X},\mathscr{A}) \to (\mathcal{X}',\mathscr{A}')$ such that $\alpha^*\mathscr{A}' = \mathscr{A} \otimes f^*\mathscr{B}$ for some $\mathscr{B} \in \mathrm{Pic}_{\mathcal{C}/S}(S)$.

Then $\mathscr{M}_{d,v,u,r}$ is a separated Deligne–Mumford stack of finite type over $\mathbb{C}$ with a coarse moduli space $M_{d,v,u,r}$ (cf. [Ols16]).

Furthermore, there exists $w > 0$ such that for any geometric point $\bar{s} \in \mathscr{M}_{d,v,u,r}$, if $\mathrm{vol}(\mathscr{A}_{\bar{s}}) = w$, then $\mathscr{A}_{\bar{s}}$ is ample and the object $(\mathcal{X}_{\bar{s}},\mathscr{A}_{\bar{s}})$ corresponding to $\bar{s}$ is specially K*-stable.*

Take w as in Theorem 3.11. We can set the *CM line bundle $\Lambda_{\mathrm{CM},w}$ on $M_{d,v,u,r}$ with respect to the volume w* as follows. As [HH25, Rem. 6.5], we can put the universal family $\pi_{\mathscr{U}}\colon (\mathscr{U},\mathscr{A}) \to \mathscr{M}_{d,v,u,r}$ for any geometric fiber $(\mathscr{U}_{\bar{s}},\mathscr{A}_{\bar{s}})$ over $\mathscr{M}_{d,v,u,r}$. Here, we note that $\mathscr{A}$ is uniquely determined up to relative $\mathbb{Q}$-linear equivalence over the universal base curve $\mathscr{C} \coloneqq \mathbf{Proj}_{\mathscr{M}_{d,v,u,r}}(\bigoplus_{l\geq 0}\pi_{\mathscr{U}*}\omega^{[-lr]}_{\mathscr{U}/\mathscr{M}_{d,v,u,r}})$. If we choose the relative linear equivalence class of $\mathscr{A}$ so that $\mathrm{vol}(\mathscr{A}_{\bar{s}}) < w$ for some geometric point $\bar{s} \in \mathscr{M}_{d,v,u,r}$, then $(\mathscr{U}_{\bar{s}},\mathscr{A}_{\bar{s}})$ might not be K-semistable or $\mathscr{A}_{\bar{s}}$ might not be ample. It is not hard to see that by adding a sufficiently relatively ample line bundle on $\mathscr{C}$ over $\mathscr{M}_{d,v,u,r}$ to $\mathscr{A}$, we may choose $\mathscr{A}$ so that $\mathscr{A}$ is relatively ample over $\mathscr{M}_{d,v,u,r}$ and $\mathrm{vol}(\mathscr{A}_{\bar{s}}) = w$. By Theorem 3.11, we have that all the members $(\mathscr{U}_{\bar{s}},\mathscr{A}_{\bar{s}})$ of $\mathscr{M}_{d,v,u,r}$ are specially K-stable. Then we can define the CM line bundle $\lambda_{\mathrm{CM},\pi_{\mathscr{U}}}$ with respect to $\mathscr{A}$ on $\mathscr{M}_{d,v,u,r}$ by the construction of $(\mathscr{U},\mathscr{A})$ (cf. [HH25, Exa. 2.13]). Let $\pi\colon \mathscr{M}_{d,v,u,r} \to M_{d,v,u,r}$ be the canonical morphism to its coarse moduli space. By [Alp13, Thm. 10.3] and [HH25, Thm. 1.4], we obtain a $\mathbb{Q}$-line bundle $\Lambda_{\mathrm{CM},w}$ on $M_{d,v,u,r}$ such that $\pi^*\Lambda_{\mathrm{CM},w} = \lambda_{\mathrm{CM},\pi_{\mathscr{U}}}$ as the argument of [CP21, Lem. 10.2]. Note that any geometric fiber $(\mathscr{U}_{\bar{s}},\mathscr{A}_{\bar{s}})$ is specially K-stable by Theorem 3.11.

§4. Nefness of the CM line bundle

We first discuss the nonnegativity of J-degree.

Proposition 4.1. *Let $\pi\colon (X,\Delta,L)\to C$ be a polarized log family, where C is a proper smooth curve, and let H be a nef $\mathbb{Q}$-line bundle on X. Suppose that for any very general closed point $s\in C$, (X_s,L_s) is J^{H_s}-semistable. Then*

$$\mathcal{J}^H((X,L)/C)\geq 0.$$

Proof. We first deal with the case when H is ample and for any very general point $s\in C$, (X_s,L_s) is J^{H_s}-semistable. Suppose that rH is very ample. Take a general member $D\in|rH|$ such that D is normal, $\pi|_D\colon D\to C$ is flat and a contraction, and D_s is compatible with F_{HN}, where $r\in\mathbb{Z}_{>0}$. We set the Harder–Narasimhan filtrations $\mathscr{F}_{\mathrm{HN}}$ on $\bigoplus_{m\geq 0}\pi_*\mathcal{O}_X(mrL)$ and $\mathscr{F}_{D,\mathrm{HN}}$ on $\bigoplus_{m\geq 0}\pi_*\mathcal{O}_D(mrL|_D)$ respectively. There exists the canonical map

$$\mathscr{F}^{\lambda}_{\mathrm{HN}}\pi_*\mathcal{O}_X(mrL)\to\mathscr{F}^{\lambda}_{D,\mathrm{HN}}\pi_*\mathcal{O}_D(mrL|_D) \tag{2}$$

for any sufficiently divisible $m\in\mathbb{Z}_{>0}$ and $\lambda\in\mathbb{Q}$ by [HL10, Lem. 1.3.3]. We consider the restricted filtration $(F_{\mathrm{HN}})|_{D_s}$ (cf. [Hat24a, Exa. 2.4]) and the induced filtration $F_{D,\mathrm{HN}}$ defined for the family $(D,rL|_D)\to S$. We see by (2) that for any sufficiently divisible $m\in\mathbb{Z}_{>0}$ and $\lambda\in\mathbb{Q}$, there exists a natural inclusion

$$(F_{\mathrm{HN}})|^{\lambda}_{D_s}H^0(D_s,\mathcal{O}_{D_s}(mL_s))\subset F^{\lambda}_{D,\mathrm{HN}}H^0(D_s,\mathcal{O}_{D_s}(mL_s)).$$

Let $w_{F_{\mathrm{HN}}}(m)$, $w_{F_{D,\mathrm{HN}}}(m)$, and $w_{(F_{\mathrm{HN}})|_{D_s}}(m)$ be the weight functions of F_{HN}, $F_{D,\mathrm{HN}}$, and $(F_{\mathrm{HN}})|_{D_s}$ respectively. Then we have by Proposition 3.9 that

$$\begin{aligned}\mathcal{J}^{rH}((X,L)/C)&=\lim_{m\to\infty}\Big(n!\frac{w_{F_{D,\mathrm{HN}}}(m)}{m^n}-\frac{nr(H_s\cdot L_s^{n-1})}{(n+1)L_s^n}\frac{(n+1)!w_{F_{\mathrm{HN}}}(m)}{m^{n+1}}\Big)\\&\geq\lim_{m\to\infty}\Big(n!\frac{w_{(F_{\mathrm{HN}})|_{D_s}}(m)}{m^n}-\frac{nr(H_s\cdot L_s^{n-1})}{(n+1)L_s^n}\frac{(n+1)!w_{F_{\mathrm{HN}}}(m)}{m^{n+1}}\Big)\\&=\mathcal{J}^{rH,\mathrm{NA}}(F_{\mathrm{HN}}).\end{aligned}$$

By Proposition 2.18 and the linearity of $\mathcal{J}^{H,\mathrm{NA}}(F_{\mathrm{HN}})$ with respect to H, we have

$$r\mathcal{J}^{H,\mathrm{NA}}(F_{\mathrm{HN}})=\mathcal{J}^{rH,\mathrm{NA}}(F_{\mathrm{HN}})\geq 0.$$

We claim that the assertion in the general case follows from what we have shown in the previous paragraph. Indeed, suppose that $L+c\pi^*P$ is ample for some $c\in\mathbb{Q}_{>0}$ and closed point $P\in C$. Then we see that

$$\mathcal{J}^{H+\varepsilon(L+c\pi^*P)}((X,L)/C)\geq 0$$

for any $\varepsilon\in\mathbb{Q}_{>0}$. Thus,

$$\mathcal{J}^H((X,L)/C)=\lim_{\varepsilon\to 0}\mathcal{J}^{H+\varepsilon(L+c\pi^*P)}((X,L)/C)\geq 0.$$

We complete the proof. ∎

Next we deal with nefness of the CM line bundle when a fiber is specially K-semistable.

Proposition 4.2. *Let $\pi\colon (X,\Delta,L)\to C$ be a polarized log $\mathbb{Q}$-Gorenstein family such that C is a proper smooth curve and $L^{n+1}=0$. Take $r\in\mathbb{Z}_{>0}$ such that rL is a line bundle. Suppose that there exist a closed point $s\in C$ and $\delta\in(0,\delta(X_s,\Delta_s,L_s)]$ such that $K_{X/C}+\Delta+\delta L$ is π-nef. Then the following hold:*

1. *For any sufficiently divisible $m\in\mathbb{Z}_{>0}$ and sufficiently small $\varepsilon\in\mathbb{Q}_{>0}$ and $\varepsilon'\in\mathbb{Q}_{>0}$, there exists $D\in|mrL+(m\varepsilon+2g(C))\pi^*P|$ such that $(X_s,\Delta_s+\frac{\delta+\varepsilon'}{mr}D_s)$ is lc, where $g(C)$ is the genus of C and P is an arbitrary closed point of C.*
2. *$K_{X/C}+\Delta+\delta L$ is globally nef.*

Proof. Consider the induced filtration F_{HN} of $R\coloneqq\bigoplus_{m\geq 0}R_m$, where we let $R_m=H^0(X_s,\mathcal{O}_{X_s}(mrL_s))$. By $L^{n+1}=0$ and Proposition 3.9, we see that $S(F_{\mathrm{HN}})=0$. By Theorem 2.14 and Lemma 2.15, for any positive sufficiently small rational numbers ε and ε', it holds that

$$\mathrm{lct}(X_s,\Delta_s;I_{m,-m\varepsilon}(F_{\mathrm{HN}}))\geq\frac{\delta(X_s,\Delta_s,L_s)+\varepsilon'}{mr}$$

for any sufficiently large $m\in\mathbb{Z}_{>0}$, where $I_{m,\lambda}(F_{\mathrm{HN}})$ is the base ideal of $F^{\lambda}_{\mathrm{HN}}R_m$. In particular, $I_{m,-m\varepsilon}(F_{\mathrm{HN}})\neq 0$. We know that $I_{m,-m\varepsilon}(F_{\mathrm{HN}})$ is the image of

$$\mathscr{F}^{-m\varepsilon}_{\mathrm{HN}}\pi_*\mathcal{O}_X(mrL)\otimes\mathcal{O}_{X_s}(-mrL_s)\to\mathcal{O}_{X_s}.$$

By [CP21, Prop. 5.7], we see that

$$\mathscr{F}^{-m\varepsilon}_{\mathrm{HN}}\pi_*\mathcal{O}_X(mrL)\subset\mathrm{Image}(H^0(X,mrL+(m\varepsilon+2g(C))\pi^*P)\otimes\mathcal{O}_C \\ \to\pi_*\mathcal{O}_X(mrL))$$

for any sufficiently divisible $m\in\mathbb{Z}_{>0}$. This means that there exists an effective divisor $D\in|mrL+(m\varepsilon+2g(C))\pi^*P|$ such that the section corresponding to D_s is contained in $I_{m,-m\varepsilon}(F_{\mathrm{HN}})$. Thus, we see that $(X_s,\Delta_s+\frac{\delta+\varepsilon'}{mr}D_s)$ is lc. Note that

$$K_{X/C}+\Delta+\frac{\delta+\varepsilon'}{mr}D\sim_{\mathbb{Q}}K_{X/C}+\Delta+(\delta+\varepsilon')\Big(L+\frac{m\varepsilon+2g(C)}{mr}\pi^*P\Big)$$

and hence $\pi_*\mathcal{O}_X(l(K_{X/C}+\Delta+\frac{\delta+\varepsilon'}{mr}D))$ is a nef vector bundle for any sufficiently large and divisible $l\in\mathbb{Z}_{>0}$ [F18, Thm. 1.11]. This means that $\mathcal{O}(1)$ of $\mathbb{P}_C(\pi_*\mathcal{O}_X(l(K_{X/C}+\Delta+\frac{\delta+\varepsilon'}{mr}D)))$ is globally nef. Since $K_{X/C}+\Delta+\frac{\delta+\varepsilon'}{mr}D$ is π-ample, we have a closed immersion $\iota\colon X\hookrightarrow\mathbb{P}_C(\pi_*\mathcal{O}_X(l(K_{X/C}+\Delta+\frac{\delta+\varepsilon'}{mr}D)))$. Then we have that

$$K_{X/C}+\Delta+\frac{\delta+\varepsilon'}{mr}D\sim_{\mathbb{Q}}\frac{1}{l}\iota^*\mathcal{O}(1)$$

is nef. Here, we take m sufficiently divisible and ε, ε' sufficiently small. Thus, $K_{X/S}+\Delta+\delta L$ is nef by taking the limit. We complete the proof. □

Theorem 4.3. *Let $\pi\colon (X,\Delta,L)\to C$ be a polarized log $\mathbb{Q}$-Gorenstein family, where C is a proper smooth curve. If there exists a closed point $s\in C$ such that (X_s,Δ_s,L_s) is specially* K*-semistable and $K_{X/C}+\Delta+\delta(X_s,\Delta_s,L_s)L$ is π-nef, then*

$$\mathrm{CM}((X,\Delta,L)/C)\geq 0.$$

Proof. By taking some $c\in\mathbb{Q}$ and replacing L by $L+c\pi^*P$, where P is a closed point of C, we may assume that $L^{n+1}=0$. Then we see that

$$\mathrm{CM}((X,\Delta,L)/C)=\mathcal{J}^{K_{X/C}+\Delta+\delta(X_s,\Delta_s,L_s)L}((X,L)/C).$$

Thus, the assertion follows from Propositions 4.1 and 4.2. □

§5. Bigness of the CM line bundle

First, we deal with Proposition 1.4. To show this, we assert the following.

Proposition 5.1. *Let $\pi\colon (X,L)\to S$ be a polarized family. Suppose that S is projective and there exists an ample $\mathbb{Q}$-line bundle H on X. If there exists a closed point $s_0\in S$ such that (X_{s_0},L_{s_0}) is* $\mathrm{J}^{H_{s_0}}$*-semistable, then $\lambda_{\mathrm{J},\pi,H}$ is big.*

Example 5.2. We remark that we do not need to assume that π has a maximal variation. Indeed, we can easily check that $\lambda_{\mathrm{J},\pi,H}$ is big even in the case when the family *pi* is a trivial test configuration as follows. Let $(X_{\mathbb{A}^1},L_{\mathbb{A}^1})$ be the trivial test configuration for a polarized manifold (X,L). Let H be an ample line bundle on X and it is trivial that $H_{\mathbb{A}^1}+aX\times\{0\}$ is ample on $X_{\mathbb{A}^1}$ for $a>0$. It is easy to see that

$$\mathrm{DF}(X_{\mathbb{A}^1},L_{\mathbb{A}^1})=\frac{1}{(n+1)L^n}\deg_{\mathbb{P}^1}\lambda_{\mathrm{CM},\pi}=0,$$

where $\pi\colon (X\times\mathbb{P}^1,L\times\mathbb{P}^1)\to\mathbb{P}^1$ is the canonical compactification of $X_{\mathbb{A}^1}$ and

$$\mathcal{J}^{H,\mathrm{NA}}(X_{\mathbb{A}^1},L_{\mathbb{A}^1})=0.$$

However, we see that

$$\frac{1}{(n+1)L^n}\deg_{\mathbb{P}^1}(\lambda_{\mathrm{J},\pi,H_{\mathbb{A}^1}+aX\times\{0\}})=\mathcal{J}^{H,\mathrm{NA}}(X_{\mathbb{A}^1},L_{\mathbb{A}^1})+a=a>0.$$

Proof of Proposition 5.1. Take an ample line bundle M on S. Take $\delta\in\mathbb{Q}_{>0}$ such that $H-\delta\pi^*M$ is also ample. By Proposition 2.21, we have that (X_s,L_s) is J^{H_s}-semistable for any very general closed point $s\in S$ since $(H-\delta\pi^*M)_s=H_s$. For any movable curve $C \to S$,

$$\lambda_{\mathrm{J},\pi,H-\delta\pi^*M} \cdot C \geq 0$$

by Propositions 3.5 and 4.1. By [BDPP13], we conclude that $\lambda_{\mathrm{J},\pi,H-\delta\pi^*M}$ is pseudo-effective. On the other hand,

$$\lambda_{\mathrm{J},\pi,\pi^*M} \cdot C = (n+1)(L_t^n)M \cdot C$$

for any movable curve $C \to S$ where t is a general point of S. Thus, $\lambda_{\mathrm{J},\pi,\pi^*M} \equiv (n+1)(L_t^n)M$ is big by [BDPP13]. Since $\lambda_{\mathrm{J},\pi,H} = \lambda_{\mathrm{J},\pi,H-\delta\pi^*M} + \delta\lambda_{\mathrm{J},\pi,\pi^*M}$, we have that $\lambda_{\mathrm{J},\pi,H}$ is big. □

Proof of Proposition 1.4. This immediately follows from Propositions 4.1 and 5.1. Indeed, we see that $\lambda_{\mathrm{J},\pi,H}|_V \sim \lambda_{\mathrm{J},\pi_V,H_V}$ is big and nef for any subvariety $V \subset S$ by Proposition 3.5. Then $\lambda_{\mathrm{J},\pi,H}$ is ample by the Nakai–Moishezon criterion. □

Next, we deal with Theorem 1.1. For this, we show the following technical result.

Proposition 5.3. *Let $\pi\colon (X,\Delta,L) \to S$ be a polarized log $\mathbb{Q}$-Gorenstein family of relative dimension n with maximal variation, where S is projective and $(X_{\bar{s}},\Delta_{\bar{s}})$ is klt for any geometric point $\bar{s} \in S$. Suppose that $\pi_*L^{n+1} \equiv 0$. Suppose that there exists $\lambda \in \mathbb{Q}_{>0}$ such that $\lambda < \delta(X_s,\Delta_s,L_s)$ for any very general closed point $s \in S$ and $K_{X/S}+\Delta+\lambda L$ is π-ample. Then the $\mathbb{Q}$-line bundle $\pi_*(K_{X/S}+\Delta+\lambda L)^{n+1}$ is big.*

Proof. We modify the argument of all the parts of the proof of [XZ20, Lem. 7.4] as follows. By taking a resolution of singularities of S, we may assume that S is smooth. Let $D = \mathrm{Supp}(\Delta)$. Take a big line bundle H on S and $r \in \mathbb{Z}_{>0}$ such that $M := r(K_{X/S}+\Delta+\lambda L)$ is a π-very ample line bundle and $H^j(X_s, M_s^{\otimes k}) = 0$ and $H^j(D_s, M_s^{\otimes k}|_{D_s}) = 0$ for any $s \in S$, $j > 0$, and $k > 0$. We may also assume that the two canonical maps

$$\mathrm{Sym}^d H^0(X_s,\mathcal{O}_{X_s}(M_s)) \to H^0(X_s,\mathcal{O}_{X_s}(dM_s))$$

and

$$H^0(X_s,\mathcal{O}_{X_s}(dM_s)) \to H^0(D_s,\mathcal{O}_{X_s}(dM_s))$$

are surjective (cf. [F+05, Lem. 5.1]) for any $d \in \mathbb{Z}_{>0}$ by taking M_s sufficiently ample. Choose $d \in \mathbb{Z}_{>0}$ such that the following further holds:

- Let I_{X_s} and I_{D_s} be ideal sheaves of $\mathbb{P}^{h^0(X_s,\mathcal{O}_{X_s}(M_s))-1}$ with respect to closed embeddings of X_s and D_s into $\mathbb{P}^{h^0(X_s,\mathcal{O}_{X_s}(M_s))-1}$ induced by M_s. Then $H^0(I_{X_s}(d))$ and $H^0(I_{D_s}(d))$ generate I_{X_s} and I_{D_s} respectively for any $s \in S$.

Indeed, since there are only finitely many possibilities for the Hilbert polynomials of X_s and D_s for all $s \in S$ with respect to M_s, we can take d such that I_{X_s} and I_{D_s} are d-regular by [F+05, Thm. 5.3] and the above condition immediately follows from [F+05, Lem. 5.1]. Then it is easy to see that X_s and D_s are cut out by homogeneous polynomials of degree at most d in $\mathbb{P}^{h^0(X_s,\mathcal{O}_{X_s}(M_s))}$. Combining them, we conclude that π and d satisfy all the conditions of [XZ20, Def. 6.1]. Let $W = \pi_*\mathcal{O}_X(M)$ and $Q := \pi_*\mathcal{O}_X(dM) \oplus (\pi|_D)_*\mathcal{O}_D(dM|_D)$ and set the ranks of them as w and q respectively. We note that Q is not locally free in general but there exists a big open subset $S^\circ \subset S$ such that $D|_{S^\circ}$ and any irreducible component of $D|_{S^\circ}$ are flat over S°. We see that $Q|_{S^\circ}$ is a locally free sheaf of rank q and set B as a Weil divisor on S such that $\det(Q|_{S^\circ}) \sim B|_{S^\circ}$. Since S is smooth, we regard B as a Cartier divisor. Due to [XZ20, Thm. 6.6], we see that there exist $m \in \mathbb{Z}_{>0}$ and a nonzero map

$$\mathrm{Sym}^{dqm}(W^{\oplus 4w})|_{S^\circ} \to \det(Q|_{S^\circ})^{\otimes m} \otimes \mathcal{O}_S(-H)|_{S^\circ}.$$

By the S_2-condition of S, the above map is uniquely extended to the nonzero map

$$\mathrm{Sym}^{dqm}(W^{\oplus 4w})|_{S^\circ} \to \mathcal{O}_S(mB - H)|_{S^\circ}.$$

For any movable curve $g\colon C \to S$, the image of C contains a very general point of S and hence g^*W is a nef vector bundle by Proposition 4.2 and [F18, Thm. 1.11]. This means that the degree of the image of $\mathrm{Sym}^{dqm}(W^{\oplus 4w}) \to \mathcal{O}_S(mB - H)$ is nonnegative since $g^*\,\mathrm{Sym}^{dqm}(W^{\oplus 4w})$ is nef (cf. [L04b, Thm. 6.1.15]). Since the map $\mathrm{Sym}^{dqm}(W^{\oplus 4w}) \to \mathcal{O}_S(mB-H)$ is also nonzero, the degree of $g^*\mathcal{O}_S(mB-H)$ is nonnegative. Therefore, B is big by [BDPP13].

In this paragraph, we show the inequality (6) below, which is a key step to show Proposition 5.3. Consider the injective maps

$$\det(\pi_*\mathcal{O}_X(dM)) \hookrightarrow \bigotimes_{i=1}^{q_1} \pi_*\mathcal{O}_X(dM)$$

and

$$\det(\pi_*\mathcal{O}_D(dM|_D)|_{S^\circ}) \hookrightarrow \bigotimes_{i=1}^{q_2} \pi_*\mathcal{O}_D(dM|_D)|_{S^\circ},$$

which are sections of the canonical surjections

$$\bigotimes_{i=1}^{q_1} \pi_*\mathcal{O}_X(dM) \to \det(\pi_*\mathcal{O}_X(dM))$$

and

$$\bigotimes_{i=1}^{q_2} \pi_*\mathcal{O}_D(dM|_D)|_{S^\circ} \to \det(\pi_*\mathcal{O}_D(dM|_D)|_{S^\circ})$$

respectively, where q_1 and q_2 are the ranks of $\pi_*\mathcal{O}_X(dM)$ and $\pi_*\mathcal{O}_D(dM|_D)$ respectively. We note that such sections indeed exist since for any vector space V of dimension l over $\mathbb{C}$, the canonical surjection

$$\bigotimes_{i=1}^{l} V \to \det(V)$$

splits in a $\mathrm{GL}(l)$-equivariant way by the linear reductivity of $\mathrm{GL}(l)$. By them, we obtain the embedding

$$\det(Q|_{S^\circ}) \hookrightarrow \bigotimes_{i=1}^{q_1} \pi_*\mathcal{O}_X(dM)|_{S^\circ} \otimes \bigotimes_{i=1}^{q_2} \pi_*\mathcal{O}_D(dM|_D)|_{S^\circ} \tag{3}$$

in a similar way. Let $Z := X^{(q_1)} \times_S D^{(q_2)}$, where $X^{(q_1)} := X \times_S X \times_S \cdots \times_S X$ means the q_1-times self fiber product of X over S. Let $M_Z := \sum_{i=1}^{q_1} p_i^* M + \sum_{j=1}^{q_2} p_j'^* M|_D$, where $p_i \colon Z \to X$ is the ith projection and $p'_j \colon Z \to D$ is the $(q_1 + j)$th projection. Let $f \colon Z \to S$ denote the canonical morphism. Then we see that (see [CP21, §2.2])

$$\bigotimes_{i=1}^{q_1} \pi_*\mathcal{O}_X(dM) \otimes \bigotimes_{i=1}^{q_2} \pi_*\mathcal{O}_D(dM|_D) \cong f_*\mathcal{O}_Z(dM_Z).$$

By the adjunction of f_* and f^* applied to (3), we have a nonzero map

$$f^*\mathcal{O}_S(B)|_{f^{-1}(S^\circ)} \to \mathcal{O}_Z(dM_Z)|_{f^{-1}(S^\circ)}.$$

This means that $(dM_Z - f^*B)|_{f^{-1}(S^\circ)}$ is effective on some irreducible component of $f^{-1}(S^\circ)$. Now, Z might not satisfy Serre's S_2-condition and $dM_Z - f^*B$ might not be effective on Z entirely. For this, we discuss as follows. Recall that any irreducible component of $D \cap \pi^{-1}(S^\circ) = D \times_S S^\circ$ is flat over S°. Thus, so is $f^{-1}(S^\circ)$ and hence we see that any irreducible component of $f^{-1}(S^\circ)$ can be denoted as $\pi^{-1}(S^\circ)^{(q_1)} \times_{S^\circ} \pi^{-1}(S^\circ) \cap D_1 \times_{S^\circ} \cdots \times_{S^\circ} \pi^{-1}(S^\circ) \cap D_{q_2}$ for some irreducible components $D_1, \ldots, D_{q_2}$ of D. We can also check that $f^{-1}(S^\circ)$ is generically reduced by the fact that f is flat over S° and a general fiber of f is a fiber product of reduced schemes over $\mathbb{C}$. Let $Z' := X^{(q_1)} \times_S D_1 \times_S \cdots \times_S D_{q_2}$ and Z'_1 be the Zariski closure in Z',

$$\overline{\pi^{-1}(S^\circ)^{(q_1)} \times_{S^\circ} \pi^{-1}(S^\circ) \cap D_1 \times_{S^\circ} \cdots \times_{S^\circ} \pi^{-1}(S^\circ) \cap D_{q_2}}.$$

Let $\iota\colon Z'_1 \hookrightarrow Z'$ be the natural inclusion and $\nu\colon Z'^{\nu} \to Z'_1$ the normalization. Since $\operatorname{codim}_Z(Z \setminus f^{-1}(S^\circ)) \geq 2$, $\operatorname{codim}_{Z'^{\nu}}(Z'^{\nu} \setminus \nu^{-1}(f^{-1}(S^\circ) \cap Z'_1)) \geq 2$. By the S_2-condition of Z'^{ν}, there exists a nonzero map

$$\nu^*\iota^* f|_{Z'}^* \mathcal{O}_S(B) \to \nu^*\iota^*\mathcal{O}_{Z'_1}(dM_Z|_{Z'}). \tag{4}$$

We denote the base change by g for any movable curve $g\colon C \to S$ of $f|_{Z'}$, $M_Z|_{Z'}$, $\iota\colon Z'_1 \hookrightarrow Z'$, and $\nu\colon Z'^{\nu} \to Z'_1$ by $f_{Z'_C}$, $M_{Z'_C}$, ι_C, and ν_C. Let $B_C := g^*B$. We note that $M_{Z'_C}$ is nef by Proposition 4.2. By the property of (4), $d\nu_C^*\iota_C^* M_{Z'_C} - \nu_C^*\iota_C^* f_{Z'_C}^* B_C$ is effective for any movable curve $C \to S$. Thus, we obtain that

$$(d\nu_C^*\iota_C^* M_{Z'_C} - \nu_C^*\iota_C^* f_{Z'_C}^* B_C) \cdot \nu_C^*\iota_C^* M_{Z'_C}^{N-1} \geq 0, \tag{5}$$

where $N = dq_1 + (d-1)q_2 = \dim Z'_{1,C} - 1$. Then we have that $\dim(Z'_C \setminus Z'_{1,C}) \leq N$ since each fiber of each $D_i \to S$ is of dimension at most $n-1$. This means that for any $N+1$ line bundles $L_1, L_2, \ldots, L_{N+1}$ on Z'_C,

$$L_1 \cdot \ldots \cdot L_{N+1} = L_1|_{Z'_{1,C}} \cdot \ldots \cdot L_{N+1}|_{Z'_{1,C}}.$$

Therefore, we have by (5) that

$$dM_{Z'_C}^{N+1} \geq (M_{Z'_C,t}^N) \deg_C B_C. \tag{6}$$

To complete the proof of Proposition 5.3, we have to show by (6) that there exists a positive constant $C_4 > 0$ such that $M_C^{n+1} \geq C_4 \deg_C B_C$ for any movable curve $C \to S$. Let

$$C_0 := \max\left\{ (M_t^n)^{q_1} \prod_{i=1}^{q_2} (M_t|_{D'_{i,t}})^{n-1} \right\} > 0$$

be a constant, where $D'_1, \ldots, D'_{q_2}$ run over all q_2 irreducible components of $D|_{\pi^{-1}(S^\circ)}$. Here, we note that $(M_{Z'_C,t}^N) = (M_t^n)^{q_1} \prod_{i=1}^{q_2} (M_t|_{D_{i,t}})^{n-1}$ and thus $(M_{Z'_C,t}^N) \geq C_0$. Next, we see as the equation [Pos22, (6.3.5.i)] that there exists a constant $C_1 > 0$ such that

$$M_C^{n+1} + M_C^n \cdot \Delta_C \geq C_1 M_{Z'_C}^{N+1}, \tag{7}$$

independent of the choice of $D_1, \ldots, D_{q_2}$ and C. Indeed, let $D_{i,C} := D_i \times_S C$ and take the Zariski closure $D_{i,C}^* := \overline{D_{i,C} \cap \pi^{-1}(S^\circ)} \subset D_{i,C}$ for each $1 \leq i \leq q_2$. It is easy to see that $D_{i,C}^*$ is flat over C. Let

$$Z'_2 := X_C^{(q_1)} \times_C D_{1,C}^* \times_C \cdots \times_C D_{q_2,C}^* \subset Z'_C.$$

Since each fiber of each $D_i \to S$ is of dimension at most $n-1$, $(M_{Z'_C})^{N+1} = (M_{Z'_C}|_{Z'_2})^{N+1}$. By applying [Pos22, Lem. 7.0.5] to Z'_2, there exists a positive constant $d_1 \in \mathbb{Q}_{>0}$ depending only on n such that

$$(M_{Z'_C})^{N+1} = q_1 d_1 (M_C^{n+1})(M_t^n)^{q_1-1} \prod_{i=1}^{q_2} (M_t|_{D_{i,t}})^{n-1} + d_1 \sum_{i=1}^{q_2} (M_C|_{D_i})^n (M_t^n)^{q_1} \prod_{j\neq i}^{q_2} (M_t|_{D_{j,t}})^{n-1}.$$

This proves the existence of such C_1. Thus, we see by (6) and (7) that there exists a positive constant $C_2 := d^{-1}C_0C_1$ independent of the choice of movable curves C such that

$$M_C^{n+1} + M_C^n \cdot \Delta_C \geq C_2 \deg_C B_C. \tag{8}$$

Now it suffices to show the following claim.

Claim 1. *There exists a positive constant $C_3 > 0$ independent of the choice of movable curves $C \to S$ such that*

$$M_C^{n+1} \geq C_3(M_C^n \cdot \Delta_C).$$

Proof of Claim 1. We mimic the proof of [XZ20, Lem. 7.6]. We note that there exists $0 < \xi < 1$ such that $K_{X_s} + (1-\xi)\Delta_s + \lambda L_s$ is big as a $\mathbb{Q}$-Weil divisor for any very general point $s \in S$. Indeed, we choose ξ such that $K_{X_\eta} + (1-\xi)\Delta_\eta + \lambda L_\eta$ is big, where η is the generic point of S. Then we see that $K_{X_s} + (1-\xi)\Delta_s + \lambda L_s$ is big for general s. For any movable curve $C \to S$, we see that (X_C, Δ_C) is klt since (X_c, Δ_c) is klt for any closed point $c \in C$. Thus, we can take a $\mathbb{Q}$-factorization Y of X_C, i.e. there exists a small projective birational morphism $g\colon Y \to X_C$ from a normal $\mathbb{Q}$-factorial variety by [BCHM10, Cor. 1.4.3]. Let $\Delta'_C := g_*^{-1}\Delta_C$ and let $\varphi\colon Y \to C$ be the canonical morphism. Fix $r' \in \mathbb{Z}_{>0}$ such that $r'L$ is a line bundle. By Proposition 4.2, we see that for any sufficiently small $\varepsilon > 0$ and sufficiently divisible $m \in \mathbb{Z}_{>0}$, there exists an effective divisor $D \in |mr'L_C + (m\varepsilon + 2g(C))f^*P|$ such that $\mathrm{lct}(X_s, \Delta_s; D_s) \geq \frac{\lambda+\varepsilon}{mr'}$, where $g(C)$ is the genus of C, $P \in C$ is a closed point, and $s \in C$ is a very general point. Thus, we see that $(X_s, \Delta_s + \frac{\lambda+\varepsilon}{mr'}D_s)$ is lc for any very general $s \in C$. Let $\Gamma := (1-\xi)\Delta'_C + \frac{\lambda+\varepsilon}{mr'}g^*D$. For any very general $s \in C$, we see that $g_s\colon Y_s \to X_s$ is a small birational morphism and $K_{Y_s} + \Gamma_s \sim_{\mathbb{Q}} (g_s)_*^{-1}(K_{X_s} + (1-\xi)\Delta_s + \lambda L_s) + \frac{\varepsilon}{mr'}g_s^*D_s$. Note that the birational map g_s^{-1} is isomorphic in codimension one and hence $(g_s)_*^{-1}(K_{X_s} + (1-\xi)\Delta_s)$ is a big $\mathbb{Q}$-divisor. Thus, $K_{Y_s} + \Gamma_s$ is big for any very general $s \in C$ and hence

$$H^0\big(Y_s, \mathcal{O}_{Y_s}(l(K_{Y_s} + \Gamma_s))\big) \neq 0 \tag{9}$$

for any sufficiently divisible $l \in \mathbb{Z}_{>0}$. Let $\psi\colon Y_{\mathrm{lc}} \to Y$ be the lc modification of (Y, Γ) by [OX12, Thm. 1.1]. In other words, ψ is a projective birational morphism of normal varieties and there exists an effective ψ-exceptional $\mathbb{Q}$-divisor G such that

$$\psi^*(K_Y + \Gamma) - G = K_{Y_{\mathrm{lc}}} + \psi_*^{-1}\Gamma + \mathrm{Ex}(\psi),$$

$K_{Y_{\mathrm{lc}}} + \psi_*^{-1}\Gamma + \mathrm{Ex}(\psi)$ is ψ-ample and $(Y_{\mathrm{lc}}, \psi_*^{-1}\Gamma + \mathrm{Ex}(\psi))$ is lc. Since $(X_s, \Delta_s + \frac{\lambda+\varepsilon}{mr'} D_s)$ is lc for sufficiently general $s \in C$, we have that G is vertical with respect to C. Therefore, there exists a coherent sheaf $\mathcal{G}_l$ on C whose support is zero-dimensional for any sufficiently divisible $l \in \mathbb{Z}_{>0}$ such that there exists an exact sequence

$$0 \to (\varphi \circ \psi)_* \mathcal{O}_{Y_{\mathrm{lc}}}\big(l(K_{Y_{\mathrm{lc}}/C} + \psi_*^{-1}\Gamma + \mathrm{Ex}(\psi))\big) \to \varphi_* \mathcal{O}_Y(l(K_{Y/C} + \Gamma)) \to \mathcal{G}_l \to 0.$$

By [F17, Thm. 1.1], we have that $(\varphi \circ \psi)_* \mathcal{O}_{Y_{\mathrm{lc}}}(l(K_{Y_{\mathrm{lc}}/C} + \psi_*^{-1}\Gamma + \mathrm{Ex}(\psi)))$ is weakly positive over C for any sufficiently divisible l. Since $\dim \mathrm{Supp}\, \mathcal{G}_l = 0$, $\varphi_* \mathcal{O}_Y(l(K_{Y/C} + \Gamma))$ is also weakly positive. This means that for any ample line bundle A on C and positive integer a, there exists $b \in \mathbb{Z}_{>0}$ such that the stalk of $\mathrm{Sym}^{ab}(g_* \mathcal{O}_Y(l(K_{Y/C} + \Gamma))) \otimes \mathcal{O}_C(bA)$ at the generic point of C is generated by $H^0(C, \mathrm{Sym}^{ab}(g_* \mathcal{O}_Y(l(K_{Y/C} + \Gamma))) \otimes \mathcal{O}_C(bA))$. By the following commutative diagram for any very general point $s \in C$,

$$\begin{array}{ccc}
\mathrm{Sym}^{ab}\big(\varphi_* \mathcal{O}_Y(l(K_{Y/C} + \Gamma))\big) \otimes \mathcal{O}_C(bA) & \longrightarrow & \varphi_* \mathcal{O}_Y(abl(K_{Y/C} + \Gamma)) \otimes \mathcal{O}_C(bA) \\
\downarrow & & \downarrow \\
\mathrm{Sym}^{ab} H^0\big(Y_s, \mathcal{O}_{Y_s}(l(K_{Y_s} + \Gamma_s))\big) & \longrightarrow & H^0\big(Y_s, \mathcal{O}_{Y_s}(abl(K_{Y_s} + \Gamma_s))\big),
\end{array}$$

(9), and the facts that

$$\mathrm{Sym}^{ab}\big(\varphi_* \mathcal{O}_Y(l(K_{Y/C} + \Gamma))\big) \otimes (\mathcal{O}_C/\mathfrak{m}_s) \cong \mathrm{Sym}^{ab} H^0\big(Y_s, \mathcal{O}_{Y_s}(l(K_{Y_s} + \Gamma_s))\big),$$
$$\varphi_* \mathcal{O}_Y(abl(K_{Y/C} + \Gamma)) \otimes (\mathcal{O}_C/\mathfrak{m}_s) \cong H^0\big(Y_s, \mathcal{O}_{Y_s}(abl(K_{Y_s} + \Gamma_s))\big),$$

where $\mathfrak{m}_s$ is the maximal ideal sheaf corresponding to s, for any very general point $s \in C$, $H^0(C, g_* \mathcal{O}_Y(abl(K_{Y/C} + \Gamma)) \otimes \mathcal{O}_C(bA)) \neq 0$. This means that $b(al(K_{Y/C} + \Gamma) + g^*A)$ is effective. By considering $a \to \infty$, we obtain that $K_{Y/C} + \Gamma$ is pseudo-effective and hence so is $K_{X_C/C} + (1 - \xi)\Delta_C + \frac{\lambda+\varepsilon}{mr'} D$. Therefore, we obtain that $K_{X_C/C} + (1 - \xi)\Delta_C + \lambda L_C$ is pseudo-effective. This means that

$$M_C^{n+1} \geq r\xi(M_C^n \cdot \Delta_C).$$

By taking $C_3 = r\xi$, we complete the proof of Claim 1. □

By (8), Claim 1, and [BDPP13], we obtain that there exists a positive constant C_4 such that $\pi_*(M^{n+1}) - C_4 B$ is pseudo-effective. Since B is big, so is $\pi_*(M^{n+1})$. We complete the proof. $\square$

Proof of Theorem 1.5. Here, we note that for any closed subvariety $V \subset S$, the restriction $\pi_V : (X_V, \Delta_V) \to V$ of π to V also has maximal variation since π has maximal variation along any curve. By the property of the Knudsen–Mumford expansion, we have that $(\pi_V)_*(M_V)^{n+1} = (\pi_*(M)^{n+1})|_V$. By Propositions 4.2 and 5.3, $(\pi_V)_*(M_V)^{n+1}$ is big and nef. Thus, Theorem 1.5 immediately follows from the Nakai–Moishezon criterion. $\square$

By applying Proposition 5.3, we show the following key ingredient to prove Theorem 1.3.

Theorem 5.4. *Let $\pi: (X, \Delta, L) \to S$ be a polarized log $\mathbb{Q}$-Gorenstein family with maximal variation, where S is projective and $(X_{\bar{s}}, \Delta_{\bar{s}})$ is klt for any geometric point $\bar{s} \in S$. Suppose that there exists a closed point $s_0 \in S$ such that $(X_{s_0}, \Delta_{s_0}, L_{s_0})$ is specially* K*-stable and $K_{X/S} + \Delta + \delta(X_{s_0}, \Delta_{s_0}, L_{s_0})L$ is π-ample. Then the CM-line bundle $\lambda_{\mathrm{CM},\pi}$ is big.*

Proof. Let n be the relative dimension of π and $v = L^n_{s_0}$. Then, for any movable curve $C \to S$, the pullback of $(n+1)vL - \pi^*(\pi_*(L^{n+1}))$ satisfies

$$\big((n+1)vL - \pi^*(\pi_*(L^{n+1}))\big)^{n+1}_C = 0.$$

Thus, we may assume that $L^{n+1}_C = 0$ for any movable curve $C \to S$ by replacing L with $(n+1)vL - \pi^*(\pi_*(L^{n+1}))$. Next we take positive rational numbers λ and ε such that for any very general point $s \in S$, $\delta_{(X_s,\Delta_s,L_s)} \geq \lambda + \varepsilon$ and (X_s, L_s) is $\mathrm{J}^{K_{X_s}+\Delta_s+\lambda L_s}$-semistable by Corollary 2.22. Here, we may assume that $K_{X/S} + \Delta + \lambda L$ is π-ample. By taking a suitable $r \in \mathbb{Z}_{>0}$, we may further assume that $M := r(K_{X/S} + \Delta + \lambda L)$ is a π-very ample line bundle.

By Proposition 5.3, $\pi_*(M^{n+1})$ is big. This means that for any movable curve $C \to S$, $M^{n+1}_C > 0$. If we choose $0 < \delta < \frac{1}{(n+1)(M^n_t)}$, then we see by [L04a, Thm. 2.2.15] that

$$M_C - \delta(\pi_C)^*(\pi_C)_*(M^{n+1}_C) = r\Big(K_{X_C/C} + \Delta_C + \lambda L_C - \frac{\delta}{r}\pi^*_C(\pi_C)_*(M^{n+1}_C)\Big)$$

is big, where t is a general closed point of S. Here, we claim the following.

Claim 2. *Let $\alpha := \inf_{t\in S} \alpha(X_t, \Delta_t; M_t)$. Then $\alpha > 0$ and*

$$K_{X_C/C} + \Delta_C + \lambda L_C - \frac{\alpha\delta\varepsilon}{\lambda + (1 + r\alpha)\varepsilon}\pi^*_C(\pi_C)_*(M^{n+1}_C) \tag{10}$$

is nef.

Proof of Claim 2. Take $D' \in |M_C - \delta\pi_C^*(\pi_C)_*(M_C^{n+1})|_{\mathbb{Q}}$ by the bigness and assume that $\operatorname{Supp} D'$ does not contain X_s for some very general closed point $s \in C$. By Proposition 4.2, we see that for any sufficiently small $\eta \in \mathbb{Q}_{>0}$ and sufficiently divisible $m \in \mathbb{Z}_{>0}$, there exists an effective divisor $D \in |mL_C + (m\eta + 2g(C))\pi_C^* P|$ such that $\mathrm{lct}(X_s, \Delta_s; D_s) \geq \frac{\lambda+\varepsilon}{m}$, where $g(C)$ is the genus of C, $P \in C$ is a closed point, and $s \in C$ is a very general point. Then we have that for any prime divisor E over X_s,

$$\frac{\lambda}{(\lambda+\varepsilon)} A_{(X_s,\Delta_s)}(E) \geq \frac{\lambda}{m} \operatorname{ord}_E(D_s).$$

On the other hand, $\alpha > 0$ by [BL22, Prop. 5.3]. Thus, we have that $(X_s, \Delta_s + \frac{\lambda}{m} D_s + \frac{\alpha\varepsilon}{\lambda+\varepsilon} D'_s)$ is lc. This means that

$$\begin{aligned} &K_{X_C/C} + \Delta_C + \frac{\lambda}{m} D + \frac{\alpha\varepsilon}{\lambda+\varepsilon} D' \\ &\quad \sim_{\mathbb{Q}} K_{X_C/C} + \Delta_C + \lambda\Big(L_C + \frac{m\eta + 2g(C)}{m} \pi_C^* P\Big) \\ &\quad\quad + \frac{r\alpha\varepsilon}{\lambda+\varepsilon}\Big(K_{X_C/C} + \Delta_C + \lambda L_C - \frac{\delta}{r} \pi_C^*(\pi_C)_*(M_C^{n+1})\Big) \end{aligned}$$

is nef by [F18, Thm. 1.11]. Since this holds for any sufficiently small η and large m, we have that (10) is nef. □

Take α as Claim 2. Then we have that for any movable curve $C \to S$,

$$\begin{aligned} \mathrm{CM}((X_C,\Delta_C,L_C)/C) = \mathcal{J}^{(K_{X_C/C}+\Delta_C+\lambda L_C - \frac{\alpha\delta\varepsilon}{\lambda+(1+r\alpha)\varepsilon}\pi_C^*(\pi_C)_*(M_C^{n+1}))}((X_C,L_C)/C) \\ + \frac{\alpha\delta\varepsilon}{\lambda+(1+r\alpha)\varepsilon}(M_C^{n+1}). \end{aligned}$$

By the choice of α, $K_{X_C/C} + \Delta_C + \lambda L_C - \frac{\alpha\delta\varepsilon}{\lambda+(1+r\alpha)\varepsilon}\pi_C^*(\pi_C)_*(M_C^{n+1})$ is absolutely nef on X_C. Since (X_s, L_s) is $\mathrm{J}^{K_{X_s}+\Delta_s+\lambda L_s}$-semistable for any very general point $s \in S$ as we stated in the first paragraph of the proof of Theorem 5.4, we have by Propositions 3.3 and 4.1 that

$$\frac{1}{(n+1)v} \lambda_{\mathrm{CM},\pi} \cdot C \geq \frac{\alpha\delta\varepsilon}{\lambda+(1+r\alpha)\varepsilon}(\pi_*(M^{n+1}) \cdot C).$$

Thus, $\lambda_{\mathrm{CM},\pi}$ is big by [BDPP13] since $\pi_*(M^{n+1})$ is big. □

Proof of Theorem 1.1. First, we assert that there exists a closed point $s_0 \in S$ such that $K_{X/S} + \Delta + \delta(X_{s_0}, \Delta_{s_0}, L_{s_0})L$ is π-ample. To show this, we note that for any closed point $s \in S$, $K_{X_s} + \Delta_s + \delta(X_s, \Delta_s, L_s)L_s$ is ample. Thus, there exists an open neighborhood U of s such that $K_{X_U/U} + \Delta_U + \delta(X_s, \Delta_s, L_s)L_U$ is $\pi|_{\pi^{-1}U}$-ample by [KM98, Prop. 1.41]. By quasi-compactness of S, there exist finitely many open subsets U_i and closed points $s_i \in U_i$ such that $\bigcup U_i = S$ and $K_{X_{U_i}/U_i} + \Delta_{U_i} + \delta(X_{s_i}, \Delta_{s_i}, L_{s_i})L_{U_i}$ is $\pi|_{\pi^{-1}U_i}$-ample. By letting s_0 be an s_i attaining $\max\{\delta(X_{s_i}, \Delta_{s_i}, L_{s_i})\}$, we see the claim holds. Then, the assertion of Theorem 1.1 immediately follows from Proposition 3.3, Theorems 4.3 and 5.4 applied to this s_0 in the same way as Proposition 1.4 by using the Nakai–Moishezon criterion. □

§6. An application to the moduli of K-stable Calabi–Yau fibrations over curves

Let $M_{d,v,u,r}$ be the coarse moduli space of $\mathscr{M}_{d,v,u,r}$, which exists by [KeMo97]. In this section, we deal with Corollary 1.3. More precisely, we prove the positivity of some CM line bundles for certain polarizations.

Theorem 6.1. *There exists $w \in \mathbb{Z}_{>0}$ such that for any proper subspace B of $M_{d,v,u,r}$, $\Lambda_{\mathrm{CM},w}|_B$ is ample. In particular, B is projective.*

First, we recall the following well-known result.

Lemma 6.2 (Cf. [Ka85, Prop. 8.3], [DG18, Prop. 4.2])**.** *Let (X, Δ) be a projective klt pair such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Then $\dim \mathrm{Aut}_0(X, \Delta) = \dim \mathrm{Pic}^0(X)$ and for any two ample line bundles A_1 and A_2 algebraically equivalent to each other, there exists $\xi \in \mathrm{Aut}_0(X, \Delta)$ such that $\xi^* A_1 \sim A_2$.*

Proof. For the reader's convenience, we show this lemma here. First, we show that $\dim \mathrm{Aut}_0(X, \Delta) \leq \dim \mathrm{Pic}^0(X)$. Fix a very ample line bundle L on X. Consider a morphism

$$\varphi_L \colon \mathrm{Aut}_0(X, \Delta) \ni g \mapsto [g^* L \otimes L^{\otimes -1}] \in \mathrm{Pic}^0(X).$$

By [M70, §4, Cor. 1] and [A05, Prop. 4.6], φ_L is a homomorphism of Abelian varieties. Thus, it suffices to show that $\mathrm{Ker}\,\varphi_L$ is a finite group scheme. Let $\iota \colon X \hookrightarrow \mathbb{P}^{h^0(X, \mathcal{O}_X(L))-1}$ be the natural embedding defined by $|L|$. Since $g \in \mathrm{Ker}\,\varphi_L$ satisfies that $g^* L \sim L$, there exists a group homomorphism $\nu \colon (\mathrm{Ker}\,\varphi_L)^0 \to \mathrm{PGL}(h^0(X, \mathcal{O}_X(L)))$ such that $(\mathrm{Ker}\,\varphi_L)^0$ acts on $\mathbb{P}^{h^0(X, \mathcal{O}_X(L))-1}$ so that ι is $(\mathrm{Ker}\,\varphi_L)^0$-equivariant, where $(\mathrm{Ker}\,\varphi_L)^0$ is the identity component of $\mathrm{Ker}\,\varphi_L$. It is easy to see that ν is trivial and $(\mathrm{Ker}\,\varphi_L)^0$ trivially acts on (X, Δ). Therefore, $\mathrm{Ker}\,\varphi_L$ is a finite group scheme. We note that if $\dim \mathrm{Aut}_0(X, \Delta) \geq \dim \mathrm{Pic}^0(X)$, then φ_L is further étale.

We prove $\dim \mathrm{Aut}_0(X, \Delta) \geq \dim \mathrm{Pic}^0(X)$ by induction on $\dim X = n$. It is well known that the assertion holds when $n = 1$. We may assume that $n > 1$. Since (X, Δ) is klt, X has only rational singularities by [KM98, Thm. 5.22]. Thus, $\dim \mathrm{Pic}^0(X) = \dim \mathrm{Alb}(X)$, where $\pi \colon X \to \mathrm{Alb}(X)$ is the Albanese morphism (cf. [Ka85, §8]). By [A05, Thm. 4.8], we have that there exist an étale morphism $A \to \mathrm{Alb}(X)$ from an Abelian variety, a projective connected klt log pair (F, Δ_F), and an isomorphism over A,

$$\Phi\colon A \times_{\mathrm{Alb}(X)} (X, \Delta) \to A \times (F, \Delta_F).$$

Note that $A \to \mathrm{Alb}(X)$ is an étale Galois covering and let $G = \mathrm{Ker}(A \to \mathrm{Alb}(X))$ be the Galois group. We see that G is a finite commutative group. By identifying (F, Δ_F) with the fiber of $\pi\colon (X, \Delta) \to \mathrm{Alb}(X)$ over 0, G acts on (F, Δ_F) naturally. Let $\psi\colon G \to \mathrm{Aut}(F, \Delta_F)$ be the natural homomorphism induced by the G-action. On the other hand, G naturally acts on $(X, \Delta) \times_{\mathrm{Alb}(X)} A$ equivariantly over A. By Φ, we obtain the induced G-action on $A \times (F, \Delta_F)$ such that

$$g \cdot (a, f) = (a + g, \phi_g(a)(f)),$$

where $g \in G$, $a \in A$, and $f \in F$ are closed points. Here, $\phi_g(a) \in \mathrm{Aut}(F, \Delta_F)$. Note that $\phi_g(0) = \Phi(g, \cdot) \circ \psi(g) \circ \Phi(0, \cdot)^{-1}$. Thus, $\phi_g(0)$ is contained in the same component of $\mathrm{Aut}(F, \Delta_F)$ as $\psi(g)$. Since $\phi_g(a)$ is continuous on $a \in A$, we can write

$$\phi_g(a) = \psi(g) \circ t_g(a),$$

where $t_g\colon A \to \mathrm{Aut}_0(F, \Delta_F)$ is a morphism of Abelian varieties.

If $\mathrm{Alb}(X)$ is a point, then $\dim \mathrm{Aut}_0(X, \Delta) = 0$ also holds by what we have shown in the first paragraph. Thus, we may assume that $\dim \mathrm{Alb}(X) > 0$ and then $\dim F < n$. Take a very ample line bundle L on X. Let $\tilde{L}$ be the pullback of L to $A \times F$ under the morphism $A \times F \to X$. Now, L_a denotes the restriction of $\tilde{L}$ to $\{a\} \times F \subset A \times F$ for any closed point $a \in A$. For any closed point $a \in A$ and $g \in G$, we have

$$\begin{aligned} L_a &= g^* \tilde{L} \otimes \mathcal{O}_{\{a\} \times F} = \phi_g(a)^* L_{g+a} \\ &= t_g(a)^* (\psi(g)^* L_{g+a}). \end{aligned} \tag{11}$$

Set

$$\rho\colon A \ni a \mapsto [L_a \otimes L_0^{\otimes -1}] \in \mathrm{Pic}^0(F);$$

ρ is indeed a morphism. We also consider the morphism

$$\varphi \coloneqq \varphi_{L_0}\colon \ \mathrm{Aut}_0(F, \Delta_F) \ni g \mapsto [g^* L_0 \otimes L_0^{\otimes -1}] \in \mathrm{Pic}^0(F).$$

By what we have shown in the first paragraph and the induction hypothesis, φ is an étale homomorphism. Furthermore, by [KM98, Lem. 1.6], we see that

$$\varphi(h) = [h^* L_b \otimes L_b^{\otimes -1}]$$

for any $h \in \mathrm{Aut}_0(F, \Delta_F)$ and $b \in A$. Thus (11) is rephrasable as

$$\rho(a) - \varphi(t_g(a)) = [\psi(g)^* L_{g+a} \otimes L_0^{\otimes -1}].$$

Since L_0 is G-invariant, we see that

$$\psi(g)^* L_{g+a} \otimes L_0^{\otimes -1} = \psi(g)^*(L_{g+a} \otimes L_0^{\otimes -1}).$$

Thus we obtain that

$$\rho(a) - \varphi(t_g(a)) = \psi(g)^* \rho(g + a). \tag{12}$$

Then, consider the cartesian diagram

$$\begin{array}{ccc} \tilde{A}_1 & \longrightarrow & \mathrm{Aut}_0(F, \Delta_F) \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ A & \xrightarrow{\ \rho\ } & \mathrm{Pic}^0(F), \end{array}$$

and let $\tilde{A}$ be the identity component of $\tilde{A}_1$. We see that $\tilde{A}$ is an Abelian variety since this is a projective algebraic group. Let $\eta \colon \tilde{A} \to A$ be the natural morphism. Then there exists a morphism $\tilde{\rho} \colon \tilde{A} \to \mathrm{Aut}_0(F, \Delta_F)$ such that $\varphi \circ \tilde{\rho} = \rho \circ \eta$. Let H be a Galois group of $p \colon \tilde{A} \to \mathrm{Alb}(X)$ and let $q \colon H \to G$ be the natural morphism. Via q, H acts on $\tilde{A} \times (F, \Delta_F)$ equivariantly over $A \times (F, \Delta_F)$. We denote the automorphism of $\tilde{A} \times (F, \Delta_F)$ by b_h induced by $h \in H$. Note that G acts on $\mathrm{Aut}_0(F, \Delta_F)$ and $\mathrm{Pic}^0(F)$ in the way that

$$g \cdot s = \psi(g) \circ s \circ \psi(g^{-1}),$$

for any $g \in G$ and $s \in \mathrm{Aut}_0(F, \Delta_F)$, and

$$g \cdot [M \otimes L_0^{\otimes -1}] = [\psi(g^{-1})^* M \otimes L_0^{\otimes -1}]$$

for any $g \in G$ and $[M] \in \mathrm{Pic}^0(F)$ respectively. We see that φ is G-equivariant. Let $\tilde{t}_h(\tilde{a}) := t_{q(h)}(p(\tilde{a}))$ for any $h \in H$ and $\tilde{a} \in \tilde{A}$. By (12), if we put

$$\theta_h(\tilde{a}) := \tilde{\rho}(\tilde{a}) - \tilde{t}_h(\tilde{a}) - \psi(q(h^{-1})) \circ \tilde{\rho}(\tilde{a} + h) \circ \psi(q(h))$$

for $h \in H$ and $\tilde{a} \in \tilde{A}$, then we have that $\theta_h(\tilde{a}) \in \mathrm{Ker}\, q = \mathrm{Ker}\, p$. Since $\theta_h \colon \tilde{A} \to \mathrm{Ker}\, p$ is a morphism and $\mathrm{Ker}\, p$ is finite, $\theta_h(\tilde{a})$ is independent of $\tilde{a}$ and we also denote $\theta_h = \theta_h(\tilde{a}) \in \mathrm{Ker}\, p$. Put an automorphism of $\tilde{A} \times (F, \Delta_F)$ over $\mathrm{Alb}(X)$ as

$$\Psi \colon (\tilde{a}, f) \mapsto (\tilde{a}, \tilde{\rho}(\tilde{a})(f))$$

and a morphism c_h for any $h \in H$ as

$$c_h \colon \tilde{A} \times F \ni (\tilde{a}, f) \mapsto (\tilde{a} + h, \psi(q(h)) \circ \theta_h^{-1}(f)) \in \tilde{A} \times F.$$

Then we see that $c_h = \Psi \circ b_h \circ \Psi^{-1}$. By Ψ, we may assume that H acts on $\tilde{A} \times (F, \Delta_F)$ by c_h and then we see that the automorphism of $\tilde{A} \times (F, \Delta_F)$,

$$\sigma_{\tilde{b}} \colon (\tilde{a}, f) \mapsto (\tilde{a} + \tilde{b}, f)$$

is H-invariant for any $\tilde{b} \in \tilde{A}$. This means that $\sigma_{\tilde{b}}$ descends to an automorphism of (X, Δ) and hence $\tilde{A}$ acts on $\mathrm{Alb}(X)$ transitively. Therefore, $\mathrm{Aut}_0(X, \Delta)$ acts on $\mathrm{Alb}(X)$ transitively. This shows that $\dim \mathrm{Pic}^0(X) = \dim \mathrm{Alb}(X) \leq \dim \mathrm{Aut}_0(X, \Delta)$. We complete the proof of $\dim \mathrm{Aut}_0(X, \Delta) = \dim \mathrm{Pic}^0(X)$.

Finally, we deal with the last assertion. Take $m \in \mathbb{Z}_{>0}$ such that $A_1^{\otimes m}$ is very ample. We see that

$$\varphi_{A_1} \colon \mathrm{Aut}_0(X, \Delta) \ni g \mapsto [g^* A_1 \otimes A_1^{\otimes -1}] \in \mathrm{Pic}^0(X)$$

is surjective since $\varphi_{A_1^{\otimes m}}$ is a surjective map and is the composition of φ_{A_1} and an étale endomorphism

$$\mathrm{Pic}^0(X) \ni [M] \mapsto [M^{\otimes m}] \in \mathrm{Pic}^0(X).$$

This is equivalent to the existence of an isomorphism $\xi \in \mathrm{Aut}_0(X, \Delta)$ such that $\xi^* A_1 \sim A_2$ for any A_2 algebraically equivalent to A_1. We complete the proof. □

To prove Theorem 6.1, we show the following by applying Lemma 6.2.

Proposition 6.3. *Let $f \colon (X, \Delta, A) \to \mathbb{P}^1$ be a uniformly adiabatically* K*-stable klt-trivial fibration with $-(K_X + \Delta)$ not numerically trivial but nef.*

Then $\dim \mathrm{Aut}_0(X, \Delta) = \dim \mathrm{Pic}^0(X)$ *and for any two ample line bundles A_1 and A_2 algebraically equivalent to each other, there exists $\varphi \in \mathrm{Aut}_0(X, \Delta)$ such that $\varphi^* A_1 \sim A_2$.*

Proof. Take $s \in \mathbb{Q}_{>0}$ such that $-(K_X + \Delta) \sim_{\mathbb{Q}} s f^* \mathcal{O}(1)$. Then we see by [Hat25, Thm. 1.1] that for any three distinct closed points $p_1, p_2, p_3 \in \mathbb{P}^1$, $(X, \Delta + \frac{s}{3} \sum_{i=1}^3 f^{-1}(p_i))$ is klt and $K_X + \Delta + \frac{s}{3} \sum_{i=1}^3 f^{-1}(p_i) \sim_{\mathbb{Q}} 0$. We claim that

$$\mathrm{Aut}_0(X, \Delta) = \mathrm{Aut}_0\left(X, \Delta + \frac{s}{3} \sum_{i=1}^{3} f^{-1}(p_i)\right). \tag{13}$$

Indeed, $\mathrm{Aut}_0(X, \Delta)$ acts on $\mathbb{P}^1$ but $\mathrm{Aut}_0(X, \Delta)$ is an Abelian variety by [Hat24a]. Let G be the image of the group homomorphism $\mathrm{Aut}_0(X, \Delta) \to \mathrm{PGL}(2)$. Since G is a proper linear algebraic group, G is a point. Thus, $\mathrm{Aut}_0(X, \Delta)$ fixes $f^{-1}(p)$ for any $p \in \mathbb{P}^1$ and (13) holds. By Lemma 6.2,

$$\dim \mathrm{Aut}_0\left(X, \Delta + \frac{s}{3} \sum_{i=1}^{3} f^{-1}(p_i)\right) = \dim \mathrm{Pic}^0(X).$$

Thus, we complete the proof of the first assertion by (13). The second assertion follows in the same way as Lemma 6.2. □

Proof of Theorem 6.1. By [Ko90, Prop. 2.7], there exist a proper normal variety B', a finite surjective morphism $g\colon B' \to B$, and a morphism of stacks $\tilde{g}\colon B' \to \mathscr{M}_{d,v,u,r}$ such that $\pi \circ \tilde{g} = \iota \circ g$, where $\iota\colon B \to M_{d,v,u,r}$ and $\pi\colon \mathscr{M}_{d,v,u,r} \to M_{d,v,u,r}$ are the natural morphisms. We set w as in Theorem 3.11. Let $f\colon (X, A) \to B'$ be the pullback of the universal family $(\mathscr{U}, \mathscr{A})$ on $\mathscr{M}_{d,v,u,r}$ via $\tilde{g}$ (cf. [HH25, Rem. 6.5]) with $\mathrm{vol}(A_{b'}) = w$ for any point $b' \in B'$. Then A is f-ample and (X_b, A_b) is specially K-stable for any closed point $b \in B'$.

Here, we claim that the family $X \to B'$ has maximal variation. To show this, assume the contrary and that there exists a proper curve $C \subset B'$ such that C passes through a very general point and for any two general closed points $p_1, p_2 \in C$, X_{p_1} and X_{p_2} are isomorphic. Then A_p and A_q are algebraically equivalent for any two very general closed points $p, q \in C$. By Proposition 6.3, we see that there exists an isomorphism $\varphi\colon X_p \to X_q$ such that $A_p \sim \varphi^* A_q$. This means that C is contained in a fiber of g by the definition of $M_{d,v,u,r}$. This contradicts the finiteness of g. Thus, the family $X \to B'$ has maximal variation.

The CM line bundle $\lambda_{\mathrm{CM},f} = g^*(\Lambda_{\mathrm{CM},w}|_B)$ on B' is big and nef by Theorem 1.1. Thus, we have that $(\Lambda_{\mathrm{CM},w}|_B)^{\dim B} > 0$. By the Nakai–Moishezon criterion [Ko90, Thm. 3.11], $\Lambda_{\mathrm{CM},w}|_B$ is ample and hence B is projective. □

Acknowledgements

The author would like to thank Professor Yuji Odaka for careful reading of a draft of the article. He would also like to thank Rei Murakami for pointing out some typos. This work is partially supported by JSPS KAKENHI 22J20059 (Grant-in-Aid for JSPS Fellows DC1).

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