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Dynamics of Fourier Multipliers on Riemannian Symmetric Spaces of Noncompact Type

by

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Abstract

Let X be a Riemannian symmetric space of noncompact type and T be a linear translation-invariant operator which is bounded on $L^p(X)$. We shall show that if T is not a constant multiple of identity then there exist complex constants z such that zT is chaotic on $L^p(X)$ when p is in the sharp range $2 < p < \infty$. This vastly generalizes the result that dynamics of the (perturbed) heat semigroup is chaotic on X proved in Ji and Weber (Ergodic Theory Dynam. Systems 30 (2010), 457–468) and Pramanik and Sarkar (J. Funct. Anal. 266 (2014), 2867–2909).

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§1. Introduction

Let $T\colon L^p(X) \to L^p(X)$ be an L^p-multiplier on a Riemannian symmetric space X of noncompact type for any fixed $1 \le p \le \infty$. We shall call T nontrivial if it is not a constant multiple of identity. The aim of this note is to show that for the range $2 < p < \infty$, given any nontrivial L^p-multiplier T, we can find complex constants $z \in \mathbb{C}$ such that the operator zT is chaotic on $L^p(X)$. This range of p will be shown to be sharp. Our definition of chaos is consistent with [4, 13], which in turn is an adaptation of the one introduced by Devaney [5].

To put the result in perspective, let us discuss the background. Let Δ be the positive Laplace–Beltrami operator on X and $T_t = e^{-t\Delta}$, $t \ge 0$ be the heat semigroup. Ji and Weber [13] showed that its perturbation $e^{ct}T_t$, $t \ge 0$ for some constant c is subspace chaotic on $L^p(X)$ when $2 < p < \infty$. Through [17] and [15]

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this result was improved by establishing that the same perturbation of the heat semigroup is actually chaotic on $L^p(X)$ with p in the same range. In this paper we shall establish that this is a particular case of a general fact. We first note that the operator $T_t = e^{-t\Delta}$ is the same as the operator $f \mapsto f * h_t$, where h_t is the heat kernel, i.e. the fundamental solution of the heat equation $(\Delta - \frac{\partial}{\partial t})f = 0$. Thus it is natural to consider the operator $f \mapsto f * \mu$, where μ is any nonatomic K-biinvariant Borel measure, a particular case of which is the heat operator. We show that such an operator is always chaotic on $L^p(X)$, $2 < p < \infty$ provided it is not a contraction (Corollary 5.0.3). Indeed, the heat operator can be substituted by any L^p-multiplier. A corollary of the main result (Theorem 4.0.1) in this paper is the following. Consider an *autonomous discretization* (see Section 2.2) of the heat semigroup $T_{t_0} = e^{-t_0\Delta}$ for any fixed $t_0 > 0$. Then there exists a constant $z \in \mathbb{C}$, such that $T = ze^{-t_0\Delta}$ is chaotic on $L^p(X)$ when p is in the range $2 < p < \infty$. It is known that a C_0-semigroup is hypercyclic if and only if it admits a hypercyclic discretization. We also note that a periodic point of a discretized semigroup is also a periodic point of the original semigroup. Thus if $T = zT_{t_0}$ is chaotic then so is the semigroup $(zT_t)_{t\geq 0}$. Thus our result in the present article accommodates the earlier results in this direction mentioned above. See Sections 2 and 5 for more details.

The paper is organized as follows. The general preliminaries are established in Section 2, while those about Riemannian symmetric spaces are given in Section 3. Section 4 contains the main result and its proof. In Section 5 we deal with some well-known multipliers and obtain some corollaries of the main results for the particular cases. In Section 6 we show the sharpness of the range of p in the main result. Finally, in Section 7 we state some open questions along with motivations.

§2. Preliminaries

In this section we shall establish notation and gather all the definitions and results required for this article.

§2.1. Generalities

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ denote respectively the set of real numbers, rational numbers and complex numbers. We use $\Re z$ and $\Im z$ to denote respectively the real and imaginary parts of $z \in \mathbb{C}$. This notation will also be used for its obvious generalization when $\mathbf{z} \in \mathbb{C}^n$. The notation $|\cdot|$ will denote the standard Euclidean norm in $\mathbb{R}^n$ and in $\mathbb{C}^n$: $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $|\mathbf{z}| = \sqrt{|\Re(\mathbf{z})|^2 + |\Im(\mathbf{z})|^2}$ for $\mathbf{z} \in \mathbb{C}^n$. We shall also use $|\cdot|$ to represent a norm on certain spaces related to the symmetric space X, but under appropriate identifications of these spaces with $\mathbb{R}^n$ or $\mathbb{C}^n$, which will make this consistent with the previous usage of this notation. (For more details see Section 3.) For a set S in a topological space, S° denotes its interior. For a function f on X, $\|f\|_p$ denotes its L^p norm. We shall mention explicitly when we use the L^p-norm of functions on spaces other than X. For any $p \in (1,\infty)$, $p' = p/(p-1)$ and for $p = 1$, $p' = \infty$. When $p = \infty$ we use p' to mean 1. We shall frequently use the notation $\gamma_p = \gamma_{p'} = |\frac{2}{p} - 1|$ for any $p \in (1,\infty)$ and $\gamma_1 = \gamma_\infty = 1$. The letters C, c will be used to denote positive constants whose values may change from one line to another. The following results of several complex variables will be used.

2.1.1. Open mapping theorem [16, Thm. 1.21, p. 17]. If $\Omega \subset \mathbb{C}^n$ is open and $f\colon \Omega \to \mathbb{C}$ is a nonconstant holomorphic function then $f(U)$ is open for every open set $U \subset \Omega$.

2.1.2. Maximum modulus principle [16, Cor. 1.22, p. 17]. Let $\Omega \subset \mathbb{C}^n$ be an open set and $f\colon \Omega \to \mathbb{C}$ a holomorphic function. If $|f|$ attains a local maximum at a point $z_0 \in \Omega$ then f is constant in the connected component of Ω containing z_0.

2.1.3. Thin sets. Let $\Omega \subset \mathbb{C}^n$ be an open set. A subset E of Ω is called *thin* if for every point $x_0 \in \Omega$ there is a ball $B(x_0, r)$ centered at x_0 with radius $r > 0$ in Ω and a nonconstant holomorphic function $f\colon B(x_0, r) \to \mathbb{C}$ such that $f(z) = 0$ for $z \in E \cap B(x_0, r)$. We quote here some well-known results related to thin sets ([16, pp. 32–33]):

(1) If $E \subset \Omega$ is not thin then no nonzero holomorphic function $f\colon \Omega \to \mathbb{C}$ can vanish on E.
(2) If $E \subset \Omega$ is thin then its closure $\bar{E}$ in Ω is also thin and E is nowhere dense.
(3) The $2n$-dimensional Lebesgue measure of a thin set $E \subset \Omega \subset \mathbb{C}^n$ is zero.
(4) If Ω is connected and $E \subset \Omega$ is thin then $\Omega \setminus E$ is also connected.

§2.2. Chaos, hypercyclicity, etc.

Let $\mathbb{B}$ be a separable Fréchet space and $T\colon \mathbb{B} \to \mathbb{B}$ be a linear dynamical system, i.e. T is a linear map from $\mathbb{B}$ to itself. For $x \in \mathbb{B}$ we call

$$\{x, Tx, T^2x, \dots\}$$

the orbit of x under T. The operator T is called hypercyclic if there is an $x \in \mathbb{B}$, such that the orbit of x under T is dense in $\mathbb{B}$. In such a case x is called a hypercyclic vector for T. (See [7, p. 37].) A point $x \in \mathbb{B}$ is called a *periodic point* of T if there is a nonzero natural number n such that $T^n x = x$. The operator T is called chaotic if T is hypercyclic and the set of all its periodic points is dense in $\mathbb{B}$.

For a C_0-semigroup $(T_t)_{t\geq 0}$ on a Fréchet space $\mathbb{B}$, and $x \in \mathbb{B}$, $\{T_t x \mid t \geq 0\}$ is called the orbit of x under $(T_t)_{t\geq 0}$. If this orbit is dense in $\mathbb{B}$, then x is called a hypercyclic vector and we say that $(T_t)_{t\geq 0}$ is hypercyclic on $\mathbb{B}$. A point $x \in \mathbb{B}$ is called a periodic point of $(T_t)_{t\geq 0}$ if $T_t x = x$ for some $t > 0$. The semigroup $(T_t)_{t\geq 0}$ is called chaotic if it is hypercyclic and the set of all its periodic points is dense in $\mathbb{B}$.

A *discretization* of a C_0-semigroup $(T_t)_{t\geq 0}$ is a sequence of operators $(T_{t_n})_n$ with $t_n \to \infty$. In particular, if $t_n = nt_0$ for some $t_0 > 0$ and $n \in \mathbb{N}$, then $(T_{t_n})_n = (T_{t_0}^n)_n$ is an *autonomous discretization* of $(T_t)_{t\geq 0}$. It is clear that any periodic point (respectively hypercyclic vector) of the operator T_{t_0} for any fixed $t_0 > 0$ is also a periodic point (respectively hypercyclic vector) of the semigroup $(T_t)_{t\geq 0}$. Thus if for some $t_0 > 0$, T_{t_0} is chaotic on a Banach space $\mathbb{B}$, then $(T_t)_{t\geq 0}$ is chaotic on $\mathbb{B}$. We also have the following result (see [7, p. 168, Thm. 6.8]).

Proposition 2.2.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space $\mathbb{B}$. If $x \in \mathbb{B}$ is a hypercyclic vector for $(T_t)_{t\geq 0}$ then it is a hypercyclic vector for each operator T_t, $t > 0$.*

For a detailed account on the relationship between dynamics of a C_0-semigroup and that of its discretization we refer to [7, Chap. 7]. This discussion in particular points out that Theorem 4.0.1 in this paper accommodates the chaoticity of the heat semigroup considered in [13, 17, 15] as a special case, which was alluded to in the introduction.

The following result due to Kitai will be used in this article ([14], [7, p. 71]).

Theorem 2.2.2 (Kitai). *Let $\mathbb{B}$ be a separable Banach space and T be a bounded linear operator from $\mathbb{B}$ to itself. Let Y_1, Y_2 be two dense subsets of $\mathbb{B}$ and $T' : Y_1 \to Y_1$ be a (not necessarily linear or continuous) map. If*

(i) $\lim_{n\to\infty} T^n y = 0 \ \forall y \in Y_2$,
(ii) $\lim_{n\to\infty} T'^n x = 0 \ \forall x \in Y_1$ *and*
(iii) $TT'x = x \ \forall x \in Y_1$,

then T is hypercyclic on $\mathbb{B}$, i.e. there is an $x \in \mathbb{B}$ such that the orbit $\{T^n x \mid n \in \mathbb{N}\}$ is dense in $\mathbb{B}$.

We conclude this section by noting that ([7, p. 167, Thm. 6.7]) if T is a hypercyclic operator on a Fréchet space $\mathbb{B}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ then T and λT have the same set of hypercyclic vectors.

§3. Riemannian symmetric spaces

Most of the notation and results in this section are standard and available for instance in [6, 10]. For convenience and for the sake of keeping the current exposition self-contained, we merely collect the relevant facts without proofs but indicate appropriate references.

§3.1. Basics

Throughout this paper, X will denote a Riemannian symmetric space of noncompact type which can be realized as a quotient space G/K where G is a connected noncompact semisimple Lie group with finite center, and K is a maximal compact subgroup of G. The group G acts naturally on X and on functions on X by left translations. Functions on X are identified with the right K-invariant functions on G and vice versa. For an element $x \in G$ and a function f on X, $\ell_x f$ is the left translation of f defined by $\ell_x f(y) = f(x^{-1}y)$. A function (or measure) on X is called K-invariant if it is invariant under left K-action. Such a function (respectively measure) can be identified naturally with a K-biinvariant function (respectively measure) on G. Frequently we shall use this identification without mentioning it.

The group G admits an Iwasawa decomposition, namely $G = KAN$, inducing a direct sum decomposition of the Lie algebra: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here, $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of G, K, A and N respectively. This decomposition fixes a system of positive roots $\Sigma^+ \subset \mathfrak{a}^*$, where $\mathfrak{a}^*$ denotes the real dual of $\mathfrak{a}$. From the collection of root spaces $\mathfrak{g}_\alpha$, parametrized by Σ^+, one obtains

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

Setting $m_\alpha = \dim(\mathfrak{g}_\alpha)$, the multiplicity of the root $\alpha \in \Sigma^+$, we define $\boldsymbol{\rho}$ as the half-sum of the elements of Σ^+ counted with multiplicities:

$$\boldsymbol{\rho} = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \cdot \alpha \quad \in \mathfrak{a}^*. \tag{3.1.1}$$

The Killing form on $\mathfrak{g}$ restricts to a positive definite form on $\mathfrak{a}$, which in turn induces a positive inner product and hence a norm $|\cdot|$ on $\mathfrak{a}^*$, so $|\boldsymbol{\rho}|$ is defined. The Killing form endows X with both a natural Riemannian metric and a corresponding G-invariant measure (denoted by dx). The positive Laplace–Beltrami operator corresponding to this Riemannian metric is denoted by Δ.

Let $\dim(\mathfrak{a}) = n$, which is by definition the rank of the space X. Using the pullback of the Killing form, we shall henceforth identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}^n$, equipped with the standard inner product $\langle\cdot,\cdot\rangle$:

$$\langle \mathbf{x},\mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i, \quad \mathbf{x}=(x_1,\dots,x_n),\ \mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n, \tag{3.1.2}$$

so that $\langle \mathbf{x},\mathbf{x}\rangle > 0$ for all nonzero $\mathbf{x}\in\mathbb{R}^n$. The complexification of $\mathfrak{a}^*$ will be denoted by $\mathfrak{a}^*_{\mathbb{C}}$ and will be naturally identified with $\mathbb{C}^n$. The real inner product (3.1.2) extends to $\mathbb{C}^n$ as a $\mathbb{C}$-bilinear form $\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ defined by

$$(\mathbf{z},\mathbf{v}) = \sum_{i=1}^{n} z_i v_i, \quad \text{where } \mathbf{z}=(z_1,\dots,z_n),\ \mathbf{v}=(v_1,\dots,v_n)\in\mathbb{C}^n. \tag{3.1.3}$$

For the action of $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ on $v\in\mathfrak{a}$ we shall use both the notation $\lambda(v)$ and the notation (λ,v).

Let W denote the Weyl group of the pair $(\mathfrak{g},\mathfrak{a})$ and $\mathfrak{a}_+$ and $\mathfrak{a}^*_+$ be the positive Weyl chambers corresponding to Σ^+ in $\mathfrak{a}$ and $\mathfrak{a}^*$ respectively.

For $p\geq 1$, we define the set ([6, p. 328])

$$\Lambda_p = \left\{\lambda\in\mathbb{C}^n \;\middle|\; |\Im(w\lambda)(H)| \leq \gamma_p \boldsymbol{\rho}(H) \text{ for all } H\in\mathfrak{a}_+,\ w\in W\right\}, \tag{3.1.4}$$

where γ_p and $\boldsymbol{\rho}$ are defined in Section 2.1 and (3.1.1) respectively. We note the following:

(a) If $p=2$ then Λ_p reduces to $\mathfrak{a}^*$, which is identified with $\mathbb{R}^n$.
(b) For $1\leq p<q\leq 2$, $\Lambda_q\subsetneq\Lambda_p$.
(c) $\Lambda_p=\Lambda_{p'}$ for $p\geq 1$.
(d) Λ_p is closed under the reflection $\lambda\mapsto-\lambda$ ([6, p. 329]).

We recall from Section 3.1 that the G-invariant measure dx on X is induced by the Killing form. On G, we fix the Haar measure dg that satisfies

$$\int_X f(x)\,dx = \int_G f(g)\,dg$$

for every function $f\in L^1(X)$ which is identified as a right K-invariant function on G in the right-hand side. Let M be the centralizer of A in K. On K we fix the normalized Haar measure dk and on K/M we fix the K-invariant normalized measure. We shall often slur the difference between the two.

§3.2. Spherical Fourier transform

Let $H\colon G\to\mathfrak{a}$ be the Iwasawa projection associated to the decomposition $G=KAN$. The *elementary spherical function* φ_λ for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ is defined by ([10, p. 200])

$$\varphi_\lambda(x) = \int_K e^{-(i\lambda+\boldsymbol{\rho})(H(x^{-1}k))}\,dk, \quad x\in G.$$

We record a few well-known facts about these functions. Some are easy to deduce. For the others, see [8, pp. 419, 427, 460] and [6, Props 3.1.4 and 3.2.2, ineqs (4.6.3), (4.6.4), (4.6.9)].

Lemma 3.2.1. *The elementary spherical functions φ_λ have the following properties:*

(a) *For each $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \equiv \mathbb{C}^n$, the function φ_λ is a K-biinvariant function on G (hence is naturally identified as a function on X) and $\int_K \varphi(xky)\,dk = \varphi(x)\varphi(y)$.*

(b) *$\varphi_\lambda = \varphi_{w\lambda}$ for all $w \in W$.*

(c) *$\varphi_{-\lambda}(x^{-1}) = \varphi_\lambda(x)$ for all $x \in G$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*

(d) *For every λ, the identity*

$$\Delta\varphi_\lambda = ((\lambda,\lambda) + |\boldsymbol{\rho}|^2)\varphi_\lambda$$

holds pointwise.

(e) *If $2 < p < \infty$ and $\lambda \in \Lambda_p^\circ$ then $\varphi_\lambda \in L^p(X)$ and for $\lambda \in \mathfrak{a}^*$, $\varphi_\lambda \in L^{2+\varepsilon}(X)$ for any $\varepsilon > 0$.*

(f) *If $\lambda \in \Lambda_1$, then $\varphi_\lambda \in L^\infty(X)$.*

(g) *If $\lambda \in \Lambda_1^\circ$, then $\varphi_\lambda \in C_0(X)$, the space of continuous functions vanishing at infinity.*

(h) *For each fixed $x \in G$, $\lambda \mapsto \varphi_\lambda(x)$ is a holomorphic function on $\mathbb{C}^n$.*

For a measurable function f of X, we define its *spherical Fourier transform* $\hat{f}$ as (see [8, p. 425])

$$\hat{f}(\lambda) = \int_X f(x)\varphi_{-\lambda}(x)\,dx, \qquad \lambda \in \mathfrak{a}^*,$$

whenever the integral makes sense. Since for all $w \in W$, $\varphi_\lambda = \varphi_{w\lambda}$ we have $\hat{f}(\lambda) = \hat{f}(w\lambda)$. Its inverse transform, again subject to convergence of the defining integral, is given by (see [8, p. 454])

$$f(x) = C\int_{\mathfrak{a}^*} \hat{f}(\lambda)\varphi_\lambda(x)|\mathrm{c}(\lambda)|^{-2}\,d\lambda, \tag{3.2.1}$$

where $\mathrm{c}(\lambda)$ is the Harish-Chandra c-function, $d\lambda$ is the Lebesgue measure on $\mathfrak{a}^*$ (and thus $|\mathrm{c}(\lambda)|^{-2}\,d\lambda$ is the spherical Plancherel measure on $\mathfrak{a}^*$) and C is a normalizing constant.

§3.3. Helgason Fourier transform

(See [10, pp. 199–203] for details.) For a function f on X, its Helgason Fourier transform is defined by

$$\tilde{f}(\xi, k) = \int_X f(x) e^{(i\xi - \boldsymbol{\rho})(H(x^{-1}k))}\, dx$$

for all $\xi \in \mathfrak{a}^*_{\mathbb{C}} \equiv \mathbb{C}^n$, $k \in K/M$, for which the integral exists. The Fourier transform $f(x) \to \tilde{f}(\xi, k)$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}^*_+ \times K/M, |\mathrm{c}(\xi)|^{-2}\, d\xi\, dk)$ where $\mathrm{c}(\xi)$ is the Harish-Chandra c-function and thus $|\mathrm{c}(\xi)|^{-2}\, d\xi\, dk$ is the Plancherel measure. We also have

$$\int_X f_1(x)\overline{f_2(x)}\, dx = \frac{1}{|W|} \int_{\mathfrak{a}^*_+ \times K/M} \widetilde{f_1}(\xi, k)\overline{\widetilde{f_2}(\xi, k)} |\mathrm{c}(\xi)|^{-2}\, d\xi\, dk,$$

where $|W|$ is the cardinality of the Weyl group W and dk is the normalized K-invariant measure on K/M. We note that if g is a K-invariant function on X, then $\tilde{g}(\xi, k) = \hat{g}(\xi)$ for all $k \in K/M$ and for f, g as above,

$$\widetilde{f * g}(\xi, k) = \tilde{f}(\xi, k)\hat{g}(\xi)$$

for $\xi \in \mathbb{C}^n$ and $k \in K/M$ whenever the quantities $f * g$, $\widetilde{f * g}$, $\tilde{f}$ and $\hat{g}$ make sense. We have the following L^p-version of the inversion formula (see [19, Thm. 3.3]).

Theorem 3.3.1 (Stanton–Tomas). *For a function $f \in L^p(X)$, $1 \leq p < 2$, if $f * \varphi_\lambda$ is in $L^1(|\mathrm{c}(\lambda)|^{-2}\, d\lambda)$, then for almost every $x \in X$,*

$$f(x) = \int_{\mathfrak{a}^*} f * \varphi_\lambda(x) |\mathrm{c}(\lambda)|^{-2}\, d\lambda.$$

In particular, if f is a K-invariant function on X and $\hat{f} \in L^1(|\mathrm{c}(\lambda)|^{-2}\, d\lambda)$, then

$$f(x) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\varphi_\lambda(x) |\mathrm{c}(\lambda)|^{-2}\, d\lambda.$$

§3.4. Herz's majorizing principle

We have the following result due to Herz ([12]) on convolution operators.

Proposition 3.4.1. *Let h be a K-biinvariant function on G, and let $T_h \colon f \mapsto f * h$ be the corresponding right convolution operator on $L^p(X)$, $p \in [1, \infty]$. Then the operator norm of $T_h \colon L^p(X) \to L^p(X)$ obeys the following bound:*

$$\|T_h\|_{L^p \to L^p} \leq \widehat{|h|}(-i\gamma_p \boldsymbol{\rho}),$$

where the equality holds if h is nonnegative.

§3.5. Fourier multipliers

We recall that for $1 < p < \infty$, $\gamma_p = |2/p - 1|$ and $\gamma_1 = \gamma_\infty = 1$. In this paper we are concerned about the bounded linear operators on $L^p(X)$, $1 \le p < \infty$ to itself which are invariant under translations by elements of G. This class of operators are called L^p-Fourier multipliers or simply L^p-multipliers and are denoted by $\mathrm{CO}_p(X)$. It is known that $\mathrm{CO}_p(X)$ is a Banach algebra. We shall briefly discuss the main points about these operators, collecting them mostly from [1]. If $T \in \mathrm{CO}_1(X)$ then $Tf = f * \mu$, where μ is a K-biinvariant finite Borel measure on G and if $T \in \mathrm{CO}_2(X)$ then for $f \in C_c^\infty(X)$,

$$\widetilde{Tf}(\lambda, k) = m(\lambda)\tilde{f}(\lambda, k), \tag{3.5.1}$$

where m is a W-invariant function in $L^\infty(\mathfrak{a}^*)$. By abuse of terminologies the function $m(\lambda)$ will also be called a Fourier multiplier. For $1 \le p_1, p_2 < \infty$ with $\gamma_{p_1} \ge \gamma_{p_2}$, $\mathrm{CO}_{p_1}(X) \subseteq \mathrm{CO}_{p_2}(X)$. In particular, $\mathrm{CO}_p(X) \subseteq \mathrm{CO}_2(X)$ for $1 \le p < \infty$ and hence they are also given by (3.5.1) for $f \in C_c^\infty(X)$. But for $1 \le p < \infty, p \ne 2$, $m(\lambda)$ extends to a W-invariant bounded holomorphic function on Λ_p°. For $p = 1$, $m(\lambda)$ is also bounded continuous on Λ_1. Henceforth we shall call a multiplier $T \in \mathrm{CO}_p(X)$ nontrivial if it is not a constant multiple of the identity operator.

We fix a p in the range $(2, \infty)$ and take a nontrivial $T \in \mathrm{CO}_p(X)$. Suppose that T is given by the function $m(\lambda)$ which by definition is W-invariant and extends to a bounded holomorphic function on Λ_p°. We have the following result for such p, T.

Proposition 3.5.1. *Let $T^*\colon L^{p'}(X) \to L^{p'}(X)$ be the adjoint operator. Then*

(i) *for any $g \in C_c^\infty(X)$, $\widetilde{T^*g}(\lambda, k) = \overline{m(\bar\lambda)}\tilde{g}(\lambda, k)$, for almost every $(\lambda, k) \in \Lambda_p^\circ \times K/M$, $\widehat{T^*g}(\lambda) = \overline{m(\bar\lambda)}\hat{g}(\lambda)$, for almost every $\lambda \in \Lambda_p^\circ$,*

(ii) *for $\lambda \in \Lambda_p^\circ$, $T\varphi_\lambda = m(\lambda)\varphi_\lambda$ and $T^*\varphi_\lambda = \overline{m(\bar\lambda)}\varphi_\lambda$.*

Proof. We take $f, g \in C_c^\infty(X)$. Then using the definition of T^* and the Plancherel theorem we have

$$\begin{aligned}\langle T^*g, f\rangle = \langle g, Tf\rangle &= \int_X g(x)\overline{Tf(x)}\,dx \\ &= \int_{\mathfrak{a}_+^* \times K/M} \tilde{g}(\lambda, k)\overline{\widetilde{Tf}(\lambda, k)}\,d\mu(\lambda)\,dk \\ &= \int_{\mathfrak{a}_+^* \times K/M} \overline{m(\lambda)}\tilde{g}(\lambda, k)\overline{\tilde{f}(\lambda, k)}\,d\mu(\lambda)\,dk.\end{aligned}$$

Since $\overline{m(\lambda)}$ is bounded, $\overline{m(\lambda)}\tilde{g}(\lambda, k) \in L^2(\mathfrak{a}_+^* \times K/M)$ and hence there exists a unique $\phi \in L^2(X)$ such that $\tilde{\phi}(\lambda, k) = \overline{m(\lambda)}\tilde{g}(\lambda, k)$. Therefore, $\int_X T^*g(x)\overline{f(x)}\,dx =$

$\int_X \phi(x)\overline{f(x)}\,dx$ which implies $T^*g = \phi$ and in particular $\widetilde{T^*g}(\lambda, k) = \overline{m(\lambda)}\tilde{g}(\lambda, k)$ for all $(\lambda, k) \in \mathfrak{a}^*_+ \times K/M$. Since T is a $p - p$ operator, by duality T^* is $p' - p'$. Hence $\lambda \mapsto \overline{m(\lambda)}$ defined on $\mathfrak{a}^*_+$ extends to a holomorphic function on Λ_p°. As $m(\lambda)$ also extends as a holomorphic function on Λ_p°, we conclude that the extension of $\overline{m(\lambda)}$ is given by $\overline{m(\bar{\lambda})}$. This proves the first part of (i). Integrating both sides of it over K/M we get the second result of (i).

We recall that $\varphi_\lambda \in L^p(X)$ for $\lambda \in \Lambda_p^\circ$ and $\overline{\varphi_\lambda(x)} = \varphi_{-\bar{\lambda}}(x)$. For a function $g \in C_c^\infty(X)$ we have

$$\langle T\varphi_\lambda, g\rangle = \langle \varphi_\lambda, T^*g\rangle = \int_G \varphi_\lambda(x)\overline{T^*g(x)}\,dx = \overline{\int_G \overline{\varphi_\lambda(x)}T^*g(x)\,dx}$$
$$= \overline{\int_G \varphi_{-\bar{\lambda}}(x)T^*g(x)\,dx}.$$

Therefore,

$$\langle T\varphi_\lambda, g\rangle = \overline{\widehat{T^*g}(\bar{\lambda})} = \overline{\overline{m(\lambda)}\hat{g}(\bar{\lambda})} = m(\lambda)\overline{\int_G g(x)\varphi_{-\bar{\lambda}}\,dx} = m(\lambda)\int_G \varphi_\lambda(x)\overline{g(x)}\,dx$$
$$= \langle m(\lambda)\varphi_\lambda, g\rangle.$$

It can be verified in a similar way that $\langle T^*\varphi_\lambda, g\rangle = \langle\overline{m(\bar{\lambda})}\phi_\lambda, g\rangle$. Thus

$$T\varphi_\lambda = m(\lambda)\varphi_\lambda \quad \text{and} \quad T^*\varphi_\lambda = \overline{m(\bar{\lambda})}\varphi_\lambda. \qquad \square$$

§4. Statement and proof of the main result

The following theorem is the main result in this paper.

Theorem 4.0.1. *Fix $p \in (2, \infty)$. Let T be a nontrivial L^p-multiplier. Then there is a constant $c > 0$ such that zT for any $z \in \mathbb{C}$ with $|z| = c$ is chaotic on $L^p(X)$.*

Suppose that T is densely defined by $\widetilde{Tf}(\lambda, k) = m(\lambda)\tilde{f}(\lambda, k)$ for $f \in C_c^\infty(X)$, $\lambda \in \mathfrak{a}^*, k \in K/M$. As T is nontrivial $m(\lambda)$ is a nonconstant function. Then $|m(\lambda)|$ is also nonconstant. Indeed, if $|m(\lambda)| = \beta$ for some $\beta > 0$ for all $\lambda \in \Lambda_p^\circ$ then the holomorphic function $\lambda \mapsto m(\lambda)$ maps the open domain Λ_p° to an arc of the circle of radius β, which is not open in $\mathbb{C}$, which violates the open mapping theorem (see Section 2).

As $|m(\lambda)|$ is not constant there exist points $\lambda_1, \lambda_2 \in \Lambda_p^\circ$ and $\alpha > 0$ such that

$$|m(\lambda_2)| < \alpha < |m(\lambda_1)|.$$

Let $c = \frac{1}{\alpha}$. We take a $z \in \mathbb{C}$ such that $|z| = c$ and define $m_1(\lambda) = zm(\lambda)$. Then $|m_1(\lambda_2)| < 1 < |m_1(\lambda_1)|$. Let $T_1 = zT$. Then T_1 is an L^p-multiplier with symbol

$m_1(\lambda)$. The proof of Theorem 4.0.1 will be completed if we show that T_1 is chaotic on $L^p(X)$. This will be done through the next two propositions.

Proposition 4.0.2. *The operator T_1 described above is hypercyclic on $L^p(X)$ for $2 < p < \infty$.*

Proof. As $\lambda \mapsto |m_1(\lambda)|$ is continuous, there exist neighborhoods N_1 and N_2 of λ_1, λ_2 respectively in Λ_p° such that $|m_1(\lambda)| > 1$ for $\lambda \in N_1$ and $|m_1(\lambda)| < 1$ for $\lambda \in N_2$. Since $m_1(\lambda)$ is W-invariant, we can and will assume that N_1 and N_2 are subsets of

$$(\Re\Lambda_p^\circ)_+ = \{\lambda \in \Lambda_p^\circ \mid \Re\lambda \in \mathfrak{a}_+^*\}.$$

We define

$$Y_1 = \operatorname{span}\{\ell_y\varphi_\lambda \mid \lambda \in N_1,\ y \in G\} \quad \text{and} \quad Y_2 = \operatorname{span}\{\ell_y\varphi_\lambda \mid \lambda \in N_2,\ y \in G\}.$$

Both Y_1 and Y_2 are dense in $L^p(X)$. Indeed if any $f \in L^{p'}(X)$ annihilates Y_1, then $f * \varphi_{-\lambda} \equiv 0$ for λ in the open set N_1. Since for every fixed $x \in X$, $\lambda \mapsto f * \varphi_{-\lambda}(x)$ is holomorphic on Λ_p° we have $f * \varphi_\lambda \equiv 0$ for all $\lambda \in \Lambda_p$. Using Theorem 3.3.1 we conclude that $f = 0$. A similar argument with the substitution of N_1 by N_2 establishes that Y_2 is also dense in $L^p(X)$.

Let

$$\eta_\lambda = a_1^\lambda \ell_{y_1^\lambda}\varphi_\lambda + \cdots + a_n^\lambda \ell_{y_n^\lambda}\varphi_\lambda \quad \in Y_1$$

be a finite linear combination of $\ell_y\varphi_\lambda$ with the same λ. We define an operator T_1' initially on such η_λ as

$$T_1'(\eta_\lambda) = m_1(\lambda)^{-1}\eta_\lambda.$$

Since elements of Y_1 are finite linear combinations of these η_λ we extend T_1' linearly on Y_1. We need to show that T_1' is well defined on Y_1. For future use we record here that

$$\eta_\lambda(e) = a_1^\lambda\varphi_{-\lambda}(y_1^\lambda) + \cdots + a_n^\lambda\varphi_{-\lambda}(y_n^\lambda).$$

Let $\xi = \sum_{i=1}^n b_i\eta_{\lambda_i}$ be a typical element of Y_1, where λ_i, $i = 1, \dots, n$ are distinct. It suffices to show that if $\xi = 0$ then $\eta_{\lambda_i} = 0$ for all $i = 1, \dots, n$, so that $T_1'(\xi) = \sum_{i=1}^n b_i T_1'(\eta_{\lambda_i}) = 0$.

Since $N_1 \subset (\Re\Lambda_p^\circ)_+$ we note that $w\lambda_i \neq \lambda_j$ for all nontrivial $w \in W$ whenever $i \neq j$. Consequently, φ_{λ_1}, φ_{λ_n} are two distinct K-invariant elements of $L^{p'}(X)$. Therefore there is a K-invariant function $f \in L^p(X)$ such that $\hat{f}(-\lambda_1) \neq 0$ and $\hat{f}(-\lambda_n) = 0$. Starting from $\xi = 0$ and noting that $\int_X f(z)\eta_{\lambda_i}(z)\,dz = \hat{f}(-\lambda_i)\eta_{\lambda_i}(e)$, we get by abuse of notation,

$$\sum_{i=1}^m b_i\hat{f}(-\lambda_i)\eta_{\lambda_i}(e) = \sum_{i=1}^m c_i\eta_{\lambda_i}(e) = 0$$

for some $m < n$. Indeed, if for any $i = 2, \dots, n-1$, $\hat{f}(-\lambda_i) = 0$ we discard it and for others write $c_i = b_i \hat{f}(-\lambda_i) \neq 0$ and relabel them as $i = 2, \dots, m$, keeping λ_1 unchanged. The assumption $\xi = 0$ also implies $\ell_x \xi = 0$ for any $x \in G$. Instead of $\xi = 0$, if we start from $\ell_{x^{-1}} \xi = 0$ then through the same steps as above, we get

$$\sum_{i=1}^{m} c_i \eta_{\lambda_i}(x) = 0.$$

Thus $\sum_{i=1}^{m} c_i \eta_{\lambda_i}(x) = 0$ for all $x \in G$. In this way we can reduce the number of η_λ's. A repeated application of this process finally yields $\eta_{\lambda_1}(x) = 0$ which was the target. Thus we have established that T_1' is a well-defined operator on Y_1.

We shall now verify that operators T_1 and T_1' satisfy the hypothesis of Theorem 2.2.2. Clearly $(T_1')^n \phi \to 0$ as $n \to \infty$ for any $\phi \in Y_1$ because $|m_1(\lambda)| > 1$ for $\lambda \in N_1$. On the other hand, as $T_1(\ell_y \varphi_\lambda) = \ell_y T_1(\varphi_\lambda) = m_1(\lambda) \ell_y \varphi_\lambda$ and on N_2, $|m_1(\lambda)| < 1$, $(T_1)^n \phi \to 0$ as $n \to \infty$ for any $\phi \in Y_2$. Lastly, $T_1 T_1'(\ell_y \varphi_\lambda) = \ell_y \varphi_\lambda$ by Proposition 3.5.1 and hence $T_1 T_1'$ is identity on Y_1. Theorem 2.2.2 now shows that T_1 is hypercyclic. □

Proposition 4.0.3. *The set of periodic points of the operator T_1 defined above is dense in $L^p(X)$ for $2 < p < \infty$.*

Proof. As there exist $\lambda_1, \lambda_2 \in \Lambda_p^\circ$ such that $|m_1(\lambda_1)| < 1 < |m_1(\lambda_2)|$, it follows from continuity of m_1 that there exists $\lambda_0 \in \Lambda_p^\circ$ such that $|m_1(\lambda_0)| = 1$. Let $S = \mathbb{T} \cap m_1(\Lambda_p^\circ)$, where $\mathbb{T}$ is the unit circle in the complex plane. By the open mapping theorem (see Section 2) $m_1(\Lambda_p^\circ)$ is an open set. Since $m_1(\lambda_0) \in S$, S is a nonempty open set of $\mathbb{T}$. We note that $m_1(\Lambda_p^\circ \setminus m_1^{-1}(S))$ is not connected. Indeed, it is the union of two nonempty sets, one inside $\mathbb{T}$ and the other outside $\mathbb{T}$ containing $m_1(\lambda_1)$ and $m_1(\lambda_2)$ respectively. We define

$$I = \left\{ r \in \mathbb{R} \;\middle|\; e^{2\pi i r} \in S \right\} \quad \text{and} \quad Z_r = \left\{ z \in \Lambda_p^\circ \;\middle|\; m_1(z) = e^{2\pi i r} \right\} \quad \text{for } r \in I.$$

Then $m_1^{-1}(S) = \bigcup_{r \in I} Z_r$. We consider the following subset of $L^p(X)$:

$$Y_3 = \operatorname{span}\left\{ \ell_y \varphi_z \;\middle|\; y \in G,\ z \in Z_\nu,\ \nu \in \mathbb{Q} \cap I \right\}.$$

The nonemptiness of Y_3 follows trivially from the fact that $m_1(\Lambda_p^\circ)$ is open in $\mathbb{C}$. If $\nu = a/b \in \mathbb{Q}$ (a, b relatively prime integers), then using $T_1(\ell_y \varphi_z) = m_1(z) \ell_y \varphi_z$, we have for $z \in Z_\nu$,

$$T_1^b(\ell_y \varphi_z) = m_1(z)^b \ell_y \varphi_z = e^{2\pi a i} \ell_y \varphi_z = \ell_y \varphi_z.$$

Thus the elements of Y_3 are periodic points of T_1.

It remains to show that Y_3 is dense in $L^p(X)$. Suppose that a nonzero function $f \in L^{p'}(X)$ annihilates Y_3. That is, $f * \varphi_{-z}(x) = 0$ for all $x \in X$ and for all $z \in Z_\nu$, $\nu \in \mathbb{Q} \cap I$. For a fixed $x \in X$ we define $F_x(z) = f * \varphi_{-z}(x)$ for $z \in \Lambda_p^\circ$. Then F_x is holomorphic on Λ_p° which vanishes on $\bigcup_{\nu \in \mathbb{Q} \cap I} Z_\nu$. We claim that F_x vanishes identically on Λ_p°. For the sake of meeting a contradiction we assume that $F_x \not\equiv 0$ on Λ_p°. Since F_x vanishes on $\bigcup_{\nu \in \mathbb{Q} \cap I} Z_\nu$, Lemma 4.0.4 implies that F_x vanishes on the set $m_1^{-1}(S) = \bigcup_{r \in I} Z_r$. But as we have assumed that F_x is a nonzero holomorphic function on Λ_p°, $m_1^{-1}(S)$ is a thin set in Λ_p°. Therefore, by the properties of thin sets (see Section 2), we conclude that the set $\Lambda_p^\circ \setminus m_1^{-1}(S)$ is connected. Since m_1 is continuous this implies that $m_1(\Lambda_p^\circ \setminus m_1^{-1}(S))$ is connected, which contradicts our early observation in this proof. Thus $F_x \equiv 0$ on Λ_p for all $x \in X$, that is, $f * \varphi_\lambda \equiv 0$ on X for all $\lambda \in \Lambda_p$. From this and Theorem 3.3.1 we conclude that $f = 0$, which establishes that Y_3 is dense. □

The following lemma will complete the proof above. We shall use the notation I and Z_r defined in the proof of the proposition above.

Lemma 4.0.4. *Let I and Z_r be as defined in the proof of Proposition 4.0.3. Fix an $r \in I$. Then for any $w \in Z_r$ and $\delta > 0$, there is a $\nu \in I \cap \mathbb{Q}$ and a $z \in Z_\nu$ such that $|w - z| < \delta$.*

Proof. Take the open ball $B_{\delta'}(w) \subset \Lambda_p^\circ$ where $\delta' < \delta$. Then $w \in B_{\delta'}(w)$. By the open mapping theorem, $m_1(B_{\delta'}(w))$ is an open set in $m_1(\Lambda_p^\circ)$ containing the point $m_1(w) = e^{2\pi i r}$. So $m_1(B_{\delta'}(w))$ will contain an arc $\{e^{2\pi i s} \mid s \in (a,b) \subset I\}$ with $r \in (a,b)$. Take a $\nu \in (a,b) \cap \mathbb{Q}$. Then the point $e^{2\pi i\nu}$ has a pre-image z in $B_{\delta'}(w)$. That is $m_1(z) = e^{2\pi i \nu}$, and hence $z \in Z_\nu$. Also as $z \in B_{\delta'}(w)$, $|w - z| < \delta' < \delta$. □

§5. Examples and remarks

Well-known examples of Fourier multipliers are spectral multipliers and convolution with suitable Borel measures. In the light of the result proved in the previous section, we shall revisit their dynamics, which will yield some interesting corollaries. The first example also relates Theorem 4.0.1 with the previous works in this direction e.g. [13, 15].

Example 5.0.1. The heat kernel h_t on X for $t > 0$ is defined as a K-invariant function in the Harish-Chandra L^p-Schwartz space $C^p(X)$, $1 \le p \le 2$, whose spherical Fourier transform is prescribed as (see [2, eq. 3.1])

$$\widehat{h_t}(\lambda) = e^{-t((\lambda,\lambda)+|\boldsymbol{\rho}|^2)} \quad \text{for all } \lambda \in \mathfrak{a}^*.$$

For a fixed $t > 0$, we consider the operator $Tf = f * h_t$, i.e. $T = e^{-t\Delta}$, where Δ is the positive Laplace–Beltrami operator on X. Then $m(\lambda) = \widehat{h_t}(\lambda)$. It is clear that T is not chaotic on $L^p(X)$ for any $1 \le p \le \infty$ since $\|T\|_{L^p - L^p} = \widehat{h_t}(-i\gamma_p\boldsymbol{\rho}) = e^{-4t|\boldsymbol{\rho}|^2/pp'} \le 1$. In general, for a multiplier given by the function $m(\lambda)$, if we define $\theta = \inf_{\lambda\in\Lambda_p} |m(\lambda)|$, $\Theta = \sup_{\lambda\in\Lambda_p} |m(\lambda)|$, then it is clear from the proof of Theorem 4.0.1 that we can choose z from the annulus $1/\Theta < |z| < 1/\theta$ where we take $1/\theta = \infty$ if $\theta = 0$. Coming back to the case in hand, $Tf = f * h_t$, we see that $\theta = 0$ and $\Theta = e^{-4t|\boldsymbol{\rho}|^2/pp'} \le 1$. So we choose $z \in \mathbb{C}$ such that $1 \le e^{4t|\boldsymbol{\rho}|^2/pp'} < |z| < \infty$. Take $z_0 = a + ib$, where $a > 4|\boldsymbol{\rho}|^2/pp'$ and $b \in \mathbb{R}$. Then $|e^{z_0 t}| = e^{at} > e^{4t|\boldsymbol{\rho}|^2/pp'}$. Thus we can take $z = e^{z_0 t}$ and by Theorem 4.0.1 $zT = e^{-t(\Delta - z_0)}$ is chaotic on $L^p(X)$, $2 < p < \infty$. A continuous semigroup version of this result is proved in [15].

Example 5.0.2. We continue to use the notation θ, Θ defined in the previous example. We consider convolution by a nonatomic and nonnegative K-invariant measure μ on X such that $\hat{\mu}(-i\gamma_p\boldsymbol{\rho}) < \infty$ for some $1 \le p \le \infty$. By Herz's majorizing principle (see Section 3.4) the operator $Tf = f * \mu$ is an L^p-multiplier. We note that in this case $\theta < 1$, because on $\lambda \in \mathfrak{a}^*$, $|\hat{\mu}(\lambda)| \to 0$ as $|\lambda| \to \infty$. Indeed, for $\lambda \in \mathfrak{a}^*$,

$$|\hat{\mu}(\lambda)| \le \int_X |\varphi_\lambda(x)|\, d\mu(x) \le \int_X \varphi_0(x)\, d\mu(x) \le \int_X \varphi_{i\gamma_p\boldsymbol{\rho}}(x)\, d\mu(x) < \infty.$$

Since, for every fixed $x \in X$, $|\varphi_\lambda(x)| \to 0$ as $|\lambda| \to \infty$, the result follows from the dominated convergence theorem.

We also note that here $\Theta = \hat{\mu}(-i\gamma_p\boldsymbol{\rho}) = \|T\|_{L^p - L^p}$. Thus if $\hat{\mu}(-i\gamma_p\boldsymbol{\rho}) \le 1$, then T is a contraction and hence not chaotic. On the other hand, if T is not a contraction, equivalently, if $\hat{\mu}(-i\gamma_p\boldsymbol{\rho}) > 1$, then it is chaotic because we can choose $z = 1$ as $1/\Theta < 1 < 1/\theta$. Precisely, we have proved the following.

Corollary 5.0.3. *Fix $2 < p < \infty$. Let μ be a nonatomic K-invariant regular nonnegative Borel measure on X and $T\colon f \mapsto f * \mu$. If $\hat{\mu}(-i\gamma_p\boldsymbol{\rho}) < \infty$ (equivalently $\|T\|_{L^p - L^p} < \infty$), then T is chaotic on $L^p(X)$ if and only if T is not a contraction.*

Corollary 5.0.4. *Let μ and T be as in Corollary 5.0.3 and $2 < p_2 < p_1 < \infty$. Suppose that μ satisfies the condition $\hat{\mu}(-i\gamma_{p_1}\boldsymbol{\rho}) < \infty$, so that $T \in \mathrm{CO}_{p_1}(X) \subset \mathrm{CO}_{p_2}(X)$. If T is chaotic on $L^{p_2}(X)$, then T is chaotic on $L^{p_1}(X)$.*

Proof. Since T is chaotic on $L^{p_2}(X)$, by Corollary 5.0.3, $\hat{\mu}(-i\gamma_{p_2}\boldsymbol{\rho}) > 1$. Therefore, by the maximum modulus principle (see Section 2), $\hat{\mu}(-i\gamma_{p_1}\boldsymbol{\rho}) > 1$ and hence again by Corollary 5.0.3, T is chaotic on $L^{p_1}(X)$. □

Instead of nonnegative K-biinvariant measure we can take a K-invariant complex measure, in particular a K-invariant measurable function g on X such that

$$1 < |\hat{g}(-i\gamma_p\boldsymbol{\rho})| < \widehat{|g|}(-i\gamma_p\boldsymbol{\rho}) < \infty.$$

Then the convolution operator $T\colon L^p(X) \to L^p(X)$, $2 < p < \infty$ given by $f \mapsto f * g$ is bounded and is chaotic on $L^p(X)$ by a similar argument.

Example 5.0.5. Fix a p in the range $2 < p < \infty$. From the proof of Theorem 4.0.1, it is clear that if an L^p-multiplier T given by the function $m(\lambda)$ is such that there exist $\lambda_1, \lambda_2 \in \Lambda_p^\circ$ with $|m(\lambda_1)| < 1 < |m(\lambda_2)|$, then T is itself chaotic on $L^p(X)$. For an element $z \in \mathbb{C}$ from the complement of the L^p-spectrum of the Laplace–Beltrami operator Δ, we consider the resolvent $T = (\Delta - z)^{-1}$. It is easy to verify that if z is sufficiently close to the spectrum (so that there are $\lambda_1, \lambda_2 \in \Lambda_p^\circ$ with $|(\lambda_1, \lambda_1) + |\boldsymbol{\rho}|^2 - z| < 1$ and $|(\lambda_2, \lambda_2) + |\boldsymbol{\rho}|^2 - z| > 1$) then T is chaotic. A description of the L^p-spectrum of Δ can be found in [15, Sect. 3.4] and the references therein.

§6. Sharpness of the range of p

The aim of this section is to show that the range of p in Theorem 4.0.1 is sharp. As preparation we gather and prove some lemmas. The first one is from [7, Prop. 5.1].

Lemma 6.0.1. *Let T be a hypercyclic operator on a Banach space $\mathbb{B}$ and T^* be the dual operator of T acting on $\mathbb{B}^*$. Then*

(i) *for any nonzero $\phi \in \mathbb{B}^*$ the orbit $\{(T^*)^n\phi \mid n \geq 0\}$ is unbounded,*
(ii) *the point spectrum of T^* is empty.*

An easy adaptation of [9, Thm. 8.1] using the fact that $\varphi_\lambda \in L^{p'}(X)$ for $\lambda \in \Lambda_q$ (see Lemma 3.2.1(e)), proves the following lemma. See also [11, 18].

Lemma 6.0.2. *For $1 \leq p < q < 2$ and $f \in L^p(X)$, there exists a subset $B \subset K/M$ of full measure such that for each $k \in B$, $\tilde{f}(\lambda, k) = \int_X f(x) e^{(i\lambda - \boldsymbol{\rho})H(x^{-1}k)}\, dx$ exists for all $\lambda \in \Lambda_q$ and is holomorphic on Λ_q. The set B may depend on the function f but does not depend on $\lambda \in \Lambda_q$.*

We also have the following results.

Lemma 6.0.3. *For $f \in L^p(X)$, $1 \leq p < 2$ and $\lambda \in \mathfrak{a}^*$, $\|\tilde{f}(\lambda, \cdot)\|_{L^2(K/M)} \leq C\|f\|_p$ for some constant $C > 0$.*

Proof. Temporarily using the notation $e_{\lambda,k}(x) = e^{(i\lambda-\boldsymbol{\rho})(H(x^{-1}k))}$ we have

$$\begin{aligned}\int_{K/M} |\tilde{f}(\lambda,k)|^2\, dk &= \int_{K/M} \overline{\tilde{f}(\lambda,k)}\tilde{f}(\lambda,k)\, dk \\ &= \int_{K/M}\int_X \overline{f(x)}\overline{e_{\lambda,k}(x)}\, dx\, \tilde{f}(\lambda,k)\, dk \\ &= \int_X \overline{f(x)} \int_{K/M} \overline{e_{\lambda,k}(x)}\tilde{f}(\lambda,k)\, dk\, dx \\ &= \int_X \overline{f(x)} f * \varphi_\lambda(x)\, dx \\ &\leq \|f\|_p \|f*\varphi_\lambda\|_{p'},\end{aligned}$$

where in the last step we have used Hölder's inequality. We recall that for $\lambda \in \mathfrak{a}^*$, $\varphi_\lambda \in L^{2+\varepsilon}(X)$ for any $\varepsilon > 0$ (see Lemma 3.2.1(e)). This implies that the operator $f \mapsto f * \varphi_\lambda$ is bounded from $L^p(X)$ to $L^{p'}(X)$ for any $1 \leq p < 2 < p' \leq \infty$ (see [3, Thm. 2.2]). That is, $\|f * \varphi_\lambda\|_{p'} \leq C\|f\|_p$ for $\lambda \in \mathfrak{a}^*$ for some constant $C > 0$ and $1 \leq p < 2$. Therefore, we have $\int_{K/M} |\tilde{f}(\lambda,k)|^2\, dk \leq C\|f\|_p^2$, which is the assertion. □

Lemma 6.0.4. *For $1 \leq p < q < 2$, let T be an L^p-multiplier given by the function $m(\lambda)$ and $f \in L^p(X)$. Then there exists a subset $B \subset K/M$ of full measure such that for each $k \in B$ and for $\lambda \in \Lambda_q$, $\widetilde{Tf}(\lambda,k) = m(\lambda)\tilde{f}(\lambda,k)$.*

Proof. Using the denseness of $C_c^\infty(X)$ in $L^p(X)$, we find a sequence $f_n \in C_c^\infty(X)$ which converges to f in $L^p(X)$. Then passing to a subsequence f_{n_i} if necessary, we have by Lemma 6.0.3,

$$\widetilde{f_{n_i}}(\lambda,k) \to \tilde{f}(\lambda,k)$$

for every fixed $\lambda \in \mathfrak{a}^*$ for almost every $k \in K/M$. We also have $Tf_{n_i} \to Tf$ in L^p and hence for a finer subsequence $\widetilde{Tf_{n_{i_k}}}(\lambda,k) \to \widetilde{Tf}(\lambda,k)$ for every fixed $\lambda \in \mathfrak{a}^*$ for almost every $k \in K/M$. By definition, $\widetilde{Tf_{n_{i_k}}}(\lambda,k) = m(\lambda)\widetilde{f_{n_{i_k}}}(\lambda,k)$ for those $k \in K/M$, and thus for every fixed $\lambda \in \mathfrak{a}^*$, $\widetilde{Tf_{n_{i_k}}}(\lambda,k)$ converges to $m(\lambda)\tilde{f}(\lambda,k)$, for almost every $k \in K/M$. This establishes that for every fixed $\lambda \in \mathfrak{a}^*$, $\widetilde{Tf}(\lambda,k) = m(\lambda)\tilde{f}(\lambda,k)$, for almost every $k \in K/M$. We note that we have a set $B \subset K/M$ of full measure in K/M, such that for every fixed $k \in B$, both $\lambda \mapsto \tilde{f}(\lambda,k)$ and $\lambda \mapsto \widetilde{Tf}(\lambda,k)$ are holomorphic on Λ_q. Therefore, the equality $\widetilde{Tf}(\lambda,k) = m(\lambda)\tilde{f}(\lambda,k)$ extends to all $\lambda \in \Lambda_q$ and $k \in B$. □

We are now ready to show the sharpness of the range of p.

Proposition 6.0.5. *Fix $1 \leq p < 2$. Let $T\colon L^p(X) \to L^p(X)$ be a nontrivial L^p-multiplier. Then T is neither hypercyclic nor has it any periodic points.*

Proof. We recall that every φ_λ with $\lambda \in \Lambda_p^\circ$ is an eigenfunction of $T^*\colon L^{p'}(X) \to L^{p'}(X)$. (See Lemma 3.2.1(e) and Proposition 3.5.1.) Therefore, by Lemma 6.0.1(ii), T is not hypercyclic.

We suppose that the multiplier T is given by the function $m(\lambda)$. We fix a $q \in (p, 2)$. If for a nonzero function $g \in L^p(X)$, $T^n g = g$ for some $n \in \mathbb{N}$, $n > 0$, then by Lemmas 6.0.2 and 6.0.4, there exists a subset $B \subset K/M$ of full measure such that for each $k \in B$, $(m(\lambda)^n - 1)\tilde{g}(\lambda, k) = 0$ for $\lambda \in \Lambda_q$. Since $\tilde{g}(\lambda, k)$, for $k \in B$, is holomorphic on Λ_q it can be zero on a thin set which has $2n$-dimensional Lebesgue measure zero. Thus $m(\lambda)^n = 1$ on Λ_q, that is, $|m(\lambda)| = 1$. This is not possible as $m(\lambda)$ is holomorphic and hence an open map. □

Proposition 6.0.6. *Let $T\colon L^2(X) \to L^2(X)$ be a nontrivial L^2-multiplier. Then T is not hypercyclic and hence not chaotic.*

Proof. Let $m \in L^\infty(\mathfrak{a}_+^*)$ and the operator T is given by $\widetilde{Tf}(\lambda, k) = m(\lambda)\tilde{f}(\lambda, k)$. Then $\|T\|_{L^2 - L^2} = \|m\|_\infty$. We assume for the sake of meeting a contradiction that T is hypercyclic, equivalently there exists a hypercyclic vector $\phi \in L^2(X)$ for T. Then there exists a sequence $\{n_k\}$ of natural numbers such that $T^{n_k}\phi \to 2\phi$ in $L^2(X)$ as $n_k \to \infty$. For convenience, by abuse of notation we write n_k as n. We have, consequently, $\|T^n\phi\|_2 \to 2\|\phi\|_2$, that is,

$$\lim_{n\to\infty} \int_{\mathfrak{a}_+^* \times K/M} |m(\lambda)|^{2n} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk = 4 \int_{\mathfrak{a}_+^* \times K/M} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk.$$

We divide the integral on the left-hand side into three parts and apply the dominated convergence theorem to get

$$\begin{aligned} \lim_{n\to\infty} &\int_{\{\lambda \in \mathfrak{a}_+^* \mid |m(\lambda)| > 1\} \times K/M} |m(\lambda)|^{2n} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk \\ &\quad + \int_{\{\lambda \in \mathfrak{a}_+^* \mid |m(\lambda)| = 1\} \times K/M} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk \\ &= 4 \int_{\mathfrak{a}_+^* \times K/M} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} &\int_{\{\lambda \in \mathfrak{a}_+^* \mid |m(\lambda)| > 1\} \times K/M} |m(\lambda)|^{2n} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk \\ &\le 4 \int_{\mathfrak{a}_+^* \times K/M} |\tilde{\phi}(\lambda, k)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda\, dk. \end{aligned}$$

By the monotone convergence theorem the left-hand side goes to infinity while the right-hand side is finite. Hence either $\tilde{\phi} \equiv 0$ on $\{\lambda \in \mathfrak{a}_+^* \mid |m(\lambda)| > 1\} \times K/M$ or the set $\{\lambda \in \mathfrak{a}_+^* \mid |m(\lambda)| > 1\} \times K/M$ has measure zero in $\mathfrak{a}_+^* \times K/M$. By the Plancherel theorem, in the first case $\|T\phi\|_2 \leq \|\phi\|_2$, hence ϕ is not a hypercyclic vector and in the second T is a contraction. Both of these conclusions contradict our assumption. □

Remark 6.0.7. A few notes on L^2-multipliers and L^1-multipliers are given in order.

(1) Following [15, Thm. 1.2] one can give a different proof of the fact that an L^2-multiplier cannot be hypercyclic. This is based on the observation that T being a multiplier preserves the left-K-types of a function ϕ. Thus a mismatch between the K-types of the possible hypercyclic vector ϕ and the target function f, will prevent the sequence $T^n\phi$ converging to f in L^2.

(2) An L^2-multiplier can have periodic points. Indeed, there are nontrivial L^2-multipliers which have a dense set of periodic points. For instance, for a rank-one symmetric space X we define a multiplier T by the following prescription: $m(\lambda) = 1$ for $\lambda \in (0, 1)$ and $m(\lambda) = -1$ otherwise. Then for any $f \in L^2(X)$, $T^2 f = f$.

(3) It is easy to find nontrivial L^∞-multipliers T such that no constant multiple of T is chaotic. For instance, if $f \in L^1(X)$ is a K-biinvariant function, then $T\colon g \mapsto g * f$ cannot be hypercyclic on $L^\infty(X)$. Indeed, for any $\phi \in L^\infty(X)$, $\phi * f$ is continuous, hence $T^n\phi$ is a sequence of continuous functions. Since its uniform limit is a continuous function, it cannot converge to an arbitrary function in $L^\infty(X)$.

§7. Open questions

The results in this article trigger some questions, which we offer to the readers. For the sake of simplicity in this section we shall restrict to rank-one symmetric spaces where Λ_p defined in (3.1.4) takes the simpler form

$$\Lambda_p = \big\{\lambda \in \mathbb{C} \bigm| |\Im\lambda| \leq \gamma_p \rho\big\},$$

where ρ is interpreted as a positive number. However, the discussion here is equally valid for arbitrary rank.

(1) We choose p_1, p_2 such that $2 < p_2 < p_1 < \infty$. Then $\mathrm{CO}_{p_1}(X) \subset \mathrm{CO}_{p_2}(X)$. It is possible to construct a linear operator $T \in \mathrm{CO}_{p_1}(X) \subset \mathrm{CO}_{p_2}(X)$ which is chaotic on $L^{p_1}(X)$ but not chaotic on $L^{p_2}(X)$. For instance, we can take $T = e^{-t(\Delta - c)}$, where Δ is the positive Laplace–Beltrami operator, $t > 0$ and c is a

constant satisfying

$$\frac{4|\boldsymbol{\rho}|^2}{p_1p_1'} < c < \frac{4|\boldsymbol{\rho}|^2}{p_2p_2'}$$

for p_1, p_2 as above. Then T will be chaotic on $L^{p_1}(X)$ but not on $L^{p_2}(X)$. To see that T will be chaotic on $L^{p_1}(X)$ we first note that T is given by the symbol $m(\lambda) = e^{-t((\lambda,\lambda)+|\boldsymbol{\rho}|^2-c)}$, $\lambda \in \Lambda_{p_1}$ and that

$$\frac{4|\boldsymbol{\rho}|^2}{pp'} = ((i\gamma_p)^2 + 1)|\boldsymbol{\rho}|^2.$$

Writing $\lambda = u + iv$ where $|v| < \gamma_{p_1}|\boldsymbol{\rho}|$, we have $|m(\lambda)| = e^{-t(|u|^2-|v|^2+|\boldsymbol{\rho}|^2-c)}$. The given condition on c implies

$$\gamma_{p_1}|\boldsymbol{\rho}| > \sqrt{|\boldsymbol{\rho}|^2 - c} > \gamma_{p_2}|\boldsymbol{\rho}|.$$

Taking u sufficiently large we have $|m(\lambda)| < 1$ for the corresponding λ. On the other hand, choosing $u = 0$ and v in the range $\gamma_{p_1}|\boldsymbol{\rho}| > |v| > \sqrt{|\boldsymbol{\rho}|^2 - c}$, we get $|m(\lambda)| > 1$. The argument in the proof of Theorem 4.0.1 now shows that T is chaotic on $L^{p_1}(X)$. It is clear that such a choice is not possible for $L^{p_2}(X)$. The result in [15, Thm. 1.3] also shows that T is neither hypercyclic nor has it *any* periodic points in $L^{p_2}(X)$.

The operator $T = e^{-t(\Delta-c)}$ considered here is indeed a convolution operator by the K-invariant measure $e^{ct}h_t$ on X. We showed in Corollary 5.0.4 that whenever $T \in \mathrm{CO}_{p_1}(X) \subset \mathrm{CO}_{p_2}(X)$ is a convolution by a nonatomic K-invariant nonnegative measure μ on X, then T is chaotic on $L^{p_2}(X)$ implies that it is chaotic on $L^{p_1}(X)$. We are thus led to ask the following question: Let $T \in \mathrm{CO}_{p_1}(X) \subset \mathrm{CO}_{p_2}(X)$ where $2 < p_2 < p_1 < \infty$. Suppose that T is chaotic (respectively hypercyclic) on $L^{p_2}(X)$. Does it follow that T is chaotic (respectively hypercyclic) on $L^{p_1}(X)$?

(2) In Corollary 5.0.3 we showed that if $T\colon f \mapsto f * \mu$ is a convolution operator initially defined for $f \in C_c^\infty(X)$, where μ is a nonatomic K-invariant measure on X which satisfies $\hat{\mu}(-i\gamma_p\boldsymbol{\rho}) < \infty$, for some $p \in (2, \infty)$, then $T \in \mathrm{CO}_p(X)$ and it is either a contraction (when $\hat{\mu}(-i\gamma_p\boldsymbol{\rho}) \leq 1$) or it is chaotic (when $\hat{\mu}(i\gamma_p\boldsymbol{\rho}) > 1$). This motivates us to ask the following question: Let $T \in \mathrm{CO}_p(X)$ for some $2 < p < \infty$ be a nontrivial multiplier on $L^p(X)$ which is not a contraction. Is T chaotic?

A related question motivated by the same (i.e. convolution with nonatomic K-invariant measure on X) is the following: Let $T\colon L^p(X) \to L^p(X)$ for some $2 < p < \infty$ be an L^p-multiplier given by the symbol $m(\lambda)$. If $|m(\lambda)| \leq 1$ on Λ_p°, then is it true that T is not hypercyclic?

(3) Let $T \in \mathrm{CO}_{p_1}(X)$ be a nontrivial multiplier with symbol $m(\lambda)$ for some $2 < p_1 < \infty$. Then $T \in \mathrm{CO}_p(X)$ for all $p \in [2, p_1]$. We note that $|m(\lambda)|$ is

nonconstant on any open set of $\Lambda_{p_1}^\circ$. Therefore, for any $\delta > 0$ such that $2+\delta < p_1$, $|m(\lambda)|$ is nonconstant on $\Lambda_{2+\delta}$. The argument of the proof of Theorem 4.0.1 shows that zT is chaotic on $L^p(X)$ for any $p \in [2+\delta, p_1]$ if we can choose two elements $\lambda_1, \lambda_2 \in \Lambda_{2+\delta}^\circ$ such that $z \in \mathbb{C}$ satisfies $|m(\lambda_1)| < 1/|z| < |m(\lambda_2)|$. This argument however prevents us from making a uniform choice for the whole range $[2, p_1]$, which can be illustrated through the following example in a rank-one symmetric space X. We define a multiplier operator T by $m(\lambda) = e^{i/(4\boldsymbol{\rho}^2+\lambda^2)}$ for $\lambda \in \Lambda_{p_1}$. It can be verified that $T \in \mathrm{CO}_{p_1}(X)$ (see [1]). Since $|m(\lambda)| = 1$ on $\mathfrak{a}^* = \mathbb{R}$, we cannot choose λ_1, λ_2 from $\mathbb{R}$ satisfying $|m(\lambda_1)| < |m(\lambda_2)|$ and proceed as above. Thus the question remains whether it is possible to find a constant $c > 0$ such that for all $z \in \mathbb{C}$ with $|z| = c$, zT is chaotic on $L^p(X)$ for all $p \in [2, p_1]$.

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