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# The Equivalence of Pseudodifferential Operators and Their Symbols via Čech–Dolbeault Cohomology

by

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**Abstract**

In this paper we construct the sheaf morphism from the sheaf of pseudodifferential operators to its symbol class. Since it is hard to construct the morphism directly, we realize it with two original ideas as follows. Firstly, to calculate cohomologies we use the theory of Čech–Dolbeault cohomology introduced by Honda, Izawa and Suwa (J. Math. Soc. Japan 75 (2023), 229–290). Secondly, we construct a new symbol class, which is called the symbols of $C^\infty$-type. These ideas enable us to construct the sheaf morphism, which is actually an isomorphism of sheaves.

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## §1. Introduction

The theory of hyperfunctions was introduced in [12] and it enables us to conduct research into systems of differential equations from a completely new perspective. The essential idea of pseudodifferential operators was given in [12], and later, Kashiwara and Kawai provided the explicit definition in [8]. Some of the fundamental results on pseudodifferential operators are presented in [8, 9].

The class of pseudodifferential operators is a sufficiently large class of differential operators and it contains truly important differential operators such as the differential operators of fractional order and those of infinite order. In order to study differential operators of infinite order, Aoki and Kataoka started to study the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators.

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Since the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators is explicitly defined by using the sheaf cohomology, for the study of $\mathscr{E}_X^{\mathbb{R}}$ in the analytic category, Kataoka [11] introduced symbols of pseudodifferential operators with the aid of Radon transformations. Moreover, Aoki [1, 3] established the symbol theory $\mathfrak{S}/\mathfrak{N}$ of $\mathscr{E}_X^{\mathbb{R}}$ and developed the study of systems of differential equations of infinite order. However, two fundamental problems are unresolved in their symbol theory:

(1) The equivalence of the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators and its symbol class $\mathfrak{S}/\mathfrak{N}$ as sheaves.

(2) The commutativity of the composition of pseudodifferential operators and the product of symbols through a symbol map.

In his theory Aoki, calculated the cohomological expression of a stalk of $\mathscr{E}_X^{\mathbb{R}}$ by using the theory of Čech cohomology. In general we have to construct the Čech coverings which consist of Stein open sets. For a global case, however, such coverings are hard to find when we manipulate the cohomological expression of $\mathscr{E}_X^{\mathbb{R}}$.

In recent years the first problem has been solved by Aoki, Honda and Yamazaki [5]. They introduce a new space with one apparent parameter and construct the sheaf morphism on it. However, their construction is complicated and a more concise solution is desired as the foundation for symbol theory.

We give two aims of this paper as follows. The first aim is to construct the map from the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators to its symbol class in the global case. In sheaf theory it is essentially important to construct the existence of a morphism on open sets which form the basis of the total space. The second aim is to realize the formulation of the symbol theory in [5] in a precise and unified way. As noted above, their method is complicated and it is not easy to even understand the symbol map $\varpi\colon \mathscr{E}_X^{\mathbb{R}} \to \mathfrak{S}/\mathfrak{N}$.

To realize the symbol theory on general open sets, we apply the theory of Čech–Dolbeault cohomology to the symbol theory introduced by Aoki. Honda, Izawa and Suwa [6] find that the local cohomology groups with coefficients in the sheaf $\mathscr{O}$ of holomorphic functions is isomorphic to the cohomology group which is induced from double complex consisting of Čech coverings and the Dolbeault complex. As the theory of Čech–Dolbeault cohomology is based on $C^\infty$-forms we can use convenient techniques such as a partition of unity, controlling the support by cutoff functions, and so on.

As mentioned above, while we can apply useful techniques to $\mathscr{E}_X^{\mathbb{R}}$ via the Čech–Dolbeault cohomology, we have still some difficulties in constructing the morphism from Čech–Dolbeault cohomology of $\mathscr{E}_X^{\mathbb{R}}$ to the symbol class $\mathfrak{S}/\mathfrak{N}$ since the symbol class $\mathfrak{S}/\mathfrak{N}$ is based on the theory of holomorphic functions. To overcome this difficulty we introduce a new symbol class $\mathfrak{S}^\infty/\mathfrak{N}^\infty$, which consists of symbols of $C^\infty$-type. Finally, we can obtain the morphism between $\mathscr{E}_X^{\mathbb{R}}$ and $\mathfrak{S}/\mathfrak{N}$ with concrete integration cycles.

The plan of this paper is as follows. Section 2 provides some notation and definitions. In Section 3 we introduce the Čech–Dolbeault cohomology of the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators. Thanks to the study of Kashiwara and Schapira [10] it is known that the section $\mathscr{E}_X^{\mathbb{R}}(V)$ on an open cone $V$ is represented by the inductive limit of local cohomology groups. We apply the theory of Čech–Dolbeault cohomology to this cohomological expression. In Section 4 we define a new symbol class and prove that the new symbol class $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ is isomorphic to the classical symbol class $\mathfrak{S}/\mathfrak{N}$ which was introduced by Aoki. While the classical symbol theory is based on holomorphic functions, the Čech–Dolbeault cohomology is based on $C^\infty$-functions. Therefore it is hard to construct the map from Čech–Dolbeault cohomology to the classical symbol class. We realize the map via a new symbol class in the next section. In Section 5 we construct the morphism $\varsigma$ from $\mathscr{E}_X^{\mathbb{R}}$ to $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ by using the Čech–Dolbeault expression of $\mathscr{E}_X^{\mathbb{R}}$. We also give the well-definedness of $\varsigma$. In the appendix we prove the commutativity of the symbol map introduced by Aoki and the morphism constructed in Section 5. For this purpose we introduced the Čech–Dolbeault cohomology with general coverings.

## §2. Preliminaries

Through this paper we shall follow the notation and definitions introduced below.

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers, respectively.

### §2.1. Notation

Let $M$ be a real analytic manifold of dimension $n$ and $X$ a complexification of $M$. We always assume that all the manifolds are countable at infinity. Set the diagonal set

$$\Delta_X = \big\{(z,z') \in X \times X \;\big|\; z = z'\big\}.$$

We write $\Delta$ instead of $\Delta_X$ if there is no risk of confusion. One denotes by $p_1$ and $p_2$ the first and the second projections from $X \times X$ to $X$, respectively.

One denotes by $\tau\colon TX \to X$ the canonical projection from the tangent bundle to $X$ and $\pi\colon T^*X \to X$ that from the cotangent bundle to $X$.

Let $\omega$ be a $(p,q)$-form with coefficients in $C^\infty$-functions, and $\partial_z$ and $\bar\partial_z$ the Dolbeault operators with respect to the variable $z$, that is, for a local coordinate $z = (z_1, z_2, \dots, z_n)$, the form $\omega$ can be written as

$$\omega = \sum_{|I|=p,|J|=q} f_{IJ}(z)\, dz^I \wedge d\bar z^J.$$

Moreover, the Dolbeault operators are written as

$$\partial_z \omega = \sum_{i=1}^{n} \sum_{|I|=p,|J|=q} \frac{\partial}{\partial z_i} f_{IJ}(z) dz_i \wedge dz^I \wedge d\bar{z}^J,$$

$$\bar{\partial}_z \omega = \sum_{i=1}^{n} \sum_{|I|=p,|J|=q} \frac{\partial}{\partial \bar{z}_i} f_{IJ}(z) d\bar{z}_i \wedge dz^I \wedge d\bar{z}^J.$$

**Definition 2.1.** We define several sheaves:

(1) Let $\mathscr{O}_X^{(p)}$ be the sheaf of holomorphic $p$-forms on $X$. In particular, $\mathscr{O}_X^{(0)} = \mathscr{O}_X$ is the sheaf of holomorphic functions on $X$.
(2) We denote by $\mathrm{or}_X$ and $\mathrm{or}_{M/X} = \mathscr{H}_M^n(\mathbb{Z}_X)$ the orientation sheaf on $X$ and the relative orientation sheaf on $M$, respectively.
(3) Set $\Omega_X^{(n)} = \mathscr{O}_X^{(n)} \otimes_{\mathbb{C}_X} \mathrm{or}_X$ and $\mathscr{O}_{X\times X}^{(0,n)} = \mathscr{O}_{X\times X} \otimes_{p_2^{-1}\mathscr{O}_X} p_2^{-1}\Omega_X^{(n)}$.
(4) One denotes by $C_X^{\infty,(p,q)}$ the sheaf of $(p,q)$-forms with coefficients in $C^\infty$ on $X$.
(5) One denotes by $\mathscr{E}_X^{\mathbb{R}}$ the sheaf of pseudodifferential operators on $T^*X$.

Let $(z;\zeta)$ be a local coordinate of $T^*X$. Set $\mathring{T}^*X = T^*X \setminus T_X^*X$, where $T_X^*X$ is the zero section. We identify $T_\Delta^*(X\times X)$ with $T^*X$ by the map

$$(z, z; \zeta, -\zeta) \mapsto (z;\zeta), \tag{2.1}$$

which is induced from the first projection $p_1 \colon X\times X \to X$.

**Definition 2.2.** Let $V$ be a set in $\mathring{T}^*X$. The set $V$ is called a cone, or equivalently called a conic set in $\mathring{T}^*X$ if and only if

$$(z;\zeta) \in V \Rightarrow (z; t\zeta) \in V \quad \text{for any } t \in \mathbb{R}_+.$$

Here, $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$.

**Remark 2.3.** Let $V$ be a set in $T^*X$. We say that $V$ is convex (resp. conic, resp. proper) if for any $z \in \pi(V)$, the set $\pi^{-1}(z) \cap V$ is convex (resp. conic, resp. proper). Recall that a cone is said to be proper if its closure contains no lines. Moreover, similar properties are defined for a set in $TX$ in the same way.

Let $V$ and $V'$ be subsets in $T^*X$. We write $V' \Subset V$ if $V'$ is a relatively compact set in $V$ for the usual topology.

**Definition 2.4.** Let $V$ be an open cone in $\mathring{T}^*X$. A set $W \subset V$ is an infinitesimal wedge of type $V$ at infinity if for any $K \Subset V$ there exists $\delta > 0$ such that

$$K_\delta = \big\{(z; t\zeta) \bigm| (z;\zeta) \in K,\ t > \delta\big\} \subset W.$$

In what follows $W$ is called the infinitesimal wedge of type $V$ for short.

**Definition 2.5.** Let $Z$ be a closed cone in $\mathring{T}^*X$. We say that a closed set $W$ is a closed infinitesimal wedge of type $Z$ if there exists an open cone $V$ in $\mathring{T}^*X$ with $Z \subset V$ such that $W^\circ$ is an infinitesimal wedge of type $V$ at infinity. Here, $W^\circ$ means the interior of $W$.

**Definition 2.6.** Let $V$ and $V'$ be cones in $\mathring{T}^*X$ with $V' \subset V$. The cone $V'$ is a relatively compact cone in $V$ if there exists a compact set $K$ of $\mathrm{int}(V)$ such that

$$\overline{V'} = \big\{(z;t\zeta) \bigm| t \in \mathbb{R}_+,\ (z;\zeta) \in K\big\}.$$

To clarify the differences, one denotes $V' \underset{\text{cone}}{\Subset} V$ if $V'$ is a relatively compact cone in $V$, and we also say that $V'$ is properly contained in $V$.

**Remark 2.7.** Let $V$ and $V'$ be cones with $V' \underset{\text{cone}}{\Subset} V$. Then by the above definition, we also have $\pi(V') \Subset \pi(V)$.

Let $\gamma$ be a closed convex cone in $TX$. We review the $\gamma$-topology on $TX$.

**Definition 2.8.** The $\gamma$-topology on $TX$ is the topology for which the open sets $U$ satisfy

(1) $U$ is open for the usual topology.
(2) $U \mathbin{\mathring{+}} \gamma = U$.

Here, $\mathring{+}$ is defined by

$$U \mathbin{\mathring{+}} \gamma = \bigsqcup_{z \in \tau(U)} (U_z + \gamma_z),$$

where $U_z = U \cap \tau^{-1}(z)$ and $\gamma_z = \gamma \cap \tau^{-1}(z)$. In particular if $\gamma_z = \emptyset$, set $U_z + \gamma_z = U_z$. An open set $V$ of $TX$ is called $\gamma$-open if $V$ is open in the sense of $\gamma$-topology.

### §2.2. The property of $C^\infty$-smooth boundaries

We introduce the approximation by a set with $C^\infty$-smooth boundary. The properties introduced here are mainly used in Section 4 and in the appendix.

Let $X = E$ be an $n$-dimensional real vector space with the norm $|\bullet|$, and $E^*$ its dual vector space. Let $\langle \bullet, \bullet \rangle\colon E \times E^* \to \mathbb{R}$ be a non-degenerate pairing of $E$ and $E^*$. For a subset $K \subset E$ we define $K^\circ \subset E^*$ by

$$K^\circ = \big\{\xi \in E^* \bigm| \langle v, \xi \rangle > 0 \text{ for all } v \in K\big\}.$$

For $\delta > 0$ and a set $K \subset X$, we define

$$K_\delta = \big\{x \in X \bigm| \mathrm{dist}(x, K) < \delta\big\},$$

where $\mathrm{dist}(x, K)$ is the distance of $x \in X$ and $K$ given by

$$\mathrm{dist}(x, K) = \inf_{y \in K} |x - y|.$$

**Remark 2.9.** Note that we define "$\gamma$-topology" and "properly contained" in the same way as in the previous subsection, which is as follows:

- For a closed cone $\gamma \subset E$ a set $U \subset E$ is $\gamma$-open if

  (1) $U$ is open for the usual topology,
  (2) $U + \gamma = U$ in $E$.

- Let $V$ and $V'$ be cones in $E$ with $V' \subset V$. The cone $V'$ is a relatively compact cone in $V$ if there exists a compact set $K$ of $\mathrm{int}(V)$ such that

$$\overline{V'} \setminus \{0\} = \{tv \mid t \in \mathbb{R}_+,\ v \in K\}.$$

  We say that $V'$ is properly contained in $V$.

**Proposition 2.10.** Let $K$ be a closed convex subset in $X$ and $\delta > 0$. Then there exists an open convex subset $W$ with $C^\infty$-smooth boundary such that

$$K \subset W \subset K_\delta$$

holds.

*Proof.* Set

$$h(x) = \mathrm{dist}(x, \overline{K_{\delta/4}}).$$

Since $\overline{K_{\delta/4}}$ is also a convex set, $h(x)$ is a convex continuous function, that is, we have

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y) \quad (0 \leq t \leq 1,\ x, y \in X).$$

Let $\varphi(x)$ be a $C^\infty$-function on $X$ satisfying

(1) $0 \leq \varphi(x) \leq 1$ and $\int_X \varphi(x)\, dx = 1$,
(2) $\mathrm{supp}(\varphi) \subset \{x \in X \mid |x| \leq 1\}$.

Set, for $\varepsilon > 0$,

$$h_\varepsilon(x) = \varepsilon^{-n} \int_X h(x - y)\varphi(y/\varepsilon)\, dy.$$

Then we can easily confirm the following properties:

(1) $K \subset \{x \in X \mid h_\varepsilon(x) = 0\}$ for $0 < \varepsilon < \delta/4$.
(2) $h_\varepsilon(x)$ is a convex $C^\infty$-function.
(3) Since $|h(x) - h(y)| \leq |x - y|$ holds for any $x, y \in X$, we have

$$h_\varepsilon \to h \quad (\varepsilon \to 0 + 0)$$

uniformly on $X$.

Due to the above property (3), we can choose $\delta/4 > \varepsilon_0 > 0$ such that

$$|h_{\varepsilon_0}(x) - h(x)| < \delta/4 \quad (x \in X)$$

holds. Hence we get for any $0 < s < \delta/4$,

$$K \subset \{x \in X \mid h_{\varepsilon_0} < s\} \subset K_{\delta/2}.$$

By the Sard theorem, the set of critical values of $h_{\varepsilon_0} \colon X \to \mathbb{R}$ is of measure zero. Thus we can choose $0 < s < \delta/4$ which is not a critical value of $h_{\varepsilon_0}$ and for such an $s$ we set

$$W = \{x \in X \mid h_{\varepsilon_0}(x) < s\}.$$

The set $W$ satisfies all the required properties. □

**Corollary 2.11.** Let $K$ be a proper convex closed cone and $V$ a conic open neighborhood of $K \setminus \{0\}$. Then there exists a proper open cone $W$ with $C^\infty$-smooth boundary except for the vertex such that

$$K \setminus \{0\} \subset W \subset V.$$

*Proof.* Take a non-zero vector $v_0 \in \operatorname{int}(K^\circ)$. Set, for some $\delta > 0$,

$$H = \{x \in X \mid \langle x, v_0\rangle = \delta\}$$

and

$$\widehat{K} = K \cap H, \quad \widehat{V} = V \cap H.$$

By Proposition 2.10, in $H$, we can find a convex open subset $\widehat{W} \subset H$ with $C^\infty$-smooth boundary such that

$$\widehat{K} \subset \widehat{W} \subset \widehat{V}.$$

Then

$$W = \{rv \mid v \in \widehat{W}, r > 0\}$$

satisfies the required conditions. □

**Corollary 2.12.** Let $G$ be a proper convex closed cone in $E$ and $S$ a convex $G$-open subset in $E$. For any $\delta > 0$ there exists a convex $G$-open subset $W$ with $C^\infty$-smooth boundary such that

$$\overline{S} \subset W \subset S_\delta.$$

*Proof.* It is enough to show that $W$ constructed in the proof of Proposition 2.10 is also $G$-open. Take a vector $v \in G$. Since $S_{\delta/4}$ is still $G$-open we have

$$h(x + y + v) \leq h(x + y).$$

Hence $W + G \subset W$ holds. □

We mention the properties of piecewise $C^\infty$-smooth boundaries. Let $L$ be a closed subset in $X$.

**Definition 2.13.** We say that $L$ has piecewise $C^\infty$-smooth boundary if and only if for any $x \in \partial L$ there exist an open neighborhood $U$ of $x$ and $C^\infty$-functions $f_1, \dots, f_\ell$ on $U$ satisfying the conditions below:

(1) We have
$$L \cap U = \{x \in U \mid f_k(x) *_k 0 \ (k = 1, 2, \dots, \ell)\},$$
where $*_k$ denotes either $=$ or $\geq$ $(k = 1, 2, \dots, \ell)$.

(2) $df_1 \wedge \cdots \wedge df_\ell \neq 0$ at any point in $U$.

Hereafter $f_1, \dots, f_\ell$ are said to be defining functions of $L$ at $x$.

Let $L_1$ and $L_2$ be closed subsets in $X$ with piecewise $C^\infty$-smooth boundaries.

**Definition 2.14.** We say that $L_1$ and $L_2$ intersect transversally if, for any $x \in \partial L_1 \cap \partial L_2$, there exist the defining functions $f_1, \dots, f_\ell$ (resp. $g_1, \dots, g_m$) of $L_1$ (resp. $L_2$) at $x$ such that
$$df_1 \wedge \cdots \wedge df_\ell \wedge dg_1 \wedge \cdots \wedge dg_m \neq 0.$$

**Remark 2.15.** We also define open subsets with piecewise $C^\infty$-smooth boundaries and their transversal intersection in the same way as those for closed subsets.

**Proposition 2.16.** Let $V$ and $W$ be non-empty proper convex open cones in $X$ with $C^\infty$-smooth boundaries except for the origin. Assume that $W$ is properly contained in $V$. Let $w$ be a non-zero vector in $W$ and set, for $t > 0$,
$$V_t = tw + V.$$
Then we have

(1) $W \setminus V_t$ is relatively compact. Furthermore, for any open neighborhood $U$ of the origin, we have $\overline{W \setminus V_t} \subset U$ if $t > 0$ is sufficiently small.

(2) $\partial W$ and $\partial V_t$ transversally intersect.

*Proof.* Take a non-zero vector $\zeta \in \operatorname{int}(V^\circ)$. We may assume $\langle w, \zeta\rangle = 1$. For $s > 0$ we set
$$H_s = \{x \in X \mid \langle x, \zeta\rangle = s\}.$$
We first show $H_s \cap V$ and $H_s \cap W$ are bounded. If $H_s \cap V$ is unbounded, then we can find $\{x_k\}$ such that $x_k \in H_s \cap V$ and $|x_k| \to \infty$. Then
$$\langle x_k/|x_k|, \zeta\rangle = s/|x_k| \to 0.$$

We may assume that $x_k/|x_k| \to x_0 \neq 0$ $(k \to \infty)$. Therefore we have $x_0 \in \overline{V}$ and

$$\langle x_0, \zeta\rangle = 0,$$

which contradicts $\zeta \in \operatorname{int}(V^\circ)$. Hence both the sets are bounded.

Set

$$\delta = \operatorname{dist}(H_1 \setminus (H_1 \cap V), H_1 \cap W).$$

Note that we have $\delta > 0$ because $\overline{H_1 \cap W} \subset H_1 \cap V$. Since $V$ and $W$ are conic, we have

$$\operatorname{dist}(H_s \setminus (H_s \cap V), H_s \cap W) \geq s\delta \quad (s > 0).$$

By noticing that $(H_t \cap W) - tw = H_0 \cap (W - tw)$, we set

$$M = \sup_{x \in (H_0 \cap (W - tw))} |x|.$$

Then it is easy to see that if $s > t + M/\delta$ we get

$$H_s \cap W \subset H_s \cap V_t.$$

Note that $M \to 0$ if $t \to 0 + 0$. Hence claim (1) follows.

Next let us show claim (2). We denote by $p_t$ the vertex of $V_t$ (i.e., $p_t = wt$). It is clear that the tangent space $(T\partial V_t)_q$ of $\partial V_t$ at $q$ contains the vector $q - p_t$. Let $(T\partial W)_q$ be the tangent space of $\partial W$ at $q$. Since $W$ is convex, $W$ and the tangent hypersurface $q + (T\partial W)_q$ of $\partial W$ at $q$ do not intersect. Hence it follows from $p_t \in W$ that the vector $q - p_t$ does not belong to $(T\partial W)_q$. Therefore we have $(T\partial W)_q \neq (T\partial V_t)_q$, which concludes that $W$ and $V_t$ transversally intersect at $q$. □

## §3. The sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators and its Čech–Dolbeault expression

First of all we briefly recall the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators. Let $X$ be a complex manifold of dimension $n$. The sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators on $T^*X$ is defined by

$$\mathscr{E}_X^{\mathbb{R}} = H^n(\mu_\Delta(\mathscr{O}_{X\times X}^{(0,n)})), \tag{3.1}$$

where $\mu_\Delta(\mathscr{O}_{X\times X}^{(0,n)})$ is the microlocalization of $\mathscr{O}_{X\times X}^{(0,n)}$ along the diagonal set $\Delta$. One denotes by $\mathscr{E}_{X,z^*}^{\mathbb{R}}$ the stalk of $\mathscr{E}_X^{\mathbb{R}}$ at a point $z^* \in T^*X$.

Let $V$ be a subset of $T^*X$. We denote by $V^\circ$ the polar set of $V$, that is, $V^\circ$ is defined by

$$V^\circ = \{y \in TX \mid \tau(y) \in \pi(V) \text{ and } \operatorname{Re}\langle x, y\rangle \geq 0 \text{ for all } x \in \pi^{-1}(\tau(y)) \cap V\}.$$

Then the following theorem is essential.

**Theorem 3.1** ([10, Thm. 4.3.2(ii)]). Let $V$ be an open convex cone in $T^*X$. We have

$$\mathscr{E}_X^{\mathbb{R}}(V) = \varinjlim_{U,G} H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X}), \tag{3.2}$$

where $U$ ranges through the family of open subsets of $X \times X$ such that $U \cap \Delta = \pi(V)$ and $G$ through the family of closed subsets of $X \times X$ such that $C_\Delta(G) \subset V^\circ$.

Here, the set $C_\Delta(G)$ is the normal cone to $G$ along $\Delta$. (See [10, Def. 4.4.1].)

Next we recall the Čech–Dolbeault cohomology introduced by Suwa [6, 13]. Let $M$ be a closed subset of $X$, $V_0 = X \setminus M$ and $V_1$ an open neighborhood of $M$ in $X$. For a covering $\mathcal{V} = \{V_0, V_1\}$ of $X$ we set

$$C_X^{\infty,(p,q)}(\mathcal{V}) = C_X^{\infty,(p,q)}(V_0) \oplus C_X^{\infty,(p,q)}(V_1) \oplus C_X^{\infty,(p,q-1)}(V_{01}), \tag{3.3}$$

where $V_{01} = V_0 \cap V_1$. We also set the differential $\bar{\vartheta}\colon C_X^{\infty,(p,q)} \to C_X^{\infty,(p,q+1)}$ as

$$\bar{\vartheta}(\omega_0, \omega_1, \omega_{01}) = (\bar{\partial}\omega_0, \bar{\partial}\omega_1, \omega_1 - \omega_0 - \bar{\partial}\omega_{01}). \tag{3.4}$$

Then $\bar{\vartheta} \circ \bar{\vartheta} = 0$ is easily shown and the pair $(C_X^{\infty,(p,\bullet)}(\mathcal{V}), \bar{\vartheta})$ is a complex.

**Definition 3.2.** The Čech–Dolbeault cohomology $H^{p,q}_{\bar{\vartheta}}(\mathcal{V})$ of $\mathcal{V}$ of type $(p,q)$ is the $q$th cohomology of the complex $(C_X^{\infty,(p,\bullet)}(\mathcal{V}), \bar{\vartheta})$.

Next we consider the subcomplex of $(C_X^{\infty,(p,\bullet)}(\mathcal{V}), \bar{\vartheta})$ defined below. Let $\mathcal{V}' = \{V_0\}$ be a covering of $X \setminus M$. We set

$$\begin{aligned} C_X^{\infty,(p,q)}(\mathcal{V}, \mathcal{V}') &= \big\{(\omega_0, \omega_1, \omega_{01}) \in C_X^{\infty,(p,q)}(\mathcal{V}) \bigm| \omega_0 = 0\big\} \\ &= C_X^{\infty,(p,q)}(V_1) \oplus C_X^{\infty,(p,q)}(V_{01}). \end{aligned}$$

Then the pair $(C_X^{\infty,(p,\bullet)}(\mathcal{V}, \mathcal{V}'), \bar{\vartheta})$ is a subcomplex of $(C_X^{\infty,(p,\bullet)}(\mathcal{V}), \bar{\vartheta})$.

**Definition 3.3.** The Čech–Dolbeault cohomology $H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ is the $q$th cohomology of the complex $(C_X^{\infty,(p,\bullet)}(\mathcal{V}, \mathcal{V}'), \bar{\vartheta})$.

We have the following proposition.

**Proposition 3.4** ([6, Prop. 4.6]). The Čech–Dolbeault cohomology $H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ is independent of the choice of $V_1$ and determined uniquely up to isomorphism.

Therefore we can choose $X$ as $V_1$, and hereafter $H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ is also denoted by $H^{p,q}_{\bar{\vartheta}}(X, X \setminus M)$.

**Theorem 3.5** ([6, Thm. 4.9]). There is a canonical isomorphism

$$H^{p,q}_{\bar\vartheta}(X, X \setminus S) \simeq H^q_S(X; \mathscr{O}^{(p)}_X). \tag{3.5}$$

Applying Theorem 3.5 to the cohomology $H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X})$ in (3.2) we get the Čech–Dolbeault expression of $\mathscr{E}^{\mathbb{R}}_X$.

**Definition 3.6.** The sheaf $C^{\infty,(p,q;r)}_{X\times X}$ is the sheaf of $(p+q, r)$-forms with coefficients in $C^\infty$-functions which are holomorphic $p$-forms with respect to the first variable, holomorphic $q$-forms with respect to the second variable and antiholomorphic $r$-forms with respect to the first and the second variables. In other words, for a local coordinate $(z_1, z_2)$ of $X \times X$ and for an open subset $V$ of $X \times X$, a form $f(z_1, z_2) \in C^{\infty,(p,q;r)}_{X\times X}(V)$ is written as

$$f(z_1, z_2) = \sum_{|I|=p,|J|=q,|K|=r} f_{IJK}(z_1, z_2)\, dz_1^I \wedge dz_2^J \wedge d\bar z^K,$$

where each $f_{IJK}(z_1, z_2)$ is a $C^\infty$-function on $V$.

Set $V_0 = U\setminus G$, $V_1 = U$ and $V_{01} = V_0 \cap V_1 = U \setminus G$. For coverings $\mathcal{V} = \{V_0, V_1\}$ of $U$ and $\mathcal{V}' = \{V_0\}$ of $U \setminus G$, we define

$$C^{\infty,(p,q;r)}_{X\times X}(\mathcal{V}, \mathcal{V}') = C^{\infty,(p,q;r)}_{X\times X}(V_1) \oplus C^{\infty,(p,q;r-1)}_{X\times X}(V_{01}).$$

The differential $\bar\vartheta\colon C^{\infty,(p,q;r)}_{X\times X}(\mathcal{V}, \mathcal{V}') \to C^{\infty,(p,q;r+1)}_{X\times X}(\mathcal{V}, \mathcal{V}')$ is also given as usual, and the pair $(C^{\infty,(p,q;\bullet)}_{X\times X}(\mathcal{V}, \mathcal{V}'), \bar\vartheta)$ is a complex.

**Definition 3.7.** The $r$th Čech–Dolbeault cohomology $H^{p,q,r}_{\bar\vartheta}(\mathcal{V}, \mathcal{V}')$ is the $r$th cohomology of the complex $(C^{\infty,(p,q;\bullet)}_{X\times X}(\mathcal{V}, \mathcal{V}'), \bar\vartheta)$.

Thanks to Proposition 3.4 and Theorem 3.5 we have the following.

**Theorem 3.8.** There is a canonical isomorphism

$$H^{0,n,n}_{\bar\vartheta}(U, U \setminus G) \simeq H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X}). \tag{3.6}$$

Thus the section of $\mathscr{E}^{\mathbb{R}}_X$ on an open convex cone $V$ is expressed by

$$\mathscr{E}^{\mathbb{R}}_X(V) = \varinjlim_{U,G} H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X}) = \varinjlim_{U,G} H^{0,n,n}_{\bar\vartheta}(U, U \setminus G),$$

where $U$ and $G$ run through the same sets as those in Theorem 3.1.

## §4. Two symbol classes

While the classical symbol theory $\mathfrak{S}/\mathfrak{N}$ of $\mathscr{E}_X^{\mathbb{R}}$ is based on holomorphic functions, the Čech–Dolbeault expression of $\mathscr{E}_X^{\mathbb{R}}$ is based on $C^\infty$-functions, and hence it is difficult to construct the map from the Čech–Dolbeault expression to the classical symbol class directly. In this section we construct a new symbol class which is of $C^\infty$-type and show that the new symbol class is isomorphic to the classical symbol class.

### §4.1. Sheaves and conic sheaves

First we introduce the relation between conic sheaves on $T^*X$ and sheaves on $\widehat{T}^*X$, which is the radial compactification of $T^*X$.

**Definition 4.1.** One defines the radial compactification $\mathbb{D}_{\mathbb{C}^n}$ of $\mathbb{C}^n$ by

$$\mathbb{D}_{\mathbb{C}^n} = \mathbb{C}^n \sqcup S^{2n-1}\infty.$$

We show the fundamental system of neighborhoods. If $z_0$ belongs to $\mathbb{C}^n$ a family of fundamental neighborhoods of $z_0$ consists of open sets

$$B_\varepsilon(z_0) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z - z_0| < \varepsilon \right\}$$

for $\varepsilon > 0$, otherwise that of $z_0\infty \in S^{2n-1}\infty$ consists of open sets

$$G_r(\Gamma) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z| > r,\ \tfrac{z}{|z|} \in \Gamma \right\} \sqcup \Gamma,$$

where $r > 0$ and $\Gamma$ is an open neighborhood of $z_0\infty$ in $S^{2n-1}\infty$ (cf. Figure 1).

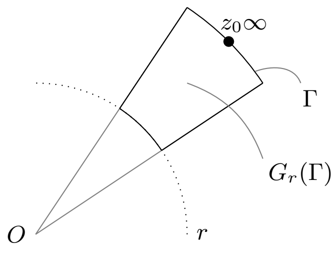



Figure 1. $G_r(\Gamma)$

**Definition 4.2.** Let $V$ be an open set in $\mathbb{C}^n$. We define the set $\widehat{V}$ in $\mathbb{D}_{\mathbb{C}^n}$ by

$$\widehat{V} = \mathbb{D}_{\mathbb{C}^n} \setminus (\overline{\mathbb{C}^n \setminus V}).$$

Here the closure $\overline{\bullet}$ is taken in $\mathbb{D}_{\mathbb{C}^n}$.

Note that we sometimes write $\widehat{\ }V$ instead of $\widehat{V}$.

**Definition 4.3.** The radial compactification $\widehat{T}^*X$ of $T^*X$ is

$$\widehat{T}^*X = \bigsqcup_{z \in X} \overline{T_z^*X}.$$

Here, $\overline{T_z^*X} \simeq \overline{\mathbb{C}_\zeta^n} = \mathbb{C}_\zeta^n \sqcup S^{2n-1}\infty$.

**Remark 4.4.** Let $V$ be an open set in $\mathring{T}^*X$. We also define the $\widehat{V}$ in the same way as that in $\mathbb{D}_{\mathbb{C}^n}$, i.e.,

$$\widehat{V} = \widehat{T}^*X \setminus (\overline{T^*X \setminus V}).$$

The topology of $\widehat{T}^*X$ is induced from that of $\mathbb{D}_{\mathbb{C}^n}$. We introduce the functor $\Psi$ from the category of sheaves on $\widehat{T}^*X$ to the one of conic sheaves on $\mathring{T}^*X$ as follows. For a sheaf $\mathscr{F}$ on $\widehat{T}^*X$, the conic sheaf $\Psi(\mathscr{F})$ on $\mathring{T}^*X$, for an open conic set $V$, is given by

$$\Psi(\mathscr{F})(V) = \varinjlim_{W} \mathscr{F}(\widehat{W}),$$

where $W$ ranges through the family of infinitesimal wedges of type $V$ at infinity.

**Remark 4.5.** We can naturally extend the conic sheaves $\Psi(\mathscr{F})$ on $\mathring{T}^*X$ to the one on $T^*X$ in the following way:

For an open set $V$ in $T^*X$ we set

$$\Psi(\mathscr{F})(V) = \varinjlim_{W} \mathscr{F}(\widehat{W} \cup \pi^{-1}(V \cap T_X^*X)),$$

where $W$ ranges through the family of infinitesimal wedges of type $V$ at infinity.

Then we have the following lemmas.

**Lemma 4.6.** The functor $\Psi$ is exact.

*Proof.* For $p = (z;\zeta) \in T^*X$ with $\zeta \neq 0$ we have

$$\Psi(\mathscr{F})_p = \mathscr{F}_{p\infty},$$

where $p\infty = \overline{\mathbb{R}_+ p} \cap (\widehat{T}^*X \setminus T^*X)$. □

Let $\mathscr{G}$ be a conic sheaf on $\mathring{T}^*X$. We say that $\mathscr{G}$ is conically soft if for any closed conic set $Z$, the restriction $\mathscr{G}(\mathring{T}^*X) \to \mathscr{G}(Z)$ is surjective. By the definition of the functor $\Psi$ we have the following lemmas.

**Lemma 4.7.** Let $\mathscr{F}$ be a soft sheaf on $\widehat{T}^*X$. Then $\Psi(\mathscr{F})$ is conically soft.

**Lemma 4.8.** Let $\mathscr{G}$ be a conically soft sheaf on $\mathring{T}^*X$ and $V$ be an open conic set in $\mathring{T}^*X$. Then we have

$$H^k(V;\mathscr{G}) = 0 \quad (k \neq 0).$$

**Proposition 4.9.** Let $\mathscr{F}$ be a sheaf on $\widehat{T}^*X$ and $V$ an open conic set in $\mathring{T}^*X$. Then we have

$$H^k(\Psi(\mathscr{F}); V) = \varinjlim_{W} H^k(\widehat{W}; \mathscr{F}),$$

where $W$ ranges through the family of infinitesimal wedges of type $V$.

*Proof.* Take a soft resolution of $\mathscr{F}$ on $\widehat{T}^*X$,

$$0 \to \mathscr{F} \to \mathscr{L}_1 \to \mathscr{L}_2 \to \cdots .$$

Since $\Psi$ is exact, we have the resolution of $\Psi(\mathscr{F})$,

$$0 \to \Psi(\mathscr{F}) \to \Psi(\mathscr{L}_1) \to \Psi(\mathscr{L}_2) \to \cdots ,$$

and we can compute $H^k(V;\Psi(\mathscr{F}))$ using this resolution. By the definition of $\Psi$ we have

$$\Gamma(V;\Psi(\mathscr{F})) = \varinjlim_{W} \Gamma(\widehat{W};\mathscr{F}).$$

Since the inductive limit and the functor $H^k(\bullet)$ commute, we obtain the conclusion. □

## §4.2. The sheaf $\mathfrak{S}/\mathfrak{N}$ of classical symbols

Let us review the classical symbol theory, which is based on the theory of Aoki [1, 3]. Let $z^* = (z;\zeta)$ be a local coordinate of $T^*X$. We construct two conic sheaves $\mathfrak{S}$ and $\mathfrak{N}$ on $\mathring{T}^*X$.

**Definition 4.10.** Let $V$ be an open cone in $\mathring{T}^*X$.

(1) A function $f(z,\zeta)$ is called a symbol on $V$ if the following conditions hold:

 (i) There exists an infinitesimal wedge $W$ of type $V$ such that

$$f(z,\zeta) \in \mathscr{O}_{T^*X}(W).$$

 (ii) For any open cone $V' \underset{\text{cone}}{\Subset} V$ there exists an infinitesimal wedge $W' \subset W$ of type $V'$ such that $f(z,\zeta)$ satisfies the following condition:
 For any constant $h > 0$, there exists a constant $C > 0$ such that

$$|f(z,\zeta)| \leq C \cdot e^{h|\zeta|} \quad \text{on } W'. \tag{4.1}$$

(2) A symbol $f(z,\zeta)$ on $V$ is called a null symbol if for any open cone $V' \underset{\text{cone}}{\Subset} V$ there exist an infinitesimal wedge $W' \subset W$ of type $V'$ and constants $h > 0$ and $C > 0$ such that

$$|f(z,\zeta)| \leq C \cdot e^{-h|\zeta|} \quad \text{on } W'. \tag{4.2}$$

(3) We denote by $\mathfrak{S}(V)$ and $\mathfrak{N}(V)$ the set of all the symbols on $V$ and the set of all the null symbols on $V$, respectively. Moreover, we set

$$\mathfrak{S}_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{S}(V), \quad \mathfrak{N}_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{N}(V),$$

where $V$ runs through the family of open conic neighborhoods of $z^* \in \mathring{T}^*X$.

We can naturally extend the sheaves $\mathfrak{S}$ and $\mathfrak{N}$ to the sheaves on $T^*X$. Define the sheaves $\mathfrak{S}|_{T^*_X X}$ and $\mathfrak{N}|_{T^*_X X}$ on the zero section $T^*_X X$ as follows.

(1) Let $U$ be an open set in $X$. The section $\mathfrak{S}|_{T^*_X X}(U)$ is a family of $f(z,\zeta) \in \mathscr{O}_{T^*X}(\pi^{-1}(U))$ which satisfies the condition below:

For any compact set $K \Subset U$ and for any constant $h > 0$ there exists a constant $C > 0$ such that

$$|f(z,\zeta)| \leq C \cdot e^{h|\zeta|} \quad \text{on } \pi^{-1}(K).$$

(2) Set $\mathfrak{N}|_{T^*_X X} = 0$.

Then the sheaves $\mathfrak{S}$ and $\mathfrak{N}$ become those on $T^*X$.

Next we construct the quotient sheaf $\mathfrak{S}/\mathfrak{N}$.

**Proposition 4.11.** Let $V$ be an open cone in $\mathring{T}^*X$. The section $\mathfrak{N}(V)$ is an ideal of $\mathfrak{S}(V)$.

*Proof.* Let $f(z,\zeta) \in \mathfrak{N}(V)$ and $g(z,\zeta) \in \mathfrak{S}(V)$. Then there exists an infinitesimal wedge $W$ of type $V$ such that $f(z,\zeta)$ and $g(z,\zeta)$ are holomorphic on $W$. By the definition of $\mathfrak{N}$ for any $V' \underset{\text{cone}}{\Subset} V$ we can find an infinitesimal wedge $W'$ of type $V'$ and the constants $h > 0$ and $C > 0$ such that

$$|f(z,\zeta)| \leq C \cdot e^{-h|\zeta|}.$$

Similarly for $V'$, $W'$ and $h > 0$ which are the same ones as above, we can find a constant $C' > 0$ such that

$$|g(z,\zeta)| \leq C' \cdot e^{\frac{1}{2}h|\zeta|}.$$

Hence we obtain

$$|f(z,\zeta) \cdot g(z,\zeta)| \leq C \cdot e^{-h|\zeta|} \cdot C' \cdot e^{\frac{1}{2}h|\zeta|} \leq CC' \cdot e^{-\frac{1}{2}h|\zeta|}. \qquad \square$$

One denotes by $(\mathfrak{S}/\mathfrak{N})^{\#P}$ the presheaf defined by the correspondence for an open cone $V$ in $\mathring{T}^*X$,

$$V \mapsto \mathfrak{S}(V)/\mathfrak{N}(V).$$

Let $\mathfrak{S}/\mathfrak{N}$ be the associated sheaf to $(\mathfrak{S}/\mathfrak{N})^{\#P}$. We have the following exact sequence of sheaves

$$0 \to \mathfrak{N} \to \mathfrak{S} \xrightarrow{\kappa_1} \mathfrak{S}/\mathfrak{N} \to 0. \tag{4.3}$$

Here $\kappa_1$ is the composition of the canonical morphisms $\mathfrak{S} \to (\mathfrak{S}/\mathfrak{N})^{\#P} \to \mathfrak{S}/\mathfrak{N}$, and (4.3) induces the long exact sequence

$$0 \to \mathfrak{N}(V) \to \mathfrak{S}(V) \to \mathfrak{S}/\mathfrak{N}(V) \to H^1(V;\mathfrak{N}) \to \cdots .$$

To treat $\mathfrak{S}/\mathfrak{N}(V)$ as it is a quotient group $\mathfrak{S}(V)/\mathfrak{N}(V)$, we claim $H^1(V;\mathfrak{N}) = 0$ for a suitable $V$.

**Theorem 4.12.** Assume $X$ to be a complex vector space and let $Z$ be a closed cone in $\mathring{T}^*X$. Moreover, assume that $Z$ satisfies the following conditions (C1), (C2) and (C3).

(C1) A family of open conic neighborhoods of $Z$ has a cofinal family which consists of Stein open cones in $\mathring{T}^*X$.

(C2) The projection $\pi(Z)$ is a compact set in $X$.

(C3) There exists $\zeta_0 \in \mathbb{C}^n \setminus \{0\}$ such that

$$Z \subset \big\{(z;\zeta) \in \mathring{T}^*X \;\big|\; z \in \pi(Z),\ \mathrm{Re}\langle \zeta, \zeta_0\rangle > 0\big\}.$$

Then $H^k(Z;\mathfrak{N}) = 0$ holds for any $k > 0$.

The conditions (C1), (C2) and (C3) are collectively called condition C.

**Example 4.13.** We can construct a closed cone $V$ satisfying the above three conditions as follows. Let $N$ be a natural number and $f_1(z), \dots, f_N(z)$ holomorphic functions on $X$. Set

$$B = \bigcap_{i=1}^{N} \{|f_i(z)| \le 1\},$$

and assume $B$ to be compact, and let $\Gamma$ be a closed proper convex cone. Then $V = B \times \Gamma$ satisfies the second and the third conditions in Theorem 4.12. A cofinal family of $B \times \Gamma$ is given in the following way. We can take a family $\{B_\varepsilon\}_{\varepsilon \in \mathbb{R}_+}$ of open neighborhoods of $B$ as follows:

$$B_\varepsilon = \bigcap_{1 \le i \le N} \{|f_i(z)| < 1 + \varepsilon\}.$$

Since $\Gamma$ is a closed proper convex cone we can take a cofinal family $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ which consists of open convex conic neighborhoods of $\Gamma$. Then the family $\{B_\varepsilon \times \Gamma_\lambda\}_{(\varepsilon,\lambda)\in\mathbb{R}_+\times\Lambda}$ is what we want.

To prove Theorem 4.12 we apply the results in the previous subsection to $\mathfrak{N}$.

**Definition 4.14.** We introduce several sheaves which are related to the sheaf $\mathfrak{N}$:

(1) Let $\tilde{L}_{2,\mathrm{loc}}$ be the sheaf of rapidly decreasing locally $L^2$-functions. That is, for an open set $\widetilde{U}$ in $\widehat{T}^*X$, a function $f(z,\zeta)$ belongs to $\tilde{L}_{2,\mathrm{loc}}(\widetilde{U})$ if $f(z,\zeta) \in L_{2,\mathrm{loc}}(\widetilde{U}\cap T^*X)$ and for any compact set $K$ in $\widetilde{U}$ there exists a constant $h>0$ such that
$$f(z,\zeta)\cdot e^{h|\zeta|} \in L^2(K\cap T^*X).$$
(2) Let $\tilde{L}^{(p,q)}_{2,\mathrm{loc}}$ be the sheaf of $(p,q)$-forms with coefficients in $\tilde{L}_{2,\mathrm{loc}}$.
(3) The sheaf $\tilde{\mathscr{L}}^{(p,q)}_{2,\mathrm{loc}}$ is the subsheaf of $\tilde{L}^{(p,q)}_{2,\mathrm{loc}}$ defined below:
For an open set $\widetilde{U}$ in $\widehat{T}^*X$ a $(p,q)$-form $f\in \tilde{L}^{(p,q)}_{2,\mathrm{loc}}(\widetilde{U})$ belongs to $\tilde{\mathscr{L}}^{(p,q)}_{2,\mathrm{loc}}(\widetilde{U})$ if $\bar{\partial} f(z,\zeta)\in \tilde{L}^{(p,q+1)}_{2,\mathrm{loc}}(\widetilde{U})$.
(4) The sheaf $\widetilde{\mathfrak{N}}$ is the sheaf of holomorphic functions of exponential decay on $\widehat{T}^*X$. That is, for an open set $\widetilde{U}$ in $\widehat{T}^*X$, a function $f(z,\zeta)$ belongs to $\widetilde{\mathfrak{N}}(\widetilde{U})$ if $f(z,\zeta)\in \mathscr{O}_{T^*X}(\widetilde{U}\cap T^*X)$ and for any compact set $K$ in $\widetilde{U}$ there exist constants $C>0$ and $h>0$ such that
$$|f(z,\zeta)| \le C\cdot e^{-h|\zeta|} \quad \text{on } K\cap T^*X.$$

Note that sheaves $\tilde{L}_{2,\mathrm{loc}}$, $\tilde{L}^{(p,q)}_{2,\mathrm{loc}}$ and $\tilde{\mathscr{L}}^{(p,q)}_{2,\mathrm{loc}}$ are soft on $\widehat{T}^*X$.

**Lemma 4.15.** We have $\Psi(\widetilde{\mathfrak{N}}) = \mathfrak{N}$ on $\mathring{T}^*X$.

*Proof.* Let $V$ be an open cone in $\mathring{T}^*X$ and $f(z,\zeta)\in\Psi(\widetilde{\mathfrak{N}})(V)$. By the definition of $\widetilde{\mathfrak{N}}$ and $\Psi$, there exists an infinitesimal wedge $W$ of type $V$ such that $f(z,\zeta)\in\Psi(\widetilde{\mathfrak{N}})(V)$ with exponential decay estimate. This $f(z,\zeta)$ is in $\mathfrak{N}$ and this correspondence gives the map $\Psi(\widetilde{\mathfrak{N}})\to\mathfrak{N}$. The inverse is also given in the same way. □

**Lemma 4.16.** Let $Z$ be a closed cone in $\mathring{T}^*X$ satisfying condition C. Then the following sequence is exact:

$$\begin{aligned} 0 \to \varinjlim_{W} \widetilde{\mathfrak{N}}(\overline{W}) \to \varinjlim_{W} \tilde{\mathscr{L}}^{(0,0)}_{2,\mathrm{loc}}(\overline{W}) \xrightarrow{\bar\partial} \varinjlim_{W} \tilde{\mathscr{L}}^{(0,1)}_{2,\mathrm{loc}}(\overline{W}) \xrightarrow{\bar\partial} \cdots \\ \xrightarrow{\bar\partial} \varinjlim_{W} \tilde{\mathscr{L}}^{(0,2n)}_{2,\mathrm{loc}}(\overline{W}) \to 0, \end{aligned} \tag{4.4}$$

where $W$ runs through the family of closed infinitesimal wedges of type $Z$ and the closure $\overline{W}$ is taken in $\widehat{T}^*X$.

The following theorem is crucial in the proof of the exactness of (4.4).

**Theorem 4.17** ([7, Thm. 4.4.2]). Let $\Omega$ be a pseudoconvex open set in $\mathbb{C}^n$ and $\varphi$ any plurisubharmonic function in $\Omega$. For every $g \in L_2^{(p,q+1)}(\Omega, \varphi)$ with $\bar{\partial} g = 0$ there is a solution $u \in L_{2,\mathrm{loc}}^{(p,q)}(\Omega)$ of the equation $\bar{\partial} u = g$ such that

$$\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-2}\, d\lambda \leq \int_\Omega |g|^2 e^{-\varphi}\, d\lambda.$$

**Remark 4.18.** In Theorem 4.17 we adopt Hörmander's notation. A form $g \in L_2^{(p,q)}(\Omega, \varphi)$ is a $(p,q)$-form on $\Omega$ with coefficients in square integrable functions with respect to the measure $e^{-\varphi}\, d\lambda$.

Now we show the exactness of (4.4). Let $V_1$ be a Stein open cone with $Z \underset{\mathrm{cone}}{\Subset} V_1$ and $W_1$ be an infinitesimal wedge of type $V_1$. Fix $\zeta_0 \in \mathbb{C}^n \setminus \{0\}$ whose existence is guaranteed by condition (C3) in Theorem 4.12. Particularly we assume $|\zeta_0| = 1$ without loss of generality. Let $H_{\zeta_0}(\delta)$ be an open half-space defined by

$$H_{\zeta_0}(\delta) = \bigsqcup_{z \in X} \bigl\{(z;\zeta) \in T_z^* X \bigm| \mathrm{Re}\langle \zeta - \delta\bar{\zeta}_0, \zeta_0\rangle > 0\bigr\}.$$

Note that we can take $V_1$ with $\overline{V}_1 \subset {}^{\frown} H_{\zeta_0}(0)$ since $Z$ satisfies condition C. By condition (C1), we can take a Stein open cone $V_2$ in $\mathring{T}^*X$ satisfying $Z \underset{\mathrm{cone}}{\Subset} V_2 \underset{\mathrm{cone}}{\Subset} V_1$. Moreover, for a sufficiently large $\delta$ the set $V_2 \cap H_{\zeta_0}(\delta)$ is an infinitesimal wedge of type $V_2$ with $V_2 \cap H_{\zeta_0}(\delta) \subset W_1$. This means that for any infinitesimal wedge $W_1$ we can take an infinitesimal wedge $W_2 = V_2 \cap H_{\zeta_0}(\delta)$ with $W_2 \subset W_1$ (cf. Figure 2). Hence it suffices to consider the infinitesimal wedge $V_2 \cap H_{\zeta_0}(\delta)$. Note that the infinitesimal wedge $V_2 \cap H_{\zeta_0}(\delta)$ is a Stein set.

Let $W_2 = V_2 \cap H_{\zeta_0}(\delta)$ be an infinitesimal wedge of type $V_2$ and $f(z,\zeta) \in \tilde{\mathscr{L}}_{2,\mathrm{loc}}^{(0,q+1)}(\widehat{W}_1)$. As $\overline{W}_2$ is compact in $\widehat{W}_1$ we can fix a small constant $0 < h' < h$ such that a function $f(z,\zeta) \cdot e^{h'\langle \zeta_0, \zeta\rangle}$ satisfies the condition of $\tilde{L}_{2,\mathrm{loc}}$ in Definition 4.14. Set $\varphi(z,\zeta) = 1$ and $F(z,\zeta) = f(z,\zeta) \cdot e^{h'\langle \zeta_0,\zeta\rangle}$. By Theorem 4.17 we can find $u(z,\zeta) \in L_{2,\mathrm{loc}}^{(0,q)}(W_2)$ such that $\bar{\partial} u = F$ and

$$\int_{W_2} (|\zeta|^2+1)^{-2} |u(z,\zeta)|^2\, d\lambda \leq \int_{W_2} |F(z,\zeta)|^2\, d\lambda < \infty. \tag{4.5}$$

Setting $g(z,\zeta) = e^{-h'\langle\zeta_0,\zeta\rangle} \cdot u(z,\zeta)$ we get

$$\bar{\partial} g = e^{-h'\langle\zeta_0,\zeta\rangle} \cdot \bar{\partial} u(z,\zeta) = e^{-h'\langle\zeta_0,\zeta\rangle} \cdot F(z,\zeta) = f(z,\zeta).$$

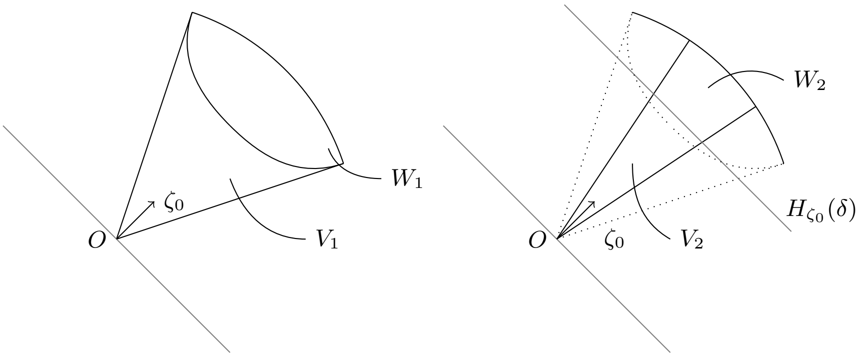



Figure 2. $W_1$ and $W_2$

Let $W_3$ be an infinitesimal wedge of type $V_2$ with $\overline{W_3} \cap T^*X \subset W_2$. By (4.5) such a $g(z, \zeta)$ belongs to $\tilde{\mathscr{L}}^{(0,q)}_{2,\mathrm{loc}}(\overline{W_3})$ because of (4.5) and the exactness of (4.4) has been proved.

Now we finish the proof of Theorem 4.12. Consider the soft resolution of $\widetilde{\mathfrak{N}}$:

$$0 \to \widetilde{\mathfrak{N}} \to \tilde{\mathscr{L}}^{(0,0)}_{2,\mathrm{loc}} \xrightarrow{\bar\partial} \tilde{\mathscr{L}}^{(0,1)}_{2,\mathrm{loc}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \tilde{\mathscr{L}}^{(0,2n)}_{2,\mathrm{loc}} \to 0.$$

Let $Z$ be a closed cone in $\mathring{T}^*X$ satisfying condition C. Applying Proposition 4.9 to the above resolution we obtain

$$H^k(Z; \mathfrak{N}) = \varinjlim_{V} H^k(V; \mathfrak{N}) = \varinjlim_{V} H^k(V; \Psi(\widetilde{\mathfrak{N}})) = \varinjlim_{V,W} H^k(\widehat{W}; \widetilde{\mathfrak{N}}) = 0,$$

where $V$ is a Stein open cone in $\mathring{T}^*X$ with $V \supset Z$ and $W$ is a Stein infinitesimal wedge of type $V$. This completes the proof of Theorem 4.12.

As an immediate consequence of the theorem we obtain the exact sequence

$$0 \to \mathfrak{N}(Z) \to \mathfrak{S}(Z) \to \mathfrak{S}/\mathfrak{N}(Z) \to 0.$$

**Corollary 4.19.** Let $Z$ be a closed cone satisfying condition C. Then an arbitrary element $f(z, \zeta) \in \mathfrak{S}/\mathfrak{N}(Z)$ is represented by some symbol $f'(z, \zeta) \in \mathfrak{S}(Z)$.

### §4.3. The sheaf $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ of symbols of $C^\infty$-type

In this subsection we introduce a new symbol class, which is called symbols of $C^\infty$-type. Let $V$ be an open cone in $\mathring{T}^*X$ and $z^* = (z; \zeta)$ a local coordinate of $T^*X$. We construct conic sheaves $\mathfrak{S}^\infty$ and $\mathfrak{N}^\infty$ on $\mathring{T}^*X$.

**Definition 4.20.** One defines the sheaf $C_z^\infty \mathscr{O}_\zeta$ as

$$f(z,\zeta) \in C_z^\infty \mathscr{O}_\zeta(V) \quad \Leftrightarrow \quad f(z,\zeta) \text{ is a } C^\infty\text{-function on } V$$
$$\text{and a holomorphic on } V \text{ in the second variable.}$$

**Remark 4.21.** The sheaf $C_z^\infty \mathscr{O}_\zeta$ is invariant under the coordinate transformation of $X$.

**Definition 4.22.** We introduce symbols of $C^\infty$ type as follows:

(1) A function $f(z,\zeta)$ is said to be a null symbol of $C^\infty$-type on $V$ if it satisfies the following conditions:

(N1) There exists an infinitesimal wedge $W$ of type $V$ such that

$$f(z,\zeta) \in C_z^\infty \mathscr{O}_\zeta(W).$$

(N2) For any open cone $V' \underset{\text{cone}}{\Subset} V$ there exists an infinitesimal wedge $W' \subset W$ of type $V'$ such that the following condition holds: For any multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$, there exist constants $h > 0$ and $C > 0$ such that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} f(z,\zeta) \right| \leq C \cdot e^{-h|\zeta|} \quad \text{on } W'.$$

(2) A function $f(z,\zeta)$ is said to be a symbol of $C^\infty$-type on $V$ if it satisfies the following conditions:

(S1) There exists an infinitesimal wedge $W$ of type $V$ such that

$$f(z,\zeta) \in C_z^\infty \mathscr{O}_\zeta(W).$$

(S2) For any open cone $V' \underset{\text{cone}}{\Subset} V$ there exists an infinitesimal wedge $W' \subset W$ of type $V'$ such that the following condition holds: For any multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$, and for any $h > 0$ there exists a constant $C > 0$ such that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} f(z,\zeta) \right| \leq C \cdot e^{h|\zeta|} \quad \text{on } W'.$$

(S3) The derivative $\frac{\partial}{\partial \bar{z}_i} f(z,\zeta)$ is a null symbol on $V$ for any $i = 1, 2, \dots, n$.

(3) We denote by $\mathfrak{S}^\infty(V)$ and $\mathfrak{N}^\infty(V)$ the set of all the symbols of $C^\infty$-type on $V$ and the set of all the null symbols of $C^\infty$-type on $V$, respectively. Moreover, we set

$$\mathfrak{S}^\infty_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{S}^\infty(V), \quad \mathfrak{N}^\infty_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{N}^\infty(V),$$

where $V$ runs through the family of open conic neighborhoods of $z^*$.

In a similar way to the discussion immediately following Definition 4.10, we can extend the sheaves $\mathfrak{S}^\infty$ and $\mathfrak{N}^\infty$ to the sheaves on $T^*X$. Set $\mathfrak{S}^\infty|_{T^*_X X} = \mathfrak{S}|_{T^*_X X}$ and $\mathfrak{N}^\infty|_{T^*_X X} = 0$. Then the sheaves $\mathfrak{S}^\infty$ and $\mathfrak{N}^\infty$ are well defined on $T^*X$.

**Proposition 4.23.** Let $V$ be an open cone in $\mathring{T}^*X$. Then $\mathfrak{N}^\infty(V)$ is an ideal of $\mathfrak{S}^\infty(V)$.

*Proof.* Let $f(z,\zeta) \in \mathfrak{S}^\infty(V)$ and $g(z,\zeta) \in \mathfrak{N}^\infty(V)$. Then we can take an infinitesimal wedge $W$ of type $V$ such that $f(z,\zeta)$ and $g(z,\zeta)$ are in $C^\infty_z \mathscr{O}_\zeta(W)$. Then we have

$$\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(f(z,\zeta)\cdot g(z,\zeta)) = \sum_{\substack{0\le\alpha'\le\alpha \\ 0\le\beta'\le\beta}} \frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^{\beta'}}{\partial \bar{z}^{\beta'}} f(z,\zeta)\cdot \frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}}\frac{\partial^{\beta-\beta'}}{\partial \bar{z}^{\beta-\beta'}} g(z,\zeta).$$

By condition (S2) of Definition 4.22, for any $V' \underset{\text{cone}}{\Subset} V$ there exists an infinitesimal wedge $W' \subset W$ of type $V'$ such that $f(z,\zeta)$ satisfies the following condition: For any $h_{\alpha'\beta'} > 0$ there exists a positive constant $C_{\alpha'\beta'}$ such that

$$\left|\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^{\beta'}}{\partial \bar{z}^{\beta'}} f(z,\zeta)\right| \le C_{\alpha'\beta'}\cdot e^{h_{\alpha'\beta'}|\zeta|}.$$

Similarly, for the same $V' \underset{\text{cone}}{\Subset} V$ and the same $W' \subset W$ there exist constants $h'_{\alpha'\beta'} > 0$ and $C'_{\alpha'\beta'} > 0$ such that

$$\left|\frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}}\frac{\partial^{\beta-\beta'}}{\partial \bar{z}^{\beta-\beta'}} g(z,\zeta)\right| \le C'_{\alpha'\beta'}\cdot e^{-h'_{\alpha'\beta'}|\zeta|}.$$

Thus by taking $h_{\alpha'\beta'} = \frac{1}{2}h'_{\alpha'\beta'}$ we obtain

$$\begin{aligned}
&\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(f(z,\zeta)\cdot g(z,\zeta))\right| \\
&\quad\le \sum_{\substack{0\le\alpha'\le\alpha \\ 0\le\beta'\le\beta}} \left|\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^{\beta'}}{\partial \bar{z}^{\beta'}} f(z,\zeta)\right|\cdot\left|\frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}}\frac{\partial^{\beta-\beta'}}{\partial \bar{z}^{\beta-\beta'}} g(z,\zeta)\right| \\
&\quad\le \sum_{\substack{0\le\alpha'\le\alpha \\ 0\le\beta'\le\beta}} C_{\alpha'\beta'}C'_{\alpha'\beta'}\cdot e^{\frac{1}{2}h'_{\alpha'\beta'}|\zeta|}\cdot e^{-h'_{\alpha'\beta'}|\zeta|} \le mCe^{-\frac{1}{2}h|\zeta|},
\end{aligned}$$

where $h = \min_{0\le\alpha'\le\alpha, 0\le\beta'\le\beta}\{h'_{\alpha'\beta'}\}$, $C = \min_{0\le\alpha'\le\alpha,0\le\beta'\le\beta}\{C_{\alpha'\beta'}C'_{\alpha'\beta'}\}$ and $m = \#\{(\alpha',\beta') \mid 0\le\alpha'\le\alpha,\ 0\le\beta'\le\beta\}$. This completes the proof. □

One denotes by $(\mathfrak{S}^\infty/\mathfrak{N}^\infty)^{\#P}$ the presheaf defined by the correspondence

$$V \mapsto \mathfrak{S}^\infty(V)/\mathfrak{N}^\infty(V),$$

where $V$ is an open cone in $\mathring{T}^*X$, and let $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ be an associated sheaf to $(\mathfrak{S}^\infty/\mathfrak{N}^\infty)^{\#P}$. We have the following exact sequence of sheaves:

$$0 \to \mathfrak{N}^\infty \to \mathfrak{S}^\infty \xrightarrow{\kappa_2} \mathfrak{S}^\infty/\mathfrak{N}^\infty \to 0. \tag{4.6}$$

Here $\kappa_2$ is the composition of the canonical morphisms $\mathfrak{S}^\infty \to (\mathfrak{S}^\infty/\mathfrak{N}^\infty)^{\#P} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty$. By the same argument as in the previous subsection we want the exactness of the sequence on $Z$,

$$0 \to \mathfrak{N}^\infty(Z) \to \mathfrak{S}^\infty(Z) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z) \to 0,$$

where $Z$ is a closed cone in $\mathring{T}^*X$ satisfying condition C, and this exactness is guaranteed by the following argument:

We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{N}(Z) & \longrightarrow & \mathfrak{S}(Z) & \xrightarrow{\kappa_1(Z)} & \mathfrak{S}/\mathfrak{N}(Z) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_2(Z)} & & \downarrow{\scriptstyle \iota_1(Z)} & & \downarrow{\scriptstyle \iota(Z)} & & \\
0 & \longrightarrow & \mathfrak{N}^\infty(Z) & \longrightarrow & \mathfrak{S}^\infty(Z) & \xrightarrow[\kappa_2(Z)]{} & \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z), & &
\end{array}$$

where $\iota_1(Z)$ and $\iota_2(Z)$ are canonical inclusions and horizontal sequences are exact. Assuming $\iota(Z)$ to be an isomorphism, the surjectivity of $\kappa_2(Z)$ follows from the fact that the composition $\iota(Z) \circ \kappa_1(Z)$ is surjective.

**Corollary 4.24.** Let $Z$ be a closed cone satisfying condition C. An arbitrary element $f(z,\zeta) \in \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z)$ is represented by some symbol $f'(z,\zeta) \in \mathfrak{S}^\infty(Z)$.

It will be proved in the next subsection that $\iota$ is an isomorphism between $\mathfrak{S}/\mathfrak{N}$ and $\mathfrak{S}^\infty/\mathfrak{N}^\infty$.

## §4.4. The equivalence of two symbol classes

In this subsection we prove the equivalence of $\mathfrak{S}/\mathfrak{N}$ and $\mathfrak{S}^\infty/\mathfrak{N}^\infty$.

By the definitions of classical symbols and symbols of $C^\infty$-type, there exist canonical inclusions

$$\iota_1 \colon \mathfrak{S} \hookrightarrow \mathfrak{S}^\infty, \quad \iota_2 \colon \mathfrak{N} \hookrightarrow \mathfrak{N}^\infty,$$

which induce the morphism

$$\iota \colon \mathfrak{S}/\mathfrak{N} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty.$$

**Theorem 4.25.** The induced morphism

$$\iota : \mathfrak{S}/\mathfrak{N} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty$$

is an isomorphism of sheaves.

Obviously we have $\mathfrak{S}/\mathfrak{N}|_{T^*_X X} = \mathfrak{S}^\infty/\mathfrak{N}^\infty|_{T^*_X X} = \mathscr{D}^\infty_X$. Hence Theorem 4.25 holds on the zero section $T^*_X X$ and it is sufficient to prove Theorem 4.25 on $\mathring{T}^* X$.

Since we have already obtained the map

$$\iota\colon \mathfrak{S}/\mathfrak{N} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty,$$

one shows $\iota_{z^*}$ to be an isomorphism of stalks at $z^* \in \mathring{T}^* X$.

For this purpose we prepare the following proposition. As the problem is local, we may assume that $T^*X \simeq \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ until the end of this subsection. In addition we can take $z^* = z^*_0 = (0; 1, 0, \ldots, 0)$ without loss of generality. Let $D = D_1(r_1, 0) \times D_2(r_2, 0) \times \cdots \times D_n(r_n, 0)$ be a polydisc in $\mathbb{C}^n_z$ where $D_i(r_i, 0)$ is an open disc in $\mathbb{C}$ whose radius is $r_i$ and the center is at the origin. Set

$$V = D \times \Gamma,$$

where $\Gamma$ is an open convex cone containing $(1, 0, \ldots, 0) \in \mathbb{C}^n_\zeta$.

We denote by $\mathfrak{N}^{\infty,(p,q)}$ the sheaf of $(p,q)$-forms with respect to the variable $z$ with coefficients in $\mathfrak{N}^\infty$. That is, we consider the variable $\zeta$ as just a holomorphic parameter.

**Proposition 4.26.** Let $V = D \times \Gamma$ be an open set defined above and let $f \in \mathfrak{N}^{\infty,(p,q)}(V)$ satisfy $\bar\partial_z f = 0$. For any polydisc $D' \Subset D$ we can find $u \in \mathfrak{N}^{\infty,(p,q-1)}(V')$ with $V' = D' \times \Gamma$ such that $\bar\partial_z u = f$ on $V'$.

*Proof.* By the induction with respect to $k$, we prove that the lemma is true if $f$ does not contain $d\bar z_{k+1}, \ldots, d\bar z_n$. If $k = 0$, it is obvious that $f = 0$. Assuming that it has been proved when $k$ is replaced by $k-1$, we write

$$f = d\bar z_k \wedge g + h,$$
$$g = \sideset{}{'}\sum_{|I|=p} \sideset{}{'}\sum_{|J|=q} g_{IJ}\, dz^I \wedge d\bar z^J.$$

Here, $g$ is a sum of $(p, q-1)$-forms on $V$ with coefficients in $C^\infty_z \mathscr{O}_\zeta$ and $h$ is a sum of $(p,q)$-forms on $V$ with coefficients in $C^\infty_z \mathscr{O}_\zeta$. Moreover, $g$ and $h$ do not contain $d\bar z_k, \ldots, d\bar z_n$ and $\sum'$ means that we sum only over increasing multi-indices. Since $\bar\partial f = 0$ holds, we have

$$\frac{\partial g_{IJ}}{\partial \bar z_j} = 0$$

for $j > k$ such that $g_{IJ}$ is analytic in these variables.

We want to construct the solution $G_{IJ}$ of the equation

$$\frac{\partial G_{IJ}}{\partial \bar z_k} = g_{IJ}.$$

For this purpose we fix a $C^\infty$-function $\psi$ on $D_k(r_k, 0)$ with compact support such that $\psi(z_k) = 1$ on a neighborhood $D'' \subset D$ of $\overline{D}'$, and set

$$\begin{aligned} G_{IJ}(z_1, \dots, z_n) &= \frac{1}{(2\pi\sqrt{-1})} \iint \frac{1}{(\tau - z_k)} \psi(\tau) g_{IJ}(z_1, \dots, z_{k-1}, \tau, z_{k+1}, \dots, z_n)\, d\tau \wedge d\bar{\tau} \\ &= \frac{-1}{(2\pi\sqrt{-1})} \iint \frac{\psi(z_k - \tau)}{\tau} g_{IJ}(z_1, \dots, z_{k-1}, z_k - \tau, z_{k+1}, \dots, z_n)\, d\tau \wedge d\bar{\tau}. \end{aligned}$$

The last integral representation shows that $G_{IJ}$ is a $C_z^\infty \mathscr{O}_\zeta$-function, and thus we just confirm that $G_{IJ}$ satisfies condition (N2) in Definition 4.22.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$ be multi-indices. Hereafter, $g_{IJ}(z_1, \dots, z_{k-1}, z_k - \tau, z_{k+1}, \dots, z_n)$ is also denoted by $g_{IJ}(z_k - \tau)$ for short. Then we have

$$\begin{aligned} \Big|\frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} G_{IJ}\Big| &= \frac{1}{2\pi} \Big| \iint \frac{1}{\tau} \cdot \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} (\psi(z_k - \tau) g_{IJ}(z_k - \tau))\, d\tau \wedge\, d\bar{\tau} \Big| \\ &= \frac{1}{2\pi} \Big| \iint \frac{1}{\tau} \cdot \frac{\partial^\alpha}{\partial \tau^\alpha} \frac{\partial^\beta}{\partial \bar{\tau}^\beta} (\psi(z_k - \tau) g_{IJ}(z_k - \tau))\, d\tau \wedge\, d\bar{\tau} \Big|. \end{aligned}$$

We can calculate the integrand as follows:

$$\begin{aligned} &\frac{1}{\tau} \frac{\partial^\alpha}{\partial \tau^\alpha} \frac{\partial^\beta}{\partial \bar{\tau}^\beta} (\psi(z_k - \tau) g_{IJ}(z_k - \tau)) \\ &\qquad = \sum_{0 \leq \alpha' \leq \alpha} \sum_{0 \leq \beta' \leq \beta} \frac{1}{\tau} \frac{\partial^{\alpha'}}{\partial \tau^{\alpha'}} \frac{\partial^{\beta'}}{\partial \bar{\tau}^{\beta'}} \psi(z_k - \tau) \cdot \frac{\partial^{\alpha - \alpha'}}{\partial \tau^{\alpha - \alpha'}} \frac{\partial^{\beta - \beta'}}{\partial \bar{z}^{\beta - \beta'}} g_{IJ}(z_k - \tau). \end{aligned}$$

Since $\psi(z_k - \tau)$ has a compact support and $g_{IJ}$ is of $\mathfrak{N}^\infty$-type,

$$\begin{aligned} &\Big| \iint \tau^{-1} \cdot \frac{\partial^{\alpha'}}{\partial \tau^{\alpha'}} \frac{\partial^{\beta'}}{\partial \bar{\tau}^{\beta'}} \psi(z_k - \tau) \cdot \frac{\partial^{\alpha - \alpha'}}{\partial \tau^{\alpha - \alpha'}} \frac{\partial^{\beta - \beta'}}{\partial \bar{z}^{\beta - \beta'}} g_{IJ}(z_k - \tau)\, d\tau \wedge \bar{\tau} \Big| \\ &\qquad \leq C_{\alpha'\beta'} e^{-h_{\alpha'\beta'}|\zeta|} \end{aligned}$$

holds for some $C_{\alpha'\beta'} > 0$ and $h_{\alpha'\beta'} > 0$. As the sets $\{\alpha \in \mathbb{Z}_{\geq 0}^n \mid 0 \leq \alpha' \leq \alpha\}$ and $\{\beta \in \mathbb{Z}_{\geq 0}^n \mid 0 \leq \beta' \leq \beta\}$ are finite we obtain

$$\Big| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} G_{IJ} \Big| \leq C e^{-h|\zeta|}$$

for some $C > 0$ and $h > 0$.

Now we construct the solution $u$. Set

$$G = \sum_{I,J}{}' G_{IJ}\, dz^I \wedge d\bar{z}^J.$$

It follows that

$$\bar{\partial} G = \sum_{I,J}{}' \sum_{j}{}' \frac{\partial G}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^I \wedge d\bar{z}^J = d\bar{z}_k \wedge g + h_1,$$

where $h_1$ is the sum when $j$ runs from 1 to $k-1$ and does not involve $d\bar{z}_k, \ldots, d\bar{z}_n$. Thus by the hypothesis of the induction we can find $v$ such that $\bar{\partial} v = f - \bar{\partial} G$ and $u = v + G$ satisfies the equation $\bar{\partial} u = f$. The proof has been completed. □

Now let us prove Theorem 4.25.

*Proof of Theorem 4.25.* It is sufficient to show that the stalks of sheaves are isomorphic to each other. One denotes by

$$\iota_{z^*} \colon \mathfrak{S}_{z^*}/\mathfrak{N}_{z^*} \to \mathfrak{S}^\infty_{z^*}/\mathfrak{N}^\infty_{z^*}$$

the induced morphism from $\iota \colon \mathfrak{S}/\mathfrak{N} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty$. Since the injectivity of $\iota_{z^*}$ is obvious, we prove the surjectivity of it.

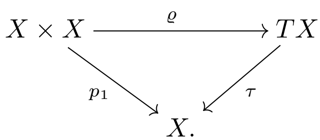
Set $F \in \mathfrak{S}_{z^*}$. There exist a neighborhood $V = D \times \Gamma$ of $z^*$ and $f \in \mathfrak{S}(V)$ such that $f$ is a representative of $F$. Then $f$ satisfies $\bar{\partial} f \in \mathfrak{N}^{\infty,(0,1)}(V)$ and $\bar{\partial}^2 f = 0$. Recalling the definition of $\mathfrak{N}^\infty$, we have $\bar{\partial}^2 f = \bar{\partial}^2_z f$, where $\bar{\partial}_z$ is the Dolbeault operator with respect to the variable $z$. Hence we can identify the operator $\bar{\partial}$ with $\bar{\partial}_z$ in this situation. By Proposition 4.26 there exist $D' \Subset D$ and $g \in \mathfrak{N}^\infty(V')$ with $V' = D' \times \Gamma$ such that $\bar{\partial} g = \bar{\partial} f$ holds. This implies $f - g \in \mathfrak{S}(V')$. Set $F' = (f-g)_{z^*}$. Then $\iota(F')_{z^*} = F$ holds and the surjectivity of $\iota_{z^*}$ has been proved. □

## §5. The equivalence of $\mathscr{E}^{\mathbb{R}}_X$ and $\mathfrak{S}^\infty/\mathfrak{N}^\infty$

In this section $X$ is assumed to be a complex vector space of dimension $n$. We identify $X \times X$ with $TX$ by the map

(5.1) $$\varrho \colon X \times X \ni (z, z') \mapsto (z, z - z') \in TX,$$

then we can see that the following diagram commutes:




Here, remark that $p_1$ is the first projection. The aim of this section is to prove the following theorem.

**Theorem 5.1.** The sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators is isomorphic to the sheaf $\mathfrak{S}/\mathfrak{N}$ of classical symbols.

## §5.1. The map $\varsigma$ from $\mathscr{E}_X^{\mathbb{R}}$ to $\mathfrak{S}^\infty/\mathfrak{N}^\infty$

Let $Z$ be a closed convex proper cone in $\mathring{T}^*X$, and let $V$ and $V'$ be open convex proper cones in $\mathring{T}^*X$ with $Z \underset{\text{cone}}{\Subset} V' \underset{\text{cone}}{\Subset} V$. Assume $\pi(Z)$ is compact, and $\pi(V')$ and $\pi(V)$ are relatively compact sets with $\overline{\pi(V')} \subset \pi(V)$. Furthermore, we assume that $V^\circ$ and $(V')^\circ$ have $C^\infty$-smooth boundaries except for the vertex. Recall that we have the cohomological expression

$$\mathscr{E}_X^{\mathbb{R}}(V) = \varinjlim_{U,G} H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X})$$

under the suitable conditions for $U$ and $G$. If we have already obtained the map

$$\tilde{\varsigma}\colon H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X}) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(V'),$$

by taking inductive limits $\varinjlim\limits_{U,G}$ and $\varinjlim\limits_{Z \underset{\text{cone}}{\Subset} V' \underset{\text{cone}}{\Subset} V}$ to $\tilde{\varsigma}$ in this order we get

$$\varsigma_Z\colon \mathscr{E}_X^{\mathbb{R}}(Z) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z).$$

Hence our aim can be rephrased to construct the map $\tilde{\varsigma}$ concretely.

To construct the domains of the integrations we introduce a new class of cones called trivializable.

**Definition 5.2.** Let $Z$ be a conic set in $\mathring{T}^*X$. We say that $Z$ is trivializable if there exist an open neighborhood $\Omega \subset X$ of $\pi(Z)$, a conic set $C_Z$ in $\mathbb{C}^n \setminus \{0\}$ and a $C^\infty$-vector bundle isomorphism $\varphi\colon \Omega \times_X T^*X \to \Omega \times \mathbb{C}^n$ such that

$$\varphi(Z) = \pi(Z) \times C_Z.$$

We call $\varphi$ the trivialization morphism and we say that $Z$ is trivialized by $\varphi$. We denote by $\varphi_z\colon T^*_zX \to \mathbb{C}^n$ the restriction of $\varphi$ to the fiber of $z$.

**Remark 5.3.** Let $Z$ be a closed convex proper cone, and $\Omega$ and $\varphi$ be the ones appearing in Definition 5.2. Then there exists a $C^\infty$-vector bundle isomorphism ${}^t\varphi^{-1}\colon \Omega \times_X TX \to \Omega \times \mathbb{C}^n$, which is the dual vector bundle morphism of $\varphi$. Moreover, $\varphi$ and ${}^t\varphi^{-1}$ preserve the inner product $\langle \bullet, \bullet \rangle$ on each fiber. Let $z \in \Omega$, $v \in T_zX$ and $\xi \in T^*_zX$, and set ${}^t\varphi_z^{-1}(v) = v'$ and $\varphi_z(\xi) = \xi'$. Then we have

$$\langle v, \xi \rangle = \langle {}^t\varphi_z^{-1}(v), \varphi_z(\xi) \rangle = \langle v', \xi' \rangle.$$

Now let us construct the map

$$\tilde{\varsigma}\colon H^n_{G\cap U}(U;\mathscr{O}^{(0,n)}_{X\times X}) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(V').$$

Since the family of closed convex proper trivializable cones in $\mathring{T}^*X$ is a basis of sets on which a conic sheaf can be defined, we assume $Z$ to be trivializable throughout this section. For such a $Z$ we fix an open set $\Omega \subset X$, a closed convex proper cone $C_Z$ and a $C^\infty$-vector bundle isomorphism $\varphi$ appearing in Definition 5.2, i.e., we have

$$\varphi(Z) = \pi(Z) \times C_Z.$$

When we consider the inductive limit $\varinjlim_{Z\subset V}$ it suffices for $V$ to run through the family of open sets $V$ which is trivializable by the common trivialization morphism $\varphi$. Hence we assume that $V$ and $V'$ are also trivialized by the same trivialization morphism $\varphi$. Furthermore, we always assume that $U$ is relatively compact.

**Remark 5.4.** Throughout this section the closure is taken in $\mathring{T}^*X$.

Set $B = \overline{\pi(V')}$. By Corollary 2.11 we take open convex proper cones $\Gamma_1$ and $\Gamma_2$ in $\mathring{T}^*X$ such that the following hold (cf. Figure 3):

(1) $V' \underset{\text{cone}}{\Subset} \Gamma_2 \underset{\text{cone}}{\Subset} \Gamma_1 \underset{\text{cone}}{\Subset} V$.

(2) For $i = 1, 2$, the cone $\Gamma_i$ can be trivialized by $\varphi$. That is, there exists an open convex proper cone $C_{\Gamma_i}$ such that

$$\varphi(\Gamma_i) = \pi(\Gamma_i) \times C_{\Gamma_i} \quad (i = 1, 2).$$

Furthermore, we assume that the dual cone $C^\circ_{\Gamma_i}$ of $C_{\Gamma_i}$ has $C^\infty$-smooth boundary except for its vertex. Note that we can always take such a cone due to Corollary 2.11.

(3) $G \cap \overline{U} \cap p_1^{-1}(B) \subset \operatorname{int}(\Gamma_1^\circ) \cup \Delta$, where $\Delta$ is the diagonal set in $X \times X$.

**Remark 5.5.** By taking $U$ sufficiently small, we can guarantee the existence of $\Gamma_1$ since $G$ is tangent to $V^\circ$ near the edge.

We construct the domains of the integrations in the following way. Set $\gamma_i = \Gamma_i^\circ$ for $i = 1, 2$. Let $D_1$ and $D_2$ be open domains in $X \times X$ with piecewise $C^\infty$-smooth boundaries such that

(D1) $D_i$ is a $\gamma_i$-open set for $i = 1, 2$.

(D2) $\Delta_X(B) \subset D_1$, where $\Delta_X\colon X \to X \times X$ is a diagonal embedding.

(D3) $\overline{D_2 \cap p_1^{-1}(z)} \subset \operatorname{int}_{p_1^{-1}(z)}(\varrho^{-1}(\overline{(V')^\circ}) \cap p_1^{-1}(z))$ for any $z \in B$, where $\operatorname{int}_{p_1^{-1}(z)}(K)$ is the set of interior points of $K$ taken in the space $p_1^{-1}(z)$.

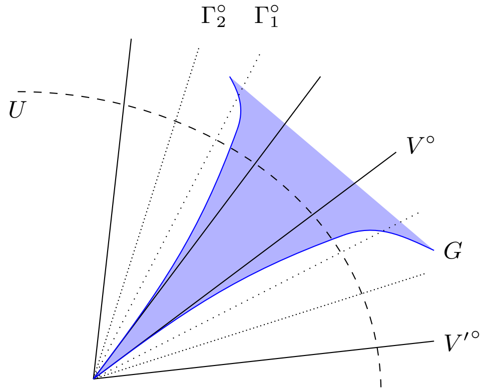



Figure 3. Geometrical relations in $X \times X$

(D4) $\overline{D} \cap p_1^{-1}(B) \subset U$ for $D = D_1 \setminus D_2$.

(D5) $\overline{E} \cap p_1^{-1}(B) \subset U \setminus G$ for $E = \partial D_1 \setminus D_2$.

(D6) $\partial D_1$ and $\partial D_2$ intersect transversally in an open neighborhood of $p_1^{-1}(B)$. Moreover, $\partial D_2$ is smooth in an open neighborhood of $\partial D_2 \cap \partial \varrho^{-1}(\gamma_1) \cap p_1^{-1}(B)$.

(D7) $p_1^{-1}(z)$ and $\partial D_1$ (resp. $\partial D_2$) intersect transversally for any $z$ in an open neighborhood of $B$.

Conditions (D1)–(D7) are collectively called condition D.

We construct the domains $D_{\varepsilon,1}$ and $D_{\varepsilon,2}$ satisfying condition D. Recall that

$$\varphi(\Gamma_i) = \pi(\Gamma_i) \times C_{\Gamma_i} \quad (i = 1, 2).$$

Since $Z \underset{\text{cone}}{\Subset} \Gamma_2 \underset{\text{cone}}{\Subset} \Gamma_1$ holds, we have $C_{\Gamma_1}^\circ \underset{\text{cone}}{\Subset} C_{\Gamma_2}^\circ \underset{\text{cone}}{\Subset} C_Z^\circ$. Here $C^\circ$ is the polar of $C$ in $\mathbb{C}^n$, i.e., for a set $C$ in $\mathbb{C}^n$ we define

$$C^\circ = \left\{ v \in \mathbb{C}^n \;\middle|\; \mathrm{Re}\langle v,\ \xi\rangle \geq 0 \text{ for any } \xi \in C \right\}.$$

Fix non-zero vectors $v_0 \in \mathrm{int}(C_{\Gamma_1}^\circ)$ and $\zeta_0 \in C_{V'}$. We define domains $\widehat{D}_{\varepsilon,1}$ and $\widehat{D}_{\varepsilon,2}$ in $\mathbb{C}^n$ (cf. Figure 4) by

$$\widehat{D}_{\varepsilon,1} = (\mathrm{int}(C_{\Gamma_1}^\circ) - \varepsilon v_0) \setminus \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z + \varepsilon v_0, \zeta_0\rangle \leq \kappa\varepsilon\},$$
$$\widehat{D}_{\varepsilon,2} = (\mathrm{int}(C_{\Gamma_2}^\circ) + \varepsilon v_0) \setminus \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z - \varepsilon v_0, \zeta_0\rangle \leq \kappa\varepsilon\},$$

where $\kappa > 0$ is taken to be sufficiently small so that

$$0 < \kappa < \frac{1}{2}\,\mathrm{Re}\langle v_0, \zeta_0\rangle$$

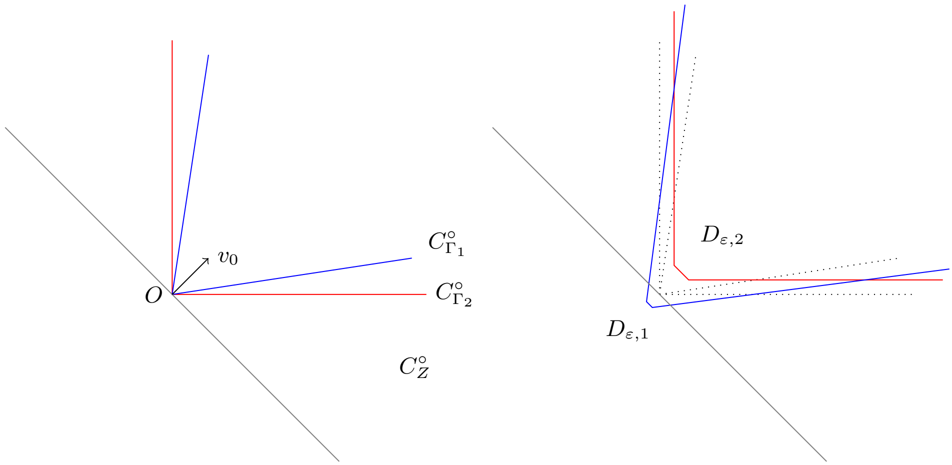



Figure 4. The figure of $D_{\varepsilon,1}$ and $D_{\varepsilon,2}$

and

$$\overline{(\mathrm{int}(C^{\circ}_{V'} + v_0)) \cap \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z - v_0, \zeta_0\rangle \le 2\kappa\}} \subset \mathrm{int}(C^{\circ}_{\Gamma_1}).$$

For $i = 1, 2$ we define the domain $D_{\varepsilon,i}$ by

$$D_{\varepsilon,i} = (\varrho^{-1} \circ {}^t\varphi)(\pi(\Gamma_i) \times \widehat{D}_{\varepsilon,i}) \quad (i = 1, 2).$$

The following lemma follows from Proposition 2.16.

**Lemma 5.6.** The domain $D_{\varepsilon,i}$ $(i = 1, 2)$ has piecewise $C^\infty$-smooth boundary and the pair $(D_{\varepsilon,1}, D_{\varepsilon,2})$ satisfies condition D. Furthermore, for any pair $(D_1, D_2)$ satisfying condition D, there exists $\varepsilon > 0$ such that

$$\overline{D_{\varepsilon,1}} \subset D_1, \quad \overline{D_{\varepsilon,1} \setminus D_{\varepsilon,2}} \subset \mathrm{int}(D_1 \setminus D_2).$$

We introduce the geometrical property of the pair $(D_{\varepsilon,1}, D_{\varepsilon,2})$, which is used in the following subsections.

**Lemma 5.7.** Let $(D_1, D_2)$ be a pair of domains satisfying condition D. Then for a sufficiently small $\varepsilon > 0$, $\partial D_{\varepsilon,1}$ and $\partial D_2$ transversally intersect in an open neighborhood of $p_1^{-1}(B)$.

*Proof.* First we show the claim for $\varepsilon = 0$. It suffices to show that $\partial C^{\circ}_{\Gamma_1}$ and $\partial\, {}^t\varphi_z^{-1}(\varrho(D_2))$ transversally intersect. Let $p \in \partial C^{\circ}_{\Gamma_1} \cap \partial\, {}^t\varphi_z^{-1}(\varrho(D_2))$. Since ${}^t\varphi_z^{-1}(\varrho(D_2))$ is $C^{\circ}_{\Gamma_2}$-open and $\partial\, {}^t\varphi_z^{-1}(\varrho(D_2))$ is smooth near $p$ by condition (D6), $(T^*\partial\, {}^t\varphi_z^{-1}(\varrho(D_2)))_p$ is contained in

$$\mathbb{R}(C^{\circ}_{\Gamma_2})^{\circ} = \mathbb{R}\overline{C_{\Gamma_2}} = \{(z, r\zeta) \mid r \in \mathbb{R},\ (z,\zeta) \in \overline{C_{\Gamma_2}}\}.$$

On the other hand, $(T^*\partial C^\circ_{\Gamma_1})_p$ is contained in

$$\mathbb{R}\partial C_{\Gamma_1} = \{(z, r\zeta) \mid r \in \mathbb{R},\ (z,\zeta) \in C_{\Gamma_1}\}.$$

As $\mathbb{R}\partial C_{\Gamma_1} \cap \mathbb{R}\overline{C_{\Gamma_2}} = \{0\}$ holds, both the hypersurfaces intersect transversally at $p$. Since the claim is true for $\varepsilon = 0$, the claim also holds for sufficiently small $\varepsilon > 0$. □

Let $D_1$ and $D_2$ be open domains in $X \times X$ satisfying condition D. Set $D = D_1 \setminus D_2$, $E = \partial D_1 \setminus D_2$, $D_z = D \cap p_1^{-1}(z)$ and $E_z = E \cap p_1^{-1}(z)$.

**Definition 5.8.** Let $u$ belong to $H^{0,n,n}_{\bar\vartheta}(U, U \setminus G)$ and let $\omega = (\omega_1, \omega_{01})$ be a representative of $u$. One defines the map

$$\tilde\varsigma\colon H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X}) = H^{0,n,n}_{\bar\vartheta}(U, U\setminus G) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(V') \tag{5.2}$$

by

$$\tilde\varsigma(\omega)(z,\zeta) = \int_{D_z} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}. \tag{5.3}$$

In the next paragraph we shall show well-definedness of $\tilde\varsigma$. More precisely we shall prove that $\tilde\varsigma$ is independent of the choices of the domains $D_1$, $D_2$ and a representative $\omega$ of $u$.

### §5.2. Well-definedness of the map $\tilde\varsigma$

We use the same notation as given in the previous subsection.

**Proposition 5.9.** Let $\omega = (\omega_1, \omega_{01})$ be a representative of $u \in H^{0,n,n}_{\bar\vartheta}(U, U \setminus G)$. The map $\tilde\varsigma$ has the following properties.

(1) The image $\tilde\varsigma(\omega)$ belongs to $\mathfrak{S}^\infty(V')$.
(2) The image $\tilde\varsigma(\omega)$ belongs to $\mathfrak{N}^\infty(V')$ if $\omega$ is equal to 0 as an element of the Čech–Dolbeault cohomology.
(3) The image $\tilde\varsigma(\omega)$ does not depend on the choices of $D_1$ and $D_2$.

To clarify the domains of the integrations, we write

$$\tilde\varsigma(\omega)(z,\zeta) = \tilde\varsigma_{[D,E]}(\omega)(z,\zeta) = \int_{D_z} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}.$$

The key to the proof of Proposition 5.9 is the transformation of the domains of the integrations. We can freely transform the domains $D_1$ and $D_2$ of the integrations as long as the difference of integral is of $\mathfrak{N}^\infty$ class.

**Definition 5.10.** Let $V$ be an open cone in $\mathring{T}^*X$, and $A$ and $B$ two symbols of $C^\infty$-type on $V$. One writes

$$A \approx B$$

if and only if $A - B$ is a null symbol of $C^\infty$-type on $V$.

Before starting the proof of Proposition 5.9 we see some lemmas.

Let $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$ be the domains of the integrations given in the previous subsection. Set

$$D_\varepsilon = D_{1,\varepsilon} \setminus D_{2,\varepsilon}, \quad E_\varepsilon = \partial D_{1,\varepsilon} \setminus D_{2,\varepsilon}.$$

**Lemma 5.11.** Let $D_1$ and $D_2$ be the domains of the integrations satisfying condition D, and $D_{\varepsilon,1}$ and $D_{\varepsilon,2}$ be the domains given in the last subsection. If we take $\varepsilon > 0$ sufficiently small so that Lemmas 5.6 and 5.7 hold, then there exist constants $h > 0$ and $C > 0$ such that

$$|\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_\varepsilon,E_\varepsilon]}(\omega)(z,\zeta)| \le C \cdot e^{-h|\zeta|}.$$

**Remark 5.12.** Let us mention that the important points are not only the statement of Lemma 5.11 but also the process of its proof. We shall use the same argument in the proof of Lemma 5.14.

*Proof of Lemma* 5.11. Recall that we identify $X \times X$ with $TX$ by the map $\varrho$, and

$$\overline{D_{\varepsilon,1}} \subset D_1, \quad \overline{D_{\varepsilon,1} \setminus D_{\varepsilon,2}} \subset \mathrm{int}(D_1 \setminus D_2)$$

hold by Lemma 5.6. Moreover, note that $(D_{\varepsilon,1}, D_2)$ satisfies condition D by Lemma 5.7. First for the pairs $(D_1, D_2)$ and $(D_{\varepsilon,1}, D_2)$ we show that Lemma 5.11 holds. By the Stokes formula we have

$$\begin{aligned}
&\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_{\varepsilon,1}\setminus D_2, \partial D_{\varepsilon,1}\setminus D_2]}(\omega)(z,\zeta) \\
&\quad = \int_{(D_1\setminus D_2)_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{(\partial D_1\setminus D_2)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&\qquad - \int_{(D_{\varepsilon,1}\setminus D_2)_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} + \int_{(\partial D_{\varepsilon,1}\setminus D_2)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&\quad = \int_{(D_1\setminus D_{\varepsilon,1})\setminus D_2} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&\qquad - \int_{(\partial D_1\setminus D_2)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle} + \int_{(\partial D_{\varepsilon,1}\setminus D_2)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle}
\end{aligned}$$

$$= \Big(\int_{(\partial D_1\setminus D_2)_z} - \int_{(\partial D_{\varepsilon,1}\setminus D_2)_z} + \int_{((D_1\setminus D_{\varepsilon,1})\cap\partial D_2)_z}\Big)\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}$$
$$- \int_{(\partial D_1\setminus D_2)_z}\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle} + \int_{(\partial D_{\varepsilon,1}\setminus D_2)_z}\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}$$
$$= \int_{((D_1\setminus D_{\varepsilon,1})\cap\partial D_2)_z}\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}.$$

By noticing that the domains of the integrations are bounded, we can take a positive number $C$ such that

$$|\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}| \le C\cdot |e^{-|\zeta|\langle z-z',\frac{\zeta}{|\zeta|}\rangle}| \le C\cdot e^{-|\zeta|\cdot\mathrm{Re}\langle z-z',\frac{\zeta}{|\zeta|}\rangle}.$$

Set $\gamma = \overline{(V')^\circ}$. Due to condition (D3) we have, for any $z\in B$,

$$\mathrm{dist}_z(\partial\gamma, \partial D_2\cap\overline{U}) > 0,$$

where $\mathrm{dist}_z(K_1,K_2)$ denotes the distance of two sets $K_1\cap p_1^{-1}(z)$ and $K_2\cap p_1^{-1}(z)$ in the space $p_1^{-1}(z)$. As $B=\overline{\pi(V')}\subset\pi(V)$, and $\overline{U}$ and $B$ are compact, there exists a positive constant $\delta$ such that

$$\inf_{z\in B}\mathrm{dist}_z(\partial\gamma,\partial D_2\cap\overline{U}) > \delta.$$

Moreover, since $\overline{D_1\cap\partial D_2}\subset U$ holds, we have

$$\inf_{z\in B}\mathrm{dist}_z(\partial\gamma,(D_1\setminus D_{\varepsilon',1})\cap\partial D_2) > \delta.$$

If we take a sufficiently small $\varepsilon'>0$, we have

$$(D_1\setminus D_{\varepsilon,1})\cap\partial D_2\subset \mathrm{int}(C(\varepsilon',V'))\quad \text{in } p_1^{-1}(B),$$

where

$$C(\varepsilon',V') = \varrho^{-1}\circ {}^t\varphi(\pi(\Omega)\times(C^\circ_{V'}+\varepsilon' v_0)).$$

Recall that $\Omega$ is an open set appearing in the definition of trivialization.

Let $S^{n-1}$ be the unit sphere in $\mathbb{C}^n$ with the center at the origin. Since $0\notin\varphi_z(S^{n-1})$ holds we have

$$(\overline{C_{V'}}\setminus\{0\})\cap\varphi_z(S^{n-1}) = \overline{C_{V'}}\cap\varphi_z(S^{n-1}),$$

which is compact. Furthermore, we have

$$\mathrm{Re}\langle v_0,\eta\rangle > 0\quad (\eta\in\overline{C_{V'}}\setminus\{0\})$$

due to $v_0\in\mathrm{int}(C^\circ_{\Gamma_1})\subset\mathrm{int}(C^\circ_{V'})$. Hence by the compactness of $\varphi_z(S^{n-1})$, we obtain

$$h = \inf_{\eta\in C_{V'}\cap\varphi_z(S^{n-1})}\mathrm{Re}\langle v_0,\eta\rangle \ge \inf_{\eta\in\overline{C_{V'}}\cap\varphi_z(S^{n-1})}\mathrm{Re}\langle v_0,\eta\rangle > 0.$$

Since ${}^t\varphi_z^{-1}(z-z')$ belongs to $C(\varepsilon', V')$,

$$\begin{aligned}\mathrm{Re}\langle z-z', \zeta/|\zeta|\rangle &= \mathrm{Re}\langle {}^t\varphi_z^{-1}(z-z'), \varphi_z(\zeta/|\zeta|)\rangle \\ &\geq \inf\bigl\{\mathrm{Re}\langle v+\varepsilon' v_0, \eta\rangle \bigm| v \in C_{V'}^{\circ},\ \eta \in C_{V'} \cap \varphi_z(S^{n-1})\bigr\} \\ &\geq \inf\bigl\{\mathrm{Re}\langle \varepsilon' v_0, \eta\rangle \bigm| \eta \in C_{V'} \cap \varphi_z(S^{n-1})\bigr\} = \varepsilon' h.\end{aligned}$$

For such $h > 0$ we have

$$\begin{aligned}&|\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_{\varepsilon,1}\setminus D_2, \partial D_{\varepsilon,1}\setminus D_2]}(\omega)(z,\zeta)| \\ &\qquad = \left|\int_{((D_1\setminus D_{\varepsilon,1})\cap\partial D_2)_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\right| \leq Ce^{-\varepsilon' h|\zeta|}.\end{aligned}$$

Next we show that Lemma 5.11 holds for the pairs $(D_{\varepsilon,1}, D_2)$ and $(D_{\varepsilon,1}, D_{\varepsilon,2})$. By the Stokes formula we obtain

$$\begin{aligned}&\tilde{\varsigma}_{[D_{\varepsilon,1}\setminus D_2, \partial D_{\varepsilon,1}\setminus D_2]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_\varepsilon, E_\varepsilon]}(\omega)(z,\zeta) \\ &\quad = \int_{(D_{\varepsilon,1}\setminus D_2)_z} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} - \int_{(\partial D_{\varepsilon,1}\setminus D_2)_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle} \\ &\qquad - \int_{(D_{\varepsilon,1}\setminus D_{\varepsilon,2})_z} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} + \int_{(\partial D_{\varepsilon,1}\setminus D_{\varepsilon,2})_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle} \\ &\quad = \int_{(D_{\varepsilon,1}\cap(D_{\varepsilon,2}\setminus D_2))_z} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} - \int_{(\partial D_1\cap(D_{\varepsilon,2}\setminus D_2))_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}.\end{aligned}$$

By the same argument as above, for sufficiently small $\varepsilon' > 0$, there exist $C > 0$ and $h > 0$ such that

$$\left|\int_{(D_{\varepsilon,1}\cap(D_{\varepsilon,2}\setminus D_2))_z} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\right| \leq Ce^{-\varepsilon' h|\zeta|}$$

and

$$\left|\int_{(\partial D_1\cap(D_{\varepsilon,2}\setminus D_2))_z} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\right| \leq Ce^{-\varepsilon' h|\zeta|}.$$

Finally we have

$$\begin{aligned}&|\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_{\varepsilon,1}, D_{\varepsilon,2}]}(\omega)(z,\zeta)| \\ &\quad \leq |\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_{\varepsilon,1}\setminus D_2, \partial D_{\varepsilon,1}\setminus D_2]}(\omega)(z,\zeta)| \\ &\qquad + |\tilde{\varsigma}_{[D_{\varepsilon,1}\setminus D_2, \partial D_{\varepsilon,1}\setminus D_2]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D_\varepsilon, E_\varepsilon]}(\omega)(z,\zeta)| \\ &\quad \leq 3Ce^{-\varepsilon' h|\zeta|},\end{aligned}$$

which is what we want. □

The next corollary immediately follows.

**Corollary 5.13.** Let $(D, E)$ and $(D', E')$ be two pairs satisfying condition D. There exists constants $h > 0$ and $C > 0$ such that

$$|\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D',E']}(\omega)(z,\zeta)| \leq C \cdot e^{-h|\zeta|}.$$

Next we show that $\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta)$ and $\tilde{\varsigma}_{[D',E']}(\omega)(z,\zeta)$ represent the same symbol in $\mathfrak{S}^\infty/\mathfrak{N}^\infty$. For this purpose we expect $\tilde{\varsigma}$ and $\frac{\partial}{\partial z}$ (resp. $\tilde{\varsigma}$ and $\frac{\partial}{\partial \bar{z}}$) to be commutative. However, $\tilde{\varsigma}$ and $\frac{\partial}{\partial z}$ (resp. $\tilde{\varsigma}$ and $\frac{\partial}{\partial \bar{z}}$) do not commute in general since the paths $D_z$ and $E_z$ of the integrations $\tilde{\varsigma}$ depend on the variables $z$. We surmount this difficulty.

For a fixed point $z_0 \in X$ and a constant $\varepsilon > 0$, we set

$$B(z_0, \varepsilon) = \{z \in X \mid |z - z_0| < \varepsilon\}.$$

We define the subsets $\widetilde{D}_1(z_0,\varepsilon)$, $\widetilde{D}_2(z_0,\varepsilon)$, $\widetilde{D}(z_0,\varepsilon)$ and $\widetilde{E}(z_0,\varepsilon)$ in $X \times X$ by

$$\begin{aligned}
\widetilde{D}_1(z_0,\varepsilon) &= B(z_0,\varepsilon) \times p_2(D_1 \cap p_1^{-1}(z_0)),\\
\widetilde{D}_2(z_0,\varepsilon) &= B(z_0,\varepsilon) \times p_2(D_2 \cap p_1^{-1}(z_0)),\\
\widetilde{D}(z_0,\varepsilon) &= B(z_0,\varepsilon) \times p_2(D \cap p_1^{-1}(z_0)),\\
\widetilde{E}(z_0,\varepsilon) &= B(z_0,\varepsilon) \times p_2(E \cap p_1^{-1}(z_0)).
\end{aligned}$$

Moreover, we write $\widetilde{D}(z_0,\varepsilon)_z = \widetilde{D}(z_0,\varepsilon) \cap p_1^{-1}(z)$ (resp. $\widetilde{E}(z_0,\varepsilon)_z = \widetilde{E}(z_0,\varepsilon) \cap p_1^{-1}(z)$) as usual.

**Lemma 5.14.** Let $z_0 \in \pi(V')$ and let $\omega = (\omega_1, \omega_{01})$ be a representative of $u \in H^{0,n,n}_{\bar{\vartheta}}(U, U \setminus G)$. The difference of the integrations

$$\tilde{\varsigma}_{[\widetilde{D}(z_0,\varepsilon),\widetilde{E}(z_0,\varepsilon)]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta)$$

is a null symbol, where

$$\begin{aligned}
&\tilde{\varsigma}_{[\widetilde{D}(z_0,\varepsilon),\widetilde{E}(z_0,\varepsilon)]}(\omega)(z,\zeta)\\
&\qquad = \int_{\widetilde{D}(z_0,\varepsilon)_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{\widetilde{E}(z_0,\varepsilon)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle}
\end{aligned}$$

and

$$\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) = \int_{D_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle}.$$

The key to the proof of this lemma is Lemma 5.11. It is possible to take a sufficiently small $\varepsilon > 0$ for which the pair $\widetilde{D}(z_0,\varepsilon)$ and $\widetilde{E}(z_0,\varepsilon)$ locally satisfies condition D near $z_0$.

*Proof of Lemma* 5.14. If we take sufficiently small $\varepsilon > 0$ and $\varepsilon' > 0$, due to Lemma 5.6 we can take the domains $\widetilde{D}_1$ and $\widetilde{D}_2$ with piecewise smooth boundaries satisfying condition D and conditions below:

(i) $(z, z) \in \widetilde{D}$.
(ii) $\widetilde{D}_1 \subset \widetilde{D}_1(z_0, \varepsilon)$ and $\widetilde{D}_1 \subset D_1$.
(iii) $D_2 \subset \widetilde{D}_2$ and $\widetilde{D}_2(z_0, \varepsilon) \subset \widetilde{D}_2$.
(iv) $\partial D_z \backslash E_z \subset C(\varepsilon', V')$, $\partial\widetilde{D}_z \backslash \widetilde{E}_z \subset C(\varepsilon', V')$ and $\partial\widetilde{D}(z_0, \varepsilon) \backslash \widetilde{E}(z_0, \varepsilon) \subset C(\varepsilon', V')$.

Here we write $\widetilde{D} = \widetilde{D}_1 \setminus \widetilde{D}_2$, $\widetilde{E} = \partial\widetilde{D}_1 \setminus \widetilde{D}_2$, $\widetilde{D}_z = \widetilde{D} \cap p_1^{-1}(z)$ and $\widetilde{E}_z = \widetilde{E} \cap p_1^{-1}(z)$. Recall that the domain $C(\varepsilon', V')$ is given by

$$C(\varepsilon', V') = \varrho^{-1} \circ {}^t\varphi(\pi(\Omega) \times (C^\circ_{V'} + \varepsilon' v_0)).$$

To complete the proof, it suffices to show that the two differences

(a) $\tilde{\varsigma}_{[\widetilde{D}(z_0,\varepsilon), \widetilde{E}(z_0,\varepsilon)]}(\omega)(z, \zeta) - \tilde{\varsigma}_{[\widetilde{D}, \widetilde{E}]}(\omega)(z, \zeta)$,
(b) $\tilde{\varsigma}_{[D,E]}(\omega)(z, \zeta) - \tilde{\varsigma}_{[\widetilde{D}, \widetilde{E}]}(\omega)(z, \zeta)$

are null symbols. Particularly since case (a) is a special case of (b), we show case (b).

Since the pairs $(D, E)$ and $(\widetilde{D}, \widetilde{E})$ are the domains of the integrations of $\varsigma(\omega)$, by the same argument as in Lemma 5.11, we can cancel out the domains of the integrations which are outside of $C(\varepsilon', V')$ by using the Stokes formula. Hence we have

$$\begin{aligned}
&\tilde{\varsigma}_{[D,E]}(\omega)(z, \zeta) - \tilde{\varsigma}_{[\widetilde{D}, \widetilde{E}]}(\omega)(z, \zeta) \\
&\quad = \left( \int_{D_z} \omega_1(z, z') \cdot e^{\langle z' - z, \zeta\rangle} - \int_{E_z} \omega_{01}(z, z') \cdot e^{\langle z' - z, \zeta\rangle} \right) \\
&\qquad - \left( \int_{\widetilde{D}_z} \omega_1(z, z') \cdot e^{\langle z' - z, \zeta\rangle} - \int_{\widetilde{E}_z} \omega_{01}(z, z') \cdot e^{\langle z' - z, \zeta\rangle} \right) \\
&\quad = \left( \int_{(D_{1,z} \backslash \widetilde{D}_{1,z}) \backslash D_{2,z}} + \int_{\widetilde{D}_{1,z} \backslash D_{2,z}} \right) \omega_1(z, z') \cdot e^{\langle z' - z, \zeta\rangle} - \int_{E_z} \omega_{01}(z, z') \cdot e^{\langle z' - z, \zeta\rangle} \\
&\qquad - \left( \int_{\widetilde{D}_{1,z} \backslash \widetilde{D}_{2,z}} \omega_1(z, z') \cdot e^{\langle z' - z, \zeta\rangle} - \int_{\partial\widetilde{D}_{1,z} \backslash \widetilde{D}_{2,z}} \omega_{01}(z, z') \cdot e^{\langle z' - z, \zeta\rangle} \right) \\
&\quad = \left( \int_{\widetilde{D}_{1,z} \backslash D_{2,z}} - \int_{\widetilde{D}_{1,z} \backslash \widetilde{D}_{2,z}} \right) \omega_1(z, z') \cdot e^{\langle z' - z, \zeta\rangle} \\
&\qquad + \left( \int_{\partial((D_{1,z} \backslash \widetilde{D}_{1,z}) \backslash D_{2,z})} - \int_{E_z} + \int_{\widetilde{E}_z} \right) \omega_{01}(z, z') \cdot e^{\langle z' - z, \zeta\rangle}
\end{aligned}$$

$$= \int_{(\widetilde{D}_{2,z}\setminus D_{2,z})\cap \widetilde{D}_{1,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}$$
$$-\Bigl(\int_{\partial D_{2,z}\cap(D_{1,z}\setminus \widetilde{D}_{1,z})} + \int_{\partial \widetilde{D}_{1,z}\cap(\widetilde{D}_{2,z}\setminus D_{2,z})}\Bigr)\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}.$$

Here we mention that the integral domains appearing in the last line above are all contained in $C(\varepsilon',V')$ by the properties (iv). We estimate the first integration on the last line of the equation above. Let $(\widetilde{D}_{2,z}\setminus D_{2,z})\cap \widetilde{D}_{1,z} = \bigsqcup_{i=1}^{N} K_{i,z}$ be a partition such that each $K_{i,z}$ is a bounded measurable subset in $p_1^{-1}(z)$. Then we have

$$\Bigl|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{(\widetilde{D}_{2,z}\setminus D_{2,z})\cap \widetilde{D}_{1,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\Bigr|$$
$$= \Bigl|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{\bigsqcup_{i=1}^{N} K_{i,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\Bigr|$$
$$\le \sum_{i=1}^{N}\Bigl|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_{i,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\Bigr|.$$

Give the local coordinate $(z,z') = (z_1,\dots,z_n,z'_1,\dots,z'_n)$ for an open neighborhood $U_i$ of $K_{i,z}$ and consider the $C^\infty$-coordinate transformation

$$\Phi_i\colon z' \mapsto \tilde{z}^i$$

such that $L_i = \Phi_i(K_{i,z})$ is independent of the variables $z$. Then we have

$$\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_{i,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} = \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{L_i} \widetilde{\omega}_1(z,\tilde{z}^i)\cdot e^{\langle \Phi_i^{-1}(\tilde{z}^i)-z,\zeta\rangle}\cdot J_{\Phi_i}.$$

Here, $\widetilde{\omega}_1(z,\widetilde{w}^i) = \omega_1(z,w)$ holds under the coordinate transform $\Phi_i$, and $J_{\Phi_i}$ is the Jacobian.

**Remark 5.15.** The existence of such a coordinate transformation $\Phi_i$ follows from the fact that $V'$ is trivializable by $\varphi$ and the domains of the integrations are contained in $C(\varepsilon',V')$.

Since the domain $L_i$ is independent of the variables $z$ we obtain

$$\Bigl|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_{i,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\Bigr|$$
$$= \Bigl|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{L_i} \widetilde{\omega}_1(z,\tilde{z}^i)\cdot e^{\langle \Phi_i^{-1}(\tilde{z}^i)-z,\zeta\rangle}\cdot J_{\Phi_i}\Bigr|$$
$$\le \int_{L_i}\Bigl|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\widetilde{\omega}_1(z,\tilde{z}^i)\cdot J_{\Phi_i}\cdot e^{\langle \Phi_i^{-1}(\tilde{z}^i)-z,\zeta\rangle})\Bigr|.$$

The absolute values of the higher derivatives of the integrand are bounded on $L_i$ since the integrands are of $C^\infty$-class. Hence, for each $i$, by the same argument as in the proof of Lemma 5.11 there exist $M_i, C_i > 0$ and $h_i > 0$ such that

$$\int_{L_i}\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\big(\widetilde{\omega}_1(z,\tilde{z}^i)\cdot J_{\Phi_i}\cdot e^{\langle \Phi^{-1}(\tilde{z}^i)-z,\zeta\rangle}\big)\right| \le M_i C_i \cdot e^{-h_i|\zeta|}.$$

Finally, for $M = \max_{1\le i\le N}\{M_i\}$, $C = \max_{1\le i\le N}\{C_i\}$ and $h = \min_{1\le i\le N}\{h_i\}$ we have

$$\sum_{i=1}^{N}\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_{i,z}}\omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\right| \le \sum_{i=1}^{N} M_i C_i e^{-h_i|\zeta|} \le CMNe^{-h|\zeta|}.$$

We can apply the same argument to

$$\left(\int_{\partial D_{2,z}\cap(D_{1,z}\setminus\widetilde{D}_{1,z})} + \int_{\partial\widetilde{D}_{1,z}\cap(\widetilde{D}_{2,z}\setminus D_{2,z})}\right)\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle},$$

and these complete the proof. □

By Lemma 5.14 we finally obtain the following proposition.

**Proposition 5.16.** Let $(D,E)$ and $(D',E')$ be two pairs satisfying condition D. Then the difference of the two symbols

$$\tilde{\varsigma}_{[D,E]}(\omega)(z,\zeta) - \tilde{\varsigma}_{[D',E']}(\omega)(z,\zeta)$$

is a null symbol.

*Proof.* Let $\widetilde{D}(z_0,\varepsilon)$, $\widetilde{E}(z_0,\varepsilon)$ and $\widetilde{D}'(z_0,\varepsilon)$, $\widetilde{E}'(z_0,\varepsilon)$ be the domains of the integrations given in the above. By Lemma 5.14 it suffices to show that the difference

$$\tilde{\varsigma}_{[\widetilde{D}(z_0,\varepsilon),\widetilde{E}(z_0,\varepsilon)]}(\omega)(z_0,\zeta) - \tilde{\varsigma}_{[\widetilde{D'}(z_0,\varepsilon),\widetilde{E'}(z_0,\varepsilon)]}(\omega)(z_0,\zeta)$$

is a null symbol. We mention that $\widetilde{D}(z_0,\varepsilon)$, $\widetilde{E}(z_0,\varepsilon)$ and $\widetilde{D}'(z_0,\varepsilon)$, $\widetilde{E}'(z_0,\varepsilon)$ satisfy condition D, and hence we can see that the difference becomes the null symbol by the same argument as in Corollary 5.13 and Lemma 5.14. □

Now we start the proof of Proposition 5.9. In the following proof the Dolbeault operator $\bar{\partial}_z + \bar{\partial}_{z'}$ is denoted by $\bar{\partial}$ without notice.

*Proof.* We start from (1).

(1) First of all note that all the symbols appearing in this proof are on $V'$. We recall the domains $D_{\varepsilon,i}$ for $i = 1,2$. The domains $\widehat{D}_{\varepsilon,1}$ and $\widehat{D}_{\varepsilon,2}$ in $\mathbb{C}^n$ are given

by

$$\widehat{D}_{\varepsilon,1} = (\mathrm{int}(C^{\circ}_{\Gamma_1}) - \varepsilon v_0) \setminus \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z + \varepsilon v_0, \zeta_0\rangle \le \kappa\varepsilon\},$$
$$\widehat{D}_{\varepsilon,2} = (\mathrm{int}(C^{\circ}_{\Gamma_2}) + \varepsilon v_0) \setminus \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z - \varepsilon v_0, \zeta_0\rangle \le \kappa\varepsilon\},$$

where $\kappa > 0$ is taken to be sufficiently small. Then we define the domains $D_{\varepsilon,i}$ for $i = 1, 2$ as follows:

$$D_{\varepsilon,i} = (\varrho^{-1} \circ {}^t\varphi)(\pi(\Gamma_i) \times \widehat{D}_{\varepsilon,i}) \quad (i = 1, 2).$$

By Proposition 5.16 it suffices to consider the case that the domains of the integrations are $D_{\varepsilon,1}$ and $D_{\varepsilon,2}$. We show that for any multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n_{\ge 0}$, and for any $h > 0$ there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$\Big|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\tilde{\varsigma}_{[D_\varepsilon, E_\varepsilon]}(\omega)\Big| = \Big|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\Big(\int_{D_{\varepsilon,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_{\varepsilon,z}} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\Big)\Big| \le Ce^{h|\zeta|}.$$

We fix $h' > 0$. Since $\Gamma_1^\circ \underset{\text{cone}}{\Subset} \Gamma_2^\circ$ holds, we can take a sufficiently small $\varepsilon > 0$ satisfying

$$D_\varepsilon \subset (\varrho^{-1} \circ {}^t\varphi)\big(\pi(\Gamma_1) \times (\mathrm{int}(C^\circ_{\Gamma_2}) \cup \{v \in \mathbb{C}^n \mid |v| \le h'\})\big).$$

Fix $z_0 \in X$ and let $\widetilde{D}(z_0,\varepsilon)$ and $\widetilde{E}(z_0,\varepsilon)$ be the same subsets for $D_\varepsilon$ and $E_\varepsilon$ appearing in the proof of Lemma 5.14. Noticing that $\widetilde{D}(z_0,\varepsilon)_z$ and $\widetilde{E}(z_0,\varepsilon)_z$ do not depend on $z$ we have

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{D_{\varepsilon,z}} \omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle} - \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{E_{\varepsilon,z}} \omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}\\
&\approx \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{\widetilde{D}(z_0,\varepsilon)_z} (\omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}) - \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{\widetilde{E}(z_0,\varepsilon)_z} (\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle})\\
&= \int_{\widetilde{D}(z_0,\varepsilon)_z} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}) - \int_{\widetilde{E}(z_0,\varepsilon)_z} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle}).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\tilde{\varsigma}(\omega)\\
&\approx \int_{\widetilde{D}(z_0,\varepsilon)_z} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}) - \int_{\widetilde{E}(z_0,\varepsilon)_z} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle})\\
&= \int_{\widetilde{D}(z_0,\varepsilon)_z} \frac{\partial^\alpha}{\partial z^\alpha}(\frac{\partial^\beta}{\partial \bar{z}^\beta}\omega_1(z,z')\cdot e^{\langle z'-z,\zeta\rangle}) - \int_{\widetilde{E}(z_0,\varepsilon)_z} \frac{\partial^\alpha}{\partial z^\alpha}(\frac{\partial^\beta}{\partial \bar{z}^\beta}\omega_{01}(z,z')\cdot e^{\langle z'-z,\zeta\rangle})
\end{aligned}$$

$$= \int_{\widetilde{D}(z_0,\varepsilon)_z} \sum_{0\le\alpha'\le\alpha} \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial^{\beta}}{\partial \bar{z}^{\beta}} \omega_1(z,z') \cdot \frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}} e^{\langle z'-z,\zeta\rangle}$$
$$- \int_{\widetilde{E}(z_0,\varepsilon)_z} \sum_{0\le\alpha'\le\alpha} \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial^{\beta}}{\partial \bar{z}^{\beta}} \omega_{01}(z,z') \cdot \frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}} e^{\langle z'-z,\zeta\rangle}.$$

We can estimate the former integral by

$$\left| \int_{\widetilde{D}(z_0,\varepsilon)_z} \sum_{0\le\alpha'\le\alpha} \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial^{\beta}}{\partial \bar{z}^{\beta}} \omega_1(z,z') \cdot \frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}} e^{\langle z'-z,\zeta\rangle} \right|$$
$$\le \sum_{0\le\alpha'\le\alpha} \int_{\widetilde{D}(z_0,\varepsilon)_z} \left| \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial^{\beta}}{\partial \bar{z}^{\beta}} \omega_1(z,z') \right| \cdot |P_{\alpha'}(\zeta)| \cdot e^{\mathrm{Re}\langle z'-z,\zeta\rangle},$$

where $P_{\alpha'}(\zeta)$ is some polynomial with respect to $\zeta$. For $v \in C^{\circ}_{\Gamma_2}$ and $\eta \in C_{V'}$, $\mathrm{Re}\langle v, \eta\rangle \ge 0$ holds. Thus we have

$$\begin{aligned}
&\mathrm{Re}\langle z'-z, \zeta/|\zeta|\rangle \\
&\quad = \mathrm{Re}\langle {}^t\varphi_z^{-1}(z'-z), \varphi_z(\zeta/|\zeta|)\rangle \\
&\quad \le \sup\{\mathrm{Re}\langle -v+\varepsilon v_0, \eta\rangle \mid v \in \mathrm{int}(C^{\circ}_{\Gamma_2}) \cup \{v\in\mathbb{C}^n \mid |v| \le h'\},\ \eta \in C_{V'} \cap \varphi_z(S^{n-1})\} \\
&\quad \le \sup\bigl\{\mathrm{Re}\langle v, \eta\rangle \bigm| |v| < h',\ \eta \in \overline{C_{V'}} \cap \varphi_z(S^{n-1})\bigr\} \\
&\qquad + \sup\bigl\{\mathrm{Re}\langle \varepsilon v_0, \eta\rangle \bigm| \eta \in \overline{C_{V'}} \cap \varphi_z(S^{n-1})\bigr\}.
\end{aligned}$$

Mentioning that $\overline{C_{V'}} \cap \varphi_z(S^{n-1})$ is compact, we have

$$\begin{aligned}
\delta_1 &= \sup\bigl\{\mathrm{Re}\langle v, \eta\rangle \bigm| |v| < 1,\ \eta \in \overline{C_{V'}} \cap \varphi_z(S^{n-1})\bigr\} < \infty, \\
\delta_2 &= \sup\bigl\{\mathrm{Re}\langle v_0, \eta\rangle \bigm| \eta \in \overline{C_{V'}} \cap \varphi_z(S^{n-1})\bigr\} < \infty.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\mathrm{Re}\langle z-z', \zeta/|\zeta|\rangle \\
&\qquad \le \sup\bigl\{\mathrm{Re}\langle v, \eta\rangle \bigm| |v| < h',\ \eta \in \overline{C_{V'}} \cap \varphi_z(S^{n-1})\bigr\} \\
&\qquad\quad + \sup\bigl\{\mathrm{Re}\langle \varepsilon v_0, \eta\rangle \bigm| \eta \in \overline{C_{V'}} \cap \varphi_z(S^{n-1})\bigr\} \\
&\qquad \le \delta_1 h' + \delta_2 \varepsilon \le h.
\end{aligned}$$

The last inequality will be justified by retaking $h' > 0$ and $\varepsilon > 0$ sufficiently small. (Note that the choice of $\varepsilon$ depends on $h'$.) Finally we get

$$\sum_{0\le\alpha'\le\alpha} \int_{\widetilde{D}(z_0,\varepsilon)_z} \left| \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial^{\beta}}{\partial \bar{z}^{\beta}} \omega_1(z,z') \right| \cdot |P_{\alpha'}(\zeta)| \cdot e^{\mathrm{Re}\langle z'-z,\zeta\rangle} \le C e^{h|\zeta|}.$$

The latter integration satisfies the same inequality by the same argument.

Next we check that $\frac{\partial}{\partial \bar{z}_i}\tilde{\varsigma}_{[D_\varepsilon,E_\varepsilon]}(\omega)$ belongs to $\mathfrak{N}^\infty(V')$ for any $i = 1, 2, \ldots, n$. Fix $z_0 \in X$ and let $\widetilde{D}(z_0, \varepsilon)$ and $\widetilde{E}(z_0, \varepsilon)$ be the same subsets as above. By the Stokes formula and the facts that $(\bar{\partial}_z + \bar{\partial}_{z'})\omega_{01} = \omega_1$ and $\bar{\partial}_z \omega_1 = -\bar{\partial}_{z'}\omega_1$, we obtain

$$\begin{aligned}
\bar{\partial}_z \tilde{\varsigma}_{[D_\varepsilon,E_\varepsilon]}(\omega) &\approx \int_{\widetilde{D}(z_0,\varepsilon)_z} \bar{\partial}_z \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{\widetilde{E}(z_0,\varepsilon)_z} \bar{\partial}_z \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&= -\int_{\widetilde{D}(z_0,\varepsilon)_z} \bar{\partial}_{z'} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&\quad - \int_{\widetilde{E}(z_0,\varepsilon)_z} (\omega_1(z,z') - \bar{\partial}_{z'}\omega_{01}(z,z')) \cdot e^{\langle z'-z,\zeta\rangle} \\
&= \int_{\partial\widetilde{D}(z_0,\varepsilon)_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&\quad + \int_{\widetilde{E}(z_0,\varepsilon)_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{\partial\widetilde{E}(z_0,\varepsilon)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&= \int_{\partial\widetilde{D}(z_0,\varepsilon)_z \setminus \widetilde{E}(z_0,\varepsilon)_z} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} + \int_{\partial\widetilde{E}(z_0,\varepsilon)_z} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle}.
\end{aligned}$$

Since the domains of the integrations are contained in $C(\varepsilon, V')$ for sufficiently small $\varepsilon > 0$, the last line above is a 1-form with respect to the variable $\bar{z}$ with coefficients in the class of null symbols.

(2) In addition to the assumption in the above proof we assume that $\omega$ is equal to 0 as an element in the relative Čech–Dolbeault cohomology. Then there exists $\tau = (\tau_1, \tau_{01}) \in C^{\infty,(0,n,n-1)}_{X\times X}(\mathcal{V}, \mathcal{V}')$ with $\bar{\vartheta}\tau = \omega$. By substituting $(\omega_1, \omega_{01})$ with $(\bar{\partial}\tau_1, \tau_1 - \bar{\partial}\tau_{01})$ we have

$$\begin{aligned}
\tilde{\varsigma}(\omega) &= \int_{D_{\varepsilon,z}} \omega_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_{\varepsilon,z}} \omega_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle} \\
&= \int_{D_{\varepsilon,z}} \bar{\partial}\tau_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_{\varepsilon,z}} (\tau_1(z,z') - \bar{\partial}\tau_{01}(z,z')) \cdot e^{\langle z'-z,\zeta\rangle}.
\end{aligned}$$

By noticing that the integrations

$$\int_{D_{\varepsilon,z}} \bar{\partial}_z \tau_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle}, \quad \int_{E_{\varepsilon,z}} \bar{\partial}_z \tau_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle}$$

vanish, we have

$$\begin{aligned}
&\int_{D_{\varepsilon,z}} \bar{\partial}\tau_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} - \int_{E_{\varepsilon,z}} (\tau_1(z,z') - \bar{\partial}\tau_{01}(z,z')) \cdot e^{\langle z'-z,\zeta\rangle} \\
&\quad = \int_{\partial D_{\varepsilon,z} \setminus E_{\varepsilon,z}} \tau_1(z,z') \cdot e^{\langle z'-z,\zeta\rangle} + \int_{\partial E_{\varepsilon,z}} \tau_{01}(z,z') \cdot e^{\langle z'-z,\zeta\rangle}.
\end{aligned}$$

By the same argument as in the proof of (1), we can find constants $h > 0$ and $C > 0$ such that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} \left( \int_{\partial D_{z,2(\varepsilon)} \setminus E_{\varepsilon,z}} \tau_1(z,z') \cdot e^{\langle z'-z,\zeta \rangle} + \int_{\partial E_{\varepsilon,z}} \tau_{01}(z,z') \cdot e^{\langle z'-z,\zeta \rangle} \right) \right| \leq C \cdot e^{-h|\zeta|}.$$

(3) The claim follows immediately from Proposition 5.16. □

Hereafter we write $\tilde{\varsigma}$ instead of $\tilde{\varsigma}_{[D,E]}$ since the map $\tilde{\varsigma}$ does not depend on the choice of $D$ and $E$ by Proposition 5.9. The corollary below also follows from Proposition 5.9.

**Corollary 5.17.** The map $\tilde{\varsigma}$ is well defined.

In the next subsection we prove the main theorem.

### §5.3. The proof of Theorem 5.1

Now we show the proof of Theorem 5.1. As a consequence of Sections 4.1 and 4.2, there exists a morphism

$$\varsigma_Z \colon \mathscr{E}_X^{\mathbb{R}}(Z) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z)$$

for any closed convex proper cone $Z$ in $\mathring{T}^*X$ with $\pi(Z)$ being compact.

Let $Z'$ be a closed convex proper cone contained in $Z$. Then it follows from Proposition 5.9 that the diagram below commutes:

$$\begin{array}{ccc} \mathscr{E}_X^{\mathbb{R}}(Z) & \xrightarrow{\varsigma_Z} & \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z) \\ \downarrow & & \downarrow \\ \mathscr{E}_X^{\mathbb{R}}(Z') & \xrightarrow{\varsigma_{Z'}} & \mathfrak{S}^\infty/\mathfrak{N}^\infty(Z'). \end{array}$$

Since the family of closed convex proper cones in $\mathring{T}^*X$ is a basis of sets on which a conic sheaf can be defined, the family $\{\varsigma_Z\}_Z$ of morphisms gives a sheaf morphism on $\mathring{T}^*X$,

$$\varsigma \colon \mathscr{E}_X^{\mathbb{R}} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty.$$

The rest of the problem is whether the map $\varsigma$ is an isomorphism or not. In particular, it suffices to show the morphism $\varsigma_{z^*} \colon \mathscr{E}_{X,z^*}^{\mathbb{R}} \to (\mathfrak{S}^\infty/\mathfrak{N}^\infty)_{z^*}$ of stalks is isomorphic. Assume that the following diagram commutes for each point $z^*$ (we show it in Theorem A.6 in Appendix A.4 since the proof is a little complicated):

$$\begin{array}{ccc} & & (\mathfrak{S}/\mathfrak{N})_{z^*} \\ & \overset{\sigma}{\nearrow} & \\ \mathscr{E}^{\mathbb{R}}_{X,z^*} & & \Big\downarrow \iota_{z^*} \\ & \underset{\varsigma_{z^*}}{\searrow} & \\ & & (\mathfrak{S}^\infty/\mathfrak{N}^\infty)_{z^*}. \end{array} \tag{5.4}$$

Here, $\sigma$ is the symbol mapping given by Aoki [2, Def. 4.4]. We also review the details of $\sigma$ in Definition A.5 in Appendix A.3. The following theorem is essential.

**Theorem 5.18** ([2, Thms 4.3 and 4.5]). The symbol mapping $\sigma$

$$\sigma\colon \mathscr{E}^{\mathbb{R}}_{X,z^*} \to (\mathfrak{S}/\mathfrak{N})_{z^*}$$

is an isomorphism of stalks.

The vertical arrow in the diagram is an isomorphism by Theorem 4.25 and $\sigma$ is also isomorphic by the result of Aoki. Therefore $\varsigma_{z^*}$ is also an isomorphism, which completes the proof of Theorem 5.1.

## Appendix A. The compatibility of two symbol maps

In the appendix we prove the commutativity of (5.4). Since the argument is local, we assume the following identification:

$$T^*X \simeq X \times \mathbb{C}^n \ni (z;\zeta) = (z_1, \dots, z_n; \zeta_1, \dots, \zeta_n).$$

Moreover, assume that $X \subset \mathbb{C}^n$ is an open set with coordinate system $z = (z_1, \dots, z_n)$.

Fix the point $z^*$ so that we can consider the stalk of $\mathscr{E}^{\mathbb{R}}_X$ on it. Then we regard $z^* = (0; \lambda, 0, \dots, 0)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ without loss of generality. Recall that the stalk of the sheaf $\mathscr{E}^{\mathbb{R}}_X$ is expressed as the inductive limit of the cohomologies with suitable subsets $U$ and $G$:

$$\mathscr{E}^{\mathbb{R}}_{X,z^*} = \varinjlim_{U,G} H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X}).$$

On the other hand, in the theory of Čech–Dolbeault cohomology, the stalk of $\mathscr{E}^{\mathbb{R}}_X$ is given in Definition 3.3 and Theorem 3.8.

To express two cohomologies in the same class we give another Čech–Dolbeault expression. In the second step we calculate the integration of two cohomology classes and finish the proof.

## Appendix A.1. Another Čech–Dolbeault expression of $\mathscr{E}_X^{\mathbb{R}}$

We construct the Čech–Dolbeault cohomology of $\mathscr{E}_X^{\mathbb{R}}$ in which two cohomologies can be embedded. (See [13].)

Let $M$ be a closed set in $X$. Moreover, let $\mathcal{W} = \{W_i\}_{i\in I}$ and $\mathcal{W}' = \{W_i\}_{i\in I'}$ be open coverings of $X$ and $X \setminus M$, respectively. Here $I$ and $I'$ are index sets with $I' \subset I$. For a sheaf $\mathscr{S}$ we set

$$C^q(\mathcal{W};\mathscr{S}) = \prod_{(\alpha_0,\alpha_1,\dots,\alpha_q)\in I^{q+1}} \mathscr{S}(W_{\alpha_0\alpha_1\cdots\alpha_q}),$$

where $W_{\alpha_0\alpha_1\cdots\alpha_q} = W_{\alpha_0}\cap W_{\alpha_1}\cap\cdots\cap W_{\alpha_q}$. Note that $\sigma_{\alpha_0\cdots\alpha_q} \in \mathscr{S}(W_{\alpha_0\alpha_1\cdots\alpha_q})$ has the orientation, that is, the section $\sigma$ satisfies the formula

$$\sigma_{\alpha_0\cdots\alpha_i\alpha_{i+1}\cdots\alpha_q} = -\sigma_{\alpha_0\cdots\alpha_{i+1}\alpha_i\cdots\alpha_q}.$$

This implies that we have $\sigma = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$. The complex $(C^\bullet(\mathcal{W};\mathscr{S}),\delta)$ is called the Čech complex with coefficients in $\mathscr{S}$. The coboundary operator $\delta$ is defined by

$$(\delta\sigma)_{\alpha_0\alpha_1\cdots\alpha_{q+1}} = \sum_{k=0}^{q+1}(-1)^k\sigma_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{q+1}}.$$

Here we set $\sigma_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{q+1}} = \sigma_{\alpha_0\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_{q+1}}$. We also define the relative Čech complex $(C^\bullet(\mathcal{W},\mathcal{W}';\mathscr{S}),\delta)$ as follows:

$$C^q(\mathcal{W},\mathcal{W}';\mathscr{S}) = \bigl\{\sigma \in C^q(\mathcal{W};\mathscr{S}) \bigm| \sigma_{\alpha_0\cdots\alpha_q} = 0 \text{ if } \alpha_0,\alpha_1,\dots,\alpha_q \in I'\bigr\}.$$

Hereafter we denote $\mathscr{F}^q = C^{\infty,(0,n;q)}_{X\times X}$ until the end of this appendix. Recall that we have the fine resolution of $\mathscr{O}^{(0,n)}_{X\times X}$:

$$0 \to \mathscr{O}^{(0,n)}_{X\times X} \to \mathscr{F}^0 \xrightarrow{\bar\partial} \mathscr{F}^1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{F}^{2n} \to 0.$$

This induces the following double complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\cdots & \xrightarrow{(-1)^{q_1}\bar\partial} & C^{q_1}(\mathcal{W},\mathcal{W}';\mathscr{F}^{q_2}) & \xrightarrow{(-1)^{q_1}\bar\partial} & C^{q_1}(\mathcal{W},\mathcal{W}';\mathscr{F}^{q_2+1}) & \xrightarrow{(-1)^{q_1}\bar\partial} & \cdots \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\cdots & \xrightarrow{(-1)^{q_1+1}\bar\partial} & C^{q_1+1}(\mathcal{W},\mathcal{W}';\mathscr{F}^{q_2}) & \xrightarrow{(-1)^{q_1+1}\bar\partial} & C^{q_1+1}(\mathcal{W},\mathcal{W}';\mathscr{F}^{q_2+1}) & \xrightarrow{(-1)^{q_1+1}\bar\partial} & \cdots \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

We consider the following associated single complex $(\mathscr{F}^\bullet(\mathcal{W},\mathcal{W}'), D)$:

$$\mathscr{F}^q(\mathcal{W},\mathcal{W}') = \bigoplus_{q_1+q_2=q} C^{q_1}(\mathcal{W},\mathcal{W}';\mathscr{F}^{q_2}), \quad D = \delta + (-1)^{q_1}\bar{\partial}.$$

**Definition A.1.** The Čech–Dolbeault cohomology $H^q(\mathcal{W},\mathcal{W}';\mathscr{F}^\bullet)$ of $(\mathcal{W},\mathcal{W}')$ with coefficients in $\mathscr{F}^\bullet$ is the $q$th cohomology of $(\mathscr{F}^\bullet(\mathcal{W},\mathcal{W}'), D)$.

We describe the differential $D$ in a bit more detail. A cochain $\xi \in \mathscr{F}^q(\mathcal{W},\mathcal{W}')$ may be expressed as $\xi = (\xi^{(q_1)})_{0\le q_1\le q}$ with $\xi^{(q_1)} \in C^{q_1}(\mathcal{W},\mathcal{W}';\mathscr{F}^{q-q_1})$. Then $D\colon \mathscr{F}^q(\mathcal{W},\mathcal{W}') \to \mathscr{F}^{q+1}(\mathcal{W},\mathcal{W}')$ is given by

$$(D\xi)^{(q_1)} = \delta\xi^{(q_1-1)} + (-1)^{q_1}\bar{\partial}\xi^{q_1}, \quad 0 \le q_1 \le q+1,$$

where we set $\xi^{(-1)} = \xi^{(q+1)} = 0$.

**Remark A.2.** Assume $\mathcal{W}$ consists of two open sets $X$ and $X\setminus M$, and $\mathcal{W}'$ consists of only one open set $X\setminus M$. Then the Čech–Dolbeault cohomology $H^q(\mathcal{W},\mathcal{W}';\mathscr{F}^\bullet)$ corresponds to the Čech–Dolbeault cohomology defined in Section 2.

## Appendix A.2. Two cohomological expressions in the Čech–Dolbeault cohomology

The goal of this subsection is to express two cohomologies in the same cohomology class. For more details of Čech representation, see [2] and [12].

By the discussion so far we can get three cohomological expressions $H^n(\mathscr{O}^{(0,n)}_{X\times X};\mathcal{W},\mathcal{W}')$, $H^{0,n,n}_{\bar{\vartheta}}(\mathcal{V},\mathcal{V}')$ and $H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W},\mathcal{W}')$ of $\mathscr{E}^{\mathbb{R}}_{X,z^*}$. Set $U_r$ and $G_{r,\varepsilon}$ as follows:

$$\begin{aligned} U_r &= \big\{(z,z') \in X\times X \;\big|\; |z|<r,\ |z'-z|<r\big\},\\ G_{r,\varepsilon} &= \{(z,z') \in U_r \mid |z'_1 - z_1| \ge \varepsilon|z'_i - z_i|\ (2\le i\le n),\\ &\qquad -\operatorname{Re}(\lambda(z'_1-z_1)) \ge \varepsilon|\operatorname{Im}(\lambda(z'_1-z_1))|\}. \end{aligned}$$

Thanks to the study of Kashiwara–Kawai [8] we have

$$\mathscr{E}^{\mathbb{R}}_{X,z^*} = \varinjlim_{\substack{r\to 0\\ \varepsilon\to 0}} H^n_{G_{r,\varepsilon}}(U_r;\mathscr{O}^{(0,n)}_{X\times X}). \tag{A.1}$$

We write $U$ and $G$ instead of $U_r$ and $G_{r,\varepsilon}$ respectively if there is no risk of confusion. We also set several open sets as follows:

$$\begin{aligned} V_0 = W_0 = U, \quad V_1 &= U\setminus G, \quad V_{01} = V_0\cap V_1 = U\setminus G,\\ W_1 &= \big\{(z,z')\in U \;\big|\; -\operatorname{Re}(\lambda(z'_1-z_1)) > \varepsilon|\operatorname{Im}(\lambda(z'_1-z_1))|\big\},\\ W_i &= \big\{(z,z')\in U \;\big|\; |z'_1-z_1| < \varepsilon|z'_i-z_i|\big\} \quad (2\le i\le n). \end{aligned}$$

We can easily see that the following are open coverings of $U$:

$$\mathcal{V} = \{V_0, V_1\}, \quad \mathcal{W} = \{W_0, W_1, \dots, W_n\},$$

and the following are open coverings of $U \setminus G$:

$$\mathcal{V}' = \{V_1\}, \quad \mathcal{W}' = \{W_1, W_2, \dots, W_n\}.$$

Fix small $r$ and $\varepsilon$. We compute the cohomology $H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X})$ by applying the Čech cohomology with respect to the Čech coverings $(\mathcal{W}, \mathcal{W}')$. Since $\mathcal{W}$ and $\mathcal{W}'$ are Stein coverings, by Leray's theorem we have the exact sequence

$$\bigoplus_{i=1}^{n} \Gamma(W_{\hat{i}}; \mathscr{O}^{(0,n)}_{X\times X}) \to \Gamma(W; \mathscr{O}^{(0,n)}_{X\times X}) \to H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X}) \to 0,$$

where $W_{\hat{i}} = \bigcap_{j\neq i} W_j$ and $W = \bigcap_{j=1}^{n} W_j$. Hence $P \in H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X})$ is represented by some holomorphic form $\psi = \psi(z, z' - z)\, dz' \in \Gamma(W; \mathscr{O}^{(0,n)}_{X\times X})$ such that

$$P = [\psi(z, z' - z)\, dz'].$$

We denote by $H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}')$ the Čech cohomology with respect to the coverings $(\mathcal{W}, \mathcal{W}')$.

Next we recall the Čech–Dolbeault expression of $H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X})$ with respect to coverings $(\mathcal{V}, \mathcal{V}')$. By the results in Section 2, we have

$$H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X}) = H^{0,n,n}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}').$$

Moreover, $P \in H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X})$ is represented by some $\omega = (\omega_0, \omega_{01}) \in C^{\infty,(0,n;n)}_{X\times X}(\mathcal{V}, \mathcal{V}')$ such that

$$P = [\omega] = [(\omega_0, \omega_{01})].$$

**Remark A.3.** While in Section 2 we set $\mathcal{V}' = \{V_0\}$ and the representative of an element of Čech–Dolbeault cohomology is represented by the pair $(\omega_1, \omega_{01})$, in this section we set $\mathcal{V}' = \{V_1\}$. Therefore the index of first term of $\omega = (\omega_0, \omega_{01})$ is different from the one in Section 3.

Finally, by applying $(\mathcal{W}, \mathcal{W}')$ to the previous subsection, we have the Čech–Dolbeault cohomology $H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}')$, which is isomorphic to the local cohomology $H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X})$:

$$\begin{array}{ccccc} & & [\omega] \in H^{0,n,n}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}') & & \\ & \nearrow & & \searrow^{\tilde{\phi}_2} & \\ P \in H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X}) & & & & H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}'). \\ & \searrow & & \nearrow_{\tilde{\phi}_1} & \\ & & [\psi] \in H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}') & & \end{array}$$

We introduce the maps $\tilde{\phi}_1$, $\tilde{\phi}_2$. Let $\phi_1$ be the inclusion map

$$\phi_1 \colon \Gamma(W; \mathscr{O}^{(0,n)}_{X\times X}) \hookrightarrow C^n(\mathcal{W}, \mathcal{W}'; \mathscr{F}^0) \subset \mathscr{F}^n(\mathcal{W}, \mathcal{W}'),$$

where $W = \bigcap_{i=1}^n W_i$, and define

$$\phi_2 \colon \mathscr{F}^n(\mathcal{V}, \mathcal{V}') \to C^0(\mathcal{W}, \mathcal{W}'; \mathscr{F}^n) \oplus C^1(\mathcal{W}, \mathcal{W}' \mathscr{F}^{n-1}) \subset \mathscr{F}^n(\mathcal{W}, \mathcal{W}')$$

by

$$\phi_2(\omega)_\alpha = \begin{cases} \omega_0|_{W_\alpha}, & \alpha = 0, \\ 0, & \alpha = 1, \dots, n, \end{cases}$$

$$\phi_2(\omega)_{\alpha\beta} = \begin{cases} \omega_{01}|_{W_{\alpha\beta}}, & (\alpha,\beta) = (0,1), \dots, (0,n), \\ -\omega_{01}|_{W_{\alpha\beta}}, & (\alpha,\beta) = (1,0), \dots, (n,0), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_1$ induces the morphism between cohomologies

$$\tilde{\phi}_1 \colon H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}') \to H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}'),$$

and $\phi_2$ induces the morphism between cohomologies

$$\tilde{\phi}_2 \colon H^{0,n,n}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}') \to H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}').$$

**Theorem A.4** ([13, Thm. 3.6, Prop. 4.3, Thm. 4.7]). *The induced morphisms $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are isomorphisms.*

Now we work in $H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}')$. Fix $P \in H^n_G(U; \mathscr{O}^{(0,n)}_{X\times X})$. Then we obtain two representatives

$$P = [\psi] \in H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}'), \quad P = [\omega] \in H^{0,n,n}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}').$$

These are clearly equivalent to each other in $H^{0,n,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}')$, and hence there exists $\eta \in \mathscr{F}^{n-1}(\mathcal{W}, \mathcal{W}')$ such that

$$\tilde{\phi}_2(\omega) - \tilde{\phi}_1(\psi) = D\eta. \tag{A.2}$$

Here, note that $(D\eta)_i = \delta\eta_i + (-1)^{i+1}\bar{\partial}\eta_{i+1}$ for $i = 1, 2, \dots, n-1$ and $\eta = (\eta_0, \eta_1, \dots, \eta_{n-1})$. That is, condition (A.2) can be described concretely as follows:

$$\begin{cases} \tilde{\phi}_2(\omega_0) = \bar{\partial}\eta_0, \\ \tilde{\phi}_2(\omega_{01}) = \delta\eta_0 - \bar{\partial}\eta_1, \\ 0 = \delta\eta_1 + \bar{\partial}\eta_2, \\ \vdots \\ 0 = \delta\eta_{n-2} + (-1)^{n-1}\bar{\partial}\eta_{n-1}, \\ -\tilde{\phi}_1(\psi) = \delta\eta_{n-1}. \end{cases}$$

## Appendix A.3. A brief review of the symbol map $\sigma$

We briefly review the symbol map $\sigma$ introduced by Aoki. See [2] and [4] for more details. In this subsection we adapt the notation in [4].

We use the same notation as introduced in the previous subsection. The stalk $\mathscr{E}^{\mathbb{R}}_{X,z^*}$ has the cohomological expression

$$H^n_{G_{r,\varepsilon}}(U_r; \mathscr{O}^{(0,n)}_{X\times X}) \simeq H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}').$$

Take $\psi = \psi(z, z'-z)\,dz' \in \Gamma(W, \mathscr{O}^{(0,n)}_{X\times X})$, which is the representative of $P \in H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}')$. Set $w = z' - z$. In order to define the symbol map $\sigma$ we set the integral paths $\gamma_i$ for $i = 1, 2, \ldots, n$ (cf. Figure 5). Fix $r > 0$ and $\varepsilon > 0$.

(1) Let $\beta_0$, $\beta_1$ be complex numbers satisfying

$$\begin{aligned} 0 &> \operatorname{Re}\lambda\beta_0 > \varepsilon \operatorname{Im}\lambda\beta_0, \\ 0 &> \operatorname{Re}\lambda\beta_1 > -\varepsilon \operatorname{Im}\lambda\beta_1. \end{aligned}$$

Then we choose $\gamma_1 \subset \mathbb{C}$ as the $C^\infty$-smooth path that goes counterclockwise around the origin from $\beta_0$ to $\beta_1$. Moreover, assume $\gamma_1$ to be a piecewise $C^\infty$-smooth curve.

(2) Fix $w_1 \in \gamma_1$. For $i = 2, \ldots, n$ we take a sufficiently small $\delta > 0$ and set

$$\gamma_i = \left\{ w_i \in \mathbb{C} \;\middle|\; |w_i| = \tfrac{|w_1|}{\varepsilon} + \delta \right\}.$$

**Definition A.5.** For a pseudodifferential operator $P = [\psi(z, z'-z)\,dz'] \in \mathscr{E}^{\mathbb{R}}_{X,z^*}$ the symbol map $\sigma$ is given by

$$\sigma(P) = \int_{\gamma_1} \oint_{\gamma_2} \cdots \oint_{\gamma_n} \psi(z, w) \cdot e^{\langle w, \zeta\rangle}\, dw.$$

## Appendix A.4. The proof of the commutativity of (5.4)

Now we show the commutativity of (5.4). Let $P$ be a pseudodifferential operator and $\omega = (\omega_1, \omega_{01})$ be its representative in the framework of Čech–Dolbeault cohomology.

**Theorem A.6.** A symbol $\sigma(P)$ is the same symbol as the one $\varsigma(\omega)$ in the $C^\infty$-symbol class $\mathfrak{S}^\infty/\mathfrak{N}^\infty$.

To calculate the difference $\sigma(P) - \varsigma(\omega)$, we work in the framework of $H^n(\mathscr{O}^{(0,n)}_{X\times X}; \mathcal{W}, \mathcal{W}')$ in what follows. Let $(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_n)$ be a local coordinate system of $X \times X \subset \mathbb{C}^n \times \mathbb{C}^n$. Since the symbol mapping is defined by integration with respect to the variable $w$, we consider $z = (z_1, \ldots, z_n)$ as the

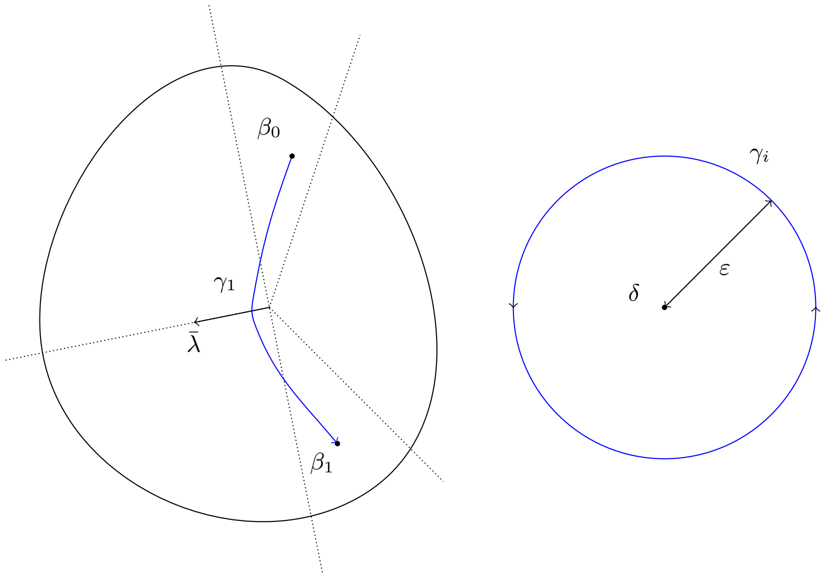



Figure 5. $\gamma_1$ and $\gamma_i$

parameter until the end of this paper. Let $U = U_r$ and $G = G_{r,\varepsilon}$ be the sets given in Section A.2. Moreover, let $q_i$ be the canonical projection given by

$$q_i\colon \mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto z_i \in \mathbb{C} \quad (i = 1, \ldots, n).$$

For sufficiently small $\varepsilon_1$ and $\varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$, we set

$$\Gamma_1^\circ = \big\{w \in \mathbb{C} \bigm| -\mathrm{Re}(\lambda w) \geq \varepsilon_1 |\mathrm{Im}(\lambda w)|\big\},$$
$$\Gamma_2^\circ = \big\{w \in \mathbb{C} \bigm| -\mathrm{Re}(\lambda w) \geq \varepsilon_2 |\mathrm{Im}(\lambda w)|\big\}.$$

Moreover, assume that $\beta_0, \beta_1 \in \Gamma_1^\circ$ and $\beta_0, \beta_1 \notin \Gamma_2^\circ$. Then we can construct the closed domains $D_1, D_2 \subset \mathbb{C}$ satisfying the following conditions (cf. Figure 6):

(1) $D_1$ and $D_2$ satisfy condition D of the 1-dimensional case.
(2) The points $\beta_0$ and $\beta_1$ are on $\partial D_1$ and $\partial D_2$.
(3) $\partial D_1 \cap D_2 \subset \Gamma_1^\circ$.

Set $B_1 = D_1$, $\gamma_1 = \partial D_1 \setminus D_2$ and $\gamma_1' = \partial D_1 \cap U$.

**Remark A.7.** The path $\gamma_1$ goes counterclockwise from $\beta_0$ to $\beta_1$.

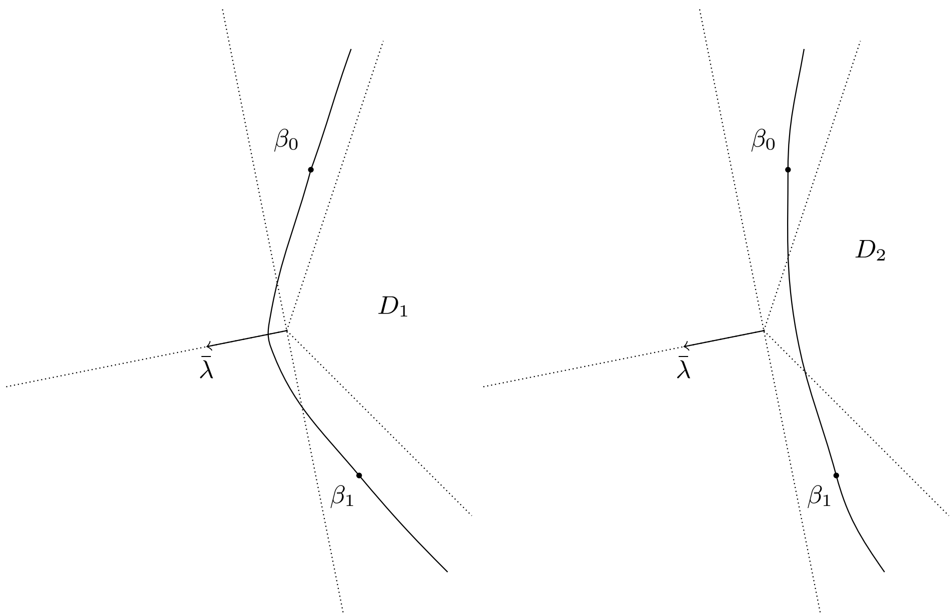



Figure 6. $D_1$ and $D_2$

Let $\gamma_i$ $(i = 2, \ldots, n)$ be the same as those that appeared in the previous subsection. For $i = 2, 3, \ldots, n$ and for $w_1 \in B_1$, set the domains $B_i \subset \mathbb{C}$ as follows:

$$B_i = \left\{ w_i \in \mathbb{C} \;\middle|\; |w_i| \le \tfrac{|w_1|}{\varepsilon} + \delta \right\}.$$

Note that the boundary of $B_i$ has the same orientation with the path $\gamma_i$ for $i = 2, \ldots, n$.

We define the domains $D$ and $E$ of integrations in the framework of Čech–Dolbeault cohomology by

$$D = q_1^{-1}(B_1) \cap \left( \bigcap_{k=2}^{n} q_i^{-1}(B_k) \right) \cap U,$$

$$E = \left( \bigcup_{i=1}^{n} N_i \right) \cap U,$$

where

$$N_i = q_i^{-1}(\gamma_i) \cap \left( \bigcap_{i=1}^{n} q_i^{-1}(B_i) \right).$$

To see the relation between the symbol mapping introduced by Aoki and the one in the framework of Čech–Dolbeault cohomology, we introduce the honeycomb system of $U$. The family $\{R_i\}_{i=0}^n$ of closed sets defined below is the honeycomb system of $U$ adapted to $\mathcal{W}$ with respect to the variable $w$ (for more details of the honeycomb system, see [13, Sect. 6]):

$$R_0 = q_1^{-1}(\overline{B_1}) \cap \left( \bigcap_{k=2}^n q_i^{-1}(B_k) \right) \cap U,$$

$$R_1 = q_1^{-1}(\overline{B_1^c}) \cap \left( \bigcap_{k=2}^n q_i^{-1}(B_k) \right) \cap U,$$

$$R_i = q_i^{-1}(\overline{B_i^c}) \cap \left( \bigcap_{k=i+1}^n q_k^{-1}(B_k) \right) \cap U \quad (i = 2, \dots, n-1),$$

$$R_n = q_n^{-1}(\overline{B_n^c}) \cap U.$$

Here, the set $K^c$ means the complement of $K$.

We set

$$I = \{1, 2, \dots, n\},$$
$$I^{(r)} = \{\alpha^{(r)} = (\alpha_1, \dots, \alpha_r) \in I^r \mid \alpha_1 \le \dots \le \alpha_r\}.$$

For simplicity we adopt the following notation. Let $k$ be an integer with $0 \le k \le n$ and $\alpha^{(r)} = (\alpha_1, \dots, \alpha_r) \in I^{(r)}$. Then $(0, \alpha_1, \dots, \alpha_r)$ is also denoted by $0\alpha^{(r)}$ and $(\alpha_1, \dots, \alpha_r, k)$ is also denoted by $\alpha^{(r)}k$. Set $\alpha^{(r)}_{\check{i}} = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_r)$.

For any $\alpha^{(r)} \in I^{(r)}$ one sets $R_{\alpha^{(r)}}$ as

$$R_{\alpha^{(r)}} = \bigcap_i R_i,$$

where $i$ ranges through all the components of $\alpha^{(r)}$. The order of the subscript in the honeycomb system means its orientation. That is, for any permutation $\rho$ we have

$$R_{\alpha^{(r)}} = \operatorname{sgn} \rho \cdot R_{\rho(\alpha^{(r)})}.$$

The key to the proof of Theorem A.6 is how to construct the domains of the integrations. By the recipe of $\{R_i\}_{i=1}^n$ we have

$$\varsigma(\omega) = \int_{R_0} \phi_2(\omega_1)^0 \cdot e^{\langle w, \zeta \rangle} + \sum_{i=1}^n \int_{R_{0i}} \phi_2(\omega_{01})^{0i} \cdot e^{\langle w, \zeta \rangle}.$$

Moreover, we have

$$R_{01\cdots n} = \bigcap_{i=1}^n R_i = \partial_1 B_1 \times \dots \times \partial_n B_n = \prod_{i=1}^n \partial B_i = \partial B_1 \times \prod_{i=2}^n \gamma_i,$$

which contains the domain $\gamma_1' \times \gamma_2 \times \dots \times \gamma_n$.

Here we recall the symbol map defined by Aoki. The symbol map $\sigma$ is given by

$$\sigma(P) = \int_{\gamma_1}\oint_{\gamma_2}\cdots\oint_{\gamma_n}\psi(z,w)\cdot e^{\langle w,\zeta\rangle}\,dw.$$

**Notation A.8.** Let $\eta\in\mathscr{F}^r(\mathcal{W},\mathcal{W}')=\bigoplus_{0\le j\le r}C^j(\mathcal{W},\mathcal{W}';\mathscr{F}^{r-j})$. For $0\le j\le r$ we write

$$\eta=(\eta_0,\dots,\eta_r),\quad \eta_j=\{(\eta_j)^{0\alpha^{(j)}}\}\in C^j(\mathcal{W},\mathcal{W}';\mathscr{F}^{r-j}),$$

where $(\eta_j)^{0\alpha^{(j)}}\in\mathscr{F}^{r-j}(W_{0\alpha^{(j)}})$.

The following theorem is crucial for the concrete computation of $\varsigma(\omega)$.

**Theorem A.9.** Let $\eta=(\eta_0,\eta_1,\dots,\eta_{n-1})$ be in $\mathrm{Ker}(\mathscr{F}^{n-1}(\mathcal{W},\mathcal{W}')\xrightarrow{D}\mathscr{F}^n(\mathcal{W},\mathcal{W}'))$. Then, for $1\le r\le n-1$, we have

$$\sum_{\alpha^{(r)}\in I^{(r)}}\int_{R_{0\alpha^{(r)}}}(\bar\partial\eta_r)^{0\alpha^{(r)}}\cdot e^{\langle w,\zeta\rangle}=-\sum_{\alpha^{(r+1)}\in I^{(r+1)}}\int_{R_{0\alpha^{(r+1)}}}(\bar\partial\eta_{r+1})^{0\alpha^{(r+1)}}\cdot e^{\langle w,\zeta\rangle},$$

where $\bar\partial$ is the Dolbeault operator.

For the proof of Theorem A.9 we show the following lemma.

**Lemma A.10.** Let $\eta\in\mathscr{F}^{n-1}(\mathcal{W},\mathcal{W}')$. We have

$$\begin{aligned}&\sum_{\alpha^{(r)}\in I^{(r)}}\sum_{i=1}^{n}\int_{R_{0\alpha^{(r)}i}}\eta_r^{0\alpha^{(r)}}\cdot e^{\langle w,\zeta\rangle}\\ &\qquad=\sum_{\beta^{(r+1)}\in I^{(r+1)}}\sum_{j=1}^{r+1}\int_{R_{0\beta^{(r+1)}}}(-1)^{r+1-j}\cdot\eta_r^{0\beta_{\check{j}}^{(r+1)}}\cdot e^{\langle w,\zeta\rangle}.\end{aligned}$$

*Proof.* Let $\{(\alpha^{(r)},i)\mid\alpha^{(r)}\in I^{(r)},\ i\in I\}$ and $\{(\beta^{(r+1)},j)\mid\beta^{(r+1)}\in I^{(r+1)},\ 1\le j\le r+1\}$ be index sets. We denote by $\alpha^{(r)}[i]$ the $i$th component of $\alpha^{(r)}$. We define the map $F$ by

$$\begin{array}{cccc}F\colon&\big\{(\alpha^{(r)},i)\mid\alpha^{(r)}\in I^{(r)},\ i\in I\big\}&\longrightarrow&\big\{(\beta^{(r+1)},j)\mid\beta^{(r+1)}\in I^{(r+1)},\ j\in I\big\}\\ &\cup&&\cup\\ &(\alpha^{(r)},i)&\longmapsto&(\gamma^{(r+1)},k),\end{array}$$

where $\gamma^{(r+1)}$ is $\alpha^{(r)}i$ sorted into increasing order and $k=\#\{\ell\mid\alpha^{(r)}[\ell]<i\}$. Remark that if $i\in\alpha^{(r)}$ we have $\alpha^{(r)}i=0$. We also define $G$ by

$$\begin{array}{cccc}G\colon&\big\{(\beta^{(r+1)},j)\mid\beta^{(r+1)}\in I^{(r+1)},\ j\in I\big\}&\longrightarrow&\big\{(\alpha^{(r)},i)\mid\alpha^{(r)}\in I^{(r)},\ i\in I\big\}\\ &\cup&&\cup\\ &(\beta^{(r+1)},j)&\longmapsto&(\beta_{\check{j}}^{(r+1)},\beta^{(r+1)}[j]).\end{array}$$

Set

$$p((\alpha^{(r)}, i)) = \int_{R_{0\alpha^{(r)}i}} \eta_r^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle},$$

$$q((\beta^{(r+1)}, j)) = \int_{R_{0\beta^{(r+1)}}} (-1)^{r+1-j} \cdot \eta_r^{0\beta_{\check{j}}^{(r+1)}} \cdot e^{\langle w,\zeta\rangle}.$$

By the definitions of $F$ and $G$ we have the following properties:

(1) $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.
(2) $q(F((\alpha(r), i))) = p((\alpha(r), i))$.

The second property can be shown as follows:

$$\begin{aligned} q\big(F((\alpha(r), i))\big) &= q((\gamma^{(r+1)}, k)) \\ &= \int_{R_{0\gamma^{(r+1)}}} (-1)^{r+1-k} \cdot \eta_r^{0\gamma_{\check{k}}^{(r+1)}} \cdot e^{\langle w,\zeta\rangle} \\ &= (-1)^{r+1-k} \int_{R_{0\alpha^{(r)}i}} (-1)^{r+1-k} \eta_r^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle} \\ &= p((\alpha(r), i)). \end{aligned}$$

Hence we have

$$\begin{aligned} &\sum_{\alpha^{(r)} \in I^{(r)}} \sum_{i=1}^{n} \int_{R_{0\alpha^{(r)}j}} \eta_r^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle} \\ &\quad = \sum_{\alpha^{(r)} \in I^{(r)}} \sum_{i=1}^{n} p((\alpha^{(r)}, i)) \\ &\quad = \sum_{\alpha^{(r)} \in I^{(r)}} \sum_{i=1}^{n} q(F((\alpha^{(r)}, i))) \\ &\quad = \sum_{\beta^{(r+1)} \in I^{(r+1)}} \sum_{j=1}^{r+1} q((\beta^{(r+1)}, j)) \\ &\quad = \sum_{\beta^{(r+1)} \in I^{(r+1)}} \sum_{j=1}^{r+1} \int_{R_{0\beta^{(r+1)}}} (-1)^{r+1-j} \cdot \eta_r^{0\beta_{\check{j}}^{(r+1)}} \cdot e^{\langle w,\zeta\rangle}. \qquad \square \end{aligned}$$

Now we start the proof of Theorem A.9.

*Proof of Theorem A.9.* By the Stokes formula we have

$$\begin{aligned}\sum_{\alpha^{(r)}\in I^{(r)}} \int_{R_{0\alpha^{(r)}}} (\bar{\partial}\eta_r)^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle} &= \sum_{\alpha^{(r)}\in I^{(r)}} \int_{\partial R_{0\alpha^{(r)}}} \eta_r^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle} \\ &= \sum_{\alpha^{(r)}\in I^{(r)}} \sum_{j=1}^{n} \int_{R_{0\alpha^{(r)}j}} \eta_r^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle}.\end{aligned}$$

By Lemma A.10 and the assumption $\delta\eta_r = (-1)^r\bar{\partial}\eta_{r+1}$ we obtain

$$\begin{aligned}&\sum_{\alpha^{(r)}\in I^{(r)}} \sum_{i=1}^{n} \int_{R_{0\alpha^{(r)}i}} \eta_r^{0\alpha^{(r)}} \cdot e^{\langle w,\zeta\rangle} \\ &= \sum_{\alpha^{(r+1)}\in I^{(r+1)}} \sum_{j=1}^{r+1} \int_{R_{0\alpha^{(r+1)}}} (-1)^{r+1-j} \cdot \eta_r^{0\alpha_{\hat{j}}^{(r+1)}} \cdot e^{\langle w,\zeta\rangle} \\ &= (-1)^{r+1} \sum_{\alpha^{(r+1)}\in I^{(r+1)}} \int_{R_{0\alpha^{(r+1)}}} (\delta\eta_r)^{0\alpha^{(r+1)}} \cdot e^{\langle w,\zeta\rangle} \\ &= (-1)^{r+1} \sum_{\alpha^{(r+1)}\in I^{(r+1)}} \int_{R_{0\alpha^{(r+1)}}} ((-1)^r\bar{\partial}\eta_{r+1})^{0\alpha^{(r+1)}} \cdot e^{\langle w,\zeta\rangle} \\ &= - \sum_{\alpha^{(r+1)}\in I^{(r+1)}} \int_{R_{0\alpha^{(r+1)}}} (\bar{\partial}\eta_{r+1})^{0\alpha^{(r+1)}} \cdot e^{\langle w,\zeta\rangle}\end{aligned}$$

and this completes the proof. □

In the framework of Čech–Dolbeault cohomology whose covering of total space consists of 2 open sets, we have

$$\varsigma(\omega) = \int_{D_z} \omega_1 \cdot e^{\langle w,\zeta\rangle} - \int_{E_z} \omega_{01} \cdot e^{\langle w,\zeta\rangle}.$$

On the other hand, in the framework of Čech–Dolbeault cohomology whose covering of total space consists of $n$ open sets, we have

$$\varsigma(\omega) = \int_{R_{0,z}} \phi_2(\omega_1)^0 \cdot e^{\langle w,\zeta\rangle} + \sum_{i=1}^{n} \int_{R_{0i,z}} \phi_2(\omega_{01})^{0i} \cdot e^{\langle w,\zeta\rangle}.$$

Thus we can calculate the image $\varsigma(\omega)$ as follows:

$$\begin{aligned}\varsigma(\omega) &= \int_{D_z} \omega_1 \cdot e^{\langle w,\zeta\rangle} - \int_{E_z} \omega_{01} \cdot e^{\langle w,\zeta\rangle} \\ &= \int_{R_0} \phi_2(\omega_1)^0 \cdot e^{\langle w,\zeta\rangle} + \sum_{i=1}^n \int_{R_{0i}} \phi_2(\omega_{01})^{0i} \cdot e^{\langle w,\zeta\rangle} \\ &= \int_{R_0} \bar\partial\eta_0^0 \cdot e^{\langle w,\zeta\rangle} + \sum_{i=1}^n \int_{R_{0i}} ((\delta\eta_0)^{0i} - (\bar\partial\eta_1)^{0i}) \cdot e^{\langle w,\zeta\rangle}.\end{aligned}$$

By the Stokes formula we have

$$\begin{aligned}&\int_{R_0} \bar\partial\eta_0^0 \cdot e^{\langle w,\zeta\rangle} + \sum_{i=1}^n \int_{R_{0i}} ((\delta\eta_0)^{0i} - (\bar\partial\eta_1)^{0i}) \cdot e^{\langle w,\zeta\rangle} \\ &\quad= \sum_{i=1}^n \int_{R_{0i}} \eta_0^0 \cdot e^{\langle w,\zeta\rangle} + \sum_{i=1}^n \int_{R_{0i}} (\delta\eta_0)^{0i} \cdot e^{\langle w,\zeta\rangle} - \sum_{i=1}^n \int_{R_{0i}} (\bar\partial\eta_1)^{0i} \cdot e^{\langle w,\zeta\rangle} \\ &\quad= \sum_{i=1}^n \int_{R_{0i}} \eta_0^0 \cdot e^{\langle w,\zeta\rangle} + \sum_{i=1}^n \int_{R_{0i}} (-\eta_0^0) \cdot e^{\langle w,\zeta\rangle} - \sum_{\alpha^{(1)}\in I^{(1)}} \int_{R_{0\alpha^{(1)}}} (\bar\partial\eta_1)^{0\alpha^{(1)}} \cdot e^{\langle w,\zeta\rangle} \\ &\quad= - \sum_{\alpha^{(1)}\in I^{(1)}} \int_{R_{0\alpha^{(1)}}} (\bar\partial\eta_1)^{0\alpha^{(1)}} \cdot e^{\langle w,\zeta\rangle}.\end{aligned}$$

By applying Theorem A.9 to the above inductively we obtain

$$\begin{aligned}&- \sum_{\alpha^{(1)}\in I^{(1)}} \int_{R_{0\alpha^{(1)},z}} (\bar\partial\eta_1)^{0\alpha^{(1)}} \cdot e^{\langle w,\zeta\rangle} \\ &\qquad= \sum_{\alpha^{(2)}\in I^{(2)}} \int_{R_{0\alpha^{(2)},z}} (\bar\partial\eta_2)^{0\alpha^{(r+1)}} \cdot e^{\langle w,\zeta\rangle} \\ &\qquad\vdots \\ &\qquad= (-1)^{n-1} \sum_{\alpha^{(n-1)}\in I^{(n-1)}} \int_{R_{0\alpha^{(n-1)},z}} (\bar\partial\eta_{n-1})^{0\alpha^{(n-1)}} \cdot e^{\langle w,\zeta\rangle}.\end{aligned}$$

By the Stokes formula we have

$$\begin{aligned}&(-1)^{n-1} \sum_{\alpha^{(n-1)}\in I^{(n-1)}} \int_{R_{0\alpha^{(n-1)},z}} (\bar\partial\eta_{n-1})^{0\alpha^{(n-1)}} \cdot e^{\langle w,\zeta\rangle} \\ &\qquad= (-1)^{n-1} \sum_{\alpha^{(n-1)}\in I^{(n-1)}} \sum_{j=1}^n \int_{R_{0\alpha^{(n-1)}j,z}} \eta_{n-1}^{0\alpha^{(n-1)}} \cdot e^{\langle w,\zeta\rangle} \\ &\qquad= (-1)^{n-1} \sum_{\alpha^{(n)}\in I^{(n)}} \sum_{j=1}^n \int_{R_{0\alpha^{(n)},z}} (-1)^{n-j} \cdot \eta_{n-1}^{0\alpha_{\hat{j}}^{(n-1)}} \cdot e^{\langle w,\zeta\rangle}\end{aligned}$$

$$
\begin{aligned}
&= - \sum_{\alpha^{(n)} \in I^{(n)}} \int_{R_{0\alpha^{(n)},z}} (\delta \eta_{n-1})^{0\alpha^{(n)}} \cdot e^{\langle w,\zeta \rangle} \\
&= - \sum_{\alpha^{(n)} \in I^{(n)}} \int_{R_{0\alpha^{(n)},z}} (-\sigma)^{0\alpha^{(n)}} \cdot e^{\langle w,\zeta \rangle} \\
&= \int_{R_{01\cdots n,z}} \psi \cdot e^{\langle w,\zeta \rangle}.
\end{aligned}
$$

To conclude the proof of the commutativity of (5.4) we show

$$
\begin{aligned}
\varsigma(\omega) = \int_{R_{01\cdots n,z}} \psi \cdot e^{\langle w,\zeta \rangle} &= \int_{\gamma_1' \times \gamma_2 \times \cdots \times \gamma_n} \psi \cdot e^{\langle w,\zeta \rangle} \\
&\approx \int_{\gamma_1 \times \gamma_2 \times \cdots \times \gamma_n} \psi \cdot e^{\langle w,\zeta \rangle} = \sigma(P).
\end{aligned}
$$

Since we have $\partial D_1 \cap D_2 \subset \Gamma_1^\circ$ by the same argument as the proofs of Lemmas 5.11 and 5.14, for any multi-indices $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$ there exist positive constants $C > 0$ and $h > 0$ such that

$$
\Big| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} (\varsigma(\omega) - \sigma(P)) \Big| \leq C e^{-h|\zeta|},
$$

and the proof has been completed.

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