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# Twisted Local Wild Mapping Class Groups: Configuration Spaces, Fission Trees and Complex Braids

by

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**Abstract**

Following the completion of the algebraic construction of the Poisson wild character varieties (B.–Yamakawa 2015) one can consider their natural deformations, generalising both the mapping class group actions on the usual (tame) character varieties, and the G-braid groups already known to occur in the wild/irregular setting. Here we study these *wild mapping class groups* in the case of arbitrary formal structure in type A. As we will recall, this story is most naturally phrased in terms of admissible deformations of wild Riemann surfaces. The main results are the following: (1) the construction of configuration spaces containing all possible local deformations, (2) the definition of a combinatorial object, the *fission forest*, of any wild Riemann surface and a proof that it gives a sharp parameterisation of all the admissible deformation classes. As an application of (1), by considering basic examples, we show that the braid groups of all the complex reflection groups known as the generalised symmetric groups appear as wild mapping class groups. As an application of (2), we compute the dimensions of all the (global) moduli spaces of type A wild Riemann surfaces (in fixed admissible deformation classes), a generalisation of the famous result known as "Riemann's count" of the dimensions of the moduli spaces of compact Riemann surfaces.

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## §1. Introduction

The appearance of braid groups in two-dimensional gauge theory is intimately related to the theory of isomonodromic deformations of linear connections and in turn to the Riemann–Hilbert problem.

The simplest picture is to fix a Lie group $G$ such as $\mathrm{GL}_n(\mathbb{C})$ and a Riemann surface $\Sigma$ and to consider the arrow

$$\Sigma \mapsto \mathcal{M}_{\mathrm{B}} := \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

attaching the $G$-character variety (or Betti moduli space) $\mathcal{M}_{\mathrm{B}}$ to the surface. This works well in families and one obtains lots of flat nonlinear connections in this way: if $\underline{\Sigma} \to \mathbb{B}$ is a family of smooth Riemann surfaces over a base $\mathbb{B}$ then the character varieties of the fibres assemble into a fibre bundle

$$\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$$

over the same base $\mathbb{B}$, and moreover this bundle comes equipped with a natural complete flat Ehresmann connection (or in other terms it is a "local system of varieties"). Integrating this nonlinear connection around loops in $\mathbb{B}$ yields an action of the fundamental group of the base on any fibre $\mathcal{M}_{\mathrm{B}}(b)$, i.e. for any basepoint $b \in \mathbb{B}$ there is a homomorphism

$$\pi_1(\mathbb{B}, b) \to \mathrm{Aut}_{\mathrm{Poisson}}(\mathcal{M}_{\mathrm{B}}(b)). \tag{1.1}$$

For example, taking $\mathbb{B}$ to be the configuration space of $m$-tuples of points of $\mathbb{C}$ leads to the usual $m$-string braid group action on the genus zero tame character varieties (Hurwitz action), or taking $\mathbb{B}$ to be the Riemann moduli space of curves yields the natural action of the mapping class group on the character varieties of compact Riemann surfaces.

This story has a vast generalisation involving new braidings, obtained by considering the monodromy data of irregular singular meromorphic connections, and their isomonodromic (monodromy-preserving) deformations. To see this generalisation, suppose $\Sigma$ above is actually a smooth complex algebraic curve. Then, under the Riemann–Hilbert correspondence, $\mathcal{M}_{\mathrm{B}}$ parameterises *regular singular* (or *tame*) algebraic connections on vector bundles on $\Sigma$. The simplest nontrivial example is to take $\Sigma$ to be a four-punctured sphere $\mathbb{P}^1 \setminus \{0, t, 1, \infty\}$, $\mathbb{B} = \mathfrak{M}_{0,4} \cong \mathbb{C} \setminus \{0, 1\}$ the moduli space of ordered four-tuples of points, and $G = \mathrm{SL}_2(\mathbb{C})$. Then the character varieties have complex dimension 6 and are foliated by symplectic leaves $\mathcal{M}_{\mathrm{B}}(\mathcal{C}) \subset \mathcal{M}_{\mathrm{B}}$ of complex dimension two (fixing the four local monodromy conjugacy classes), all preserved by the nonlinear connection, so we can restrict attention to these subbundles. On the other side of the Riemann–Hilbert correspondence these nonlinear connections can be written explicitly, whence they become the second-order nonlinear differential equations known as the Painlevé VI equations. They control the isomonodromic deformations of rank-two Fuchsian systems with four poles on $\mathbb{P}^1$. The Painlevé VI equations are the simplest examples of *nonlinear* geometric differential equations (and this story is essentially the way they were originally discovered by R. Fuchs, building on ideas of L. Fuchs and B. Riemann). The flatness of the bundle $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$ gives the definition of *isomonodromic*, as the underlying punctured surface varies.

The generalisation comes about by considering *irregular singular* (or *wild*) algebraic linear connections on vector bundles on $\Sigma$, their topological description furnished by the irregular Riemann–Hilbert correspondence (Riemann–Hilbert–Birkhoff), involving monodromy and Stokes data, and the resulting generalisation of the Betti spaces $\mathcal{M}_{\mathrm{B}}$, the *wild character varieties*. The initial motivation was simply to obtain the first 5 Painlevé equations as integrability conditions for linear connections. This was done by Garnier [36] (rewritten in a more convenient form by Jimbo–Miwa [39]), and then generalised to generic irregular connections of arbitrary rank by Jimbo–Miwa–Ueno [40] (see their article, and also those of Garnier [36, 37], for more detailed historical background). The paper [9] then rewrote part of [40] in a more moduli-theoretic language and proved all the Jimbo–Miwa–Ueno isomonodromy equations were symplectic (this involved generalising the Narasimhan, Atiyah–Bott, Goldman symplectic form to the irregular case), showing the generic wild character varieties formed a new class of holomorphic symplectic manifolds.

The key feature in the irregular case is that there are new parameters in the connections whose deformations behave exactly like the deformations of the underlying surface $\Sigma$. In brief (in the generic setting of [40]) one looks at connections locally isomorphic to connections of the form

$$\nabla = d - A, \quad A = dQ + \Lambda \frac{dz}{z} + (\text{holomorphic}),$$

where $\Lambda$ is a constant matrix and $Q$ (the *irregular type*) is a diagonal matrix of polynomials in $1/z$ (where $z$ is a local coordinate vanishing at the pole):

$$Q = \operatorname{diag}(q_1, \ldots, q_n), \quad q_i \in x\mathbb{C}[x], \ x = 1/z.$$

The generic case of [40] is when all the differences $q_i - q_j$ are polynomials of the same degree, and then the corresponding Stokes data is quite simple, as explained in [4, 40]. In effect, the story in [40] then says that, in this setting, the base space $\mathbb{B}$ of the deformations should be enriched by adding the irregular types at each pole. The genericity condition should be preserved, and so the leading coefficient at each irregular pole should be diagonalisable with distinct eigenvalues. If the structure group is $G = \mathrm{GL}_n(\mathbb{C})$ then this adds a factor of

$$\mathfrak{t}_{\mathrm{reg}} = \left\{ v \in \mathbb{C}^n \;\middle|\; v_i \neq v_j \text{ if } i \neq j \right\}$$

to $\mathbb{B}$ for each irregular pole. This gives new braidings since the fundamental group of $\mathfrak{t}_{\mathrm{reg}}$ is the (pure) $n$-string braid group.

The simplest global irregular example is to consider connections of the form

$$d - A, \quad A = \left( \frac{A_0}{z^2} + \frac{B}{z} \right) dz,$$

where $A_0$ is diagonal and $B$ is arbitrary. Then the new deformation parameters are the eigenvalues of $A_0$, appearing in the irregular type $Q = -A_0/z$ at zero. This example is especially alluring since such connections arise [6] by considering the Fourier–Laplace transform of Fuchsian systems

$$\frac{d}{dz} - C, \quad C = \sum_1^m \frac{A_i}{z - a_i},$$

whence it becomes clear that the eigenvalues of $A_0$ correspond exactly to the positions $\{a_i\}$ of the poles of the Fuchsian system. This example makes it really clear that we should be thinking of the irregular type on an equal footing to the pole positions. Of course, most irregular connections will not be related to a regular singular connection by any such integral transform. (The exact statement of [6] was clarified in [11, Appx. A], [15, Diagram 1] – see also [46, Chap. XII], [20, §2].)

The two main directions of generalisation of this story were then the following: (1) to replace the structure group $G$ by an arbitrary complex reductive group $G$ and thereby see that all the $G$-braid groups occur in two-dimensional gauge theory ([18]), and (2) to consider all the nongeneric connections and their isomonodromic deformations (basic examples of this occur in the simply-laced story [17], motivated by the increase in symmetry that occurs by allowing nongeneric connections).

For example, in the sequence of works [12, 14, 18, 24] (in increasing generality), the wild character variety $\mathcal{M}_{\mathrm{B}} = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}$ of *any* algebraic connection on a principal $G$-bundle on $\Sigma$ was constructed, as a (finite-dimensional) algebraic Poisson variety, for any complex reductive group $G$. The simpler $\mathrm{GL}_n(\mathbb{C})$ case most relevant here is reviewed in [22]. (This algebraic construction is complementary to the analytic proof [8] that the symplectic leaves of these Poisson varieties are hyperkähler manifolds in type $A$, upgrading the complex symplectic quotient in [9] to a hyperkähler quotient.)

As part of this story, the extra deformation parameters were isolated in a coordinate-free way, leading to the general definition of a *wild Riemann surface* [18, 19, 24]. The key point is that the wild character varieties form a local system of Poisson varieties over any admissible deformation of a wild Riemann surface (so we get lots of new nonlinear flat connections generalising $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$ above, and encompassing all the Painlevé and Jimbo–Miwa–Ueno examples). This viewpoint vivifies the observation [9] that the irregular isomonodromy connections generalise the nonabelian Gauss–Manin connections. The admissibility condition generalises the notion of the connections remaining generic in the generic setting.

The aim of the present work is to study the admissible deformations of an arbitrary wild Riemann surface of type $A$. In other words, we fix the structure group to be $G = \mathrm{GL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$ and allow arbitrary formal structure at each pole. In this setting a wild Riemann surface $\boldsymbol{\Sigma}$ is a triple $(\Sigma, \mathbf{a}, \Theta)$ where

- $\Sigma$ is a compact Riemann surface,
- $\mathbf{a} \subset \Sigma$ is a finite subset,
- $\Theta$ is the data of a rank-$n$ irregular class $\Theta_a$ at each point $a \in \mathbf{a}$.

In turn, an irregular class is defined as follows (cf. [24, Prop. 8]). Each point $a \in \Sigma$ canonically determines the set $\mathcal{S}_a$ of Stokes circles at $a$, and an irregular class at $a$ is a finite multiset of Stokes circles, written symbolically as a finite sum:

$$\Theta_a = \sum n_i I_i,$$

where $n_i \geq 1$ are integer multiplicities and each $I_i \in \mathcal{S}_a$ is a Stokes circle at $a$.

If we choose a local coordinate $z = 1/x$ vanishing at $a$, then each Stokes circle $I \in \mathcal{S}_a$ can be written as

$$I = \langle q \rangle, \quad q = \sum a_i x^{k_i}, \tag{1.2}$$

where the sum is finite, $a_i \in \mathbb{C}$, and each $k_i$ is a rational number $> 0$. In brief the Stokes circle $\langle q \rangle$ is the germ of the Riemann surface where the function $q$ becomes single valued, equipped with the germ of the function $q$ (a coordinate-free definition is in [24, Rmk. 3], generalising that in [18]). The *rank* of $\Theta_a = \sum n_i I_i$ is $\sum n_i \operatorname{Ram}(I_i)$ where $\operatorname{Ram}(I)$ is the ramification number of the Stokes circle $I = \langle q \rangle$ (the lowest common multiple of the denominators of the $k_i$ present in $q$). If $\operatorname{Ram}(I) > 1$ for any Stokes circle in the irregular class, we say that the irregular class is *twisted* (or *ramified*), otherwise $\Theta_a$ is *untwisted*. For example on p. 116 of Stokes' 1857 paper [53] on the Airy functions, he writes down a basis

$$u = Cx^{-1/4}\exp(-2x^{3/2})\widehat{F}(-x^{-3/2}) + Dx^{-1/4}\exp(2x^{3/2})\widehat{F}(x^{-3/2})$$

of formal solutions to the Airy equation $y'' = 9xy$ at $x = \infty$, where $\widehat{F}$ is a formal power series. In this case the Stokes circle is $\langle 2x^{3/2} \rangle$, which has rank 2 and ramification 2, so is twisted, and (still on p. 116) Stokes drew a projection of the Stokes circle to the $x$ plane, to illustrate the change in dominance of the two branches of the exponential factor $\exp(2x^{3/2})$. This trefoil-like drawing was reproduced on the title page of [24], and many more such pictures can easily be drawn [23]. In the generic setting one just has $\Theta = \sum_1^n \langle q_i \rangle$ if $Q = \operatorname{diag}(q_i)$ as above. The original tame case arises by taking each Stokes circle to be the tame circle $\langle 0 \rangle \in \mathcal{S}_a$, so the irregular class is simply $n\langle 0 \rangle$. Thus the main difference in the wild case is that we have the choice of an arbitrary finite multiset of Stokes circles at each marked point (and these give the possible essentially singular behaviours $\exp(q)$ of the solutions of the corresponding linear connections). Such functions $q$ with arbitrary ramification appeared in Fabry's 1885 thesis [34, p. 85].

The main questions motivating this paper (and the corresponding results) are as follows.

**Question (1).** Suppose we have two rank-$n$ wild Riemann surfaces $\mathbf{\Sigma}$, $\mathbf{\Sigma}'$. How do we decide if they are admissible deformations of each other?

In the tame case the answer is well known: it happens if and only if the genus $g$ of $\Sigma$ and the number $m = \#\mathbf{a}$ of marked points match up. In the wild case we will give a complete answer to this question by defining an appropriate combinatorial object, the fission tree $\mathcal{T}(\Theta)$ of any irregular class $\Theta$, so that a wild Riemann surface determines a fission forest $\mathbf{F}$ (i.e. a collection of isomorphism classes of fission trees).

**Theorem 1.1.** *Two irregular classes* $\Theta, \Theta'$ *at a point* $a \in \Sigma$ *are admissible deformations of each other if and only if their fission trees are isomorphic:* $\mathcal{T}(\Theta) \cong \mathcal{T}(\Theta')$. *Consequently, two rank-n wild Riemann surfaces* $\mathbf{\Sigma}$, $\mathbf{\Sigma}'$ *are admissible deformations of each other if and only if the corresponding pairs* $(g, \mathbf{F})$ *are equal, where* $\mathbf{F}$ *is the fission forest and* $g$ *is the genus.*

Thus, in essence, in the wild case the number $m$ of marked points gets upgraded to the fission forest $\mathbf{F}$ (with $m$ trees in it). This result will be deduced from a more precise statement about irregular types (see Corollary 3.33). The pair $(g, \mathbf{F})$ is called the *topological skeleton* of the wild Riemann surface $\mathbf{\Sigma}$. In particular, this yields a simple criterion to see if two wild character varieties are isomorphic.

**Corollary 1.2.** *Suppose that* $\mathbf{\Sigma}$, $\mathbf{\Sigma}'$ *are two rank-n wild Riemann surfaces and let* $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$, $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}')$ *be the corresponding Poisson wild character varieties (as in [24, 22], generalising [18, 12]). If* $\mathbf{\Sigma}$, $\mathbf{\Sigma}'$ *have the same genus and fission forests, then there is an algebraic Poisson isomorphism,*

$$\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}) \cong \mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}'),$$

*between the corresponding Poisson wild character varieties. In particular, their hyperkähler symplectic leaves are thus deformation equivalent.*

*Proof.* Since $\mathbf{\Sigma}$, $\mathbf{\Sigma}'$ are admissible deformations of each other, there is a local system of Poisson varieties over some base $\mathbb{B}$ having $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ and $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}')$ as two of its fibres. This statement is proved in [18, Thm. 10.2] in the untwisted case (see also [9, Prop. 3.8, Cor. 3.9] in the generic setting) and that proof works verbatim in the general setting of [24]. Thus any path $\gamma$ in $\mathbb{B}$ between the corresponding two points of $\mathbb{B}$ lifts to an algebraic Poisson isomorphism $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}) \cong \mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}')$. □

We will also give a sharp characterisation of the possible fission trees (Definition 3.18, Corollary 3.28), which thus implies a bound on the possible isomorphism classes of wild character varieties (Section 3.7).

**Question (2).** What do the admissible deformations of an arbitrary irregular class look like? What types of generalised braid groups appear in general? Are there explicit configuration spaces analogous to the simple configuration spaces $\mathbb{C}^n \setminus$ (diagonals) that appear both in the tame case and in the generic irregular case?

To answer these, we first define the notion of pointed irregular types (by adding some ordering data to an irregular class; see Definition 2.3) and then give an explicit construction of a configuration space $\mathbf{B}(Q)$ of all admissible deformations of a pointed irregular type $Q$ (with bounded slope, i.e. bounded Poincaré–Katz rank, $\mathrm{Katz}(Q)$).

The configuration space $\mathbf{B}(Q)$ is completely explicit and involves marking free coefficients on the fission tree, subject to three types of conditions (see (3.18) and Theorem 3.27). This involves a truncation $\mathcal{T}^\flat$ of the fission tree $\mathcal{T}$ just above the maximal slope (see Section 3.5). This helps understand the admissible deformations since there are natural factorisations of the configuration spaces into simple pieces.

**Theorem 1.3.** *The configuration space $\mathbf{B}(Q)$ of admissible deformations of any pointed irregular type $Q$ is homeomorphic to the following product over the vertices $\mathbb{V}^\flat$ of its truncated fission tree $\mathcal{T}^\flat$:*

$$\mathbf{B}(Q) \cong \prod_{v\in\mathbb{V}^\flat} \mathbf{B}_v(\mathcal{T}),$$

*where $\mathbf{B}_v(\mathcal{T})$ is a point if the vertex $v$ has no nonempty children, and if $v$ has $n$ nonempty children, then $\mathbf{B}_v(\mathcal{T})$ is homeomorphic to one of the following spaces:*

$$X_n \coloneqq \left\{a_1,\dots,a_n \in \mathbb{C} \;\middle|\; a_i \neq a_j \text{ for } i\neq j\right\},$$
$$X^*_{n,N} \coloneqq \left\{a_1,\dots,a_n \in \mathbb{C} \;\middle|\; a_i \neq 0,\ a_i \neq \zeta a_j \text{ for } i\neq j,\ \zeta^N = 1\right\}.$$

Compared to the untwisted case already studied in [33, 32], the second factors $X^*_{n,N}$ are new: they are hyperplane complements whose associated hyperplane arrangements are not the complexification of some real hyperplane arrangement. Interestingly, the corresponding braid groups have been studied in [27], and the corresponding Weyl groups are the generalised symmetric groups (see below and Remark 4.4).

See Example 3.25 for a somewhat involved example of how the configuration space may thus be read off from the fission tree.

As a consequence, the pure local wild mapping class groups (i.e. the fundamental groups of these configuration spaces) factorise as products of the pure braid groups associated to these hyperplane arrangements.

**Corollary 1.4.** *Let $Q$ be a pointed irregular type and let $\mathcal{T}^\flat$ be its truncated fission tree. We have*

$$\Gamma(Q) \cong \prod_{v\in\mathbb{V}^\flat} \Gamma_v(\mathcal{T}),$$

*with $\Gamma_v(\mathcal{T}) \coloneqq \pi_1(\mathbf{B}_v(\mathcal{T}))$ the pure braid group associated to the hyperplane complement $\mathbf{B}_v(\mathcal{T})$.*

In a similar way to [32], passing from these pure local wild mapping class groups to the full local wild mapping class groups (i.e. forgetting the order of the exponential factors) involves considering the automorphisms of the fission tree, leading to a finite group, the *Weyl group* $W(\mathcal{T})$ of the tree, an extension of $\mathrm{Aut}(\mathcal{T})$.

**Theorem 1.5.** *Let* $\Theta = [Q]$ *be an irregular class associated to the pointed irregular type* $Q$ *with fission tree* $\mathcal{T}$. *Then the full local wild mapping class group* $\bar{\Gamma}(\Theta)$ *is an extension of the Weyl group* $W(\mathcal{T})$ *of the fission tree by the pure wild mapping class group* $\Gamma(\Theta)$, *i.e. we have a short exact sequence*

$$1 \to \Gamma(Q) \to \bar{\Gamma}(\Theta) \to W(\mathcal{T}) \to 1.$$

In some simple examples related to the configuration spaces $X^*_{n,N}$ (see Example 4.9), the Weyl group $W(\mathcal{T})$ is isomorphic to the generalised symmetric group $S(N,n)$, which in turn is isomorphic to the complex reflection group $G(N,1,n)$ in the Shephard–Todd list [52]. Thus we have a modular interpretation (in two-dimensional gauge theory) of an infinite family of complex reflection groups and their braid groups. (This is parallel to the appearance [10] of the $G$-braid groups in two-dimensional gauge theory for any complex reductive group $G$ in the untwisted case.)

As an application of these local results we are able to make a global statement, and write down the dimension of the (global) moduli spaces $\mathfrak{M}_{g,\mathbf{F}}$ of rank-$n$, trace-free wild Riemann surfaces for any $n$, a generalisation of Riemann's count $3g-3$ of the number of moduli of a compact genus-$g$ Riemann surface. Indeed, in the trace-free case ($G = \mathrm{SL}_n(\mathbb{C})$) we expect $\mathfrak{M}_{g,\mathbf{F}}$ to be Deligne–Mumford if $2g-2+\sum \nu(\mathcal{T}) > 0$ where $\nu(\mathcal{T}) = 1 + \mathrm{Katz}(\mathcal{T})$ and we sum over all the trees $\mathcal{T}$ in the forest $\mathbf{F}$, and then its dimension is

$$\dim(\mathfrak{M}_{g,\mathbf{F}}) = 3g - 3 + \sum \mu(\mathcal{T}), \tag{1.3}$$

where $\mu(\mathcal{T})$ is the moduli number of the fission tree $\mathcal{T}$, from Definition 3.24 (equal to 1 plus the dimension of the configuration space, where the 1 corresponds to the pole position). In the tame case with $m$ marked points, this specialises to the familiar formula

$$\dim(\mathfrak{M}_{g,m}) = 3g - 3 + m.$$

## §1.1. Layout of the paper

The next section will review the notion of Stokes circles and irregular classes in more detail, before defining various convenient flavours of irregular types, adding certain ordering data to an irregular class. Then the notion of admissible deformation will be reviewed leading to the initial definition of the configuration spaces. Section 3 then gives the classification of admissible deformations, in terms of level data and fission data leading up to the precise definition of fission trees. This then yields the most convenient definition of the configuration spaces in terms of realisations of fission trees (Theorems 3.27, 3.30). Then we deduce the product decompositions (Corollary 3.31) and the classification of admissible deformations in terms of fission trees (Corollary 3.33). Then Section 3.7 deduces the global result involving fission forests/topological skeleta $(g, \mathbf{F})$, and finally Section 4 deduces the results on the local wild mapping class groups and establishes the link to the braid groups of the complex reflection groups $G(N, 1, n)$. Some possible future projects are discussed in the last section.

## §2. General setting

### §2.1. Twisted irregular types

**2.1.1. The exponential local system.** The formal data of irregular connections in the twisted case can be formulated geometrically in terms of the so-called exponential local system, which we now briefly recall (see [22] for more details). In summary, the idea is to see the exponents $q$ of the exponential factors $\exp(q)$ (occurring in formal solutions of connections, controlling their essentially singular behaviour) as sections of an intrinsic covering space $\pi\colon \mathcal{I} \to \partial$ (i.e. a local system of sets) on the circle of directions around the singularity. (We will often abuse language and refer to the exponents $q$ themselves as the exponential factors.) The connected components of $\mathcal{I}$ are the Stokes circles. The basic "extra modular parameters", the irregular class (that we eventually want to deform), is a finite multiset in the set $\mathcal{S} = \pi_0(\mathcal{I})$ of Stokes circles – this amounts to choosing a finite number of Galois orbits of exponents $q$, each with a multiplicity $\geq 1$. Note that we use the word "twisted" to refer to the case where one of the Stokes circles is a nontrivial cover of the circle of directions $\partial$ as in [24] (similarly to the theory of twisted loop groups), or equivalently that one of the exponents $q$ involves a root of the local coordinate (this is sometimes referred to as the case with "ramified formal normal form", and should not be confused with the term "ramified connection", meaning any type of singular connection).

Let $\Sigma$ be a Riemann surface and $a \in \Sigma$ a point. Let $\phi\colon \widehat{\Sigma} \to \Sigma$ be the real oriented blow-up at $a$ of $\Sigma$. The preimage $\partial \coloneqq \phi^{-1}(a)$ is a circle whose points correspond to real oriented directions in $\Sigma$ at $a$. An open interval $U \subset \partial$ determines a germ of sector $\mathrm{Sect}_U$ at $a$, and if $d \in \partial$ is a direction then $\mathrm{Sect}_d$ will denote the germ of an open sector spanning the direction $d$ (where both the opening and the radius may decrease). Strictly speaking, $\mathrm{Sect}_d$ is the tangential filter of open sets determined by the direction $d$, as in [26, Ch. 1, §6] (this type of tangential filter appears on p. 85 of [29]). In turn, functions on $\mathrm{Sect}_d$ (i.e. germs in the direction $d$) are defined as usual for germs of mappings with respect to a filter ([26, p. 66]).

Let $z$ be a local coordinate vanishing at $a$ and write $x = z^{-1}$. The exponential local system $\mathcal{I}$ is a local system of sets (that is, a covering space) on $\partial$ whose sections are germs of holomorphic functions on sectors (*Fabry functions*) that are finite sums of the form

$$q = \sum_i a_i x^{k_i},$$

where $k_i \in \mathbb{Q}_{>0}$, and $a_i \in \mathbb{C}$. More precisely, if we fix a direction $d \in \partial$ and choose a branch of $\log(z)$ on $\mathrm{Sect}_d$ then the fibre $\mathcal{I}_d = \pi^{-1}(d)$ of $\mathcal{I}$ over $d$ is the set of all such functions on $\mathrm{Sect}_d$, so that

$$\mathcal{I}_d = \left\{ q = \sum_i a_i x^{k_i} \right\} \;\cong\; \bigcup_{n\in\mathbb{N}} x^{1/n}\mathbb{C}[x^{1/n}] = \bigcup_{n\in\mathbb{N}} \mathbb{C}(\!(z^{1/n})\!)/\mathbb{C}[\![z^{1/n}]\!], \tag{2.1}$$

where $x^k := \exp(-k\log(z))$ on the left, and $x = 1/z$ is viewed as a symbol on the right. Thus they are "principal parts of Puiseux series", but viewed as actual functions on $\mathrm{Sect}_d$, via a choice of logarithm (i.e. the isomorphism from $\mathcal{I}_d$ to Puiseux principal parts depends on this choice). The intrinsic (coordinate-free) definition of $\mathcal{I}$ is in [24, Rmk. 3], whence a point $\alpha \in \mathcal{I}_d$ is an *equivalence class* of certain holomorphic functions $q_\alpha$ on $\mathrm{Sect}_d$.

The connected component of $\mathcal{I}$ of such a local section $q$ is a finite-order cover of the circle $\partial$; this covering circle, "the Stokes circle $\langle q\rangle$ of $q$", is essentially the (germ near $\partial$ of the) Riemann surface where $q$ becomes single valued.

More precisely, let $r = \mathrm{Ram}(q)$ be the smallest integer such that the expression $q = \sum_i a_i x^{k_i}$ is a polynomial in $x^{1/r}$, the *ramification order* of $q$. The corresponding holomorphic function is multivalued, and becomes single valued when passing to a finite cover $t^r = z$. Therefore, the corresponding Stokes circle, which we denote by $\langle q\rangle$, is an $r$-sheeted cover of $\partial$. As a topological space, it is homeomorphic to a circle, and $\mathcal{I}$ is thus a disjoint union of (an infinite number of) these Stokes circles. Thus $\pi\colon \mathcal{I} \to \partial$ is a covering space and if $I = \langle q\rangle \subset \mathcal{I}$ is a connected component then $\pi\colon I \to \partial$ is a degree $r = \mathrm{Ram}(q)$ covering map between two circles.

There are several polynomials in $x^{1/r}$ giving rise to the same connected component $I = \langle q\rangle$. They correspond to the Galois orbit of $q$, under the Galois group of $I \to \partial$ which is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, and are parameterised by the $r$ points of the fibre $I_d = \pi^{-1}(d)$ for any direction $d$.

Explicitly, if we write $q = \sum_{j=1}^s a_j x^{j/r}$, $r = \mathrm{Ram}(q)$, with $a_s \neq 0$, then the polynomials $q_i$ such that $\langle q_i\rangle = \langle q\rangle$ are the Galois conjugates

$$q_i = \sigma^i(q) = \sum_{j=1}^s a_j \omega^{ij} x^{j/r}, \quad i = 0, 1, \ldots, r-1,$$

where $\omega = \exp(-2\sqrt{-1}\pi/r)$. The fibre $I_d$ above $d$ of the cover $I \to \partial$ is equal to the set of germs of functions $q_0, \dots, q_{r-1}$. The monodromy $\sigma\colon I_d \to I_d$ of the cover $I \to \partial$ is given by $\sigma(q_i) = q_{i+1}$.

The degree $s$ of $q$ as a polynomial in $x^{1/r}$ is called the irregularity of $q$, which we denote by $\operatorname{Irr}(q) \in \mathbb{N}$. The slope of $q$, $\operatorname{slope}(q) := \frac{s}{r} = \operatorname{Irr}(q)/\operatorname{Ram}(q)$, is the maximal exponent present in $q$. If $r = 1$ we say that the circle $\langle q \rangle$ is untwisted/unramified. We will refer to $\langle 0 \rangle$ as the *tame circle.*

Here we view $\mathcal{I}$ as a disjoint union of circles, and the map $\pi\colon \mathcal{I} \to \partial$ as a covering space with discrete fibres. Later, below, we will deform the functions $q$, thus "remembering" the complex vector space structure of the fibres of $\pi$.

**2.1.2. Irregular classes, finite subcovers, and levels.** In this language, following [24, Prop. 8], an *irregular class* is a locally constant map $\Theta\colon \mathcal{I} \to \mathbb{N}$, assigning an integer to each component of $\mathcal{I}$, equal to zero for all but a finite number of circles. It is thus constant on each component circle, i.e. corresponds to a map $\pi_0(\mathcal{I}) \to \mathbb{N}$. An irregular class can be written as a formal sum,

$$\Theta = n_1\langle q_1 \rangle + \cdots + n_m\langle q_m \rangle,$$

where $n_1, \dots, n_m > 0$ are integers. Thus an irregular class $\Theta$ is just a finite multiset of Stokes circles, or in concrete terms a finite multiset of Galois orbits of exponential factors. (A more general definition of irregular classes, which works for other structure groups, is in [24, §3.5].) The *rank* of the irregular class $\Theta$ is the integer $\operatorname{Rank}(\Theta) := \sum_i n_i \operatorname{Ram}(q_i)$.

The (total) ramification $\operatorname{Ram}(\Theta)$ of an irregular class $\Theta = \sum n_i I_i$ is the lowest common multiple of the ramifications $\operatorname{Ram}(I_i)$ of the active circles $I_i$ in $\Theta$.

By the formal meromorphic classification of meromorphic connections (Fabry, Cope, Hukuhara, Turrittin, Levelt, Jurkat [5, Thm. II], Deligne [46, Thm. IV.2.3]), any connection on a rank-$n$ vector bundle on the formal punctured disk determines an irregular class of rank $n$, taking the Galois orbits of the exponents of the exponential factors $e^q$ that occur, repeated according to their multiplicity. For example, a regular singular connection has class $n\langle 0 \rangle$ just involving the tame circle.

A *finite subcover* is a subset $I \subset \mathcal{I}$ such that $I \to \partial$ is a finite cover, i.e. it is a finite set of Stokes circles. An irregular class determines a finite subcover consisting of the active exponents $I = \Theta^{-1}(\mathbb{N}_{>0})$. Explicitly, if $\Theta = \sum_i n_i I_i$, then $I = \bigsqcup_i I_i$. Thus an irregular class corresponds to the data of a finite subcover $I \subset \mathcal{I}$, together with a positive integer $n_i$ for each connected component.

Anyrank-$n$ irregular class $\Theta$ determines another irregular class $\operatorname{End}(\Theta)$ of rank $n^2$ (as in [22, pp. 71–72]). The nonzero slopes of the circles in $\operatorname{End}(\Theta)$ are the *levels* of $\Theta$ ([30, p. 73], [44, p. 858], [3, eq. (5.2)], [47]). Thus

$$\text{Levels}(\Theta) = \left\{\text{slope}(q_\alpha - q_\beta) \;\middle|\; \alpha, \beta \in I_d\right\} \setminus \{0\} \subset \mathbb{Q}_{>0}, \tag{2.2}$$

where $I \subset \mathcal{I}$ is the finite subcover underlying $\Theta$ and $I_d$ is any fibre of $I$. Note that the existence of irregular classes with multiple levels means there are connections whose formal fundamental solutions are not $k$-summable for any $k$ (and in particular not Borel summable), and this fact led to the theory of *multisummation* (cf. [3, §5]).

**2.1.3. Irregular types.** In the untwisted case, a distinction was made between irregular types and irregular classes, the difference being that for irregular classes the exponential factors are unordered, whereas for irregular types the order of the exponential factors matters.[1] We will now do the same for the twisted case. As a first step (as in [22, §5.3]) we can just choose an ordering of the circles in an irregular class.

**Definition 2.1.** An *irregular type* of rank $n$ is an ordered list

$$[(n_1, I_1), (n_2, I_2), \ldots, (n_m, I_m)]$$

of distinct Stokes circles $I_i \subset \mathcal{I}$ each with a multiplicity $n_i \geq 1$, such that $\sum n_i \operatorname{Ram}(I_i) = n$.

However, here it will be more convenient to work with ordered lists of exponential factors. Thus in the rest of the article we fix once and for all a direction $d \in \partial$, local coordinate $z = x^{-1}$, and a choice of logarithm around $d$, so that a section of the exponential local system $\mathcal{I}$ around the direction $d$ is identified with a Puiseux principal part $q \in x^{1/r}\mathbb{C}[x^{1/r}]$ for some $r$, as in (2.1).

**Definition 2.2.** A *full irregular type* of rank $n$ is a Galois closed ordered list $Q = (q_1, \ldots, q_n)$ of not necessarily distinct polynomials $q_i \in x^{1/r_i}\mathbb{C}[x^{1/r_i}]$ for some $r_i \in \mathbb{N}_{>0}$.

Asking for the list $(q_1, \ldots, q_n)$ to be Galois closed is equivalent to asking for it to be closed under the monodromy $\sigma$ of the exponential local system $\mathcal{I}$. Explicitly, the list $Q = (q_1, \ldots, q_n)$ is Galois closed if there exists a permutation $\widehat{\sigma} \in \operatorname{Sym}_n$ such that $\sigma(q_i) = q_{\widehat{\sigma}(i)}$ for $i = 1, \ldots, n$.

More intrinsically, given a rank-$n$ irregular class $\Theta$ and a direction $d$ then $\Theta$ determines a length-$n$ multiset in $\mathcal{I}_d$, and the full irregular types determining

[1] In the untwisted case an irregular class is the same thing as the *bare irregular type* determined by an irregular type in the sense of [18, Rmk. 10.6], with *irregular type* as defined in [18, Defn. 7.1]. (cf. also [9, Defn. 2.4] in the generic case, which gave a coordinate-free approach to [40]).

$\Theta$ correspond exactly to the $n!$ possible orderings of this multiset. If $Q$ is a full irregular type, let $I_1, \dots, I_m$ be the set of *distinct* Galois orbits of the elements of the list $(q_1, \dots, q_n)$ and $n_i$, $i = 1, \dots, m$ the multiplicity of $I_i$, i.e. the number of times that each element of $(I_i)_d$ appears in the list. Then the irregular class associated to $Q$ is $\Theta = n_1 I_1 + \cdots + n_m I_m$, and we will write $\Theta = [Q]$ for the class of $Q$. In particular, $n = \mathrm{Rank}(\Theta) = \sum n_i \,\mathrm{Ram}(I_i)$. In these terms, for any irregular class $\Theta$, the set of full irregular types determining $\Theta$ corresponds to all possible orderings of the elements of the list $(q_1, \dots, q_n)$ of one full irregular type determining $\Theta$.

It will also be useful to introduce a variant of these definitions, and consider a specific subset of full irregular types, to account for the fact that the Galois orbits are already naturally cyclically ordered.

**Definition 2.3.** A *pointed irregular type* is an ordered list

$$Q = [(n_1, q_1), \dots, (n_m, q_m)],$$

where $n_i \in \mathbb{N}_{>0}$, and the $q_i$ are Puiseux principal parts lying in distinct Galois orbits.

We identify a rank-$n$ pointed irregular type $Q$ as a full irregular type as follows:

$$\begin{aligned} Q = (\underbrace{q_1, \dots, q_1}_{n_1 \text{ times}}, \underbrace{\sigma(q_1), \dots, \sigma(q_1)}_{n_1 \text{ times}}, \dots, \underbrace{\sigma^{r_1-1}(q_1), \dots, \sigma^{r_1-1}(q_1)}_{n_1 \text{ times}}, \\ \dots, \dots, \quad \dots, \dots, \dots, \underbrace{\sigma^{r_m-1}(q_m), \dots, \sigma^{r_m-1}(q_m)}_{n_m \text{ times}}), \end{aligned}$$

where $r_i = \mathrm{Ram}(q_i)$. Note that (more intrinsically) a rank-$n$ pointed irregular type is equivalent to an ordered list

$$[(n_1, p_1, I_1), \dots, (n_m, p_m, I_m)],$$

where $n_i \in \mathbb{N}_{>0}$, the $I_i$ are distinct Stokes circles, $p_i \in (I_i)_d$ is a point of $I_i$ lying over $d$, and $n = \sum n_i \,\mathrm{Ram}(I_i)$. The correspondence is given by taking $q_i$ to be the Puiseux principal part determined by the point $p_i \in (I_i)_d$ via the logarithm choice.

Finally, observe that the notion of a pointed irregular type introduces extra discrete invariants that we do not care about (for example, $\mathrm{slope}(q_1 - q_2)$ may vary if $q_1$ is moved in its Galois orbit). To avoid this we define a notion of *compatibility* between the chosen exponential factors in different Galois orbits. For any $k \in \mathbb{Q}$

let

$$\tau_k\colon \mathcal{I}_d \to \mathcal{I}_d, \quad q = \sum a_i x^{k_i} \mapsto \tau_k(q) := \sum_{k_i \geq k} a_i x^{k_i} \tag{2.3}$$

be the truncation map, discarding all monomials of slope $< k$.

**Definition 2.4.** A pointed irregular type $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ is *compatible* if for each possible exponent $k \in \mathbb{Q}_{>0}$, and any indices $i$, $j$,

$$\langle \tau_k(q_i) \rangle = \langle \tau_k(q_j) \rangle \;\Rightarrow\; \tau_k(q_i) = \tau_k(q_j).$$

In other words, if the truncations are in the same Galois orbits, then the truncations are equal.

It is straightforward to see that for any irregular class $\Theta$, a compatible (pointed) irregular type $Q$ exists with class $\Theta = [Q]$. Up to isomorphism the configuration space $\mathbf{B}(Q)$ will not depend on the choice of irregular type with irregular class $\Theta$, so we may assume without loss of generality that the pointed irregular types we are considering are compatible.

**2.1.4. Pullback to the untwisted case.** The notions of (twisted) irregular classes and irregular types can easily be related to the corresponding untwisted notions, by passing to a finite cyclic cover. Explicitly, let $Q = (q_1, \dots, q_n)$ be any full irregular type. Let $r$ be an integer multiple of $\mathrm{Ram}(Q)$ so that $\mathrm{Ram}(q_i)$ divides $r$ for all $i$. Introduce the variable $t$ such that $t^r = x$ (so $t^{-1}$ is a coordinate on a cyclic $r$-fold ramified cover). Let $\widehat{q}_i \in t\mathbb{C}[t]$ be $q_i$ seen as a polynomial in $t$. Then $\widehat{Q} := \mathrm{diag}(\widehat{q}_1, \dots, \widehat{q}_n)$ is an untwisted irregular type associated to $\Theta$. Its (untwisted) irregular class $\widehat{\Theta} = [\widehat{Q}]$ only depends on $\Theta$ and is simply the pullback. Notice that the irregular class $\widehat{\Theta}$ is invariant under the action of $\mathbb{Z}/r\mathbb{Z}$ on the set of untwisted irregular classes obtained by replacing all polynomials $\widehat{q}_i(t) \in \mathbb{C}[t]$ by $\widehat{q}_i(e^{2\sqrt{-1}k\pi/r}t)$, for any integer $k$: we say that the untwisted irregular class $\widehat{\Theta}$ is $r$-Galois closed. Conversely, if $\widehat{Q}$ is an $r$–Galois closed untwisted irregular type, it defines a (twisted) irregular type $Q$ such that the ramification orders of all exponential factors divide $r$. The (twisted) irregular class of $Q$ only depends on the (untwisted) irregular class of $\widehat{Q}$.

## §2.2. Admissible deformations

The notion of admissible deformations was defined in [18] for arbitrary untwisted meromorphic connections in the context of any reductive group $G$, extending the generic case in [40, 45] for $\mathrm{GL}_n(\mathbb{C})$ and in [10] for other $G$. It can be extended to the twisted setting simply by saying that a family of irregular classes is an admissible deformation if and only if some (and hence any) cyclic pullback to the untwisted case is an admissible deformation. In more detail this works out as follows.

Fix $\mathcal{I}$ as above and let $\mathbb{B}$ be a connected complex manifold. Choose a rank-$n$ irregular class $\Theta_b$ on $\mathcal{I}$ for each $b \in \mathbb{B}$, thus defining a (set-theoretic) map

$$\phi\colon \mathbb{B} \to \mathrm{IC}_n(\mathcal{I}), \quad b \mapsto \Theta_b$$

to the set $\mathrm{IC}_n(\mathcal{I})$ of rank-$n$ irregular classes, i.e. length-$n$ multisets in $\pi_0(\mathcal{I})$. We will define when this collection of classes is a (holomorphic) admissible deformation.

Note that since the rank $n$ is fixed, the total ramification $\mathrm{Ram}(\Theta_b)$ is uniformly bounded[2] on $\mathbb{B}$, for example by $n!$. Thus we can choose an integer $N$ and set $x = t^N$ so that (in terms of $t$) $\Theta_b$ is a family of untwisted irregular classes, i.e. a multiset in $t\mathbb{C}[t]$ of length $n$, for each $b \in \mathbb{B}$.

By definition (see [18, Rmk. 10.6]) this is an admissible deformation if it can locally be represented as $\Theta_b = [Q_b]$ in terms of an admissible family of untwisted irregular types $Q_b = (q_1, \dots, q_n)$ with $q_i \in t\mathbb{C}[t]$ dependent on $b$. Finally (by [18, Defn. 10.1]) this is a (holomorphic) admissible deformation if each $q_i$ varies holomorphically with $b$ and the degree of the polynomial $q_i - q_j \in t\mathbb{C}[t]$ is constant (independent of $b$) for each $i \neq j$ (the degree is an integer $\geq 0$). Similarly, one can define smooth admissible deformations etc. by allowing the coefficients of the $q_i$ to vary smoothly rather than holomorphically etc. This leads to the following more direct definition, by noting that the slope multiplied by $N$ gives the degree in $t$ when pulled back, upstairs.

**Definition 2.5.**
- A holomorphic family $Q_b = (q_1, \dots, q_n)$ of full irregular types (with $q_i \in \mathcal{I}_d$) is an admissible deformation if

$$\boxed{\mathrm{slope}(q_i - q_j) \text{ is independent of } b \text{ for all } i, j.} \tag{2.4}$$

- The family $\Theta_b$ of irregular classes is a holomorphic admissible deformation if it can locally be represented as $\Theta_b = [Q_b]$ for a holomorphic family of full irregular types $Q_b = (q_1, \dots, q_n)$ with $q_i \in \mathcal{I}_d$, varying admissibly, i.e. satisfying (2.4).

If $\Theta$ and $\Theta'$ are two rank-$n$ irregular classes, we say that $\Theta'$ is an admissible deformation of $\Theta$ if there exists an admissible deformation $(\Theta_b)$ indexed by some connected manifold $\mathbb{B}$ equipped with two points $b_1, b_2 \in \mathbb{B}$ such that $\Theta_{b_1} = \Theta$ and $\Theta_{b_2} = \Theta'$. Similarly, if $Q$ and $Q'$ are two rank-$n$ full irregular types, we will say

[2] In fact, since $n = \sum n_i \,\mathrm{Ram}(I_i)$ it is bounded by the largest possible lowest common multiple of the elements of any integer partition of $n$; this is known as Landau's function $g(n)$, whose first 10000 values are listed at https://oeis.org/A000793; for example, $g(5) = 6 = \mathrm{lcm}(2,3), g(7) = 12 = \mathrm{lcm}(3,4)$.

that $Q'$ is an admissible deformation of $Q$, and we write $Q \simeq Q'$, if they are two values of an admissible family of full irregular types.

A continuity argument shows the Galois orbits in a full irregular type do not change under a holomorphic admissible deformation, i.e. the same permutation $\widehat{\sigma}$ works throughout the deformation (in particular the ramification indices of the Stokes circles are constant).

**Example 2.6.** Let us consider the holomorphic family of exponential factors $q(b) = x^{1/2} + x^{1/3} + bx^{1/6}$, which has ramification 6 for any $b \in \mathbb{C}$. The first few Galois conjugates of $q$ are

$$\begin{aligned} q = q_0(b) &= x^{3/6} + x^{2/6} + bx^{1/6}, \\ \sigma(q) = q_1(b) &= \varepsilon^3 x^{3/6} + \varepsilon^2 x^{2/6} + b\varepsilon x^{1/6}, \\ \sigma^2(q) = q_2(b) &= x^{3/6} + \varepsilon^4 x^{2/6} + b\varepsilon^2 x^{1/6}, \end{aligned}$$

where $\varepsilon = \exp(-\pi i/3)$. Considering $\mathrm{slope}(q_i - q_j)$ for $i, j = 0, \dots, 5$ shows that $\Theta_b = \langle q \rangle$ is an admissible deformation over $\mathbb{B} = \mathbb{C}$. Observe that for $b = 0$ we have $\mathrm{Ram}(q_0 - q_2) = 3$, but it is 6 for $b \neq 0$ so not everything behaves continuously.

**2.2.1. Numerical equivalence of irregular types.** We will try to guess a simple numerical criterion for (pointed) irregular types to be admissible deformations of each other. To this end consider the following relation.

**Definition 2.7.** Let $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ be a pointed irregular type. If $Q' = [(n'_1, q'_1), \dots, (n'_p, q'_p)]$ with each $q'_i \in \mathcal{I}_d$ a Puiseux polar part, then we say that $Q$, $Q'$ are *numerically equivalent*, and write

$$Q' \sim Q, \tag{2.5}$$

if $p = m$, $n'_i = n_i$ $(i = 1, \dots, m)$, and

$$\mathrm{slope}(\sigma^k(q'_i) - \sigma^l(q'_j)) = \mathrm{slope}(\sigma^k(q_i) - \sigma^l(q_j)), \tag{2.6}$$

for all $i$, $j$, $k$, $l$ with $1 \leq i, j \leq m$, $0 \leq k \leq \mathrm{Ram}(q_i)$, $0 \leq l \leq \mathrm{Ram}(q_j)$, where $\sigma$ is the Galois action.

**Lemma 2.8.** *If $Q' \sim Q$ as above, then $Q'$ is a pointed irregular type and moreover* $\mathrm{Ram}(q'_i) = \mathrm{Ram}(q_i)$ *for all $i$.*

*Proof.* Taking $j = i$, $k = \mathrm{Ram}(q_i)$, $l = 0$ shows that $\mathrm{Ram}(q'_i) \leq \mathrm{Ram}(q_i)$. Thus $\mathrm{Ram}(q'_i) = \mathrm{Ram}(q_i)$ since if $\mathrm{Ram}(q'_i) < \mathrm{Ram}(q_i)$ then there would be some identification amongst the list $\sigma^k(q'_i)$, $k = 1, 2, \dots, \mathrm{Ram}(q_i)$, but this is not possible as the differences of the slopes match those of the list $\sigma^k(q_i)$, $k = 1, 2, \dots$. Then the fact that $Q'$ is a pointed irregular type, i.e. its $m$ Galois orbits are distinct, follows from the fact that $Q$ is a pointed irregular type, so none of the slopes between two Galois orbits vanish. □

This implies $\sim$ is an equivalence relation when restricted to pairs of pointed irregular types. We will eventually see (Corollary 3.32) that for compatible pointed irregular types it is the same as the relation given by admissible deformation.

Thus it seems we should consider the simple numerical condition (2.6) applied blindly to lists of Puiseux polar parts. This leads to the following configuration spaces.

**Remark 2.9.** Note that if we just impose that $Q' = [(n_1, q'_1), \dots, (n_m, q'_m)]$ is a rank-$n$ pointed irregular type and the apparently weaker condition that (2.6) holds just for $1 \le i, j \le m$, $0 \le k \le \mathrm{Ram}(q_i) - 1$, $0 \le l \le \mathrm{Ram}(q_j) - 1$, then it follows that $Q' \sim Q$ (because this implies $\mathrm{Ram}(q'_i) \ge \mathrm{Ram}(q_i)$ and then the condition to have rank $n$ implies $\mathrm{Ram}(q'_i) = \mathrm{Ram}(q_i)$ for all $i$).

**2.2.2. Configuration spaces.** We will define a *configuration space* for each given (pointed) irregular type $Q = [(n_1, q_1), \dots, (n_m, q_m)]$, and later see it contains all the admissible deformations with bounded slope. Let $r = \mathrm{Ram}(Q) = \mathrm{lcm}\{\mathrm{Ram}(q_i)\}$ be the total ramification of $Q$ and let

$$K = \mathrm{Katz}(Q) := \max(\mathrm{slope}(q_1), \dots, \mathrm{slope}(q_m))$$

be the largest slope, which is essentially the Poincaré–Katz rank of $Q$ (cf. Poincaré [49, p. 305], Katz [42, eq. 11.9.7]). Thus all the $q_i$ can be expressed as polynomials in $t := x^{1/r}$ of degree at most $s := rK$.

Clearly, any pointed irregular type with the same number $m$ of terms, the same multiplicities and ramifications, and that has Poincaré–Katz rank $\le K$, will be of the form

$$Q_{\mathbf{a}} := \left[\left(n_1, \sum_{j=1}^{s} a_{1,j} t^j\right), \dots, \left(n_m, \sum_{j=1}^{s} a_{m,j} t^j\right)\right], \quad t = x^{1/r} \tag{2.7}$$

for some unique collection of coefficients $\mathbf{a} = (a_{i,j}) \in \mathbb{C}^{ms}$. This motivates the following definition.

**Definition 2.10.** Suppose $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ is a pointed irregular type and $r = \mathrm{Ram}(Q)$, $K = \mathrm{Katz}(Q)$, $s = rK$. The configuration space of $Q$ with bounded Poincaré–Katz rank is the topological space $\mathbf{B}(Q)$ defined by

$$\mathbf{B}(Q) := \left\{\mathbf{a} = (a_{i,j}) \in \mathbb{C}^{ms} \;\middle|\; Q_{\mathbf{a}} \sim Q\right\}, \tag{2.8}$$

with its topology being the one induced from the usual topology of $\mathbb{C}^{ms}$, where $\sim$ is from (2.5).

We will show below (Corollary 3.34) that $\mathbf{B}(Q)$ is a fine moduli space of all admissible deformations (with Poincaré–Katz rank $\le \operatorname{Katz}(Q)$) of the pointed irregular type $Q$. In the remainder of the article, our goal will then be to explicitly describe $\mathbf{B}(Q)$ and compute its fundamental group. The restriction about having bounded slopes is for the sake of convenience, since it allows us to deal with a finite number of coefficients. As was already the case [33, 32] for the untwisted situation, this entails no loss of generality as far as the topology of $\mathbf{B}(Q)$ is concerned: up to homotopy equivalence $\mathbf{B}(Q)$ does not change if we allow for coefficients associated to higher exponents.

Notice that if $Q_1$ and $Q_2$ are two pointed irregular types corresponding to the same irregular class $\Theta$, the spaces $\mathbf{B}(Q_1)$ and $\mathbf{B}(Q_2)$ are homeomorphic, an homeomorphism being given by permuting the active circles and shifting cyclically the distinguished representative of each Galois orbit by the appropriate amount. With a slight abuse of language, we may thus speak of the configuration space $\mathbf{B}(\Theta)$ that is well defined up to homeomorphism.

Similarly we define a configuration space of trace-free pointed irregular types. First define the *trace* of a full irregular type $Q = (q_1, \dots, q_n)$ to be $\operatorname{Tr}(Q) = \sum_1^n q_i \in \mathcal{I}_d$.

**Definition 2.11.** Suppose $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ is a pointed irregular type and $r = \operatorname{Ram}(Q)$, $K = \operatorname{Katz}(Q)$, $s = rK$. The traceless (or *special*) configuration space of $Q$ is the topological space $\mathbf{SB}(Q)$ defined by

$$\mathbf{SB}(Q) \coloneqq \left\{ \mathbf{a} = (a_{i,j}) \in \mathbb{C}^{ms} \;\middle|\; Q_{\mathbf{a}} \sim Q,\ \operatorname{Tr}(Q_{\mathbf{a}}) = 0 \right\}, \tag{2.9}$$

with its topology being the one induced from the usual topology of $\mathbb{C}^{ms}$, where $\sim$ is from (2.5).

If $Q = [(n_1, q)]$ just has one Galois orbit then $\operatorname{Tr}(Q) = n_1 \sum_{i=1}^{\operatorname{Ram}(q)} \sigma^i(q)$. In turn, since roots of unity sum to zero, this equals $\operatorname{Tr}(Q) = n_1 \operatorname{Ram}(q) \pi_{\mathrm{un}}(q)$, where $\pi_{\mathrm{un}} \colon \mathcal{I}_d \to x\mathbb{C}[x]$ is the linear map picking out the unramified monomials in $q$, so that $\pi_{\mathrm{un}}(x^k) = x^k$ if $k \in \mathbb{N}$ and $\pi_{\mathrm{un}}(x^k) = 0$ otherwise. It follows that the trace of any irregular type lies in the unramified part $x\mathbb{C}[x] \subset \mathcal{I}_d$. Further, there is a projection pr:

$$Q = (q_1, \dots, q_n) \mapsto \operatorname{pr}(Q) = Q - \frac{1}{n}\operatorname{Tr}(Q) = \Big(q_1 - \frac{1}{n}\operatorname{Tr}(Q), \dots, q_n - \frac{1}{n}\operatorname{Tr}(Q)\Big)$$

mapping any full irregular type to a trace-free irregular type. In particular, it makes no difference if we replace $Q$ by its trace-free projection in the definition

(2.9), and there are maps

$$\mathbf{B}(Q) \twoheadrightarrow \mathbf{SB}(Q) \hookrightarrow \mathbf{B}(Q), \tag{2.10}$$

where the first map is pr and the second is the natural inclusion. We will see below (Corollary 3.34) that $\mathbf{SB}(Q)$ is a fine moduli space of all trace-free admissible deformations of the pointed irregular type $\mathrm{pr}(Q)$. We will also show that $\mathbf{SB}(Q)$ and $\mathbf{B}(Q)$ are finite-dimensional complex algebraic manifolds (Zariski open in a complex vector space). Admitting this temporarily, one can already observe that the dimensions will differ by the integer part of the Poincaré–Katz rank and they are homotopy equivalent.

**Lemma 2.12.** *For any pointed irregular type $Q$, the configuration spaces $\mathbf{SB}(Q)$, $\mathbf{B}(Q)$ are homotopy equivalent, and*

$$\dim(\mathbf{SB}(Q)) = \dim(\mathbf{B}(Q)) - \lfloor \mathrm{Katz}(Q) \rfloor.$$

*Proof.* Two elements are in the fibre of the map $\mathrm{pr}\colon \mathbf{B}(Q) \twoheadrightarrow \mathbf{SB}(Q)$ if and only if they differ by the operation $(q_1, \dots, q_n) \mapsto (q_1 - q, \dots, q_n - q)$ for some $q \in x\mathbb{C}[x]$ of slope $\leq K$. The dimension of the space of such polynomials $q$ is $\lfloor \mathrm{Katz}(Q) \rfloor$, and this gives a retraction onto $\mathbf{SB}(Q)$. □

**Remark 2.13.** Isomonodromic deformations of a special class of twisted irregular connections were considered in [7], under a genericity condition (so that the sizes of the Galois orbits are controlled by the Jordan blocks of the leading coefficient). The relation between our general admissibility condition and the Lidskii conditions in [7], specific to their setting, are not immediately clear to us.

## §3. Classification of admissible deformations

Since an essential difference in the twisted case compared to the untwisted one is that one has to consider differences between different branches of the same exponential factor, it is worth investigating first what the admissible deformations are in the case of an irregular type corresponding to an irregular class with only one active circle.

### §3.1. A single Stokes circle

Suppose $I = \langle q \rangle \subset \mathcal{I}$ is a single Stokes circle. Recall from (2.2) that the *levels* of $I$ are the nonzero slopes of $\mathrm{End}(I)$, so that, for any $d \in \partial$,

$$\mathrm{Levels}(I) = \left\{ \mathrm{slope}(q_\alpha - q_\beta) \;\middle|\; \alpha, \beta \in I_d \right\} \setminus \{0\},$$

where $q_\alpha\colon \operatorname{Sect}_d \to \mathbb{C}$ is the function determined by $\alpha \in I_d \subset \mathcal{I}_d$. The set Levels$(I)$ is a finite, possibly empty, subset of $\mathbb{Q}_{>0}$. Suppose there are $m$ levels and write

$$\operatorname{Levels}(I) = (k_1 > k_2 > \cdots > k_m) \subset \mathbb{Q}_{>0}.$$

The key classification statement is as follows.

**Proposition 3.1.** (a) *Two Stokes circles $I, J \subset \mathcal{I}$ are admissible deformations of each other if and only if* $\operatorname{Levels}(I) = \operatorname{Levels}(J) \subset \mathbb{Q}$.

(b) *A subset $(k_1 > k_2 > \cdots > k_m) \subset \mathbb{Q}_{>0}$ is the set of levels of some circle $I \subset \mathcal{I}$ if and only if*

(3.1) $\quad k_1, k_2, \dots, k_m$ *have strictly increasing common denominators* $> 1$.

*In other words if $d_i$ is the denominator of $k_i$ (in lowest terms) and*

(3.2) $\quad r_i$ *is the lowest common multiple of* $d_1, d_2, \dots, d_i$

*for each $i$ (so that $r_i | r_{i+1}$), then* $1 < r_1 < r_2 < \cdots < r_m$.

*Proof.* Consider $I = \langle q \rangle$ and let $r = \operatorname{Ram}(q)$. Choose a local coordinate $z$ vanishing at 0, set $x = 1/z$, and suppose $x = t^r$. Then $q = \sum_{i=1}^{n} \alpha_i t^{n_i}$ is a polynomial in $t$ with each $\alpha_i$ nonzero and $n_1 > n_2 > \cdots > n_n \subset \mathbb{N}$. Let $r_0 = 1$ and let

$$r_1 < r_2 < \cdots < r_m = r$$

be the set of distinct leading common denominators $> 1$ that occur, i.e. the distinct numbers $> 1$ in the set $\{\operatorname{Ram}(\alpha_1 t^{n_1} + \cdots + \alpha_i t^{n_i}) \mid i = 1, 2, \dots, n\}$. Recall that if $b_i$ is the denominator of $n_i/r = \operatorname{slope}(\alpha_i t^{n_i})$, then $\operatorname{Ram}(\alpha_1 t^{n_1} + \cdots + \alpha_i t^{n_i}) = \operatorname{lcm}(b_1, b_2, \dots, b_i)$. Finally, for $i = 1, \dots, m$, let $k_i \in \mathbb{Q}_{>0}$ be the largest exponent such that the ramification is $r_i$, i.e.

$$k_i = \max\bigl\{ n_j/r \bigm| \operatorname{Ram}(\alpha_1 t^{n_1} + \cdots + \alpha_j t^{n_j}) = r_i \bigr\}.$$

Then we claim that the levels of $I$ are these numbers $k_1 > k_2 > \cdots > k_m$. This is an exercise, that can be done visibly by drawing a picture, as follows:[3]

For any $k \in \mathbb{R}_{\geq 0}$ define $q_k = \sum \alpha_i t^{n_i}$, where the sum is over the indices $i$ such that $n_i/r > k$. Thus $q_k$ is the leading piece of $q$ whose monomials have slope $> k$.

[3] Beware that the *naive/full fission tree* defined here will be represented by a single full branch of the precise fission trees to be defined carefully in Section 3.4 below. In brief, in the untwisted case, fission trees were defined from meromorphic connections in the quiver modularity conjecture in [13, Appx. C] (i.e. Hiroe–Yamakawa's theorem [38, 56]), and the *naive/full fission tree* here is defined similarly to those fission trees, after pulling back to the untwisted case. In turn the untwisted fission trees of [13] are essentially the same as the untwisted special case of the fission trees of Section 3.4.

Now for each $k$ consider the finite set

$$N_k := \{q_k(t), q_k(\zeta t), \dots, q_k(\zeta^{r-1}t)\} \subset t\mathbb{C}[t], \tag{3.3}$$

where $\zeta = \exp(2\pi i/r)$. If $k \in [k_{i+1}, k_i)$ then $|N_k| = r_i$ by definition (of the $r_i$ and $k_i$). Thus as $k$ varies the sets $N_k$ define a large disjoint union of copies of intervals (i.e. $r_i$ copies of $[k_{i+1}, k_i)$ for each $i$). Moreover, if $k < l$ then truncation gives a map $N_k \twoheadrightarrow N_l$, and this tells us how to glue the intervals into a tree: there is one interval, the trunk, over $[k_1, \infty)$. We glue this to the $k_1$-ends of the $r_1$ intervals over $[k_2, k_1)$. Then over $k_2$ there are $r_1$ nodes, and we glue each of them to $(r_2/r_1)$ of the $r_2$ intervals over $[k_3, k_2)$, etc., ending up with the $r$ *leaves* of the tree $N_0$ over 0. Thus the nodes where the tree branches are exactly the points $N_{k_1} \sqcup \cdots \sqcup N_{k_m}$, over $k_1 > \cdots > k_m$.

Now it is easy to see the $k_i$ are the levels: identify $I_d$ with the leaves $N_0$; thus if $i, j \in I_d$ then slope$(q_i - q_j)$ is the supremum of the unique shortest path in the tree between the leaves $i$ and $j$. Thus the levels are exactly where the branching occurs.

In turn, by definition (see (2.4)) the admissible deformations are thus exactly those which preserve the tree, as an abstract tree lying over $\mathbb{R}_{\geq 0}$ (we can choose an order of $N_0$ to get an irregular type). Consider the operation of adding to $q$ a monomial of the form $\alpha x^k$ where $k_i > k > k_{i+1}$ and $r_i k \in \mathbb{N}$, as $\alpha \in \mathbb{C}$ varies. This operation clearly gives an admissible deformation of $q$ since it does not change any of the Galois orbits (3.3). Thus we can admissibly deform $q$ to an element of the form $\sum_1^m \beta_i x^{k_i}$ with each $\beta_i \neq 0$. In turn we can admissibly deform this so that each $\beta_i = 1$. This implies (a), since we have shown that any two Stokes circles with the same levels $\{k_i\}$ can be admissibly deformed into $\langle \sum_1^m x^{k_i} \rangle$ and thus into each other (and the converse is clear). Statement (b) is also now clear: the common denominator needs to increase to get into a bigger Galois orbit. $\square$

**Remark 3.2.** Note that statement (a) includes the empty set: any two unramified circles $I, J \subset \mathcal{I}$ are admissible deformations of each other. This remark underlies the theory of Baker functions (see [16, §2.2] and the references there).

**Remark 3.3.** Note that the proof really shows that any two pointed irregular types $[(n_1, q_1)]$, $[(n_2, q_2)]$ (each with just one Galois orbit) are admissible deformations of each other if and only if Levels$(q_1)$ = Levels$(q_2)$, and $n_1 = n_2$.

For later use we will formalise the various sets of data in the proof as follows.

**Definition 3.4.** A *level datum* is a finite, possible empty, subset $L \subset \mathbb{Q}_{>0}$ satisfying the conditions (3.1) (so the numbers it contains are the possible levels of a single Stokes circle).

Thus, as above, a level datum $L = (k_1 > \cdots > k_m)$ determines ramification indices

$$\mathrm{RI}(L) = (r_1 < \cdots < r_m) \subset \mathbb{N}_{>1}$$

and, in particular, $\mathrm{Ram}(L) = r_m$. In turn, as in the proof, we can define the *inconsequential exponents* $\mathrm{Inc}(L) \subset \mathbb{Q}_{>0}$ as

$$\begin{aligned}\mathrm{Inc}(L) = \mathbb{N}_{>0} \cup \Big((k_2, k_1) \cap \frac{1}{r_1}\mathbb{N}\Big) \cup \cdots \cup \Big((k_m, k_{m-1}) \cap \frac{1}{r_{m-1}}\mathbb{N}\Big) \\ \cup \Big((0, k_m) \cap \frac{1}{r_m}\mathbb{N}\Big),\end{aligned} \tag{3.4}$$

and the *admissible exponents*,

$$A(L) \coloneqq L \sqcup \mathrm{Inc}(L) \subset \mathbb{Q}_{>0}. \tag{3.5}$$

If $L$ is empty then $\mathrm{Ram}(L) = 1$, $\mathrm{RI}(L) = \{1\}$, and $A(L) = \mathrm{Inc}(L) = \mathbb{N}_{>0}$. Note that the admissible exponents can thus also be expressed as

$$A(L) = \Big((0, k_m] \cap \frac{1}{r_m}\mathbb{N}\Big) \cup \cdots \cup \Big((0, k_2] \cap \frac{1}{r_2}\mathbb{N}\Big) \cup \Big((0, k_1] \cap \frac{1}{r_1}\mathbb{N}\Big) \cup \mathbb{N}_{>0} \subset \mathbb{Q}_{>0}. \tag{3.6}$$

Thus any Stokes circle $I \subset \mathcal{I}$ determines subsets

$$L = L(I) \subset A = A(I) = A(L) \subset \mathbb{R}_{>0}, \tag{3.7}$$

where $L = \mathrm{Levels}(I)$, and $A$ is the set of admissible exponents. The ramification index $\mathrm{Ram}(v) \in \{1, r_1, \ldots, r_m\}$ of any $v \in \mathbb{R}_{\geq 0}$ is defined to be the least common multiple of the denominators of the levels $k_i \geq v$, so that

$$\mathrm{Ram}(v) = r_i \quad \text{if } v \in (k_{i+1}, k_i]. \tag{3.8}$$

Note that (3.4), (3.5) imply the number of nonintegral admissible exponents is given by the formula

$$|A(L) \setminus \mathbb{N}| = \sum_1^m r_i k_i - \lfloor r_{i-1} k_i \rfloor, \tag{3.9}$$

where we set $r_0 = 1$.

**Example 3.5.** Let us look at a few simple examples.

- Suppose $q$ has $\mathrm{Ram}(q) = r > 1$ and $\mathrm{slope}(q) = s/r$ with $s$ and $r$ coprime. Then $q = \sum_1^s a_i x^{i/r}$ with $a_s \neq 0$, the only level of $q$ is its slope $\frac{s}{r}$, $\mathrm{Levels}(q) = (s/r)$, and $A(L) = \mathbb{N}_{>0} \cup \frac{1}{r}\{1, \ldots, s\}$.

- Consider $q = x^3+x^{5/2}+x^{3/2}+x^{1/3}$. It has ramification order $6 = \mathrm{lcm}(2,3)$. The corresponding list of ramification indices is $\mathrm{RI}(q) = (2 < 6)$, and $\mathrm{Levels}(q) = (5/2 > 1/3)$. In turn $\mathrm{Inc}(q) = \mathbb{N}_{>0} \cup \{3/2, 1/2, 1/6\}$, and $A(L) = \mathbb{N}_{>0} \cup \{5/2, 3/2, 1/2, 1/3, 1/6\}$.

**Remark 3.6.** Note similar (elementary) methods appear in the theory of curve singularities [43]. The reason for such a link to curves is the wild nonabelian Hodge correspondence, between meromorphic connections and meromorphic Higgs bundles, followed by taking the spectral curve of the Higgs field. Indeed, the dictionary in [8, p. 180] determines the irregular class from the spectral invariants at the singularity of the Higgs field. Note that we only consider *part* of the data of the corresponding curve singularity, and not all of it, i.e. the *principal part* of the singularity, determining the irregular class. This reflects the fact that fission, breaking up the curve at the pole, is about the various growth rates of the essentially singular functions $\exp(q)$ at $a \in \Sigma$, and this only involves the principal part of $q$.

**3.1.1. Single circle configuration spaces.** For any pair $q_1, q_2 \in \mathcal{I}_d$ of exponential factors we can consider the pointed irregular types $Q_1 = [(1, q_1)]$, $Q_2 = [(1, q_2)]$ and say that $q_1 \sim q_2$ if $Q_1 \sim Q_2$ in the sense of (2.5), which amounts to the condition

$$\mathrm{slope}(q_1 - \sigma^i(q_1)) = \mathrm{slope}(q_2 - \sigma^i(q_2))$$

for $i = 0, 1, \ldots, \mathrm{Ram}(q_1)$.

**Corollary 3.7.** *Let $q_1$ and $q_2$ be two exponential factors. Then $q_1 \sim q_2$ in the sense of (2.5) if and only if* $\mathrm{Levels}(q_1) = \mathrm{Levels}(q_2)$ *(i.e. if and only if they are admissible deformations of each other).*

*Proof.* Clearly, if $q_1 \sim q_2$ then $\mathrm{Levels}(q_1) = \mathrm{Levels}(q_2)$. Conversely, if they are admissible deformations of each other then they both can be admissibly deformed to $q_0 := \sum x^{k_i}$ (where the $k_i$ are the levels), and so the three lists

$$\mathrm{slope}(q_j - \sigma^i(q_j)), \quad i = 1, 2, 3, 4, \ldots$$

of rational numbers are equal, for $j = 0, 1, 2$ (since they remain equal under any small admissible deformation, and under a new choice of initial $q$). □

This motivates the definition of the configuration space $\mathbf{B}(q) := \mathbf{B}(Q)$ (from Definition 2.10, using $\sim$ from (2.5)), for any exponential factor $q$, where $Q = [(1, q)]$, since we now see $\mathbf{B}(q)$ is the set of all exponential factors that are admissible deformations of $q$ and have Poincaré–Katz rank (maximal slope) at most $K := \mathrm{Katz}(q)$. Thus we deduce an explicit description of the configuration spaces in the case of one circle.

**Proposition 3.8.** *Let $q \in \mathcal{I}_d$ be an exponential factor, and let $L = \mathrm{Levels}(q)$ be the levels of $q$. Then*

$$\mathbf{B}(q) \cong (\mathbb{C}^*)^m \times \mathbb{C}^N, \quad \mathbf{SB}(q) \cong (\mathbb{C}^*)^m \times \mathbb{C}^M,$$

*where $m = |L|$ is the number of levels, $N$ is the number $|\mathrm{Inc}^\flat(L)|$ of inconsequential exponents $\leq K$, and $M = |\mathrm{Inc}(L) \setminus \mathbb{N}|$ is the number of nonintegral inconsequential exponents. In particular, $\dim \mathbf{B}(q)$ is the number $|A^\flat(L)|$ of admissible exponents $\leq K$ and $\dim \mathbf{SB}(q) = |A(L) \setminus \mathbb{N}|$ is the number of nonintegral admissible exponents, as given by the formula (3.9).*

*Proof.* Given $I = \langle q \rangle$ we consider $L(I) \subset A(I) \subset \mathbb{R}_{>0}$ as in (3.7), consisting of the admissible exponents and the subset of levels. We can move the coefficients parameterised by $A(I)$ arbitrarily provided those from $L$ remain nonzero. The descriptions of the configuration spaces then arise (a) by not going past $K$, and (b) by only considering trace-free deformations. □

**Example 3.9.** Let us come back to our previous examples:

- If $q = a_s x^{s/r} + \cdots + a_1 x^{1/r}$, where $s$ and $r = \mathrm{Ram}(q) > 1$ are coprime, $a_s \neq 0$, then its (slope bounded) admissible deformations are of the form

  $$q' = \sum_{k=1}^{s} b_k x^{k/r},$$

  with $b_s \in \mathbb{C}^*$ nonzero, and $b_1, \ldots, b_{s-1} \in \mathbb{C}$ arbitrary. Removing the integral exponents leaves $s - \lfloor s/r \rfloor$ coefficients, agreeing with formula (3.9) for $\dim \mathbf{SB}(q)$ in this case.
- For $q = x^3 + x^{5/2} + x^{3/2} + x^{1/3}$, the set of levels is $L(q) = \{5/2, 1/3\}$, the set of admissible exponents $\leq 3$ is $\{3, 5/2, 2, 3/2, 1, 1/2, 1/3, 1/6\}$, so the (slope bounded) admissible deformations of $q$ are of the form

  $$q' = \alpha x^3 + a x^{5/2} + b x^2 + c x^{3/2} + dx + e x^{1/2} + f x^{1/3} + g x^{1/6},$$

  with $\alpha$, $a$, $b$, $c$, $d$, $e$, $f$, $g \in \mathbb{C}$, with $a$, $f$ nonzero, and the other coefficients arbitrary. The trace-free projection of $q$ is $\mathrm{pr}(q) = q - x^3$ and the trace-free admissible deformations of this are as above, but with $\alpha = b = d = 0$. This deformation space has dimension 5, agreeing with (3.9).

**Remark 3.10.** Note that if we change coordinates the subsets $L(I) \subset A(I) \subset \mathbb{R}_{>0}$ attached to any Stokes circle $I$ (as in (3.7)) do not change (and similarly for the fission trees to be defined below).

### §3.2. The case of two Stokes circles

As preparation to tackle the general case, we turn our attention to the case of two distinct active circles. We consider a pointed irregular type of the form

$$Q = [(n, q), (\widehat{n}, \widehat{q})],$$

where $q$ and $\widehat{q}$ have ramification orders $r$ and $\widehat{r}$. We denote by $q_0, \dots, q_{r-1}$ and $\widehat{q}_0, \dots, \widehat{q}_{\widehat{r}-1}$ the elements of the Galois orbit of the corresponding exponential factors. A holomorphic family defined by $Q_b = [(n, q(b)), (\widehat{n}, \widehat{q}(b))]$ is an admissible deformation if and only if $q(b)$ and $\widehat{q}(b)$ are admissible deformations and if the rational numbers $\mathrm{slope}(q_i - \widehat{q}_j)$ are constant.

We thus have to determine what the slopes of the differences $q_i - \widehat{q}_j$ are. This has been studied in [31], the results of which we now briefly recall. If $q$ and $\widehat{q}$ are two distinct exponential factors, we can decompose them into a *common part* and a *different part* in the following way. Recall that $\tau_k \colon \mathcal{I}_d \to \mathcal{I}_d$ denotes the truncation map, discarding all monomials of slope $< k$, as in (2.3). (Beware that a different truncation was used in the proof of Proposition 3.1, discarding monomials of slope $\leq k$.) Let

$$E(q) \subset \mathbb{Q}_{>0}$$

be the finite set of exponents occurring in $q$, so $q = \sum_{k \in E(q)} a_k x^k$ with each $a_k$ nonzero. Let $k \in E(q)$ be the smallest number such that

$$\langle \tau_k(q) \rangle = \langle \tau_k(\widehat{q}) \rangle$$

i.e. the Galois orbits of the truncations are equal, if such a number exists. If so then set

$$q_c = \tau_k(q), \quad \widehat{q}_c = \tau_k(\widehat{q}).$$

If there is no such $k$ then set $q_c = \widehat{q}_c = 0$. Then define $q_d = q - q_c$, $\widehat{q}_d = \widehat{q} - \widehat{q}_c$, so that we get a decomposition

$$q = q_c + q_d, \quad \widehat{q} = \widehat{q}_c + \widehat{q}_d,$$

of $q$ and $\widehat{q}$ as the sum of a *common part* $q_c$ and a *different part* $q_d$. If $q_c = \widehat{q}_c$ we say that $q$ and $\widehat{q}$ are *compatible*, as in Definition 2.4. Replacing $q$ or $\widehat{q}$ by another element of their Galois orbit if necessary, we may assume without loss of generality that this is the case. Note that if $q$ and $\widehat{q}$ do not have the same slope then they do not have the same leading term up to Galois conjugacy, so $q_c = \widehat{q}_c = 0$. If $q$ and $\widehat{q}$ have a nonzero common part, in particular they have the same slope. We call the rational number

$$f_{q,\widehat{q}} := \max(\mathrm{slope}(q_d), \mathrm{slope}(\widehat{q}_d)) \in \mathbb{Q}_{\geq 0} \tag{3.10}$$

the *fission exponent* of $q$ and $\widehat{q}$. It is zero if and only if $\langle q\rangle = \langle\widehat{q}\rangle$. Since it only depends on the circles/Galois orbits this defines the fission exponent $f_{I,\widehat{I}}$ for any Stokes circles $I = \langle q\rangle$, $\widehat{I} = \langle\widehat{q}\rangle$.

We are now in a position to describe the set of slopes we are interested in.

**Lemma 3.11.** *Let $q$ and $\widehat{q}$ be two exponential factors in distinct Galois orbits. The set of nonzero slopes among the rational numbers* $\mathrm{slope}(q_i - \widehat{q}_j)$*, for $i = 0, \dots, r-1$, $j = 0, \dots, \widehat{r}-1$ is equal to*

$$\mathrm{Levels}(q_c) \sqcup \{f_{q,\widehat{q}}\} \subset \mathbb{Q}_{>0},$$

*i.e. it consists of the levels of the common part of $q$ and $\widehat{q}$ together with their fission exponent. Furthermore, if $q$ and $\widehat{q}$ are compatible, i.e. if $q_c = \widehat{q}_c$, the map $(i,j) \mapsto$* $\mathrm{slope}(q_i - \widehat{q}_j)$ *is entirely determined by the data of* $\mathrm{Levels}(q_c)$ *and $f_{q,\widehat{q}}$.*

*Proof.* This follows directly from the proof of [31, Lem. 4.3]. The main idea is that the levels of the common part are obtained from the differences between the different Galois conjugates of the common part, while the fission exponent is the slope of all the other differences (for which the Galois conjugates of the common part are the same). □

As a consequence, the numerical equivalence relation on pointed irregular types can be clarified.

**Proposition 3.12.** *Let $Q = [(n,q),(\widehat{n},\widehat{q})]$ be a pointed irregular type with two active circles, such that $q$ and $\widehat{q}$ are compatible. Let $k \coloneqq f_{q,\widehat{q}}$ be the fission exponent of $q$ and $\widehat{q}$. Then a pointed irregular type $Q'$ satisfies $Q' \sim Q$ if and only if it is of the form $Q' = [(n,q'),(\widehat{n},\widehat{q}')]$, with $q'$ and $\widehat{q}'$ compatible and such that*

(1) *$q \sim q'$ and $\widehat{q} \sim \widehat{q}'$,*
(2) *$q'_c = \widehat{q}'_c$ satisfies $q'_c \sim q_c$,*
(3) *$f_{q',\widehat{q}'} = k$.*

*These three conditions hold if and only if* $\mathrm{Levels}(q) = \mathrm{Levels}(q')$*,* $\mathrm{Levels}(\widehat{q}) = \mathrm{Levels}(\widehat{q}')$*, and $f_{q',\widehat{q}'} = k$.*

*Proof.* Let us assume that $Q' \sim Q$. Then, considering the differences internal to the two Galois orbits, we have $q \sim q'$ and $\widehat{q} \sim \widehat{q}'$. Considering now the set of slopes of the differences between the two distinct Galois orbits for both $Q$ and $Q'$, we have from Lemma 3.11 that $L(q_c) \sqcup \{f_{q,\widehat{q}}\} = L(q'_c) \sqcup \{f_{q',\widehat{q}'}\}$, hence $f_{q',\widehat{q}'} = f_{q,\widehat{q}} = k$, and $L(q_c) = L(q'_c)$, so $q_c \sim q'_c$. Furthermore, since $q$ and $\widehat{q}$ are compatible, we have $\mathrm{slope}(q_0 - \widehat{q}_0) = k = \mathrm{slope}(q'_0 - \widehat{q}'_0)$, hence $q'$ and $\widehat{q}'$ are also compatible. For the converse, let us assume that $Q'$ is of the claimed form. Then since $q \sim q'$ and $\widehat{q} \sim \widehat{q}'$, the slopes of the internal differences are the same for $Q$ and $Q'$. We have $L(q_c) = L(q'_c)$ and $f_{q,\widehat{q}} = f_{q',\widehat{q}'}$. Since $q$ and $q'$ are compatible, as well as $\widehat{q}$ and $\widehat{q}'$, the second part of Lemma 3.11 implies that the slopes of the differences between the distinct Galois orbits are the same for $Q$ and $Q'$. □

From the case of a single circle, we know how to make the first two conditions explicit. Let us now investigate the third condition in more detail. Choose $k \in \mathbb{Q}_{>0}$ and let $q_c$ be an exponential factor whose exponents are all strictly greater than $k$. Write $k = n/m$ with $n$, $m$ coprime integers. Choose $a, \widehat{a} \in \mathbb{C}$ and consider the exponential factors

$$q \coloneqq q_c + az^k + b, \quad \widehat{q} \coloneqq q_c + \widehat{a}z^k + \widehat{b}$$

where $b$, $\widehat{b}$ are exponential factors of slope $< k$. The conditions for the fission exponent $f_{q,\widehat{q}}$ to equal $k$ are as follows.

**Proposition 3.13.** *Let $r = \mathrm{Ram}(q_c)$, $k = n/m$. Then $f_{q,\widehat{q}} = k$ if and only if either* (1) *or* (2) *holds:*

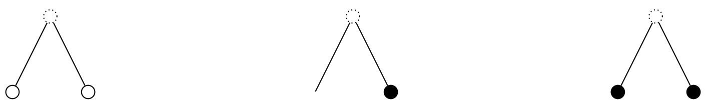
(1) *$m$ divides $r$ and $a \neq \widehat{a}$, or*

(2) *$m$ does not divide $r$ and either*

  (a) *exactly one of $a$, $\widehat{a}$ is zero, or*

  (b) *both $a \neq 0$ and $\widehat{a} \neq 0$, and furthermore $a^N \neq \widehat{a}^N$, where $N = \mathrm{lcm}(r, m)/r$.*

**Remark 3.14.** Notice that in case (1), one of $a$ and $\widehat{a}$ can be equal to zero. Also note that (2a), (2b) can be combined into the single statement that $a^N \neq \widehat{a}^N$. The three cases are distinguished since in (1) the number $k$ is in neither of the sets $\mathrm{Levels}(q)$, $\mathrm{Levels}(\widehat{q})$, for (2a) it is in just one, and for (2b) it is in both. For later use (cf. Proposition 3.26) we encode the three cases (1), (2a), (2b) pictorially as follows:



*Proof of Proposition 3.13.* Clearly we need $a \neq \widehat{a}$, and can set $b = \widehat{b} = 0$ without loss of generality. We then need to see when $\langle q \rangle \neq \langle \widehat{q} \rangle$.

(1) Let us first assume that $m$ divides $r$, so that $k$ is not a level of $\langle q \rangle$ or $\langle \widehat{q} \rangle$, and $\mathrm{Ram}(q) = \mathrm{Ram}(\widehat{q}) = \mathrm{Ram}(q_c)$. Thus the Galois orbits of $q$ and $\widehat{q}$ are in bijection with that of $q_c$ (via truncation). This implies that the Galois orbits of $q$ and $\widehat{q}$ are distinct if and only if $a \neq \widehat{a}$.

(2) (2a) is clear so we consider (2b). Let us set $r' = \operatorname{lcm}(r, m)$ and $N = r'/r$. $\operatorname{Ram}(q)$ is equal to $r'$ if $a \neq 0$, otherwise it is equal to $r$, and similarly for $\widehat{q}$. If $a \neq 0$, then in the Galois orbit of $q$ there are $N$ elements giving rise, upon truncation, to any given element of the Galois orbit of $q_c$, and their coefficients of exponent $k = n/m$ differ by an $N$th root of unity. The conclusion follows.

□

This enables us to get an explicit description of $\mathbf{B}(Q)$, as we now do for a few examples. In terms of lists of exponents attached to $I = \langle q \rangle$, $\widehat{I} = \langle \widehat{q} \rangle$ we have the subsets

$$L(I) \subset A(I) \subset \mathbb{R}_{>0}, \quad L(\widehat{I}) \subset A(\widehat{I}) \subset \mathbb{R}_{>0}$$

(as in (3.7)) which we should identify just above the fission exponent $f_{I,\widehat{I}}$, in order to get the set of exponents whose coefficients we can vary. And these coefficients can be varied arbitrarily provided those from $L(I)$ or $L(\widehat{I})$ remain nonzero and those at the fission exponent continue to satisfy the same part of Proposition 3.13.

**Example 3.15.** Let us look at a few examples illustrating the different cases:

- Consider $Q = [(1, \lambda x^{s/r}), (1, \mu x^{s/r})]$ with $r > 1$ and $s$ coprime and $\lambda \neq \mu e^{2in\pi/r}$ for any integer $n$ (so that the Galois orbits are disjoint). The common part of the two exponential factors is empty, and their fission exponent is the level $s/r$, so this fits into case (2b). The (slope bounded) admissible deformations of $Q$ are of the form $Q' = [(1, q_1'), (1, q_2')]$ with

$$q_1' = \sum_{k=1}^{s} a_k x^{k/r}, \quad q_2' = \sum_{k=1}^{s} b_k x^{k/r},$$

with $a_1, \ldots, a_s, b_1, \ldots, b_s \in \mathbb{C}$, $a_s \neq 0$, $b_s \neq 0$, and $a_s \neq b_s e^{2in\pi/r}$ for any integer $n$, i.e. we have

$$\mathbf{B}(Q) \cong \left\{(a_1, \ldots, a_s, b_1, \ldots, b_s) \in \mathbb{C}^{2s} \;\middle|\; a_s \neq 0,\ b_s \neq 0,\ \forall n \in \mathbb{N},\ a_s \neq b_s e^{2in\pi/r}\right\}.$$

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/2}$, and $q_2 = x^{3/2} + 2x^{1/2}$. The common part of $q_1$ and $q_2$ is $x^{3/2}$, and they are compatible. This fits into the first case, indeed the fission exponent $1/2$ is not a level of $q_1$ and $q_2$. The (slope bounded) admissible deformations of $Q$ are of the form $Q' = [(1, q_1'), (1, q_2')]$, with

$$q_1' = ax^{3/2} + bx + cx^{1/2}, \quad q_1' = ax^{3/2} + bx + dx^{1/2},$$

with $a, b, c, d \in \mathbb{C}$, $a \neq 0$, and $c \neq d$, i.e. we have

$$\mathbf{B}(Q) \cong \left\{(a, b, c, d) \in \mathbb{C}^4 \;\middle|\; a \neq 0,\ c \neq d\right\}.$$

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/3}$, and $q_2 = x^{3/2}$. The ramification order of $\langle q_1 \rangle$ is 6, the common part of $q_1$ and $q_2$ is $x^{3/2}$, and they are compatible. This fits into case (2a), indeed the fission exponent $1/3$ is a level of $q_1$, but does not appear in $q_2$. The (slope bounded) admissible deformations of $Q$ are of the form $Q' = [(1, q'_1), (1, q'_2)]$, with

$$q'_1 = ax^{3/2} + bx + cx^{1/2} + dx^{1/3} + ex^{1/6}, \quad q'_2 = ax^{3/2} + bx + cx^{1/2},$$

with $a, b, c, d, e \in \mathbb{C}$, with $a \neq 0$ and $d \neq 0$, i.e. we have

$$\mathbf{B}(Q) \cong \{(a, b, c, d, e) \in \mathbb{C}^5 \mid a \neq 0, \ d \neq 0\}.$$

## §3.3. Fission data

As a step towards the general case we will give a first attempt at packaging the relevant data. Recall from Definition 3.4 that a *level datum* is a finite, possibly empty, subset $L \subset \mathbb{Q}_{>0}$ satisfying the conditions (3.1) (so the numbers it contains are the possible levels of a single Stokes circle).

**Definition 3.16.** A *fission datum* is a pair $\mathcal{F} = (\mathcal{L}, f)$ where $\mathcal{L}$ is a multiset[4]

$$\mathcal{L} = L_1 + \cdots + L_m$$

of level data and $f$, the fission exponents, are the choice of a rational number $f_{ij} = f_{ji} \in \mathbb{Q}_{\geq 0}$, for all $i, j \in \{1, \dots, m\}$.

An irregular class determines a fission datum in the obvious way, as follows. If $\Theta = \sum_1^m I_i$ is a rank-$n$ irregular class (where the Stokes circles $I_i$ are not necessarily distinct) then define $L_i = L(I_i)$ to be the level datum of $I_i$ for each $i$, and define $\mathcal{L}(\Theta) := \sum_1^m L_i$ to be the corresponding multiset of level data. Then by taking $f_{ij} = f_{I_i, I_j}$ to be corresponding fission exponents (3.10), this determines the fission datum $\mathcal{F}(\Theta) = (\mathcal{L}(\Theta), f)$ of the irregular class $\Theta$. Note that the multiplicity of any given Stokes circle $I_j$ in the class $\Theta$ is determined by the fission data by the recipe

$$\Theta(I_j) = |\{i = 1, 2, \dots, m \mid f_{ij} = 0\}|.$$

In turn a *labelled* fission datum $\widehat{\mathcal{F}} = (\widehat{\mathcal{L}}, f)$ is a pair consisting of an ordered list

$$\widehat{\mathcal{L}} = [(n_1, L_1), \dots, (n_p, L_p)],$$

where the $L_i$ are not necessarily distinct level data, together with fission exponents $f_{ij} = f_{ji} \in \mathbb{Q}_{\geq 0}$ for $i, j \in \{1, \dots, p\}$ such that $f_{ij} = 0$ if and only if $i = j$. Then a pointed irregular type determines a labelled fission datum in the obvious way

[4] Recall that a multiset is a set with multiplicities. Here this means the $L_i$ are not necessarily distinct level data (but the ordering of the level data, i.e. the labelling by indices $1, \dots, m$, is not part of the data).

(and in turn a labelled fission datum determines a fission datum by forgetting the labelling).

The study in the case of one and two circles then implies one of the main statements.

**Theorem 3.17.** *Let $Q = [(n_1, q_1), \dots, (n_p, q_p)]$ be a pointed irregular type, which we assume to be compatible, and $\widehat{\mathcal{F}}$ its labelled fission datum. Then a pointed irregular type $Q'$ satisfies $Q' \sim Q$ if and only if it is compatible and its labelled fission datum equals $\widehat{\mathcal{F}}$.*

*Proof.* This follows from our study of the case of one and two active circles. To see this, notice that the fission data of $Q$ is equivalent to the data of the levels of its active circles, together with the data of the common part and the fission exponent of any pair of distinct active circles. The result now follows from Corollary 3.7 and Proposition 3.12. □

To go further we will now define fission trees (gluing just above the fission exponents, as above); this will give a way to parameterise the set of coefficients we can vary, and thus to describe the configuration spaces, leading to a proof that two irregular classes are admissible deformations of each other if and only if their fission data are equal. It will also give a way to classify the possible topological data, i.e. the set of possible admissible deformation classes (it seems difficult to write down axioms for the fission data that actually arise from irregular classes, without discussing trees).

## §3.4. Fission trees in the twisted setting

First we will describe abstractly the exact types of trees we get, and then define how to obtain such trees from irregular types/classes.

**3.4.1. Fission trees.** Consider a six-tuple $(\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$ where

- $\mathcal{T}$ is a metrised tree,[5] with vertices $\mathbb{V} \subset \mathcal{T}$,
- $\mathbb{A} \subset \mathbb{V}$ is a subset (the admissible vertices),
- $\mathbb{L} \subset \mathbb{A}$ is a finite, possibly empty, subset (the internal levels/mandatory vertices),
- $h\colon \mathcal{T} \to \mathbb{R}_{\geq 0}$ is a length-preserving map, the height map, mapping each edge isomorphically onto an interval, such that $\mathbb{V}_0 := h^{-1}(0) \subset \mathcal{T}$ is a finite set and is the set of leaves of $\mathcal{T}$,
- $n$ is a map $\mathbb{V}_0 \to \mathbb{N}_{>0}$, giving a multiplicity to each leaf.

[5] See e.g. [2] for metrised graphs, but note that ours are not compact, and recall that a tree is a special type of graph.

The edges $E = E(\mathcal{T}) = \pi_0(\mathcal{T}\setminus\mathbb{V})$ of $\mathcal{T}$ are the components of the complement of the vertices. Thus any vertex that is not a leaf is adjacent to $\geq 2$ edges, one of which is the *parent* edge, and the others are the descendant edges. The branch vertices $\mathbb{Y} \subset \mathbb{V}$ are those with $\geq 2$ descendants (where the branching of the tree occurs). The trunk of the tree is the union of all the edges and vertices above all the branch vertices. The vertices in $\mathbb{L}$ will be called *mandatory*, those in $\mathbb{I} := \mathbb{A}\setminus\mathbb{L}$ will be called *inconsequential*, and the others $(\mathbb{V}\setminus\mathbb{A})$ will be called *empty*.

The *full branch* $\mathcal{B}_i$ of any leaf $i \in \mathbb{V}_0$ is the (minimal) subset of $\mathcal{T}$ all the way from $i$ to the far end of the trunk. Let $\mathbb{L}_i = \mathbb{L}\cap\mathcal{B}_i$ denote the internal levels on the $i$th full branch, and let $\mathbb{A}_i = \mathbb{A}\cap\mathcal{B}_i$ similarly (the admissible vertices on the $i$th full branch).

**Definition 3.18.** Such a tuple $(\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$ is a *fission tree* if

(1) $\mathbb{V} = h^{-1}(\{0\}\cup h(\mathbb{A}))$; the vertices are exactly the leaves plus the points that map to $h(\mathbb{A})$,
(2) $h$ maps each full branch isomorphically onto $\mathbb{R}_{\geq 0}$,
(3) the internal levels of any full branch map to a set of levels, i.e. $L_i := h(\mathbb{L}_i) \subset \mathbb{Q}_{>0}$ satisfies the conditions (3.1), for any leaf $i$ (so they are the possible levels of a single Stokes circle),
(4) $A_i := h(\mathbb{A}_i) \subset \mathbb{Q}_{>0}$ is the set $A(L_i)$ of admissible exponents of $L_i$ for each leaf $i$, as in (3.5),
(5) the children $\mathrm{Ch}(v) \subset \mathbb{V}$ of each branch vertex $v \in \mathbb{Y}$ satisfy one of the following three conditions:
    (1) all the vertices in $\mathrm{Ch}(v)$ are inconsequential,
    (2a) one vertex in $\mathrm{Ch}(v)$ is empty and the others are mandatory,
    (2b) all the vertices in $\mathrm{Ch}(v)$ are mandatory.

Note in particular that the leaves of a fission tree are empty. Two fission trees are *isomorphic* if there is an isomorphism between the underlying trees relating all the data $\mathbb{V}$, $\mathbb{A}$, $\mathbb{L}$, $h$, $n$.

A *labelling* of a fission tree with nodes $\mathbb{V}_0$ is a total ordering of the set of leaves, i.e. a bijection $\psi\colon \{1,\dots,|\mathbb{V}_0|\} \cong \mathbb{V}_0$. Two labelled fission trees are isomorphic if there is a label-preserving isomorphism (so there is at most one isomorphism between labelled fission trees).

**Remark 3.19.** Note that the definition (3.8) extends directly to define the ramification index $\mathrm{Ram}(v)$ of any point $v \in \mathcal{T}$ of a fission tree, taking the lcm of the denominators of the heights of its mandatory ancestors $\geq v$ (i.e. in $\mathbb{L}$, of height

$\geq h(v)$ and on the same full branch). If $p \in \mathbb{V}$ is the parent of some vertex $v \in \mathbb{V}$, by definition the *relative ramification* of $p$ is $\mathrm{Ram}(v)/\mathrm{Ram}(p)$. Observe that the integer $N = \mathrm{lcm}(r, m)/r$ in part (2a) of Proposition 3.13 is an example of relative ramification.

**Remark 3.20.** Axioms (3), (4) of Definition 3.18 imply that axiom (5) could be replaced by the simpler statement "$\mathrm{Ch}(v)$ contains at most one empty vertex for any $v \in \mathbb{Y}$".

**3.4.2. Fission data of a fission tree.** Given a fission tree $\mathcal{T} = (\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$ let $A = h(\mathbb{A}) \subset \mathbb{Q}_{>0}$ be the admissible exponents. Given two distinct leaves $i, j \in \mathbb{V}_0$, let $v_{ij} \in \mathbb{Y}$ be their nearest common ancestor (i.e. the branchpoint where $\mathcal{B}_i$, $\mathcal{B}_j$ meet). Thus $\mathcal{T}$ determines a number, the fission exponent

$$f_{ij} \coloneqq \mathrm{prec}(h(v_{ij})) \in A$$

for each pair of leaves, where $\mathrm{prec}\colon A \to A \cup \{0\}$ takes $a \in A$ to the preceding element of $A$, i.e. the largest element $< a$ (or to zero if $a = \min(A)$). Axiom (5) implies $f_{ij} \neq 0$.

Thus a fission tree $\mathcal{T}$ determines a fission datum $\mathcal{F}(\mathcal{T}) = (\mathcal{L}, f)$ where $\mathcal{L} = \sum n_i L_i$ is the set of level data of each full branch (repeated according to their multiplicities) and $f$ encodes the fission exponents between the branches of the tree. The following statement is now an exercise.

**Lemma 3.21.** *Two fission trees are isomorphic if and only if their fission data are equal:*

$$\mathcal{T}_1 \cong \mathcal{T}_2 \iff \mathcal{F}(\mathcal{T}_1) = \mathcal{F}(\mathcal{T}_2).$$

Similarly, two labelled fission trees are isomorphic if and only if their labelled fission data are equal.

**3.4.3. Fission tree of an irregular class.** We now describe how to define the fission tree of an irregular class $\Theta = \sum n_i I_i$. Firstly there is a full branch $\mathcal{B}_i$ (of multiplicity $n_i$) for each distinct circle $I_i$. Thus $\mathcal{B}_i$ is a copy of $\mathbb{R}_{\geq 0}$ equipped with the subsets $\mathbb{L}_i \subset \mathbb{A}_i \subset \mathcal{B}_i$, and an isomorphism $h\colon \mathcal{B}_i \to \mathbb{R}_{\geq 0}$, defined so that the subsets $\mathbb{L}_i \subset \mathbb{A}_i$ map onto the sets $L_i = L(I_i) \subset A_i = A(I_i)$ of levels and admissible exponents (defined from $I_i$ as in (3.7)).

We then define $A = A(\Theta) \coloneqq \bigcup A_i \subset \mathbb{Q}_{>0}$ to be the union of all the admissible exponents. This is a discrete subset and for any $k \in \mathbb{Q}_{>0}$ we can define the successor $\mathrm{succ}(k) \in A$ to be the next element of $A$, i.e. the smallest element of $A$ that is $> k$.

If $f_{ij} = f_{I_i,I_j}$ is the fission exponent between $I_i, I_j$ then define the *gluing exponent*

$$g_{ij} \coloneqq \mathrm{succ}(f_{ij}) \in A$$

to be the next admissible exponent after the fission exponent.

We then glue the full branches $\mathcal{B}_i$, $\mathcal{B}_j$ over the interval $[g_{ij}, \infty)$ for each $i$, $j$, to define the tree $\mathcal{T}$ equipped with the map $h\colon \mathcal{T} \to \mathbb{R}_{\geq 0}$. The subsets $\mathbb{L}_i \subset \mathbb{A}_i$ fit together to define $\mathbb{L} \subset \mathbb{A} \subset \mathcal{T}$, and we set $\mathbb{V} = h^{-1}(A \cup \{0\}) \subset \mathcal{T}$. Let $\mathbb{V}_k = h^{-1}(k)$ denote the vertices of height $k$.

This defines the fission tree $\mathcal{T}(\Theta) = (\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$. All the axioms are clear except (5), which will follow from Proposition 3.26 below.

In case we start with a pointed irregular type $Q = [(n_1, q_1), \ldots, (n_m, q_m)]$, and not just an irregular class, then we get a labelled fission tree $\widehat{\mathcal{T}}(Q)$, by labelling the nodes $\mathbb{V}_0$ according to the labelling of the exponential factors $q_1, \ldots, q_m$.

The nodes $\mathbb{V} \subset \mathcal{T}$ may be interpreted in terms of truncated circles as follows. Recall that if $k \in \mathbb{Q}$ then $\tau_k(q)$ is the truncation, forgetting monomials of slope $< k$.

**Lemma 3.22.** *For each $k \in A \cup \{0\}$,*

$$\mathbb{V}_k \cong \{\langle \tau_k(q_1)\rangle, \ldots, \langle \tau_k(q_m)\rangle\}, \tag{3.11}$$

*i.e. the vertices of height $k$ are in bijection with the set of Galois orbits of the exponential factors truncated at $k$.*

*Proof.* Consider two distinct circles $\langle q_1\rangle$, $\langle q_2\rangle$, corresponding to two leaves. Consider the minimal element $k$ of $A$ such that $\langle \tau_k(q_1)\rangle = \langle \tau_k(q_2)\rangle$. Then, by definition, $k = g_{12} = \mathrm{succ}(f_{12})$. □

In particular, if $k > l$ are two admissible exponents (or zero), we have a surjective map $\phi_{kl}\colon \mathbb{V}_l \twoheadrightarrow \mathbb{V}_k$ defined by $\phi_{kl}(\langle \tau_l(q_i)\rangle) = \langle \tau_k(q_i)\rangle$, and this determines the structure of the tree, by defining the unique parent of each node (if $k, l \in A$ are consecutive). This also proves all the gluings of the full branches can be done consistently.

From this viewpoint, two elements $\langle \tau_l(q_i)\rangle$, $\langle \tau_l(q_j)\rangle \in \mathbb{V}_l$ are descendants of the same vertex in $\mathbb{V}_k$, where $l < k$, if they have the same truncation to exponent $k$, i.e. if $\langle \tau_k(q_i)\rangle = \langle \tau_k(q_j)\rangle$. Furthermore, if $\langle q_i\rangle$ and $\langle q_j\rangle$ are two active circles, corresponding to two leaves, in $\mathbb{V}_0$, their closest common ancestor in the tree corresponds to (the Galois orbit of) their common part (this follows immediately from the definition of the common part of two exponential factors).

**Remark 3.23.** Note that the visual image of a tree is clear by thinking about the eigenvalues of a matrix of meromorphic functions ($\sim$ a meromorphic Higgs field).

On the differential equations side, this idea is embedded in the "fission" picture, thinking about the growth/decay of the functions $\exp(q(x))$ as $x \to \infty$ along a ray, and can be traced back to the Stokes diagram in [53, p. 116], reproduced on the cover of [24] (see also the pictures in [14, 22]). However, our exact definition of fission tree is quite subtle, in order for the main results to follow cleanly (i.e. Theorem 3.27, (3.18) parameterising the configuration spaces in terms of points of the fission tree, and in turn Corollary 3.31, giving the product decomposition). In particular, the simpler definition of trees (as in [55]) on the singularity theory (spectral curve) side of the wild nonabelian Hodge correspondence is much less useful for either of these aims.[6]

## §3.5. Truncated fission trees

Since any integer is an admissible exponent of any exponential factor, the set of admissible exponents of any irregular type $Q$ is unbounded from above. For the configuration spaces we are interested in, the admissible exponents are bounded by the Poincaré–Katz rank. Thus we will consider the *truncated fission tree* $\mathcal{T}^\flat = \mathcal{T}^\flat(Q)$ by defining the *root vertex* to be that of height

$$\eta \coloneqq \lfloor \mathrm{Katz}(Q) + 1 \rfloor \tag{3.12}$$

and then removing all nodes/edges above the root (and marking the root as empty/inadmissible). Note that $\eta$ is the smallest integer greater than the Poincaré–Katz rank of $Q$ (the largest slope), so the admissible nodes $\mathbb{A}^\flat \subset \mathcal{T}^\flat$, the subset of $\mathbb{A}$ below the root, are exactly the nodes that will contribute to the configuration space (as these are the admissible nodes of height $\leq \mathrm{Katz}(Q)$). When drawing pictures of trees we will truncate as above, but also we will stop at the smallest admissible exponent, so the leaves will not be drawn. If we are just given a fission tree (and not an irregular type/class) then we will use the *minimal truncation* at $\eta = \lfloor k+1 \rfloor$, where $k = \mathrm{Katz}(\mathcal{T}) \coloneqq \max(\{0, h(\mathbb{L}), \max(h(\mathbb{Y})) - 1\})$. Indeed, one can check that in the trace-free case $\mathrm{Katz}(\mathcal{T})$ is the Poincaré–Katz rank of any irregular class with fission tree $\mathcal{T}$. Using this we can attach an integer, the moduli number, to any fission tree, as in the following definition.

[6] Specifically, if we took the definition in [55] and transposed it to our setting (shifting and truncating suitably), then the resulting definition is too local: the location of the branchpoints $\mathbb{Y}$ would then just depend on the adjacent full branches so the product decomposition will not work cleanly and moreover some of the points of the tree that should parameterise distinct coefficients are identified.

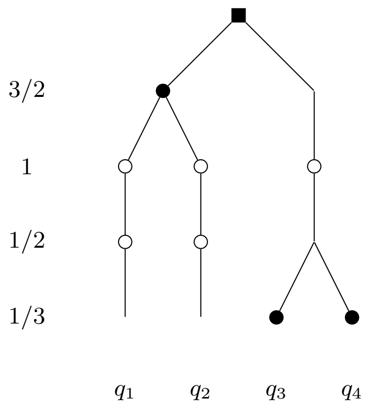



Figure 1. The fission tree $\mathcal{T}^\flat$ associated to $Q$ (not drawn isometrically). The labelling corresponds to the numbering of the $q_i$. The multiplicities of the leaves are all equal to 1.

**Definition 3.24.** The *moduli number* $\mu(\mathcal{T})$ of a fission tree is one plus the number of admissible vertices below the root, minus the number of integers below the root height:

$$\mu(\mathcal{T}) \coloneqq 1 + |\mathbb{A}^\flat| - (\eta - 1) = |\mathbb{A}^\flat| + 2 - \eta \tag{3.13}$$

$$= 1 + |A_1 \setminus \mathbb{N}| + \sum_2^m |A_i \cap [0, f_i]| \tag{3.14}$$

It is clear that $\mu$ just depends on the fission tree and not the irregular class. The dimension of the special configuration space is $\mu - 1$. In practice the second formulation is more convenient (when the tree is labelled by $1, \dots, m$) where $A_i = h(\mathbb{A}_i)$, $f_i = \min_{j<i}(f_{ij})$, and $|A_1 \setminus \mathbb{N}|$ is as in (3.9).

**Example 3.25.** Consider the pointed irregular type $Q = [(1, q_1), \dots, (1, q_4)]$ with

$$q_1 = x^{3/2} + x, \quad q_2 = x^{3/2} + 2x, \quad q_3 = x^{1/3}, \quad q_4 = 2x^{1/3}$$

having rank 10. Thus the root height $\eta = 2$ and the set of admissible exponents of $Q$ smaller than $\eta$ is $\{3/2, 1, 1/2, 1/3\}$. The labelled fission tree is drawn in Figure 1. We will always draw the mandatory vertices (the internal levels $\mathbb{L}$) as black circles, the inconsequential vertices ($\mathbb{A} \setminus \mathbb{L}$) as white circles, and the empty vertices without any decoration. The heights are indicated on the left. The root is drawn as a black square.

We will see below that the configuration space has dimension 8 (the number of nonempty nodes below the root), and further (in Theorem 3.27) that there are three types of conditions on these 8 coefficients: those from each black vertex $\bullet$ should be nonzero, those from the two inconsequential siblings $\circ$ should be distinct, and those from the two mandatory siblings $\bullet$ should have distinct $N$th powers (where $N = 3$ in this example). For the special configuration space, the dimension is $7 = 8 - 1$ since there is one positive integer height below the root. The moduli number $\mu(\mathcal{T})$ is eight.

## §3.6. Realisations of a fission tree

Suppose we are given a fission tree $\mathcal{T} = (\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$, and a map $c\colon \mathbb{A} \to \mathbb{C}$ with finite support, i.e. $c(v) = 0$ for all but a finite number of the admissible vertices $v \in \mathbb{A} \subset \mathcal{T}$.

Then for each leaf $i \in \mathbb{V}_0$ of the tree we can define an exponential factor

$$q_i = \sum_{v \in \mathbb{A}_i} c(v) x^{h(v)} \tag{3.15}$$

summing over the admissible vertices $\mathbb{A}_i = \mathcal{B}_i \cap \mathbb{A}$ on the $i$th full branch. Thus $c$ gives the coefficients of a list of exponential factors. Thus if the tree is labelled by some isomorphism $\psi\colon \{1, \dots, m\} \cong \mathbb{V}_0$ then we get an element

$$Q_c = [(n_1, q_1), \dots, (n_m, q_m)], \tag{3.16}$$

where $q_i$ is determined by $c$ as above, and $n_i = n(i)$ is the multiplicity of the leaf $i$.

We will say that the coefficient map $c$ is a *realisation* of the tree $\mathcal{T}$ if $Q_c$ is a pointed irregular type and $\mathcal{T}(Q_c) \cong \mathcal{T}$, i.e. if the fission tree of $Q_c$ is isomorphic to $\mathcal{T}$ (the labellings match up by construction).

Thus we wish to make explicit the conditions on $c$ for it to be a realisation.

In effect we just need to check that the tree determined by $Q_c$ has the desired branching and mandatory nodes. Let us focus on a single branchpoint. Let $l > k$ be two consecutive heights of the tree, and let $v \in \mathbb{V}_l$ be a vertex with $n$ children $\mathrm{Ch}(v) = \{w_1, \dots, w_n\} \subset \mathbb{V}_k$.

Let $q$ be the exponential factor determined by $c$ at the node $v$, so that $q = \tau_l(q_j)$ for any leaf $j$ that is a descendant of $v$. Let $q_i = q + c_i z^k$ (where $c_i = c(w_i)$) be the corresponding exponential factors of the children, $i = 1, \dots, n$. Thus for $c$ to be a realisation, the $c_i$ have to be such that the Galois orbits $\langle q_i \rangle$ are pairwise distinct. Proposition 3.13 allows us to characterise when this is the case. Set $r^+ := \mathrm{Ram}(q)$, and write $k = \frac{s^-}{r^-}$, with $s^-$ and $r^-$ coprime.

**Proposition 3.26.** *Let $q$ be an exponential factor with all its exponents greater than $k$, and let $c_1, \dots, c_n \in \mathbb{C}$ and consider $q_i = q + c_i z^k$ for $i = 1, \dots, n$.*

(1) *If $r^-$ divides $r^+$, then $k$ is not an internal level of any of the $q_i$. Then the $\langle q_i \rangle$ are distinct if and only if $c_i \neq c_j$ for $1 \leq i < j \leq n$.*

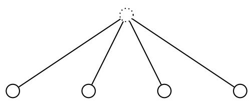
(2) *Otherwise, let us assume that $r^-$ does not divide $r^+$, and let $N := \frac{\operatorname{lcm}(r^-, r^+)}{r^+}$. Then $k$ is a level of $\langle q_i \rangle$ if and only if $c_i \neq 0$, otherwise if $c_i = 0$, $\langle q_i \rangle \in \mathbb{V}_k$ is an empty vertex. Now the $\langle q_i \rangle$ are distinct if and only if*

   (a) *either one of the $c_i$, $i = 1, \dots, n$, say $c_{i_0}$, is equal to zero and we have $c_i \neq 0$ for $i \neq i_0$ and $c_i \neq \zeta c_j$ for $i, j \in \{1, \dots, n\} \setminus \{i_0\}$ for any $N$th root of unity $\zeta$,*

   (b) *or all of the $c_i$ are nonzero, and we have $c_i \neq \zeta c_j$ for $1 \leq i < j \leq n$ for any $N$th root of unity $\zeta$.*

*Proof.* This follows immediately from the corresponding cases in Proposition 3.13. □

This implies that there are only three possible types of fission at a vertex $v$ in a fission tree, which correspond to the three cases (1), (2a), and (2b) of the proposition, and they yield axiom (5) in the definition of fission tree. In case (1), since $k$ is not an internal level of any of the $q_i$, all the corresponding vertices are inconsequential, which corresponds to the picture below (the parent vertex is dotted on the picture to indicate that it could be mandatory, inconsequential, or empty).



Otherwise, in case (2) the vertices corresponding to the $q_i$ are all mandatory provided they are nonempty. There are two possibilities: either, in case (2a), there is one empty vertex, which corresponds to the diagram on the left below, or, in case (2b), there is no empty vertex and all vertices are mandatory, which corresponds to the diagram on the right below.

Thus we can write down the exact conditions for $c$ to be a realisation. Recall that two nodes are *siblings* if they have the same parent node. In summary the result is the following.

**Theorem 3.27.** *The map* $c\colon \mathbb{A} \to \mathbb{C}$ *is a realisation of* $\mathcal{T}$ *if and only if*

(1) $c(\mathbb{L}) \subset \mathbb{C}^*$, *i.e.* $c(v) \neq 0$ *for any mandatory node* $v$,

(2) $c(u) \neq c(v)$ *for any pair* $u$, $v$ *of inconsequential siblings,*

(3) $c(u)^N \neq c(v)^N$ *for any pair* $u$, $v$ *of mandatory siblings where* $N$ *is the relative ramification of the parent of* $u$, $v$ *(defined in Remark 3.19).*

*Proof.* The first condition (and the definition of $\mathbb{A}$) implies each full branch has the correct internal levels. Then (2) and (3) show that the tree of $Q_c$ has the right branching. □

Note that (2), (3) can be combined into the following single statement: if $u$, $v$ are admissible (= nonempty) siblings then $c(u)^N \neq c(v)^N$, where $N$ is the relative ramification of the parent of $u$ or $v$ (since the relative ramification of the parent of an inconsequential vertex is 1).

**Corollary 3.28.** *Any fission tree admits a realisation.*

*Proof.* There are just a finite number of Zariski-closed conditions on the coefficients of any set of siblings, so the space of choices of $c$ is nonempty. □

This immediately gives a clearer description of the configuration spaces. First note that the definition of the fission tree implies the following.

**Lemma 3.29.** *Suppose* $Q$ *is any compatible pointed irregular type with (labelled) fission tree* $\mathcal{T} = \mathcal{T}(Q)$. *Let* $r = \mathrm{Ram}(Q)$, $x = t^r$ *so that*

$$Q \coloneqq [(n_1, q_1), \dots, (n_m, q_m)], \quad q_i = \sum_{j=1}^{s} a_{i,j} t^j, \tag{3.17}$$

*for some collection of coefficients* $\mathbf{a} = (a_{i,j}) \in \mathbb{C}$. *Then there is a unique realisation* $c = c_Q\colon \mathbb{A} \to \mathbb{C}$ *of* $\mathcal{T}$ *with* $c(v) = a_{i,k}$ *for all* $i$, $k$ *where* $v = \langle \tau_{k/r}(q_i) \rangle \in \mathbb{A}$ *is the vertex of* $\mathcal{T}$ *determined by the truncation of the exponential factor* $q_i$.

*Proof.* This amounts to verifying two conditions, which are now straightforward: (1) $a_{i,k} = 0$ if $\langle \tau_{k/r}(q_i) \rangle \in \mathbb{V} \setminus \mathbb{A}$, i.e. if the node of $\mathcal{T}$ determined by the circle $\langle \tau_{k/r}(q_i) \rangle$ is not admissible, and (2) $a_{i,k} = a_{j,k}$ if $\langle \tau_{k/r}(q_i) \rangle = \langle \tau_{k/r}(q_j) \rangle$, i.e. if the truncations determine the same node of $\mathcal{T}$. □

It follows that the configuration space $\mathbf{B}(Q)$ of any compatible pointed irregular type $Q$ is isomorphic to the space

$$\begin{aligned} \mathbf{B}(\mathcal{T}^\flat) &\coloneqq \bigl\{ c\colon \mathbb{A} \to \mathbb{C} \bigm| c \text{ is a realisation of } \mathcal{T},\ \mathrm{Katz}(c) \leq \mathrm{Katz}(Q) \bigr\} \\ &= \bigl\{ c\colon \mathbb{A}^\flat \to \mathbb{C} \bigm| (1), (2), (3) \text{ of Theorem 3.27 hold for } c \bigr\} \end{aligned} \tag{3.18}$$

of realisations of the truncated fission tree $\mathcal{T}^\flat$. Here, $\mathbb{A}^\flat$ is the set of admissible nodes of $\mathcal{T}^\flat$ and $\operatorname{Katz}(c) = \max\{h(a) \mid a \in \mathbb{A}, c(a) \neq 0\}$ is the height of the realisation. In other words we have established the following.

**Theorem 3.30.** *If $Q$ is a compatible pointed irregular type then $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T}^\flat)$, where $\mathcal{T}^\flat$ is the truncated labelled fission tree of $Q$.*

*Proof.* By Theorem 3.17, $\mathbf{B}(Q)$ is the set of compatible pointed irregular type $Q'$ with the same labelled fission data as $Q$ and $\operatorname{Katz}(Q') \leq \operatorname{Katz}(Q)$. Then as in Lemma 3.21 this is the same as saying the labelled fission tree of $Q'$ is isomorphic to that of $Q$ and $\operatorname{Katz}(Q') \leq \operatorname{Katz}(Q)$. Then by Lemma 3.29, any such $Q'$ arises uniquely as a realisation $c$ of $\mathcal{T}^\flat$. □

This immediately gives a product decomposition of the configuration space. Given a compatible pointed irregular type $Q$ with fission tree $\mathcal{T}$, for any vertex $v \in \mathbb{V}$ of $\mathcal{T}$ let $\operatorname{Ch}_{\mathbb{A}}(v) = \mathbb{A} \cap \operatorname{Ch}(v)$ be the set of admissible/nonempty children of $v$ and let $\operatorname{Ch}_\bullet(v) = \mathbb{L} \cap \operatorname{Ch}(v)$ be the subset of mandatory children of $v$ (black vertices). Define the local configuration space $\mathbf{B}_v(Q) = \mathbf{B}_v(\mathcal{T})$ for the vertex $v \in \mathbb{V}$:

$$\mathbf{B}_v(\mathcal{T}) \coloneqq \big\{c\colon \operatorname{Ch}_{\mathbb{A}}(v) \to \mathbb{C} \;\big|\; c(\operatorname{Ch}_\bullet(v)) \subset \mathbb{C}^*, \text{ and } c(u)^N \neq c(w)^N \ \forall u \neq w \in \operatorname{Ch}_{\mathbb{A}}(v)\big\}, \tag{3.19}$$

where $N$ is the relative ramification of $v$ (defined in Remark 3.19). The space $\mathbf{B}_v(\mathcal{T})$ is taken to be a point if $v$ has no nonempty children and otherwise it thus takes the form

$$\mathbf{B}_v(\mathcal{T}) \cong X_n \coloneqq \big\{a_1, \dots, a_n \in \mathbb{C} \;\big|\; a_i \neq a_j \text{ for } i \neq j\big\}, \tag{3.20}$$

if $v$ has $n$ inconsequential children, or

$$\mathbf{B}_v(\mathcal{T}) \cong X^*_{n,N} \coloneqq \big\{a_1, \dots, a_n \in \mathbb{C} \;\big|\; a_i \neq 0,\ a_i^N \neq a_j^N \text{ for } i \neq j\big\} \tag{3.21}$$

if $v$ has $n$ mandatory children and relative ramification $N$.

**Corollary 3.31.** *Let $Q$ be a compatible pointed irregular type with fission tree $\mathcal{T}$ and let $\mathbb{V}^\flat$ be the vertices of its truncated fission tree $\mathcal{T}^\flat$. The configuration space $\mathbf{B}(Q)$ admits a product decomposition*

$$\mathbf{B}(Q) \cong \prod_{v \in \mathbb{V}^\flat} \mathbf{B}_v(\mathcal{T}).$$

*In particular, the dimension of $\mathbf{B}(Q)$ is the number of admissible (nonempty) vertices of the fission tree $\mathcal{T}$ of height $\leq$ the Poincaré–Katz rank of $Q$.*

*Proof.* Since $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T}^\flat)$ this follows from the characterisation of the realisations $c\colon \mathbb{A}^\flat \to \mathbb{C}$ of $\mathcal{T}^\flat$ given in Theorem 3.27. □

In particular, it follows that the configuration spaces are *connected*, since each of the local configuration spaces $\mathbf{B}_v(\mathcal{T}) \cong X_n$ or $X^*_{n,N}$ is connected. This yields a combinatorial/topological characterisation of admissible deformations, as follows.

**Corollary 3.32.** *Two compatible pointed irregular types are admissible deformations of each other if and only if they have isomorphic labelled fission trees, if and only if they have the same labelled fission data.*

*Proof.* Given $Q$, $Q'$ suppose that $\mathrm{Katz}(Q) \geq \mathrm{Katz}(Q')$ and consider $\mathbf{B}(Q)$. If the labelled fission trees are isomorphic then $Q' = Q_c$ for some realisation $c$ of $\mathcal{T}(Q)$, by Theorem 3.30. Thus, by connectedness, $Q$, $Q'$ are admissible deformations of each other with $\mathbb{B} = \mathbf{B}(Q)$. The converse is clear. The last statement follows as in Lemma 3.21. □

**Corollary 3.33.** *Two rank-$n$ irregular classes are admissible deformations of each other if and only if they have isomorphic fission trees, if and only if they have the same fission data.*

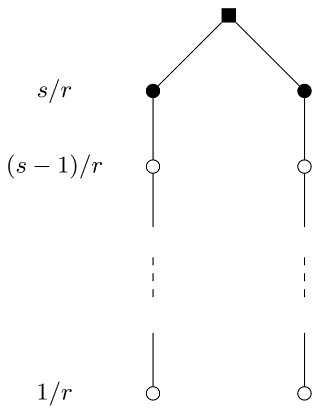
*Proof.* This follows from Corollary 3.32 by considering local lifts from irregular classes to pointed irregular types. □

**Corollary 3.34.** *Let $Q$ be a compatible pointed irregular type. Then the configuration space $\mathbf{B}(Q)$ is a fine moduli space of all pointed irregular types that are admissible deformations of $Q$, with Poincaré–Katz rank $\leq$ $\mathrm{Katz}(Q)$. Similarly, $\mathbf{SB}(Q)$ is a fine moduli space of all trace-free admissible deformations of any trace-free pointed irregular type $Q$.*

*Proof.* The main point is that the product decomposition and the formulae (3.15), (3.16) give a universal family of pointed irregular types over $\mathbf{B}(Q) = \mathbf{B}(\mathcal{T}^\flat(Q))$. □

**Example 3.35.** Let us look at a few examples, starting with those with two exponential factors studied previously.

- Consider $Q = [(1, \lambda x^{s/r}), (1, \mu x^{s/r})]$ with $r$ and $s$ coprime and $\lambda \neq \mu e^{2in\pi/r}$ for any integer $n$. The (labelled) fission tree $\mathcal{T}$ is the following:

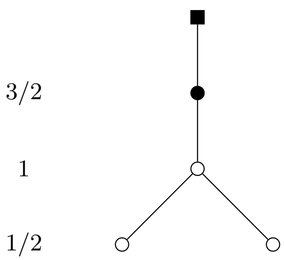



From the tree we read the space of admissible deformations: we have

$$\mathbf{B}(Q) \cong X^*_{2,r} \times \mathbb{C}^{2(s-1)}, \quad X^*_{2,r} = \big\{a_1, a_2 \in \mathbb{C} \mid a_1 a_2 \neq 0,\ a_1^r \neq a_2^r\big\}.$$

Indeed, the factor corresponding to the root vertex is $X^*_{2,r}$, the factors for the leaves are trivial, while for all other vertices $v$ the space $\mathbf{B}_v(\mathcal{T})$ is isomorphic to $\mathbb{C}$. Similarly, $\mathbf{SB}(Q) \cong X^*_{2,r} \times \mathbb{C}^N$, where $N = 2s - 2 - \lfloor s/r \rfloor$, removing one dimension for each integer below the root.

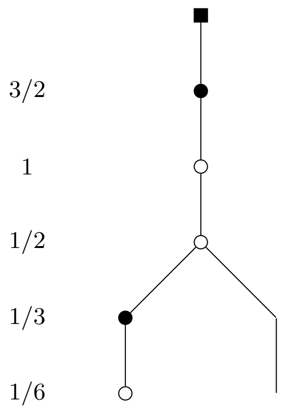
- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/2}$, and $q_2 = x^{3/2} + 2x^{1/2}$. The fission tree is drawn below.




From this we read that the space of admissible deformations satisfies

$$\mathbf{B}(Q) \cong X^*_{1,2} \times X_1 \times X_2 \cong \mathbb{C}^* \times \mathbb{C} \times \big\{a, b \in \mathbb{C} \mid a \neq b\big\}.$$

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/3}$, and $q_2 = x^{3/2}$. The fission tree is drawn below.




This yields

$$\mathbf{B}(Q) \cong X^*_{1,2} \times X^*_{1,3} \times X^3_1 \cong (\mathbb{C}^*)^2 \times \mathbb{C}^3.$$

## §3.7. Topological skeleta

Corollary 3.33 has the following immediate global consequence. Suppose $\boldsymbol{\Sigma} = (\Sigma, \mathbf{a}, \boldsymbol{\Theta})$ is a rank-$n$ wild Riemann surface, with $\Sigma$ a compact Riemann surface, $\mathbf{a} \subset \Sigma$ a finite subset, and $\boldsymbol{\Theta} = \{\Theta_a \mid a \in \mathbf{a}\}$ the data of a rank-$n$ irregular class for each marked point. For each $a \in \mathbf{a}$ let $\mathcal{T}_a = \mathcal{T}(\Theta_a)$ be the fission tree of the irregular class $\Theta_a$. Define the *topological skeleton* of $\boldsymbol{\Sigma}$ to be the pair

$$\mathrm{Sk}(\boldsymbol{\Sigma}) = (g, \mathbf{F}),$$

where $g \geq 0$ is the genus of $\Sigma$ and $\mathbf{F} = \sum_{a \in \mathbf{a}} [\mathcal{T}_a]$ is the *forest* of $\boldsymbol{\Sigma}$, i.e. the multiset of isomorphism classes of fission trees determined by all the $\mathcal{T}_a$, as $a$ ranges over the marked points $\mathbf{a} \subset \Sigma$. In general it is a multiset rather than a set as some of the fission trees at distinct points may be isomorphic. As explained above, the notion of admissible deformations of (twisted) wild Riemann surfaces follows from the untwisted case of [18] (extending the generic case in [40, 45]). In brief, a holomorphic admissible deformation of rank-$n$ wild Riemann surfaces over a base space $\mathbb{B}$ is a holomorphic family $\pi \colon \underline{\Sigma} \to \mathbb{B}$ of compact Riemann surfaces, together with a holomorphic multisection $\sigma \subset \underline{\Sigma}$ restricting to a finite subset $\mathbf{a}_b \subset \Sigma_b$ in each fibre $\Sigma_b = \pi^{-1}(b) \subset \underline{\Sigma}$, and also the choice of a rank-$n$ irregular class $\Theta_a$ at each point $a \in \mathbf{a}_b \subset \Sigma_b$ for each $b \in \mathbb{B}$. These should be such that the irregular classes vary holomorphically, and the deformation is admissible: for the pointed surfaces the admissibility condition just means that each fibre $\Sigma_b$ is smooth, and none of the points $\mathbf{a}_b$ coalesce (so we get the same number of points in each fibre). Finally we need to define what it means for the irregular classes to vary holomorphically and admissibly: these are local conditions so we can work over any small enough open subset $U$ of $\mathbb{B}$ and focus on one marked point $a \in \mathbf{a}_b$. We can then choose a local coordinate $z$ vanishing at $a$ (for all $b \in U$). Thus we reduce to the situation in the definition of holomorphic admissible deformations of irregular classes given in Definition 2.5 above. In turn, two rank-$n$ wild Riemann surfaces will be said to be admissible deformations of each other if they are related by the equivalence relation generated by the condition of being two fibres of a holomorphic admissible deformation (as defined above).

Corollary 3.33 then implies the following corollary.

**Corollary 3.36.** *Two rank-$n$ wild Riemann surfaces are admissible deformations of each other if and only if they have the same topological skeleta.*

*Proof.* We may assume the genus and the number of marked points are equal, otherwise the result is clear. Now if their topological skeleta are distinct then Corollary 3.33 implies there is no admissible deformation between them. Conversely, if they have the same topological skeleta then we can use the universal family over Teichmüller space (or its version with marked points) to admissibly deform both wild Riemann surfaces so they have the same underlying Riemann surface with marked points, and moreover we may assume (since the topological skeleta are the same) that at each marked point the two fission trees are isomorphic. Finally we use the local statement (Corollary 3.33) to deform the two irregular classes at each point in an admissible fashion, until they are equal. □

In particular, since the set of possible topological skeleta is countable, this gives control over the set of possible topological types of the wild character varieties $\mathcal{M}_{\mathrm{B}}$: as in [18] they form a local system of varieties over any admissible deformation, so up to isomorphism there is just one (Poisson) wild character variety for each possible topological skeleton (the key part of the proof in [18] is local on the circle of directions so works equally well for twisted irregular classes).

**Remark 3.37.** The irregular Deligne–Simpson problem can then be stated as follows: given a topological skeleton $(g, \mathbf{F})$, let $\mathbf{L}$ be the set of all the leaves of all the trees in the forest $\mathbf{F}$. Choose a conjugacy class $\mathcal{C}_i \subset \mathrm{GL}_{n_i}(\mathbb{C})$ for each leaf $i \in \mathbf{L}$, where $n_i \geq 1$ is the multiplicity of $i$.

**Question.** For which choices of skeleton and conjugacy classes is there an irreducible algebraic connection $(V, \nabla) \to \Sigma^\circ = \Sigma \setminus \mathbf{a}$ with the given topological skeleton and formal monodromy conjugacy classes? Passing to Stokes local systems [22] (by the Stokes version of the irregular Riemann–Hilbert correspondence), this can easily be rewritten as a linear algebra problem (as in [18, §9.4], and the graphical examples in [20, §11]).

## §4. Local wild mapping class groups

### §4.1. Pure local wild mapping class groups

Let $Q$ be a pointed irregular type. We define the *pure* local (twisted) wild mapping class group of $Q$ as the fundamental group $\Gamma(Q) \coloneqq \pi_1(\mathbf{B}(Q))$ of the configuration space of admissible deformations (with basepoint $\mathbf{a}_Q$). From the description of $\mathbf{B}(Q)$, it follows immediately that $\Gamma(Q)$ also factorises as a product of factors associated to each vertex of the fission tree. Note that the usual mapping class group may be defined in two ways, as the group of mapping classes or as the fundamental group of the moduli space/stack of Riemann surfaces; in the wild setting we only know the second type of definition, by generalising the notion of Riemann surface by incorporating nontrivial irregular classes.

**Theorem 4.1.** *Let $\mathcal{T}$ be the fission tree of $Q$ and let $\mathbb{V}^\flat$ be the nodes of its truncation. We have*

$$\Gamma(Q) \cong \prod_{v\in\mathbb{V}^\flat} \Gamma_v(\mathcal{T}),$$

*with $\Gamma_v(\mathcal{T}) \coloneqq \pi_1(\mathbf{B}_v(\mathcal{T}))$.*

Since $\mathbf{B}_v(\mathcal{T})$ is isomorphic to a hyperplane complement of the form $X_n$ or $X^*_{n,N}$, setting

$$\Gamma_n \coloneqq \pi_1(X_n), \quad \Gamma^*_{n,N} \coloneqq \pi_1(X^*_{n,N}),$$

we get that $\Gamma(Q)$ is always a product of factors of the form $\Gamma_n$ or $\Gamma^*_{n,N}$. Notice that $\Gamma_n$ is none other but the pure braid group on $n$ strands $\mathrm{PB}_n$. We recover in this way the untwisted case of [33], since it corresponds to the case where all vertices are inconsequential, hence only the factors $\Gamma_n$ will appear in the factorisation. In comparison with the untwisted case, new factors $\Gamma^*_{n,N}$ appear as factors of the local wild mapping class group.

Let us now look at a few examples. A simple case to look at is when there is only one exponential factor.

**Proposition 4.2.** *Let $q$ be an exponential factor. Then $\Gamma(q) \coloneqq \pi_1(\mathbf{B}(q))$ is isomorphic to $\mathbb{Z}^{|\mathrm{Levels}(q)|}$.*

*Proof.* This follows immediately from Proposition 3.8. □

**Example 4.3.** Let us look once again at our previous examples:

- Consider $Q = [(1, \lambda x^{s/r}), (1, \mu x^{s/r})]$ with $r$ and $s$ coprime and $\lambda \neq \mu e^{2in\pi/r}$ for any integer $n$. The space of realisations is homotopy equivalent to

$$X^*_{2,r} = \left\{a, b \in \mathbb{C}^* \;\middle|\; \forall k \in \mathbb{Z}, a \neq b e^{2ik\pi/r}\right\}.$$

The fundamental group is thus $\Gamma(Q) \cong \Gamma^*_{2,r}$.

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/2}$, and $q_2 = x^{3/2} + 2x^{1/2}$. The space of realisations is homotopy equivalent to

$$X^*_{1,2} \times X_2 = \mathbb{C}^* \times \{(a, b) \in \mathbb{C}^2 \mid a \neq b\}.$$

Its fundamental group thus satisfies $\Gamma(Q) \cong \Gamma^*_{1,2} \times \Gamma_2 \cong \mathbb{Z}^2$.

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/3}$, and $q_2 = x^{3/2}$. The space of admissible deformations is homotopy equivalent to $X^*_{1,2} \times X^*_{1,3} \cong (\mathbb{C}^*)^2$, so its fundamental group is isomorphic to $\mathbb{Z}^2$.

**Remark 4.4.** It turns out that the new building blocks $\Gamma^*_{n,N}$ which appear in the twisted case coincide with some braid groups studied in the literature on complex reflections, in particular by Broué–Malle–Rouquier [27]. More precisely, the group $\Gamma^*_{n,N}$ is the same as the group denoted $P(N, 1, n)$ there, and the hyperplane complement $X^*_{n,N}$ is equal to the one denoted by $\mathcal{M}^{\#}(N, n)$ there (introduced in their Lemma 3.3). Our study of the local wild mapping class groups thus gives a modular interpretation (in two-dimensional gauge theory) for this class of complex braid groups (coming from hyperplane arrangements of the reflecting hyperplanes of these complex/unitary reflection groups). We will see below that the corresponding complex reflection groups, the generalised symmetric groups $S(n, N) = G(N, 1, n)$, appear also in our setting, when passing from irregular types to irregular classes.

## §4.2. Full local wild mapping class groups

Given an irregular class $\Theta$ we have defined a fission tree $\mathcal{T} = \mathcal{T}(\Theta)$ and this determines a configuration space $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T}) \subset \mathrm{Map}(\mathbb{A}^\flat, \mathbb{C})$, where $\mathbb{A}^\flat$ is the finite set of admissible nodes of the truncated fission tree. Now we will define a finite group $W(\mathcal{T})$ (the *Weyl group* of the tree) and a free action of $W(\mathcal{T})$ on $\mathbf{B}(\mathcal{T})$ so that two points $Q_1, Q_2 \in \mathbf{B}(\mathcal{T})$ are in the same orbit if and only if $[Q_1] = [Q_2]$, i.e. if they determine the same irregular class. This leads to the *full* local wild mapping class group.

For certain simple examples of fission trees $\mathcal{T}$ we will then find

$$W(\mathcal{T}) \cong \mathrm{Sym}_n \ltimes (\mathbb{Z}/N\mathbb{Z})^n,$$

i.e. the Weyl group is a so-called generalised symmetric group $S(N, n)$, isomorphic to the complex reflection group denoted $G(N, 1, n)$ in the Shephard–Todd classification [52] (they are the symmetry groups of the regular complex polytopes called the generalised cubes $\gamma^N_n$ and the generalised octahedra $\beta^N_n$; see e.g. §13.4, p. 147 of Coxeter's book [28] on regular complex polytopes).

**4.2.1. The Weyl group of a fission tree.** Let $\mathcal{T}$ be a fission tree, let $p = |\mathbb{V}_0|$ be the number of leaves of $\mathcal{T}$, and choose a labelling $\psi\colon \{1,\dots,p\} \cong \mathbb{V}_0$. Let $v_i = \psi(i)$ be the $i$th leaf, and let $v_{ij} \in \mathcal{T}$ be the branchpoint where the full branches $\mathcal{B}_i$, $\mathcal{B}_j$ meet, i.e. the nearest common ancestor of $v_i$, $v_j$. Let $r_i = \mathrm{Ram}(v_i) \in \mathbb{N}$ be the ramification index of the $i$th leaf, and let $r_{ij} = \mathrm{Ram}(v_{ij})$, for all $i, j = 1, 2, \dots, p$, so that $r_{ij}$ divides both $r_i$ and $r_j$.

The group $\mathrm{Aut}(\mathcal{T})$ of automorphisms of $\mathcal{T}$ embeds in the symmetric group $\mathrm{Sym}_p = \mathrm{Aut}(\mathbb{V}_0)$ since any automorphism of the tree is determined by its action on the leaves. Thus $\mathrm{Aut}(\mathcal{T})$ acts on the product $(\mathbb{Z}/r_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/r_p\mathbb{Z})$ of cyclic groups, permuting the factors (since if two full branches are isomorphic then they have the same ramification index $r_i$). Thus we can consider the semi-direct product

$$\mathrm{Aut}(\mathcal{T}) \ltimes ((\mathbb{Z}/r_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/r_p\mathbb{Z})) \tag{4.1}$$

defined via this action. The Weyl group of $\mathcal{T}$ is the following subgroup of this semi-direct product.

**Definition 4.5.** The Weyl group of the fission tree $\mathcal{T}$ is the subgroup of (4.1) defined by

$$W(\mathcal{T}) \coloneqq \big\{(\pi, (d_1, \dots, d_p)) \in \mathrm{Aut}(\mathcal{T}) \ltimes (\mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_p\mathbb{Z}) \;\big|\; d_i \equiv d_j \bmod r_{ij}\big\}.$$

Note that since $r_{ij}|r_i$ there is a quotient map $\mathrm{pr}_i\colon \mathbb{Z}/r_i\mathbb{Z} \twoheadrightarrow \mathbb{Z}/r_{ij}\mathbb{Z}$ and the statement that $d_i \equiv d_j \bmod r_{ij}$ just means that $\mathrm{pr}_i(d_i) = \mathrm{pr}_j(d_j)$. If $p = 1$ then $W(\mathcal{T}) = \mathbb{Z}/r_1\mathbb{Z}$.

In the rest of this section we will prove the following.

**Theorem 4.6.** *Let $Q$ be a compatible pointed irregular type with fission tree $\mathcal{T}$. The Weyl group $W(\mathcal{T})$ acts freely on the configuration space $\mathbf{B}(Q) = \mathbf{B}(\mathcal{T})$ and the quotient*

$$\overline{\mathbf{B}}(\Theta) = \overline{\mathbf{B}}(\mathcal{T}) \coloneqq \mathbf{B}(Q)/W$$

*is the space of all irregular classes that are admissible deformations of $\Theta \coloneqq [Q]$ with bounded Poincaré–Katz rank.*

It follows that $\overline{\mathbf{B}}(\Theta)$ is a manifold and we can define the *full* local wild mapping class group to be

$$\overline{\Gamma}(\Theta) = \pi_1(\overline{\mathbf{B}}(\Theta)). \tag{4.2}$$

**4.2.2. Action on pointed irregular types.** The semi-direct product (4.1) is easy to understand via its action on pointed irregular types. Let

$$Q = [(n_1, q_1), \dots, (n_p, q_p)]$$

be a pointed irregular type with fission tree $\mathcal{T}$ as above. Let $G$ denote the corresponding group (4.1) defined as a semi-direct product. If $g = (\pi, \mathbf{d}) \in G$ with $\mathbf{d} = (d_1, \ldots, d_p)$ then we can obtain another pointed irregular type $g \cdot Q$ with fission tree $\mathcal{T}$ by the formula

$$\begin{aligned} g \cdot Q &= (\pi, 0) \cdot [(n_1, \sigma^{d_1}(q_1)), \ldots, (n_p, \sigma^{d_p}(q_p))] \\ &= [(n_{\pi(1)}, \sigma^{d_{\pi(1)}}(q_{\pi(1)})), \ldots, (n_{\pi(p)}, \sigma^{d_{\pi(p)}}(q_{\pi(p)}))] \end{aligned}$$

so that the cyclic groups rotate the choices of "pointing" and $\pi$ permutes the exponential factors which have isomorphic full branches.

Note that $g \cdot Q$ will always have the same fission tree as $Q$ but it may not be an admissible deformation of $Q$, i.e. it may not be a point of the configuration space $\mathbf{B}(Q)$. The Weyl group $W(\mathcal{T})$ is the subgroup characterised by this property.

**Lemma 4.7.** *Suppose $Q$ is a compatible pointed irregular type, and $g \in G$ is an element of the semi-direct product (4.1). Then $g \cdot Q$ is an admissible deformation of $Q$ (i.e. $g \cdot Q \in \mathbf{B}(Q)$) if and only if $g \in W(\mathcal{T})$.*

*Proof.* This amounts to characterising the $g \in G$ such that $g \cdot Q$ is still compatible since (1) any admissible deformation of $Q$ will still be compatible, and (2) by Lemma 3.29, any compatible $g \cdot Q$ will be in $\mathbf{B}(Q)$. Now, to see whether $g \cdot Q$ is still compatible, we need the exponential factors to "branch" like the circles they determine (i.e. their Galois orbits), and this comes down to requiring

$$\tau_k(\sigma^{d_i}(q_i)) = \tau_k(\sigma^{d_j}(q_j)),$$

where $k = g_{ij}$ is the height of the nearest common ancestor of the leaves $i$, $j$ in $\mathcal{T}$, for all indices $i \neq j$. But this just says that $\sigma^{d_i}(q_c) = \sigma^{d_j}(q_c)$, where $q_c = \tau_k(q_i) = \tau_k(q_j)$ is the common part of $q_i$, $q_j$. Now, since $r_{ij}$ is the ramification order of $q_c$, this just means that $d_i \equiv d_j$ modulo $r_{ij}$, as in the definition of $W(\mathcal{T})$. □

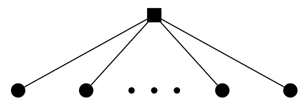
This implies that the finite group $W(\mathcal{T})$ acts on the configuration space $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T})$, and we can now prove the rest of the theorem.

*Proof of Theorem 4.6.* It remains to show the action is free, and the orbits in $\mathbf{B}(Q)$ are the subsets with the same irregular class. The action is free since (1) the circles corresponding to isomorphic full branches are indeed distinct circles (as otherwise they would be recorded in the multiplicity of the leaf of the full branch), so the permutation is trivial, and (2) the $r_i$ are indeed the exact sizes of the Galois orbits of the $q_i$, so no smaller cyclic shift will act trivially. Finally, note that the pointed irregular types with given irregular class are just related by a choice of ordering of the circles, and the pointings of each circle are related by the group $G$. Thus

Lemma 4.7 implies the $W$ orbits in $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T})$ are exactly the points with the same irregular class. □

**Remark 4.8.** The Weyl group $W(\mathcal{T})$ is thus a subgroup of the symmetric group of all permutations of the exponential factors in the corresponding full irregular type (the leaves of the corresponding full/naive fission tree, as in the proof of Proposition 3.1, closely related to the "3$d$ fission tree" in [22]).

**Example 4.9.** Suppose $Q = [(1, q_1), \dots, (1, q_p)]$ with $q_i = a_i x^{s/N}$, $i = 1, \dots, p$, where $(s, N) = 1$, $N > 1$ and the $a_i$ are generic complex numbers (in the sense that $a_i \neq 0$ and $a_i^N \neq a_j^N$ if $i \neq j$). Then the top part of $\mathcal{T}^\flat(Q)$ looks as follows, with $p$ mandatory nodes branching from the root:



Thus $\mathrm{Aut}(\mathcal{T}) = \mathrm{Sym}_p$, and in turn $W(\mathcal{T}) = \mathrm{Sym}_p \ltimes (\mathbb{Z}/N\mathbb{Z})^p$ is the generalised symmetric group $S(N, p) \cong G(N, 1, p)$, since the ramification indices $r_{ij}$ are all equal to 1. For example, any *symmetric irregular class* $I(a{:}b) := \sum_{i=1}^m \langle \varepsilon^i x^{a/b} \rangle$ falls into this setting ($p = m$, $N = b/m$). Here $a$, $b$ are positive integers with highest common factor $m$, and $\varepsilon = \exp(2\pi i/b)$. These are the classes obtained by pulling back a Stokes circle of the form $\langle w^{1/b} \rangle$, under the cyclic covering $w = x^a$, and occur for the Molins–Turrittin differential equation $y^{(n)} = x^\nu y$, which has the same exponential factors as the irregular class $I(n + \nu{:}n)$ (up to an overall scale factor of $n/(n + \nu)$) [48, 54].

As in the untwisted case [32], there is an explicit recursive description of the automorphism groups of the fission trees, and in turn the Weyl group. Define a *maximal subtree* of a fission tree $\mathcal{T}$ to be one of the trees obtained by removing the highest branch node of $\mathcal{T}$ (and all the higher edges), so the root of the subtree was the highest branch vertex of $\mathcal{T}$. The function Ram on $\mathcal{T}$ (from Remark 3.19) restricts to defining a function on any such subtree, so its Weyl group is well defined.

**Theorem 4.10.** *Let $\mathcal{T}$ be a fission tree.*

*If $\mathcal{T}$ consists only of one full branch whose leaf has ramification order $r$ then* $\mathrm{Aut}(\mathcal{T})$ *is trivial and* $W(\mathcal{T})$ *is isomorphic to* $\mathbb{Z}/r\mathbb{Z}$.

*Otherwise, let $\overline{\mathcal{T}}_1, \dots, \overline{\mathcal{T}}_s$ be the distinct isomorphism classes of decorated trees among its maximal subtrees, and for $i = 1, \dots, s$ let $n_i \in \mathbb{N}$ denote the number of such maximal subtrees having the isomorphism class $\overline{\mathcal{T}}_i$. Then* $\mathrm{Aut}(\mathcal{T})$ *is a product* *of wreath products:*

$$\mathrm{Aut}(\mathcal{T}) \cong \prod_{i=1}^{s} \mathrm{Sym}_{n_i} \wr \mathrm{Aut}(\overline{\mathcal{T}}_i).$$

*In turn, if $r$ is the ramification order of the root of all the subtrees $\overline{\mathcal{T}}_i$ then*

$$W(\mathcal{T}) \cong \Big\{(\pi_i, (g_{i,1}, \dots, g_{i,n_i})) \in \prod_{i=1}^{s} \mathrm{Sym}_{n_i} \wr W(\overline{\mathcal{T}}_i) \;\Big|\; \begin{array}{l} \delta(g_{i,k}) \equiv \delta(g_{j,l}) \bmod r \\ \forall i \neq j,\ \forall k, l \Big\}, \end{array}$$

*where $\delta(g_{i,k})$ denotes the shift at the root of the subtree induced by the automorphism $g_{i,k}$.*

*Proof.* For the untwisted automorphism group the proof is exactly the same as the one in [32] for the twisted case. For the Weyl group, the proof is similar, the only difference is that the compatibility conditions for the root shifts $\delta(g_{i,k})$ imply that we must restrict to a subgroup of the wreath product $\prod_{i=1}^{s} \mathrm{Sym}_{n_i} \wr W(\overline{\mathcal{T}}_i)$ that we would get otherwise. □

The general picture we finally arrive at is the following: the full local wild mapping class group $\overline{\Gamma}(\Theta)$ is an extension of the Weyl group $W(\mathcal{T})$ of the fission tree by the pure local wild mapping class group $\Gamma(\Theta)$, i.e. we have a short exact sequence

$$1 \to \Gamma(Q) \to \overline{\Gamma}(\Theta) \to W(\mathcal{T}) \to 1. \tag{4.3}$$

**4.2.3. The case of one active circle.** In general, the exact sequence does not split, so it is not easy to get a fully explicit description of the full local wild mapping class group. In the simple case of only one exponential factor however, we will now see it is possible to be more explicit and to determine completely the full local mapping class group.

**Theorem 4.11.** *Let $\Theta = \langle q \rangle$ be an irregular class with one circle determined by the exponential factor $q$. Then the full local wild mapping class group $\overline{\Gamma}(\Theta)$ is isomorphic to $\mathbb{Z}^{|\,\mathrm{Levels}(q)|}$.*

*Proof.* Let $r = \mathrm{Ram}(q)$ and $q_i = \sigma^i(q)$, $i = 0, \dots, r-1$ denote the Galois orbit of $q = \sum a_i x^{i/r}$. Let us write $L(q) = \mathrm{Levels}(q) = (k_1 > \cdots > k_m)$ and $k_i = n_i/r$. Let us consider the loop $\gamma_i \colon [0,1] \to \mathbf{B}$, $i = 1, \dots, m$, such that $\gamma_i$ makes the coefficient $a_{n_i}$ of $x^{k_i}$ go once around the origin, and leaves the other coefficient constant. Let us also consider the path $\nu$ in $\mathbf{B}$ such that

$$\nu(t) = \sum_j a_j e^{-2\sqrt{-1}\pi j t/r} x^{j/r},$$

so that $\nu(0) = q$, $\nu(1) = q_1$. Then $\bar{\Gamma}(q)$ is the abelian group generated by the homotopy classes determined by $\gamma_1, \ldots, \gamma_m$ and $\nu$, while the subgroup $\Gamma(q)$ is generated by $\gamma_1, \ldots, \gamma_m$. There are no relations between the generators $\gamma_1, \ldots, \gamma_m$, which recovers the fact that $\Gamma(q) \cong \mathbb{Z}^m = \mathbb{Z}^{|L(q)|}$. On the other hand, the family $\gamma_1, \ldots, \gamma_m, \nu$ is not free. Indeed, if we follow $r$ times the loop determined by $\nu$, it is the image of a loop upstairs, going from $q_0$ to itself, with the coefficient $a_{n_i}$ going around the origin a number of times equal to $\gcd(n_i, r) =: d_i$. We thus have the following relation between the $m + 1$ generators

$$d_1\gamma_1 + \cdots + d_m\gamma_m = r\nu$$

(we have used an additive notation here since the group is abelian). Using that $\gcd(r, n_1, \ldots, n_m) = 1$, the Schmidt algorithm used to classify finitely generated abelian groups transforms the vector $(d_1, \ldots, d_m, -r) \in \mathbb{Z}^{m+1}$ corresponding to this relation into $(1, 0, \ldots, 0)$, which implies that $\bar{\Gamma}(q) \cong \mathbb{Z}^m = \mathbb{Z}^{|L(q)|}$. □

In particular, the short exact sequence here reads

$$0 \to \mathbb{Z}^{|L(q)|} \to \mathbb{Z}^{|L(q)|} \to \mathbb{Z}/r\mathbb{Z} \to 0,$$

and does not split.

## §5. Outlook

Several of the directions we plan to pursue are as follows:

(1) Extend this work beyond type $A$ to any $G$: the notion of irregular class is already in [24], the analogue of fission trees for any $G$ in the untwisted case is in [33, 32], and the definition of admissible deformations will again follow from that in the untwisted case [18]. Presumably this will lead to other examples of (nonreal) complex reflection braid groups.

(2) Apply the fission trees to the Lax project [21], classifying the (wild) nonabelian Hodge spaces up to isomorphism/deformation. For example how many distinct deformations classes are there in each complex dimension $2, 4, 6, \ldots$? Can the fission trees be "combined" with the diagrams of [25, 31] (which are invariant under Fourier–Laplace) to give a refined invariant? This encompasses the question of classifying isomonodromy systems, and the Painlevé equations are amongst the dimension 2 examples.

(3) Study further the full moduli spaces/stacks $\mathfrak{M}_{g,\mathbf{F}}$ of admissible deformations of any wild Riemann surface, whose fundamental groups are the wild mapping class groups (as in [18], [19, §8]), generalising the Riemann moduli spaces $\mathfrak{M}_{g,m}$

and $\mathfrak{M}_{g,\{m\}}$ in the tame case (where $\{m\}$ means $m$ unordered marked points), as well as their universal covers (analogues of Teichmüller spaces). For example, for $g = 0$ and $\mathbf{F} = [\mathcal{T}]$ a single tree, this just amounts to quotienting the configuration space $\overline{\mathbf{B}}(\mathcal{T})$ by the two-dimensional group of Möbius transformation fixing one point of the Riemann sphere. All the Painlevé equations (in their standard Lax representations) are especially nice since their (trace-free) moduli spaces $\mathfrak{M}_{g,\mathbf{F}}$ have dimension one, so are *wild modular curves*, reflecting the fact they are ODEs not PDEs (their time variable ranges over a finite cover of this moduli space).

(4) Finally, we are interested in quantising the symplectic/Poisson local systems of wild character varieties $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B} \to \mathfrak{M}_{g,\mathbf{F}}$ (and the corresponding de Rham isomonodromy connections) to get linear representations of the wild mapping class groups (see [1, 10, 41, 57, 50, 51, 35] for some examples).

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