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On the Duality Formula for Parametrized Multiple Series

by

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Abstract

We show that a duality formula for certain parametrized multiple series yields numerous relations among them. As a result, we obtain a new relation among extended multiple zeta values, which is an extension of Ohno's relation for multiple zeta values. We perform the same study for multiple Hurwitz zeta values, and obtain a new identity for them.

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§1. Introduction

Fischler and Rivoal [5], Kawashima [21] and Ulanskiĭ [25] studied the following extension of the multiple zeta value (MZV for short):

$$(1) \qquad \sum_{\substack{0<m_1<_{c_1}\cdots<_{c_{p-1}}m_p<\infty \\ m_i\in\mathbb{Z}}} \frac{1}{m_1^{k_1}\cdots m_p^{k_p}},$$

where $1 \le p \in \mathbb{Z}$, $c_i \in \{0,1\}$, $1 \le k_i \in \mathbb{Z}$ $(i = 1,\dots,p-1)$, $2 \le k_p \in \mathbb{Z}$ and the symbols $<_{c_i}$ $(i = 1,\dots,p-1)$ denote $<$ if $c_i = 1$ and $\le$ if $c_i = 0$. For $c_i = 1$ $(i = 1,\dots,p-1)$, this multiple series becomes an MZV, which was studied in Euler [4], Hoffman [8] and Zagier [26]. The MZV has rich mathematical content. For example, the product of MZVs has two kinds of multiplication laws: one is induced from the series expression of an MZV and the other is an iterated integral expression, and they yield numerous relations over $\mathbb{Q}$. Furthermore, the set of MZVs generates a $\mathbb{Q}$-algebra with the multiplication laws. Understanding the algebra is one of the important problems in mathematics, because MZVs have

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connections with various mathematical objects, e.g., knot invariants, Feynman integrals, modular forms and mixed Tate motives over $\mathbb{Z}$. The extension (1) keeps these rich contents. Another simple case, $c_i = 0$ ($i = 1, \ldots, p-1$), becomes the multiple zeta-star value (MZSV for short). The MZSV is an interesting object itself. Indeed, it is almost the same as MZV in appearance, but aspects of its relations are fairly different: see, e.g., [18, 23, 24]. It can be seen that relations among MZSVs have brevity. All other cases of (1) are hybrids of both $<$ and $\leq$. (They can be expressed in $\mathbb{Z}$-linear combinations of MZVs or MZSVs.) These hybrid MZVs frequently appear in the study of MZVs and MZSVs. The most important point of the extension (1) is that it gives a unified expression of the three objects MZV, MZSV and the hybrid MZV, and this gives unified extensions of relations among MZVs and MZSVs: see the case $\alpha = 1$ of (7) below. Hereafter, we call the multiple series (1) the extended multiple zeta value (EMZV for short). Kawashima [21] introduced the EMZV to study a Newton series and relations among MZVs. He proved a duality relation among EMZVs [21, Prop. 5.3]. Ulanskiĭ [25] gave an algebraic formulation for the EMZV, and proved some basic properties of EMZVs: the Chen iterated integral representation, the duality formula, the shuffle and stuffle relations [25, Cor. 2, Thms 1, 2 and 3]. Fischler and Rivoal [5] used the EMZV and an extended multiple polylogarithm to study a Padé approximation problem involving multiple polylogarithms. They also proved a duality formula for EMZVs, and applied it to construction of $\mathbb{Q}$-linear forms in the Riemann zeta values (see [5, Sect. 2]). We note that multiple series of the extended form (1) naturally appear as derivatives of hypergeometric series (see [13]).

In the present paper, we also study multiple series of the extended form (1). Our interest is duality relations among them. In the first part of the paper, we study the multiple series

$$(2) \qquad \sum_{\substack{0\leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty \\ m_i \in \mathbb{Z}}} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_p!}{(\alpha)_{m_p}} \left\{ \prod_{i=1}^{p} \frac{1}{(m_i+\alpha)^{a_i}(m_i+\beta)^{b_i}} \right\},$$

where $1 \leq p \in \mathbb{Z}$, $a_i, b_i \in \mathbb{Z}$ such that $a_i + b_i \geq 1$ ($i = 1, \ldots, p-1$), $a_p + b_p \geq 2$, $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$, $\beta \notin \mathbb{Z}_{\leq 0} := \{0, -1, -2, \ldots\}$; $(a)_m$ denotes the Pochhammer symbol, i.e., $(a)_m = a(a+1)\cdots(a+m-1)$ ($1 \leq m \in \mathbb{Z}$) and $(a)_0 = 1$. The Pochhammer symbol can be expressed as a quotient of the gamma function: $(a)_m = \Gamma(a+m)/\Gamma(a)$. Therefore, Stirling's formula for $\Gamma(z)$ can be applied to the estimation of $(a)_m$. For the convergence of (2), see [12, Lem. 2.1]. The multiple series (2) is an extension of both the EMZV and our two-parameter multiple series studied in [10, 12]. The study of this kind of two-parameter extension of the MZV originated in [10] by the author. See also [12, 13] and [17, Note 2]. For other results, the author proved the cyclic sum formula for the case $c_i = 1$ $(i = 1, \dots, p-1)$ of (2) and for the case $c_i = 0$ $(i = 1, \dots, p-1)$: see [15] and also [17, Note 2(iv) and (v)].

§1.1. Definitions and notation

To describe our results concisely, we follow Ulanskiĭ's algebraic formulation for EMZVs [25], which is an extension of Hoffman's for MZVs [9]. The formulation is done by using three non-commutative variables, x_0, x_1 and x_{-1}. Hereafter, we assume that i, m, n, p, q, r, l_i, m_i, k_i, k'_i, s_i, r_i, $y_j^{(i)}$, $M_j^{(i)} \in \mathbb{Z}$. For brevity, we frequently use the notation $z_{c_{i-1}}(k_i) := x_{c_{i-1}} x_{-1}^{k_i-1}$ $(c_{i-1} \in \{0,1\},\ k_i \geq 1)$. For example, we write $x_1 x_{-1}^{k_1-1} x_{c_1} x_{-1}^{k_2-1} \cdots x_{c_{p-1}} x_{-1}^{k_p-1}$ as

$$z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p) = \prod_{i=1}^{p} z_{c_{i-1}}(k_i),$$

where $c_0 = 1$. Here we put

$$B^0 := \left\{ \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \;\middle|\; p \geq 0,\ c_0 = 1,\ c_i \in \{0,1\},\ k_i \geq 1 (i = 1, \dots, p-1),\ k_p \geq 2 \right\},$$

where $\prod_{i=1}^{0} z_{c_{i-1}}(k_i) = 1 \in \mathbb{Q}$, and denote by V^0 the $\mathbb{Q}$-vector space whose basis is B^0. (This vector space corresponds to Y^0 in [25].) We define the evaluation map $Z = Z_{(\alpha,\beta)} \colon B^0 \to \mathbb{C}$ by $Z(1; (\alpha,\beta)) = 1$ and

$$\begin{aligned} &Z(z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p); (\alpha,\beta)) \\ &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_p!}{(\alpha)_{m_p+1}} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i}} \right\} \frac{1}{(m_p+\beta)^{k_p-1}}, \end{aligned} \tag{3}$$

where $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. This map can be extended to a $\mathbb{Q}$-linear map onto the whole space V^0. To describe partial derivatives of (3), we use the evaluation map $Z^*_{(\{r_i\}_{i=1}^q)} = Z^*_{(\{r_i\}_{i=1}^q),(\beta,\alpha)} : B^0 \to \mathbb{C}$ defined by $Z^*_{(\{r_i\}_{i=1}^q)}(1; (\beta,\alpha)) = 1$ and

$$\begin{aligned} &Z^*_{(\{r_i\}_{i=1}^q)}(z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{q-1}}(k_q); (\beta,\alpha)) \\ &= \sum_{\substack{0 \leq m_1 <_{c_1} M_1^{(2)} \leq \cdots \leq M_{r_2}^{(2)} <_{1-c_2} m_2 \\ \vdots \\ m_{i-1} <_{c_{i-1}} M_1^{(i)} \leq \cdots \leq M_{r_i}^{(i)} <_{1-c_i} m_i \\ \vdots \\ m_{q-1} <_{c_{q-1}} M_1^{(q)} \leq \cdots \leq M_{r_q}^{(q)} <_{1-c_q} m_q < \infty}} \frac{(\beta)_{m_1}}{m_1!} \frac{m_q!(m_q+\alpha)}{(\beta)_{m_q+1}} \\ &\quad \times \frac{1}{(m_1+\beta)^{r_1}(m_1+\alpha)^{k_1}} \left\{ \prod_{i=2}^{q} \left(\prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\beta} \right) \frac{1}{(m_i+\alpha)^{k_i}} \right\}, \end{aligned} \tag{4}$$

where $q \geq 1$, $r_i \geq 0$ $(i = 1, \ldots, q)$, $c_q = 1$, $\alpha, \beta \in \mathbb{C}$ such that $\alpha \notin \mathbb{Z}_{\leq 0}$, $\mathrm{Re}(\beta) > 0$. This map can also be extended to a $\mathbb{Q}$-linear map onto the whole space V^0. If $r_i = 0$, we regard the inequalities $m_{i-1} <_{c_{i-1}} M_1^{(i)} \leq \cdots \leq M_{r_i}^{(i)} <_{1-c_i} m_i$ of (4) as $m_{i-1} <_{c_{i-1}} m_i$. For $r_i = 0$ $(i = 1, \ldots, q)$ and for $q = 1$, the multiple series (4) becomes

$$Z^*_{(\{0\}_{i=1}^q)}\biggl(\prod_{i=1}^{p} z_{c_{i-1}}(k_i); (\beta, \alpha)\biggr) = Z\biggl(\prod_{i=1}^{p} z_{c_{i-1}}(k_i); (\beta, \alpha)\biggr),$$

$$Z^*_{(r_1)}(z_1(k_1); (\beta, \alpha)) = \sum_{0 \leq m_1 < \infty} \frac{1}{(m_1+\beta)^{r_1+1}(m_1+\alpha)^{k_1-1}},$$

respectively; therefore the map $Z^*_{(\{r_i\}_{i=1}^q)}$ is an extension of the map Z with the additional parameters $\{r_i\}_{i=1}^q$. This gives an algebraic description of partial derivatives on α of (3): see the proof of Theorem 1.1(i). To describe another derivation aspect of our results, we need the maps $\sigma_r^{b,1}, \sigma_r \colon B^0 \to V^0$ defined by $\sigma_r^{b,1}(1) = \sigma_r(1) = 1$ and

$$\begin{aligned}
&\sigma_r^{b,1}(z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p)) \\
&\quad = \sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \biggl\{\prod_{i=1}^{p-1} \binom{k_i+r_i-1}{r_i}\biggr\} \binom{k_p+r_p-2}{r_p} \prod_{i=1}^{p} z_{c_{i-1}}(k_i+r_i),
\end{aligned}$$

$$\begin{aligned}
&\sigma_r(z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p)) \\
&\quad = \sum_{\substack{\sum_{i=1}^{p-1} c_i r_i + r_p = r \\ c_i r_i, r_p \geq 0}} \biggl\{\prod_{i=1}^{p-1} z_{c_{i-1}}(k_i + c_i r_i)\biggr\} z_{c_{p-1}}(k_p + r_p),
\end{aligned}$$

where $r \geq 0$. These are variations of the map σ_m used in [19, Sect. 6]. (For $c_i = 1$ $(i = 1, \ldots, p-1)$, the map σ_r becomes σ_m.) In the present paper, we use the following standard definition of the dual: Let τ be the map $\tau \colon B^0 \to B^0$ defined by $\tau(1) = 1$ and $\tau(x_1 x_{e_1} \cdots x_{e_{n-1}} x_{-1}) = x_1 x_{-e_{n-1}} \cdots x_{-e_1} x_{-1}$, where $n \geq 1$ and $e_i \in \{-1, 0, 1\}$ $(i = 1, \ldots, n-1)$. Then $\tau(v)$ is called the dual of v. It is obvious that $\tau^2(v) = v$. The maps $\sigma_r^{b,1}$, σ_r and τ can be extended to $\mathbb{Q}$-linear maps from the whole space V^0 to itself. Let $v \in B^0$. Then $\tau(v)$ can also be expressed as $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$, where $q \geq 0$, $c'_0 = 1$, $c'_i \in \{0, 1\}$, $k'_i \geq 1$ $(i = 1, \ldots, q-1)$, $k'_q \geq 2$. Hereafter, we assume this expression for $\tau(v)$. For any fixed real numbers a, b $(a < b \leq \infty)$, we regard the sum $\sum_{a <_{c_1} M_1 <_{c_2} \cdots <_{c_p} M_p <_{c_{p+1}} b} A_{M_1, \ldots, M_p}$ as 1 if $p = 0$.

§1.2. Main theorem

For $v = \prod_{i=1}^{p} z_1(k_i)$, we proved in [12] a large class of relations among the multiple series $Z(v; (\alpha, \alpha))$ by using the duality formula

$$(5) \qquad Z(v; (\alpha, \beta)) = Z(\tau(v); (\beta, \alpha)), \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$$

(see [12, Thm. 1.1 and Lem. 2.3]). The symmetry in α and β of (5) played an essential role for the proof (see [12, Sect. 2]). From this fact, we think that this kind of symmetry of parametrized multiple series is useful for the study of relations among multiple series (e.g., MZVs). In the present paper, we prove a large class of relations among (2) by using a duality formula for (3), which is an extension of (5) (see Lemma 2.2 below), that is, we prove the following.

Theorem 1.1. *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then*

(i) *we have*

$$(6) \qquad Z(\sigma_r^{b,1}(v); (\alpha, \beta)) = \sum_{\substack{c_1' r_1 + \sum_{i=2}^{q} r_i = r \\ c_1' r_1, r_i \geq 0}} Z^{*}_{(c_1' r_1, \{r_i\}_{i=2}^{q})}(\tau(v); (\beta, \alpha))$$

for all $r \geq 0$, $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, where c_1' and q are those of the dual $\tau(v) = \prod_{i=1}^{q} z_{c_{i-1}'}(k_i')$;

(ii) *we have*

$$(7) \qquad Z(\sigma_r(v); \alpha) = Z(\sigma_r \tau(v); \alpha)$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, where $Z(v; \alpha) := Z(v; (\alpha, \alpha))$ ($v \in B^0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$).

The identities (6) and (7) yield numerous relations among (2) and EMZVs. In fact, even the simple case $v = \prod_{i=1}^{p} z_1(k_i)$ of (7) becomes a large class of relations [12, Thm. 1.1]. We note that, for $v \in B^0$ and its dual $\tau(v)$, the identity (6) gives two different relations among (2): see Example 2.7(i) below. This is one of the features of our multi-parameter extension. The specializations $\alpha = \beta = 1$ of (6) and of (7) give a large class of relations among EMZVs. In particular, the case $\alpha = 1$ of (7) is a new extension of Ohno's relation for MZVs [22, Thm. 1], which extends Ohno's relation to a relation involving MZSVs and the hybrid MZVs. Besides this, the specialization $v = z_1(k_1)\{\prod_{i=2}^{p} z_0(k_i)\}$ and $\alpha = 1$ of (7) gives a relation between MZSVs and the hybrid MZVs: see (26) below. For a recent crucial result on (2), we note that Hirose, Murahara and Onozuka [7] determined all the linear relations among (2) with $\alpha = \beta$. In fact, they proved that the multiple series (2) with $\alpha = \beta$, $c_i = 1$ $(i = 1, \ldots, p-1)$ satisfies the same relation as the linear part of Kawashima's relation for MZVs [20, Cor. 5.5], and that any linear relation can be written as a linear combination of the linear part relation for (2) with $\alpha = \beta$, $c_i = 1$ (see [7, Thms 1–3]). It is interesting to study whether the linear part relation can be derived from another relation for (2) by using differentiation.

In the present paper, we shall also study a duality formula for multiple Hurwitz zeta values; see Section 3 for details. The result, Theorem 3.1 below, is another main theorem of the paper.

We explain our idea for the proofs of Theorems 1.1 and 3.1. It is to use symmetries in α and β of the multiple series (2), (27) and (28) below; see (9) and (32) below. These symmetries can be found by making a change of variables to an iterated integral representation of (2) and of (27): see the proof of (9) and of (32). We note that the change of variables also brings changes of the positions of the parameters of (2), (27) and (28): compare the positions of the parameters α and β on both sides of (9) and of (32). This plays an essential role for deriving various and numerous relations. Indeed, the changes of the positions allow us to show that partial differential operators act on each side of (9) and of (32) in two different ways, and this gives the relations in the theorems. (The above idea was used in our previous works [10, 12].) Another main tool for the proof is our method of computing the Pochhammer symbol $(a)_m$ used in [13], which was developed in its preprints distributed in 2013. This allows us to compute derivatives of $(a)_m$ without computing products of finite multiple harmonic sums: see, e.g., (10)–(12) below and compare them with our computation used in [11].

We shall prove Theorem 1.1 in Section 2. Our method of proof is similar to that in [10] and [12]. In Section 3 we shall apply our method to multiple Hurwitz zeta values as well, and shall obtain an identity for them similar to Theorem 1.1(i), which also yields numerous relations.

The present paper is a revised version of preprints of ours which were distributed in October 2015. See also a preprint [16, Note].

§2. Proof of Theorem 1.1

We first prove a duality formula for (3). We define the symbol $\omega_{e_i}(t)$ $(e_i \in \{-1, 0, 1\})$ by

$$\omega_{-1}(t) = \frac{1}{t}, \quad \omega_0(t) = \frac{1}{t(1-t)}, \quad \omega_1(t) = \frac{1}{1-t}.$$

Lemma 2.1. *Let $n \geq 1$ and $e_i \in \{-1, 0, 1\}$ $(i = 1, \ldots, n-1)$. Then the following iterated integral representation of* (3) *holds:*

$$(8)\quad Z(x_1 x_{e_1} \cdots x_{e_{n-1}} x_{-1}; (\alpha, \beta)) = \int \cdots \int_{0<t_0<\cdots<t_n<1} (1-t_0)^{1-\alpha} t_0^{\beta-1} \omega_1(t_0) \left\{ \prod_{i=1}^{n-1} \omega_{e_i}(t_i) \right\} \times \omega_{-1}(t_n) t_n^{1-\beta} (1-t_n)^{\alpha-1} \, \mathrm{d}t_0 \cdots \mathrm{d}t_n$$

for all $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$.

Proof. The proof is the same as that in [12, Proof of Lem. 2.2]. The integrand of the iterated integral of (8) can be rewritten as

$$\omega_1 \left\{ \prod_{i=1}^{n-1} \omega_{e_i} \right\} \omega_{-1} = \prod_{i=1}^{p} \omega_{c_{i-1}} \omega_{-1}^{k_i - 1},$$

where $p \geq 1$, $c_0 = 1$, $c_i \in \{0, 1\}$, $k_i \geq 1$ $(i = 1, \ldots, p-1)$, $k_p \geq 2$. Using this expression, we can rewrite the iterated integral of (8), which we denote by I, as

$$I = \int \cdots \int_{\substack{0<t_{11}<\cdots<t_{1k_1}< \\ \vdots \\ <t_{i1}<\cdots<t_{ik_i}< \\ \vdots \\ <t_{p1}<\cdots<t_{pk_p}<1}} (1-t_{11})^{1-\alpha} t_{11}^{\beta-1} \left\{ \prod_{i=1}^{p} \omega_{c_{i-1}}(t_{i1}) \left(\prod_{j=2}^{k_i} \omega_{-1}(t_{ij}) \right) \right\} \times t_{pk_p}^{1-\beta} (1-t_{pk_p})^{\alpha-1} \left(\prod_{i=1}^{p} \prod_{j=1}^{k_i} \mathrm{d}t_{ij} \right).$$

Further, applying the expansions

$$(1-t_{11})^{-\alpha} = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} t_{11}^m, \quad (1-t_{i1})^{-1} = \sum_{m=0}^{\infty} t_{i1}^m$$

$(i = 2, \ldots, p)$ to the integrand and integrating term by term, we also have the identities

$$I = \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \frac{(\alpha)_{m_1}}{m_1!} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i}} \right\} \frac{1}{(m_p+\beta)^{k_p-1}} \times \int_0^1 (1-t_{pk_p})^{\alpha-1} t_{pk_p}^{m_p} \, \mathrm{d}t_{pk_p}$$

$$= \sum_{0 \le m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \frac{(\alpha)_{m_1}}{m_1!} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i}} \right\} \frac{1}{(m_p+\beta)^{k_p-1}} \times \frac{\Gamma(\alpha)\Gamma(m_p+1)}{\Gamma(\alpha+m_p+1)}$$

$$= \sum_{0 \le m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_p!}{(\alpha)_{m_p+1}} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i}} \right\} \frac{1}{(m_p+\beta)^{k_p-1}}$$

$$= Z(x_1 x_{-1}^{k_1-1} x_{c_1} x_{-1}^{k_2-1} \cdots x_{c_{p-1}} x_{-1}^{k_p-1}; (\alpha,\beta))$$

for $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. The monomial $x_1 x_{-1}^{k_1-1} x_{c_1} x_{-1}^{k_2-1} \cdots x_{c_{p-1}} x_{-1}^{k_p-1}$ can be rewritten as $x_1 x_{e_1} \cdots x_{e_{n-1}} x_{-1}$ ($e_i \in \{-1, 0, 1\}$); thus we obtain (8). □

Using Lemma 2.1, we can prove the following duality formula for (3), which will play an essential role in the proof of Theorem 1.1.

Lemma 2.2 (Duality formula). *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then*

$$Z(v; (\alpha,\beta)) = Z(\tau(v); (\beta,\alpha)) \tag{9}$$

for all $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$.

Proof. Making the change of variables $t_i = 1 - u_{n-i}$ ($i = 0, 1, \ldots, n$; see [26, p. 510]) to the iterated integral on the right-hand side of (8), we obtain (9). □

Remark 2.3. Ulanskiĭ [25] proved the Chen iterated integral representation and the duality formula for EMZVs [25, Cor. 2 and Thm. 1]. The case $\alpha = \beta = 1$ of (8) and of (9) are Corollary 2 and Theorem 1 of [25], respectively. We also note that (8) and (9) are extensions of our previous results [12, Lems 2.2 and 2.3].

§2.1. Proof of Theorem 1.1(i)

Let $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$, and let $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$ be its dual. The proof is done by differentiating both sides of the duality formula (9) r times with respect to β. The left-hand side of (6) is an immediate result of differentiating that of (9). To obtain the right-hand side, we calculate the derivative $\frac{(-1)^r}{r!} \frac{\mathrm{d}^r}{\mathrm{d}\beta^r} \left(\frac{(\beta)_{m_1}}{(\beta)_{m_q+1}} \right)$ ($r \ge 0$). From the definition of c_i ($= 0, 1$), we have $(\beta)_{m_1+c'_1} = (\beta)_{m_1}(m_1+\beta)^{c'_1}$. Using this, we have the expression

$$\frac{(\beta)_{m_1}}{(\beta)_{m_q+1}} = \frac{1}{(m_1+\beta)^{c'_1}} \left(\prod_{i=2}^{q} \frac{(\beta)_{m_{i-1}+c'_{i-1}}}{(\beta)_{m_i+c'_i}} \right), \tag{10}$$

where $m_1, \ldots, m_q \in \mathbb{Z}$ such that $0 \le m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q$ and $c'_q = 1$. The derivatives of the factors on the right-hand side of (10) can be calculated as follows:

$$\begin{aligned}
&\frac{(-1)^{r_i}}{r_i!}\frac{\mathrm{d}^{r_i}}{\mathrm{d}\beta^{r_i}}\left(\frac{(\beta)_{m_{i-1}+c'_{i-1}}}{(\beta)_{m_i+c'_i}}\right) \\
&\quad = \frac{(\beta)_{m_{i-1}+c'_{i-1}}}{(\beta)_{m_i+c'_i}} \sum_{m_{i-1}+c'_{i-1} \le M_1^{(i)} \le \cdots \le M_{r_i}^{(i)} < m_i + c'_i} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\beta} \\
&\quad = \frac{(\beta)_{m_{i-1}+c'_{i-1}}}{(\beta)_{m_i+c'_i}} \sum_{m_{i-1} <_{c'_{i-1}} M_1^{(i)} \le \cdots \le M_{r_i}^{(i)} <_{1-c'_i} m_i} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\beta}
\end{aligned} \tag{11}$$

for $r_i \ge 0$ $(i = 2, \ldots, q)$. (From the definitions of the symbols c_i and $<_{c_i}$, the inequalities $m_{i-1} + c'_{i-1} \le M_1^{(i)}$ and $M_{r_i}^{(i)} < m_i + c'_i$ of (11) can be rewritten as $m_{i-1} <_{c'_{i-1}} M_1^{(i)}$ and $M_{r_i}^{(i)} <_{1-c'_i} m_i$, respectively.) Using (10) and (11), we have

$$\begin{aligned}
&\frac{(-1)^{r}}{r!}\frac{\mathrm{d}^{r}}{\mathrm{d}\beta^{r}}\left(\frac{(\beta)_{m_1}}{(\beta)_{m_q+1}}\right) \\
&\quad = \sum_{\substack{r_1+\cdots+r_q=r \\ r_i \ge 0}} \frac{\binom{r_1+c'_1-1}{r_1}}{(m_1+\beta)^{r_1+c'_1}} \\
&\qquad \times \left(\prod_{i=2}^{q} \frac{(\beta)_{m_{i-1}+c'_{i-1}}}{(\beta)_{m_i+c'_i}} \sum_{m_{i-1} <_{c'_{i-1}} M_1^{(i)} \le \cdots \le M_{r_i}^{(i)} <_{1-c'_i} m_i} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\beta} \right) \\
&\quad = \sum_{\substack{r_1+\cdots+r_q=r \\ r_i \ge 0}} \frac{\binom{r_1+c'_1-1}{r_1}}{(m_1+\beta)^{r_1}} \\
&\qquad \times \frac{(\beta)_{m_1}}{(\beta)_{m_q+1}} \prod_{i=2}^{q} \left(\sum_{m_{i-1} <_{c'_{i-1}} M_1^{(i)} \le \cdots \le M_{r_i}^{(i)} <_{1-c'_i} m_i} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\beta} \right) \\
&\quad = \sum_{\substack{c'_1 r_1+\sum_{i=2}^{q} r_i=r \\ c'_1 r_1, r_i \ge 0}} \frac{1}{(m_1+\beta)^{c'_1 r_1}} \\
&\qquad \times \frac{(\beta)_{m_1}}{(\beta)_{m_q+1}} \prod_{i=2}^{q} \left(\sum_{m_{i-1} <_{c'_{i-1}} M_1^{(i)} \le \cdots \le M_{r_i}^{(i)} <_{1-c'_i} m_i} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\beta} \right)
\end{aligned} \tag{12}$$

for $r \geq 0$: the last identity of (12) follows from the identity

$$\binom{r_i + c'_i - 1}{r_i} = \begin{cases} 1 & \text{if } c'_i = r_i = 0 \text{ or } c'_i = 1,\ r_i \geq 0, \\ 0 & \text{if } c'_i = 0,\ r_i \geq 1. \end{cases} \tag{13}$$

Therefore, using (12), we have

$$\begin{aligned} &\frac{(-1)^r}{r!}\frac{\partial^r}{\partial\beta^r}\left(\frac{(\beta)_{m_1}}{m_1!}\frac{m_q!}{(\beta)_{m_q+1}}\left\{\prod_{i=1}^{q-1}\frac{1}{(m_i+\alpha)^{k'_i}}\right\}\frac{1}{(m_q+\alpha)^{k'_q-1}}\right) \\ &= \sum_{\substack{c'_1 r_1 + \sum_{i=2}^{q} r_i = r \\ c'_1 r_1, r_i \geq 0}} \sum_{\substack{m_1 <_{c'_1} M_1^{(2)} \leq \cdots \leq M_{r_2}^{(2)} <_{1-c'_2} m_2 \\ \vdots \\ m_{q-1} <_{c'_{q-1}} M_1^{(q)} \leq \cdots \leq M_{r_q}^{(q)} <_{1-c'_q} m_q}} \frac{(\beta)_{m_1}}{m_1!}\frac{m_q!(m_q+\alpha)}{(\beta)_{m_q+1}} \\ &\quad \times \frac{1}{(m_1+\beta)^{c'_1 r_1}(m_1+\alpha)^{k'_1}}\left\{\prod_{i=2}^{q}\left(\prod_{j=1}^{r_i}\frac{1}{M_j^{(i)}+\beta}\right)\frac{1}{(m_i+\alpha)^{k'_i}}\right\} \end{aligned} \tag{14}$$

for $r \geq 0$, $m_1, \ldots, m_q \in \mathbb{Z}$ such that $0 \leq m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q$ and $c'_q = 1$. Differentiating the right-hand side of (9) r times with respect to β and using (14), we obtain the right-hand side of (6). This completes the proof of Theorem 1.1(i).

§2.2. Proof of Theorem 1.1(ii) and a related theorem

For brevity, we put $\mathbf{y}_m^{(i)} := \sum_{j=1}^{m} y_j^{(i)}$ $(i, m \geq 1,\ y_j^{(i)} \geq 0)$: we regard $\mathbf{y}_0^{(i)}$ as 0. For any $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$ and its dual $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$, we define the following two monomials:

$$v_{\mathbf{y}} := \left\{\prod_{i=1}^{p-1} z_{c_{i-1}}(k_i + \mathbf{y}_{k_i-c_i}^{(i)})\right\} z_{c_{p-1}}(k_p + \mathbf{y}_{k_p-2}^{(p)}), \tag{15}$$

$$\begin{aligned} v'_{(\{l_i\}_{i=1}^{q-1})} &:= x_1 x_{-1}^{k'_1-1}\left\{\prod_{i=2}^{q} x_{c'_{i-1}} x_1^{l_{i-1}} x_{-1}^{k'_i-1}\right\} \\ &= z_1(k'_1)\left\{\prod_{i=2}^{q} z_{c'_{i-1}}(1) z_1(1)^{l_{i-1}-1} z_1(k'_i)\right\}, \end{aligned} \tag{16}$$

where $\mathbf{y} := (\{\mathbf{y}^{(i)}_{k_i-c_i}\}_{i=1}^{p-1}, \mathbf{y}^{(p)}_{k_p-2})$ and $l_i \geq 0$ $(i = 1, \ldots, q-1)$. If $l_{i-1} = 0$ in (16), the factor becomes

$$x_{c'_{i-1}} x_1^{l_{i-1}} x_{-1}^{k'_i-1} = x_{c'_{i-1}} x_{-1}^{k'_i-1} = z_{c'_{i-1}}(k'_i);$$

therefore we regard $z_{c'_{i-1}}(1) z_1(1)^{-1} z_1(k'_i)$ as $z_{c'_{i-1}}(k'_i)$.

Theorem 1.1(ii) can be proved in the same way as in [12, Sect. 2]. We first prove the following identity.

Proposition 2.4. *Let $v \in B^0$. Then*

$$\sum_{l=0}^{r} \sum_{\substack{\sum_{i=1}^{p-1} \mathbf{y}^{(i)}_{k_i-c_i} + \mathbf{y}^{(p)}_{k_p-2} = l \\ y_j^{(i)} \geq 0}} Z(\sigma_{r-l}(v_{\mathbf{y}}); \alpha)$$

$$= \sum_{l=0}^{r} \sum_{\substack{l_1+\cdots+l_{q-1}=l \\ l_i \geq 0}} Z(\sigma_{r-l}(v'_{(\{l_i\}_{i=1}^{q-1})}); \alpha) \tag{17}$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$.

Proof. The proof is similar to that of [12, Lem. 2.5]. Let $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$, and let $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$ be its dual. Here we note that the identities

$$\frac{(-1)^r}{r!} \frac{\mathrm{d}^r}{\mathrm{d}\beta^r} \left(\left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i}} \right\} \frac{1}{(m_p+\beta)^{k_p-1}} \right)$$

$$= \sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \left(\prod_{i=1}^{p-1} \frac{\binom{k_i+r_i-1}{r_i}}{(m_i+\beta)^{k_i+r_i}} \right) \frac{\binom{k_p+r_p-2}{r_p}}{(m_p+\beta)^{k_p+r_p-1}}$$

$$= \sum_{l=0}^{r} \sum_{\substack{\sum_{i=1}^{p-1} \mathbf{y}^{(i)}_{k_i-c_i} \\ + \mathbf{y}^{(p)}_{k_p-2} = l \\ y_j^{(i)} \geq 0}} \sum_{\substack{\sum_{i=1}^{p-1} c_i r_i \\ + r_p = r-l \\ c_i r_i, r_p \geq 0}} \left(\prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i + \mathbf{y}^{(i)}_{k_i-c_i} + c_i r_i}} \right)$$

$$\times \frac{1}{(m_p+\beta)^{k_p + \mathbf{y}^{(p)}_{k_p-2} + r_p - 1}} \tag{18}$$

hold for $r \geq 0$ and $m_1, \ldots, m_p \in \mathbb{Z}$ such that $0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p$. Therefore, using (18), we have

$$\frac{(-1)^r}{r!} \frac{\partial^r}{\partial \beta^r} \left(\frac{(\alpha)_{m_1}}{m_1!} \frac{m_p!}{(\alpha)_{m_p+1}} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+\beta)^{k_i}} \right\} \frac{1}{(m_p+\beta)^{k_p-1}} \right) \Bigg|_{\beta=\alpha}$$

$$= \sum_{l=0}^{r} \sum_{\substack{\sum_{i=1}^{p-1} \mathbf{y}^{(i)}_{k_i-c_i} \\ +\mathbf{y}^{(p)}_{k_p-2}=l \\ y^{(i)}_j \geq 0}} \sum_{\substack{\sum_{i=1}^{p-1} c_i r_i \\ +r_p=r-l \\ c_i r_i, r_p \geq 0}} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_p!}{(\alpha)_{m_p}} \left(\prod_{i=1}^{p-1} \frac{1}{(m_i+\alpha)^{k_i+\mathbf{y}^{(i)}_{k_i-c_i}+c_i r_i}} \right)$$

$$\times \frac{1}{(m_p+\alpha)^{k_p+\mathbf{y}^{(p)}_{k_p-2}+r_p}} \tag{19}$$

for $r \geq 0$ and $m_1, \ldots, m_p \in \mathbb{Z}$ such that $0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p$. Differentiating the left-hand side of (9) r times with respect to β at $\beta = \alpha$ ($\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$) and using (19), we obtain the left-hand side of (17). To obtain the right-hand side of (17), we use the same computation as in the proof of Theorem 1.1(i). From the definition of c_i $(= 0, 1)$, we have $(\beta)_{m_i+c'_i} = (\beta)_{m_i}(m_i+\beta)^{c'_i}$. Using this, we have the expression

$$\frac{(\beta)_{m_1}}{(\beta)_{m_q+1}} = \left(\prod_{i=1}^{q-1} \frac{(\beta)_{m_i+c'_i}}{(\beta)_{m_{i+1}}} \right) \left(\prod_{i=1}^{q-1} \frac{1}{(m_i+\beta)^{c'_i}} \right) \frac{1}{m_q+\beta}, \tag{20}$$

where $m_1, \ldots, m_q \in \mathbb{Z}$ such that $0 \leq m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q$. The derivatives of the factors on the right-hand side of (20) can be calculated as follows:

$$\frac{(-1)^{s_i}}{s_i!} \frac{\mathrm{d}^{s_i}}{\mathrm{d}\beta^{s_i}} \left(\frac{(\beta)_{m_i+c'_i}}{(\beta)_{m_{i+1}}} \right)$$

$$= \frac{(\beta)_{m_i+c'_i}}{(\beta)_{m_{i+1}}} \sum_{m_i+c'_i \leq M^{(i)}_1 \leq \cdots \leq M^{(i)}_{s_i} < m_{i+1}} \prod_{j=1}^{s_i} \frac{1}{M^{(i)}_j+\beta}$$

$$= \frac{(\beta)_{m_i+c'_i}}{(\beta)_{m_{i+1}}} \sum_{m_i <_{c'_i} M^{(i)}_1 \leq \cdots \leq M^{(i)}_{s_i} < m_{i+1}} \prod_{j=1}^{s_i} \frac{1}{M^{(i)}_j+\beta}$$

$$= \frac{(\beta)_{m_i+c'_i}}{(\beta)_{m_{i+1}}} \sum_{l_i=0}^{s_i} \sum_{\substack{\sum_{j=1}^{l_i} y^{(i)}_j = s_i-l_i \\ y^{(i)}_j \geq 0}} \sum_{m_i <_{c'_i} M^{(i)}_1 < \cdots < M^{(i)}_{l_i} < m_{i+1}} \prod_{j=1}^{l_i} \frac{1}{(M^{(i)}_j+\beta)^{y^{(i)}_j+1}} \tag{21}$$

for $s_i, m_i, m_{i+1} \geq 0$ such that $m_i <_{c'_i} m_{i+1}$ $(i = 1, \ldots, q-1)$, where we regard

$$\sum_{\substack{0=s_i\\ y_j^{(i)}\geq 0}} = \begin{cases} 1 & \text{if } s_i = 0,\\ 0 & \text{otherwise.}\end{cases}$$

(From the definitions of the symbols c_i and $<_{c_i}$, the inequality $m_i + c'_i \leq M_1^{(i)}$ of (21) can be rewritten as $m_i <_{c'_i} M_1^{(i)}$.) Using (13), (20) and (21), we have

$$\frac{(-1)^r}{r!}\frac{\mathrm{d}^r}{\mathrm{d}\beta^r}\left(\frac{(\beta)_{m_1}}{(\beta)_{m_q+1}}\right)$$

$$= \sum_{\substack{\sum_{i=1}^{q} r_i + \sum_{i=1}^{q-1} s_i = r\\ r_i, s_i \geq 0}} \left(\prod_{i=1}^{q-1} \frac{\binom{r_i + c'_i - 1}{r_i}}{(m_i+\beta)^{r_i + c'_i}}\right) \frac{1}{(m_q+\beta)^{r_q+1}}$$

$$\times \prod_{i=1}^{q-1}\left(\frac{(\beta)_{m_i+c'_i}}{(\beta)_{m_{i+1}}} \sum_{l_i=0}^{s_i} \sum_{\substack{\sum_{j=1}^{l_i} y_j^{(i)} = s_i - l_i\\ y_j^{(i)}\geq 0}} \sum_{m_i <_{c'_i} M_1^{(i)} < \cdots < M_{l_i}^{(i)} < m_{i+1}} \prod_{j=1}^{l_i} \frac{1}{(M_j^{(i)}+\beta)^{y_j^{(i)}+1}}\right)$$

$$= \sum_{\substack{\sum_{i=1}^{q-1} c'_i r_i + r_q\\ + \sum_{i=1}^{q-1} s_i = r\\ c'_i r_i, r_q, s_i \geq 0}} \frac{(\beta)_{m_1}}{(\beta)_{m_q}} \left(\prod_{i=1}^{q-1} \frac{1}{(m_i+\beta)^{c'_i r_i}}\right) \frac{1}{(m_q+\beta)^{r_q+1}}$$

$$\times \prod_{i=1}^{q-1}\left(\sum_{l_i=0}^{s_i} \sum_{\substack{\sum_{j=1}^{l_i} y_j^{(i)} = s_i - l_i\\ y_j^{(i)}\geq 0}} \sum_{m_i <_{c'_i} M_1^{(i)} < \cdots < M_{l_i}^{(i)} < m_{i+1}} \prod_{j=1}^{l_i} \frac{1}{(M_j^{(i)}+\beta)^{y_j^{(i)}+1}}\right)$$

$$= \sum_{l=0}^{r} \sum_{\substack{l_1+\cdots+l_{q-1}=l\\ l_i\geq 0}} \sum_{\substack{\sum_{i=1}^{q-1} c'_i r_i + r_q\\ + \sum_{i=1}^{q-1}\sum_{j=1}^{l_i} y_j^{(i)} = r-l\\ c'_i r_i, r_q, y_j^{(i)} \geq 0}} \frac{(\beta)_{m_1}}{(\beta)_{m_q}} \left(\prod_{i=1}^{q-1} \frac{1}{(m_i+\beta)^{c'_i r_i}}\right) \frac{1}{(m_q+\beta)^{r_q+1}}$$

$$\times \prod_{i=1}^{q-1}\left(\sum_{m_i <_{c'_i} M_1^{(i)} < \cdots < M_{l_i}^{(i)} < m_{i+1}} \prod_{j=1}^{l_i} \frac{1}{(M_j^{(i)}+\beta)^{y_j^{(i)}+1}}\right)$$

for $r \geq 0$. Therefore, using this result, we have

$$\frac{(-1)^r}{r!}\frac{\partial^r}{\partial\beta^r}\left(\frac{(\beta)_{m_1}}{m_1!}\frac{m_q!}{(\beta)_{m_q+1}}\left\{\prod_{i=1}^{q-1}\frac{1}{(m_i+\alpha)^{k'_i}}\right\}\frac{1}{(m_q+\alpha)^{k'_q-1}}\right)\Bigg|_{\beta=\alpha}$$

$$= \sum_{l=0}^{r}\sum_{\substack{l_1+\cdots+l_{q-1}=l\\ l_i\geq 0}}\sum_{\substack{\sum_{i=1}^{q-1}c'_ir_i+r_q\\ +\sum_{i=1}^{q-1}\sum_{j=1}^{l_i}y_j^{(i)}=r-l\\ c'_ir_i,r_q,y_j^{(i)}\geq 0}}\sum_{\substack{m_1<_{c'_1}M_1^{(1)}<\cdots<M_{l_1}^{(1)}<m_2\\ \vdots\\ m_{q-1}<_{c'_{q-1}}M_1^{(q-1)}<\cdots<M_{l_{q-1}}^{(q-1)}<m_q}}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_q!}{(\alpha)_{m_q}}$$

$$(22)\qquad \times\left\{\prod_{i=1}^{q-1}\frac{1}{(m_i+\alpha)^{k'_i+c'_ir_i}}\left(\prod_{j=1}^{l_i}\frac{1}{(M_j^{(i)}+\alpha)^{y_j^{(i)}+1}}\right)\right\}\frac{1}{(m_q+\alpha)^{k'_q+r_q}}$$

for $r \geq 0$ and $m_1, \ldots, m_q \in \mathbb{Z}$ such that $0 \leq m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q$. Differentiating the right-hand side of (9) r times with respect to β at $\beta = \alpha$ ($\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$) and using (22), we obtain the right-hand side of (17) This completes the proof of (17). □

Proposition 2.5. *Theorem 1.1*(ii) *and Proposition 2.4 are equivalent.*

Proof. Let $v \in B^0$, and let $v_{\mathbf{y}}$ be the monomial defined by (15). From the definition of the dual map τ, it can be seen that the dual $\tau(v_{\mathbf{y}})$ takes the form

$$(23)\qquad x_1x_{-1}^{k'_1-1}\left\{\prod_{i=2}^{q}x_{c'_{i-1}}x_1^{l_{i-1}}x_{-1}^{k'_i-1}\right\} = z_1(k'_1)\left\{\prod_{i=2}^{q}z_{c'_{i-1}}(1)z_1(1)^{l_{i-1}-1}z_1(k'_i)\right\}$$

with

$$l_1+\cdots+l_{q-1} = \sum_{i=1}^{p-1}\mathbf{y}_{k_i-c_i}^{(i)} + \mathbf{y}_{k_p-2}^{(p)},$$

where $l_i \geq 0$ ($i = 1, \ldots, q-1$) and the parameters q, c'_{i-1} and k'_i are those of $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$. The monomial (23) is just $v'_{(\{l_i\}_{i=1}^{q-1})}$, the monomial defined by (16). The equivalence can be proved by using these facts on monomials and the same argument as in [12, Proof of Prop. 2.7]. □

We can also prove the following theorem, which shows that the same equivalence as in [12, Thm. 1.2] holds for the extended multiple series.

Theorem 2.6. *The following assertion* (A) *is equivalent to Theorem 1.1*(ii)*:*

(A) *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the identity (7) holds for all "even" integers $r \geq 0$ and $\alpha \in \mathbb{C}$ with* $\mathrm{Re}(\alpha) > 0$.

Proof. This can be proved in the same way as in [12, Sect. 3]. □

Applying Theorem 2.6 to the case $v = \prod_{i=1}^{p} z_1(k_i)$, we have Theorem 1.2 of [12].

Example 2.7. We give some examples of Theorem 1.1:

(i) We put $v_0 := z_1(k_1)\{\prod_{i=2}^{p} z_0(k_i)\}$ $(p, k_i \geq 1$ $(i = 1, \dots, p-1)$, $k_p \geq 2)$. Then we have

$$\tau(v_0) = \Big\{\prod_{i=p}^{2} x_1^{k_i-1}x_0\Big\}x_1^{k_1-1}x_{-1} = \Big\{\prod_{i=1}^{q-1} z_{c'_{i-1}}(1)\Big\}z_{c'_{q-1}}(2),$$

where $q \geq 1$, $c'_0 = 1$, $c'_i \in \{0,1\}$ $(i = 1, \dots, q-1)$. Because $\tau^2(v_0) = v_0$, the dual of $\tau(v_0)$ becomes v_0. For these monomials, the identity (6) gives the following two different relations among (2), which come from the symmetry in α and β of (9) (compare (14) with (19)): by taking $v = v_0$ in (6),

$$\begin{aligned}\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\geq 0}} &\Big\{\prod_{i=1}^{p-1}\binom{k_i+r_i-1}{r_i}\Big\}\binom{k_p+r_p-2}{r_p}\\ &\times Z\Big(z_1(k_1+r_1)\Big\{\prod_{i=2}^{p} z_0(k_i+r_i)\Big\};(\alpha,\beta)\Big)\\ &= \sum_{\substack{c'_1r_1+\sum_{i=2}^{q} r_i=r\\ c'_1r_1, r_i\geq 0}} Z^*_{(c'_1r_1,\{r_i\}_{i=2}^{q})}\Big(\Big\{\prod_{i=1}^{q-1} z_{c'_{i-1}}(1)\Big\}z_{c'_{q-1}}(2);(\beta,\alpha)\Big)\end{aligned} \tag{24}$$

and, by taking $v = \tau(v_0)$ in (6),

$$\begin{aligned}\sum_{\substack{r_1+\cdots+r_q=r\\ r_i\geq 0}} &Z\Big(\Big\{\prod_{i=1}^{q-1} z_{c'_{i-1}}(1+r_i)\Big\}z_{c'_{q-1}}(2+r_q);(\alpha,\beta)\Big)\\ &= \sum_{\substack{\varepsilon(p)r_1+\sum_{i=2}^{p} r_i=r\\ \varepsilon(p)r_1, r_i\geq 0}} Z^*_{(\varepsilon(p)r_1,\{r_i\}_{i=2}^{p})}\Big(z_1(k_1)\Big\{\prod_{i=2}^{p} z_0(k_i)\Big\};(\beta,\alpha)\Big)\end{aligned} \tag{25}$$

for all $r \geq 0$, $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, where $\varepsilon(p) = 1$ if $p = 1$ and $\varepsilon(p) = 0$ otherwise. If $p = 1$, the right-hand side of (25) becomes $Z^*_{(r)}(z_1(k_1);(\beta,\alpha)) = \sum_{m=0}^{\infty}(m+\beta)^{-r-1}(m+\alpha)^{-k_1+1}$; therefore, the case $p = 1$ of (25) is the sum formula proved in [10] (see also [12, Rem. 2.4] and [13, (**R2**)]). We note that, for the monomials v_0 and $\tau(v_0)$, the identity (7) gives only the following relation among (2) with $\alpha=\beta$:

$$Z\left(z_1(k_1)\left\{\prod_{i=2}^{p-1} z_0(k_i)\right\}z_0(k_p+r);\alpha\right)$$

$$= \sum_{\substack{\sum_{i=1}^{q-1} c_i' r_i + r_q = r \\ c_i' r_i, r_q \geq 0}} Z\left(\left\{\prod_{i=1}^{q-1} z_{c_{i-1}'}(1+c_i' r_i)\right\} z_{c_{q-1}'}(2+r_q);\alpha\right) \tag{26}$$

for all $r\geq 0$, $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$.

(ii) Since the dual of $\prod_{i=1}^{p} z_1(k_i)$ is $\prod_{i=1}^{q} z_1(k_i')$, the identity (6) with $v=\prod_{i=1}^{p} z_1(k_i)$ becomes

$$\sum_{\substack{r_1+\cdots+r_p=r \\ r_i\geq 0}} \left\{\prod_{i=1}^{p-1}\binom{k_i+r_i-1}{r_i}\right\}\binom{k_p+r_p-2}{r_p} Z\left(\prod_{i=1}^{p} z_1(k_i+r_i);(\alpha,\beta)\right)$$

$$= \sum_{\substack{r_1+\cdots+r_q=r \\ r_i\geq 0}} Z^{*}_{(\{r_i\}_{i=1}^{q})}\left(\prod_{i=1}^{q} z_1(k_i'+r_i);(\beta,\alpha)\right)$$

for all $r\geq 0$, $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$.

§3. Duality of multiple Hurwitz zeta values

In the present section, we apply our method used in Section 2 to deriving duality relations for multiple Hurwitz zeta values. Our result in the present section is also formulated in the same way as in the introduction. We define the evaluation map $\zeta=\zeta_\alpha\colon B^0\to\mathbb{C}$ by $\zeta(1;\alpha)=1$ and

$$\zeta(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{p-1}}(k_p);\alpha) = \sum_{0\leq m_1<_{c_1}\cdots<_{c_{p-1}} m_p<\infty} \prod_{i=1}^{p} \frac{1}{(m_i+\alpha)^{k_i}}, \tag{27}$$

where $p\geq 1$ and $\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. This map can be extended to $\mathbb{Q}$-linear maps onto the whole space V^0. We call the multiple series (27) the multiple Hurwitz zeta value (MHZV for short). In [10] and [13], we studied relations for MHZVs in some different ways. Our results were described as relations between MHZVs and the multiples series

$$\sum_{0\leq m_1<_{c_1}\cdots<_{c_{p-1}} m_p<\infty} z^{m_p}\frac{m_p!}{(\alpha)_{m_p}}\left\{\prod_{i=1}^{p}\frac{1}{(m_i+\alpha)^{a_i}(m_i+1)^{b_i}}\right\}, \tag{28}$$

where $z\in\{-1,1\}$, $p\geq 1$, $a_i,b_i\in\mathbb{Z}$ such that $a_i+b_i\geq 1$ $(i=1,\ldots,p-1)$, $a_p+b_p\geq 2$, $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$, $(a)_m$ is the Pochhammer symbol. (For related works, see Remark 3.3 below.) Our result in the present section, a duality of MHZVs, is also described as such a relation. To formulate it, we introduce the evaluation map $H^*_{(\{r_i\}_{i=1}^q)} = H^*_{(\{r_i\}_{i=1}^q),\alpha}\colon B^0 \to \mathbb{C}$ defined by $H^*_{(\{r_i\}_{i=1}^q)}(1;\alpha) = 1$ and

$$\begin{aligned} &H^*_{(\{r_i\}_{i=1}^q)}(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);\alpha) \\ &= \sum_{\substack{0\le M_1^{(1)}\le\cdots\le M_{r_1}^{(1)}<_{1-c_1}m_1 \\ \vdots \\ m_{i-1}<_{c_{i-1}}M_1^{(i)}\le\cdots\le M_{r_i}^{(i)}<_{1-c_i}m_i \\ \vdots \\ m_{q-1}<_{c_{q-1}}M_1^{(q)}\le\cdots\le M_{r_q}^{(q)}<_{1-c_q}m_q<\infty}} \frac{(m_q+1)!}{(\alpha)_{m_q+1}}\left\{\prod_{i=1}^{q}\left(\prod_{j=1}^{r_i}\frac{1}{M_j^{(i)}+\alpha}\right)\frac{1}{(m_i+1)^{k_i}}\right\}, \end{aligned} \tag{29}$$

where $q \ge 1$, $r_i \ge 0$ $(i = 1,\dots,q)$, $c_q = 1$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. If $r_i = 0$, we regard the inequalities $m_{i-1} <_{c_{i-1}} M_1^{(i)} \le \cdots \le M_{r_i}^{(i)} <_{1-c_i} m_i$ of (29) as $m_{i-1} <_{c_{i-1}} m_i$. This map can be extended to $\mathbb{Q}$-linear maps onto the whole space V^0. We also use the map $\sigma_r^{b,2}\colon B^0 \to V^0$ defined by $\sigma_r^{b,2}(1) = 1$ and

$$\begin{aligned} &\sigma_r^{b,2}(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{p-1}}(k_p)) \\ &\quad= \sum_{\substack{r_1+\cdots+r_p=r \\ r_i\ge 0}} \left\{\prod_{i=1}^{p}\binom{k_i+r_i-1}{r_i}\right\}\prod_{i=1}^{p} z_{c_{i-1}}(k_i+r_i), \end{aligned}$$

where $r \ge 0$. This can be extended to a $\mathbb{Q}$-linear map from the whole space V^0 to itself. Using the same method as in Section 2, we can prove the following new relation between MHZVs and (28) with $z = 1$, which yields numerous relations.

Theorem 3.1. *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then*

$$\zeta(\sigma_r^{b,2}(v);\alpha) = \sum_{\substack{r_1+\cdots+r_q=r \\ r_i\ge 0}} H^*_{(\{r_i\}_{i=1}^q)}(\tau(v);\alpha) \tag{30}$$

for all $r \ge 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$.

Proof. Let $v = \prod_{i=1}^p z_{c_{i-1}}(k_i) \in B^0$, and let $\tau(v) = \prod_{i=1}^q z_{c'_{i-1}}(k'_i)$ be its dual. In the same way as in the proof of Lemma 2.1 with $\alpha = 1$, we have the following iterated integral representation of (27):

$$\begin{aligned} &\zeta(x_1x_{e_1}\cdots x_{e_{n-1}}x_{-1};\alpha) \\ &\quad= \int\cdots\int_{0<t_0<\cdots<t_n<1} t_0^{\alpha-1}\omega_1(t_0)\left\{\prod_{i=1}^{n-1}\omega_{e_i}(t_i)\right\}\omega_{-1}(t_n)\,\mathrm{d}t_0\cdots\mathrm{d}t_n \end{aligned} \tag{31}$$

for $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, where $n \geq 1$ and $e_i \in \{-1, 0, 1\}$ $(i = 1, \ldots, n-1)$. Further, making the change of variables $t_i = 1 - u_{n-i}$ $(i = 0, 1, \ldots, n)$ to the above iterated integral, we have the duality formula

$$\begin{aligned}\zeta(v;\alpha) &= \sum_{0\leq m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q < \infty} \frac{(m_q+1)!}{(\alpha)_{m_q+1}} \left\{ \prod_{i=1}^{q} \frac{1}{(m_i+1)^{k'_i}} \right\} \\ &= H^*_{(\{0\}_{i=1}^q)}(\tau(v);\alpha) \qquad (32)\end{aligned}$$

for $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. The left-hand side of (30) can be obtained by differentiating that of (32) r times. The right-hand side of (30) can also be obtained in the same way as in the proof of Theorem 1.1(i). Indeed, dividing both sides of (10) by $(\beta)_{m_1}$, we have the expression

$$\frac{1}{(\alpha)_{m_q+1}} = \frac{1}{(\alpha)_{m_1+c'_1}} \left(\prod_{i=2}^{q} \frac{(\alpha)_{m_{i-1}+c'_{i-1}}}{(\alpha)_{m_i+c'_i}} \right),$$

where $m_1, \ldots, m_q \in \mathbb{Z}$ such that $0 \leq m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q$ and $c'_q = 1$. Using this and a computation similar to that in the proof of (12), we have

$$\begin{aligned}&\frac{(-1)^r}{r!} \frac{\mathrm{d}^r}{\mathrm{d}\alpha^r} \left(\frac{(m_q+1)!}{(\alpha)_{m_q+1}} \left\{ \prod_{i=1}^{q} \frac{1}{(m_i+1)^{k'_i}} \right\} \right) \\ &= \sum_{\substack{r_1+\cdots+r_q=r \\ r_i\geq 0}} \left(\frac{(m_q+1)!}{(\alpha)_{m_1+c'_1}} \sum_{0\leq M_1^{(1)}\leq\cdots\leq M_{r_1}^{(1)} < m_1+c'_1} \prod_{j=1}^{r_1} \frac{1}{M_j^{(1)}+\alpha} \right) \\ &\quad\times \left(\prod_{i=2}^{q} \frac{(\alpha)_{m_{i-1}+c'_{i-1}}}{(\alpha)_{m_i+c'_i}} \sum_{\substack{m_{i-1}+c'_{i-1}\leq M_1^{(i)}\leq\cdots \\ \leq M_{r_i}^{(i)} < m_i+c'_i}} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\alpha} \right) \left\{ \prod_{i=1}^{q} \frac{1}{(m_i+1)^{k'_i}} \right\} \\ &= \sum_{\substack{r_1+\cdots+r_q=r \\ r_i\geq 0}} \frac{(m_q+1)!}{(\alpha)_{m_q+1}} \left(\sum_{0\leq M_1^{(1)}\leq\cdots\leq M_{r_1}^{(1)} <_{1-c'_1} m_1} \prod_{j=1}^{r_1} \frac{1}{M_j^{(1)}+\alpha} \right) \\ &\quad\times \left\{ \prod_{i=2}^{q} \left(\sum_{\substack{m_{i-1} <_{c'_{i-1}} M_1^{(i)}\leq\cdots \\ \leq M_{r_i}^{(i)} <_{1-c'_i} m_i}} \prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\alpha} \right) \right\} \left\{ \prod_{i=1}^{q} \frac{1}{(m_i+1)^{k'_i}} \right\}\end{aligned}$$

$$(33)\qquad = \sum_{\substack{r_1+\cdots+r_q=r\\ r_i\geq 0}} \sum_{\substack{0\leq M_1^{(1)}\leq\cdots\leq M_{r_1}^{(1)} <_{1-c'_1} m_1\\ \vdots \\ m_{i-1}<_{c'_{i-1}} M_1^{(i)}\leq\cdots\leq M_{r_i}^{(i)} <_{1-c'_i} m_i\\ \vdots \\ m_{q-1}<_{c'_{q-1}} M_1^{(q)}\leq\cdots\leq M_{r_q}^{(q)} <_{1-c'_q} m_q}} \frac{(m_q+1)!}{(\alpha)_{m_q+1}} \left\{ \prod_{i=1}^{q} \left(\prod_{j=1}^{r_i} \frac{1}{M_j^{(i)}+\alpha} \right) \frac{1}{(m_i+1)^{k'_i}} \right\}$$

for $r \geq 0$, $m_1, \ldots, m_q \in \mathbb{Z}$ such that $0 \leq m_1 <_{c'_1} \cdots <_{c'_q-1} m_q$ and $c'_q = 1$. Therefore, differentiating the right-hand side of (32) r times and using (33), we obtain the right-hand side of (30). This completes the proof. □

Note. I note that the case $v = \prod_{i=1}^p z_1(k_i)$ of (30) was proved in my manuscript submitted to a journal on March 9, 2015 and its preprint distributed in February 2015 in the same way as in the proof of (30): the case $v = \prod_{i=1}^p z_1(k_i)$ of the proof of (30) is just the proof written therein. As regards the duality formula (32), I stated the case $v = \prod_{i=1}^p z_1(k_i)$ of it in my talk at the Seminar on Analytic Number Theory, Graduate School of Mathematics, Nagoya University, Japan, February 13, 2008. (See also [12, Acknowledgments, p. 578] and [16, Note (ii)].) Therefore Theorem 3.1 and its proof are extensions of my previous works on MHZVs.

Example 3.2. For the monomials v_0 and $\tau(v_0)$ used in Example 2.7(i), the identity (30) also gives the following two different relations between MHZVs and (29), which are similar to (24) and (25): by taking $v = v_0$ in (30),

$$\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\geq 0}} \left\{ \prod_{i=1}^{p} \binom{k_i+r_i-1}{r_i} \right\} \zeta\left(z_1(k_1+r_1) \left\{ \prod_{i=2}^{p} z_0(k_i+r_i) \right\}; \alpha \right)$$
$$= \sum_{\substack{r_1+\cdots+r_q=r\\ r_i\geq 0}} H^*_{(\{r_i\}_{i=1}^q)} \left(\left\{ \prod_{i=1}^{q-1} z_{c'_{i-1}}(1) \right\} z_{c'_{q-1}}(2); \alpha \right)$$

and, by taking $v = \tau(v_0)$ in (30),

$$\sum_{\substack{r_1+\cdots+r_q=r\\ r_i\geq 0}} (1+r_q) \zeta\left(\left\{ \prod_{i=1}^{q-1} z_{c'_{i-1}}(1+r_i) \right\} z_{c'_{q-1}}(2+r_q); \alpha \right)$$
$$= \sum_{\substack{r_1+\cdots+r_p=r\\ r_i\geq 0}} H^*_{(\{r_i\}_{i=1}^p)} \left(z_1(k_1) \left\{ \prod_{i=2}^{p} z_0(k_i) \right\}; \alpha \right)$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$.

Remark 3.3. Coppo [1], Coppo and Candelpergher [2], Émery [3] and Hasse [6] proved relations between the case $c_i = 0$ (or $c_i = 1$) ($i = 1, \ldots, p-1$) of (28) and the single Hurwitz(–Lerch) zeta values $\zeta(z_1(k_1); \alpha)$, $\sum_{m=0}^{\infty} z^m (m+\alpha)^{-k}$. We proved in [10] relations between the above case of (28) with $z = 1$ and the case $c_i = 1$ ($i = 1, \ldots, p-1$) of (27): see [10, Prop. 2 and its examples] and [13, (**R3**)]. See also [17, Note 2].

Corrections to [14]. (i) Page 223, line 7: "the idea" should be "our idea". (ii) Page 223, line 15 from the bottom and page 234, line 12: "former" should be "previous". (iii) Page 235, line 7: "sort" should be "kind".

Corrections to [12]. (i) Page 575, lines 2–3 from the bottom: "what we noted" should be "a note on". (ii) Page 578, line 23 from the bottom: "March 2007" should be "February 3, 2007".

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