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Rationally AF Algebras and KMS States of $\mathcal{Z}$-Absorbing C*-Algebras

In memory of Huzihiro Araki

by

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Abstract

In order to realize all possible KMS-bundles on the Jiang–Su algebra, we introduce a class of C*-algebras which we call rationally approximately finite-dimensional (RAF). Using these, we show that for a given proper simplex bundle (S, π) with a singleton $\pi^{-1}(\{0\})$ and a unital separable monotracial C*-algebra A absorbing the Jiang–Su algebra tensorially (for instance, the irrational rotation algebra), there exists a flow on A whose KMS-bundle is isomorphic to (S, π).

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§1. Introduction

Approximately finite-dimensional separable C*-algebras (AF algebras) were classified, roughly 50 years ago, in [34, 16, 5, 20], in terms of the Murray–von Neumann semigroup of equivalence classes of projections – equivalently, the universal enveloping ordered group that has come to be known (since Murray and von Neumann) as K_0 (and in the setting of AF algebras, the dimension group).

A very general class of simple separable C*-algebras, assumed to be well behaved but satisfying very simple abstract axioms, has now been classified by

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a generalization of this invariant; see [45, 46, 47, 59, 40, 41, 21, 23, 27, 26, 72, 12, 25, 38, 39], all of which follow on substantial earlier work over the last 30 to 40 years. The axioms are amenability, absorption of the Jiang–Su algebra $\mathcal{Z}$, and the universal coefficient theorem (conceivably redundant in the amenable case). Recall (see [44]) that the tensor product of any C*-algebra with the Jiang–Su algebra absorbs the Jiang–Su algebra tensorially. In many examples (see [28]; see also e.g. [32, 67]), this property already holds. The invariant, in the stable case, is $K_* = K_0 \oplus K_1$ together with the natural pairing with traces, where both K_0 and K_1 are arbitrary countable abelian groups, and the traces constitute an arbitrary metrizable simplicial cone paired in an arbitrary way with K_0 (the natural order in which is determined by this pairing).

At the same time, considerable progress has been made in the last 30 years in the classification of non-simple C*-algebras beyond the case of AF algebras. In the infinite ($\mathcal{O}_\infty$-stable) case an ideal-related KK-equivalence-based isomorphism theorem was outlined in [46]; a different proof of this was given in [33]. In the finite (non-simple) case, after earlier results, a K-theoretical classification of inductive limits of finite direct sums of matrix algebras over commutative C*-algebras (AH, or approximately homogeneous, algebras) with no dimension growth in the spectra and with the ideal property (every closed two-sided ideal generated by projections) was given in [35]. The next step in this direction would be to classify inductive limits of sequences of C*-subalgebras of matrix algebras over commutative C*-algebras (ASH, or approximately subhomogeneous, algebras), with no dimension growth and with the ideal property.

Unexpectedly, in an investigation of KMS-state behavior of one-parameter automorphism groups of a C*-algebra, along the lines of [7] and [8], a class of ASH algebras arose which it was possible to classify. This is the basis of the present paper, which uses this new classification theorem to construct specified KMS-state phenomena in the Jiang–Su C*-algebra, and hence in the tensor product of this algebra with any other monotracial C*-algebra.

The class of algebras in question (see Section 4), to be named rationally AF algebras (following the terminology of [40] and [41]), or RAF algebras, can be characterized as those C*-algebras such that the tensor product with every infinite-dimensional UHF (Glimm uniformly hyperfinite) algebra is AF.

By [36], every C*-algebra which is both RAF and AH with no dimension growth is AF. (To see this, one may assume that the algebra is stable, and then as shown in the proof of Corollary 5.4, below, it has the ideal property. By [36] it is AT, and since by Lemma 4.2, it has zero K_1 group, by [70] it is AI. By [6, Cor. 1.3] any AH algebra with no dimension growth has real rank zero if it has

small eigenvalue variation, which holds if the tensor product with a UHF algebra has real rank zero. It is well known that an AI algebra of real rank zero is AF. AT algebras of real rank zero were classified in [23].)

The present class, thus, projects directly into the unknown territory of the non-simple ASH class.

Interestingly (see Theorem 5.3), the classifying invariant for this class – or, rather, the Jiang–Su stable members of the class – at least up to stable isomorphism, is exactly the same as for the class of AF algebras, namely, the ordered K_0-group, which can be an arbitrary countable ordered abelian group whose tensor product with every dense subgroup of the rational numbers is a dimension group. (We use the terminology rational dimension group, following [53] which deals with the simple case. See Definition 3.2, Lemma 3.1(v), and Theorem 4.4.)

The classification up to isomorphism for non-stable algebras is also almost the same as for AF algebras, and in the case that there is an approximate unit consisting of projections (for instance if the algebras are stable or unital), it is exactly the same (namely, the dimension range). The general case (appearing already with the non-unital algebras stably isomorphic to the Jiang–Su algebra) is slightly more subtle. See Corollaries 4.8 and 5.4, which introduce what we shall call the matrix dimension range.

The proof of Theorem 5.3, the $\mathcal{Z}$-absorbing RAF-algebra classification theorem, consists of an application of the Winter deformation technique [75, Prop. 4.5], extended in a simple way beyond the unital case in which it is at present couched.

Our result concerning KMS-state structure, which follows from the Jiang–Su-absorbing RAF classification (Corollaries 4.8 and 5.4) is Theorem 6.5. (A precise statement is given in the abstract.) The method is similar to that of [7] and [8], and to that of [31] and [30], and there is some overlap in the results. In [7] and [8] the C*-algebras on which the actions are constructed are not precisely identified. In [30], the present setting but with only the case of a compact bundle, i.e., bounded set of admissible inverse temperatures, is dealt with – this uses the known simple C*-algebra classification referred to above. The case of an infinite Kirchberg algebra is also dealt with in [30], without the compactness restriction. In [31], the only overlap is the case of a monotracial simple unital AF algebra.

As a straightforward application of the present paper, we obtain in particular the following result.

Corollary 1.1. *For any irrational rotation algebra, there exists a flow which realizes any given proper simplex bundle with singleton fiber over* 0 *as a KMS-bundle on it.*

§2. Preliminaries

Let us start with some basic terminology and a fact for ordered abelian groups with the Riesz interpolation property.

In this paper, a (*partially*) *ordered abelian group* (G, G^+) will mean an abelian group G with a positive cone G^+ satisfying $G^+ + G^+ \subset G^+$ and $G^+ \cap -G^+ = \{0\}$. If just the first condition is satisfied, (G, G^+) will be called a *pre-ordered abelian group* (see [42]). If, in addition to both conditions, $G = G^+ - G^+$, then (G, G^+) will be called an *ordered abelian group* (see [64, Def. 5.1.3]). For a partially ordered (or pre-ordered) abelian group (G, G^+), we shall denote by $\leq$ the order relation defined by $g \leq h$ if $h - g \in G^+$.

A partially ordered abelian group (G, G^+) will be said to have the *Riesz (or Birkhoff–Riesz) interpolation property* (RIP) if for g_0, g_1, h_0, $h_1 \in G$ satisfying $g_i \leq h_j$ for all $i, j \in \{0, 1\}$, there exists $a \in G$ such that $g_i \leq a \leq h_j$ for all $i, j \in \{0, 1\}$. (See [1].)

For a supernatural number $\mathfrak{n}$ (i.e., a formal infinite product of finite or infinite powers of prime numbers; see [16, 62, 64]), denote by $\mathbb{D}_\mathfrak{n}$ the set of all rational numbers p/q given by $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $q|\mathfrak{n}$, and fix the positive cone $\mathbb{D}_\mathfrak{n}^+ = \{d \in \mathbb{D}_\mathfrak{n} : d \geq 0\}$. Denote by $\mathbb{D}_\mathfrak{n}[e^x, e^{-x}]$ the group of Laurent polynomials on $\mathbb{R}$ with coefficients in $\mathbb{D}_\mathfrak{n}$. For a closed subset $F \subset \mathbb{R}$, equip $\mathbb{D}_\mathfrak{n}[e^x, e^{-x}]$ with the strict pointwise order, making it an ordered abelian group:

$$\mathbb{D}_\mathfrak{n}[e^x, e^{-x}]_F^+ = \{0\} \cup \big\{f \in \mathbb{D}_\mathfrak{n}[e^x, e^{-x}] : f(x) > 0 \text{ for any } x \in F\big\}.$$

We shall have occasion to use the following fundamental lemma in the sequel (in Lemma 3.5).

Lemma 2.1. *Let $F \subset \mathbb{R}$ be a closed subset of $\mathbb{R}$ and $\mathfrak{n}$ an infinite supernatural number. Then the ordered abelian group $(\mathbb{D}_\mathfrak{n}[e^x, e^{-x}], \mathbb{D}_\mathfrak{n}[e^x, e^{-x}]_F^+)$ has the RIP if and only if F is semibounded (i.e., either bounded below or bounded above).*

Proof. The "if" part of the statement is a variant of the argument of [71, Sect. 5]. We include a proof for the reader's convenience. Suppose that F is semibounded, and set $-F = \{x \in \mathbb{R} : -x \in F\}$. In the case that F is bounded, the Stone–Weierstrass theorem can be applied directly to show the RIP. So we may assume that F is unbounded. To simplify notation, set $G = \mathbb{D}_\mathfrak{n}[e^x, e^{-x}]$ and $G_F^+ = \mathbb{D}[e^x, e^{-x}]_F^+$. It is immediate that (G, G_F^+) has the RIP if and only if (G, G_{-F}^+) has. Thus, we may assume that F is bounded below.

Let p_i, $q_j \in G$, $i, j \in \{0, 1\}$, be such that $p_i \leq q_j$ for all $i, j \in \{0, 1\}$. If $p_{i_0} = q_{j_0}$ for some $i_0, j_0 \in \{0, 1\}$, then $a = p_{i_0} = q_{j_0}$ satisfies $p_i \leq a \leq q_j$ for all $i, j \in \{0, 1\}$. Therefore, we may suppose that $p_i(x) < q_j(x)$ for all $i, j \in \{0, 1\}$ and

$x \in F$. As the first step in the argument which follows, note that for sufficiently large $N \in \mathbb{N}$ and $R > 0$, there exists $a_0 \in G$ such that

$$p_i(x) + e^{-Nx} < a_0(x) < q_j(x) - e^{-Nx},$$

for all $i, j \in \{0, 1\}$ and $x \in F \cap [R, \infty)$.

Let $\mathbb{D}_{\mathfrak{n}}^{\infty}$ denote the infinite direct sum $\bigoplus_{n\in\mathbb{Z}} \mathbb{D}_{\mathfrak{n}}$ of copies of $\mathbb{D}_{\mathfrak{n}}$ over $\mathbb{Z}$, and equip $\mathbb{D}_{\mathfrak{n}}^{\infty}$ with the reverse lexicographic order $\leq_{\text{lex}}$, which means that for $g = (g_n)_{n\in\mathbb{Z}}$ and $h = (h_n)_{n\in\mathbb{Z}} \in \mathbb{D}_{\mathfrak{n}}^{\infty}$, $g \leq_{\text{lex}} h$ if $g_n = h_n$ for all $n \in \mathbb{Z}$ or $g_{n_0} < h_{n_0}$ for $n_0 = \max\{n \in \mathbb{Z} : g_n \neq h_n\}$. If $g \leq_{\text{lex}} h$ and $g \neq h$, write $g <_{\text{lex}} h$. Since F is unbounded (above), it follows that $(g_n)_{n\in\mathbb{Z}} <_{\text{lex}} (h_n)_{n\in\mathbb{Z}}$ if, and only if, there exists $r > 0$ such that $\sum_{n\in\mathbb{Z}} g_n e^{nx} < \sum_{n\in\mathbb{Z}} h_n e^{nx}$ for any $x \in F \cap [r, \infty)$. Let $p_i^{\infty} = (p_{i,n})_{n\in\mathbb{Z}}$, $q_j^{\infty} = (q_{j,n})_{n\in\mathbb{Z}} \in \mathbb{D}_{\mathfrak{n}}^{\infty}$ denote the coefficient sequences of $p_i(x) = \sum_{n\in\mathbb{Z}} p_{i,n} e^{nx}$ and $q_j(x) = \sum_{n\in\mathbb{Z}} q_{j,n} e^{nx}$, $i, j \in \{0, 1\}$. Since the reverse lexicographic order is a total order, reindexing, we may assume that $p_0^{\infty} \leq_{\text{lex}} p_1^{\infty} <_{\text{lex}} q_0^{\infty} \leq_{\text{lex}} q_1^{\infty}$. Set $n_0 = \max\{n \in \mathbb{Z} : p_{1,n} \neq q_{0,n}\}$ and note that $p_{1,n_0} < q_{0,n_0}$. For $k \in \mathbb{Z}$, set $\delta_k = (\delta_{k,n})_{n\in\mathbb{Z}} \in \mathbb{D}_{\mathfrak{n}}^{\infty}$, where $\delta_{k,n}$ is the Kronecker delta, and set $a_0^{\infty} = p_1^{\infty} + \delta_{n_0-1}$. Then we have $p_1^{\infty} <_{\text{lex}} a_0^{\infty} <_{\text{lex}} q_0^{\infty}$. Let $N \in \mathbb{N}$ chosen above be chosen sufficiently large that $p_0^{\infty} + \delta_{-N} \leq_{\text{lex}} p_1^{\infty} + \delta_{-N} <_{\text{lex}} a_0^{\infty} <_{\text{lex}} q_0^{\infty} - \delta_{-N} \leq q_1^{\infty} - \delta_{-N}$ also, which is the analogue in $\mathbb{D}_{\mathfrak{n}}^{\infty}$ of the required condition.

Since by the choice of a_0, $e^{Nx}(p_i(x) - a_0(x)) < -1 < 1 < e^{Nx}(q_j(x) - a_0(x))$ for all $i, j \in \{0, 1\}$ and $x \in F \cap [R, \infty)$, there exist $d > 0$ and a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with compact support such that $e^{Nx}(p_i(x) - a_0(x)) + d < f(x) < e^{Nx}(q_j(x) - a_0(x)) - d$ for any $x \in F$. By the Stone–Weierstrass theorem (applied to the compact space $F \cup \{\infty\}$ and the subalgebra of Laurent polynomials bounded on this space), we obtain $a_1 \in G$ such that $\sup_{x\in F} |f(x) - a_1(x)| < d$. Defining $a \in G$ by $a(x) = e^{-Nx} a_1(x) + a_0(x)$, we have $p_i \leq a \leq q_j$ for all $i, j \in \{0, 1\}$.

To show the "only if" part, assume that F is not semibounded. Let $N \in \mathbb{N}$ and $\beta \in F$ be such that $N < e^{\beta} < 2N$. Define p_i, $q_j \in G$, $i, j \in \{0, 1\}$, by $p_0(x) = -1$, $p_1(x) = (N - e^x)(e^x - 2N)$, $q_0(x) = e^{2x}$, and $q_1(x) = N^2$ for $x \in \mathbb{R}$. Then it follows that $p_i \leq q_j$ for all $i, j \in \{0, 1\}$. If there existed $a \in G$ with $p_i \leq a \leq q_j$ for all $i, j \in \{0, 1\}$, then, as $p_0 \leq a \leq q_1$, a would be a bounded function on F. Since F is not semibounded, a must be a constant function, in contradiction with $p_1 \leq a \leq q_0$. □

§3. Rational dimension groups and simplex bundles

The notion of a rationally Riesz group was introduced in [53], and it was shown that a weakly unperforated simple ordered abelian group G is a rationally Riesz group if and only if $G \otimes \mathbb{D}_{\mathfrak{p}}$ has the Riesz interpolation property for every infinite

supernatural number $\mathfrak{p}$. (See [62] for the definition of the supernatural numbers, which extend the natural numbers, which we shall refer to as finite.) In this section, we shall consider a modified definition for non-simple ordered abelian groups (Definition 3.2), and construct an example which gives rise in a natural way to a given proper simplex bundle (see below) in Proposition 3.6. In the sequel (see Theorem 6.5), proper simplex bundles will be shown to give rise to KMS-bundles on the Jiang–Su algebra.

Recall that the positive cone $(G \otimes H)^+$ of the tensor product of two partially ordered (or pre-ordered) abelian groups (G, G^+) and (H, H^+) is defined as the set of all finite sums of $\{g \otimes h : g \in G^+,\ h \in H^+\}$, and note that $(G \otimes H, (G \otimes H)^+)$ is then a partially ordered (or pre-ordered) abelian group; see [43]. For convenience, we shall just write $G \otimes H^+$. For an abelian group G we shall denote by $G \otimes 1_{\mathbf{k}}$ the subgroup of $G \otimes \mathbb{D}_{\mathbf{k}}$ consisting of $\{g \otimes 1_{\mathbf{k}} : g \in G\}$, where $1_{\mathbf{k}}$ denotes the number $1 \in \mathbb{D}_{\mathbf{k}}$.

Lemma 3.1. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be relatively prime supernatural numbers. For an abelian group G, the following fundamental statements hold:*

(i) *If $G \otimes \mathbb{D}_{\mathfrak{p}}$ and $G \otimes \mathbb{D}_{\mathfrak{q}}$ are torsion-free abelian groups, then so also is G.*

(ii) *If G is torsion-free, then the subgroup $(G \otimes \mathbb{D}_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}) \cap (G \otimes 1_{\mathfrak{p}} \otimes \mathbb{D}_{\mathfrak{q}})$ of $G \otimes \mathbb{D}_{\mathfrak{p}} \otimes \mathbb{D}_{\mathfrak{q}}$ is isomorphic to G as an abelian group.*

(iii) *If a partially ordered abelian group (G, G^+) is such that $(G \otimes \mathbb{D}_{\mathfrak{p}}, G \otimes \mathbb{D}_{\mathfrak{p}}{}^+)$ and $(G \otimes \mathbb{D}_{\mathfrak{q}}, G \otimes \mathbb{D}_{\mathfrak{q}}{}^+)$ are torsion-free ordered abelian groups, then (G, G^+) is an ordered abelian group (i.e., $G = G^+ - G^+$).*

(iv) *If (G, G^+) is an unperforated ordered abelian group (i.e., $ng \geq 0$ for some $n \in \mathbb{N}$ implies $g \geq 0$; see [2, Def. 6.7.1]), then so also is $(G \otimes \mathbb{D}_{\mathfrak{n}}, G \otimes \mathbb{D}_{\mathfrak{n}}{}^+)$ for any supernatural number $\mathfrak{n}$.*

(v) *If (G, G^+) is an unperforated partially ordered abelian group such that $(G \otimes \mathbb{D}_{\mathfrak{p}}, G \otimes \mathbb{D}_{\mathfrak{p}}{}^+)$ and $(G \otimes \mathbb{D}_{\mathfrak{q}}, G \otimes \mathbb{D}_{\mathfrak{q}}{}^+)$ are dimension groups, then (G, G^+) is an ordered abelian group and $(G \otimes \mathbb{D}_{\mathfrak{n}}, G \otimes \mathbb{D}_{\mathfrak{n}}{}^+)$ is a dimension group for any infinite supernatural number $\mathfrak{n}$.*

Proof. (i) Assume that $g \in G$ is a torsion element and $n \in \mathbb{N}$ is the first number such that $ng = 0$. Since $G \otimes \mathbb{D}_{\mathfrak{p}}$ is torsion-free, we have $g \otimes 1_{\mathfrak{p}} = 0$ in $\langle g \rangle \otimes \mathbb{D}_{\mathfrak{p}}$. Since $\langle g \rangle \otimes \mathbb{D}_{\mathfrak{p}} \cong (\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{D}_{\mathfrak{p}} \cong \mathbb{D}_{\mathfrak{p}}/n\mathbb{D}_{\mathfrak{p}}$, it follows that $1 \in n\mathbb{D}_{\mathfrak{p}}$. Similarly, we have $1 \in n\mathbb{D}_{\mathfrak{q}}$. Because $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime, this implies that $n = 1$ and so $g = ng = 0$.

(ii) Since G is torsion-free, the canonical embedding $\varphi\colon G \to G \otimes 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$ defined by $\varphi(g) = g \otimes 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$ for $g \in G$ is a group isomorphism. Therefore, it suffices

to show that $(G \otimes \mathbb{D}_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}) \cap (G \otimes 1_{\mathfrak{p}} \otimes \mathbb{D}_{\mathfrak{q}}) = G \otimes 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$. If $x \in (G \otimes \mathbb{D}_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}) \cap (G \otimes 1_{\mathfrak{p}} \otimes \mathbb{D}_{\mathfrak{q}})$, then there exist $k, l \in \mathbb{N}$ such that $1/k \in \mathbb{D}_{\mathfrak{p}}$, $1/l \in \mathbb{D}_{\mathfrak{q}}$, and $kx, lx \in G \otimes 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$. Since k and l are relatively prime, there exist $a, b \in \mathbb{Z}$ such that $1 = ak + bl$. It follows that $x = akx + blx \in G \otimes 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$. The converse inclusion $(G \otimes \mathbb{D}_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}) \cap (G \otimes 1_{\mathfrak{p}} \otimes \mathbb{D}_{\mathfrak{q}}) \supset G \otimes 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$ is trivial.

(iii) Fix $x \in G$. For $\mathfrak{i} = \mathfrak{p}, \mathfrak{q}$, since $(G \otimes \mathbb{D}_{\mathfrak{i}}, G \otimes \mathbb{D}_{\mathfrak{i}}{}^{+})$ is an ordered group, there exist $a_{\mathfrak{i}}, b_{\mathfrak{i}} \in G \otimes \mathbb{D}_{\mathfrak{i}}{}^{+}$ such that $x \otimes 1_{\mathfrak{i}} = a_{\mathfrak{i}} - b_{\mathfrak{i}}$. Let $n_{\mathfrak{i}}$ be such that $1/n_{\mathfrak{i}} \in \mathbb{D}_{\mathfrak{i}}$ and $n_{\mathfrak{i}} a_{\mathfrak{i}}$, $n_{\mathfrak{i}} b_{\mathfrak{i}} \in G \otimes 1_{\mathfrak{i}}$ for $\mathfrak{i} = \mathfrak{p}, \mathfrak{q}$. Since $a_{\mathfrak{i}}$, $b_{\mathfrak{i}} \in G \otimes \mathbb{D}_{\mathfrak{i}}{}^{+}$, there exist $g_{a_{\mathfrak{i}}}$, $g_{b_{\mathfrak{i}}} \in G^{+}$, $\mathfrak{i} = \mathfrak{p}, \mathfrak{q}$, such that $n_{\mathfrak{i}} a_{\mathfrak{i}} = g_{a_{\mathfrak{i}}} \otimes 1_{\mathfrak{i}}$ and $n_{\mathfrak{i}} b_{\mathfrak{i}} = g_{b_{\mathfrak{i}}} \otimes 1_{\mathfrak{i}}$. By (i), G is torsion-free, and so from $n_{\mathfrak{i}} x \otimes 1_{\mathfrak{i}} = (g_{a_{\mathfrak{i}}} - g_{b_{\mathfrak{i}}}) \otimes 1_{\mathfrak{i}}$ it follows that $n_{\mathfrak{i}} x = g_{a_{\mathfrak{i}}} - g_{b_{\mathfrak{i}}}$. Since $n_{\mathfrak{p}}$ and $n_{\mathfrak{q}}$ are relatively prime, there exist $c, d \in \mathbb{N}$ such that $1 = cn_{\mathfrak{p}} - dn_{\mathfrak{q}}$. Then we have

$$x = (cg_{a_{\mathfrak{p}}} + dg_{b_{\mathfrak{q}}}) - (cg_{b_{\mathfrak{p}}} + dg_{a_{\mathfrak{q}}}) \in G^{+} - G^{+}.$$

(iv) Since (G, G^{+}) and $(\mathbb{D}_{\mathfrak{n}}, \mathbb{D}_{\mathfrak{n}}{}^{+})$ are torsion-free ordered abelian groups, it is straightforward to check that $(G \otimes \mathbb{D}_{\mathfrak{n}}, G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+})$ is also a torsion-free ordered abelian group. Therefore, it suffices to show that there exists $d \in \mathbb{N}$ such that $1/d \in \mathbb{D}_{\mathfrak{n}}$ and $dx \in G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+}$ if $x \in G \otimes \mathbb{D}_{\mathfrak{n}}$ and $n \in \mathbb{N}$ satisfy $nx \in G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+}$. Let $g_j \in G$ and $d_j \in \mathbb{D}_{\mathfrak{n}}$, $j = 1, 2, \ldots, N$, be such that $\sum_{j=1}^{N} g_j \otimes d_j = x$. Since $nx \in G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+}$, there exist $h_m \in G_{+}$, and $e_m \in \mathbb{D}_{\mathfrak{n}}{}^{+}$, $m = 1, 2, \ldots, M$, such that $nx = \sum_{m=1}^{M} h_m \otimes e_m$. Choose $d \in \mathbb{N}$ such that $1/d \in \mathbb{D}_{\mathfrak{n}}$ and dd_j, $de_m \in \mathbb{Z}$ for all $j = 1, 2, \ldots, N$ and $m = 1, 2, \ldots, M$. Then it follows that $(\sum_{j=1}^{N} ndd_j g_j) \otimes 1_{\mathfrak{n}} = (\sum_{m=1}^{M} de_m h_m) \otimes 1_{\mathfrak{n}}$. Since G is torsion-free, we have $\sum_{j=1}^{N} ndd_j g_j = \sum_{m=1}^{M} de_m h_m \in G^{+}$. Finally, since G is unperforated, we conclude that $dx = (\sum_{j=1}^{N} dd_j g_j) \otimes 1_{\mathfrak{n}} \in (G \otimes \mathbb{D}_{\mathfrak{n}})^{+}$.

(v) By (i) and (iii), we see that (G, G^{+}) is an unperforated ordered abelian group, and by (iv) so also is $(G \otimes \mathbb{D}_{\mathfrak{n}}, G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+})$. Because the Riesz interpolation property in an ordered abelian group is equivalent to the Riesz decomposition property of [61] (see [1]; see also [15, Thm. IV 6.2]), it suffices to show that for x, a_0, $a_1 \in G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+}$ with $x \leq a_0 + a_1$ there exist two elements $x_i \in G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+}$, $i = 0, 1$, such that $x = x_0 + x_1$ and $x_i \leq a_i$ for both $i = 0, 1$.

Choose $N \in \mathbb{N}$ such that $1/N \in \mathbb{D}_{\mathfrak{n}}$ and Nx, $Na_i \in G \otimes 1_{\mathfrak{n}}$, and denote by g_x, $g_{a_i} \in G^{+}$, $i = 0, 1$, the elements such that $g_x \otimes 1_{\mathfrak{n}} = Nx$ and $g_{a_i} \otimes 1_{\mathfrak{n}} = Na_i$ for $i = 0, 1$. Since $g_x \otimes 1_{\mathfrak{n}} \leq (g_{a_0} + g_{a_1}) \otimes 1_{\mathfrak{n}}$, we have $g_x \leq g_{a_0} + g_{a_1}$, as G is unperforated. By the Riesz decomposition property of $G \otimes \mathbb{D}_{\mathfrak{p}}$ and $G \otimes \mathbb{D}_{\mathfrak{q}}$, for $\mathfrak{j} = \mathfrak{p}, \mathfrak{q}$ there are $y_i^{(\mathfrak{j})} \in G \otimes \mathbb{D}_{\mathfrak{j}}{}^{+}$, $i = 0, 1$, such that $g_x \otimes 1_{\mathfrak{j}} = y_0^{(\mathfrak{j})} + y_1^{(\mathfrak{j})}$ and $y_i^{(\mathfrak{j})} \leq g_{a_i} \otimes 1_{\mathfrak{j}}$ for both $i = 0, 1$. Choose $N_{\mathfrak{j}} \in \mathbb{N}$, $\mathfrak{j} = \mathfrak{p}, \mathfrak{q}$, such that $1/N_{\mathfrak{j}} \in \mathbb{D}_{\mathfrak{j}}$ and $N_{\mathfrak{j}} y_i^{(\mathfrak{j})} \in G \otimes 1_{\mathfrak{j}}$ for both $i = 0, 1$, and denote by $g_i^{(\mathfrak{j})} \in G^{+}$ the element such that $N_{\mathfrak{j}} y_i^{(\mathfrak{j})} = g_i^{(\mathfrak{j})} \otimes 1_{\mathfrak{j}}$. Thus we have $N_{\mathfrak{j}} g_x = g_0^{(\mathfrak{j})} + g_1^{(\mathfrak{j})}$ for $\mathfrak{j} = \mathfrak{p}, \mathfrak{q}$. Since

$N_{\mathfrak{p}}$ and $N_{\mathfrak{q}}$ are relatively prime, if $\mathfrak{n}$ is infinite, there exist natural numbers c, d, and M $(> N_{\mathfrak{p}}N_{\mathfrak{q}})$ such that $1/(MN) \in \mathbb{D}_{\mathfrak{n}}$ and $M = cN_{\mathfrak{p}} + dN_{\mathfrak{q}}$. The elements $x_i = (cg_i^{(\mathfrak{p})} + dg_i^{(\mathfrak{q})}) \otimes 1/(MN) \in G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+}$, $i = 0, 1$, then satisfy the desired conditions. □

Definition 3.2. We shall call a partially ordered abelian group (G, G^+) a *rational dimension group* if (G, G^+) is unperforated and, further, $(G \otimes \mathbb{D}_{\mathfrak{p}}, G \otimes \mathbb{D}_{\mathfrak{p}}{}^{+})$ and $(G \otimes \mathbb{D}_{\mathfrak{q}}, G \otimes \mathbb{D}_{\mathfrak{q}}{}^{+})$ are dimension groups for some pair of relatively prime infinite supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$. Note that a partially ordered abelian group is not necessarily unperforated even if $G \otimes \mathbb{D}_{\mathfrak{n}}$ is a dimension group for every infinite supernatural number $\mathfrak{n}$. For example, the ordered abelian group $(\mathbb{Z}, S_n)$ with $S_n = \{0, n, n+1, \ldots\}$ is not unperforated for $n \in \mathbb{N} \setminus \{1\}$; however, $(\mathbb{Z} \otimes \mathbb{D}_{\mathfrak{n}}, S_n \otimes \mathbb{D}_{\mathfrak{n}}{}^{+})$ is isomorphic to $(\mathbb{D}_{\mathfrak{n}}, \mathbb{D}_{\mathfrak{n}}{}^{+})$ as an ordered abelian group for any infinite supernatural number $\mathfrak{n}$.

Remark 3.3. For simple ordered abelian groups, it is enough for the tensor product of an unperforated ordered group with $\mathbb{D}_{\mathfrak{n}}$ for a single supernatural number $\mathfrak{n}$ to be a dimension group in order for it to be a rational dimension group (see [53, Prop. 5.7]). However, for non-simple ordered abelian groups, the following example indicates the necessity of the two tensor products in the definition above.

Let p and q be two relatively prime numbers, and let $\mathfrak{u}$ denote the universal supernatural number, with $\mathbb{D}_{\mathfrak{u}} = \mathbb{Q}$. Set $H = \mathbb{Q}^2$, $H^+ = \{(x, y) \in H : x \geq 0, y \geq 0\}$, $a_0 = (0, 0)$, $b_0 = (1, 0)$, $a_1 = (1/p, 1/p - 1)$, and $b_1 = (1/p, 1/p)$ in H. Define (G, G^+) as the ordered subgroup of (H, H^+) generated by $\{a_1, b_1\}$, which is unperforated and non-simple. Note that, for any supernatural number $\mathfrak{n}$, the ordered abelian group $(G \otimes \mathbb{D}_{\mathfrak{n}}, G \otimes \mathbb{D}_{\mathfrak{n}}{}^{+})$ is isomorphic to $(\mathbb{D}_{\mathfrak{n}}a_1 + \mathbb{D}_{\mathfrak{n}}b_1, (\mathbb{D}_{\mathfrak{n}}a_1 + \mathbb{D}_{\mathfrak{n}}b_1) \cap H^+)$, and that a_0, $b_0 \in G$. Then the tensor product $(G \otimes \mathbb{D}_{\mathfrak{u}}, G \otimes \mathbb{D}_{\mathfrak{u}}{}^{+}) \cong (H, H^+)$ is a dimension group. However, if there exists $x \in G \otimes \mathbb{D}_{q^\infty}$ such that $a_i \otimes 1_{q^\infty} \leq x \leq b_j \otimes 1_{q^\infty}$, for $i, j \in \{0, 1\}$, then x must correspond to $(1/p, 0)$ via the identification of $G \otimes \mathbb{D}_{q^\infty}$ with $\mathbb{D}_{q^\infty}a_1 + \mathbb{D}_{q^\infty}b_1$. Since p and q are relatively prime, x cannot be contained in $G \otimes \mathbb{D}_{q^\infty}$. This shows that $(G \otimes \mathbb{D}_{q^\infty}, G \otimes \mathbb{D}_{q^\infty}{}^{+})$ does not have the RIP.

Let us recall a few notions concerning simplex bundles, introduced in [7] and [8] (see also [31, 30]). A pair (S, π) consisting of a second countable locally compact Hausdorff space S and a continuous map $\pi\colon S \to \mathbb{R}$ is called a *simplex bundle* if $\pi^{-1}(\{\beta\})$ is compact and has a structure of Choquet simplex for every $\beta \in \mathbb{R}$. To simplify notation, we shall write $\pi^{-1}(\beta)$ for $\pi^{-1}(\{\beta\})$. Two simplex bundles (S, π) and (S', π') are said to be *isomorphic* if there exists a homeomorphism $\Phi\colon S \to S'$ such that $\pi' \circ \Phi = \pi$ and the restriction of Φ to $\pi^{-1}(\beta)$ is affine for all $\beta \in \mathbb{R}$.

For a simplex bundle (S,π), we shall denote by $A(S)$ the set of all continuous functions f from S to $\mathbb{R}$ such that each restriction of f to $\pi^{-1}(\beta)$ is affine for any $\beta\in\mathbb{R}$. We also set

$$A_0(S)=\{f\in A(S): f \text{ vanishes at infinity}\},$$
$$A_{00}(S)=\{f\in A(S): \mathrm{supp}(f) \text{ is compact and } \mathrm{supp}(f)\cap\pi^{-1}(0)=\emptyset\}.$$

A simplex bundle (S,π) will be called *proper* if π is a proper map (i.e., $\pi^{-1}(K)$ is compact for any compact set $K\subset\mathbb{R}$), and $A(S)$ separates points of S.

Lemma 3.4. *Let (S,π) be a simplex bundle, S' a locally compact second countable Hausdorff space, and $\pi':S'\to\mathbb{R}$ a continuous map. Suppose that π is proper and Φ is a bijective continuous map from S to S' such that $\pi=\pi'\circ\Phi$. Then Φ is a homeomorphism. In particular, if (S,π) is a proper simplex bundle and the restriction of Φ to $\pi^{-1}(\beta)$ is affine for any $\beta\in\mathbb{R}$, then (S',π') is also a proper simplex bundle.*

Proof. For any compact subset K of S', since π' is continuous, there exists a compact interval $I\subset\mathbb{R}$ such that $\pi'(K)\subset I$, which implies that $\Phi^{-1}(K)\subset\pi^{-1}(I)$. Since $\pi^{-1}(I)$ is compact, so also is $\Phi^{-1}(K)$. This shows that Φ is a proper map. Let F be a closed subset of S. To show Φ^{-1} is continuous, it suffices to show that $\Phi(F)$ is closed. For any compact subset K of S', it follows that $\Phi(F)\cap K=\Phi(F\cap\Phi^{-1}(K))$ is compact. Since S' is locally compact and Hausdorff, this shows that $\Phi(F)$ is closed.

As $\pi=\pi'\circ\Phi$, it follows that ${\pi'}^{-1}(B)=\Phi(\pi^{-1}(B))$ for any subset B of $\mathbb{R}$. Then π' is also a proper map. If each restriction of Φ to $\pi^{-1}(\beta)$ is affine, then we see that $A(S')=\{f\circ\Phi^{-1}: f\in A(S)\}$ separates points of S'. □

The concept of a proper simplex bundle was presented as an abstract characterization of the KMS-bundle of a C*-algebra in [7, 8, 31, 30]. On the one hand, it was shown in [7] and [8] (see also [9, Sect. 5.3]) that a KMS-bundle for a unital separable C*-algebra is a proper simplex bundle, and on the other hand, in [31], it was shown that any proper simplex bundle can be realized as a KMS-bundle on a given unital simple infinite-dimensional AF-algebra. The construction in this paper is a variant of [31, Sect. 4.2] for rational dimension groups (instead of just dimension groups). In the rest of this section, we shall consider a proper simplex bundle (S,π) such that $\pi^{-1}(0)$ is a singleton $\{\tau_S\}$.

Let $\mathcal{G}$ be a countable additive subgroup of $\mathbb{R}$ with $1\in\mathcal{G}$ and let $\mathcal{G}^\infty$ denote the infinite direct sum $\bigoplus_{n\in\mathbb{Z}}\mathcal{G}$ of copies of $\mathcal{G}$ over $\mathbb{Z}$. Given a sequence $g=(g_n)_{n\in\mathbb{Z}}$

in $\mathcal{G}^\infty$, define a continuous function $L(g) \in A(S)$ by

$$L(g)(s) = \sum_{n\in\mathbb{Z}} g_n e^{n\pi(s)} \quad \text{for } s \in S.$$

From now on, we shall denote by $e^{-\pi}$ the function $e^{-\pi(\cdot)} \in A(S)$, so that $L(g) = \sum_{n\in\mathbb{Z}} g_n e^{n\pi}$. Let σ_S denote the automorphism of $A(S)$ defined by $\sigma_S(f) = e^{-\pi} f$ for $f \in A(S)$, and $\sigma_{\mathcal{G}^\infty}$ the automorphism of $\mathcal{G}^\infty$ defined by $\sigma_{\mathcal{G}^\infty}((g_n)_{n\in\mathbb{Z}}) = (g_{n+1})_{n\in\mathbb{Z}}$ for $(g_n)_{n\in\mathbb{Z}} \in \mathcal{G}^\infty$. Note that

$$\sigma_S \circ L = L \circ \sigma_{\mathcal{G}^\infty}.$$

For $k \in \mathbb{N}$, choose positive continuous functions ψ_k, ψ_{k+}, ψ_{k-} from $\mathbb{R}$ into $[0,1]$ such that $(-\infty, -1/k] \subset \psi_{k-}^{-1}(\{1\})$, $[-1/2k, 1/2k] \subset \psi_k^{-1}(\{1\})$, $[1/k, \infty) \subset \psi_{k+}^{-1}(\{1\})$, and $\psi_{k-} + \psi_k + \psi_{k+} = 1_{C(\mathbb{R})}$. Denote by $\mathcal{G}[e^{-\pi}, (1-e^{-\pi})]$ the additive group generated by the functions $ge^{n\pi}(1-e^{-\pi})^m$, $g \in \mathcal{G}$, $m, n \in \mathbb{Z}$ defined on $S \setminus \pi^{-1}(0)$. For $h \in \mathcal{G}[e^{-\pi}, (1-e^{-\pi})]$, we regard $h(\psi_{k\pm} \circ \pi)$ as elements of $A(S)$ in a canonical way. Consider the countable subgroup of $A(S)$,

$$\mathcal{G}_k^\infty = \mathcal{G}[e^{-\pi}, (1-e^{-\pi})]\psi_{k-} \circ \pi + L(\mathcal{G}^\infty)\psi_k \circ \pi + \mathcal{G}[e^{-\pi}, (1-e^{-\pi})]\psi_{k+} \circ \pi.$$

In the same way as in [8, Lem. 2.2] and [31, Prop. 4.5], choose a countable subgroup $\mathcal{G}_{00}$ of $A_{00}(S)$ satisfying the following conditions:

(1) For any $f \in A_{00}(S)$, $\varepsilon > 0$, and $N \in \mathbb{N}$ with $\operatorname{supp}(f) \subset \pi^{-1}((-N, N) \setminus \{0\})$, there exists $g \in \mathcal{G}_{00}$ such that $\sup_{s\in S} |f(x) - g(x)| < \varepsilon$ and $\operatorname{supp}(g) \subset \pi^{-1}((-N, N) \setminus \{0\})$,
(2) $\mathcal{G}_k^\infty + \mathcal{G}_{00} \subset \mathcal{G}_{k+1}^\infty + \mathcal{G}_{00}$ for any $k \in \mathbb{N}$, and
(3) $\sigma_S(\mathcal{G}_{00}) = (\mathrm{id}_{A(S)} - \sigma_S)(\mathcal{G}_{00}) = \mathcal{G}_{00}$.

Consider the countable subgroup of $A(S)$,

$$\mathcal{R}(\mathcal{G}) = \bigcup_{k=1}^\infty \mathcal{G}_k^\infty + \mathcal{G}_{00}.$$

Define positive cones $A(S)^+$ and $\mathcal{R}(\mathcal{G})^+$ of $A(S)$ and $\mathcal{R}(\mathcal{G})$ by

$$A(S)^+ = \{0\} \cup \{f \in A(S) : f(x) > 0 \text{ for any } x \in S\},$$
$$\mathcal{R}(\mathcal{G})^+ = \mathcal{R}(\mathcal{G}) \cap A(S)^+.$$

Lemma 3.5. *With $\mathcal{G}$ and $\mathcal{R}(\mathcal{G})$ as above, the following statements hold:*

(i) $(\mathcal{R}(\mathcal{G}), \mathcal{R}(\mathcal{G})^+)$ *is an unperforated ordered group.*
(ii) *In the case $\mathcal{G} = \mathbb{Z}$, $(\mathcal{R}(\mathbb{Z}), \mathcal{R}(\mathbb{Z})^+)$ is a rational dimension group.*

Proof. (i) It is straightforward to see that $\mathcal{R}(\mathcal{G})^+ \cap -\mathcal{R}(\mathcal{G})^+ = 0$ and that the partially ordered abelian group $(\mathcal{R}(\mathcal{G}), \mathcal{R}(\mathcal{G})^+)$ is unperforated. We must show that $\mathcal{R}(\mathcal{G}) = \mathcal{R}(\mathcal{G})^+ - \mathcal{R}(\mathcal{G})^+$. For $k \in \mathbb{N}$, let $g \in \mathcal{G}_k^\infty + \mathcal{G}_{00} \subset \mathcal{R}(\mathcal{G})$ be given, so that

$$g = h_- \psi_{k-} \circ \pi + h_0 \psi_k \circ \pi + h_+ \psi_{k+} \circ \pi + g_0,$$

with h_+, $h_- \in \mathcal{G}[e^{-\pi}, (1-e^{-\pi})]$, $h_0 \in L(\mathcal{G}^\infty)$, and $g_0 \in \mathcal{G}_{00}$. For $N \in \mathbb{N}$, let us fix the notation $-[N,\infty) \coloneqq (-\infty, -N]$ and $+[N,\infty) \coloneqq [N,\infty)$. With M a large enough natural number, we have $h_\pm(x) \le e^{\pm M\pi(x)}$ for any $x \in \pi^{-1}(\pm[M,\infty))$. Since the supports of $h_0\psi_k \circ \pi$ and g_0 are compact, and π is proper, there exists $c > 0$ such that the function $f = c(e^{-M\pi}\psi_{k-} \circ \pi + \psi_k \circ \pi + e^{M\pi}\psi_{k+} \circ \pi) \in \mathcal{R}(\mathcal{G})^+$ satisfies $f - g \in \mathcal{R}(\mathcal{G})^+$. Thus, $g = f - (f-g) \in \mathcal{R}(\mathcal{G})^+ - \mathcal{R}(\mathcal{G})^+$.

(ii) Let $\mathfrak{n}$ be an infinite supernatural number and denote by $\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$ the subgroup of $A(S)$ defined by

$$\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n}) = \left\{ \textstyle\sum_{i=1}^N d_i g_i \in A(S) : N \in \mathbb{N},\ d_i \in \mathbb{D}_\mathfrak{n},\ g_i \in \mathcal{R}(\mathbb{Z}) \right\}.$$

We define a positive cone $\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+$ by $\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+ = \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n}) \cap A(S)^+$. First we show that the partially ordered abelian group $(\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n}), \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+)$ is isomorphic to $(\mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}, \mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}{}^+)$ (as a partially ordered abelian group). Note that, by Lemma 3.1(iv), $(\mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}, \mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}{}^+)$ is an unperforated ordered abelian group. Let $\Phi\colon \mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n} \to \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$ denote the group homomorphism determined by $\Phi(g \otimes d) = dg$ for $g \in \mathcal{R}(\mathbb{Z})$ and $d \in \mathbb{D}_\mathfrak{n}$, which is obviously surjective. To show the injectivity of Φ, let g_i, $g_i' \in \mathcal{R}(\mathbb{Z})$ and d_i, $d_i' \in \mathbb{D}_\mathfrak{n}$, $i = 1, 2, \dots, N$, be such that $\sum_{i=1}^N d_i g_i = \sum_{i=1}^N d_i' g_i'$ in $\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$. Choose $d \in \mathbb{N}$ such that dd_i, $dd_i' \in \mathbb{Z}$ for all $i = 1, 2, \dots, N$ and $1/d \in \mathbb{D}_\mathfrak{n}$. Then it follows that $\sum_{i=1}^N dd_i g_i = \sum_{i=1}^N dd_i' g_i'$ in $\mathcal{R}(\mathbb{Z})$. Thus we have $d(\sum_{i=1}^N g_i \otimes d_i) = d(\sum_{i=1}^N g_i' \otimes d_i')$ in $\mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}$. Since $\mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}$ is torsion-free, we have $\sum_{i=1}^N g_i \otimes d_i = \sum_{i=1}^N g_i' \otimes d_i'$. It is trivial to see that $\Phi(\mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}{}^+) \subset \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+$. Since $(\mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}, \mathcal{R}(\mathbb{Z}) \otimes \mathbb{D}_\mathfrak{n}{}^+)$ is unperforated, we also have the converse inclusion.

What remains to be shown is the RIP of $(\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n}), \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+)$. The following argument is essentially same as that in the proof of [31, Lem. 4.6]. Let f_i, $g_j \in \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$, $i, j \in \{0,1\}$, be such that $f_i(x) < g_j(x)$ for any $x \in S$ and $i, j \in \{0,1\}$. At $\tau_S \in \pi^{-1}(0)$, since $\mathbb{D}_\mathfrak{n}$ is dense in $\mathbb{R}$, we obtain $d \in \mathbb{D}_\mathfrak{n}$ such that $f_i(\tau_S) < d < g_j(\tau_S)$ for all $i, j \in \{0,1\}$. By the definition of $\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$, there exist $N \in \mathbb{N}$ and $\tilde{f}_{i+}$, $\tilde{f}_{i-}$, $\tilde{g}_{j+}$, $\tilde{g}_{j-} \in \mathbb{D}_\mathfrak{n}[e^{-\pi}, (1-e^{-\pi})]$, $i, j \in \{0,1\}$ such that $\tilde{f}_{i\pm}(x) = f_i(x)$ and $\tilde{g}_{j\pm}(x) = g_j(x)$ for any $i, j \in \{0,1\}$ and $x \in \pi^{-1}(\pm[N,\infty))$. Note that $\pi(\pi^{-1}(\pm[N,\infty)))$ are semibounded closed subsets of $\mathbb{R}$, since π is a proper map. By Lemma 2.1, we obtain e_-, $e_+ \in \mathbb{D}_\mathfrak{n}[e^x, e^{-x}]$ such that

$$f_i(x) < e_\pm(\pi(x)) < g_j(x) \quad \text{for } s \in \pi^{-1}(\pm[N,\infty)).$$

Set

$$\bar{h}(x) = (e_-\psi_{k-} + d\psi + e_+\psi_{k+}) \circ \pi(x) \quad \text{for } x \in S;$$

then it follows that $f_i(x) < \bar{h}(x) < g_j(x)$ for any $i, j \in \{0,1\}$ and $x \in \pi^{-1}((-\infty, -N] \cup \{0\} \cup [N, \infty))$. By [8, Lem. 2.3] and [31, Lem. 4.4], there exists $\tilde{h} \in A(S)$ such that $\tilde{h}(x) = \bar{h}(x)$ for $x \in \pi^{-1}((-\infty, -N] \cup \{0\} \cup [N, \infty))$, and $f_i < \tilde{h} < g_j$ on S for all $i, j \in \{0,1\}$. Because $\pi^{-1}([-N, N])$ is compact, there exists $\varepsilon > 0$ such that

$$f_i(x) + \varepsilon < \tilde{h}(x) < g_j(x) - \varepsilon,$$

for any $i, j \in \{0,1\}$ and $x \in \pi^{-1}([-N, N])$. Set $\bar{r}(x) = \tilde{h}(x) - \bar{h}(x)$ for $x \in S$. Then it follows that $\mathrm{supp}(\bar{r}) \subset \pi^{-1}((-N, N) \setminus \{0\})$. By (1) of the conditions for $\mathcal{G}_{00}$, we have $r \in \mathcal{G}_{00}$ such that $r(x) \approx_\varepsilon \bar{r}(x)$ for $x \in S$ and $\mathrm{supp}(r) \subset \pi^{-1}((-N, N) \setminus \{0\})$. The element $h = \bar{h} + r \in \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$ satisfies $f_i < h < g_j$ for all $i, j \in \{0,1\}$. □

For an infinite supernatural number $\mathfrak{n}$, consider the ordered abelian group $(\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n}), \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+)$ introduced in the proof above. Define an ordered group $G_{\mathbb{D}_\mathfrak{n}}$ and a subgroup $G_\mathbb{Z} \subset G_{\mathbb{D}_\mathfrak{n}}$ by

$$G_{\mathbb{D}_\mathfrak{n}} = \big\{\xi \oplus g \in (\mathbb{D}_\mathfrak{n}^\infty \oplus \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})) : \exists\, \varepsilon > 0, \text{for any } x \in \pi^{-1}((-\varepsilon, \varepsilon)),\ L(\xi)(x) = g(x)\big\},$$
$$G_\mathbb{Z} = (\mathbb{Z}^\infty \oplus \mathcal{R}(\mathbb{Z})) \cap G_{\mathbb{D}_\mathfrak{n}}.$$

Define positive cones for $G_\mathbb{Z}$ and $G_{\mathbb{D}_\mathfrak{n}}$ by

$$G^+_{\mathbb{D}_\mathfrak{n}} = \{0\} \cup \big\{\xi \oplus g \in G_{\mathbb{D}_\mathfrak{n}} : g \in \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})^+ \setminus \{0\}\big\},$$
$$G^+_\mathbb{Z} = G_\mathbb{Z} \cap G^+_{\mathbb{D}_\mathfrak{n}}.$$

Note that $(G_\mathbb{Z}, G^+_\mathbb{Z})$ is independent of the choice of infinite supernatural number $\mathfrak{n}$. It is straightforward to check that $(G_{\mathbb{D}_\mathfrak{n}}, G^+_{\mathbb{D}_\mathfrak{n}})$ and $(G_\mathbb{Z}, G^+_\mathbb{Z})$ are torsion-free ordered abelian groups such that $(G_{\mathbb{D}_\mathfrak{n}}, G^+_{\mathbb{D}_\mathfrak{n}})$ is isomorphic to $(G_\mathbb{Z} \otimes \mathbb{D}_\mathfrak{n}, G_\mathbb{Z} \otimes \mathbb{D}^+_\mathfrak{n})$ (as ordered abelian groups). Since $\widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$ is a dimension group (see the proof of Lemma 3.5), so also is $G_{\mathbb{D}_\mathfrak{n}}$, for any infinite supernatural number $\mathfrak{n}$, which means that $(G_\mathbb{Z}, G^+_\mathbb{Z})$ is a rational dimension group.

Define an automorphism σ of $(G_{\mathbb{D}_\mathfrak{n}}, G^+_{\mathbb{D}_\mathfrak{n}})$ by

$$\sigma(\xi \oplus g) = (\sigma_{\mathbb{D}^\infty_\mathfrak{n}}(\xi) \oplus \sigma_S(g)), \quad \text{for } \xi \oplus g \in G_{\mathbb{D}_\mathfrak{n}}.$$

From $\sigma_S(\mathcal{R}(\mathbb{Z})^+) = \mathcal{R}(\mathbb{Z})^+$, it follows that $\sigma(G^+_\mathbb{Z}) = G^+_\mathbb{Z}$. Then, using the same symbol, we may regard σ as an automorphism of $(G_\mathbb{Z}, G^+_\mathbb{Z})$.

Set $1_0 = (\delta_{0,n})_{n\in\mathbb{Z}} \in \mathbb{Z}^\infty$, using the Kronecker delta δ_{mn} for $m, n \in \mathbb{Z}$, and set $u = (1_0 \oplus 1) \in G_\mathbb{Z}$. Although u is not necessarily an order unit, we may consider a kind of state space $S_u(H)$ for an ordered abelian subgroup (H, H^+) of $(G_{\mathbb{D}_\mathfrak{n}}, G^+_{\mathbb{D}_\mathfrak{n}})$

with $u \in H^+$ defined by

$$S_u(H) = \{\varphi\colon H \to \mathbb{R} : \text{a positive group homomorphism with } \varphi(u) = 1\}.$$

Consider $S_u(H)$ with the relative topology from the Cartesian product $\mathbb{R}^H$ (the topology of pointwise convergence). Assuming $\sigma(H^+) = H^+$, for $\alpha = e^{-\beta} \in \mathbb{R}$, consider the subset $S_\sigma^\alpha(H)$ of $S_u(H)$ and the bundle $S_\sigma(H)$ over $\mathbb{R}$ defined by

$$S_\sigma^\alpha(H) = \{\varphi \in S_u(H) : \varphi \circ \sigma = \alpha\varphi\},$$
$$S_\sigma(H) = \{(\varphi, \beta) \in S_u(H) \times \mathbb{R} : \varphi \in S_\sigma^\alpha(H)\}.$$

Although $\mathbb{R}^H$ is not locally compact, we shall show that $S_\sigma(G_\mathbb{Q})$ is locally compact, in Proposition 3.6(i) below. Define a projection $\pi_{\sigma H}\colon S_\sigma(H) \to \mathbb{R}$ by $\pi_{\sigma H}(\varphi, \beta) = \beta$. To simplify notation, let us write $\pi_\mathbb{Z}$ and $\pi_{\mathbb{D}_\mathfrak{n}}$ for $\pi_{\sigma G_\mathbb{Z}}$ and $\pi_{\sigma G_{\mathbb{D}_\mathfrak{n}}}$.

For a supernatural number $\mathfrak{n}$, let $\iota_\mathfrak{n}$ denote the canonical embedding of $G_\mathbb{Z}$ into $G_{\mathbb{D}_\mathfrak{n}}$, and $\iota_\mathfrak{n}^*$ the induced map from $S_u(G_{\mathbb{D}_\mathfrak{n}})$ to $S_u(G_\mathbb{Z})$ defined by $\iota_\mathfrak{n}^*(\varphi) = \varphi \circ \iota_\mathfrak{n}$ for $\varphi \in S_u(G_{\mathbb{D}_\mathfrak{n}})$. Define a continuous map $\iota_{\mathfrak{n}\sigma}^*$ from $S_\sigma(G_{\mathbb{D}_\mathfrak{n}})$ to $S_\sigma(G_\mathbb{Z})$ by $\iota_{\mathfrak{n}\sigma}^*(\varphi, \beta) = (\iota_\mathfrak{n}^*(\varphi), \beta)$ for $(\varphi, \beta) \in S_\sigma(G_{\mathbb{D}_\mathfrak{n}})$. For $s \in S$ and $g \in G_{\mathbb{D}_\mathfrak{n}}$ with $g = \xi \oplus f$ for some $\xi \in \mathbb{D}_\mathfrak{n}^\infty$ and $f \in \widetilde{\mathcal{R}}(\mathbb{D}_\mathfrak{n})$, set $\hat{s}(g) = f(s)$ and regard $\hat{s}$ as an element of $S_\sigma^\alpha(G_{\mathbb{D}_\mathfrak{n}})$ for $\alpha = e^{-\pi(s)}$. Define a continuous map $\Psi_{\mathbb{D}_\mathfrak{n}}$ from S to $S_\sigma(G_{\mathbb{D}_\mathfrak{n}})$ by $\Psi_{\mathbb{D}_\mathfrak{n}}(s) = (\hat{s}, \pi(s))$ for $s \in S$.

Proposition 3.6. *As in Remark 3.3, let* $\mathfrak{u}$ *denote the supernatural number corresponding to the universal UHF algebra, i.e.,* $\mathbb{D}_\mathfrak{u} = \mathbb{Q}$.

(i) *The pair* $(S_\sigma(G_\mathbb{Q}), \pi_\mathbb{Q})$ *is a proper simplex bundle.*

(ii) *The induced map* $\iota_{\mathfrak{u}\sigma}^*\colon S_\sigma(G_\mathbb{Q}) \to S_\sigma(G_\mathbb{Z})$ *is a homeomorphism such that* $\pi_\mathbb{Z} \circ \iota_{\mathfrak{u}\sigma}^* = \pi_\mathbb{Q}$ *and the restriction* $\iota_{\mathfrak{u}\sigma}^*|_{\pi_\mathbb{Q}^{-1}(\beta)}$ *is affine for each* $\beta \in \mathbb{R}$. *Therefore, the pair* $(S_\sigma(G_\mathbb{Z}), \pi_\mathbb{Z})$ *is also a proper simplex bundle that is isomorphic to* $(S_\sigma(G_\mathbb{Q}), \pi_\mathbb{Q})$.

(iii) $\Psi_\mathbb{Z}\colon S \to S_\sigma(G_\mathbb{Z})$ *is an isomorphism of simplex bundles.*

To prove the proposition, we shall consider the subgroup of $G_\mathbb{Q}$ for which u is an order unit, and its elements of compact support. Let $\rho\colon G_\mathbb{Q} \to \widetilde{\mathcal{R}}(\mathbb{Q})$ denote the standard projection defined by $\rho(\xi \oplus f) = f$. Set $G_\mathbb{Q}^{u+} = \{g \in G_\mathbb{Q}^+ :$ there exists $N \in \mathbb{N}$ such that $x \le Nu\}$, $G_\mathbb{Q}^{c+} = \{g \in G_\mathbb{Q}^+ : \operatorname{supp}(\rho(g))$ is compact$\}$, $G_\mathbb{Q}^u = G_\mathbb{Q}^{u+} - G_\mathbb{Q}^{u+}$, and $G_\mathbb{Q}^c = G_\mathbb{Q}^{c+} - G_\mathbb{Q}^{c+}$. It is straightforward to check that $(G_\mathbb{Q}^u, G_\mathbb{Q}^{u+})$ is an ordered abelian group with order unit u. Since $(G_\mathbb{Q}, G_\mathbb{Q}^+)$ is a dimension group, we also see that $(G_\mathbb{Q}^u, G_\mathbb{Q}^{u+})$ has the RIP. Let $S(G_\mathbb{Q}^u)$ denote the state space of $G_\mathbb{Q}^u$, a Choquet simplex (see for example [42, Thm. 10.17], [17]).

To simplify notation, set $Q_0 = \mathbb{Q}[e^x, e^{-x}]$ and $Q = \mathbb{Q}[e^{-x}, (1 - e^{-x})]$. For $\xi = (\xi_n)_{n\in\mathbb{Z}} \in \mathbb{Q}^\infty$, define $\Sigma\xi \in Q_0$ by $\Sigma\xi(x) = \sum_{n\in\mathbb{Z}} \xi_n e^{nx}$. Let $\varphi \in S(G_\mathbb{Q}^u)$. Note

that, since $\varphi(Ng) = N\varphi(g)$ for any $g \in G_{\mathbb{Q}}^u$ and $N \in \mathbb{N}$, we have $\varphi(\lambda g) = \lambda\varphi(g)$ for $\lambda \in \mathbb{Q}$ and $g \in G_{\mathbb{Q}}^u$.

Lemma 3.7. *Suppose that $\varphi, \psi \in S(G_{\mathbb{Q}}^u)$, $\lambda > 0$, and $\beta \in \mathbb{R}$ satisfy $\psi(g) \le \lambda\varphi(g)$ for any $g \in G_{\mathbb{Q}}^{u+}$, and $\varphi(\sigma(g)) = e^{-\beta}\varphi(g)$ for any $g \in G_{\mathbb{Q}}^c$. Then it follows that $\psi(\sigma(g)) = e^{-\beta}\psi(g)$ for any $g \in G_{\mathbb{Q}}^c$.*

Proof. For $\eta = (\eta_n)_{n\in\mathbb{Z}}$ and $\xi = (\xi_n)_{n\in\mathbb{Z}} \in \mathbb{Q}^\infty$, let $\eta * \xi$ denote the convolution product of η and ξ (i.e., $\eta*\xi = (\sum_{k\in\mathbb{Z}} \eta_k\xi_{n-k})_n$). For $q \in Q_0$ and $g = (\xi \oplus f) \in G_{\mathbb{Q}}^c$, setting $q^\infty = (q_n)_n \in \mathbb{Q}^\infty$ such that $\Sigma q^\infty = q$, we define $q \cdot g \in G_{\mathbb{Q}}^c$ by $q \cdot g = (q^\infty * \xi \oplus q \circ \pi f)$. Note that, since $\varphi(e^{-x} \cdot g) = e^{-\beta}\varphi(g)$ for any $g \in G_{\mathbb{Q}}^c$, it follows that $\varphi(q \cdot g) = q(\beta)\varphi(g)$ for any $q \in Q_0$ and $g \in G_{\mathbb{Q}}^c$.

Fix $g \in G_{\mathbb{Q}}^{c+}$ and define $\psi_g(q) = \psi(q \cdot g)$ for $q \in Q_0$. Thus one obtains the Cauchy–Schwarz inequality

$$\psi_g(qr)^2 \le \psi_g(q^2)\psi_g(r^2)$$

for any $q, r \in Q_0$. Choose a sequence $\alpha_n \in \mathbb{Q}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \alpha_n = e^{-\beta}$, and set $\tilde{q}_n = e^{-x} - \alpha_n \in Q_0$. Then we have

$$\begin{aligned}\psi_g(\tilde{q}_n)^2 &\le \psi_g(\tilde{q}_n^2)\psi_g(1) \le \lambda^{-1}\varphi(\tilde{q}_n^2 \cdot g)\psi_g(1)\\ &= \lambda^{-1}\tilde{q}_n^2(\beta)\varphi(g)\psi(g) \to 0 \quad (n \to \infty),\end{aligned}$$

which implies that $\lim_{n\to\infty} \psi(e^{-x} \cdot g) - \alpha_n\psi(g) = 0$. Since $G_{\mathbb{Q}}^c = G_{\mathbb{Q}}^{c+} - G_{\mathbb{Q}}^{c+}$, we have $\psi(e^{-x} \cdot g) = e^{-\beta}\psi(g)$ for any $g \in G_{\mathbb{Q}}^c$. □

Proof of Proposition 3.6. (i) First we show that $\pi_{\mathbb{Q}}\colon S_\sigma(G_{\mathbb{Q}}) \to \mathbb{R}$ is a proper map. Let $K \subset \mathbb{R}$ be a compact set, and $s_\lambda = (\varphi_\lambda, \beta_\lambda) \in \pi_{\mathbb{Q}}^{-1}(K)$, $\lambda \in \Lambda$ be a universal net (see [58] for the definition). For $g \in G_{\mathbb{Q}}$ such that $\rho(g)$ is bounded on S, there exists $c \in \mathbb{N}$ such that $|\varphi_\lambda(g)| < c$ for any $\lambda \in \Lambda$. Then the net $\varphi_\lambda(g)$, $\lambda \in \Lambda$, converges to a point in $[-c, c]$. For a general $g \in G_{\mathbb{Q}}$, there are g_0, g_+, $g_- \in G_{\mathbb{Q}}$ such that $g = g_- + g_0 + g_+$, and $\rho(g_0)$, $\rho(\sigma^N(g_-))$, and $\rho(\sigma^{-N}(g_+))$ are bounded on S. By $(\varphi_\lambda, \beta_\lambda) \in \pi_{\mathbb{Q}}^{-1}(K)$ and $\varphi_\lambda(g_\pm) = e^{\pm N\beta_\lambda}\varphi_\lambda(\sigma^{\mp N}(g_\pm))$, it follows that $\varphi_\lambda(g)$, $\lambda \in \Lambda$, converges in $\mathbb{R}$. Set $\varphi(g) = \lim_{\lambda\to\infty} \varphi_\lambda(g)$. Setting $\beta = \lim_{\lambda\to\infty} \beta_\lambda \in K$, we have $\varphi(\sigma(g)) = e^{-\beta}\varphi(g)$ for any $g \in G_{\mathbb{Q}}$, which implies that s_λ converges to $(\varphi, \beta) \in \pi_{\mathbb{Q}}^{-1}(K)$. Thus, $\pi_{\mathbb{Q}}^{-1}(K)$ is compact.

For $s = (\varphi, \beta) \in S_\sigma(G_{\mathbb{Q}})$, we see that $\pi_{\mathbb{Q}}^{-1}([\beta-\varepsilon, \beta+\varepsilon])$ for $\varepsilon > 0$ is a compact neighborhood of S, which means that $S_\sigma(G_{\mathbb{Q}})$ is locally compact.

It remains to show that $\pi_{\mathbb{Q}}^{-1}(\beta)$ is a Choquet simplex for any $\beta \in \mathbb{R}$. Fix $\beta \in \mathbb{R}$. Denote by ι_u the canonical embedding of $G_{\mathbb{Q}}^u$ into $G_{\mathbb{Q}}$ and define the induced map $\iota_u^*\colon \pi_{\mathbb{Q}}^{-1}(\beta) \to S(G_{\mathbb{Q}}^u)$ by $\iota_u^*(s) = \varphi \circ \iota_u$ for $s = (\varphi, \beta) \in \pi_{\mathbb{Q}}^{-1}(\beta)$. From the definition, it follows that ι_u^* is affine and continuous in the topology of pointwise convergence.

We see that points of $\pi_{\mathbb{Q}}^{-1}(\beta)$ are separated by $G_{\mathbb{Q}}^u$, because any $g \in G_{\mathbb{Q}}$ can be decomposed as $g = \sigma^{-N}(g_-) + g_0 + \sigma^N(g_+)$ for some g_0, g_-, $g_+ \in G_{\mathbb{Q}}^u$ and $N \in \mathbb{N}$. Thus ι_u^* is injective. Since a compact face of a Choquet simplex is also a Choquet simplex (see [42, Thm. 10.9]), it suffices to show that $\mathrm{Im}(\iota_u^*)$ is a face of $S(G_{\mathbb{Q}}^u)$.

Let $\varphi \in \mathrm{Im}(\iota_u^*)$, ψ, $\eta \in S(G_{\mathbb{Q}}^u)$, and $\lambda \in (0,1)$ satisfy $\varphi = \lambda\psi + (1-\lambda)\eta$. In the following argument, we first consider a specific $g \in G_{\mathbb{Q}}^u$ with the property $\rho(g) = \psi_{k+} \circ \pi$, $\psi_{k-} \circ \pi$, $(\psi_{k+} - \psi_{l+}) \circ \pi$, or $(\psi_{k-} - \psi_{l-}) \circ \pi$ for k, $l \in \mathbb{N}$ with $k > l$. Set $f_g = \rho(g)$, $Q_g = \{q \in Q : 0 \oplus (q \circ \pi) f_g \in G_{\mathbb{Q}}^u\}$, and $\psi_g(q) = \psi(0 \oplus (q \circ \pi) f_g)$ for $q \in Q_g$. Thus we obtain the Cauchy–Schwarz inequality, $\psi_g(qr)^2 \le \psi_g(q^2)\psi_g(r^2)$, for any q, $r \in Q_g$ satisfying qr, q^2, $r^2 \in Q_g$. By an argument similar to that in the proof of Lemma 3.7, we have

$$\psi(0 \oplus (q \circ \pi) f_g) = q(\beta)\psi(0 \oplus f_g) \quad \text{for any } q \in Q_g. \tag{1}$$

For general $g = \xi \oplus f \in G_{\mathbb{Q}}$ represented as $f = (f_- \psi_{k-}) \circ \psi + L(\xi)\psi_k \circ \pi + (f_+ \psi_{k+}) \circ \pi + f_0$ with f_+, $f_- \in Q$, and $f_0 \in \mathcal{G}_{00}$, define

$$\widetilde{\psi}(g) = f_-(\beta)\psi(0 \oplus \psi_{k-} \circ \pi) + \psi(\xi \oplus L(\xi)\psi_k \circ \pi) + f_+(\beta)\psi(0 \oplus \psi_{k+} \circ \pi) + \psi(0 \oplus f_0).$$

Then $\widetilde{\psi} \colon G_{\mathbb{Q}} \to \mathbb{R}$ is well defined (single valued). Indeed, if two elements g_1, $g_2 \in G_{\mathbb{Q}}$ are written as

$$g_i = \xi^{(i)} \oplus \big((f_-^{(i)} \psi_{k_i -}) \circ \pi + L(\xi^{(i)})(\psi_{k_i} \circ \pi) + (f_+^{(i)} \psi_{k_i +}) \circ \pi + f_0^{(i)}\big), \quad i = 1, 2,$$

for some $f_-^{(i)}$, $f_+^{(i)} \in Q$, and $f_0^{(i)} \in G_{00}$, and if $g_1 = g_2$, then we may assume that $k_1 \le k_2$ and $f_\pm^{(1)} = f_\pm^{(2)}$ without loss of generality. Applying (1) to $f_g = (\psi_{k_2 \pm} - \psi_{k_1 \pm}) \circ \pi$ we have

$$\psi(0 \oplus f_\pm(\psi_{k_2 \pm} - \psi_{k_1 \pm}) \circ \pi) = f_\pm(\beta)\psi(0 \oplus (\psi_{k_2 \pm} - \psi_{k_1 \pm}) \circ \pi),$$

which implies $\widetilde{\psi}(g_2) - \widetilde{\psi}(g_1) = \psi(g_2 - g_1) = 0$. Similarly, applying (1) to $f_g = \psi_{k\pm} \circ \pi$ we see that $\widetilde{\psi}(g) = \psi(g)$ for any $g \in G_{\mathbb{Q}}^u$. It is not hard to see that $\widetilde{\psi} \colon G_{\mathbb{Q}} \to \mathbb{R}$ is a group homomorphism such that $\widetilde{\psi}(u) = 1$. In the following paragraph we shall show that $\widetilde{\psi}$ is positive.

Let $g = \xi \oplus f \in G_{\mathbb{Q}}$ be such that $f > 0$ and let $k \in \mathbb{N}$, f_+, $f_- \in Q$, and $f_0 \in \mathcal{G}_{00}$ satisfy $f = (f_- \psi_{k-}) \circ \pi + L(\xi)(\psi_k \circ \pi) + (f_+ \psi_{k+} \circ \pi) + f_0$. Let $N \in \mathbb{N}$ and continuous functions e_+, e_- on $\mathbb{R}$ satisfy $\mathrm{supp}(f_0) \subset [-N, N]$, $0 \le e_\pm \le 1$, $\mathrm{supp}(e_\pm) \subset \pm[N, \infty)$, and $e_\pm|_{\pm[N+1,\infty)} = 1$. Set

$$z = ((1 - e_-) f_- \psi_{k-}) \circ \pi + L(\xi)\psi_k \circ \pi + ((1 - e_+) f_+ \psi_{k+}) \circ \pi + f_0 \in A(S).$$

Since $f > 0$, we may assume that $N \in \mathbb{N}$, e_+, and e_- also satisfy $f_\pm e_\pm \ge 0$, and that $z \ge 0$. Since there exists (a large) $M \in \mathbb{N}$ such that $e^{\mp Mx} f_\pm$ are bounded

on $\pm[0,\infty)$, by approximating $e^{\pm Mx}(1-e_\pm)$ on $\pm[0,\infty)$ uniformly, we obtain sequences d_{+n}, $d_{-n} \in Q$ such that

$$\sup_{x\in\pm[0,\infty)} |d_{\pm n} f_\pm(x) - (1-e_\pm) f_\pm(x)| < 1/n.$$

Set

$$y_n = (f_-(1-d_{-n})\psi_{k-}) \circ \pi + (f_+(1-d_{+n})\psi_{k\pm}) \circ \pi \in \widetilde{\mathcal{R}}(\mathbb{Q}),$$
$$z_n = (f_- d_{-n}\psi_{k-}) \circ \pi + L(\xi)\psi_k \circ \pi + (f_+ d_{+n}\psi_{k+}) \circ \pi + f_0 \in \widetilde{\mathcal{R}}(\mathbb{Q}).$$

Then it follows that $0 \oplus y_n$, $\xi \oplus z_n \in G_\mathbb{Q}$ and $g = (0\oplus y_n) + (\xi \oplus z_n)$ for any $n \in \mathbb{N}$. Since $f_\pm(1-d_\pm)$ converge uniformly to $e_\pm f_\pm$ on $\pm[0,\infty)$, we have that

$$\begin{aligned}\liminf_{n\to\infty} \widetilde{\psi}(0\oplus y_n) = \liminf_{n\to\infty} & f_-(1-d_{-n})(\beta)\psi(0\oplus\psi_{k-}\circ\pi) \\ & + f_+(1-d_{+n})(\beta)\psi(0\oplus\psi_{k+}\circ\pi) \geq 0.\end{aligned}$$

Since z_n converges uniformly to $z \geq 0$ on S, and $\mathrm{supp}(z)$ is compact, we have $\xi \oplus z_n \in G_\mathbb{Q}^u$ and $\liminf_{n\to\infty} \widetilde{\psi}(\xi\oplus z_n) = \liminf_{n\to\infty} \psi(\xi\oplus z_n) \geq 0$. Thus we conclude that $\widetilde{\psi}(g) \geq 0$.

By Lemma 3.7 and $\psi(g) \leq \lambda^{-1}\varphi(g)$ for $g \in G_\mathbb{Q}^{u+}$, we see that $\widetilde{\psi}(\sigma(g)) = e^{-\beta}\widetilde{\psi}(g)$ for any $g \in G_\mathbb{Q}$. From $\widetilde{\psi}(g) = \psi(g)$ for any $g \in G_\mathbb{Q}^u$, it follows that $\psi = \iota_u^*(\widetilde{\psi}) \in \mathrm{Im}(\iota_u^*)$. By the same argument, we also see that $\eta \in \mathrm{Im}(\iota_u^*)$, which means that $\mathrm{Im}(\iota_u^*)$ is a face of $S(G_\mathbb{Q}^u)$, as asserted, and the proof of (i) is complete.

(ii) For any $g \in G_\mathbb{Q}$, consider the smallest number $N_g \in \mathbb{N}$ such that $N_g g \in G_\mathbb{Z}$. To show injectivity of $\iota_{\mathrm{u}\sigma}^*$, let $s = (\varphi, \pi_\mathbb{Q}(\varphi))$, $t = (\psi, \pi_\mathbb{Q}(\psi)) \in S_\sigma(G_\mathbb{Q})$ satisfy $\iota_{\mathrm{u}\sigma}^*(s) = \iota_{\mathrm{u}\sigma}^*(t)$. Then

$$N_g\varphi(g) = \varphi\circ\iota_\mathrm{u}(N_g g) = \psi\circ\iota_\mathrm{u}(N_g g) = N_g\psi(g),$$

for any $g \in G_\mathbb{Q}$, and it follows that $s = t$. To prove surjectivity of $\iota_{\mathrm{u}\sigma}^*$, let $s = (\varphi,\beta) \in S_\sigma(G_\mathbb{Z})$ and define a map $\widetilde{\varphi}\colon G_\mathbb{Q} \to \mathbb{R}$ by $\widetilde{\varphi}(g) = \frac{1}{N_g}\varphi(N_g g)$ for $g \in G_\mathbb{Q}$. From $\widetilde{\varphi}(g+h) = \frac{1}{N_{g+h}N_gN_h}\varphi(N_{g+h}N_gN_h(g+h)) = \widetilde{\varphi}(g) + \widetilde{\varphi}(h)$ for any g, $h \in G_\mathbb{Q}$, it follows that $\widetilde{\varphi}$ is a group homomorphism. It is straightforward to see that $\widetilde{\varphi}$ is a positive group homomorphism satisfying $\widetilde{\varphi}(u) = 1$. By $\widetilde{\varphi}(\sigma(g)) = \frac{1}{N_gN_{\sigma(g)}}\varphi(N_gN_{\sigma(g)}\sigma(g)) = \frac{1}{N_gN_{\sigma(g)}}e^{-\beta}\varphi(N_gN_{\sigma(g)}g) = e^{-\beta}\widetilde{\varphi}(g)$ for any $g \in G_\mathbb{Q}$, we have $(\widetilde{\varphi},\beta) \in S_\sigma(G_\mathbb{Q})$ and $\iota_{\mathrm{u}\sigma}^*((\widetilde{\varphi},\beta)) = (\varphi,\beta)$.

By the same argument as in (i), we see that $S_\sigma(G_\mathbb{Z})$ is also locally compact. From the definition of ι_u^*, it follows that $\iota_{\mathrm{u}\sigma}^*$ is a bijective continuous map such that $\pi_\mathbb{Z}\circ\iota_{\mathrm{u}\sigma}^* = \pi_\mathbb{Q}$ and $\iota_{\mathrm{u}\sigma}^*|_{\pi_\mathbb{Q}^{-1}(\beta)}$ is affine for any $\beta \in \mathbb{R}$. Because of Lemma 3.4, $\iota_{\mathrm{u}\sigma}^*$ is an isomorphism of simplex bundles and $(S_\sigma(G_\mathbb{Z}), \pi_\mathbb{Z})$ is also proper.

(iii) As $\Psi_{\mathbb{Z}} = \iota^*_{\mathfrak{u}\sigma} \circ \Psi_{\mathbb{Q}}$, it suffices to show that $\Psi_{\mathbb{Q}} \colon S \to S_\sigma(G_{\mathbb{Q}})$ is an isomorphism of simplex bundles. From the definition, it is straightforward to check that $\Psi_{\mathbb{Q}}$ is injective and continuous, and the restriction of $\Psi_{\mathbb{Q}}$ to $\pi^{-1}(\beta)$ is affine for any $\beta \in \mathbb{R}$. By Lemma 3.4, it is enough to show that $\Psi_{\mathbb{Q}}$ is surjective. However, this is done in the proof of [31, Lem. 4.12]. We sketch a proof for the convenience of the reader.

Let (φ, β) be in $S_\sigma(G_{\mathbb{Q}})$. If $g = \xi \oplus f \in G_{\mathbb{Q}}$ satisfies $f = 0 \in \widetilde{\mathcal{R}}(\mathbb{Q})$, then it follows that $ng \leq u$ for any $n \in \mathbb{N}$, which implies $\varphi(g) = 0$. Thus we obtain a positive group homomorphism $\varphi' \colon \widetilde{\mathcal{R}}(\mathbb{Q}) \to \mathbb{R}$ such that $\varphi' \circ \rho = \varphi$. Denote by $A_{\mathbb{R}}(S)$ the real Banach space, with the supremum norm, consisting of functions in $A(S)$ which have a limit at infinity. Since $\rho(G^u_{\mathbb{Q}}) \cap A_{\mathbb{R}}(S)$ is uniformly dense in $A_{\mathbb{R}}(S)$ (see [31, Lem. 4.11]), we can extend φ' to $A_{\mathbb{R}}(S)$ in such a way that $|\varphi'(f)| \leq \sup_{s\in S} |f(s)|$ for any $f \in A_{\mathbb{R}}(S)$. By the Hahn–Banach theorem, we can further extend φ' to the Banach space $C^u(S) = \{f \in C(S, \mathbb{R}) : \text{there exists } \lim_{s\to\pm\infty} f(s)\}$. Since $\varphi'(1) = 1$, the extension of φ' is also a positive bounded linear functional on $C^u(S)$. Thus we obtain a Borel measure m_φ on S such that

$$\varphi'(f) = \int_S f(x)\, dm_\varphi(x) \quad \text{for any } f \in C_0(S).$$

For $y \in \mathrm{Aff}(\pi^{-1}(\beta))$ such that $y > 0$, by [31, Lem. 4.4(2) and Lem. 4.11], we obtain a sequence $g_{yn} = (\xi_{yn} \oplus f_{yn}) \in G_{\mathbb{Q}}$, $n \in \mathbb{N}$, such that $\mathrm{supp}(f_{yn})$ is compact, $f_{yn} \geq 0$ for any $n \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \sup_{s\in S} |f_{yn}(s)| < \infty$, and $f_{yn}|_{\pi^{-1}(\beta)}$ converges to y uniformly. Let ω be a free ultrafilter on $\mathbb{N}$. For $y \in \mathrm{Aff}(\pi^{-1}(\beta))$ with $y > 0$, we define $\bar\varphi(y) = \lim_{n\to\omega} \varphi'(f_{yn})$. Then $\bar\varphi$ is independent of the choice of ξ_{yn} and f_{yn}. Indeed, if we suppose that $g'_{yn} = (\xi'_{yn} \oplus f'_{yn}) \in G_{\mathbb{Q}}$ is another choice, then for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sup_{s\in\pi^{-1}(\beta)} |f_{yn}(s) - f'_{yn}(s)| < \varepsilon$, for any $n > N$. Set $X_n = \mathrm{supp}(f_{yn}) \cup \mathrm{supp}(f'_{yn})$, which is compact for any $n \in \mathbb{N}$, and let $q_m \in \mathbb{Q}[e^x, e^{-x}]$, $m \in \mathbb{N}$, be a sequence such that $q_m|_{X_n}$ converges to the characteristic function χ_β at $\{\beta\}$ and such that $\sup_{s\in X_n} |q_m(s)|$, $m \in \mathbb{N}$, is a bounded sequence. Then, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{m\to\infty} \varphi'((q_m \circ \pi) f_{yn}) &= \int_S (\chi_\beta \circ \pi) f_{yn}\, dm_\varphi \\ &\approx_\varepsilon \int_S (\chi_\beta \circ \pi) f'_{yn}\, dm_\varphi = \lim_{m\to\infty} \varphi'((q_m \circ \pi) f'_{yn}),\end{aligned}$$

for any $n \geq N$. It follows that

$$|\varphi'(f_{yn} - f'_{yn})| \leq \limsup_{m\to\infty} (|\varphi'(q_m \circ \pi(f_{yn} - f'_{yn}))| + |(1 - q_m(\beta))\varphi'(f_{yn} - f'_{yn})|) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n\to\omega} \varphi'(f_{yn}) = \lim_{n\to\omega} \varphi'(f'_{yn})$.

Since $\bar{\varphi}$ is well defined, we see that $\bar{\varphi}(x+y) = \bar{\varphi}(x) + \bar{\varphi}(y)$ for any $x, y \in \mathrm{Aff}(\pi^{-1}(\beta))$ with $x, y > 0$. Because $\mathrm{Aff}(\pi^{-1}(\beta))$ is an ordered abelian group in the strict ordering, we can extend $\bar{\varphi}$ to a positive group homomorphism from $\mathrm{Aff}(\pi^{-1}(\beta))$ to $\mathbb{R}$. Thus there exist $s \in \pi^{-1}(\beta)$ and $\lambda \in (0, \infty)$ such that $\bar{\varphi}(y) = \lambda y(s)$ for any $y \in \mathrm{Aff}(\pi^{-1}(\beta))$; see [42, Thm. 7.1], for example. From the definition of $\bar{\varphi}$ it follows that

$$\lambda f(s) = \bar{\varphi}(f|_{\pi^{-1}(\beta)}) = \varphi(\xi \oplus f),$$

for any $g = \xi \oplus f \in G_{\mathbb{Q}}$ such that $\mathrm{supp}(f)$ is compact. For $g = \xi \oplus f \in G_{\mathbb{Q}}$ with $f \in A_0(S)$ and $\xi = 0$, by property (1) of $\mathcal{G}_{00}$, there exists a sequence $f_n \in \mathcal{G}_{00}$, $n \in \mathbb{N}$, which converges to f uniformly. Then it follows that $\varphi(0 \oplus f) = \lim_{n\to\infty} \varphi(0 \oplus f_n) = \lambda f(s)$. For $f \in Q$, there exists a large $N \in \mathbb{N}$ such that $(fe^{\mp Nx}\psi_{k\pm}) \circ \pi \in A_0(S)$ for any $k \in \mathbb{N}$. Then we have that $\varphi(0 \oplus (f\psi_{k\pm}) \circ \pi) = e^{\pm N\beta}\varphi(0 \oplus (e^{\mp Nx} f\psi_{k\pm}) \circ \pi) = \lambda f\psi_{k\pm}(\beta)$, which implies that $\varphi(g) = \lambda f(s)$ for any $g = \xi \oplus f \in G_{\mathbb{Q}}$. Since $\varphi(u) = 1$, it is clear that $\lambda = 1$, and thus we conclude that $\Psi_{\mathbb{Q}}(s) = (\varphi, \beta)$. □

§4. Rationally AF algebras

In order to realize a given rational dimension group at the level of K_0-groups for operator algebras, it seems plausible that a corresponding C*-algebra should become an AF-algebra after tensoring with a UHF-algebra. This tensoring procedure is called rationalization in [3, 76]. While the rationalization was determined by a single UHF-algebra in these previous works, we would like to use the term "rationally" to refer to the tensor products with two UHF-algebras, that are relatively prime in the natural sense. (Cf. Remark 3.3.)

We prepare some facts and notation concerning C*-algebras. For a C*-algebra C, we let $C^{\sim}$ denote the unitization of C, and C^1 denote the closed unit ball of C. We denote by id_C the identity automorphism of C. For a subset $F \subset C$, let $P(F)$ denote the set of all projections in F, and F^+ the set of all positive elements in F. When C is a unital C*-algebra, the symbol 1_C means the unit of C. For two elements x, $y \in C$, and $\varepsilon > 0$, we use the notation $x \approx_\varepsilon y$ if $\|x - y\| < \varepsilon$. For subsets F, G in C, and $\varepsilon > 0$, the notation $F \subset_\varepsilon G$ means that for any $x \in F$ there exists $y \in G$ with $y \approx_\varepsilon x$.

For a supernatural number $\mathfrak{n}$, let $M_{\mathfrak{n}}$ denote the uniformly hyperfinite (UHF) algebra of type $\mathfrak{n}$ (i.e., $(K_0(M_{\mathfrak{n}}), K_0(M_{\mathfrak{n}})^+, [1_{M_{\mathfrak{n}}}]_0) \cong (\mathbb{D}_{\mathfrak{n}}, \mathbb{D}_{\mathfrak{n}}^+, 1_{\mathbb{D}_{\mathfrak{n}}})$), and $\mathcal{K}$ the C*-algebra of all compact operators on a separable Hilbert space. For a natural number $n \in \mathbb{N}$, we let $\{e_{ij}^{(n)}\}_{i,j=1}^n$ or $\{e_{ij}\}_{i,j=1}^n$ denote the set of canonical matrix units of M_n.

For two C*-algebras C, D, and a $*$-homomorphism φ from C to D, denote the induced map from $K_0(C)$ to $K_0(D)$, a homomorphism of pre-ordered abelian groups (the positive cone of K_0 being the canonical image of the Murray–von Neumann semigroup – see [64]) by φ_*.

Definition 4.1. Given a C*-algebra A, we shall say that A is *rationally AF* (RAF) if $A \otimes M_{\mathfrak{p}}$ and $A \otimes M_{\mathfrak{q}}$ are AF-algebras for some relatively prime pair of supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$.

Lemma 4.2. *The following two statements hold:*

(i) *If A is an RAF-algebra, then $A \otimes M_{\mathfrak{n}}$ is approximately finite-dimensional for every infinite supernatural number $\mathfrak{n}$.*

(ii) *Any RAF-algebra has trivial K_1-group.*

Proof. (i) Since $A \otimes M_n$ is also an RAF-algebra for any $n \in \mathbb{N}$, to show $A \otimes M_{\mathfrak{n}}$ is approximately finite-dimensional, it suffices to show that for any finite subset F of A and $\varepsilon > 0$ there exists a finite-dimensional C*-subalgebra B of $A \otimes M_{\mathfrak{n}}$ such that $F \otimes 1_{M_{\mathfrak{n}}} \subset_\varepsilon B$.

Taking large natural numbers $m_{\mathfrak{p}}$ and $m_{\mathfrak{q}}$ with $m_{\mathfrak{p}} | \mathfrak{p}$ and $m_{\mathfrak{q}} | \mathfrak{q}$, we have finite-dimensional C*-subalgebras $A_{\mathfrak{i}}$ of $A \otimes M_{\mathfrak{i}}$ for $\mathfrak{i} = \mathfrak{p}, \mathfrak{q}$ such that $F \otimes 1_{M_{\mathfrak{i}}} \subset_\varepsilon A_{\mathfrak{i}}$ for both $\mathfrak{i} = \mathfrak{p}, \mathfrak{q}$. Since $m_{\mathfrak{p}}$ and $m_{\mathfrak{q}}$ are relatively prime, there exist natural numbers a, b, and N such that $am_{\mathfrak{p}} + bm_{\mathfrak{q}} = N$ and $N | \mathfrak{n}$ (cf. proof of Lemma 3.1). Therefore, there exists a unital embedding Φ of $M_{m_{\mathfrak{p}}} \oplus M_{m_{\mathfrak{q}}}$ into $M_{\mathfrak{n}}$. Identifying $A \otimes (M_{m_{\mathfrak{p}}} \oplus M_{m_{\mathfrak{q}}})$ with $(A \otimes M_{m_{\mathfrak{p}}}) \oplus (A \otimes M_{m_{\mathfrak{q}}})$, we obtain a finite-dimensional C*-subalgebra $B = (\mathrm{id}_A \otimes \Phi)(A_{\mathfrak{p}} \oplus A_{\mathfrak{q}})$ of $A \otimes M_{\mathfrak{n}}$ which satisfies the above condition.

(ii) Because of the Künneth theorem [69, Thm. 2.14] (see also [2, Thm. 23.1.3]), we see that $K_1(A) \otimes K_0(M_{\mathfrak{i}}) \cong K_1(A \otimes M_{\mathfrak{i}}) = 0$ for both $\mathfrak{i} = \mathfrak{p}, \mathfrak{q}$. From (i) of Lemma 3.1, it follows that $K_1(A)$ is torsion-free. Therefore the canonical embedding of $K_1(A)$ into $K_1(A) \otimes \mathbb{D}_{\mathfrak{p}} = 0$ is injective, and so $K_1(A) = 0$. □

Note that any RAF-algebra is AF-embeddable, and so it is stably finite. Hence the ordered K_0-group of any RAF-algebra is a partially ordered abelian group. In combination with Lemma 3.1(iii), the following results show that the ordered K_0-group of a $\mathcal{Z}$-absorbing RAF-algebra is an ordered abelian group.

A typical example of an RAF-algebra is the Jiang–Su algebra $\mathcal{Z}$, which is constructed as a unital separable simple monotracial RAF-algebra whose ordered K_0-group is $(\mathbb{Z}, \mathbb{Z}^+)$. Gong, Jiang, and Su [37, Thm. 1] showed that the ordered K_0-group of a unital simple $\mathcal{Z}$-absorbing C*-algebra is weakly unperforated. Part (ii) of the following Proposition 4.3 is a variant of their argument for RAF-algebras in the absence of simplicity.

In contrast, it is noteworthy that Villadsen algebras of the first kind are examples of non-type-I simple RAF-algebras which do not absorb the Jiang–Su algebra $\mathcal{Z}$ tensorially [74, 73].

Proposition 4.3. *The following two statements hold:*

(i) *Let A be a C*-algebra and $\mathfrak{n}$ a supernatural number. Then the pre-ordered abelian groups $(K_0(A\otimes M_{\mathfrak{n}}), K_0(A\otimes M_{\mathfrak{n}})^+)$ and $(K_0(A)\otimes \mathbb{D}_{\mathfrak{n}}, K_0(A)\otimes \mathbb{D}_{\mathfrak{n}}{}^+)$ are isomorphic (in a natural way).*

(ii) *Let A be a $\mathcal{Z}$-absorbing RAF-algebra. Then the partially ordered abelian group $(K_0(A), K_0(A)^+)$ is unperforated, and, furthermore, is a rational dimension group.*

Proof. (i) It is enough to note that, for any natural number n, the Murray–von Neumann semigroup of $A\otimes M_n$ is the same (by definition – with respect to the map $A\to A\otimes e_{11}\subset A\otimes M_n$) as that of A (the image of which in $K_0(A)$ is the positive cone), and the semigroup endomorphism corresponding to the map $A\to A\otimes 1_{M_n}\subset A\otimes M_n$ is just multiplication by n. (Recall that both the Murray–von Neumann semigroup functor and the K_0 functor preserve inductive limits.)

(ii) If A is an RAF-algebra, then so also is $A\otimes\mathcal{K}$. Since the ordered K_0-groups $(K_0(A), K_0(A)^+)$ and $(K_0(A\otimes\mathcal{K}), K_0(A\otimes\mathcal{K})^+)$ are isomorphic, without loss of generality we may assume that A is stable.

Let ι denote the canonical embedding of A into $A\otimes\mathcal{Z}$ defined by $\iota(a)=a\otimes 1_{\mathcal{Z}}$ for $a\in A$, and ι_* the induced map by ι from $K_0(A)$ to $K_0(A\otimes\mathcal{Z})$. As in the proof of [37, Thm. 1], for $g\in K_0(A)$ it is enough to show that $\iota_*(g)\in K_0(A\otimes\mathcal{Z})^+$ if and only if $ng\in K_0(A)^+$ for some $n\in\mathbb{N}$. Indeed, if $x\in K_0(A\otimes\mathcal{Z})$ satisfies $nx\in K_0(A\otimes\mathcal{Z})^+$ for some $n\in\mathbb{N}$, then, since (by [69]) ι_* is a group isomorphism, there exists $g_x\in K_0(A)$ such that $\iota_*(g_x)=x$. Then we obtain $m\in\mathbb{N}$ such that $mng_x\in K_0(A)^+$ which implies $x\in K_0(A\otimes\mathcal{Z})^+$.

Suppose that $g\in K_0(A)$ satisfies $\iota_*(g)\in K_0(A\otimes\mathcal{Z})^+$. Recall that the Jiang–Su algebra is constructed as the inductive limit C*-algebra $\varinjlim(\mathcal{Z}_{p_nq_n},\varphi_n)$ of prime dimension drop algebras $\mathcal{Z}_{p_nq_n}$, $n\in\mathbb{N}$, and the connecting maps $\varphi_n\colon \mathcal{Z}_{p_nq_n}\to\mathcal{Z}_{p_{n+1}q_{n+1}}$, $n\in\mathbb{N}$, where p_n and q_n are relatively prime numbers; see [44]. Note that $A\otimes\mathcal{Z}_{p_nq_n}$ is stably finite for each $n\in\mathbb{N}$: then $(K_0(A\otimes\mathcal{Z}_{p_nq_n}), K_0(A\otimes\mathcal{Z}_{p_nq_n})^+)$ is a partially ordered abelian group. By the continuity of the functor K_0 (see [64, Thm. 6.3.2], for example), there exist embeddings $\psi_n\colon A\otimes\mathcal{Z}_{p_nq_n}\to A\otimes\mathcal{Z}$ such that $K_0(A\otimes\mathcal{Z})^+=\bigcup_{n=1}^{\infty}\psi_{n*}(K_0(A\otimes\mathcal{Z}_{p_nq_n})^+)$. Then there exist $N\in\mathbb{N}$ and $x\in K_0(A\otimes\mathcal{Z}_{p_Nq_N})^+$ such that $\iota_*(g)=\psi_{N*}(x)$. Let $\iota_N\colon A\to A\otimes\mathcal{Z}_{p_Nq_N}$ denote the canonical embedding defined by $\iota_N(a)=a\otimes 1_{\mathcal{Z}_{p_Nq_N}}$ for $a\in A$ and $\mathrm{ev}_0\colon A\otimes\mathcal{Z}_{p_Nq_N}\to A\otimes M_{p_N}$ the evaluation map determined by $\mathrm{ev}_0(f)\otimes 1_{M_{q_N}}=$

$f(0)$ for $f \in A \otimes \mathcal{Z}_{p_N q_N}$. Since $\mathrm{ev}_0 \circ \iota_N(a) = a \otimes 1_{M_{p_N}}$ for any $a \in A$, it follows that $\mathrm{ev}_{0*} \circ \iota_{N*}(g) = p_N g$. Since $\iota_{N*}(g) = x \in K_0(A \otimes \mathcal{Z}_{p_N q_N})^+$, we have $p_N g = \mathrm{ev}_{0*} \circ \iota_{N*}(g) \in K_0(A)^+$.

Conversely, suppose that $g \in K_0(A)$ satisfies $ng \in K_0(A)^+$ for some $n \in \mathbb{N}$. Let p and $q \in \mathbb{N} \setminus \{1\}$ be relatively prime natural numbers, and denote by $\iota_i \colon A \to A \otimes M_{i^\infty}$, $i = p, q$, the canonical embeddings defined by $\iota_i(a) = a \otimes 1_{M_{i^\infty}}$ for $a \in A$. Since $K_0(A \otimes M_{i^\infty})$, $i = p, q$, are ordered abelian groups, there exist a_i, $b_i \in K_0(A \otimes M_{i^\infty})^+$ such that $\iota_{i*}(g) = a_i - b_i$ for both $i = p, q$. Since $K_0(A \otimes M_{i^\infty})$ is unperforated and $n(a_i - b_i) = \iota_{i*}(ng) \in K_0(A \otimes M_{i^\infty})^+$ for $i = p, q$, it follows that $a_i - b_i \in K_0(A \otimes M_{i^\infty})^+$. Because $A \otimes M_{i^\infty}$, $i = p, q$, are stable AF-algebras, we obtain projections $y_i \in A \otimes M_{i^\infty}$, $i = p, q$, such that $[y_i]_0 = a_i - b_i$. Identifying $A \otimes M_{p^\infty} \otimes M_{q^\infty}$ with $A \otimes M_{q^\infty} \otimes M_{p^\infty}$ canonically, we have $[y_p \otimes 1_{M_{q^\infty}}]_0 = [y_q \otimes 1_{M_{p^\infty}}]_0$ in $K_0(A \otimes M_{p^\infty} \otimes M_{q^\infty})$. Since $A \otimes M_{p^\infty} \otimes M_{q^\infty}$ is an AF-algebra, there exists a projection $z \in C([0,1]) \otimes A \otimes M_{p^\infty} \otimes M_{q^\infty}$ such that $z(0) = y_p \otimes 1_{M_{q^\infty}}$ and $z(1) = y_q \otimes 1_{M_{p^\infty}}$. Considering the C*-algebra

$$\mathcal{Z}_\infty = \left\{ f \in C([0,1]) \otimes M_{p^\infty} \otimes M_{q^\infty} : f(0) \in M_{p^\infty} \otimes 1_{M_{q^\infty}},\ f(1) \in 1_{M_{p^\infty}} \otimes M_{q^\infty} \right\},$$

introduced in [65], we can regard z as a projection in $A \otimes \mathcal{Z}_\infty$. Denote by $\iota_\infty \colon A \to A \otimes \mathcal{Z}_\infty$ the canonical embedding defined by $\iota_\infty(a) = a \otimes 1_{\mathcal{Z}_\infty}$ for $a \in A$, and by $\mathrm{Ev}_i \colon A \otimes \mathcal{Z}_\infty \to A \otimes M_{i^\infty}$, $i = p, q$, the evaluation maps determined by $\mathrm{Ev}_p(f) \otimes 1_{M_{q^\infty}} = f(0)$ and $\mathrm{Ev}_q(f) \otimes 1_{M_{p^\infty}} = f(1)$ for $f \in A \otimes \mathcal{Z}_\infty$. Since $K_1(A \otimes M_{p^\infty} \otimes M_{q^\infty}) = 0$, the induced map $(\mathrm{Ev}_p \oplus \mathrm{Ev}_q)_* \colon K_0(A \otimes \mathcal{Z}_\infty) \to K_0(A \otimes M_{p^\infty}) \oplus K_0(A \otimes M_{q^\infty})$ is injective. Since

$$(\mathrm{Ev}_p \oplus \mathrm{Ev}_q)_*([z]_0 - \iota_{\infty *}(g)) = ([y_p]_0 \oplus [y_q]_0) - (\iota_{p*}(g) \oplus \iota_{q*}(g)) = 0,$$

considering the embedding $\iota_{\mathcal{Z}_\infty} \colon A \otimes \mathcal{Z}_\infty \to A \otimes \mathcal{Z}$ we conclude that $\iota_*(g) = \iota_{\mathcal{Z}_\infty *} \circ \iota_{\infty *}(g) \in K_0(A \otimes \mathcal{Z})^+$.

This shows that $(K_0(A), K_0(A)^+)$ is an unperforated partially ordered abelian group. On using once more that A is RAF, the second statement follows from (i). □

Our goal in this section is to provide a construction of RAF-algebras which exhausts all countable rational dimension groups as K_0-groups. Here we emphasize that the non-simple cases of RAF-algebras are required in order to realize the rational dimension group $(G_{\mathbb{Z}}, G_{\mathbb{Z}}^+)$ of Section 3. The construction is somewhat analogous to the construction of simple C*-algebras in [24] and [44].

As a corollary, we also show that a natural addition to the invariant in the non-stable case is exhausted (Corollary 4.8). In Section 5, we shall show that the augmented invariant (K_0 alone in the stable case) is complete.

Theorem 4.4. *For any countable rational dimension group (G, G^+), there exists a separable stable RAF-algebra A_G such that $A_G \otimes \mathcal{Z} \cong A_G$ and $(K_0(A_G), K_0(A_G)^+)$ is isomorphic to (G, G^+) as an ordered abelian group.*

Let (G, G^+) be a countable rational dimension group. For $i = 0, 1$, we obtain a stable AF-algebra $\mathscr{F}_i$ such that $(K_0(\mathscr{F}_i), K_0(\mathscr{F}_i)^+)$ is isomorphic to $(G \otimes \mathbb{D}_{(2+i)^\infty}, G \otimes \mathbb{D}_{(2+i)^\infty}{}^+)$ as an ordered abelian group. Because of the Elliott classification theorem [20] and Proposition 4.3(i), we can see that

$$\mathscr{F}_0 \otimes M_{3^\infty} \cong \mathscr{F}_1 \otimes M_{2^\infty}.$$

To simplify notation, we suppress the isomorphism between $\mathscr{F}_0 \otimes M_{3^\infty}$ and $\mathscr{F}_1 \otimes M_{2^\infty}$ in the rest of this section, and set $\mathscr{F} = \mathscr{F}_0 \otimes M_{3^\infty} = \mathscr{F}_1 \otimes M_{2^\infty}$. We denote by $\iota_\infty^{(i)}$ the canonical embedding of $\mathscr{F}_i$ into $\mathscr{F}$ defined by $\iota_\infty^{(i)}(a) = a \otimes 1_{M_{(3-i)^\infty}}$ for $i = 0, 1$ and $a \in \mathscr{F}_i$. The desired RAF-algebra A_G will be constructed as an inductive limit whose building blocks are type-I C*-subalgebras of the following generalized dimension drop algebra I_G.

We define a C*-algebra I_G by

$$I_G = \big\{ f \in C([0,1]) \otimes \mathscr{F} : f(i) \in \mathrm{Im}(\iota_\infty^{(i)}) \text{ for both } i = 0, 1 \big\}.$$

Proposition 4.5. *Let A_i, $i = 0, 1$, be two AF-embeddable C*-algebras. Suppose that there exist an AF-algebra A and embeddings $\iota^{(i)}$, $i = 0, 1$, of A_i into A. Then the C*-algebra J defined by*

$$J = \big\{ f \in C([0,1]) \otimes A : f(i) \in \mathrm{Im}(\iota^{(i)}) \text{ for both } i = 0, 1 \big\}$$

has the following properties:

(i) *Suppose that the induced maps $\iota_*^{(i)} \colon K_0(A_i) \to K_0(A)$, $i = 0, 1$, are injective. Let $\mathrm{ev}_i \colon J \to \mathrm{Im}(\iota^{(i)})$, $i = 0, 1$, denote the evaluation maps at these two points: $\mathrm{ev}_i(f) = f(i)$. Then, on identifying $K_0(\mathrm{Im}(\iota^{(i)}))$ with $\mathrm{Im}(\iota_*^{(i)})$, the induced maps $\mathrm{ev}_{i*} \colon K_0(J) \to \mathrm{Im}(\iota_*^{(i)})$, $i = 0, 1$, satisfy $\mathrm{ev}_{0*} = \mathrm{ev}_{1*}$ and $\mathrm{ev}_{0*} \colon K_0(J) \to \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)})$ is a group isomorphism.*

(ii) *If $\iota^{(i)}$, $i = 0, 1$, satisfy the assumption of* (i) *and $\iota_*^{(i)}(K_0(A_i)^+) = \iota_*^{(i)}(K_0(A_i)) \cap K_0(A)^+$ for both $i = 0, 1$, then ev_{0*} is an isomorphism of ordered abelian groups from $(K_0(J), K_0(J)^+)$ to $(\mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}), \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) \cap K_0(A)^+)$.*

(iii) *In particular, for $A = \mathscr{F}$, $A_i = \mathscr{F}_i$, and $\iota^{(i)} = \iota_\infty^{(i)}$, $i = 0, 1$, it follows that $(K_0(I_G), K_0(I_G)^+)$ is isomorphic to (G, G^+) as an ordered abelian group.*

Proof. (i) Denote by ι the canonical embedding of $C_0((0,1)) \otimes A$ into J. From the exact sequence

$$0 \to C_0((0,1)) \otimes A \xrightarrow{\iota} J \xrightarrow{\mathrm{ev}_0 \oplus \mathrm{ev}_1} \mathrm{Im}(\iota^{(0)}) \oplus \mathrm{Im}(\iota^{(1)}) \to 0,$$

we obtain the six-term exact sequence of Bott periodicity which implies the following exact sequence:

$$0 \to K_0(J) \xrightarrow{(\mathrm{ev}_0 \oplus \mathrm{ev}_1)_*} \mathrm{Im}(\iota_*^{(0)}) \oplus \mathrm{Im}(\iota_*^{(1)}) \xrightarrow{\partial} K_0(A) \to 0,$$

where ∂ is the exponential map from $K_0(\mathrm{Im}(\iota^{(0)})) \oplus K_0(\mathrm{Im}(\iota^{(1)}))$ to $K_1(C_0((0,1)) \otimes A) \cong K_0(A)$. Then it is not hard to check that $\partial(x \oplus y) = x - y$ for $x \in \mathrm{Im}(\iota_*^{(0)})$ and $y \in \mathrm{Im}(\iota_*^{(1)})$. Set $\Phi\colon \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) \to \mathrm{Im}(\iota_*^{(0)}) \oplus \mathrm{Im}(\iota_*^{(1)})$ by $\Phi(x) = x \oplus x$ for $x \in \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)})$. Then the group homomorphism $(\mathrm{ev}_0 \oplus \mathrm{ev}_1)_*$ is an isomorphism onto $\mathrm{Im}(\Phi)$. Thus we see that $\mathrm{ev}_{0*} = \Phi^{-1} \circ (\mathrm{ev}_0 \oplus \mathrm{ev}_1)_* = \mathrm{ev}_{1*}$ is a group isomorphism.

(ii) From the assumption $\iota_*^{(i)}(K_0(A_i)^+) = \iota_*^{(i)}(K_0(A_i)) \cap K_0(A)^+$, for $x \in \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) \cap K_0(A)^+$, there exist $N \in \mathbb{N}$ and projections $p_i \in A_i \otimes M_N$, $i = 0, 1$, such that $\iota_*^{(i)}([p_i]_0) = x$ for both $i = 0, 1$. Since $A \otimes M_N$ is an AF-algebra and $[\iota^{(0)} \otimes \mathrm{id}_{M_N}(p_0)]_0 = [\iota^{(1)} \otimes \mathrm{id}_{M_N}(p_1)]_0$ in $K_0(A)$, there exists a projection $\tilde{p} \in C([0,1]) \otimes A \otimes M_N$ such that $\tilde{p}(i) = \iota^{(i)} \otimes \mathrm{id}_{M_N}(p_i)$ for $i = 0, 1$. Regarding $\tilde{p}$ as a projection in $J \otimes M_N$, we have that $(\mathrm{ev}_0 \oplus \mathrm{ev}_1)_*([\tilde{p}]_0) = \iota_*^{(0)}([p]_0) \oplus \iota_*^{(1)}([p]_0) = x \oplus x$. Then it follows that $x = \Phi^{-1} \circ (\mathrm{ev}_0 \oplus \mathrm{ev}_1)_*([\tilde{p}]_0) = \mathrm{ev}_{0*}([\tilde{p}]_0) \in \mathrm{ev}_{0*}(K_0(J)^+)$, which implies that $\mathrm{ev}_{0*}(K_0(J)^+) = \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) \cap K_0(A)^+$.

(iii) For $i = 0, 1$, since $K_0(A_i)$ is torsion-free, it follows that the induced map $\iota_*^{(i)}\colon K_0(A_i) \to K_0(A) \cong K_0(A_i) \otimes \mathbb{D}_{(3-i)^\infty}$ is injective. By Proposition 4.3(i), it follows that

$$\begin{aligned}(K_0(A), K_0(A)^+) &\cong (K_0(A_0) \otimes \mathbb{D}_{3^\infty}, K_0(A_0) \otimes \mathbb{D}_{3^\infty}{}^+) \\ &\cong (K_0(A_1) \otimes \mathbb{D}_{2^\infty}, K_0(A_1) \otimes \mathbb{D}_{2^\infty}{}^+),\end{aligned}$$

as ordered abelian groups. Then, for $i = 0, 1$ and $x \in \mathrm{Im}(\iota_*^{(i)}) \cap K_0(A)^+$, there exists $d \in \mathbb{N}$ such that $(3-i)^d x \in \iota_*^{(i)}(K_0(A_i)^+)$. Since $(K_0(A_i), K_0(A_i)^+)$ is unperforated, it follows that $x \in \iota_*^{(i)}(K_0(A_i)^+)$, which implies that $\mathrm{Im}(\iota_*^{(i)}) \cap K_0(A)^+ = \iota_*^{(i)}(K_0(A_i)^+)$ for both $i = 0, 1$. Applying (ii), we see that $(K_0(I_G), K_0(I_G)^+)$ is isomorphic to $(\mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}), \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) \cap K_0(A)^+)$.

By Lemma 3.1(i), any rational dimension group is torsion-free, and by Lemma 3.1(ii), we have $\mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) = G \otimes \mathbb{D}_{2^\infty} \otimes 1_{\mathbb{D}_{3^\infty}} \cap G \otimes 1_{\mathbb{D}_{2^\infty}} \otimes \mathbb{D}_{3^\infty} \cong G$. Let $\Psi\colon G \to G \otimes \mathbb{D}_{2^\infty} \otimes \mathbb{D}_{3^\infty}$ denote the positive group homomorphism defined by $\Psi(g) = g \otimes 1_{\mathbb{D}_{2^\infty}} \otimes 1_{\mathbb{D}_{3^\infty}}$ for $g \in G$. The group isomorphism of (the proof

of) Lemma 3.1(ii) is just the map Ψ. Furthermore, since G is unperforated it is straightforward to check that $\Psi(G^+) = \Psi(G) \cap (G \otimes \mathbb{D}_{2^\infty} \otimes \mathbb{D}_{3^\infty})^+$. Therefore we conclude that

$$\begin{aligned}(\mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}), \mathrm{Im}(\iota_*^{(0)}) \cap \mathrm{Im}(\iota_*^{(1)}) \cap K_0(A)^+) \\ \cong (\Psi(G), \Psi(G) \cap (G \otimes \mathbb{D}_{2^\infty} \otimes \mathbb{D}_{3^\infty})^+) \cong (G, G^+).\end{aligned}$$ □

Before going into the proof, we collect some necessary well-known facts concerning finite-dimensional C*-algebras.

Lemma 4.6. *The following five statements hold:*

(i) *If natural numbers P, Q, R and a projection $p \in M_Q \otimes M_R$ satisfy $R \le \mathrm{rank}(p)$, $R | \mathrm{rank}(p)$, $\frac{\mathrm{rank}(p)}{R} < Q$, and $P | (Q - \frac{\mathrm{rank}(p)}{R})$, then there exists a unitary $u \in M_P \otimes M_Q \otimes M_R$ such that for any C*-algebra A, $a \in A$, and $b \in A \otimes M_P$,*

$$\mathrm{Ad}(1_{A^\sim} \otimes u)\big((a \otimes 1_{M_P} \otimes p) \oplus (b \otimes (1_{M_Q \otimes M_R} - p))\big) \in A \otimes 1_{M_P} \otimes M_Q \otimes 1_{M_R}.$$

(ii) *For $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $\delta > 0$ such that if A and B are C*-subalgebras of a unital C*-algebra $\mathscr{A}$ such that $\dim(B) \le N$ and $B^1 \subset_\delta A$, then there exists a unitary $u \in \mathscr{A}$ such that $u \approx_\varepsilon 1_{\mathscr{A}}$ and $\mathrm{Ad}\, u(B) \subset A$.*

(iii) *For $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $\delta > 0$ such that if a finite-dimensional C*-algebra A and two embeddings ι_0 and ι_1 of A into a unital C*-algebra $\mathscr{A}$ satisfy $\dim(A) \le N$ and $\|\iota_0(a) - \iota_1(a)\| < \delta$ for all $a \in A^1$, then there exists a unitary $u \in \mathscr{A}$ such that $\mathrm{Ad}\, u \circ \iota_0 = \iota_1$ and $\|u - 1_{\mathscr{A}}\| < \varepsilon$.*

(iv) *For $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $\delta > 0$ such that if A, B, and C are C*-subalgebras of a unital C*-algebra $\mathscr{A}$ satisfying $C \subset A \cap B$, $B^1 \subset_\delta A$, and $\dim(B), \dim(C) \le N$, then there exists a unitary $y \in \mathscr{A}$ such that $y \in C' \cap \mathscr{A}$, $\mathrm{Ad}\, y(B) \subset A$, and $\|y - 1_{\mathscr{A}}\| < \varepsilon$.*

(v) *Let A and B be C*-subalgebras of a unital C*-algebra $\mathscr{A}$. Suppose that A and B are finite-dimensional C*-algebras and unitaries u, $v \in \mathscr{A}$ satisfy $\{u, v\} \subset_{1/16} B + \mathbb{C}1_{\mathscr{A}}$ and $\mathrm{Ad}\, u(A) \cup \mathrm{Ad}\, v(A) \subset B$. Then there exists a unitary $w \in B + \mathbb{C}1_{\mathscr{A}}$ such that $\mathrm{Ad}\, wu(a) = \mathrm{Ad}\, v(a)$ for any $a \in A$.*

Proof. (i) Set a projection $e \in M_Q$ with $\mathrm{rank}(e) = \frac{\mathrm{rank}(p)}{R}$. By $P | (Q - \mathrm{rank}(e))$, we obtain a unital embedding $\iota\colon M_P \to (1_{M_Q} - e) M_Q (1_{M_Q} - e)$. Since $P\,\mathrm{rank}(p) = PR\,\mathrm{rank}(e)$, there exists a partial isometry $v \in M_P \otimes M_Q \otimes M_R$ such that $v^*v = 1_{M_P} \otimes p$ and $vv^* = 1_{M_P} \otimes e \otimes 1_{M_R}$. Denote by η the canonical embedding of M_P into $M_P \otimes (1_{M_Q \otimes M_R} - p)$ defined by $\eta(a) = a \otimes (1_{M_Q \otimes M_R} - p)$ for $a \in M_P$. Since the multiplicity of η is $QR - \mathrm{rank}(p)$, the same as that of $1_{M_P} \otimes \iota \otimes 1_{M_R}$,

there exists a partial isometry $w \in M_P \otimes M_Q \otimes M_R$ such that $w^*w = \eta(1_{M_P})$, $ww^* = 1_{M_P} \otimes \iota(1_{M_P}) \otimes 1_{M_R}$, and $\operatorname{Ad} w \circ \eta = 1_{M_P} \otimes \iota \otimes 1_{M_R}$. The unitary $u = v + w$ in $M_P \otimes M_Q \otimes M_R$ then satisfies the desired conditions. Indeed, for any C*-algebra A, a, $b' \in A$, and $x \in M_P$,

$$\begin{aligned}&\operatorname{Ad}(1_{A^\sim} \otimes u)\big((a \otimes 1_{M_P} \otimes p) \oplus (b' \otimes x \otimes (1_{M_Q \otimes M_R} - p))\big)\\ &\quad = a \otimes 1_{M_P} \otimes e \otimes 1_{M_R} + b' \otimes 1_{M_P} \otimes \iota(x) \otimes 1_{M_R} \in A \otimes 1_{M_P} \otimes M_Q \otimes 1_{M_R}.\end{aligned}$$

(ii) See [15, Lem. III 3.2].

(iii) Since A is finite-dimensional, we may identify A with $\bigoplus_{l=1}^{L} M_{k_l}$ for some $k_1, k_2, \dots, k_L \in \mathbb{N}$. For given $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $\delta \in (0, \varepsilon/4N)$ such that if two projections p and q in $\mathscr{A}$ satisfy $\|p - q\| < \delta$ then there exists a partial isometry $w \in \mathscr{A}$ such that $w^*w = p$, $ww^* = q$, and $w \approx_{\varepsilon/4N} p$. Let us show that δ is as required. Suppose that A, ι_0, and ι_1 satisfy the hypotheses. Then there are partial isometries $w_l \in \mathscr{A}$, $l = 1, 2, \dots, L$, such that $w_l^* w_l = \iota_0(e_{11}^{(l)})$, $w_l w_l^* = \iota_1(e_{11}^{(l)})$, and $w_l \approx_{\varepsilon/4N} \iota_0(e_{11}^{(l)})$. Consider the partial isometry $w = \sum_{l=1}^{L} \sum_{j=1}^{k_l} \iota_1(e_{j,1}^{(l)}) w_l \iota_0(e_{1,j}^{(l)}) \in \mathscr{A}$. We have $\operatorname{Ad} w \circ \iota_0 = \iota_1$ and $w \approx_{\varepsilon/2} \iota_0(1_A)$. Since $1_{\mathscr{A}} - \iota_0(1_A) \approx_{\delta} 1_{\mathscr{A}} - \iota_1(1_A)$, there exists a partial isometry $v \in \mathscr{A}$ such that $v^*v = 1_{\mathscr{A}} - \iota_0(1_A)$, $vv^* = 1_{\mathscr{A}} - \iota_1(1_A)$, and $v \approx_{\varepsilon/2} 1_{\mathscr{A}} - \iota_0(1_A)$. The unitary $u = v + w \in \mathscr{A}$ satisfies the stipulated conditions.

(iv) Applying (iii), we obtain $\delta' \in (0, \varepsilon)$ satisfying the condition of (iii) for $\varepsilon/2 > 0$ and $N \in \mathbb{N}$. Applying (ii), we also obtain $\delta > 0$ satisfying the condition of (ii) for $\delta'/2 > 0$ and $N \in \mathbb{N}$. This δ satisfies the desired condition. Indeed, if A, B, and C satisfy the assumption of (iv) for $\delta > 0$ and $N \in \mathbb{N}$, then there exists a unitary $u \in \mathscr{A}$ such that $u \approx_{\delta'/2} 1_{\mathscr{A}}$ and $\operatorname{Ad} u(B) \subset A$, which implies that $\operatorname{Ad} u(c) \approx_{\delta'} c$ for any $c \in C^1$. Thus we obtain a unitary $w \in A + \mathbb{C}1_{\mathscr{A}}$ such that $\operatorname{Ad} wu(c) = c$ for any $c \in C$ and $w \approx_{\varepsilon/2} 1_{\mathscr{A}}$. The unitary $y = wu \in \mathscr{A}$ satisfies the conditions.

(v) Let A and $\bigoplus_{l=1}^{L} M_{k_l}$ be as in the proof of (iii). Define a unitary $w'' \in \mathscr{A}$ by $w'' = vu^*$. By hypothesis, there exists a unitary w' in $B + \mathbb{C}1_{\mathscr{A}}$ such that $w' \approx_{1/2} w''$. It follows that $\operatorname{Ad} w'u(e_{11}^{(l)}) \approx_1 \operatorname{Ad} v(e_{11}^{(l)})$ for all $l = 1, 2, \dots, L$. Thus there exists a partial isometry $w_l \in B$ such that $w_l^* w_l = \operatorname{Ad} w'u(e_{11}^{(l)})$ and $w_l w_l^* = \operatorname{Ad} v(e_{11}^{(l)})$. Set $\overline{w} = (\sum_{l=1}^{N} \sum_{j=1}^{k_l} \operatorname{Ad} v(e_{j,1}^{(l)}) w_l \operatorname{Ad} w'u(e_{1j}^{(l)}))w'$; $\overline{w}$ is a partial isometry in B such that $\operatorname{Ad} \overline{w}u(a) = \operatorname{Ad} v(a)$ for any $a \in A$. Since B is finite-dimensional, we can extend $\overline{w}$ to a unitary w in $B + \mathbb{C}1_{\mathscr{A}}$ such that $\operatorname{Ad} wu(a) = \operatorname{Ad} v(a)$ for any $a \in A$. □

In the following proof, for a given C*-algebra A and a Lipschitz continuous function $f \in C([0,1]) \otimes A$ we denote by $\operatorname{Lip}(f)$ the Lipschitz constant of f. We shall denote by $1_{\mathscr{F}}$ and $1_{\mathscr{F}_i}$ the units of the unitizations $\mathscr{F}^\sim$ and $\mathscr{F}_i^\sim$.

Proof of Theorem 4.4. Let F_n, $n \in \mathbb{N}$, be an increasing sequence of finite subsets of G^+ whose union is G^+ with $F_1 = \{0\}$, and $\varepsilon_n > 0$, $n \in \mathbb{N}$, be a decreasing sequence such that $\sum_{n\in\mathbb{N}} \varepsilon_n < 1$. Let F_{An} (resp. $F_{A_i n}$), $n \in \mathbb{N}$, be an increasing sequence of finite subsets of $\mathscr{F}^1$ (resp. $\mathscr{F}_i^1$) whose union is dense in $\mathscr{F}^1$ (resp. $\mathscr{F}_i^1$) and $F_{A1} = \{0\}$ (resp. $F_{A_i 1} = \{0\}$ for $i = 0, 1$). For $n \in \mathbb{N}$, we shall inductively construct numbers L_n, Λ_n, $\tilde{L}_n \in \mathbb{N} \cup \{0\}$, finite-dimensional C*-subalgebras $A_n^{(i)}, B_n^{(i)} \subset \mathscr{F}_i$, $i = 0, 1$, and $A_n \subset \mathscr{F}$, embeddings $\epsilon_1 \colon \{0\} \to \mathscr{F}$, $\epsilon_n \colon M_{2^{L_n}} \otimes M_{3^{L_n}} \to A'_{n-1} \cap \mathscr{F}$ (for $n \geq 2$), $\iota_n^{(i)} \colon A_n^{(i)} \to A_n$, $\kappa_n^{(i)} \colon B_n^{(i)} \otimes M_{(3-i)^{\Lambda_n}} \to \mathscr{F}$ for $i = 0, 1$, a C*-subalgebra $\mathscr{A}_n$ of $C([0,1]) \otimes A_n$, and an injective $*$-homomorphism $\varphi_n \colon \mathscr{A}_{n-1} \to \mathscr{A}_n$ (where $\mathscr{A}_0 = \{0\}$) satisfying the following conditions:

$(1,n)$ $\Lambda_n > L_n + \Lambda_{n-1}$, $\tilde{L}_n > L_n \geq \Lambda_{n-1}$,

$(2,n)$ for $i = 0, 1$, on regarding $(K_0(\mathscr{F}_i), K_0(\mathscr{F}_i)^+)$ as $(G \otimes \mathbb{D}_{(2+i)^\infty}, G \otimes \mathbb{D}_{(2+i)^\infty}{}^+)$, one has $F_n \otimes 1_{(2+i)^\infty} \subset K_0(A_n^{(i)})^+$, $A_{n-1}^{(i)} \subset A_n^{(i)} \subset B_n^{(i)}$, and $F_{A_i n} \subset_{\varepsilon_n} A_n^{(i)}$,

$(3,n)$ if $n = 2n' + i_n$ for some $n' \in \mathbb{N} \cup \{0\}$ and $i_n \in \{0, 1\}$, then there exists a unitary U_n in $\mathscr{F}^\sim$ such that

$$U_n \approx_{\varepsilon_n} 1_{\mathscr{F}} \text{ and } (A_{n-1} + \mathbb{C}1_{\mathscr{F}}) \operatorname{Im}(\epsilon_n) \subset \operatorname{Ad} U_n(A_n^{(1-i_n)} \otimes M_{(2+i_n)^{\tilde{L}_n}}) = A_n,$$

$(4,n)$ $\epsilon_n(1_{(M_{2^{L_n}} \otimes M_{3^{L_n}})})a = a$ for any $a \in A_{n-1}$, and $F_{An} \subset_{\varepsilon_n} A_n^1$,

$(5,n)$ $\mathscr{A}_n$ is defined by

$$\mathscr{A}_n = \left\{ f \in C([0,1]) \otimes A_n : f(i) \in \operatorname{Im}(\iota_n^{(i)}) \text{ for } i = 0, 1 \right\},$$

$(6,n)$ there are unitaries $V_n^{(i)}, W_n^{(i)} \in \mathscr{F}^\sim$, $i = 0, 1$, such that

$$\iota_n^{(i)} = \operatorname{Ad} V_n^{(i)} \circ \iota_\infty^{(i)}|_{A_n^{(i)}}, \quad \kappa_n^{(i)}(b \otimes 1_{M_{(3-i)^{\Lambda_n}}}) = \operatorname{Ad} W_n^{(i)}(b \otimes 1_{M_{(3-i)^\infty}}),$$

for any $i = 0, 1$ and $b \in B_n^{(i)}$, and

$$V_n^{(i)} \in_{\varepsilon_n} A_n + \mathbb{C}1_{\mathscr{F}} \quad \text{and} \quad W_n^{(i)} \in_{\frac{\varepsilon_n}{4}} \operatorname{Im}(\kappa_n^{(i)}) + \mathbb{C}1_{\mathscr{F}},$$

$(7,n)$ $A_n \subset \operatorname{Im}(\kappa_n^{(i)})$ for $i = 0, 1$, and

$$\iota_n^{(i)}(a) = \kappa_n^{(i)}(a \otimes 1_{M_{(3-i)^{\Lambda_n}}}) \quad \text{for any } i = 0, 1, \text{ and } a \in A_n^{(i)},$$

$(8,n)$ one has

$$\operatorname{Lip}(\varphi_n(f)) \leq \operatorname{Lip}(f)/2 + 2\varepsilon_n$$

for any Lipschitz continuous function $f \in \mathscr{A}_{n-1}$ with $\|f\| \leq 1$.

For $n = 1$ and $i = 0, 1$, setting $\Lambda_0 = 0$, $A_0^{(i)} = \{0\} \subset \mathscr{F}_i$, and $A_0 = \{0\} \subset \mathscr{F}$, define $L_1 = 0$, $\Lambda_1 = \tilde{L}_1 = 1$, $A_1 = \{0\} \subset \mathscr{F}$, $A_1^{(i)} = B_1^{(i)} = \{0\} \subset \mathscr{F}_i$, $U_1 = V_1^{(i)} = W_1^{(i)} = 1_{\mathscr{F}}$, and $\mathscr{A}_0 = \mathscr{A}_1 = \{0\}$. These choices satisfy (1, 1)–(8, 1).

For the induction, assume that L_n, Λ_n, $\tilde{L}_n$, $A_n^{(i)}$, $B_n^{(i)}$, A_n, ϵ_n, $\iota_n^{(i)}$, $\kappa_n^{(i)}$, $\mathscr{A}_n$, and φ_n satisfying (1, n)–(8, n) have been constructed. Choose $L_{n+1} \in \mathbb{N}$ large enough that $2^{L_{n+1}} > 3^{\Lambda_n}$. Then there are k'_{n+1}, $l'_{n+1} \in \mathbb{N}$ such that $k'_{n+1} < 3^{\Lambda_n}$, $l'_{n+1} < 2^{\Lambda_n}$, $3^{\Lambda_n} | 2^{L_{n+1}} - k'_{n+1}$, and $2^{\Lambda_n} | 3^{L_{n+1}} - l'_{n+1}$. Set $N_{n+1} = L_{n+1} + \Lambda_n$, $k_{n+1} = 3^{L_{n+1}} k'_{n+1}$, $l_{n+1} = 2^{L_{n+1}} l'_{n+1}$, and $m_{n+1} = 6^{L_{n+1}}$. Note that from $k'_{n+1} < 3^{\Lambda_n} \le 2^{L_{n+1}} - k'_{n+1}$ and $l'_{n+1} < 2^{\Lambda_n} \le 3^{L_{n+1}} - l'_{n+1}$ it follows that

$$m_{n+1} - (k_{n+1} + l_{n+1}) = (3^{L_{n+1}} - l'_{n+1})(2^{L_{n+1}} - k'_{n+1}) - l'_{n+1} k'_{n+1} > 0.$$

Set $N = (\dim(B_n^{(0)}) + \dim(B_n^{(1)}) + \dim(A_n)) 6^{2L_{n+1}} \in \mathbb{N}$, and $\delta_{n+1}^{(0)} = \varepsilon_{n+1}/8$. Applying Lemma 4.6(ii), (iii), (iv) inductively to $\delta_{n+1}^{(k)}$ $(= \varepsilon) > 0$ and N, we obtain $\delta_{n+1}^{(k+1)}$ $(= \delta) > 0$ verifying the estimates of (ii), (iii), (iv) of Lemma 4.6. Refining the choice of $\delta_{n+1}^{(k)} > 0$, $k \in \mathbb{N}$, making each one smaller in turn, we may suppose that $\delta_{n+1}^{(k+1)} < \delta_{n+1}^{(k)}$ for $k \in \mathbb{N}$ and

$$N\left(\sum_{j=1}^{\infty} 8^j \delta_{n+1}^{(k+j)} \right) < \delta_{n+1}^{(k)} \quad \text{for any } k \in \mathbb{N}.$$

Since $\mathscr{F}_i \otimes M_{(2+i)^\infty} \cong \mathscr{F}_i$ for $i = 0, 1$, there are embeddings $\epsilon_{n+1}^{(i)} \colon M_{(2+i)^{L_{n+1}}} \to (B_n^{(i)})' \cap \mathscr{F}_i$, $i = 0, 1$, such that

$$(\mathrm{Im}(\epsilon_{n+1}^{(1)}) \otimes 1_{M_{2^\infty}})^1 \subset_{\delta_{n+1}^{(10)}} (\mathrm{Im}(\epsilon_{n+1}^{(0)}) \otimes 1_{M_{3^\infty}})' \cap \mathscr{F},$$

$$b \epsilon_{n+1}^{(i)}(1_{M_{(2+i)^{L_{n+1}}}}) \approx_{\frac{\varepsilon_{n+1}}{4}} b \quad \text{for any } i = 0, 1 \text{ and } b \in (B_n^{(i)})^1,$$

and

$$a(\epsilon_{n+1}^{(i)}(1_{M_{(2+i)^{L_{n+1}}}}) \otimes 1_{M_{(3-i)^\infty}}) \approx_{\delta_{n+1}^{(10)}} a \quad \text{for any } a \in A_n^1 \cup \bigcup_{j=0,1} (B_n^{(j)} \otimes 1_{M_{(3-j)^\infty}})^1.$$

By Lemma 4.6(ii), we obtain a unitary E_1 in $\mathscr{F}^\sim$ such that $E_1 \approx_{\delta_{n+1}^{(9)}} 1_{\mathscr{F}}$ and

$$\mathrm{Ad}\, E_1(\mathrm{Im}(\epsilon_{n+1}^{(1)}) \otimes 1_{M_{2^\infty}}) \subset (\mathrm{Im}(\epsilon_{n+1}^{(0)}) \otimes 1_{M_{3^\infty}})' \cap \mathscr{F}.$$

Set $\bar{\epsilon}_{n+1}(a \otimes b) = \epsilon_{n+1}^{(0)}(a) \otimes 1_{M_{3^\infty}} \mathrm{Ad}\, E_1(\epsilon_{n+1}^{(1)}(b) \otimes 1_{M_{2^\infty}})$ for $a \in M_{2^{L_{n+1}}}$, $b \in M_{3^{L_{n+1}}}$. Choosing $\epsilon_{n+1}^{(0)}$ and $\epsilon_{n+1}^{(1)}$ almost commuting with $\mathrm{Im}(\kappa_n^{(i)})^1$ and $W_n^{(i)}$, we may further assume that

$$\mathrm{Im}(\kappa_n^{(i)})^1 \cup \{W_n^{(i)}\} \subset_{\delta_{n+1}^{(8)}} \mathrm{Im}(\bar{\epsilon}_{n+1})' \cap \mathscr{F},$$

for both $i = 0, 1$. Because of the inclusion $A_n \subset \mathrm{Im}(\kappa_n^{(i)})$ for $i = 0, 1$ in (7, n), we have an embedding $\epsilon_{n+1} \colon M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}} \to A_n' \cap \mathscr{F}$ such that $\epsilon_{n+1}(x) \approx_{\delta_{n+1}^{(6)}} \bar{\epsilon}_{n+1}(x)$ for all $x \in (M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}})^1$. Then it follows that $\mathrm{Im}(\epsilon_{n+1})^1 \subset_{\delta_{n+1}^{(5)}}$

$\mathrm{Im}(\kappa_n^{(i)})' \cap \mathscr{F}$ for $i = 0, 1$. Note that, since $1_{A_n}\epsilon_{n+1}(1_{M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}}) \approx_{\varepsilon_{n+1}} 1_{A_n}$ and $\epsilon_{n+1}(1_{M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}}) \in A_n'$, we have $\epsilon_{n+1}(1_{M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}})1_{A_n} = 1_{A_n}$, which implies $\epsilon_{n+1}(1_{M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}})a = a$ for any $a \in A_n$ in $(4,\ n+1)$. For the same reason, from $1_{A_n^{(i)}}\epsilon_{n+1}^{(i)}(1_{M_{(2+i)^{L_{n+1}}}}) \approx_{\varepsilon_{n+1}} 1_{A_n^{(i)}}$ and $\epsilon_{n+1}^{(i)}(1_{M_{(2+i)^{L_{n+1}}}}) \in A_n^{(i)\prime}$, it follows that $\epsilon_{n+1}^{(i)}(1_{M_{(2+i)^{L_{n+1}}}})a = a$ for all $i = 0, 1$ and $a \in A_n^{(i)}$.

For $i = 0, 1$, since $\mathscr{F}_i \otimes M_{(3-i)^\infty} = \mathscr{F}$, there exist $\tilde{L}_{n+1} \in \mathbb{N}$ and a finite-dimensional C*-subalgebra $A_{n+1}^{(i)}$ of $\mathscr{F}_i$ satisfying $F_{n+1} \otimes 1_{(2+i)^\infty} \subset K_0(A_{n+1}^{(i)})^+$ in $(2,\ n{+}1)$, $\tilde{L}_{n+1} > L_{n+1}$, $A_n^{(i)} \subset A_{n+1}^{(i)}$, $F_{A_i n+1} \subset_{\varepsilon_{n+1}} A_{n+1}^{(i)}$, $A_n \subset_{\delta_{n+1}^{(10)}} A_{n+1}^{(i)} \otimes M_{(3-i)^{\tilde{L}_{n+1}}}$, and $B_n^{(i)}\,\mathrm{Im}(\epsilon_{n+1}^{(i)}) \subset A_{n+1}^{(i)}$. When n is an even number (resp. an odd number), we have $i_{n+1} = 1$ (resp. $i_{n+1} = 0$). By Lemma 4.6(ii), there exists $\tilde{\delta} > 0$ satisfying the condition for $\delta_{n+1}^{(10)}$ $(= \varepsilon) > 0$ and $\dim(A_{n+1}^{(i_{n+1})})6^{2L_{n+1}} \in \mathbb{N}$. Enlarging the choice of $\tilde{L}_{n+1} \in \mathbb{N}$ and $A_{n+1}^{(1-i_{n+1})}$ further, we may choose $\tilde{L}_{n+1}$ and $A_{n+1}^{(1-i_{n+1})}$ such that

$$\begin{aligned}(A_{n+1}^{(i_{n+1})} \otimes M_{(3-i_{n+1})^{L_{n+1}}})^1 \cup (A_n + \mathbb{C}1_{\mathscr{F}})\,\mathrm{Im}(\epsilon_{n+1})^1 \cup \mathrm{Im}(\kappa_n^{(0)})^1 \\ \cup\, \mathrm{Im}(\kappa_n^{(1)})^1 \cup F_{An+1} \subset_{\min\{\delta_{n+1}^{(10)},\tilde{\delta}\}} A_{n+1}^{(1-i_{n+1})} \otimes M_{(2+i_{n+1})^{\tilde{L}_{n+1}}}\end{aligned} \tag{2}$$

and

$$\{W_n^{(0)}, W_n^{(1)}\} \subset_{\delta_{n+1}^{(10)}} A_{n+1}^{(1-i_{n+1})} \otimes M_{(2+i_{n+1})^{\tilde{L}_{n+1}}} + \mathbb{C}1_{\mathscr{F}}. \tag{3}$$

By Lemma 4.6(ii) for $\delta_{n+1}^{(9)} > 0$ and $N \in \mathbb{N}$, there exists a unitary $U_{n+1} \in \mathscr{F}^\sim$ satisfying the condition of $(3,\ n+1)$ and $U_{n+1} \approx_{\delta_{n+1}^{(9)}} 1_{\mathscr{F}}$. Define

$$A_{n+1} = \mathrm{Ad}\,U_{n+1}(A_{n+1}^{(1-i_{n+1})} \otimes M_{(2+i_{n+1})^{\tilde{L}_{n+1}}}) \subset \mathscr{F}.$$

Note that from $F_{An+1} \subset_{\delta_{n+1}^{(10)}} A_{n+1}^{(1-i_{n+1})} \otimes M_{(2+i_{n+1})^{\tilde{L}_{n+1}}}$, we see that $F_{An+1} \subset_{\varepsilon_{n+1}} A_{n+1}^1$ as required for $(4,\ n{+}1)$. By the conditions of $(3,\ n{+}1)$ and $(4,\ n{+}1)$, it follows that $A_n \subset A_{n+1}$. Since $\bigcup_{i=0,1}\mathrm{Im}(\kappa_n^{(i)})^1 \subset_{\delta_{n+1}^{(8)}} A_{n+1}$, applying Lemma 4.6(iv) to $A = A_{n+1}$, $B = \mathrm{Im}(\kappa_n^{(i)})$, and $C = A_n$, we obtain unitaries $y_n^{(i)} \in \mathscr{F}^\sim \cap A_n'$, $i = 0, 1$, such that $\mathrm{Ad}\,y_n^{(i)}(\mathrm{Im}(\kappa_n^{(i)})) \subset A_{n+1}$ and $y_n^{(i)} \approx_{\delta_{n+1}^{(7)}} 1_{\mathscr{F}}$. Define embeddings $\bar{\kappa}_n^{(i)} \colon B_n^{(i)} \otimes M_{(3-i)^{\Lambda_n}} \to A_{n+1}$, $i = 0, 1$, by $\bar{\kappa}_n^{(i)} = \mathrm{Ad}\,y_n^{(i)} \circ \kappa_n^{(i)}$. Note that, by $(7,\ n)$, we also see that $\bar{\kappa}_n^{(i)}(a \otimes 1_{M_{(3-i)^{\Lambda_n}}}) = \iota_n^{(i)}(a)$ for any $i = 0, 1$ and $a \in A_n^{(i)}$. Now we have $A_n \subset \mathrm{Im}(\bar{\kappa}_n^{(i)}) \subset A_{n+1}$ for both $i = 0, 1$, $\mathrm{Im}(\epsilon_{n+1}) \subset A_{n+1}$, and $\mathrm{Im}(\bar{\kappa}_n^{(i)})^1 \subset_{\delta_{n+1}^{(4)}} A_{n+1} \cap \mathrm{Im}(\epsilon_{n+1})'$ for $i = 0, 1$. Applying Lemma 4.6(iv) to $A = A_{n+1} \cap \mathrm{Im}(\epsilon_{n+1})'$, $B = \mathrm{Im}(\bar{\kappa}_n^{(i)})$, and $C = A_n$, we obtain unitaries $\tilde{y}_n^{(i)} \in A_n' \cap \mathscr{F}^\sim$, $i = 0, 1$, such that $\tilde{y}_n^{(i)} \approx_{\delta_{n+1}^{(3)}} 1_{\mathscr{F}}$ and $\mathrm{Ad}\,\tilde{y}_n^{(i)}(\mathrm{Im}(\bar{\kappa}_n^{(i)})) \subset A_{n+1} \cap \mathrm{Im}(\epsilon_{n+1})'$. Define

embeddings $\tilde{\kappa}_n^{(i)}\colon B_n^{(i)} \otimes M_{(3-i)^{\Lambda_n}} \to A_{n+1}$, $i = 0, 1$, by $\tilde{\kappa}_n^{(i)} = \operatorname{Ad} \tilde{y}_n^{(i)} \circ \bar{\kappa}_n^{(i)}$, which satisfy $\tilde{\kappa}_n^{(i)}(a \otimes 1) = \iota_n^{(i)}(a)$ for any $i = 0, 1$ and $a \in A_n^{(i)}$, and also $A_n \subset \operatorname{Im}(\tilde{\kappa}_n^{(i)}) \subset A_{n+1} \cap \operatorname{Im}(\epsilon_{n+1})'$ for both $i = 0, 1$.

To simplify notation, let $\{e_{ij}\}_{i,j=1}^{m_{n+1}}$ be a system of matrix units for $M_{m_{n+1}} \cong M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}$. Note that $m_{n+1} > k_{n+1} + l_{n+1}$. Define an embedding ξ from $C([0,1]) \otimes A_n$ into $C([0,1]) \otimes A_{n+1}$ by

$$\begin{aligned}\xi(f)(t) = &\sum_{j=1}^{k_{n+1}} f\Big(\frac{t}{2}\Big)\epsilon_{n+1}(e_{jj}) + \sum_{j=1+k_{n+1}}^{m_{n+1}-l_{n+1}} f\Big(\frac{1}{2}\Big)\epsilon_{n+1}(e_{jj}) \\ &+ \sum_{j=1+m_{n+1}-l_{n+1}}^{m_{n+1}} f\Big(\frac{1+t}{2}\Big)\epsilon_{n+1}(e_{jj}),\end{aligned}$$

for $f \in C([0,1]) \otimes A_n$ and $t \in [0,1]$. Set $p = \epsilon_{n+1}(\sum_{j=1}^{k_{n+1}} e_{jj})$. Since $\tilde{\kappa}_n^{(0)}(a \otimes 1_{M_{3^{\Lambda_n}}}) = \iota_n^{(0)}(a)$ for any $a \in A_n^{(0)}$ and $A_n \subset \operatorname{Im}(\tilde{\kappa}_n^{(0)})$, for any $f \in \mathscr{A}_n$ there exist $a_f \in A_n^{(0)} \subset B_n^{(0)}$ and $b_f \in B_n^{(0)} \otimes M_{3^{\Lambda_n}}$ such that

$$\begin{aligned}\xi(f)(0) &= \sum_{j=1}^{k_{n+1}} f(0)\epsilon_{n+1}(e_{jj}) + \sum_{j=1+k_{n+1}}^{m_{n+1}} f\Big(\frac{1}{2}\Big)\epsilon_{n+1}(e_{jj}) \\ &= \tilde{\kappa}_n^{(0)}(a_f \otimes 1)p + \tilde{\kappa}_n^{(0)}(b_f)(\epsilon_{n+1}(1) - p).\end{aligned}$$

Since $3^{L_{n+1}} | 3^{L_{n+1}} k'_{n+1} = k_{n+1}$ and $3^{\Lambda_n} | 2^{L_{n+1}} - k'_{n+1}$, applying Lemma 4.6(i) to $P = 3^{\Lambda_n}$, $Q = 2^{L_{n+1}}$, $R = 3^{L_{n+1}}$, and p, we obtain a unitary $U_{n+1}^{(0)}$ in $\tilde{\kappa}_n^{(0)}(1_{B_n^{(0)}} \otimes M_{3^{\Lambda_n}})\epsilon_{n+1}(M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}) + \mathbb{C}1_{\mathscr{F}} \subset A_{n+1} + \mathbb{C}1_{\mathscr{F}}$ such that

$$\operatorname{Ad} U_{n+1}^{(0)}(\xi(f)(0)) \in \tilde{\kappa}_n^{(0)}(B_n^{(0)} \otimes 1_{M_{3^{\Lambda_n}}})\epsilon_{n+1}(M_{2^{L_{n+1}}} \otimes 1_{M_{3^{L_{n+1}}}}),$$

for any $f \in \mathscr{A}_n$. By (6, n), now we have $\operatorname{Ad} W_n^{(0)}(b \otimes 1_{M_{3^\infty}}) = \kappa_n^{(0)}(b \otimes 1_{M_{3^{\Lambda_n}}})$ for any $\in B_n^{(0)}$ and $W_n^{(0)} \in_{\varepsilon_n/4} \operatorname{Im}(\kappa_n^{(0)}) + \mathbb{C}1_{\mathscr{F}}$. Since $W_n^{(0)} \in_{\delta_{n+1}^{(8)}} \operatorname{Im}(\bar{\epsilon}_{n+1})' \cap \mathscr{F}^\sim$ (from (3)), and $\bar{\epsilon}_{n+1}(x) \approx_{\delta_{n+1}^{(6)}} \epsilon_{n+1}(x)$ for all $x \in (M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}})^1$, we obtain a unitary $\widetilde{W}_n^{(0)} \in \operatorname{Im}(\epsilon_{n+1})' \cap \mathscr{F}^\sim$ such that $\widetilde{W}_n^{(0)} \approx_{\delta_{n+1}^{(5)}} W_n^{(0)}$. Then it follows that

$$\begin{aligned}&\operatorname{Ad} W_n^{(0)*}(\tilde{\kappa}_n^{(0)}(b \otimes 1_{M_{3^{\Lambda_n}}})\epsilon_{n+1}(a \otimes 1_{M_{3^{L_{n+1}}}})) \\ &\quad\approx_{2\delta_{n+1}^{(5)}} \operatorname{Ad} W_n^{(0)*}(\tilde{\kappa}_n^{(0)}(b \otimes 1_{M_{3^{\Lambda_n}}}))\epsilon_{n+1}(a \otimes 1_{M_{3^{L_{n+1}}}}) \\ &\quad= (b \otimes 1_{M_{3^\infty}})\epsilon_{n+1}(a \otimes 1_{M_{3^{L_{n+1}}}}) \\ &\quad\approx_{\delta_{n+1}^{(6)} + 2\delta_{n+1}^{(9)}} \big((b\epsilon_{n+1}^{(0)}(a)) \otimes 1_{M_{3^\infty}}\big)(\epsilon_{n+1}^{(1)}(1) \otimes 1_{M_{2^\infty}}) \\ &\quad\approx_{\delta_{n+1}^{(10)}} b\epsilon_{n+1}^{(0)}(a) \otimes 1_{M_{3^\infty}},\end{aligned}$$

for any $b \in B_n^{(0)1}$ and $a \in M_{2^{L_{n+1}}}^1$. By Lemma 4.6(iii), we obtain a unitary $u^{(0)}$ in $\mathscr{F}^\sim$ such that $u^{(0)} \approx_{\delta_{n+1}^{(3)}} 1_{\mathscr{F}}$ and

$$\operatorname{Ad} u^{(0)} W_n^{(0)*}(\tilde{\kappa}_n^{(0)}(b \otimes 1_{M_{3^{\Lambda_n}}})\epsilon_{n+1}(a \otimes 1_{M_{3^{L_{n+1}}}})) = (b\epsilon_{n+1}^{(0)}(a)) \otimes 1_{M_{3^\infty}},$$

for any $b \in B_n^{(0)}$ and $a \in M_{2^{L_{n+1}}}$. Consider the unitary $z_n^{(0)} = W_n^{(0)} u^{(0)*}$ in $\mathscr{F}^\sim$, which satisfies $z_n^{(0)} \approx_{\delta_{n+1}^{(3)}} W_n^{(0)}$ and

$$\operatorname{Ad} z_n^{(0)}(b\epsilon_{n+1}^{(0)}(a) \otimes 1_{M_{3^\infty}}) = \tilde{\kappa}_n^{(0)}(b \otimes 1_{M_{3^{\Lambda_n}}})\epsilon_{n+1}(a \otimes 1_{M_{3^{L_{n+1}}}})$$

for any $b \in B_n^{(0)}$ and $a \in M_{2^{L_{n+1}}}$. Define an embedding $\bar{\iota}_{n+1}^{(0)}$ of $A_{n+1}^{(0)}$ into $\mathscr{F}$ by

$$\bar{\iota}_{n+1}^{(0)}(a) = \operatorname{Ad} {U_{n+1}^{(0)}}^* z_n^{(0)}(a \otimes 1_{M_{3^\infty}}) \quad \text{for } a \in A_{n+1}^{(0)}.$$

From $B_n^{(0)} \operatorname{Im}(\epsilon_{n+1}^{(0)}) \subset A_{n+1}^{(0)}$, it follows that $\xi(f)(0) \in \operatorname{Im}(\bar{\iota}_{n+1}^{(0)})$ for any $f \in \mathscr{A}_n$.

In the same way as for $i = 0$, we also obtain unitaries $u^{(1)}$, $U_{n+1}^{(1)}$, $z_n^{(1)}$ in $\mathscr{F}^\sim$, and an embedding $\bar{\iota}_{n+1}^{(1)}$ of $A_{n+1}^{(1)}$ into $\mathscr{F}$ such that

$$\begin{aligned} U_{n+1}^{(1)} &\in \tilde{\kappa}_n^{(1)}(1_{B_n^{(1)}} \otimes M_{2^{\Lambda_n}})\epsilon_{n+1}(M_{2^{L_{n+1}}} \otimes M_{3^{L_{n+1}}}) + \mathbb{C}1_{\mathscr{F}}, \\ z_n^{(1)} &= W_n^{(1)} u^{(1)*} \approx_{\delta_{n+1}^{(3)}} W_n^{(1)}, \\ \bar{\iota}_{n+1}^{(1)}(a) &= \operatorname{Ad} {U_{n+1}^{(1)}}^* z_n^{(1)}(a \otimes 1_{M_{2^\infty}}) \end{aligned}$$

for $a \in A_{n+1}^{(1)}$, and $\xi(f)(1) \in \operatorname{Im}(\bar{\iota}_{n+1}^{(1)})$ for any $f \in \mathscr{A}_n$.

Note that, since $(A_{n+1}^{(i)} \otimes 1_{M_{(3-i)^{L_{n+1}}}})^1 \subset_{\delta_{n+1}^{(8)}} A_{n+1}$ for both $i = 0, 1$, and since $z_n^{(i)} \approx_{\delta_{n+1}^{(3)}} W_n^{(i)} \in_{\delta_{n+1}^{(8)}} A_{n+1} + \mathbb{C}1_{\mathscr{F}}$ (from (3)), we have

$$\operatorname{Im}(\bar{\iota}_{n+1}^{(i)})^1 = \operatorname{Ad} {U_{n+1}^{(i)}}^* z_n^{(i)}((A_{n+1}^{(i)})^1 \otimes 1_{M_{(3-i)^{L_{n+1}}}}) \subset_{\delta_{n+1}^{(2)}} A_{n+1}.$$

Although $\dim(A_{n+1}^{(i)})$ can be bigger than N, the following argument allows us to obtain unitaries $\widetilde{U}_{n+1}^{(i)} \in \mathscr{F}^\sim$, $i = 0, 1$, such that $\widetilde{U}_{n+1}^{(i)} \approx_{\delta_{n+1}^{(2)}} 1_{\mathscr{F}}$ and $\operatorname{Ad} \widetilde{U}_{n+1}^{(i)} \circ \bar{\iota}_{n+1}^{(i)}(A_{n+1}^{(i)}) \subset A_{n+1}$. In the case $i = 1 - i_{n+1}$, we fix a unitary

$$\overline{U}_{n+1}^{(i)} = U_{n+1} u^{(i)} W_n^{(i)*} U_{n+1}^{(i)} \in \mathscr{F}^\sim.$$

Since $W_n^{(i)} \in_{\delta_{n+1}^{(8)}} A_{n+1} + \mathbb{C}1_{\mathscr{F}}$ and $U_{n+1}^{(i)} \in A_{n+1} + \mathbb{C}1_{\mathscr{F}}$, there exists a unitary $\bar{y}^{(i)} \in A_{n+1} + \mathbb{C}1_{\mathscr{F}}$ such that $\bar{y}^{(i)} \approx_{\delta_{n+1}^{(2)}} \overline{U}_{n+1}^{(i)}$. Consider the unitary $\widetilde{U}_{n+1}^{(i)} = \bar{y}^{(i)*} \overline{U}_{n+1}^{(i)}$, which satisfies $\widetilde{U}_{n+1}^{(i)} \approx_{\delta_{n+1}^{(2)}} 1_{\mathscr{F}}$ and

$$\operatorname{Ad} \widetilde{U}_{n+1}^{(i)} \circ \bar{\iota}_{n+1}^{(i)}(a) = \operatorname{Ad} \bar{y}^{(i)*} U_{n+1}(a \otimes 1_{M_{(3-i)^\infty}}) \in A_{n+1}$$

for any $a \in A_{n+1}^{(i)}$. In the case $i = i_{n+1}$, from (2) it follows that $(A_{n+1}^{(i)} \otimes M_{(3-i)^{L_{n+1}}})^1 \subset_{\tilde{\delta}} A_{n+1}^{(1-i)} \otimes M_{(2+i)^{\tilde{L}_{n+1}}}$ and $\tilde{\delta} > 0$ is chosen for $\dim(A_{n+1}^{(i)})6^{2L_{n+1}}$. By Lemma 4.6(ii), there exists a unitary $V \in \mathscr{F}^\sim$ such that $V \approx_{\delta_{n+1}^{(10)}} 1_{\mathscr{F}}$ and

$$\operatorname{Ad} V(A_{n+1}^{(i)} \otimes M_{(3-i)^{L_{n+1}}}) \subset A_{n+1}^{(1-i)} \otimes M_{(2+i)^{\tilde{L}_{n+1}}}.$$

It follows that

$$\operatorname{Ad} U_{n+1} V(A_{n+1}^{(i)} \otimes M_{(3-i)^{L_{n+1}}}) \subset A_{n+1}.$$

Set $\overline{U}_{n+1}^{(i)} = U_{n+1} V u^{(i)} W_n^{(i)^*} U_{n+1}^{(i)}$ which is a unitary in $\mathscr{F}^\sim$ such that $\overline{U}_{n+1}^{(i)} \in_{2\delta_{n+1}^{(3)}} A_{n+1} + \mathbb{C}1_{\mathscr{F}}$. Then there exists a unitary $\bar{y}^{(i)} \in A_{n+1} + \mathbb{C}1_{\mathscr{F}}$ such that $\bar{y}^{(i)} \approx_{\delta_{n+1}^{(2)}} \overline{U}_{n+1}^{(i)}$. Setting $\widetilde{U}_{n+1}^{(i)} = \bar{y}^{(i)^*} \overline{U}_{n+1}^{(i)}$, we also see that

$$\operatorname{Ad} \widetilde{U}_{n+1}^{(i)} \circ \bar{\iota}_{n+1}^{(i)}(a) = \operatorname{Ad} \bar{y}^{(i)^*} U_{n+1} V(a \otimes 1_{M_{(3-i)^\infty}}) \in A_{n+1}$$

for any $a \in A_{n+1}^{(i)}$.

Define embeddings $\iota_{n+1}^{(i)}$ of $A_{n+1}^{(i)}$ into A_{n+1}, $i = 0, 1$, by $\iota_{n+1}^{(i)} = \operatorname{Ad} \widetilde{U}_{n+1}^{(i)} \circ \bar{\iota}_{n+1}^{(i)}$. Then it follows that $\operatorname{Ad} \widetilde{U}_{n+1}^{(i)}(\xi(f)(i)) \in \operatorname{Im}(\iota_{n+1}^{(i)})$ for any $i = 0, 1$ and $f \in \mathscr{A}_n$. Define unitaries $V_{n+1}^{(i)} = \widetilde{U}_{n+1}^{(i)} U_{n+1}^{(i)^*} z_n^{(i)}$, $i = 0, 1$, in $\mathscr{F}^\sim$. Since $U_{n+1}^{(i)} \in A_{n+1} + \mathbb{C}1_{\mathscr{F}}$ for both $i = 0, 1$, we see that $V_{n+1}^{(i)}$, $i = 0, 1$, satisfy the conditions of $(6, n+1)$. We define a C*-subalgebra $\mathscr{A}_{n+1}$ of $C([0,1]) \otimes A_{n+1}$ as required for $(5, n+1)$. Define finite-dimensional C*-subalgebras $E^{(i)}$, $i = 0, 1$, of A_{n+1} by $E^{(i)} = \{\xi(f)(i) : f \in \mathscr{A}_n\}$. Note that $\dim(E^{(i)}) \le \dim(A_n^{(i)}) + \dim(B_n^{(i)})(3-i)^{2\Lambda_n} < N$ for both $i = 0, 1$ and that $\xi(f)(i) \approx_{2\delta_{n+1}^{(2)}} \operatorname{Ad} \widetilde{U}_{n+1}^{(i)}(\xi(f)(i))$ for any $f \in \mathscr{A}_n^1$. By Lemma 4.6(iii) with $(3, n+1)$, we obtain unitaries $\mathcal{U}^{(i)}$ in $A_{n+1} + \mathbb{C}1_{\mathscr{F}}$, $i = 0, 1$, such that $\mathcal{U}^{(i)} \approx_{\delta_{n+1}^{(0)}} 1_{\mathscr{F}}$ and $\operatorname{Ad}\mathcal{U}^{(i)}(\xi(f)(i)) = \operatorname{Ad} \widetilde{U}_{n+1}^{(i)}(\xi(f)(i))$ for any $i = 0, 1$ and $f \in \mathscr{A}_n$. Thus we obtain a unitary $\widetilde{\mathcal{U}} \in C([0,1]) \otimes (A_{n+1} + \mathbb{C}1_{\mathscr{F}})$ which is a Lipschitz continuous function such that $\widetilde{\mathcal{U}}(i) = \mathcal{U}^{(i)}$ for $i = 0, 1$, $\widetilde{\mathcal{U}}(t) \approx_{\varepsilon_{n+1}} 1_{\mathscr{F}}$ for all $t \in [0,1]$ and $\operatorname{Lip}(\widetilde{\mathcal{U}}) \le \varepsilon_{n+1}$. We define an embedding φ_{n+1} of $\mathscr{A}_n$ into $\mathscr{A}_{n+1}$ by $\varphi_{n+1}(f) = \operatorname{Ad}\widetilde{\mathcal{U}} \circ \xi(f)$ for $f \in \mathscr{A}_n$. Because of the construction of ξ and the inequality $\operatorname{Lip}(\widetilde{\mathcal{U}}) \le \varepsilon_{n+1}$, it follows that φ_{n+1} satisfies $(8, n+1)$.

In the rest of the induction, we construct Λ_{n+1}, $B_{n+1}^{(i)}$, and $\kappa_{n+1}^{(i)}$ satisfying the conditions $(1, n+1)$, $(2, n+1)$, $(6, n+1)$, $(7, n+1)$. Set $\bar{\delta}_{n+1}^{(0)} = \varepsilon_{n+1}/8$. By applying Lemma 4.6(ii), (iii), (iv) to $\bar{\delta}_{n+1}^{(k)}$ $(= \varepsilon)$ and $\overline{N} = \dim(A_{n+1})$ inductively, there exists $\bar{\delta}_{n+1}^{(k+1)}$ $(= \delta) > 0$ satisfying the conditions of (ii), (iii), and (iv). Taking smaller $\bar{\delta}_{n+1}^{(k)}$, $k \in \mathbb{N}$, as above, we may assume that $\bar{\delta}_{n+1}^{(k+1)} < \bar{\delta}_{n+1}^{(k)}$ for $k \in \mathbb{N}$ and $N \sum_{j=1}^{\infty} 8^j \bar{\delta}_{n+1}^{(k+j)} < \bar{\delta}_{n+1}^{(k)}$. Since $\mathscr{F} = \mathscr{F}_i \otimes M_{(3-i)^\infty}$ for both $i = 0, 1$, there exist $\Lambda_{n+1} \in \mathbb{N}$ with the condition of $(1, n+1)$ and finite-dimensional C*-subalgebras

$B_{n+1}^{(i)} \subset \mathscr{F}_i$, $i = 0, 1$, such that $A_{n+1}^{(i)} \subset B_{n+1}^{(i)}$, $A_{n+1}^1 \subset_{\bar{\delta}_{n+1}^{(4)}} B_{n+1}^{(i)} \otimes M_{(3-i)^{\Lambda_{n+1}}}$, and $\{V_{n+1}^{(0)}, V_{n+1}^{(1)}\} \subset_{\bar{\delta}_{n+1}^{(4)}} B_{n+1}^{(i)} \otimes M_{(3-i)^{\Lambda_{n+1}}} + \mathbb{C}1_{\mathscr{F}}$.

By Lemma 4.6(ii), there exist unitaries $\overline{W}_{n+1}^{(i)}$, $i = 0, 1$, in $\mathscr{F}^\sim$ such that $\overline{W}_{n+1}^{(i)} \approx_{\bar{\delta}_{n+1}^{(3)}} 1_{\mathscr{F}}$ and $\operatorname{Ad}\overline{W}_{n+1}^{(i)}(B_{n+1}^{(i)} \otimes M_{(3-i)^{\Lambda_{n+1}}}) \supset A_{n+1}$ for $i = 0, 1$. Define embeddings $\hat{\kappa}_{n+1}^{(i)} \colon B_{n+1}^{(i)} \otimes M_{(3-i)^{\Lambda_{n+1}}} \to \mathscr{F}$, $i = 0, 1$, by $\hat{\kappa}_{n+1}^{(i)} = \operatorname{Ad}\overline{W}_{n+1}^{(i)}$. By Lemma 4.6(v) and $V_{n+1}^{(i)} \in_{\bar{\delta}_{n+1}^{(2)}} \operatorname{Im}(\hat{\kappa}_{n+1}^{(i)}) + \mathbb{C}1_{\mathscr{F}}$, there exist unitaries $\widehat{W}_{n+1}^{(i)} \in \operatorname{Im}(\hat{\kappa}_{n+1}^{(i)}) + \mathbb{C}1_{\mathscr{F}}$ such that

$$\operatorname{Ad}\widehat{W}_{n+1}^{(i)} V_{n+1}^{(i)}(a \otimes 1_{M_{(3-i)^\infty}}) = \operatorname{Ad}\overline{W}_{n+1}^{(i)}(a \otimes 1_{M_{(3-i)^\infty}})$$

for any $a \in A_{n+1}^{(i)}$. Define unitaries $W_{n+1}^{(i)}$, $i = 0, 1$, in $\mathscr{F}^\sim$ and embeddings $\kappa_{n+1}^{(i)} \colon B_{n+1}^{(i)} \otimes M_{(3-i)^{\Lambda_{n+1}}} \to \mathscr{F}$ by

$$W_{n+1}^{(i)} = (\widehat{W}_{n+1}^{(i)})^* \overline{W}_{n+1}^{(i)}, \quad \kappa_{n+1}^{(i)}(b \otimes a) = \operatorname{Ad} W^{(i)}(b \otimes a),$$

for $b \in B_{n+1}^{(i)}$ and $a \in M_{(3-i)^{\Lambda_{n+1}}} \subset M_{(3-i)^\infty}$. These choices satisfy the conditions of $(6, n+1)$ and $(7, n+1)$.

Define A_G as the inductive limit C*-algebra $\varinjlim(\mathscr{A}_n, \varphi_{n+1})$. In the proof of Proposition 4.5(iii), we saw that $(G, G^+) \cong (\operatorname{Im}(\iota_{\infty *}^{(0)}) \cap \operatorname{Im}(\iota_{\infty *}^{(1)}), \operatorname{Im}(\iota_{\infty *}^{(0)}) \cap \operatorname{Im}(\iota_{\infty *}^{(1)}) \cap K_0(\mathscr{F})^+)$, as ordered abelian groups. To show that the right-hand side is isomorphic to $(K_0(A_G), K_0(A_G)^+)$, we need the following observation. Let η_n denote the canonical embedding of A_n into A_{n+1} which is obtained from $(3, n+1)$ for $n \in \mathbb{N}$. Write $\operatorname{ev}_n^{(i)} \colon \mathscr{A}_n \to \operatorname{Im}(\iota_n^{(i)})$, $i = 0, 1$, to denote the evaluation maps and $\bar{\eta}_n^{(i)} \colon \operatorname{Im}(\iota_n^{(i)}) \to A_n$ for the canonical embeddings.

Observation 4.7. *For $i = 0, 1$ and $n \in \mathbb{N}$, the two $*$-homomorphisms $\operatorname{ev}_{n+1}^{(i)} \circ \varphi_{n+1}$ and $\operatorname{Ad}\widetilde{\mathcal{U}}(i) \circ \eta_n \circ \bar{\eta}_n^{(i)} \circ \operatorname{ev}_n^{(i)} \colon \mathscr{A}_n \to \operatorname{Im}(\iota_{n+1}^{(i)})$ are homotopic.*

Proof. Let $\Phi_t^{(i)}$, $t \in [0, 1]$, $i = 0, 1$, be pointwise continuous paths of automorphisms on $\mathscr{A}_n$ such that $\Phi_0^{(0)} = \Phi_0^{(1)} = \mathrm{id}_{\mathscr{A}_n}$, $\Phi_1^{(0)}(f)(t) = f(0)$, $t \in [0, 1/2]$, $\Phi_1^{(0)}(f)(t) = f(2t-1)$, $t \in [1/2, 1]$, $\Phi_1^{(1)}(f)(t) = f(2t)$, $t \in [0, 1/2]$, and $\Phi_1^{(1)}(f)(t) = f(1)$, $t \in [1/2, 1]$, for any $f \in \mathscr{A}_n$. Then it follows that $\operatorname{ev}_{n+1}^{(i)} \circ \varphi_{n+1}$ and $\operatorname{ev}_{n+1}^{(i)} \circ \varphi_{n+1} \circ \Phi_1^{(i)}$ are homotopic for both $i = 0, 1$. From the construction of $\varphi_{n+1} = \operatorname{Ad}\widetilde{\mathcal{U}} \circ \xi$ and $\Phi_1^{(i)}(f)(1/2) = f(i)$, we have

$$\operatorname{ev}_{n+1}^{(i)} \circ \varphi_{n+1} \circ \Phi_1^{(i)}(f) = \operatorname{Ad}\widetilde{\mathcal{U}}(i)(f(i)\epsilon_{n+1}(1_{M_{m_{n+1}}})) = \operatorname{Ad}\widetilde{\mathcal{U}}(i) \circ \eta_n \circ \bar{\eta}_n^{(i)} \circ \operatorname{ev}_n^{(i)}(f),$$

for any $i = 0, 1$ and $f \in \mathscr{A}_n$. □

Denote by $\eta_{\infty n}$ the canonical embedding of A_n into $\mathscr{F}$. For $i = 0, 1$, denote by $\operatorname{Ev}_n^{(i)} \colon \mathscr{A}_n \to A_n$ the evaluation map at i. Note that $\bar{\eta}_n^{(i)} \circ \operatorname{ev}_n^{(i)} = \operatorname{Ev}_n^{(i)}$. Let

$\eta_n^{(i)}\colon A_n^{(i)} \to A_{n+1}^{(i)}$ denote the canonical embeddings of $(2,\, n+1)$ and $\eta_{\infty n}^{(i)}\colon A_n^{(i)} \to \mathscr{F}_i$ the canonical embeddings. For $m,\, n \in \mathbb{N}$ with $m > n$, set $\varphi_{m,n} = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_{n+1}\colon \mathscr{A}_n \to \mathscr{A}_m$, $\eta_{m,n} = \eta_{m-1} \circ \eta_{m-2} \circ \cdots \circ \eta_n\colon A_n \to A_m$, and $\eta_{m,n}^{(i)} = \eta_{m-1}^{(i)} \circ \eta_{m-2}^{(i)} \circ \cdots \circ \eta_n^{(i)}\colon A_n^{(i)} \to A_m^{(i)}$. By Observation 4.7, $\mathrm{Ev}_{n+1}^{(i)} \circ \varphi_{n+1}$ and $\mathrm{Ad}\,\widetilde{\mathcal{U}}(i) \circ \eta_n \circ \mathrm{Ev}_n^{(i)}$ are homotopic as $*$-homomorphisms from $\mathscr{A}_n$ into A_{n+1}. Since $\widetilde{\mathcal{U}}(i) \in A_{n+1} + \mathbb{C}1_{\mathscr{F}}$, it follows that $\mathrm{Ev}_{n+1*}^{(i)} \circ \varphi_{n+1*} = \eta_{n*} \circ \mathrm{Ev}_{n*}^{(i)}$. From $(6,\, n)$ it follows that

$$\iota_{n+1}^{(i)} \circ \eta_n^{(i)} = \mathrm{Ad}\, V_{n+1}^{(i)} V_n^{(i)*} \circ \iota_n^{(i)}, \quad \text{and} \quad V_{n+1}^{(i)} V_n^{(i)*} \in_{2\varepsilon_n} A_{n+1} + \mathbb{C}1_{\mathscr{F}}.$$

Then we have $\mathrm{Ad}\, V_{n+1}^{(i)} V_n^{(i)*}|_{\mathrm{Im}(\iota_n^{(i)})*} = \eta_n|_{\mathrm{Im}(\iota_n^{(i)})*}$ as group homomorphisms from $K_0(\mathrm{Im}(\iota_n^{(i)}))$ to $K_0(A_{n+1})$, which implies that $\iota_{n+1*}^{(i)} \circ \eta_{n*}^{(i)} = \eta_{n*} \circ \iota_{n*}^{(i)}$. It follows that

$$\eta_{n*}(\mathrm{Im}(\iota_{n*}^{(0)}) \cap \mathrm{Im}(\iota_{n*}^{(1)})) \subset \mathrm{Im}(\iota_{n+1*}^{(0)}) \cap \mathrm{Im}(\iota_{n+1*}^{(1)}).$$

Denote by (I, I^+) the inductive limit of ordered groups $\varinjlim(\mathrm{Im}(\iota_{n*}^{(0)}) \cap \mathrm{Im}(\iota_{n*}^{(1)}), \eta_{n*})$, where the order in $\mathrm{Im}(\iota_{n*}^{(0)}) \cap \mathrm{Im}(\iota_{n*}^{(1)})$ is determined by $\mathrm{Im}(\iota_{n*}^{(0)}) \cap \mathrm{Im}(\iota_{n*}^{(1)}) \cap K_0(A_n)^+$. By $(6,\, n)$, we also see that $\eta_{\infty n*} \circ \iota_{n*}^{(i)} = \iota_{\infty *}^{(i)} \circ \eta_{\infty n*}^{(i)}$. Since $K_0(\mathscr{F}) = \bigcup_{n=1}^{\infty} \eta_{\infty n*}(K_0(A_n))$ (from $F_{An} \subset_{\varepsilon_n} A_n^1$ in $(4,\, n)$), we have

$$I \cong \bigcup_{n=1}^{\infty} \eta_{\infty,n*}(\mathrm{Im}(\iota_{n*}^{(0)}) \cap \mathrm{Im}(\iota_{n*}^{(1)})) = \bigcup_{n=1}^{\infty} \mathrm{Im}(\iota_{\infty *}^{(0)} \circ \eta_{\infty n*}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)} \circ \eta_{\infty n*}^{(1)}).$$

As $K_0(\mathscr{F}_i) = \bigcup_{n=1}^{\infty} \eta_{\infty n*}^{(i)}(K_0(A_n^{(i)}))$ (since $F_{A_i n} \subset A_n^{(i)}$ by $(2,\, n)$), it follows that

$$\bigcup_{n=1}^{\infty} \mathrm{Im}(\iota_{\infty *}^{(0)} \circ \eta_{n*}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)} \circ \eta_{n*}^{(1)}) = \mathrm{Im}(\iota_{\infty *}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)}).$$

For the same reason, it is straightforward to check that I^+ corresponds to $\mathrm{Im}(\iota_{\infty *}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)}) \cap K_0(\mathscr{F})^+$. Then it follows that (I, I^+) is isomorphic to

$$(\mathrm{Im}(\iota_{\infty *}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)}), \mathrm{Im}(\iota_{\infty *}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)}) \cap K_0(\mathscr{F})^+).$$

Since $\mathrm{Ev}_{n+1*}^{(0)} \circ \varphi_{n+1*} = \eta_{n*} \circ \mathrm{Ev}_{n*}^{(0)}$ and $\mathrm{Im}(\mathrm{Ev}_{n*}^{(0)}) \subset \mathrm{Im}(\iota_{n,*}^{(0)}) \cap \mathrm{Im}(\iota_{n,*}^{(1)})$, we obtain a positive group homomorphism $\mathrm{Ev}_*^{(0)}$ from $K_0(A_G) = \varinjlim(K_0(\mathscr{A}_n), \varphi_{n+1*})$ into (I, I^+). We may regard $\mathrm{Ev}_*^{(0)}$ as a positive group homomorphism from $K_0(A_G)$ to $\mathrm{Im}(\iota_{\infty *}^{(0)}) \cap \mathrm{Im}(\iota_{\infty *}^{(1)})$. To show the injectivity of $\mathrm{Ev}_*^{(0)}$, let $x \in \varphi_{\infty n*}(K_0(\mathscr{A}_n)) \subset K_0(A_G)$ be such that $\mathrm{Ev}_*^{(0)}(x) = 0$ and $\bar{x} \in K_0(\mathscr{A}_n)$ be such that $x = \varphi_{\infty n*}(\bar{x})$. Since $0 = \mathrm{Ev}_*^{(0)}(x) = \eta_{\infty n*} \circ \mathrm{Ev}_{n*}^{(0)}(\bar{x}) = \eta_{\infty n*} \circ \mathrm{Ev}_{n*}^{(1)}(\bar{x})$, there exists $m > n+1$ such that $\eta_{m,n*} \circ \mathrm{Ev}_{n*}^{(0)}(\bar{x}) = \eta_{m,n*} \circ \mathrm{Ev}_{n*}^{(1)}(\bar{x}) = 0$. Then it follows that $\mathrm{Ev}_{m*}^{(0)} \circ \varphi_{m,n*}(\bar{x}) = \mathrm{Ev}_{m*}^{(1)} \circ \varphi_{m,n*}(\bar{x}) = 0$. Fix $i = 1 - i_m$. From the construction of $\iota_m^{(i)}$, there exists a unitary $\bar{y}_m^{(i)} \in A_m + \mathbb{C}1_{\mathscr{F}}$ such that $\iota_m^{(i)}(a) = \mathrm{Ad}\, \bar{y}_m^{(i)} U_m(a \otimes 1_{(3-i)^\infty})$ for $a \in A_m^{(i)}$.

From $(3,\ m)$ for $i = 1 - i_m$, it follows that $\bar{\eta}^{(i)}_{m*}\colon K_0(\mathrm{Im}(\iota^{(i)}_m)) \to K_0(A_m)$ is injective. Then we have $\mathrm{ev}^{(i)}_{m*} \circ \varphi_{mn*}(\bar{x}) = 0$. By Observation 4.7, we see that $\mathrm{ev}^{(i)}_{m+1*} \circ \varphi_{m+1,n*}(\bar{x}) = 0$. By the same argument as for $1-i_{m+1}$ $(= i_m)$ in $(3, m+1)$, we also have $\mathrm{ev}^{(1-i)}_{m+1*} \circ \varphi_{m+1,n*}(\bar{x}) = 0$. Because $\mathrm{ev}^{(0)}_{m+1*} \oplus \mathrm{ev}^{(1)}_{m+1*}\colon K_0(\mathscr{A}_{m+1}) \to K_0(\mathrm{Im}(\iota^{(0)}_{m+1})) \oplus K_0(\mathrm{Im}(\iota^{(1)}_{m+1}))$ is injective, it follows that $\varphi_{m+1,n*}(\bar{x}) = 0$, which implies that $x = \varphi_{\infty,n*}(\bar{x}) = 0$.

Let $y \in \mathrm{Im}(\iota^{(0)}_{\infty*}) \cap \mathrm{Im}(\iota^{(1)}_{\infty*}) \cap K_0(\mathscr{F})^+ \cong G^+$. To show the surjectivity of $\mathrm{Ev}^{(0)}_*$, since $(\mathrm{Im}(\iota^{(0)}_{\infty*}) \cap \mathrm{Im}(\iota^{(1)}_{\infty*}), \mathrm{Im}(\iota^{(0)}_{\infty*}) \cap \mathrm{Im}(\iota^{(1)}_{\infty*}) \cap K_0(\mathscr{F})^+)$ is an ordered abelian group, it suffices to show that $y \in \mathrm{Ev}^{(0)}_*(K_0(A_G)^+)$. Since $G^+ = \bigcup_{n\in\mathbb{N}} F_n$, by $(2,\ n)$, there exists $m \in \mathbb{N}$ such that $y \in \iota^{(i)}_{\infty*} \circ \eta^{(i)}_{\infty m*}(K_0(A^{(i)}_m)^+)$ for both $i = 0, 1$. Thus we have projections $p^{(i)}_m$ in $A^{(i)}_m \otimes M_N$, $i = 0, 1$, for some $N \in \mathbb{N}$ such that $y = \iota^{(i)}_{\infty*} \circ \eta^{(i)}_{\infty m*}([p^{(i)}_m]_0)$. Since $y = \eta_{\infty,m*} \circ \iota^{(i)}_{m*}([p^{(i)}_m]_0)$ for $i = 0, 1$, there exists $l > m$ such that $\eta_{l,m*} \circ \iota^{(0)}_{m*}([p^{(0)}_m]_0) = \eta_{l,m*} \circ \iota^{(1)}_{m*}([p^{(1)}_m]_0)$. Choose projections $p^{(i)}_l = \eta^{(i)}_{l,m} \otimes \mathrm{id}_{M_N}(p^{(i)}_m) \in A^{(i)}_l \otimes M_N$, $i = 0, 1$, which satisfy $\iota^{(0)}_{l*}([p^{(0)}_l]_0) = \iota^{(1)}_{l*}([p^{(1)}_l]_0)$ in $K_0(A_l)$. Then there exists a projection $\tilde{p}$ in $C([0,1]) \otimes A_l \otimes M_N$ such that $\tilde{p}(i) = \iota^{(i)}_l \otimes \mathrm{id}_{M_N}(p^{(i)}_l)$ for $i = 0, 1$. Regarding $\tilde{p}$ as a projection in $\mathscr{A}_l \otimes M_N$, we have $\mathrm{Ev}^{(0)}_{l*}([\tilde{p}]_0) = \iota^{(0)}_{l*}([p^{(0)}_l]_0)$ and

$$y = \eta_{\infty l*} \circ \iota^{(0)}_{l*}([p^{(0)}_l]_0) = \eta_{\infty l*} \circ \mathrm{Ev}^{(0)}_{l*}([\tilde{p}]_0) = \mathrm{Ev}^{(0)}_*(\varphi_{\infty l*}([\tilde{p}]_0)) \in \mathrm{Ev}^{(0)}_*(K_0(A_G)^+).$$

Thus, $\mathrm{Ev}^{(0)}_*$ is surjective and ${\mathrm{Ev}^{(0)}_*}^{-1}$ is also a positive group homomorphism.

In the rest of the proof, we show that A_G is a rationally AF algebra. We only show that $A_G \otimes M_{2^\infty}$ is approximately finite-dimensional, because by replacing even numbers by odd numbers the same argument allows us to see that $A_G \otimes M_{3^\infty}$ is approximately finite-dimensional. Since any separable local AF-algebra is exactly approximately finite-dimensional ([5, Thm. 2.2]), it suffices to show that for a given finite subset F of $\mathscr{A}^1_n$ and $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a finite-dimensional C*-subalgebra E of $\mathscr{A}_{n+N} \otimes M_{2^\infty}$ such that $\varphi_{n+N,n}(F) \otimes 1_{M_{2^\infty}} \subset_\varepsilon E$. Because the set of Lipschitz continuous functions is dense in $\mathscr{A}^1_n$, we may assume that F consists of Lipschitz continuous functions. Set $L = \max_{f\in F} \mathrm{Lip}(f)$, and let $m \in \mathbb{N}$ be such that $\frac{L+2}{2^m} + 4\sum_{j=m+1}^{\infty} \varepsilon_j < \varepsilon$. Because of $(8,\ n)$ and the inequality $\sum_{n\in\mathbb{N}} \varepsilon_n < 1$, for any $f \in F$, $s, t \in [0,1]$, and $l > m$, it follows that

$$\begin{aligned}\|\varphi_{n+m+l,n}(f)(s) - \varphi_{n+m+l,n}(f)(t)\| &\le \frac{\mathrm{Lip}(f)}{2^{l+m}} + 2\left(\sum_{j=n+1}^{n+m+l} \frac{\varepsilon_j}{2^{n+m+l-j}}\right) \\ &< \frac{L+2}{2^{m+1}} + 2\sum_{j=m+1}^{\infty} \varepsilon_j < \frac{\varepsilon}{2}.\end{aligned}$$

Choose $l \in \mathbb{N}$ such that, in addition to $l > m$, the number $n+m+l$ is even (odd for the case $A_G \otimes M_{3^\infty}$). Set $N = m+l$, $k = n+N$, and

$$\overline{E} = 1_{C([0,1])} \otimes \iota_k^{(0)}(A_k^{(0)}) \subset C([0,1]) \otimes A_k.$$

From the calculation above, it follows that $\varphi_{k,n}(F) \subset_{\varepsilon/2} \overline{E}$. Let us now modify $\overline{E}$ to a finite-dimensional C*-algebra E in $\mathscr{A}_k \otimes M_{2^\infty}$. By $(3,k)$ with $i_k = 0$, we obtain a unitary U_k in $\mathscr{F}^\sim$ such that $U_k \approx_{\varepsilon_k} 1_{\mathscr{F}}$ and $\operatorname{Ad} U_k(A_k^{(1)} \otimes M_{2^{\tilde{L}_k}}) = A_k$. By $(6,k)$ there exists a unitary $V_k^{(1)} \in \mathscr{F}^\sim$ such that $\iota_k^{(1)}(a) = \operatorname{Ad} V_k^{(1)}(a \otimes 1_{M_{2^\infty}}) \in A_k$ for any $a \in A_k^{(1)}$, and $V_k^{(1)} \in_{\varepsilon_k} A_k + \mathbb{C}1_{\mathscr{F}}$. Applying Lemma 4.6(v), we obtain a unitary $W \in A_k + \mathbb{C}1_{\mathscr{F}}$ such that $\operatorname{Ad} WU_k(a \otimes 1_{M_{2^\infty}}) = \operatorname{Ad} V_k^{(1)}(a \otimes 1)$ for $a \in A_k^{(1)}$. Define an isomorphism $\tilde{\epsilon}\colon M_{2^{\tilde{L}_k}} \to A_k \cap \operatorname{Im}(\iota_k^{(1)})'$ by $\tilde{\epsilon}(a) = \operatorname{Ad} WU_k(1_{A_k^{(1)}} \otimes a)$ for $a \in M_{2^{\tilde{L}_k}}$, and note that $A_k = \operatorname{Im}(\iota_k^{(1)}) \operatorname{Im}(\tilde{\epsilon})$. Denote by s the self-adjoint unitary in $\operatorname{Im}(\tilde{\epsilon}) \otimes M_{2^{\tilde{L}_k}}$ such that $\operatorname{Ad} s(\tilde{\epsilon}(x) \otimes y) = \tilde{\epsilon}(y) \otimes x$ for any $x, y \in M_{2^{\tilde{L}_k}}$. Choose a path of unitaries $\tilde{s} \in C([0,1]) \otimes A_k \otimes M_{2^{\tilde{L}_k}}$ such that $\tilde{s}(t) \in \operatorname{Im}(\tilde{\epsilon}) \otimes M_{2^{\tilde{L}_k}}$ for all $t \in [0,1]$, $\tilde{s}(0) = 1$, and $\tilde{s}(1) = s$. Define a finite-dimensional C*-algebra E by

$$E = \operatorname{Ad} \tilde{s}(\overline{E} \otimes 1_{M_{2^{\tilde{L}_k}}}) \subset C([0,1]) \otimes A_k \otimes M_{2^{\tilde{L}_k}}.$$

We can regard E as a finite-dimensional C*-subalgebra of $\mathscr{A}_k \otimes M_{2^{\tilde{L}_k}}$. Indeed, for $e \in E$, there exists $a_e \in \operatorname{Im}(\iota_k^{(0)})$ such that $e(0) = a_e \otimes 1_{M_{2^{\tilde{L}_k}}}$. Since $a_e \in A_k = \operatorname{Im}(\iota_k^{(1)}) \operatorname{Im}(\tilde{\epsilon})$, there are $x_l \in A_k^{(1)}$, $y_l \in M_{2^{\tilde{L}_k}}$, $l = 1, 2, \dots, L$, such that $a_e = \sum_{l=1}^{L} \iota_k^{(1)}(x_l)\tilde{\epsilon}(y_l)$. Then it follows that

$$e(1) = \operatorname{Ad} s(a_e \otimes 1_{M_{2^{\tilde{L}_k}}}) = \sum_{l=1}^{L} \iota_k^{(1)}(x_l) \otimes y_l \in \operatorname{Im}(\iota_k^{(1)}) \otimes M_{2^{\tilde{L}_k}}.$$

We conclude that for $f \in F$ there exists $x_f \in (A_k^{(1)})^1$ and $y_f \in (A_k^{(0)})^1$ such that

$$\begin{aligned} \varphi_{k,n}(f) \otimes 1_{M_{2^{\tilde{L}_k}}} &\approx_{\varepsilon/2} (1_{C([0,1])} \otimes \iota_k^{(1)}(x_f)) \otimes 1_{M_{2^{\tilde{L}_k}}} \\ &= \operatorname{Ad} \tilde{s}(1_{C([0,1])} \otimes \iota_k^{(1)}(x_f) \otimes 1_{M_{2^{\tilde{L}_k}}}) \\ &\approx_{\varepsilon/2} \operatorname{Ad} \tilde{s}(1_{C([0,1])} \otimes \iota_k^{(0)}(y_f) \otimes 1_{M_{2^{\tilde{L}_k}}}) \in E. \end{aligned}$$

In order to adjoin the property of $\mathcal{Z}$-absorption to A_G, we only need to consider $A_G \otimes \mathcal{Z}$ instead, which is also an RAF-algebra (as a UHF algebra is $\mathcal{Z}$-absorbing). Indeed, the ordered K_0-group $(K_0(A_G \otimes \mathcal{Z}), K_0(A_G \otimes \mathcal{Z})^+)$ is isomorphic to (G, G^+), because, in the proof of Proposition 4.3(ii), for an RAF-algebra A and the map $\iota_*\colon K_0(A) \to K_0(A \otimes \mathcal{Z})$ induced by the canonical embedding ι of A into $A \otimes \mathcal{Z}$, we saw that for $g \in K_0(A)$, $\iota_*(g) \in K_0(A \otimes \mathcal{Z})^+$ if and only if $ng \in K_0(A)^+$ for some $n \in \mathbb{N}$. Applying this fact to A_G, since $(K_0(A_G), K_0(A_G)^+) \cong$

(G, G^+) is an unperforated ordered abelian group, we see that the induced map from $K_0(A_G)$ to $K_0(A_G \otimes \mathcal{Z})$ is an isomorphism of ordered abelian groups.

Finally, to make $A_G \otimes \mathcal{Z}$ stable, we just need to replace it by the tensor product $A_G \otimes \mathcal{Z} \otimes \mathcal{K}$ by the algebra $\mathcal{K}$ of compact operators on a separable infinite-dimensional Hilbert space. □

Note that the invariant considered in Theorem 4.4, the ordered K_0-group, is shown in Theorem 5.3 below to be complete, for stable, separable, $\mathcal{Z}$-absorbing RAF-algebras. Theorem 4.4 is therefore a range of the invariant theorem for the class in question.

As a consequence of Theorem 4.4, together with its proof, and also the isomorphism result Theorem 5.3, we can determine the range of a generalization of the invariant of [20] for non-stable AF-algebras – referred to in [20] as the *dimension range*. In Corollary 5.4 below, we shall show (using Theorem 5.3) that this analogue of the non-stable AF invariant, what might now be called the matrix dimension range, is complete – for not necessarily stable, separable, $\mathcal{Z}$-absorbing RAF-algebras.

Recall that in [20] the invariant for general (separable) AF-algebras, the range of the Murray–von Neumann dimension – the local abelian semigroup of equivalence classes of projections in the algebra, or, equivalently, the subset of the K_0-group consisting of those equivalence classes – was characterized as an upward directed, hereditary, generating subset of the positive cone of the ordered K_0-group – and, more abstractly, as an arbitrary such subset of a dimension group (by [21] and [18], an unperforated countable ordered abelian group with the interpolation property of [1], equivalent to the decomposition property of [61]).

To extend this invariant to RAF-algebras, since there are fewer projections, we must keep track of the increasing sequence of dimension ranges of matrix algebras over the algebra; let us consider these as subsets of the ordered K_0-group. Of course, in the stable case, each of these will be the whole positive cone of the K_0-group. Let us call this structure the *matrix dimension range*.

Corollary 4.8. *The matrix dimension range of a $\mathcal{Z}$-absorbing separable RAF-algebra can be described in terms of the order-unit K_0-group of the algebra with unit adjoined, tensored with $\mathcal{Z}$, as follows. The* nth *level of the matrix dimension range, $n \in \mathbb{N}$, is the set of positive elements of the K_0-group of the given algebra which, with respect to the embedding of this in the K_0-group of the algebra with unit adjoined, are majorized by n times the class of the unit. (A $\mathcal{Z}$-absorbing RAF-algebra has cancellation, so this is the same as comparison in the algebra.) The matrix dimension range in fact determines the larger ordered abelian group. Hence for any countable rational dimension group with specified order unit for which there*

exists a positive map onto $\mathbb{Z}$ *taking the order unit into* $1 \in \mathbb{Z}$*, there exists a* $\mathcal{Z}$*-absorbing separable RAF-algebra with ordered* K_0*-group the kernel of the map onto* $\mathbb{Z}$*, and with matrix dimension range as described above with respect to the specified order unit.*

Proof. First, let us check that a $\mathcal{Z}$-absorbing separable RAF-algebra A has cancellation of projections – the Murray–von Neumann semigroup of A is mapped injectively into $K_0(A)$. It is enough to consider the stable case. By Proposition 4.3(ii), $K_0(A)$ is a rational dimension group. By Theorem 4.4 together with its proof there exists a stable, $\mathcal{Z}$-absorbing, RAF-algebra B with $K_0(B)$ isomorphic to $K_0(A)$ as an ordered abelian group, and such that B is the inductive limit of a sequence of point-line algebras (called one-dimensional non-commutative CW complexes in [19]), each tensored with $\mathcal{Z}$. It is straightforward to show (using that the Murray–von Neumann semigroup of $\mathcal{Z}$ is the same as that of the complex numbers – recall that $\mathcal{Z}$ has stable rank one and therefore cancellation) that the Murray–von Neumann semigroup of a point-line algebra after tensoring with $\mathcal{Z}$ is the same as before this operation, namely, a certain subsemigroup of the finite direct sum of copies of the (cancellative) semigroup of natural numbers (including zero), indexed by the points at infinity in the spectrum – see the discussion of point-line algebras in [24]. Thus, B has cancellation. By Theorem 5.3 below, A is isomorphic to B.

Now, given a countable rational dimension group G, by Theorem 4.4 (now just the statement), there exists a stable, separable, $\mathcal{Z}$-absorbing RAF-algebra B with $K_0(B)$ isomorphic to G as an ordered abelian group. In particular, consider the case that, as an ordered abelian group, $G = H + H'$ where H is an order ideal and H' is isomorphic in the relative order to the ordered group $\mathbb{Z}$. In this case, since (see proof of Corollary 5.4 below) B has the ideal property (closed two-sided ideals generated as such by projections), and also (as shown above) has cancellation (so that the equivalence classes of projections are the same as their K_0-classes), closed two-sided ideals of B are in exact correspondence with the order ideals of $K_0(B)$. With A_0 the ideal of B corresponding to the order ideal $H \subset H + H' = K_0(B)$, denote by A the hereditary C*-subalgebra $eA_0e \subset A_0 \subset B$ where $e \in B$ is a projection with $K_0(e) = 1 \in H' \cong \mathbb{Z}$. Then A is as desired, i.e., $K_0(A^\sim \otimes \mathcal{Z}) \cong G$. □

§5. Classification of $\mathcal{Z}$-absorbing RAF-algebras

In this section, we prove that the invariant considered in Section 4, the ordered K_0-group together with what we propose to call the matrix dimension range (see

Corollary 4.8), is complete (Theorem 5.3 in the stable case and Corollary 5.4 in the non-stable case).

We begin with the observation that the hypothesis of a unit in the otherwise completely general deformation isomorphism theorem for $\mathcal{Z}$-absorbing separable C*-algebras due to Winter – [75, Prop. 4.5] (see also [51]) – is not used, as is seen on replacing all unitaries appearing in the statement and in the proof by the corresponding quasiunitaries, with the convention that the inner automorphism determined by a quasiunitary is just that determined by the corresponding unitary in the unitization, obtained (by definition) by adding the unit to it.

Proposition 5.1 (Essentially [75, Prop. 4.5], cf. proof of [38, Thm. 14.3]). *Let $\mathfrak{p}$ and $\mathfrak{q}$ be relatively prime supernatural numbers. Suppose that A and B are separable $\mathcal{Z}$-absorbing C*-algebras and let $\varphi\colon A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be a quasiunitarily suspended $C([0,1])$-isomorphism (as in [75, Def. 4.2] with unitaries replaced by quasiunitaries). Then there is an isomorphism $\widetilde{\varphi}\colon A \to B \otimes \mathcal{Z}$ such that*

$$\widetilde{\varphi} \approx_{\mathrm{au}} (\mathrm{id}_B \otimes \bar{\sigma}_{\mathfrak{p},\mathfrak{q}}) \circ \varphi \circ (\mathrm{id}_A \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}),$$

where $\bar{\sigma}_{p,q}$ is the standard embedding $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \hookrightarrow \mathcal{Z}$ of [75, Prop. 3.4].

Proof. The proof is exactly the same as the proof of [75, Prop. 4.5], with unitaries replaced by the quasiunitaries which in the present context (quasiunitarily suspended $C([0,1])$-isomorphism) they correspond to. Note that, when in the proof of [75, Prop. 4.5] (spread over [75, Sects 4.3, 4.4, and 4.5]), the product of two (or three) unitaries appears, these unitaries and therefore also the product correspond to quasiunitaries, which they should be replaced by. □

Lemma 5.2 (Essentially [4, Thm. 2.3]). *Let A and B be (separable) AF-algebras, and let φ_0 and φ_1 be C*-algebra homomorphisms from A to B that agree on the ordered K_0-group. It follows that φ_0 and φ_1 are (one-parameter) asymptotically quasiunitarily equivalent: there exists a one-parameter family of unitaries u_t, $t \in [0,1)$, in $B^\sim$ such that $u_t - 1_{B^\sim} \in B$, $0 \le t < 1$ (i.e., $u_t - 1_{B^\sim}$ is a quasiunitary in B), $u_0 = 1_{B^\sim}$, and $\lim_{t\to 1} \mathrm{Ad}\, u_t \circ \varphi_0 = \varphi_1$.*

Proof. This holds by the proof of [4, Thm. 2.3]. □

Theorem 5.3 (Cf. [75, Prop. 4.6]). *Suppose that A and B are stable separable RAF-algebras absorbing the Jiang–Su algebra tensorially. If there is an ordered group isomorphism γ from $(K_0(A), K_0(A)_+)$ to $(K_0(B), K_0(B)_+)$, then A is isomorphic to B and there exists an isomorphism α from A to B such that $\alpha_* = \gamma$ at the level of K_0-groups.*

Proof. By Definition 4.1, there are relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ such that $A \otimes M_{\mathfrak{p}}$ and $A \otimes M_{\mathfrak{q}}$ are AF. We may suppose (enlarging them if necessary) that $\mathfrak{p}$ and $\mathfrak{q}$ are infinite. Hence by Lemma 4.2(i), $B \otimes M_{\mathfrak{p}}$ and $B \otimes M_{\mathfrak{q}}$ are AF. It is immediate that the maps

$$\gamma \otimes \mathrm{id}_{\mathbb{D}_{\mathfrak{p}}} : K_0(A) \otimes \mathbb{D}_{\mathfrak{p}} \to K_0(B) \otimes \mathbb{D}_{\mathfrak{p}} \quad \text{and} \quad \gamma \otimes \mathrm{id}_{\mathbb{D}_{\mathfrak{q}}} : K_0(A) \otimes \mathbb{D}_{\mathfrak{q}} \to K_0(B) \otimes \mathbb{D}_{\mathfrak{q}}$$

are isomorphisms of ordered groups, and by Proposition 4.3(i) these maps may be viewed as ordered group isomorphisms

$$\gamma_{\mathfrak{p}} : K_0(A \otimes M_{\mathfrak{p}}) \to K_0(B \otimes M_{\mathfrak{p}}) \quad \text{and} \quad \gamma_{\mathfrak{q}} : K_0(A \otimes M_{\mathfrak{q}}) \to K_0(B \otimes M_{\mathfrak{q}}).$$

By [20], there are isomorphisms of AF-algebras

$$\varphi_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \to B \otimes M_{\mathfrak{p}} \text{ with } \varphi_{\mathfrak{p}*} = \gamma_{\mathfrak{p}} \quad \text{and} \quad \varphi_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \to B \otimes M_{\mathfrak{q}} \text{ with } \varphi_{\mathfrak{q}*} = \gamma_{\mathfrak{q}}.$$

Since $\varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ and $\varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ give rise to the same K_0-map (in the obvious sense), $\gamma \otimes \mathrm{id}_{\mathbb{D}_{\mathfrak{p}}} \otimes \mathrm{id}_{\mathbb{D}_{\mathfrak{q}}} : K_0(A \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) \to K_0(B \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}})$, by Lemma 5.2 there is a quasiunitarily suspended $C([0,1])$-isomorphism $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ agreeing with $\varphi_{\mathfrak{p}}$ and $\varphi_{\mathfrak{q}}$ at the endpoints of the interval $[0,1]$.

Hence by Proposition 5.1, there exists an isomorphism $\widetilde{\varphi}\colon A \to B \otimes \mathcal{Z}$ such that $\widetilde{\varphi} \approx_{\mathrm{au}} (\mathrm{id}_B \otimes \bar{\sigma}_{\mathfrak{p},\mathfrak{q}}) \circ \varphi \circ (\mathrm{id}_A \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})$. Hence (as in [75, Prop. 4.6]), $\widetilde{\varphi}_* = \gamma$. □

Corollary 5.4. *Let A and B be $\mathcal{Z}$-absorbing, separable RAF-algebras (not necessarily stable). Suppose that the invariants of A and B described in Corollary 4.8 are isomorphic. Then this isomorphism is induced by an isomorphism of the* C*-*algebras A and B.*

Proof. Note first that $A^{\sim} \otimes \mathcal{Z}$ and $B^{\sim} \otimes \mathcal{Z}$ are RAF, as (see [11]) an extension of one AF-algebra by another is AF. By definition – see Corollary 4.8 – there is an isomorphism of order-unit groups $K_0(A^{\sim} \otimes \mathcal{Z})$ and $K_0(B^{\sim} \otimes \mathcal{Z})$, respecting the subgroups $K_0(A)$ and $K_0(B)$. By Theorem 5.3, there is an isomorphism of the stabilizations $A^{\sim} \otimes \mathcal{Z} \otimes \mathcal{K}$ and $B^{\sim} \otimes \mathcal{Z} \otimes \mathcal{K}$ giving rise to the given isomorphism of K_0-groups. In particular, it takes the cutdown of $A^{\sim} \otimes \mathcal{Z} \otimes \mathcal{K}$ by $1_{A^{\sim} \otimes \mathcal{Z}} \otimes e_{11}$ into the cutdown of $B^{\sim} \otimes \mathcal{Z} \otimes \mathcal{K}$ by a projection with K_0-class equal to the class of $1_{B^{\sim} \otimes \mathcal{Z}} \otimes e_{11}$, and therefore (by cancellation – Corollary 4.8) Murray–von Neumann equivalent to it – in fact unitarily equivalent to it as the C*-algebra is stable. So we may assume that the isomorphism takes $A^{\sim} \otimes \mathcal{Z} = A^{\sim} \otimes \mathcal{Z} \otimes e_{11}$ onto $B^{\sim} \otimes \mathcal{Z} = B^{\sim} \otimes \mathcal{Z} \otimes e_{11}$, and reproduces the given isomorphism of $K_0(A^{\sim} \otimes \mathcal{Z})$ with $K_0(B^{\sim} \otimes \mathcal{Z})$.

Now note that any stable RAF algebra has the ideal property – any closed two-sided ideal (let us just say "ideal") is generated (as an ideal) by its projections.

This holds because it holds in an AF algebra. More precisely, by [5] (see also [15]), every ideal in an inductive limit C*-algebra is the inductive limit of its finite-stage inverse limits, which shows that the ideals of the tensor product of any C*-algebra with a UHF algebra are just the tensor products of the ideals of the given algebra with the UHF algebra (given that this is true for a finite matrix algebra in place of the UHF). If the given algebra is RAF, so that the tensor product is AF, the ideal in the tensor product corresponding to a given ideal is generated (as an ideal) by projections in the AF algebra. By Proposition 4.3(i), a multiple of any K_0-element of the tensor product belongs to the K_0-group of the canonical image of the given algebra. Since the tensor product is AF, and the given algebra is stable, this says that any projection in the tensor product is equivalent to a projection in the given algebra, and of course in the same ideal. Such projections therefore generate the given ideal.

(In fact, this argument shows that, in the stable case, every ideal of an RAF algebra has an approximate unit consisting of projections. Indeed, in the case of an ideal with compact spectrum, the corresponding ideal in the (AF) tensor product with a UHF algebra (which has the same spectrum) is generated by a single projection – as is seen by looking at the finite-dimensional finite stages in an inductive limit decomposition – and therefore the given ideal of the RAF algebra is generated by a single projection. Hence by Brown's theorem ([10]) the given ideal (assumed to be separable, as well as stable) is isomorphic to the stabilization of the cutdown by this projection, which has an approximate unit consisting of projections. Since the spectrum of a (separable) AF algebra, and therefore of an RAF-algebra, is an increasing union of compact open sets (the spectra of ideals generated by a single projection), and so the algebra is the closure of the corresponding increasing sequence of ideals, it follows that the whole (stable) RAF-algebra has an approximate unit consisting of projections. This property could also be used to prove Theorem 5.3 above, using [75, Prop. 4.5] directly (for the unital case), provided that one also established a uniqueness theorem.)

The proof of the present non-stable isomorphism theorem is now in hand. The isomorphism of the stabilized algebras $A^\sim \otimes \mathcal{Z} \otimes \mathcal{K}$ and $B^\sim \otimes \mathcal{Z} \otimes \mathcal{K}$, giving rise to the given isomorphism of the order-unit groups $K_0(A^\sim \otimes \mathcal{Z})$ and $K_0(B^\sim \otimes \mathcal{Z})$, since this respects the canonical order ideals $K_0(A) = K_0(A \otimes \mathcal{Z})$ and $K_0(B) = K_0(B \otimes \mathcal{Z})$, and since the ideals $A \otimes \mathcal{Z} \otimes \mathcal{K}$ and $B \otimes \mathcal{Z} \otimes \mathcal{K}$ are generated by their projections (and because of cancellation – see the proof of Corollary 4.8 – which implies that equivalence classes of projections are the same as their K_0-classes), restricts to an isomorphism of $A = A \otimes \mathcal{Z}$ with $B = B \otimes \mathcal{Z}$ giving rise to the given isomorphism of $K_0(A)$ with $K_0(B)$. □

§6. KMS states of $\mathcal{Z}$-absorbing C*-algebras

In order to show the main result, Theorem 6.5, let us prepare some facts for ideals and traces of RAF-algebras. By an ideal of a C*-algebra, we shall mean a closed two-sided ideal. For a C*-algebra A, we shall denote by $\mathcal{I}_A$ the set of all ideals in A. For a supernatural number $\mathfrak{n}$, we shall consider the map $\Phi_{\mathfrak{n}}$ from $\mathcal{I}_A$ to $\mathcal{I}_{A\otimes M_{\mathfrak{n}}}$ defined by $\Phi_{\mathfrak{n}}(I) = I \otimes M_{\mathfrak{n}} \subset A \otimes M_{\mathfrak{n}}$ for $I \in \mathcal{I}_A$.

Let (G, G^+) be an ordered abelian group and let γ_j, $j \in J$, be automorphisms of G as an ordered abelian group. If no order ideal of (G, G^+) other than 0 or G is invariant under all γ_j, $j \in J$, we shall say that (G, G^+) is $\{\gamma_j\}_{j\in J}$*-simple*. For a C*-algebra A and automorphisms α_j, $j \in J$, of A, if no ideal of A other than 0 or A is invariant under all α_j, $j \in J$, we shall say that A is $\{\alpha_j\}_{j\in J}$*-simple*.

Lemma 6.1. *The following two statements hold:*

(i) *For any* C*-*algebra* A *and supernatural number* $\mathfrak{n}$*, the map* $\Phi_{\mathfrak{n}} \colon \mathcal{I}_A \to \mathcal{I}_{A\otimes M_{\mathfrak{n}}}$ *is bijective.*

(ii) *Suppose that* A *is a* $\mathcal{Z}$*-absorbing RAF-algebra and* α_j*,* $j \in J$*, are automorphisms of* A*. If the ordered abelian group* $(K_0(A), K_0(A)^+)$ *is* $\{\alpha_{j*}\}_{j\in J}$*-simple, then* A *is* $\{\alpha_j\}_{j\in J}$*-simple.*

Proof. (i) Given an ideal of I of $A \otimes M_{\mathfrak{n}}$, set $I_A = \{a \in A : a \otimes 1_{M_{\mathfrak{n}}} \in I\}$, which is an ideal of A. Then $I_A \otimes M_N = I \cap (A \otimes M_N)$ for any $N \in \mathbb{N}$ with $N|\mathfrak{n}$. Indeed, if $x \in I \cap (A \otimes M_N)$ is written as $x = \sum_{i,j=1}^N a_{ij} \otimes e_{ij}^{(N)}$ for some $a_{ij} \in A$ and system of matrix units $\{e_{ij}^{(N)}\}_{i,j=1}^N$ of M_N, then, using an approximate unit h_λ, $\lambda \in \Lambda$, of A we have

$$a_{ij} \otimes e_{ij}^{(N)} = \lim_{\lambda\to\infty} (h_\lambda \otimes e_{ii}^{(N)}) x (h_\lambda \otimes e_{jj}^{(N)}) \in I \cap (A \otimes M_N).$$

It follows that $a_{ij} \otimes 1_{M_N} \in I$, which means $a_{ij} \in I_A$ for all $i, j = 1, 2, \dots, N$. The converse inclusion $I_A \otimes M_N \subset I \cap (A \otimes M_N)$ is trivial. Thus it follows that

$$I = \overline{\bigcup_{N|\mathfrak{n}} I \cap (A \otimes M_N)} = \overline{\bigcup_{N|\mathfrak{n}} I_A \otimes M_N} = \Phi_{\mathfrak{n}}(I_A)$$

(see [5, Lem. 3.1], and see also [15, Lem. III 4.1]). It is straightforward to show the injectivity of $\Phi_{\mathfrak{n}}$. Indeed, for I_A, $J_A \in \mathcal{I}_A$ with $\Phi_{\mathfrak{n}}(I_A) = \Phi_{\mathfrak{n}}(J_A)$, choosing an approximate unit k_λ, $\lambda \in \Lambda$, of I_A, we have

$$\lim_{\lambda\to\infty} (ak_\lambda) \otimes 1_{M_{\mathfrak{n}}} = \lim_{\lambda\to\infty} (a \otimes 1_{M_{\mathfrak{n}}})(k_\lambda \otimes 1_{M_{\mathfrak{n}}}) = a \otimes 1_{M_{\mathfrak{n}}},$$

for any $a \in J_A$. Then it follows that $a = \lim_{\lambda\to\infty} ak_\lambda \in I_A$. This shows that $J_A \subset I_A$.

(ii) Let $\mathfrak{n}$ be an infinite supernatural number and let $I \in \mathcal{I}_A$ be invariant under all α_j, $j \in J$. Then the ideal $\Phi_{\mathfrak{n}}(I) \in \mathcal{I}_{A\otimes M_{\mathfrak{n}}}$ is also invariant under all $\alpha_j \otimes \mathrm{id}_{M_{\mathfrak{n}}}$, $j \in J$. By (i), it suffices to show that $A \otimes M_{\mathfrak{n}}$ is $\{\alpha_j \otimes \mathrm{id}_{M_{\mathfrak{n}}}\}_{j\in J}$-simple. In the case that $A \otimes M_{\mathfrak{n}}$ is an AF-algebra, it is well known that $\mathcal{I}_{A\otimes M_{\mathfrak{n}}}$ corresponds to the set of all order ideals of $(K_0(A\otimes M_{\mathfrak{n}}), K_0(A\otimes M_{\mathfrak{n}})^+)$. Hence it is enough to show that $(K_0(A \otimes M_{\mathfrak{n}}), K_0(A \otimes M_{\mathfrak{n}})^+)$ is $\{(\alpha_j \otimes \mathrm{id}_{M_{\mathfrak{n}}})_*\}_{j\in J}$-simple. By Proposition 4.3(i), $(K_0(A \otimes M_{\mathfrak{n}}), K_0(A \otimes M_{\mathfrak{n}})^+)$ is isomorphic to $(K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}, K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}{}^+)$ as an ordered abelian group, and in such a way that $(\alpha_j \otimes \mathrm{id}_{M_{\mathfrak{n}}})_*$, $j \in J$, corresponds to $\alpha_{j*} \otimes \mathrm{id}_{\mathbb{D}_{\mathfrak{n}}}$, $j \in J$. Hence we only need to show that $(K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}, K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}{}^+)$ is $\{\alpha_{j*} \otimes \mathrm{id}_{\mathbb{D}_n}\}_{j\in J}$-simple.

Let H be an order ideal of $(K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}, K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}{}^+)$ which is invariant under all $\alpha_{j*} \otimes \mathrm{id}_{D_{\mathfrak{n}}}$, $j \in J$. Set $H_A = \{g \in K_0(A) : g \otimes 1_{\mathbb{D}_{\mathfrak{n}}} \in H\}$, and $H_A^+ = H_A \cap K_0(A)^+$. Since $H^+ = H \cap (K_0(A) \otimes \mathbb{D}_{\mathfrak{n}})^+$ is hereditary (i.e., if $g \in K_0(A) \otimes \mathbb{D}_n{}^+$ and $h \in H^+$ satisfy $g \le h$, then $g \in H^+$), it is trivial to see that H_A^+ is also hereditary. To show that (H_A, H_A^+) is an ideal of $(K_0(A), K_0(A)^+)$, we must show that $H_A = H_A^+ - H_A^+$. We shall use the fact that the ordered abelian group $(K_0(A), K_0(A)^+)$ is unperforated, by Proposition 4.3(ii). For $x \in H_A$, since $H = H^+ - H^+$ there exist y, $z \in H^+$ such that $x \otimes 1_{\mathbb{D}_{\mathfrak{n}}} = y - z$. From y, $z \in K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}{}^+$, we obtain $N \in \mathbb{N}$ with $N|\mathfrak{n}$ and $y_A, z_A \in K_0(A)^+$ such that $Ny = y_A \otimes 1_{\mathbb{D}_{\mathfrak{n}}}$ and $Nz = z_A \otimes 1_{\mathbb{D}_{\mathfrak{n}}}$. Since $Nx = y_A - z_A \le Ny_A$, it follows that $x \le y_A$, which implies that $x = y_A - (y_A - x) \in H_A^+ - H_A^+$. Since H is invariant under all $\alpha_{j*} \otimes \mathrm{id}_{\mathbb{D}_{\mathfrak{n}}}$, $j \in J$, we see that H_A is also invariant under all α_{j*}, $j \in J$, which implies that $H_A = 0$ or $K_0(A)$. Then it follows that $H = 0$ or $K_0(A) \otimes \mathbb{D}_{\mathfrak{n}}$, as required. □

Proposition 6.2. *Let A be a $\mathcal{Z}$-absorbing RAF-algebra, α an automorphism of A, and σ an automorphism of $\mathcal{Z}$. Suppose that the ordered abelian group $(K_0(A), K_0(A)^+)$ is α_*-simple and σ has the weak Rohlin property (see [66, Def. 1.1]). Then the crossed product C*-algebra $(A \otimes \mathcal{Z}) \rtimes_{\alpha\otimes\sigma} \mathbb{Z}$ is simple.*

Proof. Since $(K_0(A), K_0(A)^+)$ is α_*-simple, it follows that $(K_0(A \otimes \mathcal{Z}), K_0(A \otimes \mathcal{Z})^+)$ $(\cong (K_0(A), K_0(A)^+))$ is $(\alpha\otimes\sigma)_*$-simple. By Lemma 6.1(ii), therefore $A \otimes \mathcal{Z}$ is $\alpha \otimes \sigma$-simple. Denote by u the implementing unitary of $\alpha \otimes \sigma$. As in a similar argument in the proof of [22, Thm. 3.2] (see also [48]), it suffices to show that for any $x \in A\otimes\mathcal{Z}$, any finite subset F of $\mathbb{Z}\backslash\{0\}$, and any finite family $\{y_i\}_{i\in F} \subset A\otimes\mathcal{Z}$,

$$\|x\| \le \|x + \sum_{i\in F} y_i u^i\|.$$

Set $k = \max\{|i| : i \in F\}$, $k_0 = k_1 = k$, and $k_2 = k + 1$. Applying [50, Thm. 6.4] to $\sigma \in \mathrm{Aut}(\mathcal{Z})$, we obtain positive contractions $f_{j,n}^{(l)} \in \mathcal{Z}$, $l = 0, 1, 2$,

$j = 0, 1, \ldots, k_l$, $n \in \mathbb{N}$ such that

$$\sum_{l=0}^{2} \sum_{j=0}^{k_l} f_{j,n}^{(l)} = 1_{\mathcal{Z}},$$

each $(f_{j,n}^{(l)})_{n\in\mathbb{N}}$ is a central sequence in $\mathcal{Z}$ for all $l = 0, 1, 2$ and $j = 0, 1, \ldots, k_l$, $\lim_{n\to\infty} \|\sigma(f_{j,n}^{(l)}) - f_{j+1,n}^{(l)}\| = 0$ for all $l = 0, 1, 2$ and $0 \le j \le k_l \bmod (k_l + 1)$, and $\lim_{n\to\infty} \|f_{i,n}^{(l)} f_{j,n}^{(l)}\| = 0$ for all $l = 0, 1, 2$ and $0 \le i \ne j \le k_l$. Let h_n, $n \in \mathbb{N}$, be an increasing approximate unit of A. For any $n \in \mathbb{N}$, define a completely positive map Φ_n from $(A \otimes \mathcal{Z}) \rtimes_{\alpha\otimes\sigma} \mathbb{Z}$ to $(A \otimes \mathcal{Z}) \rtimes_{\alpha\otimes\sigma} \mathbb{Z}$ by

$$\Phi_n(a) = \sum_{l=0}^{2} \sum_{j=0}^{k_l} (h_n \otimes f_{j,n}^{(l)})^{1/2} a (h_n \otimes f_{j,n}^{(l)})^{1/2},$$

for $a \in (A \otimes \mathcal{Z}) \rtimes_{\alpha\otimes\sigma} \mathbb{Z}$. Note that Φ_n is a contraction for each $n \in \mathbb{N}$. Since $(h_n \otimes f_{j,n}^{(l)})_{n\in\mathbb{N}}$ is a central sequence in $A \otimes \mathcal{Z}$, it follows that $\lim_{n\to\infty} \|\Phi_n(a) - a\| = 0$ for any $a \in A \otimes \mathcal{Z}$. Since

$$\lim_{n\to\infty} \|\sigma^i(f_{j,n}^{(l)}) f_{j,n}^{(l)}\| = \lim_{n\to\infty} \|f_{i+j,n}^{(l)} f_{j,n}^{(l)}\| = 0,$$

where $i + j$ is considered $\bmod(k_l + 1)$ for $i \in F$ and $j \in \{0, 1, \ldots, k_l\}$, we have $\lim_{n\to\infty} \|\Phi_n(u^i)\| = 0$ for any $i \in F$ (where $\Phi_n(u^i)$ is defined in a natural way). Then we have

$$\begin{aligned} \|x\| &\le \limsup_{n\to\infty} \left\| \Phi_n\left(x + \sum_{i\in F} y_i u^i \right) \right\| + \left\| \Phi_n\left(\sum_{i\in F} y_i u^i \right) \right\| \\ &\le \left\| x + \sum_{i\in F} y_i u^i \right\| + \limsup_n \sum_{i\in F} \|y_i \Phi_n(u^i)\| \\ &= \left\| x + \sum_{i\in F} y_i u^i \right\|. \end{aligned}$$

□

For a C*-algebra A, we denote by $\mathcal{T}(A)$ the cone of densely defined lower semicontinuous traces of A. With $\mathrm{Ped}(A)$ the Pedersen ideal of A [57], it is well known that $\tau(p) < \infty$ and $p \in \mathrm{Ped}(A)$ for any $\tau \in \mathcal{T}(A)$ and projection p in A. Set $\mathrm{Ped}(A)^+ = \mathrm{Ped}(A) \cap A^+$. For $n \in \mathbb{N}$, taking the normalized trace tr_n of M_n we set $\tau \otimes \mathrm{tr}_n(a \otimes b) = \tau(a)\mathrm{tr}_n(b)$ for $\tau \in \mathcal{T}(A)$, and $a \otimes b \in \mathrm{Ped}(A) \otimes M_n$. By the following Lemma 6.3(i), we shall see that $\mathrm{Ped}(A) \otimes M_n = \mathrm{Ped}(A \otimes M_n)$, so that $\tau \otimes \mathrm{tr}_n$ can be regarded as an element of $\mathcal{T}(A \otimes M_n)$. To simplify, we use the same symbol τ for $\tau \otimes \mathrm{tr}_n$.

For an ordered abelian group (G, G^+), we define $S_0(G)$ as the set of positive group homomorphisms from G to $\mathbb{R}$. In the case that $(K_0(A), K_0(A)^+)$ is an ordered abelian group, and A has cancellation, we define the standard affine map $\Phi_{\mathcal{T}}\colon \mathcal{T}(A) \to S_0(K_0(A))$ by $\Phi_{\mathcal{T}}(\tau)([p]_0) = \tau(p)$ for $[p]_0 \in K_0(A)^+$. Note that, by the cancellation property of A, projections p and q in $A \otimes M_n$, $n \in \mathbb{N}$, are Murray–von Neumann equivalent if and only if $[p]_0 = [q]_0$ in $K_0(A)$. Then $\Phi_{\mathcal{T}}(\tau)$ is well defined for $\tau \in \mathcal{T}(A)$. We shall often use the notation τ_* for $\Phi_{\mathcal{T}}(\tau)$.

Consider the subset A_0 of the C*-algebra A defined by $A_0 = \{a \in A : a \in pAp \text{ for some projection } p \in A\}$. Equip $\mathcal{T}(A)$ with the smallest topology such that $\mathcal{T}(A) \ni \tau \mapsto \tau(a)$ is continuous for any $n \in \mathbb{N}$ and $a \in (A \otimes M_n)_0$, and equip $S_0(G)$ with the topology of pointwise convergence.

Lemma 6.3. *The following two statements hold:*

(i) *For a* C*-*algebra* A *and* $n \in \mathbb{N}$, *it follows that* $\mathrm{Ped}(A) \otimes M_n = \mathrm{Ped}(A \otimes M_n)$.

(ii) *If* A *is a real rank zero* C*-*algebra with cancellation, and* $(K_0(A), K_0(A)^+)$ *is an ordered abelian group, then a net* $\tau_\lambda \in \mathcal{T}(A)$, $\lambda \in \Lambda$, *converges to* $\tau \in \mathcal{T}(A)$ *if and only if the net* $\Phi_{\mathcal{T}}(\tau_\lambda)$, $\lambda \in \Lambda$, *converges to* $\Phi_{\mathcal{T}}(\tau)$ *in* $S_0(K_0(A))$.

Proof. (i) Let us first check the inclusion $\mathrm{Ped}(A) \otimes M_n \subset \mathrm{Ped}(A \otimes M_n)$. From the construction of the Pedersen ideal, one sees that $\mathrm{Ped}(A) \otimes e_{ii}^{(n)} \subset \mathrm{Ped}(A \otimes M_n)$, $i = 1, 2, \ldots, n$. Since the Pedersen ideal is equal to its square, it is an ideal of the multiplier algebra of the C*-algebra. Applying this to $\mathrm{Ped}(A \otimes M_n)$, we see that also $\mathrm{Ped}(A) \otimes e_{ij}^{(n)} \subset \mathrm{Ped}(A \otimes M_n)$, $i, j = 1, 2, \ldots, n$.

Since the Pedersen ideal is the smallest dense two-sided ideal, the two ideals are equal.

(ii) The "only if" part of the statement is obvious. Assume that $\Phi_{\mathcal{T}}(\tau_\lambda)$, $\lambda \in \Lambda$, converges to $\Phi_{\mathcal{T}}(\tau)$ in $S_0(K_0(A))$. Let x be a self-adjoint element in $p(A \otimes M_n)p$ for some $n \in \mathbb{N}$ and a projection $p \in A \otimes M_n$. Without loss of generality, we may assume that $\tau(p) > 0$. Since $p(A \otimes M_n)p$ has real rank zero, for $\varepsilon > 0$ there exists a self-adjoint element $\bar{x} \in p(A \otimes M_n)p$ with finite spectrum such that $\|x - \bar{x}\| < \frac{\varepsilon}{4\tau(p)}$. By

$$\bar{x} - \frac{\varepsilon}{4\tau(p)} p \le x \le \bar{x} + \frac{\varepsilon}{4\tau(p)} p,$$

it follows that $|\tau_\lambda(x) - \tau_\lambda(\bar{x})| \le \varepsilon \tau_\lambda(p)/(4\tau(p))$ for any $\lambda \in \Lambda$. By $\lim_{\lambda\to\infty} \tau_\lambda(\bar{x}) = \tau(\bar{x})$ and $\lim_{\lambda\to\infty} \tau_\lambda(p) = \tau(p)$, we have that $\limsup_{\lambda\to\infty} |\tau_\lambda(x) - \tau(x)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, a net τ_λ, $\lambda \in \Lambda$, converges to τ in the topology of $\mathcal{T}(A)$. □

In the case of an RAF-algebra A, we note that for $\tau \in \mathcal{T}(A)$ the map τ_* on $K_0(A)$ is also well defined. Indeed, for $N \in \mathbb{N} \setminus \{1\}$ and two projections p, $q \in A \otimes M_N$, if $[p]_0 = [q]_0$ in $K_0(A)$, then $[p \otimes 1_{M_{N^\infty}}]_0 = [q \otimes 1_{M_{N^\infty}}]_0$ in $K_0(A \otimes M_{N^\infty})$.

Since $A \otimes M_{N^\infty}$ is an AF algebra, there exists a partial isometry $r \in A \otimes M_{N^\infty}$ such that $r^*r = p \otimes 1_{M_{N^\infty}}$ and $rr^* = q \otimes 1_{M_{N^\infty}}$. Taking a large $k \in \mathbb{N}$ we can obtain such an r in $A \otimes M_{N^k}$. Then it follows that $\tau(p) = \tau \otimes \mathrm{tr}_{N^k}(p \otimes 1_{M_{N^k}}) = \tau \otimes \mathrm{tr}_{N^k}(q \otimes 1_{M_{N^k}}) = \tau(q)$. Thus, for an RAF-algebra A, we can define $\Phi_{\mathcal{T}} \colon \mathcal{T}(A) \to S_0(K_0(A))$.

Proposition 6.4. *If A is a $\mathcal{Z}$-absorbing RAF-algebra, then the map $\Phi_{\mathcal{T}} \colon \mathcal{T}(A) \to S_0(K_0(A))$ is an affine homeomorphism.*

Proof. Note that, by Proposition 4.3(ii), $(K_0(A), K_0(A)^+)$ is an ordered abelian group. Let $\mathfrak{n}$ be an infinite supernatural number and ι the embedding of A into $A \otimes M_{\mathfrak{n}}$ defined by $\iota(a) = a \otimes 1_{M_{\mathfrak{n}}}$ for $a \in A$. Since $A \otimes M_{\mathfrak{n}}$ is an AF-algebra, we see that the canonical map $\Phi_{\mathcal{T}, A\otimes M_{\mathfrak{n}}} : \mathcal{T}(A \otimes M_{\mathfrak{n}}) \to S_0(K_0(A \otimes M_{\mathfrak{n}}))$ is bijective; see the last paragraph of [13] and [71, Lem. 3.5], for example.

Denote by $\tau_{\mathfrak{n}}$ the unique tracial state of $M_{\mathfrak{n}}$. Regarding $(K_0(A \otimes M_{\mathfrak{n}}), K_0(A \otimes M_{\mathfrak{n}})^+)$ as $(K_0(A) \otimes K_0(M_{\mathfrak{n}}), K_0(A) \otimes K_0(M_{\mathfrak{n}})^+)$ by Proposition 4.3(i), define $\tau_* \otimes \tau_{\mathfrak{n}*} \in S_0(K_0(A \otimes M_{\mathfrak{n}}))$ for $\tau \in \mathcal{T}(A)$ by $\tau_* \otimes \tau_{\mathfrak{n}*}([p]_0 \otimes [q]_0) = \tau(p)\tau_{\mathfrak{n}}(q)$ for l, $m \in \mathbb{N}$ and projections $p \in A \otimes M_l$, $q \in A \otimes M_m$. The expression $\tau \otimes \tau_{\mathfrak{n}}$ denotes the densely defined lower semicontinuous trace on $A \otimes M_{\mathfrak{n}}$ such that $\Phi_{\mathcal{T}, A\otimes M_{\mathfrak{n}}}(\tau \otimes \tau_{\mathfrak{n}}) = \tau_* \otimes \tau_{\mathfrak{n}*}$. In the following argument, we shall show that $\tau \otimes \tau_{\mathfrak{n}}(a \otimes 1_{M_{\mathfrak{n}}}) = \tau(a)$ for any $a \in \mathrm{Ped}(A)$, where we note that $a \otimes 1_{M_{\mathfrak{n}}} \in \mathrm{Ped}(A \otimes M_{\mathfrak{n}})$ for $a \in \mathrm{Ped}(A)$.

For $\varepsilon > 0$, consider the continuous function f_ε defined by

$$f_\varepsilon(t) = \max\{0, \min\{1, (t - \varepsilon)\}\} \quad \text{for } t \in \mathbb{R},$$

and set $a_\varepsilon = f_\varepsilon(a)$ for $a \in \mathrm{Ped}(A)^+$. Since $\tau \otimes \tau_{\mathfrak{n}}$ and τ are lower semicontinuous, it suffices to see that $\tau \otimes \tau_{\mathfrak{n}}(a_\varepsilon \otimes 1_{M_{\mathfrak{n}}}) = \tau(a_\varepsilon)$ for any $\varepsilon > 0$ and a contraction $a \in \mathrm{Ped}(A)^+$. Fix $\varepsilon > 0$ and a contraction $a \in \mathrm{Ped}(A)^+$. Then there exists $\bar{a} \in \mathrm{Ped}(A)^+$ such that $\bar{a}a_\varepsilon = a_\varepsilon$. Since $\overline{a_\varepsilon A a_\varepsilon} \otimes M_{\mathfrak{n}} = \overline{(a_\varepsilon \otimes 1_{M_{\mathfrak{n}}})A \otimes M_{\mathfrak{n}}(a_\varepsilon \otimes 1_{M_{\mathfrak{n}}})}$ is an AF-algebra, for $n \in \mathbb{N}$ there exist $N_n \in \mathbb{N}$ with $N_n | \mathfrak{n}$ and a positive element $a_n \in \overline{a_\varepsilon A a_\varepsilon} \otimes M_{N_n}$ with finite spectrum such that $\|a_n - a_\varepsilon \otimes 1_{M_{\mathfrak{n}}}\| < 1/n$. Since $a_n - \frac{1}{n}(\bar{a} \otimes 1_{M_{\mathfrak{n}}}) \le a_\varepsilon \otimes 1_{M_{\mathfrak{n}}} \le a_n + \frac{1}{n}(\bar{a} \otimes 1_{M_{\mathfrak{n}}})$, we have $\tau \otimes \tau_{\mathfrak{n}}(a_\varepsilon \otimes 1_{M_{\mathfrak{n}}}) = \lim_{n\to\infty} \tau \otimes \tau_{\mathfrak{n}}(a_n)$. On the other hand, since $\tau \otimes \tau_{\mathfrak{n}} \in \mathcal{T}(A \otimes B)$ corresponds to $\tau_* \otimes \tau_{\mathfrak{n}*} \in S_0(K_0(A))$, it follows that $\tau \otimes \tau_{\mathfrak{n}}(p) = \tau \otimes \mathrm{tr}_{N_n}(p)$ for any projection $p \in \overline{a_\varepsilon A a_\varepsilon} \otimes M_{N_n}$. Thus we have

$$\tau(a_\varepsilon) = \lim_{n\to\infty} \tau \otimes \mathrm{tr}_{N_n}(a_n) = \lim_{n\to\infty} \tau \otimes \tau_{\mathfrak{n}}(a_n) = \tau \otimes \tau_{\mathfrak{n}}(a_\varepsilon \otimes 1_{M_{\mathfrak{n}}}).$$

Define a map $\Psi \colon \mathcal{T}(A) \to \mathcal{T}(A \otimes M_{\mathfrak{n}})$ by $\Psi(\tau) = \tau \otimes \tau_{\mathfrak{n}}$ for $\tau \in \mathcal{T}(A)$. From $\tau(a) = \Psi(\tau)(a \otimes 1_{M_{\mathfrak{n}}})$ for any $a \in \mathrm{Ped}(A)$, it follows that Ψ is injective and affine. For $\varphi \in \mathcal{T}(A \otimes M_{\mathfrak{n}})$, setting $\varphi_A(a) = \varphi(a \otimes 1_{M_{\mathfrak{n}}})$ for $a \in \mathrm{Ped}(A)$ we obtain

$\varphi_A \in \mathcal{T}(A)$. Since $\tau_{\mathfrak{n}}$ is a unique tracial state, it follows that $\varphi(p\otimes x) = \varphi_A(p)\tau_{\mathfrak{n}}(x)$ for any projection $p \in A$ and $x \in M_{\mathfrak{n}}$. Thus we see that $\Psi(\varphi_A) = \varphi$.

Let $\iota_*\colon K_0(A) \to K_0(A \otimes M_{\mathfrak{n}})$ denote the induced map defined by $\iota\colon A \to A \otimes M_{\mathfrak{n}}$. Let us show that the map $\hat{\iota}_*\colon S_0(K_0(A \otimes M_{\mathfrak{n}})) \to S_0(K_0(A))$ defined by $\hat{\iota}_*(\varphi) = \varphi \circ \iota_*$ is bijective. Indeed, for any $x \in K_0(A \otimes M_{\mathfrak{n}}) \cong K_0(A) \otimes K_0(M_{\mathfrak{n}})$ there exists $N \in \mathbb{N}$ such that $Nx \in \iota_*(K_0(A))$. Thus we see that $\hat{\iota}_*$ is injective. Since $K_0(A)$ is torsion-free, ι_* is injective, which implies the surjectivity of $\hat{\iota}_*$. Since $\Phi_{\mathcal{T}}$ can be decomposed as $\Phi_{\mathcal{T}} = \hat{\iota}_* \circ \Phi_{\mathcal{T},A\otimes M_{\mathfrak{n}}} \circ \Psi$, it follows that $\Phi_{\mathcal{T}}$ is a bijective affine map.

It is straightforward to check that $\Phi_{\mathcal{T}}$ is continuous. It remains to show that ${\Phi_{\mathcal{T}}}^{-1}$ is also continuous. Suppose that τ_λ, $\lambda \in \Lambda$ is a net in $\mathcal{T}(A)$ such that $\Phi_{\mathcal{T}}(\tau_\lambda)$ converges to $\Phi_{\mathcal{T}}(\tau)$ for some $\tau \in \mathcal{T}(A)$. Then it follows that the net $\tau_{\lambda *}\otimes\tau_{\mathfrak{n}*}$, $\lambda \in \Lambda$ in $S_0(K_0(A\otimes M_{\mathfrak{n}}))$ converges to $\tau_* \otimes \tau_{\mathfrak{n}*}$ in the topology of pointwise convergence. Applying Lemma 6.3(ii) to $A\otimes M_{\mathfrak{n}}$, we have that $\tau_\lambda \otimes \tau_{\mathfrak{n}}$, $\lambda \in \Lambda$ converges to $\tau\otimes\tau_{\mathfrak{n}}$ in $\mathcal{T}(A \otimes M_{\mathfrak{n}})$. Since $(A \otimes M_N)_0 \otimes 1_{M_{\mathfrak{n}}} \subset (A \otimes M_N \otimes M_{\mathfrak{n}})_0$ for any $N \in \mathbb{N}$, we conclude that $\lim_{\lambda\to\infty} \tau_\lambda(a) = \tau(a)$ for any $a \in (A \otimes M_N)_0$. □

We are now ready to give our application of the RAF-algebra classification.

Theorem 6.5. *Let (S, π) be a proper simplex bundle such that $\pi^{-1}(0)$ is a singleton. Then there exists a 2π-periodic flow θ on the Jiang–Su algebra whose KMS-bundle is isomorphic to (S, π).*

For a given proper simplex bundle (S, π) with $\pi^{-1}(0) = \{\tau_S\}$, in Section 3 we constructed the rational dimension group $(G_{\mathbb{Z}}, G_{\mathbb{Z}}^+)$ and its shift automorphism σ such that $(S_\sigma(G_{\mathbb{Z}}), \pi_{\mathbb{Z}})$ is isomorphic to (S, π) as a simplex bundle (Proposition 3.6(iii)).

We define a positive group homomorphism $\Sigma_0\colon G_{\mathbb{Z}} \to \mathbb{Z}$ by $\Sigma_0((z_n)_{n\in\mathbb{Z}} \oplus g) = \sum_{n\in\mathbb{Z}} z_n = g(\tau_S)$ for $(z_n)_n \oplus g \in G_{\mathbb{Z}}$. By the same arguments as in the proofs of [31, Lem. 4.10], [30, Lem. 3.7], and [31, Lem. 4.14], the rational dimension group $(G_{\mathbb{Z}}, G_{\mathbb{Z}}^+)$ has the following properties.

Lemma 6.6. *The following two statements hold:*

(i) $(G_{\mathbb{Z}}, G_{\mathbb{Z}}^+)$ *is σ-simple and*
(ii) $(\mathrm{id} - \sigma)(G_{\mathbb{Z}}) = \ker(\Sigma_0)$.

From Theorems 4.4 and 5.3, we obtain a stable $\mathcal{Z}$-absorbing RAF-algebra A and an automorphism $\alpha_{\mathbb{Z}}$ on A such that

$$(K_0(A), K_0(A)^+) \cong (G_{\mathbb{Z}}, G_{\mathbb{Z}}^+) \quad \text{and} \quad \alpha_{\mathbb{Z}*} = \sigma \text{ on } K_0(A).$$

By Proposition 6.2 and the same argument in [54, Rem. 2.8] (see also [30, Lem. 3.6]), A and $\alpha_{\mathbb{Z}}$ can be chosen to have the following properties.

Lemma 6.7. *The following three statements hold:*

(i) $A \rtimes_{\alpha_{\mathbb{Z}}} \mathbb{Z}$ *is a simple* C*-*algebra,*

(ii) *the restriction map* $\tau \mapsto \tau|_A$ *from* $\mathcal{T}(A \rtimes_{\alpha_{\mathbb{Z}}} \mathbb{Z})$ *onto the* $\alpha_{\mathbb{Z}}$*-invariant traces in* $\mathcal{T}(A)$ *is bijective, and*

(iii) $A \rtimes_{\alpha_{\mathbb{Z}}} \mathbb{Z}$ *absorbs the Jiang–Su algebra tensorially.*

Set $C = A \rtimes_{\alpha_{\mathbb{Z}}} \mathbb{Z}$, and denote by P the conditional expectation from C onto A. Since A is a stable C*-algebra, for $u \in G_{\mathbb{Z}}^+ \cong K_0(A)^+$ defined as $u = (1_0, 1)$ in Section 3, there exists a projection p in A such that $[p]_0 = u$. Then the unital C*-algebra pCp has the following properties.

Proposition 6.8. *The following four statements hold:*

(i) pCp *has a unique tracial state,*

(ii) $(K_0(pCp), K_0(pCp)^+) \cong (\mathbb{Z}, \mathbb{Z}^+)$, $[p]_0 = 1 \in \mathbb{Z}$,

(iii) $K_1(pCp) \cong 0$, *and*

(iv) *therefore* pCp *is isomorphic to* $\mathcal{Z}$.

Proof. (i) Set $\mathcal{T}^0_{\alpha_{\mathbb{Z}}}(A) = \{\tau \in \mathcal{T}(A) : \tau \circ \alpha_{\mathbb{Z}} = \tau, \text{ and } \tau(p) = 1\}$. By Proposition 6.4, $\mathcal{T}^0_{\alpha_{\mathbb{Z}}}(A)$ corresponds to $S^1_\sigma(G_{\mathbb{Z}}) = \{\tau_S\}$ via $\Phi_{\mathcal{T}}$. Because of [14, Prop. 4.7], the set of tracial states $T(pCp)$ of pCp corresponds to $\{\tau \in \mathcal{T}(C) : \tau(p) = 1\}$ via the restriction on pCp. By Lemma 6.7(ii), we know that $\{\tau \in \mathcal{T}(C) : \tau(p) = 1\}$ is a singleton, which implies that $T(pCp)$ is also singleton.

(ii) By Lemma 4.2(ii), we see that $K_1(A) = 0$. Then the Pimsner–Voiculescu six-term exact sequence allows us to see that

$$K_0(C) \cong G_{\mathbb{Z}}/(\mathrm{id} - \sigma)(G_{\mathbb{Z}}) = G_{\mathbb{Z}}/\ker(\Sigma_0) \cong \mathbb{Z}.$$

Consider the quotient map $q\colon G_{\mathbb{Z}} \to G_{\mathbb{Z}}/(\mathrm{id} - \sigma)(G_{\mathbb{Z}})$ and denote by $\iota_*\colon K_0(A) \to K_0(C)$ the map induced by the inclusion $\iota\colon A \to C$. In the exact sequence, precisely, ι_* corresponds to q and the above group isomorphism $\Sigma\colon K_0(C) \to \mathbb{Z}$ is determined by $\Sigma \circ q = \Sigma_0$; see also the paragraph after [30, Lem. 3.7]. Set $\tau^0 \in \mathcal{T}^0_{\alpha_{\mathbb{Z}}}(A)$ which corresponds to $\tau_S \in S^1_\sigma(G_{\mathbb{Z}})$ (determined by $\tau^0_* = \widehat{\tau_S}$). For $x \in K_0(C)^+$, now we obtain $(y_n)_{n\in\mathbb{Z}} \oplus f \in G_{\mathbb{Z}}$ such that $q((y_n)_n \oplus f) = x$. Then it follows that

$$0 \le (\tau^0 \circ P)_*(x) = \tau^0_*((y_n)_n \oplus f) = f(\tau_S) = \sum_{n\in\mathbb{Z}} y_n \in \mathbb{Z},$$

which implies that $\Sigma(x) \in \mathbb{Z}^+$. The converse inclusion $\Sigma(K_0(C)^+) \supset \mathbb{Z}^+$ follows from the properties $nu \in K_0(A)^+$ and $n = \Sigma \circ q(nu) \in \Sigma(K_0(C)^+)$ for any $n \in \mathbb{Z}^+$. Then we have that $(K_0(pCp), K_0(pCp)^+) \cong (K_0(C), K_0(C)^+) \cong (\mathbb{Z}, \mathbb{Z}^+)$. In particular, since $1 = \Sigma \circ q(u)$, $[p]_0$ corresponds to $1 \in \mathbb{Z}$.

(iii) For the same reason as in [30, Lem. 3.11], we see that the map $\mathrm{id} - \sigma$ is injective on $K_0(A) \cong G_{\mathbb{Z}}$. From the Pimsner–Voiculescu exact sequence, we see that $K_1(C) = 0$.

(iv) By Lemma 6.7, we have now shown that pCp is a unital separable simple nuclear C*-algebra which absorbs the Jiang–Su algebra tensorially and satisfies the universal coefficient theorem. By (i), (ii), (iii) above, we see that pCp is monotracial and the Elliott invariant of pCp is same as for $\mathcal{Z}$. From [63, Cor. 4.6], we also see that pCp has strict comparison. Then by [56, Cor. 6.2] (see also [68] and [52]), we conclude that $pCp \cong \mathcal{Z}$. □

Proof of Theorem 6.5. Denote by $\tilde{\theta}$ the dual action of $\alpha_{\mathbb{Z}}$ on C and by θ the restriction of $\tilde{\theta}$ to $pCp \cong \mathcal{Z}$. Denote by (S_θ, π_θ) the KMS-bundle of the dynamical system $(\mathcal{Z}, \theta)$. By Proposition 3.6(iii), $(S_\sigma(G_{\mathbb{Z}}), \pi_{\mathbb{Z}})$ is a proper simplex bundle isomorphic to (S, π). It remains to show that $(S_\sigma(G_{\mathbb{Z}}), \pi_{\mathbb{Z}})$ is isomorphic to (S_θ, π_θ). Set

$$\mathcal{T}_{\alpha_{\mathbb{Z}}}(A) = \big\{(\tau, \beta) \in \mathcal{T}(A) \times \mathbb{R} : \tau \circ \alpha_{\mathbb{Z}} = e^{-\beta}\tau,\ \tau(p) = 1\big\},$$

equipped with the product topology on $\mathcal{T}(A) \times \mathbb{R}$, and denote by $\pi_{\alpha_{\mathbb{Z}}} \colon \mathcal{T}_{\alpha_{\mathbb{Z}}}(A) \to \mathbb{R}$ the projection. Define a continuous map $\Phi_{\alpha_{\mathbb{Z}}} \colon \mathcal{T}_{\alpha_{\mathbb{Z}}}(A) \to S_\sigma(G_{\mathbb{Z}})$ by $\Phi_{\alpha_{\mathbb{Z}}}(\tau, \beta) = (\Phi_{\mathcal{T}}(\tau), \beta)$ for $(\tau, \beta) \in \mathcal{T}_{\alpha_{\mathbb{Z}}}(A)$. By Proposition 6.4, $\Phi_{\alpha_{\mathbb{Z}}}$ is a homeomorphism, and $\Phi_{\alpha_{\mathbb{Z}}}^{-1}|_{\pi_{\mathbb{Z}}^{-1}(\beta)}$ is affine for each $\beta \in \mathbb{R}$.

It follows that the projection $p \in A$ (of Proposition 6.8) is full in C, i.e., $\overline{CpC} = C$, by the simplicity of C. Define a map $\Psi \colon \mathcal{T}_{\alpha_{\mathbb{Z}}}(A) \to S_\theta$ by $\Psi((\tau, \beta)) = (\tau \circ P|_{pCp}, \beta)$ for $(\tau, \beta) \in \mathcal{T}_{\alpha_{\mathbb{Z}}}(A)$. By [31, Lem. 4.1], we see that Ψ is bijective and $\Psi|_{\pi_{\alpha_{\mathbb{Z}}}^{-1}(\beta)}$ is an affine homeomorphism from $\pi_{\alpha_{\mathbb{Z}}}^{-1}(\beta)$ onto $\pi_\theta^{-1}(\beta)$. Since $pAp \subset A_0$, if a sequence $\tau_n \in \mathcal{T}(A)$, $n \in \mathbb{N}$, converges to $\tau \in \mathcal{T}(A)$ then $\tau_n \circ P|_{pCp} \to \tau \circ P|_{pCp}$ in the topology of pointwise convergence. Thus, Ψ is continuous. We define a bijective continuous map $\Phi \colon S_\sigma(G_{\mathbb{Z}}) \to S_\theta$ by $\Phi = \Psi \circ \Phi_{\alpha_{\mathbb{Z}}}^{-1}$. By Lemma 3.4, we conclude that $(S_\sigma(G_{\mathbb{Z}}), \pi_{\mathbb{Z}})$ is isomorphic to (S_θ, π_θ), as desired. □

Theorem 6.9. *Let A be a unital separable* C*-*algebra with a unique tracial state. Suppose that A absorbs the Jiang–Su algebra tensorially. Then for any proper simplex bundle (S, π) such that $\pi^{-1}(0)$ is singleton, there exists a 2π-periodic flow on A whose KMS-bundle is isomorphic to (S, π).*

Proof. Let $\tilde{\iota}_t = \mathrm{id}_A$, $t \in \mathbb{R}$, denote the trivial flow on A and τ the unique tracial state of A. For a given (S, π), by Theorem 6.5 we obtain a flow θ on $\mathcal{Z}$ whose

KMS-bundle is isomorphic to (S, π). We denote by $A \otimes \mathcal{Z}$ the C*-algebra tensor product and define a flow α on $A \otimes \mathcal{Z}$ as the tensor product action which is defined by $\alpha_t(a \otimes b) = \tilde{\iota}_t(a) \otimes \theta_t(b)$ for $a \in A$ and $b \in \mathcal{Z}$. Let (S_α, π_α) denote the KMS-bundle of α and $S_\theta^\beta(\mathcal{Z})$ (resp. $S_\alpha^\beta(A \otimes \mathcal{Z})$) the set of β-KMS states for θ (resp. α). Define an affine map Φ from $S(\mathcal{Z})$ to $S(A \otimes \mathcal{Z})$ by $\Phi(\varphi) = \tau \otimes \varphi$. Then it is straightforward to check that $\Phi(\varphi)$ is a β-KMS state for α for any $\varphi \in S_\theta^\beta(\mathcal{Z})$, and $\Phi|_{S_\theta^\beta(\mathcal{Z})} \colon S_\theta^\beta(\mathcal{Z}) \to S_\alpha^\beta(A \otimes \mathcal{Z})$ is affine continuous and injective for any $\beta \in \mathbb{R}$. Define a continuous map $\Phi_A \colon S_\theta \to S_\alpha$ by $\Phi_A((\varphi, \beta)) = (\Phi(\varphi), \beta)$ for $(\varphi, \beta) \in S_\theta$.

In order to apply Lemma 3.4 to Φ_A, it remains to show surjectivity of $\Phi|_{S_\theta^\beta(\mathcal{Z})}$. Given $\psi \in S_\alpha^\beta(A \otimes \mathcal{Z})$, $x \in \mathcal{Z}$, and $a, b \in A$, note that $a \otimes 1_\mathcal{Z}$ and $b \otimes 1_\mathcal{Z}$ are analytic elements for α and $\psi(ab \otimes x) = \psi((b \otimes x)\alpha_{i\beta}(a \otimes 1_\mathcal{Z})) = \psi(ba \otimes x)$. Since τ is a unique tracial state, we have $\tau(a)\psi(1_A \otimes x) = \psi(a \otimes x)$, for any $a \in A$ and $x \in \mathcal{Z}$. Set $\psi_\mathcal{Z}(x) = \psi(1_A \otimes x)$, for $x \in \mathcal{Z}$. Then $\psi_\mathcal{Z}$ is contained in $S_\theta^\beta(\mathcal{Z})$, as $\psi(1_A \otimes xy) = \psi((1_A \otimes y)\alpha_{i\beta}(1_A \otimes x)) = \psi(1_A \otimes y(\theta_{i\beta}(x)))$ for any analytic elements $x, y \in \mathcal{Z}$ for θ. Thus we see that $\Phi(\psi_\mathcal{Z})(a \otimes x) = \psi(a \otimes x)$ for any $a \in A$ and $x \in \mathcal{Z}$. □

From [60, Thm. 3.2], [49, Prop. 2.1], and [55, Thm. 1.1], we obtain uncountably many flows which are not approximately inner on the following classifiable class of C*-algebras (cf. the explicit automorphism group computation in [29]).

Corollary 6.10 (Cf. [30, Cor. 4.2]). *Any unital separable simple amenable monotracial* C*-*algebra with strict comparison has uncountably many flows which are not approximately inner, up to cocycle conjugacy and the trivial scaling equivalence.*

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