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On the effects of small perturbation on low energy Laplace eigenfunctions

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Abstract. We investigate several aspects of the nodal geometry and topology of Laplace eigenfunctions, with particular emphasis on the low-frequency regime. This includes investigations in and around the well-known nodal line conjecture, opening angle estimates of nodal domains, saturation of (fundamental) spectral gaps etc., and behaviour of all of the above under small-scale perturbations. We aim to highlight interesting aspects of spectral theory and nodal phenomena tied to ground state/low energy eigenfunctions, as opposed to asymptotic results.

1. Introduction and preliminaries

Let (M, g) be a compact Riemannian manifold. Consider the eigenequation

$$-\Delta\varphi = \lambda\varphi, \tag{1.1}$$

where Δ is the Laplace–Beltrami operator given by (using the Einstein summation convention)

$$\Delta f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f),$$

where $|g|$ is the determinant of the metric tensor g_{ij}. In the Euclidean space, this reduces to the usual $\Delta = \partial_1^2 + \cdots + \partial_n^2$. Observe that we are using the analyst's sign convention for the Laplacian, namely that $-\Delta$ is positive semidefinite.

If M has a boundary, we will consider either the Dirichlet boundary condition

$$\varphi(x) = 0, \quad x \in \partial M,$$

or the Neumann boundary condition

$$\partial_\eta\varphi(x) = 0, \quad x \in \partial M,$$

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where η denotes the outward pointing unit normal on ∂M. Recall that if M has a reasonably regular boundary, $-\Delta_g$ has a discrete spectrum

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty,$$

repeated with multiplicity with corresponding (real-valued L^2 normalised) eigenfunctions φ_k. Also, let $\mathcal{N}_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\}$ denote the nodal set of the eigenfunction φ_λ. Sometimes for ease of notation, we will also denote the nodal set by $\mathcal{N}(\varphi_\lambda)$. Recall that any connected component of $M \setminus \mathcal{N}_{\varphi_\lambda}$ is known as a nodal domain of the eigenfunction φ_λ denoted by Ω_λ. These are domains where the eigenfunction is not sign-changing (this follows from the maximum principle).

In this paper, we study the nodal topology and geometry associated to Laplace eigenfunctions, for both curved and flat domains, and with both Dirichlet and/or Neumann boundary conditions. Most of our results are related to the low frequency regime (low energy eigenfunctions), though some will also apply to the high frequency asymptotics, but no result is purely asymptotic in nature.

1.1. Notational convention

When two quantities X and Y satisfy $X \leq c_1 Y$ ($X \geq c_2 Y$) for constants c_1, c_2 dependent on the geometry (M, g), we write $X \lesssim_{(M,g)} Y$ (respectively $X \gtrsim_{(M,g)} Y$). Unless otherwise mentioned, these constants will in particular be independent of eigenvalues λ. Throughout the text, the quantity $\frac{1}{\sqrt{\lambda}}$ is referred to as the wavelength and any quantity (e.g., distance) is said to be of *sub-wavelength* (*super-wavelength*) *order* if it is $\lesssim_{(M,g)} \frac{1}{\sqrt{\lambda}}$ (respectively $\gtrsim_{(M,g)} \frac{1}{\sqrt{\lambda}}$).

First, we recall a few basic facts and results that we would need in the sequel.

1.2. Characterisation of eigenvalues

We note that the Sobolev space $H^1(M)$ can be defined as the completion of $C^\infty(M)$ with respect to the inner product

$$\langle f, g \rangle_{H^1} := \langle f, g \rangle_{L^2(M)} + \langle \nabla f, \nabla g \rangle_{L^2(M)}$$

where $f, g \in C^\infty(M)$. Next, we introduce a bilinear form in $H^1(M)$. Consider the bilinear form on $C^\infty(M) \times C^\infty(M)$,

$$D(f, g) := \langle \nabla f, \nabla g \rangle_{L^2(M)}.$$

Since $H^1(M)$ is the completion of $C^\infty(M)$ in the induced norm, given $f, g \in H^1(M)$, there exists sequence $\{f_i\}, \{g_i\} \in C^\infty(M)$ converging to f, g in H^1 norm. Then we can define the bilinear form on $H^1(M) \times H^1(M)$ as

$$D(f, g) = \lim_{i \to \infty} \langle \nabla f_i, \nabla g_i \rangle_{L^2(M)}.$$

Now, we have the following characterisation of Laplacian eigenvalues.

Theorem 1.1. *For any $k \in \mathbb{N}$, let $\{\varphi_1, \varphi_2, \dots, \varphi_{k-1}\}$ be the first $k-1$ orthonormal eigenfunctions. Then for any $f \in H^1(M)$, $f \neq 0$ such that*

$$\langle f, \varphi_1 \rangle = \cdots = \langle f, \varphi_{k-1} \rangle = 0$$

we have

$$\lambda_k \leq \frac{D(f, f)}{\|f\|^2}. \tag{1.2}$$

Moreover, the equality holds if and only if f is an eigenfunction corresponding to λ_k.

Note that for characterising the Dirichlet Laplacian eigenvalues, the admissible function class in the above theorem is $H_0^1(M)$, which is the completion of $C_0^\infty(M)$ with respect to the induced norm from the above defined inner product. But, for Neumann boundary condition and manifolds without boundary, we use the above characterisation as it is.

Moreover, we also note that, in case of a boundaryless manifold or a manifold with Neumann boundary condition we have that $\lambda_1 = 0$ and for a manifold with Dirichlet boundary, we have that $\lambda_1 > 0$. This follows from the above characterisation.

1.3. Eigenfunctions and Fourier synthesis

In a certain sense, the study of eigenfunctions of the Laplace–Beltrami operator is the analogue of Fourier analysis in the setting of compact Riemannian manifolds. Recall that the Laplace eigenequation is the standing state for a variety of partial differential equations modelling physical phenomena like heat diffusion, wave propagation or Schrödinger problems. Below, we note down this well-known method of "Fourier synthesis":

$$\begin{aligned} &\text{heat equation} && (\partial_t - \Delta)u = 0, && u(t,x) = e^{-\lambda t}\varphi(x), \\ &\text{wave equation} && (\partial_t^2 - \Delta)u = 0, && u(t,x) = e^{i\sqrt{\lambda}t}\varphi(x), \\ &\text{Schrödinger equation} && (i\partial_t - \Delta)u = 0, && u(t,x) = e^{i\lambda t}\varphi(x). \end{aligned}$$

Further, note that $u(t,x) = e^{\sqrt{\lambda}t}\varphi(x)$ solves the harmonic equation $(\partial_t^2 + \Delta)u = 0$ on $\mathbb{R} \times M$. In the interest of completeness, we include one last useful heuristic: if one considers the eigenequation (1.1) on metric balls of radius $\frac{\sqrt{\epsilon}}{\sqrt{\lambda}}$ and rescale to a ball of radius 1, it produces an "almost harmonic" function (see [50, Section 2] for more details).

A motivational perspective of the study of Laplace eigenfunctions then comes from quantum mechanics (via the correspondence with Schrödinger operators), where the L^2-normalised eigenfunctions induce a probability density $\varphi^2(x)dx$, i.e., the probability density of a particle of energy λ to be at $x \in M$. Another physical (real-life) motivation of studying the eigenfunctions, dated back to the late 18th century, is based on the acoustics experiments done by E. Chladni which were in turn inspired from the observations of R. Hooke in the late 17th century. But what is surprising (at least to the present authors!) is that the earliest observation of these vibration patterns were made by G. Galileo in early 17th century. The experiments of Chladni consist of drawing a bow over a piece of metal plate whose surface is lightly covered with sand. When resonating, the plate is divided into regions that vibrate in opposite directions causing the sand to accumulate on parts with no vibration. The study of the patterns formed by these sand particles (Chladni figures) were of great interest which led to the study of nodal sets and nodal domains.

1.4. Elementary facts about eigenvalues and eigenfunctions

We now collect some elementary facts about the eigenvalues and eigenfunctions of the Laplace–Beltrami operator. One well-known global property of the eigenfunctions is the following theorem which gives an upper bound on the number of nodal domains corresponding to the kth eigenfunction φ_k.

Theorem 1.2 (Courant's nodal domain theorem). *The number of nodal domains of φ_k can be at most k. In other words, the total number of connected components of $M \setminus \mathcal{N}_{\varphi_k}$ is strictly less than $k + 1$.*

Remark 1.3. φ_1 is always non sign changing. In the case of Laplace eigenfunctions, this can be easily observed by replacing φ_1 by $|\varphi_1|$, which is non-negative and using the variational characterisation (1.2) above.

Remark 1.4. The multiplicity of λ_1 is always 1 i.e., λ_1 is simple (for a manifold without boundary, $\lambda_1 = 0$ corresponding to the constant eigenfunctions). If not, then φ_2 has a constant sign, from the previous remark. This contradicts the fact that $\langle \varphi_1, \varphi_2 \rangle = 0$, since φ_1 has a constant sign as well. As a result, λ_1 is characterised as being the only eigenvalue with eigenfunction of constant sign. For significantly more general operators (like Schrödinger operators), the result is still true, but this requires the use of the Krein–Rutman theorem.

Remark 1.5. The above two remarks in conjunction with the Courant nodal domain theorem imply that φ_2 has exactly two nodal domains. Moreover, any φ_k has at least two nodal domains for $k \geq 2$.

Theorem 1.6 (Domain monotonicity). *Suppose $\Omega_1 \subseteq \Omega_2 \subseteq M$. Then their fundamental Dirichlet eigenvalues satisfy*

$$\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1),$$

and the above inequality is strict if the set $\Omega_2 \setminus \Omega_1$ has positive capacity.

Theorem 1.7 (Wavelength density). *For any (M, g), there exists a constant $C > 0$ (depending on g) such that every ball of radius bigger that $C/\sqrt{\lambda}$ intersects with the nodal set corresponding to φ_λ.*

We end this section by giving an overview of the paper.

1.5. Overview of the paper

Here we take the space to list some of our main results. First, we discuss the stability of topological properties of the first Dirichlet nodal set (or, nodal set for any of the second Dirichlet eigenfunctions) under small perturbations. Among other results, we prove that satisfying the strong Payne property[1] (or not satisfying) are both open conditions in a one-parameter family of perturbations of a given bounded domain $\Omega \subseteq \mathbb{R}^n$: this is Proposition 2.6 below. Observe that the results hold in all dimensions, and not restricted to planar domains only.

After a general laying of foundations, we discuss the position and topology of the nodal set corresponding to the second Dirichlet eigenfunction in the connector or "handle" region of thin dumbbell domains (for a proper definition, see Section 3.1 below). In particular, we check the validity of the Payne property (see Definition 2.5 below). We quote the result.

Theorem 1.8. *Consider two bounded domains $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ $(n \geq 2)$ with C^∞ boundary and a one parameter family of smooth dumbbells Ω_ϵ (as described in Section 3.1) whose connector widths go to zero as $\epsilon \to 0$. Assume that $\Omega_i, i = 1, 2$ have simple second eigenvalues. Let $\lambda_2^{\Omega_i}, \lambda_{2,\epsilon}$ denote the second eigenvalues of Ω_i, Ω_ϵ corresponding to eigenfunctions $\varphi_2^{\Omega_i}$, $\varphi_{2,\epsilon}$ respectively, $i = 1, 2$. Assume that the connector does not intersect $\mathcal{N}(\varphi_2^{\Omega_i}) \cap \partial\Omega_i$ and Ω_i do not have the same first or second Dirichlet eigenvalues. If $\lambda_{2,\epsilon} \to \lambda_2^{\Omega_1}$ (without loss of generality) and Ω_1 satisfies the strong Payne property, then for sufficiently small $\epsilon > 0$, Ω_ϵ satisfies the strong Payne property as well.*

[1] See Definition 2.5 below.

We note that the simplicity assumption of the second eigenvalue is true up to generic perturbations (see [66], and also Theorem 4.1 below).

Next, we start a discussion about negative results surrounding the Payne property. In Section 3.2, we first describe the two-dimensional counterexample in [34] and the higher-dimensional counterexample in [14]. Next, we give a new counterexample to the Payne property in higher dimensions, which has the merit of being simply connected as well. Our example is a perturbation of the base domain provided by Fournais in [14]. We believe that conceptually our example might be simpler than the counterexample in [40]. We then finally indicate how to jazz it up to get a domain with prescribed topological complexity which violates the Payne property.

Theorem 1.9. *Let $G = \pi_1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is some bounded domain with smooth boundary. Then, we can construct a domain $\Omega' \subset \mathbb{R}^n$ with fundamental group G such that Ω' does not satisfy the Payne property.*

Two remarks are in order. Firstly, since our methods are based on perturbation theory, we are actually able to get infinitely many (indeed, a one-parameter family) of domains violating the Payne property. Secondly, in the above theorem, G can be any finitely presented group, provided we are content with looking for counterexamples in dimensions $n \geq 4$. In other words, given a finitely presented group G, one can construct a bounded domain $\Omega' \subset \mathbb{R}^n$ for any $n \geq 4$ such that $\pi_1(\Omega') = G$ and Ω' does not satisfy the Payne property. Recall the fact that for any finitely presented group G and any $n \geq 4$, one can construct a finite 2-complex X which has an embedding $i : X \to \mathbb{R}^n$ such that $i(X)$ is a retract of its neighbourhood in $\mathbb{R}^n$. [27, Corollary A.10] proves the easier estimate $n \geq 5$, and we refer the reader to [18, Theorem 3.5] and the discussion therein for the sharp estimate of $n \geq 4$ (the statement that we need also follows from a classical result of Stallings from 1965, unpublished). The rest follows from the proof of Theorem 1.9.

Next, we begin an investigation into some geometric properties of the nodal set of low energy eigenfunctions. Our discussion is mainly centred around the angular properties of the nodal set where it meets the boundary. Apart from being of independent interest, one motivation to consider this question comes from the Payne problem itself. In the usual perturbative approach to the Payne conjecture, one considers a domain with embedded first nodal set, if possible, and perturbs this domain to a convex domain (or a more general domain that satisfies the strong Payne property). Somewhere along the perturbation process, one encounters a configuration where the nodal set meets the boundary in a "degenerate" way. In dimension $n = 2$, this looks like a "teardrop," where the first nodal set cuts the boundary exactly at one point. In such a situation, one can ask what are the possible angles formed at the boundary between the nodal set and the boundary. It was shown by Melas [53] that the nodal domain for the second Dirichlet eigenfunction which intersects the boundary of a planar domain

Ω cannot have an "opening angle" of 0 or π at the point of intersection. First, to set the stage, we prove the following for the case of the nodal set intersecting the boundary (the interior case has already been addressed in [21]).

Theorem 1.10. *Suppose $\mathcal{N}_\varphi$ intersects the boundary of the manifold ∂M at a point p. When $\dim M = 3$, the nodal set $\mathcal{N}_\varphi$ at p satisfies an interior cone condition (see Definition 3.5 below) with angle $\gtrsim \frac{1}{\sqrt{\lambda}}$. When $\dim M = 4$, $\mathcal{N}_\varphi$ at p satisfies an interior cone condition with angle $\gtrsim \frac{1}{\lambda^{7/8}}$. Lastly, when $\dim M \geq 5$, $\mathcal{N}_\varphi$ at p satisfies an interior cone condition with angle $\gtrsim \frac{1}{\lambda}$.*

The proof of Theorem 1.10 is essentially the same as that of [21, Theorem 1.6], with the added observation that the classical Bers expansion of eigenfunctions around nodal critical points (Theorem 3.6 below) also extends to points on the boundary.

In [26] it is established that the nodal set is the union of a smooth hypersurface and a set of singular points that is countably $(n-2)$-rectifiable. In particular, in dimension 2, the nodal set is the union of a finite collection of embedded arcs and finitely many nodal critical points. A result by Cheng [7] further states that on a compact Riemannian surface, the nodal lines form an equiangular system at these nodal critical points, the angle depending on the order of vanishing of the eigenfunction at the nodal critical point. In higher dimensions, the nodal geometry around nodal critical points might have a more complicated structure since the local picture is obtained from multiple intersections of $(n-1)$-dimensional submanifolds and there is no ready notion of an angle of intersection at such points. So, we focus on finding the angles between any two nodal hypersurfaces intersecting at a nodal critical point. When restricted to the latter case, our result below can be regarded as a sharpened version of Theorem 1.10, as also as an extension of Cheng's result in higher dimensions.

Theorem 1.11. *Let $x \in M$, where M is a compact manifold of dimension $n \geq 3$. Let x lie at the intersection of two nodal hypersurfaces M_1, M_2. Let $\eta_1, \eta_2 \in S^{n-1}$ be two unit normal vectors to M_1 and M_2 at x. If the order of vanishing of φ_λ at x is n_0, then the angle between M_1 and M_2 at x is $\arccos\langle \eta_1, \eta_2\rangle \in P$, where*

$$P = \left\{ \frac{p}{q}\pi : q = 1, 2, \ldots, n_0,\, p = 0, 1, \ldots, q \right\}.$$

Roughly, since the angle of intersection cannot change continuously, this result should be seen as a "perturbation resistant" nodal geometry phenomenon and puts a constraint on the possible Payne configurations before the possible bifurcation phenomenon, which is the main obstruction in the usual perturbative proofs of the Payne conjecture.

In the last section of our paper, we use our perturbation theoretic tools in conjunction with well-known facts about the Payne problem for long convex domains to discuss the somewhat related problem of saturation of the fundamental gap, namely, whether the fundamental gap $\lambda_2 - \lambda_1$ is attained on convex domains. We prove the following. Let $\mathfrak{P}$ denote the class of strictly convex C^2-planar domains. Then, we have the following.

Theorem 1.12. *Let $\Omega \in \mathfrak{P}$ with diameter $D = 1$ and inner radius ρ. There exists a universal constant $C \ll 1$ such that if $\rho \leq C$, then Ω cannot be the minimiser of the fundamental gap functional in $\mathfrak{P}$.*

The proof uses some of the ideas on perturbation theory developed in the preceding parts of the paper, along with Jerison's investigations into the Payne problem [35–37]. The popular belief in the community seems that the fundamental gap is not saturated in the class of all convex domains, and any infimising sequence for $\lambda_2 - \lambda_1$ (under the normalisation $D = 1$) should degenerate to a line segment. This intuitively indicates that, as far as ruling out minimisers of the fundamental gap is concerned, the "difficult regime" to look for is the class of narrow convex domains, and the case of domains with large inner radius might be easier. However, we are able to get a result only in the former case and not in the latter, which might (somewhat counterintuitively) be the harder case. This is not completely surprising, as our methods are perturbative in nature, and we are using the deep results of Jerison (which hold in the setting of long narrow domains only) about the Payne problem, which seems a priori unrelated to the fundamental gap problem. We close Section 4 by showing that for a broad class of non-convex simply-connected domains, the multiplicity of the second Dirichlet eigenvalue is ≤ 2 (this is Theorem 4.9 below). This is directly related to the Payne property via an insight from [47], and might be of some value in the future studies of the Payne property.

2. Stability, Payne property and some preliminary results

In this section, we will first look into a certain aspect of the nodal sets that remains stable under perturbation. Note that many aspects of nodal sets of Laplace eigenfunctions are rather unstable under perturbation, which normally disallows perturbative techniques (like Ricci flow and related geometric flows etc.) in the study of nodal geometry. However, one is inclined to ask the question that if the perturbations are "small enough," are there certain "soft" properties of the nodal set that are still reasonably stable? This was answered in [55], where we proved that if the perturbation is of subwavelength scale, then the nodal sets do not "see it." The one-parameter family of perturbations considered in [55] was inspired by the construction in [64]. Below we provide a more general setting under which the stability arguments go through.

2.1. Stability under general perturbations

Consider a domain Ω_{t_0} with a simple spectrum and a one-parameter family of domains (not necessarily a continuous family) Ω_t. Let $\{t_i\} \to t_0$ be a sequence such that for all $k = 1, \ldots,$ we have

$$\lim_{t_i \to t_0} \lambda_k(\Omega_{t_i}) = \lambda_k(\Omega_{t_0}). \tag{2.1}$$

Define

$$\widetilde{\Omega} := \Big\{ x \in \Omega_{t_0} : \text{there exists } \varepsilon_x > 0 \text{ such that } x \in \bigcap_{t_i} \Omega_t \text{ where } t_i \in (t_0 - \varepsilon_x, t_0 + \varepsilon_x) \Big\}.$$

Equivalently, let

$$\widetilde{\Omega}_p := \bigcap_{i \geq p} \Omega_{t_i} \quad \text{and} \quad \widetilde{\Omega} := \lim_{p \to \infty} \widetilde{\Omega}_p.$$

Heuristically, $\widetilde{\Omega}$ is the *eventually unperturbed* part of Ω_{t_0}. Consider the following C^∞–convergence[2] of eigenfunctions $\varphi_{k,t} \in C^\infty(\Omega_t)$, where $\varphi_{k,t}$ is defined as the k-th Dirichlet eigenfunction of Ω_t:

$$\lim_{t \to t_0} \varphi_{k,t} = \varphi_{k,t_0} \quad \text{on } \widetilde{\Omega}. \tag{2.2}$$

Note that all the above conditions on the sub-family Ω_{t_i} are met when considering generic perturbations of a given domain Ω_{t_0}. But our results below are applicable for any perturbations family as long as the above criteria are met. All the eigenfunctions involved are assumed to be L^2-normalised, i.e.,

$$\|\varphi_{k,t}\|_{L^2(\Omega_t)} = \|\varphi_{k,t_0}\|_{L^2(\Omega_{t_0})} = 1.$$

Then we have the following.

Lemma 2.1. *Let $\mathcal{N}(\varphi_{k,t})$ and $\mathcal{N}(\varphi_{k,t_0})$ denote the nodal sets corresponding to $\varphi_{k,t}$ and φ_{k,t_0} respectively. Consider a sequence of points $\{x_i\}$ such that for each i, $x_i \in \mathcal{N}(\varphi_{k,t_i}) \cap \widetilde{\Omega}$. If a limit point x of $\{x_i : i \in \mathbb{N}\}$ exists, then $x \in \mathcal{N}(\varphi_{k,t_0})$.*

Proof. Denote $\tilde{\varphi}_{k,t} = \varphi_{k,t}|_{\widetilde{\Omega}}$. We apply the convergence

$$\varphi_{k,t} \to \varphi_{k,t_0} \quad \text{in } C^0(\widetilde{\Omega})$$

to obtain

$$|\varphi_{k,t_0}(x_i)| = |\varphi_{k,t_0}(x_i) - \tilde{\varphi}_{k,t_i}(x_i)| \leq \|\varphi_{k,t_0} - \tilde{\varphi}_{k,t_i}\|_{C^0(\widetilde{\Omega})} \to 0 \quad \text{as } i \to \infty.$$

[2]Actually, it turns out that for almost all of our applications, only C^0-convergence would suffice.

Now, φ_{k,t_0} being a continuous function, $x_i \to x$ implies $\varphi_{k,t_0}(x_i) \to \varphi_{k,t_0}(x)$. Therefore, $\varphi_{k,t_0}(x) = 0$, i.e., $x \in \mathcal{N}(\varphi_{k,t_0})$. ■

Now, we have the following.

Lemma 2.2. *Suppose in addition to* (2.1) *and* (2.2), *we have that* $|\Omega_{t_i} \setminus \widetilde{\Omega}| \to 0$. *If* $\mathcal{N}(\varphi_{k,t_0}) \subset \widetilde{\Omega}$, *then for* t_i *close enough to* t_0, *the nodal set* φ_{k,t_i} *is fully contained inside* $\widetilde{\Omega}$.

The proof is a variant of the ideas in [42,64] and follows from a limiting argument.

Proof. Our goal is to show that $\mathcal{N}(\varphi_{k,t}) \subset \widetilde{\Omega}$ for t close enough to t_0. If possible, let there exist a sequence $t_i \to t_0$ for which $\mathcal{N}(\varphi_{k,t_i}) \not\subseteq \widetilde{\Omega}$. It can happen in the following three ways.

(A) The nodal set $\mathcal{N}(\varphi_{k,t_i})$ passes from $\widetilde{\Omega}$ to $\Omega_{t_i} \setminus \widetilde{\Omega}$ in a continuous path and no other disconnected component of $\mathcal{N}(\varphi_{k,t_i})$ is contained in $\Omega_{t_i} \setminus \widetilde{\Omega}$.

(B) A component of the nodal set passes from $\widetilde{\Omega}$ to $\Omega_{t_i} \setminus \widetilde{\Omega}$ in a continuous path but some other disconnected component of $\mathcal{N}(\varphi_{k,t_i})$ is also contained in $\Omega_{t_i} \setminus \widetilde{\Omega}$.

(C) There is no continuous path as a part of the nodal set from $\widetilde{\Omega}$ to $\Omega_{t_i} \setminus \widetilde{\Omega}$ but the nodal set of $\mathcal{N}(\varphi_{k,t_i})$ contains at least one disconnected component in $\Omega_{t_i} \setminus \widetilde{\Omega}$.

First, we show that as $t_i \to t_0$ any disconnected component of $\mathcal{N}(\varphi_{k,t_i})$ lies inside $\widetilde{\Omega}$, i.e., cases B and C hold only for finitely many i's. Note that it is enough to show that case C can hold only for finitely many i's. If not, let there exist a subsequence $\{j\} \subset \{i\}$ such that C holds. By our assumption, as $t_j \to t_0$, we have that $|\Omega_{t_i} \setminus \widetilde{\Omega}| \to 0$. By Weyl's law, any nodal domain contained in $\Omega_{t_i} \setminus \widetilde{\Omega}$ has the ground state Dirichlet eigenvalue go to infinity. However, note that

$$\lambda_1(\Omega_{t_i} \setminus \widetilde{\Omega}) = \lambda_k(\Omega_{t_i}) \to \lambda_k(\Omega),$$

which contradicts $\lambda_1(\Omega_{t_i} \setminus \widetilde{\Omega}) \to \infty$. So, B and C cannot hold for infinitely many j's.

From above, if there exists a subsequence $\{j\} \subset \{i\}$ such that for some $y_j \in \mathcal{N}(\varphi_{k,t_j})$, $y_j \notin \widetilde{\Omega}$, then $\mathcal{N}(\varphi_{k,t_j})$ satisfies only condition A. Now, consider a sequence of points $\{x_j\}$ such that $x_j \in \mathcal{N}(\varphi_{k,t_j}) \cap \widetilde{\Omega}$ for each j. Then for each j, we get a continuous path from x_j to y_j contained in $\mathcal{N}(\varphi_{k,t_j})$ whose one end is in $\widetilde{\Omega}$ and the other end in $\Omega_{t_j} \setminus \widetilde{\Omega}$. Then there exists a point $z_j \in \mathcal{N}(\varphi_{k,t_j}) \cap \partial\widetilde{\Omega}$. Choosing a convergent subsequence of $\{z_j\}$ (and renaming it $\{z_j\}$ with mild abuse of notation), we have a limit $z \in \partial\widetilde{\Omega}$. Now, using Lemma 2.1 we have that $z \in \mathcal{N}(\varphi_{k,t_0})$.

This implies that the intersection $\mathcal{N}(\varphi_{k,t_0}) \cap \partial\widetilde{\Omega}$ is non-empty. But we have assumed that the nodal set $\mathcal{N}(\varphi_{k,t_0})$ is away from the perturbation of Ω_{k,t_0}, in other words, away from $\partial\widetilde{\Omega}$. This is a contradiction, and it completes the proof. ■

Next, we also recall the following convergence theorem from [31] which will play a crucial role in the upcoming discussion.

Theorem 2.3 ([31, Theorem 2.2.25]). *Let K_n be a sequence of compact sets contained in a fixed compact set B. Then there exists a compact set K contained in B and a subsequence K_{n_k} that converges in the sense of Hausdorff to K as $k \to \infty$.*

Note that the above lemma can be interpreted in terms of "positional stability" of the nodal sets. Next we discuss the "topological stability" of the first nodal set and a celebrated conjecture of Payne which is related to the topology type of the first nodal set for bounded planar domains. A substantial portion of our discussion in this paper will revolve around this conjecture, which states the following.

Conjecture 2.4 ([58, Conjecture 5]). *For a bounded domain $\Omega \subset \mathbb{R}^2$, the second eigenfunction of the Laplacian with Dirichlet boundary condition does not have a closed nodal line.*

We will refer this conjecture as the Payne conjecture or the nodal line conjecture throughout our text. The second nodal domains represent a 2-partition of Ω minimising the spectral energy, that is

$$\lambda_2(\Omega) = \inf\{\max\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\} : \Omega_1, \Omega_2 \subset \Omega \text{ open},$$
$$\Omega_1 \cap \Omega_2 = \emptyset, \overline{\Omega_1 \cup \Omega_2} = \bar{\Omega}\},$$

where the infimum is attained only when Ω_1, Ω_2 are the nodal domains of some φ_2. One idea behind the above conjecture is that it would be suboptimal from the perspective of energy minimisation to have one nodal domain concentrated somewhere in the interior of Ω, with the other occupying its boundary. Liboff, in [45], conjectured that the nodal surface of the first excited state of a three-dimensional convex domain intersects its boundary in a single simple closed curve. The conjecture is analogous to that of Payne in dimension 3.

2.2. Some previous work on topology of first nodal sets

Now, let us look at some progress made on the above conjecture in a chronological order.

Payne [59] addressed the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is symmetric with respect to one line and convex with respect to the direction vertical to this line. Lin [47] following a similar approach, proved the conjecture provided the domain

$\Omega \subseteq \mathbb{R}^2$ is smooth, convex and invariant under a rotation with angle $2\pi \frac{p}{q}$, where p and q are positive integers. Both the proofs rely heavily on the symmetry of the domain. In [48], Lin and Ni provided a counterexample of the nodal domain conjecture for the Dirichlet Schrödinger eigenvalue problem. For each $n \geq 2$, they construct a radially symmetric potential V in a ball so that the nodal domain conjecture is violated. In 1991, Jerison proved in [36] that the conjecture is true for long thin convex sets in $\mathbb{R}^2$. More specifically, there is an absolute constant C such that given a convex domain $\Omega \subset \mathbb{R}^2$ with $\frac{\text{diam}(\Omega)}{\text{inrad}(\Omega)} \geq C$, we have that the nodal set corresponding to the second eigenfunction intersects the boundary at exactly two points. Here $\text{inrad}(\Omega)$ denotes the radius of the largest ball that can be inscribed in Ω and $\text{diam}(\Omega)$ denotes the diameter. In the following year, Melas relaxed the condition of "long and thin" in [53] and proved the conjecture for any bounded convex domain Ω in $\mathbb{R}^2$ with C^∞ boundary. Alessandrini further relaxed the C^∞-boundary condition and proved the conjecture for general convex planar domains in [3].

To the extent of our knowledge, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and Nadirashvili in [34] provided the first counterexample of the Payne conjecture in $\mathbb{R}^2$ for the case of Dirichlet Laplacian. We outline the basic idea of their construction in Section 3.2 below. We mention in passing that the boundedness of the domain is crucial for results of the Payne type (see [15]).

Regarding the topological properties of the first nodal set in higher dimensions, Jerison [37] extended his result for long and thin convex sets in higher dimensions In [14], Fournais extended the result of [34] in higher dimensions and proved that the first nodal set does not intersect the boundary (we outline his construction in Section 3.2 below). The domain constructed by Fournais was not topologically simple, which was later addressed in [40]. Recently, Kiwan, in [41], proved the nodal domain conjecture for domains which are of the form $A \setminus B$ where A and B have sufficient symmetry and convexity.

Let φ be a Dirichlet eigenfunction for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. For any $p \in \partial\Omega$, $p \in \mathcal{N}_\varphi$ if and only if $\frac{\partial \varphi}{\partial \eta} = 0$, where η denotes the outward normal at p. The proof for dimension $n = 2$ is covered in [47, Lemma 1.2], and one can check that a similar proof is true in higher dimensions as well. Let $x = (x_1, \dots, x_n) =: (x', x_n)$, and let the domain Ω be tangent to the x'-hyperplane at the origin. If $\frac{\partial \varphi}{\partial \eta}(0) \neq 0$, then by the implicit function theorem, x_n is uniquely solvable as a function of x' in a neighbourhood of 0, which means that the only zeros of φ near the origin occur on $\partial\Omega$. The converse case is addressed by a variant of the Hopf boundary principle (see [23, Lemma H]).

Consider any Dirichlet eigenfunction φ whose nodal set $\mathcal{N}_\varphi$ divides Ω into exactly two nodal domains. In particular, any first nodal set (nodal set corresponding to some second eigenfunction) always divides the domain Ω into exactly two components. Then we have the following three cases.

(SP) If $\frac{\partial \varphi}{\partial \eta}$ changes sign on the boundary, then $\bar{\mathcal{N}}_\varphi \cap \partial\Omega \neq \emptyset$ and $\bar{\mathcal{N}}_\varphi$ divides at least one component of $\partial\Omega$ into exactly two components.

(WP) If $\frac{\partial \varphi}{\partial \eta} \geq 0$ (without loss of generality) on the boundary with at least one point $x \in \partial\Omega$ such that $\frac{\partial \varphi}{\partial \eta}(x) = 0$, then $\bar{\mathcal{N}}_\varphi \cap \partial\Omega \neq \emptyset$ but $\partial\Omega \setminus \bar{\mathcal{N}}_\varphi$ has the same number of connected components as $\partial\Omega$.

(NP) If $\frac{\partial \varphi}{\partial \eta} > 0$ (without loss of generality) on the boundary, then $\bar{\mathcal{N}}_\varphi \cap \partial\Omega = \emptyset$.

Definition 2.5. We say that any second eigenfunction φ *satisfies the Payne property* if the nodal set of φ intersects the boundary $\partial\Omega$, that is either (SP) or (WP) is true. We say that φ *satisfies the strong Payne property* if only (SP) is true. Also, we say that Ω *satisfies the (strong) Payne property* if every second eigenfunction φ of Ω satisfies the (strong) Payne property.

Now, we will prove below that certain nodal configurations of the first nodal set remain stable under small enough perturbations.

Proposition 2.6. *In the setting of Lemma 2.2, satisfying the property* (SP) *or* (NP) *is an open condition, in the sense that if Ω_{t_0} satisfies* (SP) *(or* (NP)*), so do Ω_{t_i} for t_i sufficiently close to t_0.*

Proof. By Lemma 2.2, we know that $\mathcal{N}(\varphi_{2,t})$ is eventually inside $\widetilde{\Omega} \subset \Omega_{t_0}$. By precompactness in Hausdorff metric as quoted in Theorem 2.3, one can extract a subsequence called $\mathcal{N}(\varphi_{2,t_i})$, which converges to a set $X \subset \Omega_{t_0}$ in the Hausdorff metric. By Lemma 2.1, we already know that $X \subset \mathcal{N}(\varphi_{2,t_0})$. It follows that for i large enough, $\mathcal{N}(\varphi_{2,t_i})$ is within any δ-tubular neighbourhood of $\mathcal{N}(\varphi_{2,t_0})$.

Let $\mathcal{N}(\varphi_{2,t_0})$ satisfy (NP). Then $\mathcal{N}(\varphi_{2,t_0})$ does not intersect the boundary $\partial\widetilde{\Omega}$. For small enough δ, the δ-tubular neighbourhood of $\mathcal{N}(\varphi_{2,t_0})$ does not intersect $\partial\widetilde{\Omega}$. This implies that given such a δ, for large enough i, $\mathcal{N}(\varphi_{2,t_i})$ does not intersect $\partial\widetilde{\Omega}$. The above argument is schematically described in Figure 1 below.

Now, assume that $\mathcal{N}(\varphi_{2,t_0})$ satisfies (SP). If possible, let (SP) be not an open condition, that is there exists a subsequence $\{k\} \subset \{i\}$ such that $\mathcal{N}(\varphi_{2,t_k})$ does not satisfy (SP). This means that one of nodal domains of the second Dirichlet eigenfunction of Ω_{t_k} is within any δ-tubular neighbourhood of $\mathcal{N}(\varphi_{2,t_0})$ and the volume of such a tubular neighbourhood is going to 0 as $\delta \searrow 0$ (see Figure 2 below). This contradicts the Faber–Krahn inequality (or the inner radius estimate for the second nodal domain of Ω_{ϵ_k}), and implies that for large enough i, $\mathcal{N}(\varphi_{2,t_i})$ intersects the boundary. Moreover, if the first nodal set is a submanifold, then using Thom's isotopy theorem (see [1, Section 20.2]) one can conclude that for large enough i, $\mathcal{N}(\varphi_{2,t_i})$ is diffeomorphic to $\mathcal{N}(\varphi_{2,t_0})$. ∎

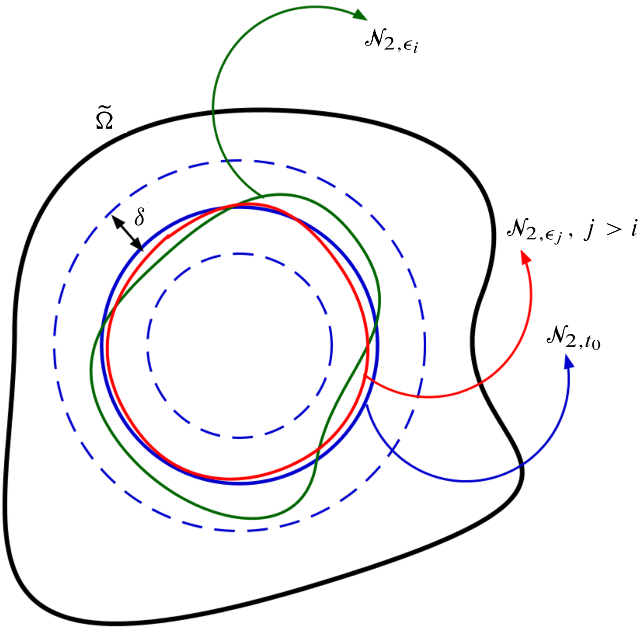


Figure 1. Property (NP) is an open condition.

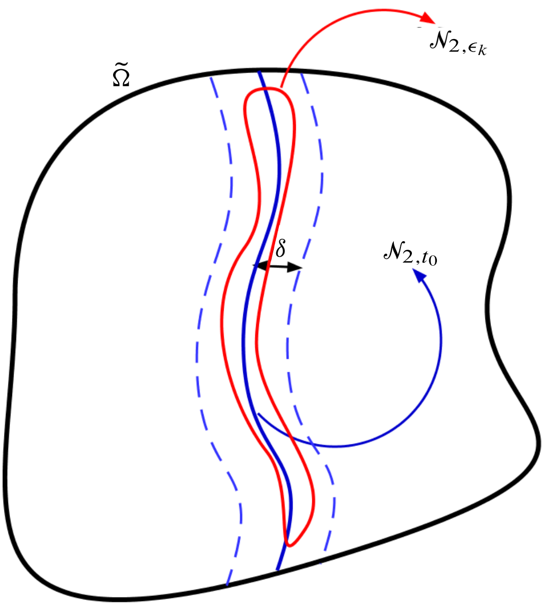


Figure 2. Property (SP) is not an open condition.

In essence, the goal of the proof described above is two fold: to prove the stability of the boundary Neumann data in case that the limiting domain satisfies (SP) or (NP), and to show that the first nodal set of the perturbed domains cannot perturb (in Hausdorff sense) too much from the limiting nodal set. More precisely, for sufficiently small perturbations the first nodal set of the perturbed domains should lie in a δ-neighbourhood (δ however small) of the first nodal set of the limiting domain given that the limiting nodal set satisfies (NP) or (SP). We finish this section with the following.

Remark 2.7. Let $\Omega \subset \mathbb{R}^n$ be a domain which can be realised as a one-parameter family of real-analytic perturbations of the ball. Let the unit ball be denoted by Ω_0 and $\Omega_1 = \Omega$. Then $\{t \in [0,1] : \Omega_t \text{ satisfies (SP)}\}$ is an open set.

3. Applications: Stability on low energy nodal sets

In [55], the present authors proved that certain simply-connected perturbations of convex planar domains and perforated domains with sufficiently small perforations satisfy the nodal line conjecture. In what follows, we continue that discussion with several other classes of domains. Moreover, we also look at several other applications of our stability results and look at certain "perturbation resistant" features of angle estimates of the nodal domains in higher dimensions.

3.1. Payne property of domains with narrow connector

As is already pointed out, by the work in [53], the strong Payne property is known to hold on convex domains. By further work in [55], it is also known to hold on domains obtained from small perturbations of (strictly) convex domains. Somehow a natural approach would be to investigate the validity of the conjecture on domains which are in some sense both "very far" from being convex, or being small perturbations thereof. A natural class of such domains would be the so-called *dumbbell domains*.

We consider dumbbells as constructed in [38]. Consider two bounded disjoint open sets Ω_1 and Ω_2 in $\mathbb{R}^n, n \geq 2$ with smooth boundary such that the boundary of each domain has a flat region. More precisely, for some positive constant $\xi > 0$,

$$\bar{\Omega}_1 \cap \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 \geq -1; |x'| < 3\xi\} = \{(-1, x') \in \partial\Omega_1 : |x'| < 3\xi\},$$

and

$$\bar{\Omega}_2 \cap \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 \leq 1; |x'| < 3\xi\} = \{(1, x') \in \partial\Omega_2 : |x'| < 3\xi\}.$$

Let Q be a line segment joining the flat segments (as described above) of $\partial\Omega_1$ and $\partial\Omega_2$. For some small enough fixed $\epsilon > 0$, consider the dumbbell domain Ω_ϵ obtained

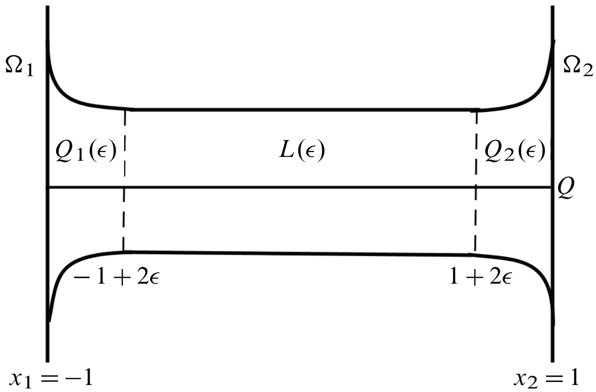


Figure 3. Thin connector Q_ϵ.

by joining Ω_1 and Ω_2 with a connector Q_ϵ denoted by

$$\Omega_\epsilon := \Omega_1 \cup \Omega_2 \cup Q_\epsilon.$$

Here

$$Q_\epsilon = Q_1(\epsilon) \cup L(\epsilon) \cup Q_2(\epsilon)$$

is given by

$$Q_1(\epsilon) = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : -1 \le x_1 \le -1 + 2\epsilon; |x'| < \epsilon\rho\left(\frac{-1-x_1}{\epsilon}\right)\right\},$$
$$Q_2(\epsilon) = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : 1 - 2\epsilon \le x_1 \le 1; |x'| < \epsilon\rho\left(\frac{x_1-1}{\epsilon}\right)\right\},$$
$$L(\epsilon) = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : -1 + 2\epsilon \le x_1 \le 1 - 2\epsilon; |x'| < \epsilon\},$$

where $\rho \in C^\infty((-2,0)) \cap C^0((-2,0])$ is a positive bump function satisfying $\rho(0) = 2$ and $\rho(q) = 1$ for $q \in (-2,-1)$ (see Figure 3).

Considering now the Dirichlet boundary condition on Ω_ϵ, as pointed out in [25, Chapter 7] (see also [10] and [29, Chapter 2]),

$$\lambda_k(\Omega_\epsilon) \to \lambda_k(\Omega_1 \cup \Omega_2) \quad \text{as } \epsilon \to 0.$$

Here, $\lambda_k(\Omega_1 \cup \Omega_2)$ denotes the k-th element after rearranging the Dirichlet eigenvalues of Ω_1 and Ω_2 non-decreasingly. Let Λ_i denote the spectrum of Ω_i ($i = 1, 2$), and $\varphi_k^{\Omega_i}$ denote the k-th Dirichlet eigenfunction of Ω_i corresponding to the eigenvalue $\lambda_k^{\Omega_i}$. If $\Lambda_1 \cap \Lambda_2 = \emptyset$, then each eigenfunction $\varphi_{k,\epsilon}$ on the domain Ω_ϵ approaches in L^2-norm an eigenfunction $\varphi_{k,0} := \varphi_{k'}^{\Omega_i}$ (for some $i = 1, 2$ and $k' \le k$) which is fully localised in one subdomain Ω_i and zero in the other. The fact that the spectra of Ω_1 and Ω_2 do not intersect is important for localisation of the eigenfunctions to exactly one subdomain Ω_i.

We are interested in looking at the nodal sets of the second eigenfunctions of these dumbbell domains with narrow connectors. Without loss of generality, let the eigenfunctions φ_{2,ϵ_i} localise (in the sense described above) on Ω_1 as $\epsilon_i \to 0$. Then, we can rearrange $\Lambda_1 \sqcup \Lambda_2$ in the following two ways:

Case I: $\lambda_1^{\Omega_2} < \lambda_1^{\Omega_1} < \lambda_j^{\Omega_i} \leq \cdots$, for some $i = 1, 2$, and $j \geq 2$;

Case II: $\lambda_1^{\Omega_1} < \lambda_2^{\Omega_1} < \lambda_j^{\Omega_i} \leq \cdots$, for some $i = 1, 2$, and $j \in \mathbb{N}$.

In general, we label the above arrangement as $\lambda_{1,0} \leq \lambda_{2,0} \leq \cdots$. Now, redefine $\varphi_{2,\epsilon_i}, \varphi_{2,0}$ on $\mathbb{R}^n$ as

$$\varphi_{2,\epsilon_i} = \begin{cases} \varphi_{2,\epsilon_i} & \text{on } \Omega_{\epsilon_i}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varphi_{2,0} = \begin{cases} \varphi_1^{\Omega_1} (\text{or } \varphi_2^{\Omega_1} \text{ for Case II}) & \text{on } \Omega_1, \\ 0 & \text{otherwise.} \end{cases}$$

Since we have assumed that the second eigenfunction localises on Ω_1, we have that

$$\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as } \epsilon_i \to 0.$$

We now begin proving Theorem 1.8 which deals with Case II.

Proof. Consider a smooth hypersurface $\Gamma' \subset \overline{\Omega}_1$ such that there exists a δ-tubular neighbourhood of Γ' (denoted by $T_{\Gamma',\delta}$) contained in Ω_1 and away from the perturbation. Moreover, Γ' is chosen in such a way that

- Γ' divides every Ω_ϵ into exactly two components with one of the components being $\Omega' \subset \Omega_1$ (see Figure 4);
- $\partial\Omega_1$ is not a subset of the closure of either components (in other words, we are not considering Γ' to be a closed smooth hypersurface completely contained inside Ω_1).

Now, define $\Omega = \Omega' \cup T_{\Gamma',\delta}$, and $\Gamma = \Gamma' + \delta$ (the outer boundary of $T_{\Gamma',\delta}$ with respect to Ω'). Note that the boundary of Ω can be divided into two parts, namely Γ and $\Gamma^* := \overline{\Omega} \cap \partial\Omega_1$.

We want to prove that $\varphi_{2,\epsilon} \to \varphi_{2,0}$ in $C^0(\Omega')$. We divide Ω' into two regions, Ω_1' and Ω_2', such that $\overline{\Omega'} = \overline{\Omega_1'} \cup \overline{\Omega_2'}$. Here, $\Omega_1' := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$, the "inner δ-shell" of Ω_1, and $\Omega_2' := \Omega' \setminus \overline{\Omega_1'}$ (see Figure 4).

We first prove that $\varphi_{2,\epsilon} \to \varphi_{2,0}$ in $C^0(\Omega_1')$. Consider

$$(\Delta + \lambda_{2,0})[\varphi_{2,\epsilon_i} - \varphi_{2,0}] = (\lambda_{2,0} - \lambda_{2,\epsilon_i})\varphi_{2,\epsilon_i} \quad \text{on } \Omega_1'$$

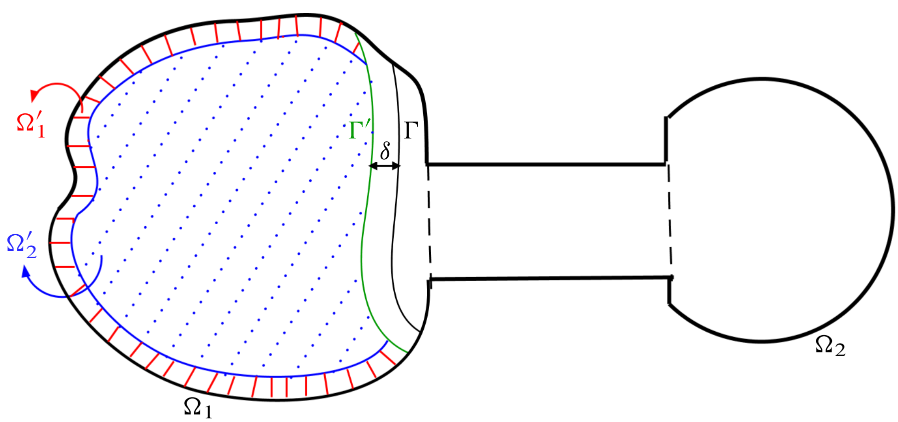


Figure 4. $\Omega' :=$ interior of $\overline{\Omega_1'} \cup \overline{\Omega_2'}$.

and

$$(\Delta + \lambda_{2,\epsilon_i})\varphi_{2,\epsilon_i} = 0 \quad \text{on } \Omega_{\epsilon_i}.$$

Observe that Ω_1' is compactly contained inside $\Omega \subset \Omega_1$. Now, applying [24, Theorem 8.24 and Theorem 8.15] in the above two equations consecutively, we have that for some $q > n$ and $\nu > \sup\{\lambda_{2,0}, \lambda_{2,\epsilon_i}\}$,

$$\begin{aligned}
&\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^\infty(\Omega_1')} \\
&\quad \leq C(\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\Omega)} + \|(\lambda_{2,0} - \lambda_{2,\epsilon_i})\varphi_{2,\epsilon_i}\|_{L^{q/2}(\Omega_1')}) \\
&\quad \leq C(\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\Omega)} + C^*|(\lambda_{2,0} - \lambda_{2,\epsilon_i})| \cdot \|\varphi_{2,\epsilon_i}\|_{L^\infty(\Omega_1')}) \\
&\quad \leq C(\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\Omega)} + C^*|(\lambda_{2,0} - \lambda_{2,\epsilon_i})| \cdot \|\varphi_{2,\epsilon_i}\|_{L^\infty(\Omega_{\epsilon_i})}) \\
&\quad \leq C(\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\Omega)} + C'|\lambda_{2,0} - \lambda_{2,\epsilon_i}| \cdot \|\varphi_{2,\epsilon_i}\|_{L^2(\Omega_{\epsilon_i})}) \\
&\quad \leq C(\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} + C'|\lambda_{2,0} - \lambda_{2,\epsilon_i}| \cdot \|\varphi_{2,\epsilon_i}\|_{L^2(\mathbb{R}^n)}),
\end{aligned}$$

where C, C' depends on q, ν, δ and $|\Omega_{\epsilon_i}|$. For each i, $|\Omega_{\epsilon_i}|$ is uniformly bounded which implies that the constants on the right are independent of i.

Now, using $\lambda_{2,\epsilon_i} \to \lambda_{2,0}$ and the fact that $\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} \to 0$, we have that as $i \to \infty$,

$$\|\varphi_{2,\epsilon_i} - \varphi_{2,0}\|_{L^\infty(\Omega_1')} \to 0.$$

Now, considering the other part Ω_2', note that $\varphi_{2,\epsilon_i}, \varphi_{2,0} = 0$ on $\partial\Omega_2' \cap \partial\Omega_1$. Using [68, Theorem 1.1], we have that the supremum norm of the gradients of φ_{2,ϵ_i} and $\varphi_{2,0}$ are bounded above uniformly, which in turn implies that $\varphi_{2,\epsilon_i}, \varphi_{2,0}$ is sufficiently close to 0. This combined with the above uniform convergence gives us our required C^0-convergence on Ω'.

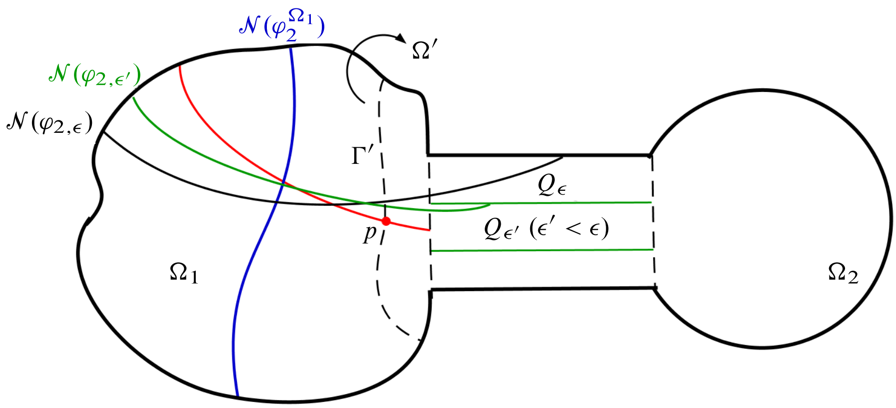


Figure 5. Behaviour of nodal line as $\epsilon \to 0$.

From our assumption, we have $\lambda_{2,0} = \lambda_2^{\Omega_1}$ with $\varphi_{2,\epsilon}|_{\Omega_1} \to \varphi_2^{\Omega_1}$ in $L^2(\Omega_1)$. Additionally, choose the hypersurface $\Gamma' \subset \Omega_1$ such that $\mathcal{N}(\varphi_2^{\Omega_1})$ lies in Ω'.

If possible, let $\mathcal{N}(\varphi_{2,\epsilon_i})$ intersect the connector Q_{ϵ_i} for every $\epsilon_i > 0$, and let $\mathcal{N}(\varphi_{2,\epsilon_i}) \cap \Gamma' = \{p_i\} \in \Omega_1$. Then there exists a subsequence $\{p_j\} \subset \{p_i\}$ and some $p \in \Omega_1$ such that $\{p_j\} \to p$, and from Lemma 2.1, we have that $p \in \mathcal{N}(\varphi_2^{\Omega_1})$. Recall that we have assumed that Q_ϵ is away from $\mathcal{N}(\varphi_2^{\Omega_i})$ $(i = 1, 2)$, that is, Q_ϵ does not intersect $\mathcal{N}(\varphi_2^{\Omega_1})$ for any $\epsilon > 0$. This leads to a contradiction which implies that, there exists $\epsilon_0 > 0$ such that $\mathcal{N}(\varphi_{2,\epsilon}) \subset \Omega'$ for any $\epsilon < \epsilon_0$ (see Figure 5).

Now, using Theorem 2.3 one can extract a subsequence called $\mathcal{N}_{\varphi_{2,\epsilon_i}}$ $(\epsilon_i < \epsilon_0)$ which converges to a set $X \subset \Omega_1$ in the Hausdorff metric and by Lemma 2.1 we know that $X \subset \mathcal{N}(\varphi_2^{\Omega_1})$. Now, following the argument as in Proposition 2.6, we conclude the proof. ■

Remark 3.1. With some obvious modifications, the above proof also tells us that if Ω_1 satisfies (NP), then so does Ω_ϵ for sufficiently small ϵ.

So far, we have only looked at the nodal sets of the dumbbells under Case II. Now, we turn our attention to the location of nodal sets under Case I.

Theorem 3.2. *Consider a family of dumbbells as described in Theorem 1.8 with the condition that $\lambda_{2,\epsilon} \to \lambda_1^{\Omega_1}$. Then, for sufficiently small ϵ, $\mathcal{N}(\varphi_{2,\epsilon})$ does not enter $\Omega' \subset \Omega_1$.*

We would like to point out that the only restriction on the choice of Γ' in this case is that Γ' divides every Ω_ϵ into exactly two components as described in the beginning of the proof of Theorem 1.8. In other words, one can choose $\Gamma' \subset \Omega_1$ sufficiently close to the connectors.

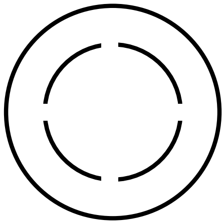


Figure 6. $N = 4$ ([34, Figure 1]).

Proof. Considering the case when $\lambda_{2,0} = \lambda_1^{\Omega_1}$ and $\varphi_{2,\epsilon_i} \to \varphi_1^{\Omega_1}$ in $L^2(\Omega_1)$, note that $\varphi_1^{\Omega_1}$ does not change sign in Ω_1. Without loss of generality, assume that $\varphi_1^{\Omega_1} > 0$ in Ω_1. Moreover, we know that $\mathcal{N}(\varphi_{2,\epsilon})$ divides Ω_ϵ into two components. If possible, let $\mathcal{N}(\varphi_{2,\epsilon})$ enters Ω' for every $\epsilon > 0$. Then $\mathcal{N}(\varphi_{2,\epsilon})$ intersects Γ' for every ϵ. Since $\varphi_1^{\Omega_1} > 0$ in Ω_1 and $\|\varphi_{2,\epsilon} - \varphi_1^{\Omega_1}\|_{C^0(\overline{\Omega'})} \to 0$, it is clear that for small enough ϵ, $\varphi_{2,\epsilon} > 0$ on Γ', which is a contradiction. ■

A natural follow-up to the above theorem is that, under Case I, *can we say that for sufficiently small* ϵ, $\mathcal{N}(\varphi_{2,\epsilon})$ *does not enter* Ω_1 *at all?*

We comment in passing that dumbbell domains have many interesting properties that are of interest to spectral theorists. For example, they provide examples of domains which have arbitrarily low second Neumann eigenvalue, and almost satisfy a weak version of the hot spot conjecture (see [56]). There is also a significant literature on mass concentration questions in the connector of the dumbbell, for example see [5, 11, 22, 67].

3.2. Counterexample to the Payne property in higher dimensions

We begin by discussing the planar counterexample as given in [34]. First, we choose two concentric balls B_{R_1} and B_{R_2} in $\mathbb{R}^2$ such that

$$\lambda_1(B_{R_1}) < \lambda_1(B_{R_2} \setminus \overline{B_{R_1}}) < \lambda_2(B_{R_1}).$$

Next, we carve out holes into ∂B_{R_1}. Let $N \in \mathbb{N}$ and $\epsilon < \pi/N$. The domain $\Omega_{N,\epsilon}$ is defined as

$$\Omega_{N,\epsilon} = B_{R_1} \cup (B_{R_2} \setminus \overline{B_{R_1}}) \cup \Big(\bigcup_{j=0}^{N-1} \Big\{ x \in \mathbb{R}^2 : r = R_1, \omega \in \Big(\frac{2\pi j}{N} - \epsilon, \frac{2\pi j}{N} + \epsilon \Big) \Big\} \Big).$$

Then the first nodal line does not intersect the boundary for sufficiently large N and small ϵ (see Figure 6 above).

In higher dimensions ($n \geq 3$), the domain constructed by Fournais was motivated from the example in [34] described above and defined as follows:

$$\Omega^\epsilon = B_{R_1} \cup (B_{R_2} \setminus \overline{B_{R_1}}) \cup \Big(\bigcup_{i=1}^{N} B(x_i, \epsilon) \Big),$$

where B_R is a ball of radius R centred at 0, and $x_1, \dots, x_N \in S^{n-1}_{R_1}$ are chosen in such a way that the "patches" $B(x_i, \epsilon) \cap S^{n-1}_{R_1}$ are evenly distributed over $S^{n-1}_{R_1}$, the sphere with centre at 0 and radius R_1. For convenience, moving forward we will refer to the sphere $S^{n-1}_{R_1}$ as S. Also, R_1 and R_2 are chosen such that

$$\lambda_1(B_{R_1}) < \lambda_1(B_{R_2} \setminus \overline{B_{R_1}}) < \lambda_2(B_{R_1}).$$

Then for small enough ϵ, the second eigenfunction $\varphi_{2,\epsilon}$ satisfies

$$\mathcal{N}(\varphi_{2,\epsilon}) \cap \partial\Omega^\epsilon = \emptyset.$$

The main idea in [14] is to prove that for small enough $\delta > 0$, there is $\epsilon > 0$ such that $\varphi_{2,\epsilon}(x) > 0$ on $|x| = R_1 - \delta$. Then using various assumptions made during the construction along with certain topological restrictions of the first nodal set $\mathcal{N}(\varphi_{2,\epsilon})$, one concludes that $\mathcal{N}(\varphi_{2,\epsilon})$ is contained inside $B_{R_1-\delta}$.

Note that the domain Ω^ϵ described above is not simply connected. Also, Ω^ϵ has a simple spectrum. Our goal in this section is to produce a simply-connected domain whose nodal set does not intersect the boundary.

Let $\Omega_0 := \Omega^\epsilon$, where the nodal set is contained in $B_{R_1-\delta}$. Throughout the rest of the proof, the above ϵ and δ will remain fixed. From Ω_0, we can construct simply-connected domains by adding $(n-1)$-dimensional "tunnels" or "strips" T_η along S in between the "patches" $B(x_i, \epsilon)$ such that every patch is connected to the neighbouring patches by tunnels (see Figure 7). Our idea is to make these tunnels narrow enough so that the nodal set of Ω_0 does not get sufficiently perturbed.

Let $\eta > 0$. For any $i, j \in \{1, \dots, N\}$ $(i \neq j)$, let $p_{ij}(t)\colon [0,1] \to S$ be a path between x_i and x_j along S such that the length of p_{ij} is $\text{dist}_S(x_i, x_j)$, the geodesic distance between x_i and x_j on S. Let t_0 and $t_1 \in [0,1]$ be such that $p_{ij}(t_0) \in \partial B(x_i, \epsilon)$ and $p_{ij}(t_1) \in \partial B(x_j, \epsilon)$. Now, consider the path segment $P_{ij} = [p_{ij}(t_0), p_{ij}(t_1)]$. Let $\tau_{ij}(\eta)$ denote the η-tubular neighbourhood of P_{ij}. Define the tunnel $T^{i,j}_\eta := S \cap \tau_{ij}(\eta)$. Let there be k_N tunnels in total. Denote

$$T_\eta := \bigcup_{i=1}^{k_N} T^{i,j}_\eta.$$

Now, we define a family of domains Ω_η as

$$\Omega_\eta := B_{R_1} \cup (B_{R_2} \setminus \overline{B_{R_1}}) \cup \Big(\bigcup_{i=1}^{N} B(x_i, \epsilon) \Big) \cup T_\eta = \Omega_0 \cup T_\eta.$$

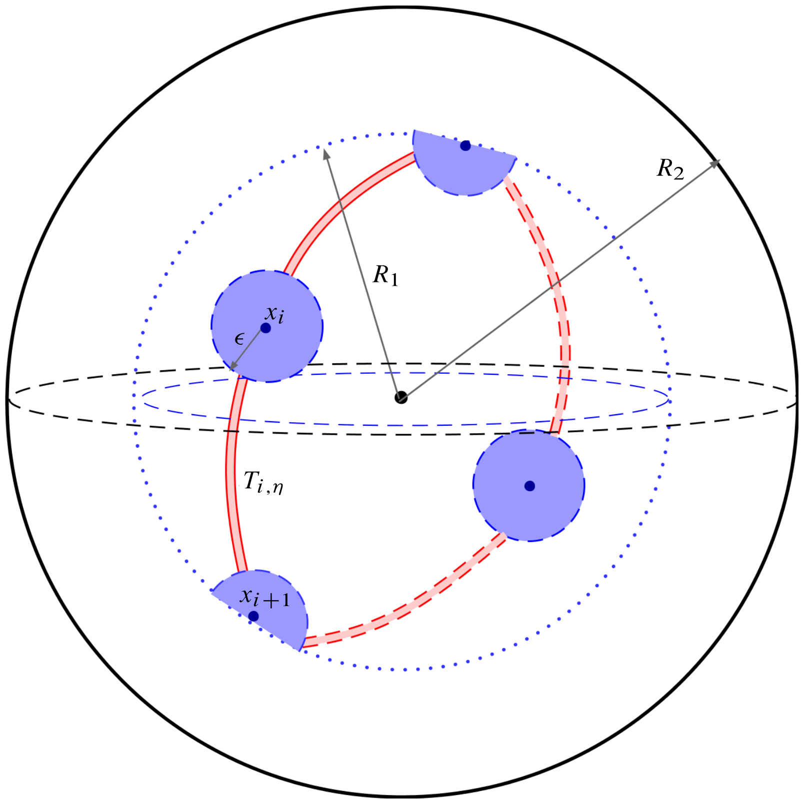


Figure 7. Topologically simple counterexample to Payne.

It is easy to check that Ω_η is simply connected and given any sequence $\eta_q \searrow 0$ there is a subsequence $\{\eta_p\} \subseteq \{\eta_q\}$ such that Ω_{η_p} converges to Ω_0 in Hausdorff metric. Now, we check that the perturbation to the nodal set is controlled.

Let $\varphi_{j,\eta}, \varphi_{j,0}$ denote the eigenfunction corresponding to eigenvalues $\lambda_{j,\eta}, \lambda_{j,0}$ of the Dirichlet Laplacian $-\Delta_\eta, -\Delta_0$. We assume that the eigenfunctions are L^2-normalised. Also note that, from our assumption, we have that $\varphi_{2,0} > 0$ in $\Omega_0 \setminus B_{R_1-\delta}$.

Let $\{\eta_p\} \searrow 0$ be any strictly monotonically decreasing sequence and

$$X_p := B_{R_2} \setminus \Omega_{\eta_p}.$$

Note that $\{X_p\}$ is an increasing family of compact sets. Define

$$P_p := \bigcup_{k \geq p}^{\infty} X_k \quad \text{and} \quad Q_p := \bigcap_{k \geq p}^{\infty} X_k.$$

Using the convention from [63, p. 278], we have that $P_p \nearrow X = \overline{\lim} X_p$ and $Q_p \searrow X = \underline{\lim} X_p$ where $X = B_{R_2} \setminus \Omega_0'$ and $\Omega_0' := \Omega_0 \cup \big(\bigcup P_{ij}\big)$. Also, for any $p, m \in \mathbb{N}$, $X_p \triangle X_m \subset T_{\eta_1}$ which has finite capacity and $\operatorname{cap}(\Omega_0 \triangle \Omega_0') = 0$. Then using [63, Theorem 2.2], we have that $-\Delta_{\eta_p}$ converges to $-\Delta_0$ as $p \to \infty$ in norm resolvent sense (recall that if $\{T_p\}_{n=1}^{\infty}$ and T are unbounded self-adjoint operators, then $T_p \to T$ in norm resolvent sense means that for some $z \in \mathbb{C} \setminus \mathbb{R}$, $\|(zI - T_p)^{-1} - (zI - T)^{-1}\| \to 0$ as $p \to \infty$). In particular, for $\lambda_{2,0}$ there exists a sequence $\eta_p \to 0$ such that

$$\lambda_{2,\eta_p} \to \lambda_{2,0}.$$

Redefining $\varphi_{2,\eta_p}, \varphi_{2,0}$ by 0 on $\mathbb{R}^n \setminus \Omega_{\eta_p}, \mathbb{R}^n \setminus \Omega_0$ we also have that $\varphi_{2,\eta_p} \to \varphi_{2,0}$ in $L^2(\mathbb{R}^n)$.

We know that $\mathcal{N}(\varphi_{2,0})$ is completely contained inside $B_{R_1-\delta}$. Now, we would like to show that $\varphi_{2,\eta_p} \to \varphi_{2,0}$ in $C^0(B_{R_1-\delta})$. Consider

$$(\Delta + \lambda_{2,0})[\varphi_{2,\eta_p} - \varphi_{2,0}] = (\lambda_{2,0} - \lambda_{2,\eta_p})\varphi_{2,\eta_p} \quad \text{on } B_{R_1-\delta'},$$

and

$$(\Delta + \lambda_{2,\eta_p})\varphi_{2,\eta_p} = 0 \quad \text{on } \Omega_{\eta_p}$$

where $0 < \delta' < \delta$. Now, using [24, Theorem 8.24 and Theorem 8.15] consecutively on the above equations as done in the proof of Theorem 1.8 we have that for some $q > n$ and $\nu > \sup\{\lambda_{2,0}, \lambda_{2,\eta_p}\}$,

$$\begin{aligned}&\|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^\infty(B_{R_1-\delta})}\\ &\quad \le C(\|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} + C'|\lambda_{2,0} - \lambda_{2,\eta_p}| \cdot \|\varphi_{2,\eta_p}\|_{L^2(\mathbb{R}^n)}),\end{aligned}$$

where C, C' depends on n, q, ν, and $|\Omega_{\eta_p}|$. For each p, $|\Omega_{\eta_p}|$ is uniformly bounded which implies that the constants on the right hand are independent of p. Now, using $\lambda_{2,\eta_p} \to \lambda_{2,0}$ and $\|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} \to 0$, we have that as $p \to \infty$,

$$\|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^\infty(B_{R_1-\delta})} \to 0,$$

which gives our desired $C^0(B_{R_1-\delta})$ convergence.

Finally, using Lemma 2.1 and Proposition 2.6, we know that $\mathcal{N}(\varphi_{2,\eta_n})$ converges to $\mathcal{N}(\varphi_{2,0})$. So, for sufficiently large $n_0 \in \mathbb{N}$, we have

$$\mathcal{N}(\varphi_{2,\eta_{n_0}}) \subset\subset B_{R_1-\delta}.$$

In other words, we have a simply-connected domain $\Omega_{\eta_{n_0}} \in \mathbb{R}^n (n \ge 3)$ for which

$$\mathcal{N}(\varphi_{2,\eta_p}) \cap \partial\Omega_{\eta_{n_0}} = \emptyset \quad \text{for every } p \le n_0.$$

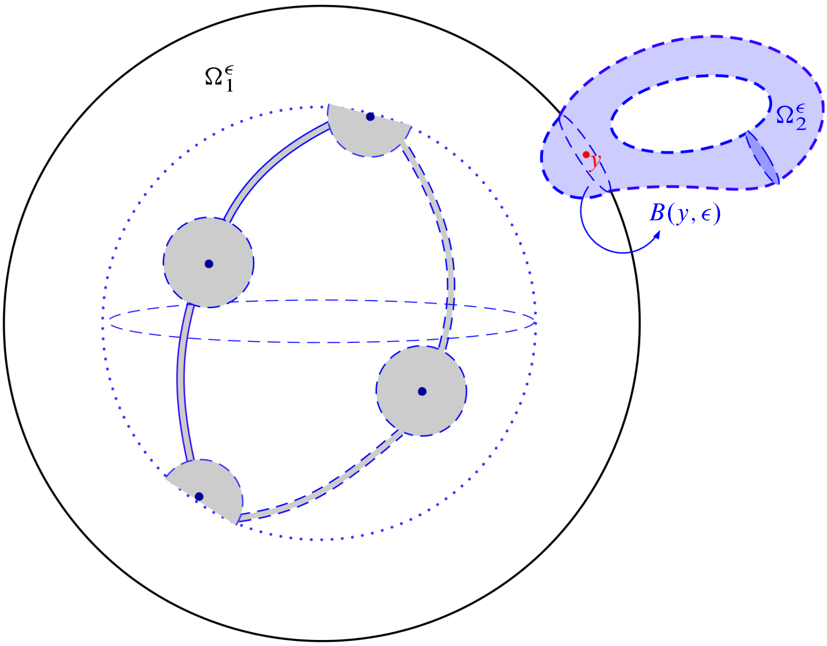


Figure 8. Counterexample of Payne property with any prescribed topology.

Proof of Theorem 1.9. Given $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), the way to construct the required domains is by taking the connected sum of $\Omega_{\eta_{n_0}}$ with the given domain Ω. But we note that one cannot blindly attach one domain with another since the attached metric will not be Euclidean (or even flat) in general. So the family of metrics is to be designed precisely to ensure that each deformation $\Omega_\epsilon = \Omega_{\eta_{n_0}} \#_\epsilon \Omega$ is a Euclidean domain.

Consider a one-parameter family of deformations Ω_ϵ, where Ω_ϵ can be written as a disjoint union $\Omega_1^\epsilon \sqcup \Omega_2^\epsilon$ (see Figure 8), where

- $\Omega_1^\epsilon := \Omega_{\eta_{n_0}} \setminus B(\xi, \epsilon)$, for some $\xi \in \partial B_{R_2}$;
- $\Omega_2 \subset \mathbb{R}^n$ is another domain defined as $\Omega \cup B(y, 1)$ with $y \in \partial\Omega$, and Ω_2^ϵ is obtained from Ω_2 by scaling $g|_{\Omega_2^\epsilon} = \epsilon^2 g|_{\Omega_2}$.

Then, with some obvious modifications to the proof of [42, Theorem 3.4], we have that for sufficiently small ϵ, $\varphi_{2,\epsilon} \to \varphi_2(\Omega_{\eta_{n_0}})$ in C^0-norm. Then applying Lemma 2.1 and Proposition 2.6 again, we have that Ω_ϵ satisfies (NP) for sufficiently small ϵ. Moreover, since $\Omega_{\eta_{n_0}}$ is simply connected, we have that Ω_1^ϵ is simply connected which implies that Ω_ϵ has the same fundamental group as that of Ω. ∎

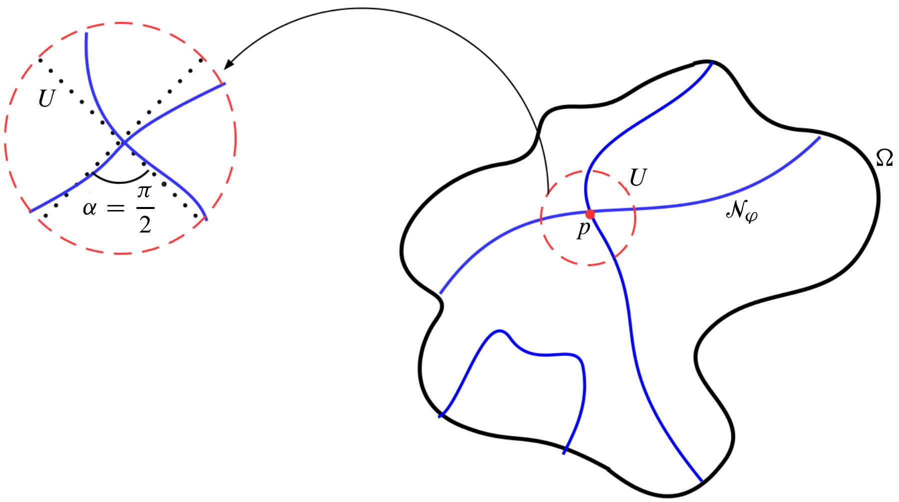


Figure 9. Four equiangular "rays" from p.

3.3. Angle estimates of nodal sets

A consequence of the fact that (SP) is open (as proved in Proposition 2.6) is that the angle at the nodal critical points of the first nodal set with order of vanishing 2 and satisfying (SP) remains stable under perturbation.

3.3.1. Opening angles nodal domains in the interior and the boundary. It was shown by Melas [53] that the nodal domain for the second Dirichlet eigenfunction which intersects the boundary $\partial\Omega$ cannot have an "opening angle" of 0 or π at the point of intersection. Here, we provide a generalisation of this result from a different perspective, one that was introduced in [21].

Interior cone conditions. In dimension $n = 2$, a well-known result of Cheng [7] says the following (see also [62] for a proof using Brownian motion).

Theorem 3.3. *For a compact Riemannian surface M, the nodal set $\mathcal{N}_{\varphi_\lambda}$ satisfies an interior cone condition with opening angle $\alpha \gtrsim \frac{1}{\sqrt{\lambda}}$.*

Furthermore, in dimension 2, the nodal lines form an equiangular system at a singular point of the nodal set (see Figure 9). The idea behind Cheng's proof is the following: using a local power series expansion due to Bers (see Theorem 3.6 below), near any point of vanishing the eigenfunction "looks like" a homogeneous harmonic polynomial whose degree matches the order of vanishing at that point. If the order of vanishing is k, then in two dimensions such a function would be a linear combination of $r^k \cos k\theta$ and $r^k \sin k\theta$. This gives an equiangular nodal junction.

The situation is significantly more complicated in higher dimensions. Setting $\dim M \geq 3$, we discuss the question whether at the singular points of the nodal set $\mathcal{N}_\varphi$, the nodal set can have arbitrarily small opening angles, or even "cusp"-like situations, or the nodal set has to self-intersect "sufficiently transversally." We observe that in dimension $n \geq 3$ the nodal sets satisfies an appropriate "interior cone condition," and give an estimate on the opening angle of such a cone in terms of the eigenvalue λ.

Now, in order to properly state or interpret such a result, one needs to define the concept of "opening angle" in dimension $n \geq 3$. We start by defining precisely the notion of tangent directions in our setting.

Definition 3.4. Let Ω_λ be a nodal domain and $x \in \partial\Omega_\lambda$, which means that $\varphi_\lambda(x) = 0$. Consider a sequence $x_n \in \mathcal{N}_\varphi$ such that $x_n \to x$. Let us assume that in normal coordinates around x, $x_n = \exp(r_n v_n)$, where r_n are non-negative real numbers, and $v_n \in S(T_x M)$, the unit sphere in $T_x M$. Then, we define the space of tangent directions at x, denoted by $\mathcal{S}_x \mathcal{N}_\varphi$ as

$$\mathcal{S}_x \mathcal{N}_\varphi = \{v \in S(T_x M) : v = \lim v_n, \text{ where } x_n \in \mathcal{N}_\varphi, x_n \to x\}.$$

Observe that there are more well-studied variants of the above definition, for example, as due to Clarke or Bouligand (for more details, see [60]). With that in place, we now give the following definition of "opening angle."

Definition 3.5. We say that the nodal domain Ω_λ satisfies an *interior cone condition with opening angle* α *at* $x \in \mathcal{N}_\varphi \subset \partial\Omega_\lambda$, if any connected component of $S(T_x M) \setminus \mathcal{S}_x \partial\Omega_\varphi$ has an inscribed ball of radius $\gtrsim \alpha$.

We will use Bers scaling of eigenfunctions near zeros (see [6]). We quote the version which appears in [69, Section 3.11].

Theorem 3.6 (Bers). *Assume that φ_λ vanishes to order k at x_0. Let $\varphi_\lambda(x) = \varphi_k(x) + \varphi_{k+1}(x) + \cdots$ denote the Taylor expansion of φ_λ into homogeneous terms in normal coordinates x centred at x_0. Then $\varphi_\kappa(x)$ is a Euclidean harmonic homogeneous polynomial of degree k.*

We also use the following inradius estimate for real analytic metrics (see [19]).

Theorem 3.7. *Let (M, g) be a real-analytic closed manifold of dimension at least* 3. *If Ω_λ is a nodal domain corresponding to the eigenfunction φ_λ, then there exist constants λ_0, c_1 and c_2 which depend only on (M, g), such that*

$$\frac{c_1}{\lambda} \leq \operatorname{inrad}(\Omega_\lambda) \leq \frac{c_2}{\sqrt{\lambda}}, \quad \lambda \geq \lambda_0.$$

Since the statement of Theorem 3.7 is asymptotic in nature, we need to justify that if $\lambda < \lambda_0$, a nodal domain corresponding to λ will still satisfy $\operatorname{inrad}(\Omega_\lambda) \geq \frac{c_3}{\lambda}$ for some constant c_3. This follows from the inradius estimates of Mangoubi in [50], which hold for all frequencies. Consequently, we can assume that every nodal domain Ω on S^n corresponding to the spherical harmonic $\varphi_k(x)$, as in Theorem 3.6, has inradius $\gtrsim \frac{1}{\lambda}$.

Now, we start proving Theorem 1.10.

Proof. Since the eigenequation $-\Delta\varphi_\lambda = \lambda\varphi_\lambda$ is satisfied at p, one can check that the proof of Theorem 3.6 above still works at p, for all $p \in M \cup \partial M$. We observe that Theorem 3.7 applies to spherical harmonics, and in particular the function $\exp^*(\varphi_k)$, restricted to $S(T_{x_0}M)$, where $\varphi_k(x)$ is the homogeneous harmonic polynomial given by expanding φ at p in terms of $x \in M \cup \partial M$ given by Theorem 3.6. Also, a nodal domain for any spherical harmonic on S^2 (respectively, S^3) corresponding to eigenvalue λ has inradius $\sim \frac{1}{\sqrt{\lambda}}$ (respectively, $\gtrsim \frac{1}{\lambda^{7/8}}$).

With that in place, it suffices to prove that

$$\mathcal{S}_{x_0}\mathcal{N}_\varphi \subseteq \mathcal{S}_{x_0}\mathcal{N}_{\varphi_k}. \tag{3.1}$$

Now, by definition, $v \in \mathcal{S}_{x_0}N_\varphi$ if there exists a sequence $x_n \in N_\varphi$ such that $x_n \to x_0$, $x_n = \exp(r_n v_n)$, where r_n are positive real numbers and $v_n \in S(T_{x_0}M)$, and $v_n \to v$.

This gives us

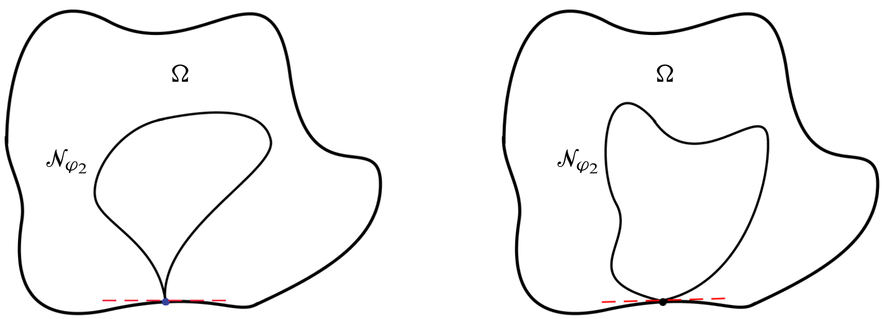
$$\begin{aligned} 0 = \varphi_\lambda(x_n) &= \varphi_\lambda(r_n \exp v_n) \\ &= r_n^k \varphi_k(\exp v_n) + \sum_{m>k} r_n^m \varphi_m(\exp v_n) \\ &= \varphi_k(\exp v_n) + \sum_{m>k} r_n^{m-k} \varphi_m(\exp v_n) \\ &\to \varphi_k(\exp v), \quad \text{as } n \to \infty. \end{aligned}$$

Observing that $\varphi_k(x)$ is homogeneous, this proves (3.1). ■

Observe that Theorem 1.10 above tells us that the following two situations in Figure 10 can never happen at the boundary for the nodal set of any eigenfunction (there is nothing specific about the second eigenfunction).

Remark 3.8. In dim $M = 2$, since any point $p \in \partial M$ satisfies the eigenequation $-\Delta\varphi = \lambda\varphi$, the local expansion of Bers is true on the boundary as well. Then using the above ideas of Cheng on the boundary, we have that if $p \in \partial M$ has k-th order of vanishing then $\mathcal{N}_\varphi$ forms an equiangular junction at p with respect to the tangent at p.

3.3.2. More precise estimates on opening angles. We are now going to investigate in more detail the angle between two nodal hypersurfaces at a point of intersection. In some sense, our results here are going to be higher-dimensional analogues of Cheng's



(a) Opening angle=0. **(b)** Opening angle=π.

Figure 10. Impermissible angle of intersections for any bounded domain in $\mathbb{R}^2$.

result outlined in Remark 3.8. Very interestingly, such problems have been investigated from a completely different viewpoint in classical Fourier analysis, namely, the existence of Heisenberg uniqueness pairs.

Now, we begin proving Theorem 1.11. Our proof is a modification of ideas in [13].

Proof. Let

$$P = \left\{ \frac{p}{q}\pi : q = 1, 2, \ldots, n_0, p = 0, 1, \ldots, q \right\}.$$

If possible, let $\cos^{-1} \langle \eta_1, \eta_2 \rangle \notin P$ where η_i is a unit normal to M_i at x. Without loss of generality, we think of x as the origin in $\mathbb{R}^n$. Consider the spherical coordinates (r, θ, φ) in $\mathbb{R}^n$, where $r \geq 0$, $\theta := (\theta_1, \ldots, \theta_{n-2}) \in [0, \pi)^{n-2}$, $\varphi \in [0, 2\pi)$.

It is known that $\{Y_\alpha : \alpha \in \mathcal{S} := \mathbb{N}_0^{n-2} \times \mathbb{Z}\}$ forms a basis of spherical harmonics, where

$$Y_\alpha(r, \theta, \varphi) = r^{|\alpha|} \exp(i\alpha_{n-1}\varphi)\widetilde{Y}_\alpha(\theta),$$

with $\widetilde{Y}_\alpha(\theta) := \prod_{i=1}^{n-2}(\sin\theta_{n-i})^{|\alpha|^{i+1}} C_{\alpha_i}^{\gamma_i}(\cos\theta_i)$, and $|\alpha|^i = \alpha_i + \alpha_{i+1+\cdots+\alpha_{n-1}}$, $\gamma_i = |\alpha|^{i+1} + \frac{1}{2}(n - i - 1)$, where $C_{\alpha_i}^{\gamma_i}$ are the Gegenbauer polynomials. From the orthogonality of C_n^γ, for each $n \geq 1$, notice that the set

$$\{\widetilde{Y}_{(\beta,m)} : (\beta, m) \in \mathbb{N}_0^{n-1}, |\beta| + m = n\}, \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

is linearly independent.

Observe that $M_i \cap B(0, \epsilon)$ can be parametrised in polar coordinates as

$$M_i \cap B(0, \epsilon) = \{(r, \theta, \psi_i(r, \theta)), 0 \leq r < \epsilon, \theta \in [0, \pi)^{n-2}\},$$

where $\psi_i(r, \theta) \in S^1$ and ψ_i's are smooth functions. Defining

$$\varphi_i(\theta) := \lim_{r \to \theta} \psi_i(r, \theta),$$

from our assumption $\arccos\langle \eta_1, \eta_2 \rangle \notin P$, it follows that $\varphi_1 - \varphi_2 \notin P$.

Since the order of vanishing of φ_λ at 0 is n_0, using Theorem 3.6, the solution φ_λ of $(\Delta+\lambda)\varphi_\lambda = 0$ in $M_i \cap B(0,\epsilon)$ can be expressed in spherical coordinates in the form

$$\varphi_\lambda(r,\theta,\psi_i(r,\theta)) = r^{n_0}\sum_{m=-n_0}^{n_0}\Big(\sum_{|\beta|+|m|=n_0} c_{\beta,m}\widetilde{Y}_{\beta,m}(\theta)\Big)e^{im\varphi_i} + o(r^{n_0}).$$

Since, $\varphi_\lambda = 0$ on $M_i \cap B(0,\epsilon)$ as $r \to 0$, it follows that

$$\sum_{m=-n_0}^{n_0}\Big(\sum_{|\beta|+|m|=n_0} c_{\beta,m}\widetilde{Y}_{\beta,m}(\theta)\Big)e^{im\varphi_i} = 0,$$

that is

$$\sum_{|\beta|=n_0} c_{\beta,0}\widetilde{Y}_{\beta,0}(\theta) + \sum_{m=1}^{n_0}\Big(\sum_{|\beta|+m=n_0}(c_{\beta,m}e^{im\varphi_i} + c_{\beta,-m}e^{-im\varphi_i})\widetilde{Y}_{\beta,m}(\theta)\Big) = 0$$

Since $\{\widetilde{Y}_{(\beta,m)}\}$ is linearly independent, $c_{\beta,0} = 0$ whenever $|\beta| = n_0$ and for each $m = 1,2,\ldots,n_0$, we have n_0 system of equations

$$c_{\beta,m}e^{im\varphi_1} + c_{\beta,-m}e^{-im\varphi_1} = 0,$$
$$c_{\beta,m}e^{im\varphi_2} + c_{\beta,-m}e^{-im\varphi_2} = 0.$$

Notice that the determinant of each of the above systems is $2i\sin m(\varphi_1-\varphi_2)$, $m = 1,2,\ldots,n_0$. If

$$(\varphi_1-\varphi_2) \notin \Big\{\frac{p}{q}\pi : q = 1,2,\ldots,n_0,\, p = 0,1,\ldots,q\Big\},$$

then each of above n_0 determinants is non-zero, which forces each $c_{\beta,m} = c_{\beta,-m} = 0$, which implies that the coefficient of r^{n_0} is zero. But this contradicts the fact that φ_λ has n_0 order of vanishing at 0. So,

$$\cos^{-1}\langle\eta_1,\eta_2\rangle \in P. \qquad \blacksquare$$

Remark 3.9. Recall the celebrated result of [12] that any λ-eigenfunction φ_λ vanishes to at most order $c(M,g)\sqrt{\lambda}$ for any point in M. Then, using our result above, we have that whenever two nodal hypersurfaces intersect, the admissible angles between such intersecting hypersurfaces is from the set

$$P = \Big\{\frac{p}{q}\pi : q = 1,2,\ldots,[c\sqrt{\lambda}],\ p = 0,1,\ldots,q\Big\}. \tag{3.2}$$

Also, using Theorem 1.10, we can rule out the cases when $p = 0, q$ for every $q = 1,2,\ldots,c\sqrt{\lambda}$. Then one sees that the minimum angle (in the sense of Theorem 1.11) between two nodal hypersurfaces is $\gtrsim \frac{1}{\sqrt{\lambda}}$.

To sum up the discussion so far, consider a point $x \in \mathcal{N}_\varphi$. Then the opening angle of a nodal domain at x will in general be given by Definition 3.5. However, if x happens to lie at the intersection of some nodal hypersurfaces, then the angle between any pair of such nodal hypersurfaces will come from the set P in (3.2).

4. Further applications: Perturbation theory and low spectral gaps

We study the variation of the Dirichlet Laplace spectrum and corresponding eigenfunctions with C^2 variations of bounded Euclidean domains, particularly those domains Ω which are critical points of $\lambda_2 - \lambda_1$.

Let the smooth deformation space of Ω be given by a Banach manifold $\mathcal{B}$. First, we prove the following.

Theorem 4.1. *The set of points inside $\mathcal{B}$ (each represented by a perturbation of our starting domain Ω) such that the Dirichlet Laplacian has simple spectrum is a residual set.*

We note that Theorem 4.1 is not new, for example see [66, Example 3, Section 4]. The ideas involved in our proof are based on [17] and are well known by now, though we have not seen this exact proof in the literature. More importantly, it sets up some crucial heuristics and ideas used later.

Results of the nature of Theorem 4.1 are ultimately based on transversality phenomena (as illustrated in [66]). Loosely speaking, they can be considered infinite-dimensional analogues of the following statement: generically, all symmetric matrices have non-repeated eigenvalues. At a more basic level, an equivalent statement is the fact that single variable polynomials generically have non-repeated roots.

4.1. Proof of Theorem 4.1

The topic of variation of spectra under perturbation has a long history starting with the analytic perturbation theory of Kato (see [39]). In the case of rather generic families of elliptic operators, see pioneering work in [3, 66]. In case the perturbation is non-generic, such results have been recently studied in, for example, [32, 33, 54] etc. In this note, we give a slight variant of a proof for Theorem 4.1. To set up the stage, we start by considering a bounded domain $\Omega \subset \mathbb{R}^n$, and consider a vector field V defined on $\mathbb{R}^n$, whose coordinates we denote by $(V_1, \dots, V_n)$, and whose regularity we assume to be C^2 for immediate purposes. Now, consider the perturbation of the domain Ω along the vector field V to the domain Ω_ε, defined by $\{x^\varepsilon = x + \varepsilon V : x \in \Omega\}$. We wish to study the variation of the eigenequation

$$-\Delta\varphi = \lambda\varphi$$

along the parameter ε. However, to fit the language of perturbation theory, instead of dealing with a one-parameter family of domains, it is much more convenient to pull back all Ω_ε to the original domain Ω, so that we get a one-parameter family of elliptic PDEs on Ω whose coefficients are dependent on ε. This lands us in the familiar framework of a family of self-adjoint operators with common domain of definition varying over a Banach manifold. Upon computation following [17, Sections 4, 5 (particularly p. 299), and 6], we see that the eigenequation

$$-\Delta_\varepsilon \varphi_\varepsilon = \lambda_\varepsilon \varphi_\varepsilon$$

on Ω_ε, when pulled back to Ω, becomes

$$A_\varepsilon u = \sum_{j,k} -\partial_k (J \beta_{kj} \partial_j u) = \lambda_\varepsilon J u,$$

with the Dirichlet boundary condition being preserved, and where J is the determinant of the Jacobian matrix of the transformation $x \mapsto x^\varepsilon$, and $\beta_{jk} = \sum_l \frac{\partial x_j}{\partial x_l^\varepsilon} \frac{\partial x_k}{\partial x_l^\varepsilon}$. To see this, write

$$\begin{aligned}
(A_\varepsilon u, v)_{L^2(\Omega)} &= (-\Delta_\varepsilon u^\varepsilon, v^\varepsilon)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} -\Delta_\varepsilon u^\varepsilon v^\varepsilon dx^\varepsilon \\
&= \sum_k \int_{\Omega_\varepsilon} \partial_{x_k^\varepsilon} u^\varepsilon \partial_{x_k^\varepsilon} v^\varepsilon dx^\varepsilon = \sum_{i,j,k} \int_\Omega \partial_{x_j} u \frac{\partial x_j}{\partial x_k^\varepsilon} \partial_{x_i} v \frac{\partial x_i}{\partial x_k^\varepsilon} J dx \\
&= \int_\Omega -\partial_{x_i} \Big(J \frac{\partial x_i}{\partial x_k^\varepsilon} \frac{\partial x_j}{\partial x_k^\varepsilon} \partial_{x_j} u \Big) v dx.
\end{aligned}$$

The main idea behind the computation is that since

$$\frac{\partial x_j^\varepsilon}{\partial x_i} = \delta_{ij} + \varepsilon \frac{\partial V_j}{\partial x_i}$$

up to first order errors in ε, we can write that

$$\frac{\partial x_j}{\partial x_k^\varepsilon} = \delta_{jk} - \varepsilon \frac{\partial V_j}{\partial x_k} + O(\varepsilon),$$

whereas J can be expressed as

$$J = 1 + \varepsilon \Big(\sum_j \frac{\partial V_j}{\partial x_j} \Big) + O(\varepsilon),$$

and

$$\beta_{jk} = \delta_{jk} - \varepsilon \Big(\frac{\partial V_j}{\partial x_k} + \frac{\partial V_k}{\partial x_j} \Big) + O(\varepsilon),$$

and β_{jk} has a power series expansion

$$\beta_{jk} = \delta_{jk} + \sum_{i=1}^{\infty} \varepsilon^i \beta_{jk}^i,$$

where $\beta_{jk}^1 = -(\frac{\partial V_j}{\partial x_k} + \frac{\partial V_k}{\partial x_j})$, as mentioned before. For details on the above, see [17, Section 6].

All told then, the perturbation A_ε can be expressed as

$$A_\varepsilon u = -\Delta u + \varepsilon\Big(\sum_{j,k} \partial_k((\partial_k V_j + \partial_j V_k)\partial_j u) + \frac{1}{2}\sum_j \partial_j u \Delta(V_j)\Big) + O(\varepsilon).$$

Now, we bring in the Sard–Smale transversality formalism used by Uhlenbeck. We first quote the following theorem.

Theorem 4.2. *Let Φ: $H \times B \to E$ be a C^k map, where H, B and E are Banach manifolds with H and E separable. If* 0 *is a regular value of Φ and $\Phi_b := \Phi(., b)$ is a Fredholm map of index $< k$, then the set $\{b \in B :$* 0 *is a regular value of $\Phi_b\}$ is residual in B.*

Here, we wish to check Theorem 4.2 for our domain perturbations in the particular setting that $H = E = \mathcal{D}(\Delta)$ and B is the collection of parameters for domain perturbation. For starters, all A_ε are self-adjoint by the Kato–Rellich theorem, being relatively bounded perturbations of $A_0 = \Delta_\Omega$. Also, ellipticity in such cases implies the Fredholm property, as is well known.

Now, if 0 is not a regular value as above, we have that for all perturbations given by V and ε small, we have Laplace eigenfunctions φ, ψ (ψ corresponding to the eigenvalue λ) such that

$$\begin{aligned} -\int_\Omega \sum_{jk}((\partial_k V_j + \partial_j V_k)\partial_j \varphi \partial_k \psi) + \frac{\varphi\psi}{2}\Delta\Big(\sum_j \partial_j V_j\Big) &= -\int_{\partial\Omega} \frac{\partial \varphi}{\partial \eta}\frac{\partial \psi}{\partial \eta}\Big(\sum_k V_k \eta_k\Big) \\ &= 0. \end{aligned}$$

This basically means that by Holmgren's uniqueness theorem, φ and ψ are identically zero, establishing our claim.

4.2. Fundamental gap, narrow convex domains and small perturbations

Now, we take a look at the problem of minimising the fundamental gap. In this regard, recall the main result from [4]: for any convex domain $\Omega \subset \mathbb{R}^n$,

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{D^2},$$

where $D = \operatorname{diam} \Omega$. Now, the following question is natural.

Question 4.3. Is the above inequality saturated by some domain?

The popular belief in the community seems that it is not, and any infimising sequence for $\lambda_2 - \lambda_1$ (under the normalisation $D = 1$) should degenerate to a line segment. In particular, the correct regime to look for in the search for minimisers is the class of narrow convex domains. This problem seems quite difficult, as standard precompactness ideas (e.g., see recent work in [44, 52]) do not apply directly. Also, it is quite resistant to perturbative techniques, as generic perturbations (even small ones) might destroy convexity. In addition, the problem seems quite sensitive to the class of domains: it might demonstrate a markedly different behaviour if the overall class of domains is changed, for example see [49].

Recall that $\mathfrak{P}$ denotes the class of strictly convex C^2-planar domains. Now, we begin proving Theorem 1.12. We finish the proof in two steps. First, we prove the following.

Theorem 4.4. *Let $\Omega \in \mathfrak{P}$ with diameter $D = 1$ and inner radius ρ which minimises the fundamental gap functional $\lambda_2 - \lambda_1$ in $\mathfrak{P}$. There exists a universal constant $C \ll 1$ such that if $\rho \leq C$, then $\lambda_2(\Omega)$ is not simple.*

Proof. Recall the Hadamard formula (see [29, Section 2.5.2]) which expresses the evolution of Laplace spectrum with respect to perturbation of a domain Ω by a vector field V:

$$\lambda_k'(0) = -\int_{\partial\Omega} \left(\frac{\partial \varphi}{\partial \eta}\right)^2 V \cdot \eta \, dS.$$

Suppose λ_i, λ_j are Dirichlet eigenvalues of Ω with corresponding eigenfunctions φ_i, φ_j respectively. If $\lambda_j - \lambda_i$ considered as a function of domains has a critical point at a domain Ω, then we must have that

$$0 = (\lambda_j - \lambda_i)'(0) = -\int_{\partial\Omega} \left(\left(\frac{\partial \varphi_j}{\partial \eta}\right)^2 - \left(\frac{\partial \varphi_i}{\partial \eta}\right)^2\right) V \cdot \eta dS,$$

for all perturbation vector fields V. Now, we specify to the special case $j = 2, i = 1$, and the above calculation with the Hadamard formula holds true under the assumption that λ_2 is simple.

Now, consider a small perturbation vector field V such that $V = 0$ at x_1, x_2 (see Figure 11 below) and $V \cdot \eta$ is non sign-changing away from x_1, x_2. Note that the aforementioned V constrains the diameter to be fixed along the perturbation. Then, this implies that

$$\left|\frac{\partial \varphi_1}{\partial \eta}\right| = \left|\frac{\partial \varphi_2}{\partial \eta}\right| \quad \text{on } \partial\Omega \text{ away from } \{x_1, x_2\}.$$

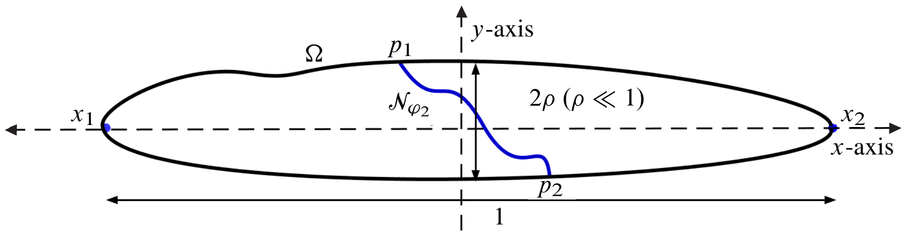


Figure 11. Small perturbation of a narrow convex domain.

Using the maximum principles and the Hopf Lemma, we find that without loss of generality, $\frac{\partial \varphi_1}{\partial \eta} > 0$ on $\partial\Omega$. Moreover, since convex domains or their small perturbations satisfy the strong Payne property (as proved in [3,53,55]), using [47, Lemma 1.2] there exist exactly two points $p_1, p_2 \in \partial\Omega$ such that

$$\varphi_2(p_1) = \varphi_2(p_2) = 0, \quad \text{and} \quad \frac{\partial \varphi_2}{\partial \eta}(p_1) = \frac{\partial \varphi_2}{\partial \eta}(p_2) = 0.$$

From work in [35–37] (and perturbation arguments based on [55]), it is known that p_1, p_2 cannot be near x_1, x_2 once the domain Ω is long and narrow enough (which is encoded in the statement by the universal constant C). This is a contradiction since

$$0 = \left|\frac{\partial \varphi_2}{\partial \eta}(p_i)\right| = \left|\frac{\partial \varphi_1}{\partial \eta}(p_i)\right| > 0. \qquad \blacksquare$$

Question 4.5. The following interesting question comes up in connection to the proof of the last theorem. On a domain Ω, can there be two Dirichlet eigenfunctions (corresponding to different eigenvalues) such that they also have the same Neumann data? One is tempted to speculate that such an event should not happen unless Ω is a ball. As pointed out by Antoine Henrot, the first and sixth eigenfunctions on the planar disc are both radially symmetric, so they can be scaled to have the same Neumann data. This is in turn related to a conjecture due to Schiffer.

We augment the above result by the following observation.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a C^2-domain. Assume that Ω has a multiple eigenvalue of the Dirichlet Laplacian*

$$\lambda_{k+1}(\Omega) = \lambda_{k+2}(\Omega) = \cdots = \lambda_{k+m}(\Omega).$$

Then, for each fixed $1 \le l \le m$, there exists a deformation field Ω_t passing through $\Omega_0 := \Omega$ generated by a C^2-vector field V such that for small enough t,

$$\lambda_{k+1}(\Omega_t) < \lambda_{k+1}(\Omega_0), \ldots, \lambda_{k+l}(\Omega_t) < \lambda_{k+l}(\Omega_0),$$

and

$$\lambda_{k+l+1}(\Omega_t) > \lambda_{k+l+1}(\Omega_0), \ldots, \lambda_{k+m}(\Omega_t) > \lambda_{k+m}(\Omega_0).$$

Furthermore, we can ensure that

$$|\Omega_0| = |\Omega_t|.$$

Suppose $\Omega_0 \in \mathfrak{P}$ *is a long narrow domain as in Theorem* 4.4 *above. Then, we can additionally ensure that* $\Omega_t \in \mathfrak{P}$ *and*

$$\operatorname{diam}(\Omega_0) = \operatorname{diam}(\Omega_t).$$

The proof is based on some ideas in [30, Lemma 1].

Proof. We begin by observing that because of the existence of multiple eigenvalues, $\lambda_{k+p}(\Omega_t)$ is not differentiable at $t = 0$ in the usual Frechet sense, but there is a nice formula giving directional derivatives, in the sense of the limit $\frac{\lambda_{k+p}(\Omega_t) - \lambda_{k+p}(\Omega_t)}{t}$ as $t \to 0$. Such directional derivatives are precisely the eigenvalues of the $m \times m$-matrix

$$\mathcal{M} = \left(- \int\limits_{\partial\Omega} \frac{\partial u_i}{\partial \eta} \frac{\partial u_j}{\partial \eta} V \cdot \eta \, d\sigma \right), \quad p = 1, 2, \ldots, m, \tag{4.1}$$

where u_i, $1 \leq i \leq m$ denotes the eigenspace for the repeated eigenvalue λ_{k+1}.

Let us consider points $A_1, A_2, \ldots, A_m \in \partial\Omega$ the choice of which will be explained below. Also, consider a deformation vector field V such that $V \cdot \eta = 1$ in a ε-neighbourhood of $A_1, A_2, \ldots, A_l$ and $V \cdot \eta = -1$ in a ε-neighbourhood of $A_{l+1}, \ldots, A_m$, $V.\eta = 0$ outside a 2ϵ-neighbourhood of $A_1, \ldots, A_m$, and regularised in a 2ε-neighbourhood around each such point maintaining $|\Omega_0| = |\Omega_t|$. To preserve the diameter also, one just needs to choose the points A_j sufficiently away from x_1, x_2 (see the figure above).

By (4.1) above, it suffices to prove that the symmetric matrix $\mathcal{M}$ has signature $(l, m-l)$. When $\varepsilon \to 0$, $\mathcal{M}$ converges to the matrix

$$M = \Big(- \sum_{k=1}^{l} \frac{\partial u_i}{\partial \eta}(A_k) \frac{\partial u_j}{\partial \eta}(A_k) + \sum_{k=l+1}^{m} \frac{\partial u_i}{\partial \eta}(A_k) \frac{\partial u_j}{\partial \eta}(A_k) \Big).$$

Consider the column vectors

$$v_{A_k} := \Big(\frac{\partial u_1}{\partial \eta}(A_k), \ldots, \frac{\partial u_m}{\partial \eta}(A_k) \Big)^T.$$

Note that $M = V \cdot W$, where

$$V = (v_{A_1}, \ldots, v_{A_m}) \quad \text{and} \quad W = (-v_{A_1}, \ldots, -v_{A_l}, v_{A_{l+1}}, \ldots, v_{A_m})^T.$$

It is enough to ensure that the vectors $\{v_{A_k} : k = 1, \dots, m\}$ are linearly independent. Then the signature of M is $(l, m - l)$.

If the columns in the matrix V are not independent, then they satisfy a homogeneous linear equation, which means in turn that there is a homogeneous linear relation among the rows of V, namely that, on an open set $S \subset \partial\Omega$ away from x_1, x_2, we have that

$$\sum_{p=1}^{m} c_p \frac{\partial u_p}{\partial \eta} = \frac{\partial}{\partial \eta}\Big(\sum_{p=1}^{m} c_p u_p\Big) = 0 \quad \text{on } S.$$

This is a contradiction from Hölmgren's uniqueness theorem. ∎

The following question seems interesting.

Question 4.7. Could we also ensure that $\lambda_{k+l}(\Omega_t) = \lambda_{k+l}(\Omega_0) < \lambda_{k+l+1}(\Omega_t)$ at the expense of changing the volume of Ω_t?

Proof of Theorem 1.12. Now, if we observe the way that the vector field was chosen in the proof of Theorem 4.6, it is clear that if one wants to fix only the diameter, one can choose such a V easily such that $\lambda_1(\Omega_t) > \lambda_1(\Omega_0)$ for small enough t. This reduces the gap even further, contradicting that Ω_0 is a minimiser. Finally, putting Theorems 4.4 and 4.6 together, we conclude the proof. ∎

Remark 4.8. Theorem 4.6 is not essential for the proof of Theorem 1.12, but it might be of independent interest to few. The topological restrictions on the first nodal set imposed by Jerison in [36] is enough to prove Theorem 1.12, and the idea of the proof is as follows: from Theorem 4.4, we know that the second eigenvalue of any minimising domain Ω cannot be simple. Let φ_1, φ_2 be any two linearly independent second eigenfunctions of Ω. From choosing any point $p \in \Omega$ sufficiently close to the "ends" x_1 or x_2, there exists a second eigenfunction $\varphi = c_1\varphi_1 + c_2\varphi_2$ (for some $c_1, c_2 \in \mathbb{R}$) such that $\varphi(p) = 0$. This leads to contradiction, since from [36] we know that the first nodal set stays away from the ends x_1, x_2. This in turn implies that Ω cannot be a minimiser, which concludes the proof of Theorem 1.12.

In line with Theorems 4.4 and 4.6, we make some general remarks about the multiplicity of eigenvalues. An important component in the proof of [53] is a result from [47] which states that if the Neumann data of a second Dirichlet eigenfunction is non sign-changing on the boundary of a convex planar domain Ω, then the second eigenvalue of Ω is simple. So, studying the multiplicity of eigenvalues might prove important for future work on the Payne conjecture. It is clear that on a simply-connected domain, satisfaction of the strong Payne property implies that the multiplicity of the second Dirichlet eigenvalue is at most two. Suppose there are three eigenfunctions $\varphi_j, j = 1, 2, 3$ corresponding to λ_2. Then picking any two points

$p, q \in \partial\Omega$, one can find an eigenfunction $\psi_{pq} = \sum_j \alpha_j \varphi_j$ such that ψ intersects $\partial\Omega$ exactly at p, q. Now, consider a sequence p_n, q_n approaching a common point $o \in \partial\Omega$. Then in the limit one gets an eigenfunction ψ which intersects $\partial\Omega$ at exactly o, and vanishes to order at least 2 there, which would be a contradiction to the strong Payne property.

It is natural to wonder when the Laplace–Beltrami operator of a compact manifold has repeated spectrum. Firstly, it is a well-known fact that *generic* spaces have simple spectrum. This is a well-known transversality phenomenon investigated in [2,66] (also Theorem 4.1 above). On the other hand, it is a well-known heuristic (by now folklore) that the presence of symmetries of the space M leads to repetitions in the spectrum. An explicit proof of this heuristic using a variant of the Peter–Weyl argument has been recorded in [65]. The main claim is that the presence of a non-commutative group G of isometries of the space will lead to infinitely many repeated eigenvalues of the Laplace–Beltrami operator.

With that in place, we look at the following multiplicity result.

Theorem 4.9. *Consider a bounded simply-connected domain $\Omega \subseteq \mathbb{R}^2$ and let $(0,0) \notin \Omega$ satisfying the following:*

- *the boundary $\partial\Omega$ contains exactly two distinct points P, Q dividing $\partial\Omega$ into two components Γ_j, $j = 1, 2$ such that the outward unit normal at P (respectively, Q) is in the direction of the vector joining $(0,0)$ to P (respectively, vector joining Q to $(0,0)$);*
- *at every point $(x, y) \in \Gamma_1$, the outward normal η makes an acute angle with $(-y, x)$ and at every point $(x, y) \in \Gamma_2$, the outward normal η makes an obtuse angle with $(-y, x)$.*

In such domains, the multiplicity of the second Dirichlet eigenvalue is at most 2.

Remark 4.10. Observe that the above theorem includes in particular domains which are convex in one direction, when the origin is taken arbitrarily far from the domain (point at infinity).

Proof. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. Let $(0,0) \notin \Omega$ be a point such that the boundary $\partial\Omega$ contains exactly two distinct points P, Q dividing $\partial\Omega$ into two components Γ_j, $j = 1, 2$ in a way that the unit outward normal at P is in the direction to the vector joining $(0,0)$ to P and the unit outward normal (blue arrows in Figure 12) at Q is opposite to the vector joining $(0,0)$ to Q (green arrows in Figure 12). In particular, for the points $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$, we have

$$\left\langle \eta_P, \frac{(P_1, P_2)}{|P|} \right\rangle = 1 \quad \text{and} \quad \left\langle \eta_Q, \frac{(Q_1, Q_2)}{|Q|} \right\rangle = -1.$$

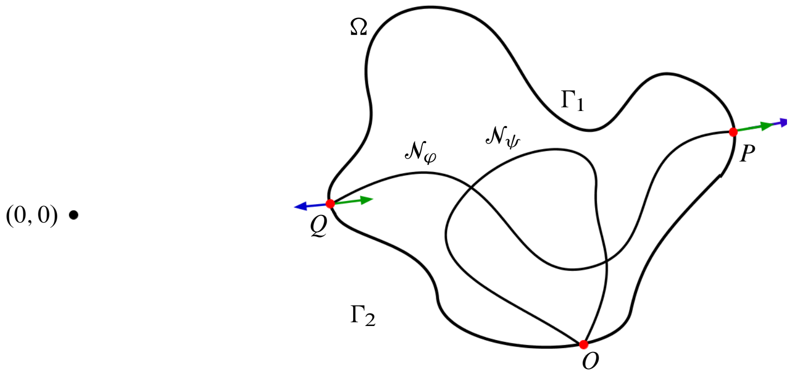


Figure 12. A non-convex domain satisfying the assumptions of Theorem 4.9.

Considering the rotational vector field

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

from our second assumption, we have that $\langle \eta_{(x,y)}, X_{(x,y)} \rangle > 0$ (respectively, < 0) when $(x, y) \in \Gamma_1$ (respectively, Γ_2), and $\langle \eta_P, X_P \rangle = \langle \eta_Q, X_Q \rangle = 0$.

Suppose to the contrary, that the multiplicity of λ_2 is at least 3. Up to forming linear combinations, let ψ be a second eigenfunction whose nodal set intersects $\partial\Omega$ at P, Q. Also, given $O \in \partial\Omega$, one can find a second eigenfunction φ whose nodal set intersects $\partial\Omega$ at O, as discussed above.

One can check that $[\Delta, X] = 0$, which gives us that

$$\begin{aligned} \int_\Omega \varphi\Delta(X\psi) - X\psi\Delta\varphi &= \int_\Omega \varphi X\Delta\psi - X\psi\Delta\varphi = \int_\Omega -\lambda_2\varphi X\psi - X\psi\Delta\varphi \\ &= \int_\Omega \Delta\varphi X\psi - \Delta\varphi X\psi = 0. \end{aligned}$$

Then we have

$$0 = \int_\Omega \varphi\Delta(X\psi) - X\psi\Delta\varphi = \int_{\partial\Omega} X\psi \frac{\partial\varphi}{\partial\eta} = \int_{\partial\Omega} \langle X, \eta\rangle \frac{\partial\psi}{\partial\eta}\frac{\partial\varphi}{\partial\eta}.$$

Since $\langle X, \eta\rangle \frac{\partial\psi}{\partial\eta}$ does not change sign on $\partial\Omega$, $\frac{\partial\varphi}{\partial\eta}$ must, which leads to a contradiction. ■

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