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Asymptotics for the spectral function on Zoll manifolds

Yaiza Canzani, Jeffrey Galkowski, and Blake Keeler

Abstract. On a smooth, compact, Riemannian manifold without boundary (M, g), let Δ_g be the Laplace–Beltrami operator. We define the orthogonal projection operator

$$\Pi_{I_\lambda} : L^2(M) \to \bigoplus_{\lambda_j \in I_\lambda} \ker(\Delta_g + \lambda_j^2)$$

for an interval I_λ centered around $\lambda \in \mathbb{R}$ of a small, fixed length. The Schwartz kernel, $\Pi_{I_\lambda}(x, y)$, of this operator plays a key role in the analysis of monochromatic random waves, a model for high energy eigenfunctions. It is expected that $\Pi_{I_\lambda}(x, y)$ has universal asymptotics as $\lambda \to \infty$ in a shrinking neighborhood of the diagonal in $M \times M$ (provided I_λ is chosen appropriately) and hence that certain statistics for monochromatic random waves have universal behavior. These asymptotics are well known for the torus and the round sphere, and were recently proved to hold near points in M with few geodesic loops by Canzani–Hanin. In this article, we prove that the same universal asymptotics hold in the opposite case of Zoll manifolds (manifolds all of whose geodesics are closed with a common period) under an assumption on the volume of loops with length incommensurable with the minimal common period.

1. Introduction

Let (M, g) be a compact, Riemannian manifold without boundary and let Δ_g be the associated, negative definite, Laplace–Beltrami operator. Denote the eigenvalues of $-\Delta_g$ by $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots$ repeated according to multiplicity. For $I \subset \mathbb{R}$, let

$$\mathcal{N}(I) := \#\{j : \lambda_j \in I\}.$$

The Weyl law states that

$$\mathcal{N}([0, \lambda]) = (2\pi)^{-n} \operatorname{vol}(\mathbb{B}_n) \operatorname{vol}_g(M)\lambda^n + R(\lambda),$$

where $R(\lambda) = \mathcal{O}(\lambda^{n-1})$ as $\lambda \to \infty$ [1, 20, 25, 37]. This remainder term is sharp and is saturated, for example, on the round sphere, $\mathbb{S}^n$. However, when the set of

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closed geodesics has measure zero in S^*M, the remainder, $R(\lambda)$, can be improved to $o(\lambda^{n-1})$ [16, 22]. The improved remainder also allows for asymptotics on short windows: for $\mathrm{w} > 0$,

$$\mathcal{N}([\lambda - \mathrm{w}, \lambda + \mathrm{w}]) = 2\mathrm{w}(2\pi)^{-n} \operatorname{vol}(\mathbb{S}^{n-1}) \operatorname{vol}_g(M)\lambda^{n-1} + o_{\mathrm{w}}(\lambda^{n-1}). \quad (1.1)$$

A *Zoll manifold* (M, g) is a smooth, compact, Riemannian manifold without boundary such that all of its geodesics are periodic with a common period. This is a rich class of manifolds that includes compact rank one symmetric spaces. Indeed, while the most well-known example of a Zoll manifold is the round sphere, $\mathbb{S}^2$, the moduli space of Zoll metrics on $\mathbb{S}^2$ is infinite-dimensional [19].

It is well known that, like on the sphere of radius $T/(2\pi)$, the eigenvalues of $-\Delta_g$ on a Zoll manifold with minimal common period T are strongly clustered near the sequence

$$\nu_\ell := \frac{2\pi}{T}\Big(\ell + \frac{\mathfrak{a}}{4}\Big), \quad \ell = 0, 1, 2, \ldots, \quad (1.2)$$

where $\mathfrak{a}$ is the common Maslov index of the closed geodesics [14–16, 35, 36]. The remainder estimate $R(\lambda) = O(\lambda^{n-1})$ is saturated on any Zoll manifold. As proved in [16], if the set of periodic trajectories with period $< T$ has zero measure, then a modified version of (1.1) which takes into account the clustering holds: for all $0 < \mathrm{w} < (2\pi)/T$,

$$\mathcal{N}([\nu_\ell - \mathrm{w}, \nu_\ell + \mathrm{w}]) = \frac{2\pi}{T}(2\pi)^{-n} \operatorname{vol}(\mathbb{S}^{n-1}) \operatorname{vol}_g(M)\nu_\ell^{n-1} + o_{\mathrm{w}}(\nu_\ell^{n-1}). \quad (1.3)$$

A Zoll manifold all of whose geodesics do not self-intersect before time T is called *simply closed,* or SC_T, in the language of [41]. In particular, (1.3) (and much stronger estimates) hold for an SC_T manifold. Many Zoll manifolds are SC_T manifolds. Indeed, all known smooth Zoll metrics on simply connected manifolds yield SC_T manifolds. However, as far as the authors are aware, the only topological manifold on which all Zoll metrics are known to be SC_T metrics is $\mathbb{S}^2$ [18].

The example of the disjoint union of two simply connected Zoll manifolds with different, rationally related minimal common periods shows that (1.3) cannot hold without additional assumptions. One would like to know whether all connected, smooth, compact, Zoll manifolds satisfy (1.3). However, we do not know whether such metrics must have a zero measure set of periodic geodesics of period $< T$. Nevertheless, an analog of (1.3) holds for *every* Zoll manifold provided that one is willing to sum over a finite number of small windows. In what follows, $\operatorname{inj}(M)$ denotes the injectivity radius of M.

Theorem 1. *Let (M, g) be a smooth, compact, Zoll manifold of dimension $n \geq 2$ with minimal common period $T > 0$. Then, there is an integer $0 < N_0 < T/\operatorname{inj}(M)$ such* *that for all* $N \geq N_0$ *and* $0 < \mathrm{w} < (2\pi)/T$,

$$\sum_{j=0}^{N-1} \mathcal{N}([\nu_{\ell+j} - \mathrm{w}, \nu_{\ell+j} + \mathrm{w}]) \\ = \frac{2\pi N}{T}(2\pi)^{-n} \operatorname{vol}(\mathbb{S}^{n-1}) \operatorname{vol}_g(M)\nu_\ell^{n-1} + o_{\mathrm{w}}(\nu_\ell^{n-1}), \quad \text{as } \ell \to \infty. \tag{1.4}$$

We note that N_0 can be taken to be the smallest integer such that the set of trajectories with period smaller than T/N_0 has zero Liouville measure on S^*M. Indeed, for any SC_T manifold, $N_0 = 1$, and this recovers (1.3).

Although it is not stated there, Theorem 1 can be derived from [29, Theorem 1] (see also [30, Theorem 1.7.6]). These references handle much more general geometries than the Zoll manifolds considered here. We choose to include Theorem 1 and state it as shown because it helps to motivate Conjecture 1.1 below. We give a complete proof of Theorem 1 in Section 7 because it is an easy consequence of the analysis leading to Theorem 2. In Theorem 2, we require uniform estimates on the derivatives of the spectral function in small neighborhoods of the diagonal. Because of this, the analysis leading to Theorem 2 necessarily differs from that used in the references above, where the authors consider on-diagonal estimates.

We also note that the estimate (1.4) captures the majority of the eigenvalues in the window $[\nu_\ell - \mathrm{w}, \nu_{\ell+N-1} + \mathrm{w}]$, since

$$\sum_{j=0}^{N-1} \mathcal{N}([\nu_{\ell+j} - \mathrm{w}, \nu_{\ell+j} + \mathrm{w}]) = \mathcal{N}([\nu_\ell - \mathrm{w}, \nu_{\ell+N-1} + \mathrm{w}]) + o(\nu_\ell^{n-1}). \tag{1.5}$$

In fact, substantially stronger estimates than (1.5) hold (see e.g. [15, 16]).

Next, we describe a refinement of Theorem 1 with applications to the theory of random waves. For this, we let $\{\varphi_j\}_{j=0}^\infty$ be an orthonormal basis of $L^2(M)$ such that

$$-\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad j = 0, 1, 2, \ldots, \tag{1.6}$$

and for $I \subset \mathbb{R}$ consider the orthogonal projection operator

$$\Pi_I : L^2(M) \to \bigoplus_{\lambda_j \in I} \ker(\Delta_g + \lambda_j^2).$$

The Schwartz kernel of Π_I takes the form

$$\Pi_I(x, y) = \sum_{\lambda_j \in I} \varphi_j(x)\overline{\varphi_j(y)}, \quad x, y \in M.$$

Since $\operatorname{trace} \Pi_I = \mathcal{N}(I)$, the operator Π_I plays a crucial role in studying both Weyl laws and monochromatic random waves.

We study the asymptotics as $\lambda \to \infty$ of spectral projectors of the form $\Pi_{I_\lambda}(x, y)$, where I_λ is an interval, centered at λ, with length uniformly bounded from above and below. These spectral projectors appear as the covariance kernels of monochromatic random waves (see (1.12)). The asymptotics of $\Pi_{I_\lambda}(x, y)$ are intimately connected to the dynamics of the geodesic flow on (M, g).

The most classical random wave studies occur on the round sphere, $\mathbb{S}^n$, and flat torus, $\mathbb{T}^n$. In the case of the sphere,

$$\lambda_\ell^2 = \ell(\ell + n - 1) \qquad \ell = 0, 1, \dots,$$

and it is known that, with $\nu_\ell := \ell + (n-1)/2$ and $0 < \mathrm{w} < 1$ for $x, y \in \mathbb{S}^n$, with $d_g(x, y) \le r_\ell$ and $\lim_{\ell\to\infty} r_\ell = 0$,

$$\begin{aligned}\Pi_{\{\lambda_\ell\}}(x, y) &= \Pi_{[\nu_\ell - \mathrm{w}, \nu_\ell + \mathrm{w}]}(x, y) \\ &= \frac{\nu_\ell^{n-1}}{(2\pi)^{n/2}} \frac{J_{(n-2)/2}(|\nu_\ell d_g(x, y)|)}{(\nu_\ell d_g(x, y))^{(n-2)/2}} + o(\nu_\ell^{n-1}), \quad \ell \to \infty. \end{aligned} \tag{1.7}$$

Here, we write $d_g(x, y)$ for the Riemannian distance between x and y and J_α for the Bessel function of the first kind with index α.

Despite the fact that the dynamics of the geodesic flow on the n-dimensional flat torus are dramatically different than those on the sphere, we also have for $\mathrm{w} > 0$, $x, y \in \mathbb{T}^n$ with $d_g(x, y) \le r_\nu$, and $\lim_{\nu\to\infty} r_\nu = 0$,

$$\Pi_{[\nu - \mathrm{w}, \nu + \mathrm{w}]}(x, y) = \frac{2\mathrm{w}\nu^{n-1}}{(2\pi)^{n/2}} \frac{J_{(n-2)/2}(|\nu d_g(x, y)|)}{(\nu d_g(x, y))^{(n-2)/2}} + o(\nu^{n-1}), \quad \nu \to \infty. \tag{1.8}$$

Indeed, one expects that the local behavior of Π_{I_λ} is, in some sense, universal.

Conjecture 1.1. *Let (M, g) be a smooth, compact, Riemannian manifold of dimension n without boundary and $x_0 \in M$. Then, there exist $C > 0$, a sequence $\nu_\ell \to \infty$, and a sequence $0 < \mathrm{w}_\ell < C$ such that for any positive sequence $r_\ell \to 0$, $\alpha, \beta \in \mathbb{N}^d$,*

$$\begin{aligned}\sup_{x, y \in B(x_0, r_\ell)} \Bigg| \nu_\ell^{-|\alpha|+|\beta|} \partial_x^\alpha \partial_y^\beta \Bigg(&\frac{\Pi_{[\nu_\ell - \mathrm{w}_\ell, \nu_\ell + \mathrm{w}_\ell]}(x, y)}{\mathcal{N}([\nu_\ell - \mathrm{w}_\ell, \nu_\ell + \mathrm{w}_\ell])} \\ &- \frac{(2\pi)^{n/2}}{\mathrm{vol}(\mathbb{S}^{n-1})} \frac{J_{(n-2)/2}(|\nu_\ell d_g(x, y)|)}{(\nu_\ell d_g(x, y))^{(n-2)/2}} \Bigg) \Bigg| = o(1)_{\ell\to\infty}. \end{aligned} \tag{1.9}$$

Observe that for any $0 < \mathrm{w} < 1$ on the round sphere, we have

$$\mathcal{N}([\nu_\ell - \mathrm{w}, \nu_\ell + \mathrm{w}]) = \frac{\mathrm{vol}(\mathbb{S}^{n-1})}{(2\pi)^n} \nu_\ell^{n-1} + o(\nu_\ell^{n-1}),$$

and on the torus we have

$$\mathcal{N}([\nu_\ell - \mathrm{w}, \nu_\ell + \mathrm{w}]) = 2\mathrm{w} \frac{\mathrm{vol}(\mathbb{S}^{n-1}) \nu_\ell^{n-1}}{(2\pi)^n} + o(\nu_\ell^{n-1}).$$

Hence, in both cases, (1.7) and (1.8) yield

$$\frac{\Pi_{[\nu_\ell-\mathrm{w},\nu_\ell+\mathrm{w}]}(x,y)}{\mathcal{N}([\nu_\ell-\mathrm{w},\nu_\ell+\mathrm{w}])}=\frac{(2\pi)^{n/2}}{\mathrm{vol}(\mathbb{S}^{n-1})}\frac{J_{(n-2)/2}(|\nu_\ell d_g(x,y)|)}{(\nu_\ell d_g(x,y))^{(n-2)/2}}+o(1),$$

and the conjecture holds in these examples.

In [10, 11], Canzani and Hanin showed that the asymptotics (1.9) hold whenever x is a non-self focal point. That is, the set of directions $\xi \in S^*_xM$ that generate a geodesic loop that returns to x has Liouville measure zero. As for the flat torus, in the case of non-self focal points, one can take any sequence $\nu_\ell \to \infty$ and $\mathrm{w}_\ell = 1$.

On a Zoll manifold every point is, in some sense, the opposite of non-self focal. Because of the sphere-like clustering of the spectrum, it is too much to hope that (1.9) holds for any choice of $\nu_\ell \to \infty$ and, as in the case of the Weyl law, we should instead work with spectral projectors for a well-chosen sequence ν_ℓ. In particular, we take ν_ℓ as in (1.2).

Our goal is to show that Conjecture 1.1 holds at certain points on Zoll manifolds. As discussed above, many Zoll manifolds of period T are SC_T manifolds. However, it is possible that some may have geodesics with shortest periods T/N for some $N > 1$ or closed geodesics of length T that are not simple (i.e., that pass over the same base point more than once). Indeed, the (albeit trivial) example of the disjoint union of two Zoll manifolds with rationally related periods shows that, at least in principle, there may be a large set of closed geodesics of period smaller than T. However, these must have period T/N for some fixed N.

In order to handle this type of situation, we formulate our next theorem in a way that allows for large sets of loops at times rationally related to T, as well as a zero-volume set of loops with non-rationally related looping time. For $N \in \mathbb{N}$, $T > 0$, $\varepsilon > 0$, define the sets

$$K_N^T := \Big\{\frac{p}{q}T : p,q \in \mathbb{N},\ 0 \le p \le N,\ 0 < q \le N\Big\},$$
$$K_{N,\tau}^T := ((-\tau,\tau) + K_N^T) \cup (-\infty,0) \cup (T,\infty).$$

Let (M,g) be a smooth Zoll manifold of dimension $n \ge 2$ with minimal common period $T > 0$ and $\varphi_t\colon S^*M \to S^*M$, $\varphi_t := \exp((t/2)H_{|\xi|_g^2})$ denote the geodesic flow for time t. Fix a metric on T^*M, and define

$$\mathcal{L}_{N,\tau}(x_0) := \{\rho \in S^*_{x_0}M : \varphi_t(\rho) \in S^*_{x_0}M \text{ for some } t \in (K_{N,\tau}^T)^{\mathrm{c}}\}.$$

Note that a direction $\rho \in S^*_{x_0}M$ is in $\mathcal{L}_{N,\tau}(x_0)$ if there is a time $t \in [0,T]$ that is at least τ-far from every element in K_N^T such that $\pi_M(\varphi_t(\rho)) = x_0$.

Our next result gives pointwise estimates on the spectral projector near any $x_0 \in M$ such that for each $\tau > 0$, $\mu_{S^*M}(\mathcal{L}_{N,\tau}(x_0)) = 0$.

Theorem 2. *Let (M,g) be a smooth, compact, Zoll manifold of dimension $n \geq 2$ with minimal common period $T > 0$ and ν_ℓ defined in (1.2). For $x, y \in M$ and $\mathrm{w} > 0$, define*

$$R_{N,\mathrm{w}}(\ell; x, y) := \frac{1}{N}\sum_{j=0}^{N-1} \Pi_{[\nu_{\ell+j}-\mathrm{w}, \nu_{\ell+j}+\mathrm{w}]}(x,y) - \frac{2\pi}{T}\frac{\nu_\ell^{n-1}}{(2\pi)^{n/2}}\frac{J_{(n-2)/2}(\nu_\ell d_g(x,y))}{(\nu_\ell d_g(x,y))^{(n-2)/2}}$$

Let $N > 0$ and $x_0 \in M$ such that for each $\tau > 0$

$$\mu_{S^*M}(\mathcal{L}_{N,\tau}(x_0)) = 0. \tag{1.10}$$

Then, for any $0 < \mathrm{w} < (2\pi)/T$ and $\alpha, \beta \in \mathbb{N}^n$,

$$\lim_{\delta\to 0^+} \limsup_{\ell\to\infty} \sup_{x,y\in B(x_0,\delta)} |\nu_\ell^{1-n-|\alpha|-|\beta|}\partial_x^\alpha \partial_y^\beta R_{N,\mathrm{w}}(\ell; x, y)| = 0.$$

Our next theorem gives two examples where the assumptions of Theorem 2 hold.

Theorem 3. *Let (M,g) be a smooth, compact, Zoll manifold of dimension $n \geq 2$.*

(1) *If (M,g) is an SC_T manifold, then for every $x_0 \in M$ (1.10) holds with $N = 1$.*
(2) *If (M,g) is a real analytic, then there is N such that for every $x_0 \in M$ (1.10) holds.*

For SC_T manifolds, there are no sub-periodic loops and hence it easy to see that (1.10) holds with $N = 1$. We recall that even though all known smooth Zoll metrics on simply connected manifolds yield SC_T manifolds, the only topological manifold on which all Zoll metrics are known to be SC_T is the 2-sphere.

We also note that, for SC_T manifolds, the on-diagonal version of Theorem 2, without derivatives, was also proved in [41, Theorem 2]. Part (2) of Theorem 3 is proved in Section 8.

As a corollary of Theorems 1 and 2, we obtain that Conjecture 1.1 holds whenever (1.10) holds on a Zoll manifold.

Corollary 1.2. *Let (M,g) be a smooth, compact, Zoll manifold of dimension $n \geq 2$ with minimal common period $T > 0$ and $x_0 \in M$, $N > 0$ satisfy $\mu_{S^*M}(\mathcal{L}_{N,\tau}(x_0)) = 0$ for each $\tau > 0$. Then Conjecture 1.1 holds at x_0.*

As discussed briefly before, a motivation for proving Theorem 2 is its application to the theory of random waves on manifolds. A *monochromatic random wave* on (M,g) is a Gaussian random field of the form

$$\psi_{\lambda,\mathrm{w}}(x) := (\mathcal{N}([\lambda-\mathrm{w}, \lambda+\mathrm{w}]))^{-1/2}\sum_{\lambda_j\in[\lambda-\mathrm{w},\lambda+\mathrm{w}]} a_j\varphi_j(x), \tag{1.11}$$

where the a_j are i.i.d. standard Gaussian random variables and the φ_j are the eigenfunctions in (1.6).

Monochromatic random waves were created to model eigenfunction behavior. Although $\psi_{\lambda,\mathrm{w}}$ is not an actual eigenfunction, it is expected to behave like one. (For a careful account of the history, see [8, 38] and references there.) In particular, much research has been dedicated to understanding the behavior of the zero sets and critical points of random waves. The corresponding features of deterministic eigenfunctions are very difficult to study, and their analysis becomes much more tractable for the monochromatic random counterparts.

The statistics of $\psi_{\lambda,\mathrm{w}}$ are completely determined by the associated two-point correlation function

$$K_{\lambda,\mathrm{w}}(x,y) := \mathrm{Cov}(\psi_{\lambda,\mathrm{w}}(x), \psi_{\lambda,\mathrm{w}}(y)) = \frac{\Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}(x,y)}{\mathcal{N}([\lambda-\mathrm{w},\lambda+\mathrm{w}])}, \quad x,y \in M. \tag{1.12}$$

Most research is typically done on the round sphere or the flat torus since $K_{\lambda,\mathrm{w}}$ is well understood for these spaces [2, 3, 5–7, 24, 26, 28]. Studying features like the zero sets and critical points of $\psi_{\lambda,\mathrm{w}}$ relies on having asymptotics for $K_{\lambda,\mathrm{w}}(x,y)$ when $x,y \in B(x_0, 1/\lambda)$ with x_0 fixed. Although treating $K_{\lambda,\mathrm{w}}$ on general manifolds is quite challenging, Conjecture 1.1 would imply that, when the eigenvalue intervals defining the sum in (1.11) are appropriately chosen,

$$\begin{aligned}\lim_{\ell\to\infty} \sup_{|u|,|v|\le r_\ell} \Big| \partial_u^\alpha \partial_v^\beta \Big(K_{\nu_\ell,\mathrm{w}}\Big(\exp_{x_0}\Big(\frac{u}{\nu_\ell}\Big), \exp_{x_0}\Big(\frac{v}{\nu_\ell}\Big)\Big) \\ - \frac{(2\pi)^{n/2}}{\mathrm{vol}(S^*M)} \frac{J_{(n-2)/2}(|u-v|)}{(|u-v|)^{(n-2)/2}}\Big)\Big| = 0. \end{aligned}\tag{1.13}$$

Here, $\exp_{x_0}\colon T^*_{x_0}M \to M$ denotes the exponential map with footpoint at x_0. Corollary 1.2 shows that for appropriately chosen intervals these asymptotics do, in fact, hold at points on a Zoll manifold where (1.10) is satisfied.

Results about Conjecture 1.1 yield corresponding asymptotics for the covariance function of monochromatic random waves. Indeed, for a general manifold (M,g), when the interval in (1.11) is $[\lambda - 1/2, \lambda + 1/2]$, the asymptotics from [10, 11] show that (1.13) holds when the point x_0 is non-self-focal. In the case where (M,g) has no conjugate points [23] (or more generally there are 'very' few loops, see [9]), the asymptotics in (1.13) hold at every point with a logarithmic improvement on the rate of decay to 0.

In the language of Nazarov and Sodin [27], if the asymptotics in (1.13) hold at every $x_0 \in M$, then the random waves $\psi_{\lambda,\mathrm{w}}$ have translation invariant local limits. For ensembles with such translation invariant local limits, Zelditch [42], Nazarov and Sodin [27], Sarnak and Wigman [31], Gayet and Welschinger [17], Canzani and

Sarnak [13], Canzani and Hanin [12], as well as others, prove detailed results on non-integral statistics of the nodal sets of random waves. Such nodal set statistics include the number of connected components, Betti numbers, and topological types.

1.1. Comments on the proof

Since our long term goal is to approach Conjecture 1.1, we aim to implement a method that uses only the dynamical information obtained from the fact that (M, g) is Zoll. In particular, to prove Theorem 2, we avoid using the fact that one can find Q, a pseudodifferential operator of order -1, such that $\sqrt{-\Delta_g + Q}$ has spectrum contained in $\cup_\ell\{\nu_\ell\}$ [15]. This extra structure was used in [41] to obtain a full asymptotic expansion of $\Pi_{[\nu_\ell-\mathrm{w},\nu_\ell+w]}(x, x)$ in the SC_T case.

Using standard Tauberian arguments, the analysis reduces to understanding the singularities of $e^{it\sqrt{-\Delta_g}}(x, y)$ for $t \in [-\sigma^{-1}, \sigma^{-1}]$ with $\sigma \to 0$ very slowly as $\lambda \to \infty$. These singularities are located at times t when there is a geodesic loop from x to y. We analyze these singularities in three steps. (1) We use the periodicity of the flow to study the singularities near kT, $k \in \mathbb{Z}$. (2) We show that zero measure sets of loops do not to contribute to the main asymptotics using methods similar to those in [11, 33, 34, 39, 40]. (3) By summing of the windows $[\nu_{\ell+j} - \mathrm{w}, \nu_{\ell+j} + \mathrm{w}]$, $j = 0, \dots, N - 1$, we are able to incorporate the function $\sin(\pi Nt/T)/\sin(\pi t/T)$ into the amplitude multiplying $e^{-it\sqrt{-\Delta_g}}$ (see (3.5)). Using this extra structure, we then show that even positive measure sets of loops at times jT/N do not contribute to the leading term of the asymptotics. Note that, when considering asymptotics for the counting function Theorem 1, only periodic trajectories need to be analyzed. Thus, since periodic trajectories must have minimal period T/N for some N, we need no extra assumption to obtain asymptotics for the counting function.

1.2. Organization of the paper

We begin in Section 2 by analyzing the implications of the assumption (1.10), namely that it allows us to construct a pair of microlocal cutoffs which localize near, and respectively away from, the measure zero set of geodesics which have looping times outside of K_N. Section 3 proceeds with an analysis of the asymptotic contributions of the smooth spectral projector microlocalized away from all subperiodic loops. This is complemented by Section 4 which studies the contributions near both types of subperiodic looping times: those which lie near K_N (which may have positive measure), and those which do not (and must therefore have zero measure by assumption). We take a brief detour in Section 5 to prove some estimates on the spectral projector restricted to the diagonal, which are necessary for estimating the difference between the smooth and rough projectors. These on-diagonal estimates do not depend on the subperiodic loops assumption and slightly generalize similar results in [16]. In Section 6, we assemble the pieces produced in the previous sections to complete the proof of Theorem 2. Theorem 1 is proved in Section 7.

2. Microlocalization near subperiodic loops

In this section, we discuss a technical construction that is essential for the asymptotic analysis in the subsequent parts of the proof of Theorem 2. Our assumption on $\mathcal{L}_{N,0,\tau}(x_0)$ allows for the existence of subperiodic loops with looping times which do not lie near K_N, as long as the set of such loops is sufficiently small in measure. It is therefore crucial to construct a pair of pseudodifferential cutoffs which localize near and away from these loops. This idea is analogous to the constructions done in [10, 34], although our procedure is somewhat different because we cannot rely solely on the upper semicontinuity of the reciprocal of the return-time function. Define

$$\mathcal{L}_{N,\varepsilon,\tau}(x_0) := \Big\{\rho \in S^*_{x_0}M : \bigcup_{t\in(K^T_{N,\tau})^c} \varphi_t(B_{S^*M}(\rho,\varepsilon)) \cap S^*_{B(x_0,\varepsilon)}M \neq \emptyset\Big\}.$$

and write

$$\mathcal{L}_{N,0,\tau}(x_0) := \mathcal{L}_{N,\tau}(x_0).$$

We start by showing that we can relate $\mu_{S^*_{x_0}M}(\mathcal{L}_{N,\varepsilon,\tau}(x_0))$ to $\mu_{S^*_{x_0}M}(\mathcal{L}_{N,0,\tau}(x_0))$ as $\varepsilon \to 0$.

Lemma 2.1. *For all $\tau > 0$, we have*

$$\mu_{S^*_{x_0}M}(\mathcal{L}_{N,0,\tau}(x_0)) = \lim_{\varepsilon\to 0+} \mu_{S^*_{x_0}M}(\mathcal{L}_{N,\varepsilon,\tau}).$$

Proof. Observe that $\mathcal{L}_{N,\varepsilon',\tau}(x_0) \subset \mathcal{L}_{N,\varepsilon,\tau}(x_0)$ for $0 \le \varepsilon' < \varepsilon$. Hence, we need only show that

$$\mu_{S^*_{x_0}M}(\mathcal{L}_{N,0,\tau}(x_0)) \geq \lim_{\varepsilon\to 0+} \mu_{S^*_{x_0}M}(\mathcal{L}_{N,\varepsilon,\tau}(x_0)).$$

To do this, observe that

$$\lim_{\varepsilon\to 0+} \mu_{S^*_{x_0}M}(\mathcal{L}_{N,\varepsilon,\tau}(x_0)) = \mu_{S^*_{x_0}M}\Big(\bigcap_{\varepsilon>0} \mathcal{L}_{N,\varepsilon,\tau}(x_0)\Big).$$

Suppose that $\rho \in \bigcap_{\varepsilon>0} \mathcal{L}_{N,\varepsilon,\tau}(x_0)$. Then, there are $\rho_n \in S^*_{x_0}M$ with $d(\rho,\rho_n) < 1/n$ and $t_n \in (K^T_{N,\tau})^c$ such that

$$d(\pi_M(\varphi_{t_n}(\rho_n)), x_0) < \frac{1}{n}.$$

Since $(K^T_{N,\tau})^c$ is compact, we may assume that $t_n \to t_\infty \in (K^T_{N,\tau})^c$ and hence

$$\varphi_{t_\infty}(\rho) = \lim_{n\to\infty} \varphi_{t_n}(\rho_n) = x_0.$$

In particular, $\rho \in \mathcal{L}_{N,0,\tau}(x_0)$ and hence $\bigcap_{\varepsilon>0} \mathcal{L}_{N,\varepsilon,\tau}(x_0) \subset \mathcal{L}_{N,0,\tau}(x_0)$. This finishes the proof of the lemma. ■

Next, we prove the following lemma, which shows that the assumption (1.10) implies a more concrete fact in local coordinates. Below, m denotes the Lebesgue measure on S^{n-1}.

Lemma 2.2. *Fix $T > 0$, let $K \Subset \mathbb{R}$ be a closed set, and define*

$$K_\tau := ((-\tau, \tau) + K) \cup (-\infty, 0) \cup (T, \infty).$$

Let

$$A_{\varepsilon,\tau}(x_0) := \Big\{\rho \in S^*_{x_0}M : \bigcup_{t\in K^c_\tau} \varphi_t(B_{S^*M}(\rho, \varepsilon)) \cap S^*_{B(x_0,\varepsilon)}M \neq \emptyset\Big\}.$$

Fix $x_0 \in M$, let γ be a diffeomorphism from a neighborhood of x_0 into $\mathbb{R}^n$, and set

$$\begin{aligned} L_{\varepsilon,\tau}(x_0) := \{\xi \in S^{n-1} : \exists x \in B(x_0, \varepsilon), t \in K^c_\tau,\\ \Pi_M(\varphi_t(\gamma^{-1}(x), (\partial\gamma(x))^t\xi)) \in B(x_0, \varepsilon)\}. \end{aligned} \tag{2.1}$$

Then,

$$\lim_{\varepsilon\to 0+} \mu_{S^*_{x_0}M}(A_{\varepsilon,\tau}(x_0)) = 0 \implies \lim_{\varepsilon\to 0+} m(L_{\varepsilon,\tau}(x_0)) = 0.$$

Proof. Let $\alpha > 0$ to be chosen small and consider $\{\xi_j\}_{j=1}^{N_\varepsilon}$ an $\alpha\varepsilon$-maximal separated set in S^{n-1}. Then there is $\mathfrak{D} > 0$, depending only on n, and $\{\mathcal{J}_\ell\}_{\ell=1}^{\mathfrak{D}}$ such that

$$S^{n-1} \subset \bigcup_{j=1}^{N_\varepsilon} B(\xi_j, \alpha\varepsilon), \quad \{1, \dots, N_\varepsilon\} = \bigcup_{\ell=1}^{\mathfrak{D}} \mathcal{J}_\ell,$$
$$B(\xi_j, 10\alpha\varepsilon) \cap B(\xi_k, 10\alpha\varepsilon) = \emptyset, \quad i \neq k,\ i, k \in \mathcal{J}_\ell.$$

First, we claim there exists $\alpha_0 > 0$ such that if $\alpha < \alpha_0$, then

$$B(\xi_j, \alpha\varepsilon) \cap L_{\alpha\varepsilon,\tau}(x_0) \neq \emptyset \implies B\Big(\iota(\xi_j), \frac{1}{2}\varepsilon\Big) \subset A_{\varepsilon,\tau}(x_0), \tag{2.2}$$

where $\iota\colon S^{n-1} \to S^*_{x_0}M$ is the map $\iota(\xi) := (x_0, (\partial\gamma(0))^t\xi)$.

Indeed, let $\eta \in L_{\alpha\varepsilon,\tau}(x_0)$ with $|\eta - \xi_j| < \alpha\varepsilon$. Then, there are $|x| < \alpha\varepsilon$ and $t \in K^c_\tau$ such that

$$\varphi_t(\gamma(y), (\partial\gamma(y))^t\eta) \in B(x_0, \alpha\varepsilon).$$

Now, $d(\gamma(y), x_0) \le C\alpha\varepsilon$, and $d(\iota(\eta), (\partial\gamma(y))^t\eta) < C\alpha\varepsilon$. Similarly, $d(\iota(\xi_j), \iota(\eta)) < C\alpha\varepsilon$, so that

$$(\gamma(y), (\partial\gamma(y))^t\eta) \in B(\iota(\xi_j), C\alpha\varepsilon).$$

Choosing $\alpha_0 < 1/(2C)$, then implies that for any $\rho \in B(\iota(\xi_j), 1/2\varepsilon) \cap S^*_{x_0}M$,

$$(\gamma(y), (\partial\gamma(y))^t\eta) \in B(\rho, \varepsilon)$$

which, in turn, implies that $B(\iota(\xi_j), \varepsilon/2) \subset A_{\varepsilon,\tau}(x_0)$. This proves the claim in (2.2).

Notice that there is a $C > 0$ such that for all ℓ and any $\rho \in S^*_{x_0}M$

$$|\{j \in \mathcal{J}_\ell : B(\iota(\xi_j), \alpha\varepsilon) \cap B(\rho, \varepsilon)\}| \le C\alpha^{1-n}. \tag{2.3}$$

Now, define

$$\mathcal{I} := \{j \in \{1, \dots, N_\varepsilon\} : B(\xi_j, \alpha\varepsilon) \cap L_{\alpha\varepsilon,\tau}(x_0) \neq \emptyset\},$$
$$\mathcal{I}_\ell := \{j \in \mathcal{J}_\ell : B(\xi_j, \alpha\varepsilon) \cap L_{\alpha\varepsilon,\tau}(x_0) \neq \emptyset\}.$$

Then, by (2.3),

$$\mu_{S^*_{x_0}M}(A_{\varepsilon,\tau}(x_0)) \ge \max_\ell \Big| \bigcup_{j \in \mathcal{I}_\ell} B\Big(\rho_j, \frac{1}{2}\varepsilon\Big) \Big| \ge c\alpha^{n-1} \max_\ell |\mathcal{I}_\ell| \varepsilon^{n-1}$$
$$\ge c\alpha^{n-1}\varepsilon^{n-1}|\mathcal{I}|/\mathfrak{D} \ge c\frac{m(L_{\alpha\varepsilon,\tau}(x_0))}{\mathfrak{D}}. \qquad \blacksquare$$

With Lemma 2.2 in hand, we construct the desired pseudodifferential cutoffs. Fix a point $x_0 \in M$ which satisfies (1.10) and choose a diffeomorphism γ from a neighborhood U of x_0 into $\mathbb{R}^n$. Then, by Lemma 2.1 and Lemma 2.2, the measure of the sets $L_{\varepsilon,\tau}(x_0)$ tends to 0 as $\varepsilon \to 0^+$ and then $\tau \to 0^+$. Thus, for any $\tau > 0$ and $r > 0$ there is $\varepsilon > 0$ and an open set $\mathcal{O}_{\varepsilon,\tau} \subset S^{n-1}$ such that $L_{\varepsilon,\tau}(x_0) \subseteq \mathcal{O}_{\varepsilon,\tau}$ and $m(\mathcal{O}_{\varepsilon,\tau}) < r$. Hence, we can find some $\tilde{b}_{\varepsilon,\tau} \in C^\infty(S^{n-1})$ that is identically 1 on $\mathcal{O}_{\varepsilon,\tau}$ and zero outside of a slightly larger open set $V_{\varepsilon,\tau}$ with $m(V_{\varepsilon,\tau}) < r$. Now, let $\chi \in C^\infty(M)$ be supported in the coordinate neighborhood U and equal to 1 on a slightly smaller neighborhood, and choose some $\beta \in C^\infty(\mathbb{R})$ which vanishes on a neighborhood of 0 and is equal to 1 outside $[-1/2, 1/2]$. Then, setting

$$b_{\varepsilon,\tau}(x, \xi) = \chi(x)\beta(|\xi|)\tilde{b}_{\varepsilon,\tau}\Big(\frac{\xi}{|\xi|}\Big), \quad c_{\varepsilon,\tau}(x, \xi) := 1 - b_{\varepsilon,\tau}(x, \xi),$$

we define the pseudodifferential operators $B_{\varepsilon,\tau}$ and C_ε:

$$B_{\varepsilon,\tau} f(x) = \frac{1}{2\pi} \int e^{i\langle\gamma(x)-\gamma(y),\xi\rangle} b_{\varepsilon,\tau}(x, \xi) f(y)\, dy\, d\xi,$$

$$C_{\varepsilon,\tau} f(x) = \frac{1}{2\pi} \int e^{i\langle\gamma(x)-\gamma(y),\xi\rangle} c_{\varepsilon,\tau}(x, \xi) f(y)\, dy\, d\xi.$$

Note that

$$B_{\varepsilon,\tau} + C_{\varepsilon,\tau} = I. \tag{2.4}$$

Observe that

$$\operatorname{supp} c_{\varepsilon,\tau} \cap \overline{L_{\varepsilon,\tau}(x_0)} = \emptyset, \tag{2.5}$$

$$\lim_{\varepsilon\to 0^+} \sup_{x\in M} \|1 - c_{\varepsilon,\tau}(x,\cdot)\|_{L^1(\mathbb{S}^{n-1})} = 0. \tag{2.6}$$

3. Analysis of the smoothed projector away from subperiodic loops

By the construction in the preceding section, for any fixed $\varepsilon > 0$, we have a microlocal partition of unity near x_0 in the form of $B_{\varepsilon,\tau}$ and $C_{\varepsilon,\tau}$. By (2.4),

$$\frac{1}{N}\sum_{j=0}^{N-1} \Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}\lambda+\mathrm{w}]}(x,y) = \frac{1}{N}\sum_{j=0}^{N-1} \Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}\lambda+\mathrm{w}]}(B^*_{\varepsilon,\tau} + C^*_{\varepsilon,\tau})(x,y).$$

Since $B^*_{\varepsilon,\tau}$ has small microsupport, we expect the contribution from this term to be negligible from the perspective of the asymptotics. We prove this rigorously in Section 4. The bulk of our analysis is dedicated to studying the $C^*_{\varepsilon,\tau}$ term. In fact, we will study a smoothed version of this object, which involves a convolution with a suitably chosen Schwartz-class function.

We introduce $\rho \in \mathcal{S}(\mathbb{R})$ with the property that $\hat{\rho}$ is supported in $[-2,2]$ and equal to one on $[-1,1]$. Then, for any $\sigma > 0$, let $\rho_\sigma(\mu) = (1/\sigma)\rho(\mu/\sigma)$, so that

$$\hat{\rho}_\sigma(t) = \hat{\rho}(\sigma t) \tag{3.1}$$

is supported in $[-2/\sigma, 2/\sigma]$ and equal to one on $[-1/\sigma, 1/\sigma]$. The goal of this section is to study the asymptotic behavior of

$$\frac{1}{N}\sum_{j=0}^{N-1} \rho_\sigma * \Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}+\mathrm{w}]} C^*_{\varepsilon,\tau}.$$

This is done in Proposition 3.4 below. In preparation for this result, in Section 3.1 we first rewrite $\rho_\sigma * \Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}$ in terms of the kernel of the half wave operator and its singularities. Later, in Section 3.2, we find the asymptotic behavior of the kernel when localized to each singularity. We finally state and prove Proposition 3.4 which combines these estimates to obtain asymptotics for the full projector.

3.1. Singularities of the half-wave operator

To study the smoothed projector, for any $\mathrm{w}, \sigma > 0$ we define

$$\psi_\sigma(\mu) := \rho_\sigma * \mathbb{1}_{[-\mathrm{w},\mathrm{w}]}(\mu),$$

which is Schwartz-class and has Fourier transform

$$\hat{\psi}_\sigma(t) = \hat{\rho}_\sigma(t)\frac{2\sin(t\mathrm{w})}{t}. \tag{3.2}$$

Then, if $U_t(x, y)$ denotes the kernel of the half-wave operator $U_t = e^{-it\sqrt{-\Delta_g}}$, we have

$$\rho_\sigma * \Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}C^*_{\varepsilon,\tau}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x, y)\,dt \tag{3.3}$$

for all $\varepsilon > 0$, by Fourier inversion. Note that on the left-hand side of (3.3), the convolution is taken with respect to the λ variable. From [16], we have that U_t is a Fourier integral operator of class $I^{-1/4}(\mathbb{R} \times M, M; \mathcal{C})$, where the canonical relation $\mathcal{C}$ is given by

$$\begin{aligned}\mathcal{C} = \{((t, \tau), (x, \xi), (y, \eta)) : (t, \tau) \in T^*\mathbb{R} \setminus \{0\},\\ (x, \xi), (y, \eta) \in T^*M \setminus \{0\},\ \tau + |\xi_g| = 0,\ (x, \xi) = \Phi^t(y, \eta)\},\end{aligned} \tag{3.4}$$

where $\Phi^t\colon T^*M \to T^*M$ denotes the geodesic flow. For any $\lambda > 0$,

$$\begin{aligned}&\sum_{j=0}^{N-1} \rho_\sigma * \Pi_{[\lambda+2\pi j/T-\mathrm{w},\lambda+2\pi j/T+\mathrm{w}]}C^*_{\varepsilon,\tau}(x, y)\\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=0}^{N-1} e^{it(\lambda+2\pi j/T)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x, y)\,dt\\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda+(N-1)\pi/T)}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x, y)\,dt,\end{aligned} \tag{3.5}$$

where the final equality follows from the Dirichlet kernel identity

$$\sum_{j=0}^{N-1} e^{ijx} = e^{i(N-1)x/2}\frac{\sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

Later, we will set $\lambda = \nu_\ell$, for ν_ℓ defined as in (1.2). We have

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda+(N-1)\pi/T)}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x, y)\,dt\\ &\quad= \mathcal{A}_{\varepsilon,\tau}(\lambda, \sigma; x, y) + \mathcal{B}_{\varepsilon,\tau}(\lambda, \sigma; x, y)\end{aligned}$$

for

$$\mathcal{A}_{\varepsilon,\tau}(\lambda,\sigma;x,y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda+\frac{(N-1)\pi}{T})}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x,y)\sum_{k\in\mathbb{Z}}\hat{\rho}(t-kT)\,dt, \quad (3.6)$$

$$\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda+\frac{(N-1)\pi}{T})}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x,y)\Big(1-\sum_{k\in\mathbb{Z}}\hat{\rho}(t-kT)\Big)\,dt. \quad (3.7)$$

We can think of $\mathcal{A}$ and $\mathcal{B}$ as being localized near to and away from times which are integer multiples of T, respectively. We first consider $\mathcal{A}_{\varepsilon,\tau}(\lambda,\sigma;x,y)$. Changing variables, $t\mapsto t+kT$,

$$\begin{aligned}\mathcal{A}_{\varepsilon,\tau}(\lambda,\sigma;x,y) &= \sum_{k\in\mathbb{Z}}\frac{e^{ikT(\lambda+(N-1)\pi/T)}}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda+(N-1)\pi/T)}(-1)^{(N-1)k} \\ &\qquad \times\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t+kT)\hat{\rho}(t)U_{t+kT}C^*_{\varepsilon,\tau}(x,y)\,dt \\ &= \sum_{k\in\mathbb{Z}}\frac{e^{ikT\lambda}}{2\pi}\mathcal{F}^{-1}_{t\mapsto\lambda}(\hat{f}_k(t)U_{t+kT}C^*_{\varepsilon,\tau}(x,y)),\end{aligned} \quad (3.8)$$

where we define

$$\hat{f}_k(t) = e^{it\frac{(N-1)\pi}{T}}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t+kT)\hat{\rho}(t), \quad (3.9)$$

and $\mathcal{F}^{-1}_{t\mapsto\lambda}$ is the inverse Fourier transform mapping t to λ. Then, we can use that $U_s\varphi_j = e^{-is\lambda_j}\varphi_j$ to obtain

$$\begin{aligned}\mathcal{F}^{-1}_{t\mapsto\lambda}(\hat{f}_k(t)U_{t+kT}C^*_{\varepsilon,\tau}(x,y)) &= \mathcal{F}^{-1}_{t\mapsto\lambda}\Big(\hat{f}_k(t)\sum_{j=0}^{\infty}e^{-i\lambda_j(t+kT)}\varphi_j(x)\overline{C_{\varepsilon,\tau}\varphi_j(y)}\Big) \\ &= f_k * \Big(\sum_{j=0}^{\infty}\delta(\lambda-\lambda_j)e^{-ikT\lambda_j}\varphi_j(x)\overline{C_{\varepsilon,\tau}\varphi_j(y)}\Big) \\ &= f_k * \partial_\lambda\Big(\sum_{\lambda_j\le\lambda}\varphi_j(x)\overline{C_{\varepsilon,\tau}U_{-kT}\varphi_j(y)}\Big) \\ &= \partial_\lambda(f_k * \Pi_{[0,\lambda]}U_{kT}C^*_{\varepsilon,\tau}(x,y)).\end{aligned}$$

Therefore, if $d(x, y) \leq \delta$, (3.8) yields

$$\mathcal{A}_{\varepsilon,\tau}(\lambda, \sigma; x, y) = \sum_{k \in \mathbb{Z}} e^{ikT\lambda} \partial_\lambda (f_k * \Pi_{[0,\lambda]} U_{kT} C^*_{\varepsilon,\tau}(x, y)).$$

By [16, p. 53], with $\mathfrak{a}$ as in (1.2) and

$$\mathfrak{b} := \frac{\pi \mathfrak{a}}{2T}, \tag{3.10}$$

we have that $U_t - e^{i\mathfrak{b}T} U_{t+T}$ is a Fourier integral operator of one order lower than U_t, namely $-1/4 - 1$. In particular, we have that $U_0 - e^{i\mathfrak{b}T} U_T$ is a pseudodifferential operator of order -1, and

$$U_0 - e^{ik\mathfrak{b}T} U_{kT} \in \Psi^{-1}(M),$$

for any $k \in \mathbb{Z}$. Since U_0 is the identity map, we can write

$$U_{kT} = e^{-ik\mathfrak{b}T}(I + Q_k)$$

for $Q_k \in \Psi^{-1}(M)$ with polyhomogeneous symbol. Thus, we obtain

$$\mathcal{A}_{\varepsilon,\tau}(\lambda, \sigma; x, y) = \sum_{k \in \mathbb{Z}} e^{ikT(\lambda - \mathfrak{b})} \partial_\lambda (f_k * \Pi_{[0,\lambda]}(I + Q_k) C^*_{\varepsilon,\tau})(x, y). \tag{3.11}$$

Therefore, we must determine the asymptotic behavior of

$$\partial_\lambda (f_k * \Pi_{[0,\lambda]}(I + Q_k) C^*_{\varepsilon,\tau}).$$

Remark 3.1. Note that for each fixed $\sigma, \delta > 0$, the $\hat{f}_k$ are identically 0 for sufficiently large k. Therefore, the sum in (3.11) is finite for each $\sigma, \delta > 0$.

3.2. Pseudodifferential perturbations of the spectral projector

The goal of this section is to find the asymptotic behavior of

$$\partial_\lambda (f_k * \Pi_{[0,\lambda]}(I + Q_k) C^*_{\varepsilon,\tau})(x, y)$$

for each k. We are interested in working with points $x, y \in M$ for which $d_g(x, y)$ is small. Therefore, we will assume that we work with coordinates $y = (y_1, \ldots, y_n)$ on M and dual coordinates $(\xi_1, \ldots, \xi_n)$ on $T^*_y M$. The Riemannian volume form in these coordinates takes the form $\sqrt{|g_y|}dy$, where $|g_y|$ denotes the determinant of the matrix representation of $g(y)$. We also define the function

$$\Theta(x, y) := |\det_g D_{\exp_x^{-1}(y)} \exp_x|,$$

where the subscript g means that we use the metric to choose an orthonormal basis on $T_{\exp_x^{-1}(y)}(T_xM)$ and T_y^*M (cf. [4, Chapter 2, Proposition C.III.2]). The determinant is then independent of the choice of such a basis. We note that $\Theta(x,y) = \sqrt{|g_x|}$ in normal coordinates centered at y.

If $\xi \in T_y^*M$ is represented as $\xi = r\omega$ with $(r,\omega) \in (0,+\infty) \times S_y^*M$, then we endow S_y^*M with the measure $d\omega$ such that $d\xi = r^{n-1}\, d\omega\, dr$.

Remark 3.2. We note that $d\omega$ is not a coordinate invariant measure, but it behaves like a density in y under changes of coordinates. Thus, $d\omega$ should be regarded as a measure taking values in the space of densities on M. Despite this, we note that for $v \in \mathbb{R}^n$

$$\frac{1}{(2\pi)^{n/2}} \frac{J_{(n-2)/2}(|v|)}{|v|^{(n-2)/2}} = \frac{1}{(2\pi)^n} \int_{S^{n-1}} e^{i\langle v,\omega\rangle}\, d\sigma_{\mathbb{S}^{n-1}}(\omega).$$

Hence,

$$\frac{1}{(2\pi)^n} \int_{S_y^*M} e^{i\lambda\langle \exp_y^{-1}(x),\omega\rangle_g} \frac{d\omega}{\sqrt{|g_y|}} = \frac{1}{(2\pi)^{n/2}} \frac{J_{(n-2)/2}(|\lambda d_g(x,y)|)}{(\lambda d_g(x,y))^{(n-2)/2}},$$

and the right-hand side is clearly coordinate invariant. Here, we used that

$$d\omega = |g_y|^{1/2}\, d\sigma_{S^{n-1}}$$

and that in local coordinates

$$\langle \exp_y^{-1}(x), \omega\rangle_g = \langle g_y^{-1/2} \exp_y^{-1}(x), g_y^{-1/2}\omega\rangle_{\mathbb{R}^n}$$

with $g_y^{-1/2}\omega \in \mathbb{S}^{n-1}$ and $|g_y^{-1/2} \exp_y^{-1}(x)|_{\mathbb{R}^n} = d_g(x,y)$.

Proposition 3.3. *Let (M,g) be a compact, smooth Riemannian manifold of dimension $n \geq 2$ without boundary. Let C and Q be pseudodifferential operators with polyhomogeneous symbols c and q of orders 0 and -1, respectively. Fix $\delta \leq \mathrm{inj}(M,g)/2$. Then, for each pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$, there exist constants $C_1, C_2, \mu_0 > 0$, such that for any function $f \in C^\infty(\mathbb{R})$ with $\hat{f}$ smooth and compactly supported, and any $x, y \in M$ with $d_g(x,y) \leq \delta$ we have*

$$\begin{aligned}&\Theta^{1/2}(x,y)\partial_\mu(f * \Pi_{[0,\mu]}(I+Q)C)(x,y)\\ &\quad = \frac{\mu^{n-1}\hat{f}(0)}{(2\pi)^n} \int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} c(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} + R(\mu,x,y),\end{aligned}$$

with

$$\sup_{d_g(x,y)\le\delta} |\partial_x^\alpha \partial_y^\beta R(\mu,x,y)| \le C_1\delta\|\partial_t \hat{f}\|_{L^\infty([-\delta,\delta])}\mu^{n-1+|\alpha|+|\beta|} + C_2\mu^{n-2+|\alpha|+|\beta|} \tag{3.12}$$

for all $\mu \ge \mu_0$. Here, C_1 is independent of δ, Q and f.

Proof. We prove the statement first in the case where $\alpha = \beta = 0$. Observe that

$$\partial_\mu(f * \Pi_{[0,\mu]}(I+Q)C)(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\mu}\hat{f}(t)U_t(I+Q)C(x,y)\,dt. \tag{3.13}$$

Using the parametrix for U_t constructed in [10, Proposition 8], we have that if

$$d_g(x,y) \le \frac{1}{2}\operatorname{inj}(M,g),$$

then

$$U_t(x,y) = \frac{\Theta^{-1/2}(x,y)}{(2\pi)^n}\int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - it|\xi|_{g_y}} A(t,y,\xi)\frac{d\xi}{\sqrt{|g_y|}} \tag{3.14}$$

modulo smoothing kernels, for some symbol $A \in S^0$ with a polyhomogeneous expansion

$$A \sim \sum_{j=0}^{\infty} A_{-j}.$$

In particular, $A_0(t,y,\xi) \equiv 1$ for all t, and when $t = 0$, $A_{-j}(0,y,\xi) = 0$ for all $j \ge 1$. Since C and Q are pseudodifferential, we can use the same parametrix construction to write

$$U_t(I+Q)C(x,y) = \frac{\Theta^{-1/2}(x,y)}{(2\pi)^n}\int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - it|\xi|_{g_y}} D(t,y,\xi)\frac{d\xi}{\sqrt{|g_y|}} \tag{3.15}$$

for some $D \in S^0$. Note that since the principal symbol of U_t is identically 1 and C, Q are pseudodifferential, the principal symbols of U_tC and U_tQC are each independent of t. At $t = 0$, we have $U_0C = C$ and $U_0QC = QC$, and hence the principal symbol of U_tC is $c_0(y,\xi)$ for all t. Furthermore, since the subprincipal symbol of C is identically zero and all lower order terms of A vanish at $t = 0$, we have that the symbol of $U_t(I+Q)C$ satisfies

$$D(t,y,\xi) - c_0(y,\xi) - D_{-1}(t,y,\xi) \in S^{-2},$$

where $D_{-1} \in S^{-1}$ is homogeneous degree -1. From (3.13) and (3.15), we obtain

$$\begin{aligned}\partial_\mu(f * \Pi_\mu(I+Q)C)(x,y) \\ = \frac{\Theta^{-1/2}(x,y)}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{T_y^*M} e^{it\mu} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - it|\xi|_{g_y}} \hat{f}(t) D(t,y,\xi) \frac{d\xi dt}{\sqrt{|g_y|}} \\ + \mathcal{O}(\mu^{-\infty}).\end{aligned} \tag{3.16}$$

To control the integral on the right-hand side above, we change variables via $\xi \mapsto \mu r \omega$ for $(r,\omega) \in \mathbb{R}^+ \times S_y^*M$, which yields that the left-hand side of (3.16) is

$$\begin{aligned}\frac{\mu^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_0^{\infty} \hat{f}(t) e^{i\mu t(1-r)} r^{n-1} \\ \times \left(\int_{S_y^*M} e^{i\mu r\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D(t,y,\mu r\omega) \frac{d\omega}{\sqrt{|g_y|}} \right) dr\, dt.\end{aligned} \tag{3.17}$$

Noting that since the phase is nonstationary for $r \neq 1$ we may introduce a cutoff function $\zeta \in C_c^\infty(\mathbb{R})$ which is equal to one on a neighborhood of $r = 1$, and supported in $[1/2, 3/2]$. This results in an error which is $\mathcal{O}(\mu^{-\infty})$ as $\mu \to \infty$.

Let $S(t,y,\xi) = c_0(y,\xi) + D_{-1}(t,y,\xi)$ be the first two terms in the polyhomogeneous expansion of D. Since $D - S$ is a symbol of order -2, we have

$$|D(t,y,\mu r\omega) - S(t,y,\mu r\omega)| \le C\mu^{-2}$$

uniformly for all t, y. Combining this fact with an application of stationary phase in (t,r), we see that the left-hand side of (3.16) is equal to

$$\begin{aligned}\frac{\mu^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(t) e^{i\mu t(1-r)} r^{n-1} \zeta(r) \\ \times \left(\int_{S_y^*M} e^{i\mu r\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} S(t,y,\mu r\omega) \frac{d\omega}{\sqrt{|g_y|}} \right) dr\, dt \\ + \mathcal{O}(\mu^{n-3}),\end{aligned}$$

where $\zeta \in C_c^\infty(\mathbb{R})$ is a cut-off function that is equal to 1 near $r = 1$ and vanishes for $r \notin [1/2, 3/2]$. Notice that by homogeneity in the fiber variable, we have that for any $(y,\eta) \in T^*M$,

$$\begin{aligned}\int_{S_y^*M} e^{i\langle \eta,\omega\rangle_{g_y}} S(t,y,\mu r\omega) \frac{d\omega}{\sqrt{|g_y|}} \\ = \int_{S_y^*M} e^{i\langle \eta,\omega\rangle_{g_y}} \left(c_0(y,\omega) + \frac{1}{\mu r} D_{-1}(t,y,\omega)\right) \frac{d\omega}{\sqrt{|g_y|}}.\end{aligned}$$

Then, following the proof of [32, Theorem 1.2.1], there exist smooth functions $a_\pm \in C^\infty(T^*M)$ and $b_\pm \in C^\infty(\mathbb{R} \times T^*M)$ such that

$$\int_{S_y^*M} e^{i\langle \eta,\omega\rangle_{g_y}} c_0(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} = \sum_\pm e^{\pm i|\eta|_{g_y}} a_\pm(y,\eta), \tag{3.18}$$

and

$$\int_{S_y^*M} e^{i\langle \eta,\omega\rangle_{g_y}} D_{-1}(t,y,\omega) \frac{d\omega}{\sqrt{|g_y|}} = \sum_\pm e^{\pm i|\eta|_{g_y}} b_\pm(t,y,\omega), \tag{3.19}$$

satisfying the estimates

$$|\partial_\eta^\gamma a_\pm(y,\eta)| \le C_\gamma (1+|\eta|_{g_y})^{-(n-1)/2-|\gamma|}, \tag{3.20a}$$

$$|\partial_t^k \partial_\eta^\gamma b_\pm(t,y,\eta)| \le C_{\gamma,k} (1+|\eta|_{g_y})^{-(n-1)/2-|\gamma|} \tag{3.20b}$$

for any multi-index γ, any integer $k \ge 0$, and some constants C_γ, $C_{\gamma,k}$ which are independent of t, y, and η. Therefore, by (3.13), (3.14), (3.15), (3.18), and (3.19),

$$\partial_\mu (f * \Pi_{[0,\lambda]}(I+Q)C)(x,y) = \frac{\mu^n}{(2\pi)^{n+1}} \sum_\pm \int_{\mathbb{R}} \int_0^\infty e^{i\mu\psi_\pm(t,r,x,y)} g_\pm(t,r,x,y,\mu)\, dr\, dt,$$

where

$$\psi_\pm(t,r,x,y) = t(1-r) \pm r d_g(x,y)$$

and

$$g_\pm(t,r,x,y,\mu) = r^{n-1}\zeta(r)\hat{f}(t)\Big(a_\pm(y,\mu r \exp_y^{-1}(x)) + \frac{1}{\mu r} b_\pm(t,y,\mu r \exp_y^{-1}(x))\Big). \tag{3.21}$$

Observe that for any fixed $x, y \in M$, the critical points of $\psi_\pm$ occur at $(t_c^\pm, r_c^\pm) = (\pm d_g(x,y), 1)$, and that

$$\det(\operatorname{Hess} \psi_\pm(t_c^\pm, r_c^\pm, x, y)) = 1.$$

Therefore, by the method of stationary phase, we see that

$$\partial_\mu (f * \Pi_{[0,\lambda]}(I+Q)C)(x,y) = \frac{\mu^{n-1}}{(2\pi)^n} \sum_\pm e^{\pm i\mu d_g(x,y)} \Big(g_\pm(t_c^\pm, r_c^\pm, x, y, \mu) - \frac{i}{\mu} \partial_r \partial_t g_\pm(t_c^\pm, r_c^\pm, x, y, \mu)\Big) + \mathcal{O}(\mu^{n-3}).$$

From (3.21) and (3.20), we have that

$$
\begin{aligned}
&|\partial_r \partial_t g_\pm(t_c^\pm, r_c^\pm, x, y, \mu)| \\
&\quad \le C_1 |\partial_t \hat{f}(\pm d_g(x,y))| + \frac{C_2}{\mu}(|\hat{f}(\pm d_g(x,y))| + |\partial_t \hat{f}(\pm d_g(x,y))|) \\
&\quad \le C_1 \|\partial_t \hat{f}\|_{L^\infty([-\delta,\delta])} + \frac{C_2}{\mu}\|\hat{f}\|_{C^1([-\delta,\delta])},
\end{aligned}
$$

and we remark that C_1 is independent of Q due to the definition of $a_\pm$. Therefore,

$$
\begin{aligned}
&\Theta^{1/2}(x,y)\partial_\mu(f * \Pi_{[0,\mu]}(I+Q)C)(x,y) \\
&\quad = \frac{\mu^{n-1}}{(2\pi)^n} \sum_\pm e^{\pm i\mu d_g(x,y)} \hat{f}(\pm d_g(x,y)) \\
&\qquad\qquad\qquad \times \Big(a_\pm(y, \mu \exp_y^{-1}(x)) + \frac{1}{\mu} b_\pm(t_c^\pm, y, \mu \exp_y^{-1}(x)) \Big) \\
&\qquad + R_1(\mu, x, y),
\end{aligned}
$$

where

$$
\sup_{d_g(x,y)\le\delta} |R_1(\mu,x,y)| \le C_1 \|\hat{f}\|_{\dot{C}^1([-\delta,\delta])} \mu^{n-2} + C_2 \|\hat{f}\|_{C^1([-\delta,\delta])} \mu^{n-3} + \mathcal{O}(\mu^{n-3}),
$$

with C_1 independent of Q. Next, let us Taylor expand $\hat{f}$ near 0, which yields

$$
\hat{f}(\pm d_g(x,y)) = \hat{f}(0) \pm d_g(x,y) \partial_t \hat{f}(s_\pm)
$$

for some $s_\pm$ between 0 and $\pm d_g(x,y)$. Combining this with the fact that

$$
\begin{aligned}
&\sum_\pm e^{\pm i\mu d_g(x,y)} a_\pm(y, \mu \exp_y^{-1}(x)) \\
&\qquad = \int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x), \omega\rangle} c_0(y,\omega) \frac{d\omega}{\sqrt{|g_y|}},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\Theta^{1/2}(x,y)\partial_\mu(\hat{f} * \Pi_{[0,\mu]}(I+Q)C)(x,y) \\
&\quad = \frac{\mu^{n-1}\hat{f}(0)}{(2\pi)^n} \Big(\int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} c_0(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} \\
&\qquad\qquad + \sum_\pm e^{\pm i\mu d_g(x,y)} b_\pm(t_c^\pm, y, \mu \exp_y^{-1}(x)) \Big) \\
&\qquad + R_1(\mu,x,y) + R_2(\mu,x,y),
\end{aligned} \tag{3.22}
$$

where R_1 is as above, and R_2 satisfies

$$\sup_{d_g(x,y)\le\delta} |R_2(\mu,x,y)| \le \delta \|\partial_t \hat{f}\|_{L^\infty([-\delta,\delta])}(C_0\mu^{n-1} + C_1\mu^{n-2})$$

for some $C_0 > 0$ which is independent of Q and $C_1 > 0$. Next, we Taylor expand

$$b_\pm(t_c^\pm, y, \mu \exp_y^{-1}(x)) = b_\pm(0, y, \mu \exp_y^{-1}(x)) \pm d_g(x,y)\partial_t b_\pm(s'_\pm, y, \mu \exp_y^{-1}(x))$$

for some $s'_\pm$ between 0 and $t_c^\pm = \pm d_g(x,y)$. Recalling (3.20), we have that

$$|\partial_t b_\pm(s_\pm, y, \mu \exp_y^{-1}(x))| \le C_2(1 + \mu d_g(x,y))^{-(n-1)/2},$$

since $|s_\pm| \le d_g(x,y)$. Therefore, we obtain

$$\begin{aligned}
&\frac{\mu^{n-2}\hat{f}(0)}{(2\pi)^m}\sum_{\pm} e^{\pm i\mu d_g(x,y)} b_\pm(t_c^\pm, y, \mu\exp_y^{-1}(x)) \\
&\quad = \frac{\mu^{n-2}\hat{f}(0)}{(2\pi)^n}\int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle} D_{-1}(0,y,\omega)\frac{d\omega}{\sqrt{|g_y|}} + R_3(\mu,x,y), \qquad (3.23)
\end{aligned}$$

where

$$\sup_{d_g(x,y)\le\delta} |R_3(\mu,x,y)| \le C_2\delta\hat{f}(0)\mu^{n-2},$$

after potentially increasing C_2. Therefore, we have that (3.22) and (3.23) yield

$$\begin{aligned}
&\Theta^{1/2}(x,y)\partial_\mu(\hat{f} * \Pi_{[0,\mu]}(I+Q)C)(x,y) \\
&\quad = \frac{\mu^{n-1}\hat{f}(0)}{(2\pi)^n}\int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} c_0(y,\omega)\frac{d\omega}{\sqrt{|g_y|}} + \widetilde{R}(\mu,x,y),
\end{aligned}$$

where $\widetilde{R}$ satisfies

$$\begin{aligned}
\sup_{d_g(x,y)\le\delta} |\widetilde{R}(\mu,x,y)| &\le C_1\delta\|\hat{f}\|_{\dot{C}^1([-\delta,\delta])}\mu^{n-1} + C_2\|\hat{f}\|_{\dot{C}^1([-\delta,\delta])}\mu^{n-2} \\
&\quad + C_3\delta\hat{f}(0)\mu^{n-2} + C_4\|\hat{f}\|_{C^1([-\delta,\delta])}\mu^{n-3} + \mathcal{O}(\mu^{n-3}),
\end{aligned}$$

for some $C_1, C_2, C_3, C_4 > 0$, with C_1 independent of δ, f, and Q. This completes the proof in the case where $\alpha = \beta = 0$.

To include derivatives in x, y, we observe that

$$\partial_x^\alpha \partial_y^\beta e^{i\langle \exp_y^{-1}(x),\xi\rangle} = \mathcal{O}(|\xi|^{|\alpha|+|\beta|})$$

as $|\xi| \to \infty$. Therefore, we can repeat the preceding argument where the orders of the symbols involved are increased by at most $|\alpha| + |\beta|$ to obtain the desired result. ∎

3.3. Asymptotics for $\mathcal{A}_{\varepsilon,\tau}(\lambda, \sigma; x, y)$

With Proposition 3.3 in hand, we are equipped to prove the main result of this section, namely the asymptotic behavior of $\mathcal{A}_{\varepsilon,\tau}(\lambda, \sigma; x, y)$, which accounts for the contributions near each multiple of the period T. In particular, we set $\lambda = \nu_\ell$ for $\ell = 0, 1, 2, \ldots$. Then, we can define

$$R_{\varepsilon,\tau}(\ell, \sigma; x, y) := \mathcal{A}_{\varepsilon,\tau}(\nu_\ell, \sigma; x, y) - \frac{2\pi N}{T} \cdot \frac{\nu_\ell^{n-1}}{(2\pi)^n} \int_{S_y^*M} e^{i\nu_\ell \langle \exp_y^{-1}(x), \omega\rangle_g} \frac{d\omega}{\sqrt{|g_y|}}. \tag{3.24}$$

Proposition 3.4. *Let (M, g) be a smooth Zoll manifold with minimal common period $T > 0$. Fix $0 < \mathrm{w} < (2\pi)/T$. Let $\mathcal{A}_{\mathrm{w},\varepsilon}$ as in (3.6) with C_ε satisfying (2.6). Then, for any multi-indices $\alpha, \beta \in \mathbb{N}^n$,*

$$\lim_{\sigma\to 0^+} \lim_{\tau\to 0} \lim_{\varepsilon\to 0^+} \lim_{\delta\to 0^+} \limsup_{\ell\to\infty} \sup_{d_g(x,y)\le\delta} \left| \frac{1}{\nu_\ell^{n-1+|\alpha|+|\beta|}} \partial_x^\alpha \partial_y^\beta R_{\varepsilon,\tau}(\ell, \sigma; x, y) \right| = 0.$$

Proof. Fix two multi-indices $\alpha, \beta \in \mathbb{N}^n$. First, note that for $\mathfrak{b}$ as in (3.10) we have that for all $k \in \mathbb{Z}$

$$e^{ikT(\mathfrak{b}-\nu_\ell)} = e^{ikT(-2\pi\ell/T)} = e^{-2\pi i k\ell} = 1.$$

Combine (3.11) with Proposition 3.3 to obtain

$$\mathcal{A}_{\varepsilon,\tau}(\lambda, \sigma; x, y) = \nu_\ell^{n-1} A_{\varepsilon,\tau}(\ell, x, y) \sum_{k\in\mathbb{Z}} \hat{f}_k(0) + \nu_\ell^{n-2} \sum_{k\in\mathbb{Z}} \hat{f}_k(0) W_{k,\varepsilon,\tau}(\ell, x, y)$$
$$+ \sum_{k\in\mathbb{Z}} R_{k,\varepsilon,\tau}(\ell, x, y), \tag{3.25}$$

where

$$A_{\varepsilon,\tau}(\ell, x, y) = \frac{1}{(2\pi)^n \Theta^{1/2}(x, y)} \int_{S_y^*M} e^{i\nu_\ell \langle \exp_y^{-1}(x), \omega\rangle_g} c_{\varepsilon,\tau}^0(y, \omega) \frac{d\omega}{\sqrt{|g_y|}},$$

$$W_{k,\varepsilon,\tau}(\ell, x, y) = \frac{1}{(2\pi)^n \Theta^{1/2}(x, y)} \int_{S_y^*M} e^{i\nu_\ell \langle \exp_y^{-1}(x), \omega\rangle_g} c_{\varepsilon,\tau}^0(y, \omega) \sigma(Q_k)(y, \omega) \frac{d\omega}{\sqrt{|g_y|}},$$

and $R_{k,\varepsilon,\tau}$ satisfies

$$\sup_{d_g(x,y)\le\delta} |\partial_x^\alpha \partial_y^\beta R_{k,\varepsilon,\tau}(\ell, x, y)| \le C_1 \delta \|\partial_t \hat{f}_k\|_{L^\infty([-\delta,\delta])} \nu_\ell^{n-1+|\alpha|+|\beta|} + C_2 \nu_\ell^{n-2+|\alpha|+|\beta|}$$

with C_1 independent of δ and k. Recalling that the summation in k is actually finite and that $\sup_{\{\sigma>0,\delta<1,k\in\mathbb{Z}\}}\|\partial_t \hat{f}_k\|_{L^\infty([-\delta,\delta])}<\infty$ (see Remark 3.1) we have that if we define

$$F_{\varepsilon,\tau}(\ell,x,y) := \frac{1}{\nu_\ell^{n-1+|\alpha|+|\beta|}}\Big(\nu_\ell^{n-2}\sum_{k\in\mathbb{Z}}\hat{f}_k(0)\partial_x^\alpha\partial_y^\beta W_{k,\varepsilon,\tau}(\nu_\ell,x,y) + \sum_{k\in\mathbb{Z}}\partial_x^\alpha\partial_y^\beta R_{k,\varepsilon,\tau}(\nu_\ell,x,y)\Big),$$

then we have for each fixed $\sigma>0$

$$\lim_{\delta\to 0+}\limsup_{\ell\to\infty}\sup_{d_g(x,y)\le\delta}|F_{\varepsilon,\tau}(\ell,x,y)| = 0. \tag{3.26}$$

Define

$$A(\ell,x,y) := \frac{1}{(2\pi)^n\Theta^{1/2}(x,y)}\int_{S_y^*M} e^{i\nu_\ell\langle \exp_y^{-1}(x),\omega\rangle_g}\frac{d\omega}{\sqrt{|g_y|}}.$$

To deal with the first term in (3.25), we claim that

$$\lim_{\sigma\to 0+}\lim_{s\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\limsup_{\ell\to\infty}\sup_{d_g(x,y)<\delta}\nu_\ell^{-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta A_{\varepsilon,\tau}(\ell,x,y)-\partial_x^\alpha\partial_y^\beta A(\ell,x,y)| = 0. \tag{3.27}$$

and

$$\lim_{\sigma\to 0+}\lim_{s\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\limsup_{\ell\to\infty}\sup_{d_g(x,y)<\delta}\nu_\ell^{-|\alpha|-|\beta|}\Big|\partial_x^\alpha\partial_y^\beta A(\ell,x,y)\sum_{k\in\mathbb{Z}}\hat{f}_k(0) - \frac{2\pi N}{T}\partial_x^\alpha\partial_y^\beta A(\ell,x,y)\Big| = 0. \tag{3.28}$$

Observe that (3.27) follows from the fact that

$$\limsup_{\ell\to\infty}\nu_\ell^{-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta(e^{i\nu_\ell\langle\exp_y^{-1}(x),\omega\rangle_g}(1-c^0_{\varepsilon,\tau}(y,\omega)))| \le C_{\alpha\beta}|(1-c^0_{\varepsilon,\tau}(y,\omega))|$$

and that

$$\|1-c^0_{\varepsilon,\tau}(y,\omega)\|_{L^1(\mathbb{S}^{n-1}_\omega)}\xrightarrow[\varepsilon\to 0]{} 0.$$

To prove (3.28), first note that, by (3.9) and (3.1), we have

$$\sum_{k\in\mathbb{Z}}\hat{f}_k(0) = \sum_{k\in\mathbb{Z}}\lim_{t\to 0}e^{i\pi t(N-1)/T}\frac{\sin\big(\frac{\pi N t}{T}\big)}{\sin\big(\frac{\pi t}{T}\big)}\hat{\psi}_\sigma(t+kT)\hat{\rho}_\delta(t) = N\sum_{k\in\mathbb{Z}}\hat{\psi}_\sigma(kT).$$

Using the Poisson summation formula and the definition of $\psi_{\mathrm{w},\sigma} := \psi_\sigma$ in (3.2),

$$N\sum_{k\in\mathbb{Z}}\hat{\psi}_\sigma(kT) = N\sum_{k\in\mathbb{Z}}\frac{\sin(\mathrm{w}kT)}{kT}\hat{\rho}(\sigma kT) = \frac{2\pi N}{T}\sum_{k\in\mathbb{Z}}\mathbb{1}_{[-1,1]} * \rho_{\sigma/\mathrm{w}}\Big(\frac{2\pi k}{T\mathrm{w}}\Big)$$
$$= \frac{2\pi N}{T}\sum_{k\in\mathbb{Z}}\psi_{1,\sigma/\mathrm{w}}\Big(\frac{2\pi k}{T\mathrm{w}}\Big).$$

Motivated by the form of the above expression, we replace σ by $\mathrm{w}\sigma$, which is permitted since w is fixed throughout this argument. Thus,

$$N\sum_{k\in\mathbb{Z}}\hat{\psi}_{\mathrm{w},\mathrm{w}\sigma}(kT) = \frac{2\pi N}{T}\sum_{k\in\mathbb{Z}}\psi_{1,\sigma}\Big(\frac{2\pi k}{T\mathrm{w}}\Big).$$

Since $\psi_{1,\sigma} = \mathbb{1}_{[-1,1]} * \rho_\sigma$ and $0 < \mathrm{w} < 2\pi/T$, we have that for $k \neq 0$,

$$\Big|\frac{1}{T}\psi_{1,\sigma}\Big(\frac{k}{T\mathrm{w}}\Big)\Big| \leq \frac{C_{N'}}{T}\Big(1 + \frac{|k|}{T\mathrm{w}\sigma}\Big)^{-N'} \quad \text{for any } N'.$$

Thus, if we choose $N' \geq 2$, we obtain

$$\Big|\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\frac{1}{T}\psi_{1,\sigma}\Big(\frac{k}{T\mathrm{w}}\Big)\Big| \leq C_{N'}\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}(\mathrm{w}\sigma)^{N'}T^{-1}\Big(\mathrm{w}\sigma + \frac{|k|}{T}\Big)^{-N'}$$
$$\leq C_{N'}(\mathrm{w}\sigma)^{N'}T^{-1}\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\Big(\frac{|k|}{T}\Big)^{-N},$$

which converges to 0 as $\sigma \to 0$. Also, when $k = 0$, we have

$$\psi_{1,\sigma}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin t}{t}\hat{\rho}(\sigma t)\,dt \to \psi(0) = 1$$

as $\sigma \to 0$, and this finishes the proof of the claim in (3.28).

Combining (3.24), (3.25), (3.26), and (3.28) yields that the final step in the proof is to eliminate the factor of $\Theta^{-1/2}(x, y)$ implicit in the definition of L. For this, we observe that $\Theta^{-1/2}(x, x) = 1$ and its differential vanishes on the diagonal in $M \times M$. Hence, for small $d_g(x, y)$, we have

$$\Theta^{-1/2}(x, y) = 1 + d_g(x, y)^2 G(x, y)$$

for some smooth, bounded function G. Thus, it suffices to show that

$$\lim_{\delta\to 0^+}\limsup_{\ell\to\infty}\sup_{d_g(x,y)\leq\delta}\Big|\frac{1}{\nu_\ell^{|\alpha|+|\beta|}}\partial_x^\alpha\partial_y^\beta\Big(d_g(x, y)^2\int_{S_y^*M} e^{i\nu_\ell\langle\exp_y^{-1}(x),\omega\rangle}\frac{d\omega}{\sqrt{|g_y|}}\Big)\Big| = 0. \tag{3.29}$$

In the case where at most one derivative falls on the factor of $d_g(x,y)^2$, the above statement holds trivially. If two or more derivatives fall on this factor, then at most $|\alpha|+|\beta|-2$ factors of ν_ℓ can appear from differentiating the integral over $S^*_y M$, and so (3.29) also holds in this case. ∎

4. The contributions of subperiodic loops

4.1. Subperiodic loops near K_N

In this section, we analyze the asymptotic behavior of the contributions to the spectral projector from times which are bounded away from integer multiples of T, which are characterized by the quantity

$$\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x,y)\Big(1-\sum_{k\in\mathbb{Z}}\hat{\rho}(t-kT)\Big)\,dt. \tag{4.1}$$

We can rewrite this as

$$\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y) = \sum_{k\in\mathbb{Z}}\frac{1}{2\pi}\int_{kT}^{(k+1)T} e^{it\lambda}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t C^*_{\varepsilon,\tau}(x,y) \times (1-\hat{\rho}(t-kT)-\hat{\rho}(t-(k+1)T))\,dt.$$

Changing variables via $t\mapsto t+kT$, we obtain

$$\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y) = \sum_{k\in\mathbb{Z}}\frac{e^{ikT\lambda}}{2\pi}\int_0^T e^{it\lambda}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t+kT)U_{t+kT} C^*_{\varepsilon,\tau}(x,y) \times (1-\hat{\rho}(t)-\hat{\rho}(t-T))\,dt.$$

Similarly to Section 3, we use the fact that we can write

$$U_{t+kT} = e^{-ik\mathfrak{b}T}(U_t + Q_k(t)) \tag{4.2}$$

for some $Q_k(t)$ which is an FIO of order $-1/4-1$. Thus,

$$\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y) = \sum_{k\in\mathbb{Z}}\frac{e^{ikT(\lambda-\mathfrak{b})}}{2\pi}\int_0^T e^{it\lambda}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t+kT)(U_t+Q_k(t)) \times C^*_{\varepsilon,\tau}(x,y)(1-\hat{\rho}(t)-\hat{\rho}(t-T))\,dt.$$

Note that due to the support properties of $\hat{\rho}$, we can extend the integral over $[0,T]$ to be performed over the whole real line. Let us define

$$\hat{g}_{k,N}(t) = \frac{\sin\left(\frac{N\pi t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t+kT)(1-\hat{\rho}(t)-\hat{\rho}(t-T))1_{[0,T]}(t) \tag{4.3}$$

so that

$$\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y) = \sum_{k\in\mathbb{Z}} \frac{e^{ikT(\lambda-\mathfrak{b})}}{2\pi} \mathcal{F}^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}(U_t + Q_k(t))C^*_{\varepsilon,\tau}]. \tag{4.4}$$

Lemma 4.1. *Suppose that in some coordinate chart,*

$$U_t(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\varphi(x,y,t,\xi)} a(x,y,\xi)\, d\xi$$

for some nondegenerate homogeneous phase function φ and some symbol $a \in S^0$. Then,

$$U_t C^*_{\varepsilon,\tau}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\varphi(x,y,t,\xi)} a_{\varepsilon,\tau}(x,y,\xi)\, d\xi,$$

where

$$a_{\varepsilon,\tau}(x,y,\xi) - a^0(x,y,\xi)\overline{c^0_{\varepsilon,\tau}(y,-\partial_y\varphi)} \in S^{-1}.$$

This lemma follows from the standard FIO calculus (cf. [21, Chapter 25]). With this in hand, we have the following proposition.

Proposition 4.2. *Let (M,g) be a smooth, compact, Zoll manifold with minimal common period T. Suppose that $x_0 \in M$ and for each s satisfies*

$$\lim_{\varepsilon\to 0} \mu_{S^*M}(\mathcal{L}_{N,\varepsilon,\tau}(x_0)) = 0,$$

and let $c_{\varepsilon,\tau}$ satisfy (2.5) and $\mathcal{B}_{\varepsilon,\tau}$ be as in (4.1). Then, for all $\alpha,\beta \in \mathbb{N}$, $\sigma > 0$,

$$\lim_{\tau\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0^+}\limsup_{\lambda\to\infty}\sup_{x,y\in B(x_0,\delta)} \lambda^{1-n-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta \mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y)| = 0.$$

Proof. We claim that for w > 0, $\sigma > 0$, and any $k \in \mathbb{Z}$ fixed, we have

$$F^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}U_t C^*_{\varepsilon,\tau}(x,y)] = D_{\varepsilon,\tau,k}(\lambda,x,y)\lambda^{n-1} + R_{\varepsilon,\tau,k}(\lambda,x,y), \tag{4.5}$$

where $D_{\varepsilon,\tau,k}$ and $R_{\varepsilon,\tau,k}$ are functions satisfying

$$\lim_{s\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\limsup_{\lambda>0}\sup_{x,y\in B(x_0,\delta)} \sum_k \lambda^{-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta D_{\varepsilon,\tau,k}(\lambda,x,y)| = 0$$

$$\lim_{\lambda\to\infty}\sup_{x,y\in B(x_0,\delta)} \lambda^{1-n-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta R_{\varepsilon,\tau,k}(\lambda,x,y)| = 0.$$

Moreover, we claim that if Q_k is as in (4.2), then

$$\lim_{\lambda\to\infty}\sup_{x,y\in B(x_0,\delta)} |\lambda^{1-n-|\alpha|-|\beta|}\partial_x^\alpha\partial_y^\beta F^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}Q_k C^*_{\varepsilon,\tau}(x,y)]| = 0. \tag{4.6}$$

We start proving the proposition given the claim. Notice that, since $\sigma > 0$, the sum in (4.4) is finite; we have

$$\begin{aligned}\limsup_{\lambda\to\infty} \sup_{x,y\in B(x_0,\delta)} & \lambda^{1-n-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta \mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma;x,y)| \\ & \leq \sum_k \limsup_{\lambda\to\infty} \sup_{x,y\in B(x_0,\delta)} \lambda^{-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta D_{\varepsilon,\tau,k}(\lambda,x,y)|.\end{aligned}$$

The proposition then follows since

$$\begin{aligned}&\lim_{\tau\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0+}\sum_k \limsup_{\lambda\to\infty} \sup_{x,y\in B(x_0,\delta)} \lambda^{-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta D_{\varepsilon,\tau,k}(\lambda,x,y)| \\ &= \sum_k \lim_{\tau\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0+} \limsup_{\lambda\to\infty} \sup_{x,y\in B(x_0,\delta)} \lambda^{-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta D_{\varepsilon,\tau,k}(\lambda,x,y)| = 0.\end{aligned}$$

Fix $\varepsilon, \tau > 0$. Note that since $C_{\varepsilon,\tau}$ is a pseudodifferential operator, the canonical relation of $U_t C_{\varepsilon,\tau}^*$ is identical to that of U_t, which we denote by

$$\mathcal{C} = \{(x,\xi,y,\eta,t,\tau) : |\tau| = |\xi|_{g_y}, \Phi^t(y,\eta) = (x,\xi)\},$$

and hence $U_t C_{\varepsilon,\tau}^*(x,y)$ can be represented as a locally finite sum of expressions of the form

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\varphi(x,y,t,\xi)} a_{\varepsilon,\tau}(x,y,t,\xi)\,d\xi,$$

where $a_{\varepsilon,\tau}^0(x,y,t,\xi) = a^0(x,y,\xi)\overline{c_{\varepsilon,\tau}^0(y,-d_y\varphi)}$ with a^0 and c_ε^0 being the principal symbols of U_t and C_ε, respectively. Here, φ is some nondegenerate phase function parameterizing $\mathcal{C}$. Thus, we have that

$$F_{t\mapsto\lambda}^{-1}[\hat{g}_{k,N} U_t C_{\varepsilon,\tau}^*(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\hat{g}_{k,N}(t) U_t C_{\varepsilon,\tau}^*(x,y)\,dt$$

can be expressed as a locally finite sum of terms of the form

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{it\lambda+i\varphi(x,y,t,\xi)}\hat{g}_{k,N}(t) a_{\varepsilon,\tau}(x,y,t,\xi)\,d\xi\,dt \\ &\quad = \frac{\lambda^n}{(2\pi)^n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i\lambda(t+\varphi(x,y,t,\xi))}\hat{g}_{k,N}(t) a_{\varepsilon,\tau}^0(x,y,t,\xi)\,d\xi\,dt + \mathcal{O}(\lambda^{n-2}),\end{aligned}$$

since the subprincipal symbol of $C_{\varepsilon,\tau}$ is zero in a neighborhood of x_0. To see this, observe that the principal symbol is independent of x in a neighborhood of x_0 and is homogeneous degree 0 for $|\xi|$ large enough. Let us convert to polar coordinates via $\xi = r\omega$ for $r > 0$ and $\omega \in \mathbb{R}^n$, which gives

$$\frac{\lambda^n}{(2\pi)^n} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{i\lambda(t+\varphi(x,y,t,\xi))} \hat{g}_{k,N}(t) a^0_{\varepsilon,\tau}(x,y,t,\xi)\, d\xi\, dt$$
$$= \frac{\lambda^n}{(2\pi)^n} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-1}} e^{i\lambda(t+r\varphi(x,y,t,\omega))} \hat{g}_{k,N}(t) a^0_{\varepsilon,\tau}(x,y,t,\omega) r^{n-1}\, dr\, dt\, d\omega.$$

Let $\chi_{\delta'} \in C^\infty(\mathbb{S}^{n-1})$ be such that $|\nabla_\omega \varphi(x,y,t,\omega)| < \delta'$ for all $\omega \in \operatorname{supp} \chi_{\delta'}$. Then,

$$\frac{\lambda^n}{(2\pi)^n} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-1}} e^{i\lambda(t+r\varphi(x,y,t,\omega))} \hat{g}_{k,N}(t) a^0_{\varepsilon,\tau}(x,y,t,\omega) r^{n-1}\, dr\, dt\, d\omega$$
$$= \frac{\lambda^n}{(2\pi)^n} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-1}} e^{i\lambda(t+r\varphi(x,y,t,\omega))} \chi_{\delta'}(\omega) \hat{g}_{k,N}(t) a^0_{\varepsilon,\tau}(x,y,t,\omega) r^{n-1}\, dr\, dt\, d\omega$$
$$+ \mathcal{O}(\lambda^{-\infty}), \tag{4.7}$$

since $|\nabla_\omega \varphi|$ is bounded below on the support of $1 - \chi_{\delta'}$, and so we may integrate by parts arbitrarily many times in ω using the operator $(\nabla_\omega \varphi \cdot \nabla_\omega)/(i\lambda r |\nabla_\omega \varphi|^2)$. Now, for each fixed ω, we aim to perform stationary phase in (t, r). The critical points of the phase function $\tilde{\varphi} = t + r\varphi(x,y,t,\omega)$ occur when

$$\partial_t \varphi(x,y,t,\omega) = -\frac{1}{r} \quad \text{and} \quad \varphi(x,y,t,\omega) = 0.$$

The Hessian at such points is given by

$$\begin{pmatrix} r\partial_t^2 \varphi & \partial_t \varphi \\ \partial_t \varphi & 0 \end{pmatrix},$$

and hence the critical points are nondegenerate. Since φ is homogeneous of degree 1 in the fiber variable, we have that $\partial_r \varphi(x,y,t,r\omega) = \varphi(x,y,t,\omega) = 0$ at any of the critical values of t. Also, since $|\nabla_\omega \varphi(x,y,t,r\omega)| < \delta'$ on $\operatorname{supp} \chi_{\delta'}$, we have that $|d_\xi \varphi| < \delta'$ at the critical points, and hence the points

$$(x, d_x\varphi, y, -d_y\varphi, t, \partial_t \varphi)$$

are very close to the canonical relation $\mathcal{C}$. Thus, for δ' sufficiently small, we have that at each critical point (t_c, r_c),

$$d_g(\Phi^{t_c}(y, -d_y\varphi), (x, d_x\varphi)) < \min\Big(\frac{\varepsilon}{2}, \delta\Big).$$

Thus, if $(x, y) \in B(x_0, \varepsilon/2)$, this implies that $\Phi^{t_c}(y, -d_y\varphi) \in B(x_0, \varepsilon)$. Due to the support properties of $c_{\varepsilon,\tau}(y, -d_y\varphi)$ (see (2.5)), we have that $-d_y\varphi \notin L_{\varepsilon,\tau}(x_0)$, (see (2.1) for the definition of $L_{\varepsilon,\tau}$). In particular, the only critical points which contribute a nonzero term to the sum are those for which $|t_c - (p/q)T| < \tau$ for some $0 < p < q \leq N$. Therefore, by stationary phase, the leading term in (4.7) can be expressed as a finite sum of terms of the form

$$\frac{\lambda^{n-1}}{(2\pi)^n} \int_{S^{n-1}} \sum \frac{1}{|\partial_t \varphi|} e^{i\lambda t_c + i\pi|\operatorname{sgn}\operatorname{Hess}\tilde{\varphi}|/4} \chi_{\delta'}(\omega) \hat{g}_{k,N}(t_c) a^0_{\varepsilon,\tau}(x, y, t_c, \omega) r_c^{n-1}\, d\omega + \mathcal{O}_{\delta',\varepsilon,\tau}(\lambda^{n-2}), \tag{4.8}$$

where the sum is taken over all critical points $(t_c(\omega), r_c(\omega))$ for which

$$a^0_{\varepsilon,\tau}(x, y, t_c, \omega) \neq 0.$$

Since, for $d_g(x, y) < \varepsilon/2$ small enough, $|t_c - (p/q)T| < \tau$ for some $0 < p < q \leq N$, we have that $|\sin(N\pi t_c/T)| \leq N\pi\tau/T$, and hence

$$\begin{aligned} &\lim_{s\to 0} \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \sup_{d_g(x,y)<\delta} \hat{g}_{k,N}(t_c) a^0_{\varepsilon,\tau}(x, y, t_c, \omega) \\ &= \lim_{s\to 0} \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \sup_{d_g(x,y)<\delta} \frac{\sin\left(\frac{N\pi t_c}{T}\right)}{\sin\left(\frac{\pi t_c}{T}\right)} \hat{\psi}_\sigma(t_c + kT) \\ &\qquad\qquad \times (1 - \hat{\rho}(t_c) - \hat{\rho}(t_c - T)) a^0_{\varepsilon,\tau}(x, y, t_c, \omega) \\ &= 0, \end{aligned}$$

which completes the proof of (4.5). The estimate (4.6) follows by a similar argument if we note that $Q_k(t)$ is an FIO of one order lower than U_t with the same canonical relation.

This completes the proof for $|\alpha| = |\beta| = 0$. To handle derivatives, observe that if a derivative falls on the amplitude, then the kernel is smaller than the main term by a power of λ and hence does not contribute after taking the $\lambda \to \infty$. Therefore, the only term we need to consider is when all of the derivatives fall on the exponential. In this case, we obtain (4.8) with λ^{n-1} replaced $\lambda^{n-1+|\alpha|+|\beta|}$ and the symbol a^0_{w} replaced by another (uniformly bounded) symbol $\tilde{a}^0_{\mathrm{w}}$ with the same support properties. Hence, the proof is completed in the same way as for $|\alpha| = |\beta| = 0$. ∎

4.2. Looping times outside of K_N

By the assumptions of Theorem 2, we know that there must be at most a small measure set of subperiodic loops with lengths which are outside of $K_N = \{(p/q)T : 1 \leq p < q \leq N\}$, and thus we expect their contributions to be negligible. In this section, we demonstrate this rigorously by utilizing analysis similar to [10, 34].

Proposition 4.3. *Let $B_{\varepsilon,\tau} \in \Psi^0(M)$ be any pseudodifferential operator such that the principal symbol $B^0_{\varepsilon,\tau}(x,\xi)$ satisfies*

$$\lim_{\varepsilon\to 0} \sup_{x\in M} \|B^0_{\varepsilon,\tau}(x,\cdot)\|_{L^1(\mathbb{S}^{n-1})} = 0.$$

Then, for any $a > 0$, $\alpha, \beta \in \mathbb{N}$, there exist constants $\mu_0, C > 0$ such that

$$\sup_{x,y\in M} |\partial_x^\alpha \partial_y^\beta \Pi_{[\mu,\mu+a]} B^*_{\varepsilon,\tau}(x,y)| \leq c_{1,\varepsilon,\tau}\mu^{n-1+|\alpha|+|\beta|} + c_{2,\varepsilon,\tau}\mu^{n-2+|\alpha|+|\beta|},$$

with $\lim_{\varepsilon\to 0} c_{1,\varepsilon,\tau} = 0$.

Remark 4.4. Note that this proposition holds on *any* Riemannian manifold, not only Zoll manifolds.

First, we claim that for any $\rho \in \mathcal{S}(\mathbb{R})$ with Fourier transform $\hat{\rho}$ supported in $[-\operatorname{inj}(M,g)/2, \operatorname{inj}(M,g)/2]$ with $\hat{\rho} \equiv 1$ in a neighborhood of 0, we have

$$\begin{aligned}\Theta^{1/2}&(x,y)\partial_\mu(\Pi_{[0,\mu]}B^*_{\varepsilon,\tau})(x,y)\\ &= \frac{\mu^{n-1}}{(2\pi)^n}\int_{S^*_yM} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle}\,\overline{B^0_{\varepsilon,\tau}(y,\omega)}\frac{d\omega}{\sqrt{|g_y|}} + R_{\varepsilon,\tau}(\mu,x,y),\end{aligned}$$

where

$$|\partial_x^\alpha \partial_y^\beta R_{\varepsilon,\tau}(\mu,x,y)| \leq C_0\mu^{n-2+|\alpha|+|\beta|}$$

for some $C_0 > 0$. This follows by a repetition of the proof of Proposition 3.3 with $(I+Q)C$ replaced by $B^*_{\varepsilon,\tau}$ and f replaced by ρ. Since $\hat{\rho} \equiv 1$ near 0, the first term in the remainder estimate (3.12) vanishes. Then, since $B^0_{\varepsilon,\tau}$ is supported in a set of small we have that

$$|\partial_\mu \partial_x^\alpha \partial_y^\beta(\rho * \Pi_{[0,\mu]}B^*_{\varepsilon,\tau}(x,y))| \leq c_{1,\varepsilon,\tau}\mu^{n-1+|\alpha|+|\beta|} + c_{2,\varepsilon,\tau}\mu^{n-2+|\alpha|+|\beta|} \quad (4.9)$$

for sufficiently large μ, where $\lim_{\varepsilon\to 0} c_{1,\varepsilon,\tau} = 0$.

To control the difference $\partial_\mu\Pi_{[0,\mu]}B^*_{\varepsilon,\tau} - \partial_\mu(\rho * \Pi_{[0,\mu]}B^*_{\varepsilon,\tau})$, we invoke general Tauberian theorems in the exact same fashion as in [10].

Lemma 4.5 (Tauberian theorem for non-monotone functions). *Let θ be a piecewise continuous function such that there exists $A > 0$ with $\hat{\theta}(t) = 0$ for $|t| \leq A$. Suppose further that there exist constants $m \in \mathbb{N}$ and $c_1, c_2 > 0$ such that for all $\mu \in \mathbb{R}$*

$$|\theta(\mu+s) - \theta(\mu)| \leq c_1(1+|\mu|)^m + c_2(1+|\mu|)^{m-1} \quad \textit{for all } s \in [0,1]. \quad (4.10)$$

Then, there exists a positive constant $c_{m,A}$, depending only on m and A, such that for all μ we have

$$|\theta(\mu)| \leq c_{m,A}(c_1(1+|\mu|)^m + c_2(1+|\mu|)^{m-1}).$$

Proof. Let ρ be a Schwartz function with $\hat\rho \in C_c^\infty(-A, A)$ and $0 \notin \operatorname{supp}(1-\hat\rho)$. Then, $\rho * \theta = 0$ and

$$
\begin{aligned}
|\theta(\mu)| = |\theta(\mu) - \rho * \theta(\mu)| &\le \sum_{j\in\mathbb{Z}} \int_j^{j+1} |\rho(s)(\theta(\mu - s) - \theta(\mu))|\, ds \\
&\le C_{\rho,N} \sum_{j\ge 0} \langle j\rangle^{-N} \int_j^{j+1} |\theta(\mu - s) - \theta(\mu - j)|\, ds \\
&\quad + \langle j\rangle^{-N} \sum_{k=0}^{j-1} |\theta(\mu - k) - \theta(\mu - (k+1))| \\
&\quad + C_{\rho,N} \sum_{j<0} \langle j\rangle^{-N} \int_j^{j+1} |\theta(\mu - s) - \theta(\mu - j)|\, ds \\
&\quad + \langle j\rangle^{-N} \sum_{k=0}^{|j|-1} |\theta(\mu + k) - \theta(\mu + k + 1)| \\
&\le C_{\rho,N} \sum_{j} \langle j\rangle^{-N} \sum_{k=-j}^{j} (c_1(1 + |\mu - k|)^m + c_2(1 + |\mu - k|)^{m-1}) \\
&\le C_{\rho,m}(c_1(1 + |\mu|)^m + c_2(1 + |\mu|)^{m-1}).
\end{aligned}
$$
■

To apply this lemma, we first set

$$\theta_{\varepsilon,\tau}(x, y, \mu) = \partial_x^\alpha \partial_y^\beta (\Pi_{[0,\mu]} B^*_{\varepsilon,\tau}(x, y) - \rho * \Pi_{[0,\mu]} B^*_{\varepsilon,\tau}(x, y)).$$

We must then demonstrate that $\theta_{\varepsilon,\tau}$ satisfies the hypotheses listed above. Note that

$$\mathcal{F}_{\mu\to t}(\theta_{\varepsilon,\tau}(x, y, \cdot)(t)) = (1 - \hat\rho(t))\mathcal{F}_{\mu\to t}(\partial_x^\alpha \partial_y^\beta (\Pi_{[0,\cdot]} B^*_{\varepsilon,\tau}(x, y)))(t).$$

Since $\hat\rho \equiv 1$ near 0, we therefore have that

$$\mathcal{F}_{\mu\mapsto t}(\theta_{\varepsilon,\tau}(x, y, \cdot)) = 0$$

for t in some interval around 0. Then, $\mathcal{F}_{\mu\mapsto t}(\theta_{\varepsilon,\tau}(x, y, \cdot))$ vanishes in a neighborhood of $t = 0$. We now verify the hypothesis (4.10) for θ_ε.

Next, let $s \in [0, 1]$ and $\mu \in \mathbb{R}$, and notice that

$$
\begin{aligned}
&\theta_{\varepsilon,\tau}(x, y, \mu + s) - \theta_{\varepsilon,\tau}(x, y, \mu) \\
&\quad = \partial_x^\alpha \partial_y^\beta (\Pi_{[0,\mu]} B^*_{\varepsilon,\tau}(x, y, \mu + s) - \Pi_{[0,\mu]} B^*_{\varepsilon,\tau}(x, y, \mu)) \\
&\qquad + \partial_x^\alpha \partial_y^\beta (\hat\rho * \Pi_{[0,\mu]} B^*_{\varepsilon,\tau}(x, y, \mu + s) - \hat\rho * \Pi_{[0,\mu]} B^*_{\varepsilon,\tau}(x, y, \mu)).
\end{aligned}
$$

To control the first difference, we apply Cauchy–Schwartz, which gives

$$\begin{aligned}
&|\partial_x^\alpha \partial_y^\beta (\Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu+s) - \Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu))| \\
&= \Big| \sum_{\mu<\lambda_j\le\mu+s} \partial_x^\alpha \varphi_j(x) \overline{\partial_y^\beta B_{\varepsilon,\tau}\varphi_j(y)} \Big| \\
&\le \Big(\sum_{\mu<\lambda_j\le\mu+s} |\partial_x^\alpha \varphi_j(x)|^2 \Big)^{1/2} \Big(\sum_{\mu<\lambda_j\le\mu+s} |\partial_y^\beta B_{\varepsilon,\tau}\varphi_j(y)|^2 \Big)^{1/2}.
\end{aligned}$$

Applying the local Weyl law (cf. [32, Theorem 5.2.3]), we have

$$\begin{aligned}
&|\partial_x^\alpha \partial_y^\beta (\Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu+s) - \Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu))| \\
&\quad \le c_{1,\varepsilon,\tau}\mu^{n-1+|\alpha|+|\beta|} + c_{2,\varepsilon,\tau}\mu^{n-2+|\alpha|+|\beta|}
\end{aligned}$$

with $\lim_{\varepsilon\to 0} c_{1,\varepsilon,\tau} = 0$, for $\mu \ge 1$ and $s \in [0,1]$. To estimate the derivatives of

$$\hat{\rho} * \Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu+s) - \hat{\rho} * \Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu),$$

we simply integrate (4.9) from μ to $\mu + s$ to obtain

$$\begin{aligned}
&|\partial_x^\alpha \partial_y^\beta (\hat{\rho} * \Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu+s) - \hat{\rho} * \Pi_{[0,\mu]} B_{\varepsilon,\tau}^*(x,y,\mu))| \\
&\quad \le c_{1,\varepsilon,\tau}\lambda^{n-1+|\alpha|+|\beta|} + c'_{2,\varepsilon,\tau}\lambda^{n-2+|\alpha|+|\beta|}
\end{aligned}$$

for all $s \in [0,1]$, all μ sufficiently large, and some new constants $c'_{1,\varepsilon,\tau}, c'_{2,\varepsilon,\tau}$ where $\lim_{\varepsilon\to 0} c'_{1,\varepsilon,\tau} = 0$. Therefore, we have that

$$|\theta_\varepsilon(x,y,\mu+s) - \theta_\varepsilon(x,y,\mu)| \le c_{1,\varepsilon,\tau}\mu^{n-1+|\alpha|+|\beta|} + c_{2,\varepsilon,\tau}\mu^{n-2+|\alpha|+|\beta|}$$

after potentially increasing $c_{1,\varepsilon,\tau}$ and $c_{2,\varepsilon,\tau}$ (but still with $\lim_{\varepsilon\to 0} c_{1,\varepsilon,\tau} = 0$). Applying Lemma 4.5 with $m = n - 1 + |\alpha| + |\beta|$, $A = \mathrm{inj}(M,g)/2$, and $\theta = \theta_{\varepsilon,\tau}$, we obtain, using $n \ge 2$, that

$$|\theta_{\varepsilon,\tau}(\mu,x,y)| \le c'_{1,\varepsilon,\tau}\mu^{n-1+|\alpha|+|\beta|} + c_{2,\varepsilon,\tau}\mu^{n-2+|\alpha|+|\beta|},$$

where $\lim_{\varepsilon\to 0} c'_{1,\varepsilon,\tau} = 0$, which completes the proof of Proposition 4.3.

5. On-diagonal analysis of the spectral projector

The goal of this section is to establish a lower bound for the spectral function restricted to the diagonal, which is critical for the purposes of comparing the smoothed projector to the original. In particular, we show that most of the "mass" of the spectral function is concentrated near

$$\bigcup_{\ell\in\mathbb{N}} [\nu_\ell - r\ell^{-1/2}, \nu_\ell + r\ell^{-1/2}],$$

with ν_ℓ as defined in (1.2). This is similar to the original eigenvalue clustering result of [16, Theorem 3.1]. We expect that a stronger cluster estimate with $r\ell^{-1/2}$ replaced with $r\ell^{-1}$ should hold, but we do not prove this here as the refined statement is not needed. We also note that the results of this section do not depend on any assumptions about superiodic loops. We need only that all geodesics are periodic with minimal common period T.

Proposition 5.1. *Let (M,g) be a Zoll manifold with minimal common period $T>0$ and let $\{\varphi_j\}_j$ be the corresponding Laplace eigenfunctions defined in (1.6). Let $r>0$ and fix a multi-index $\alpha\in\mathbb{N}^n$. Then, there exist $K,C,\lambda_0>0$ so that for all $x\in M$ and $\lambda\geq\lambda_0$*

$$\sum_{\lambda_j\in\mathcal{A}(K,r,\lambda)}|\partial_x^\alpha\varphi_j(x)|^2\geq(1-Cr^{-2})\sum_{|\lambda_j-\lambda|\leq K}|\partial_x^\alpha\varphi_j(x)|^2,$$

where

$$\mathcal{A}(K,r,\lambda)=\Big\{\lambda_j:|\lambda_j-\lambda|\leq K,\lambda_j\in\bigcup_{\ell\in\mathbb{N}}[\nu_\ell-r\ell^{-1/2},\nu_\ell+r\ell^{-1/2}]\Big\}.$$

Proof. We begin by considering the case where $\alpha=0$ separately. For this, we proceed in close analogy to the proof of [16, Theorem 3.1]. Let $\chi\in\mathcal{S}(\mathbb{R})$ with $\chi\geq0$ and $\hat{\chi}\in C_c^\infty(\mathbb{R})$ with $\hat{\chi}(0)>0$. Repeating previous calculations, we have that for $x\in M$

$$\sum_{j=0}^{\infty}\chi(\lambda-\lambda_j)|\varphi_j(x)|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{it\lambda}\hat{\chi}(t)U_t(x,x)\,dt. \tag{5.1}$$

Similarly,

$$\sum_{j=0}^{\infty}e^{i(\mathfrak{b}-\lambda_j)T}\chi(\lambda-\lambda_j)|\varphi_j(x)|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{it\lambda}\hat{\chi}(t)e^{i\mathfrak{b}T}U_{t+T}(x,x)\,dt. \tag{5.2}$$

Recalling that $U_t-e^{i\mathfrak{b}T}U_{t+T}$ is an FIO defined by $\mathcal{C}$ of order $-1/4-1$ (see (3.4)), we know that we can write

$$U_t(x,x)-e^{i\mathfrak{b}T}U_{t+T}(x,x)=\frac{1}{(2\pi)^n}\int_{T_y^*M}e^{i\phi(t,x,x,\xi)}B(t,x,x,\xi)\,d\xi,$$

where B is a symbol of order -1 and ϕ is any admissible phase function which parametrizes $\mathcal{C}$ (cf. [16, p. 45]). As in the proof of Proposition 3.3, we can use the phase function

$$\phi(t,x,y,\xi)=\langle\exp_y^{-1}(x),\xi\rangle_{g_y}-t|\xi|_{g_y}.$$

Hence,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\hat{\chi}(t)(U_t(x,x)-e^{i\mathfrak{b}T}U_{t+T}(x,x))\,dt\\
&=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{T_y^*M} e^{it(\lambda-|\xi|)}\hat{\chi}(t)B(t,x,x,\xi)\,d\xi\,dt\\
&=\frac{\lambda^n}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{S_y^*M}\hat{\chi}(t)e^{i\lambda t(1-s)}s^{n-1}B(t,x,x,\lambda s\omega)\,ds\,d\omega\,dt\\
&=\mathcal{O}(\lambda^{n-2}).
\end{aligned}
\tag{5.3}
$$

Here, to obtain the bound in the last line we used the fact that B is a symbol of order -1 and repeated the calculations from the proof of Proposition 3.3 that follow (3.17). From (5.1) and (5.2), it follows that

$$\sum_{j=0}^{\infty}\chi(\lambda-\lambda_j)(1-e^{i(\mathfrak{b}-\lambda_j)T})|\varphi_j(x)|^2=\mathcal{O}(\lambda^{n-2}). \tag{5.4}$$

Thus, we can take real parts to obtain that

$$\sum_{j=0}^{\infty}(1-\cos(T(\mathfrak{b}-\lambda_j)))\chi(\lambda-\lambda_j)|\varphi_j(x)|^2=\mathcal{O}(\lambda^{n-2}) \tag{5.5}$$

as $\lambda\to\infty$. For any $r,\ell>0$, define the set

$$\mathcal{E}(\ell,r)=\{\lambda_j\in\operatorname{Spec}(\sqrt{-\Delta_g}):r\ell^{-1/2}\le T|\lambda_j-\nu_\ell|\le\pi\}$$

Recall that $\nu_\ell=2\pi\ell/T+\mathfrak{b}$ by (3.10). Thus, if $\lambda_j\in\mathcal{E}(\ell,r)$, we have that

$$1-\cos(T(\mathfrak{b}-\lambda_j))=1-\cos(T(\nu_\ell-\lambda_j)-2\pi\ell)\ge\frac{1}{2}r^2\ell^{-1}-\frac{1}{24}r^4\ell^{-2},$$

since $1-\cos(\theta-2\pi\ell)\ge(1/2)\theta^2-(1/24)\theta^4$ for $\theta\in[-\pi,\pi]$ and all $\ell\in\mathbb{N}$. Therefore, using that $\nu_\ell\ge c\ell$ for ℓ large enough, together with (5.5), we obtain that for every $r>0$ there exist $C,\ell_0>0$ such that for all $\ell\ge\ell_0$, we have

$$
\begin{aligned}
&\sum_{\lambda_j\in\mathcal{E}(\ell,r)}\frac{1}{2}r^2\ell^{-1}\min\Big(\chi(\mu):|\mu|\le\frac{\pi}{T}\Big)|\varphi_j(x)|^2\\
&\quad\le C\sum_{\lambda_j\in\mathcal{E}(\ell,r)}(1-\cos((\mathfrak{b}-\lambda_j)))\chi(\nu_\ell-\lambda_j)|\varphi_j(x)|^2\le C\ell^{n-2}.
\end{aligned}
$$

If we adjust χ so that $\chi(\mu) > 0$ for all $|\mu| \leq \pi/T$, we obtain that

$$\sum_{\lambda_j \in \mathcal{E}(\ell,r)} |\varphi_j(x)|^2 \leq C r^{-2} \ell^{n-1} \tag{5.6}$$

for all $r > 0$ and all ℓ large enough.

Next, observe that for any $K, r > 0$,

$$\mathcal{A}(K,r,\lambda) = \{\lambda_j : |\lambda_j - \lambda| \leq K\} \cap \bigcap_{\ell=1}^{\infty} \mathcal{E}(\ell,r)^{\mathrm{c}}.$$

Therefore,

$$\sum_{\lambda_j \in \mathcal{A}(K,r,\lambda)} |\varphi_j(x)|^2 = \sum_{|\lambda_j - \lambda| \leq K} |\varphi_j(x)|^2 - \sum_{\ell=1}^{\infty} \sum_{\lambda_j \in \{|\lambda_j - \lambda| \leq K\} \cap \mathcal{E}(\ell,r)} |\varphi_j(x)|^2. \tag{5.7}$$

Note that

$$\{\lambda_j : |\lambda_j - \lambda| \leq K\} \cap \mathcal{E}(\ell,r) = \emptyset \quad \text{if } |\nu_\ell - \lambda| > K + \pi.$$

Thus, if we define

$$\mathcal{V}(\lambda, K) = \{\ell : |\nu_\ell - \lambda| \leq K + \pi\},$$

by (5.7)

$$\sum_{\lambda_j \in \mathcal{A}(K,r,\lambda)} |\varphi_j(x)|^2 = \sum_{|\lambda_j - \lambda| \leq K} |\varphi_j(x)|^2 - \sum_{\ell \in \mathcal{V}(\lambda,K)} \sum_{\lambda_j \in \{|\lambda_j - \lambda| \leq K\} \cap \mathcal{E}(\ell,r)} |\varphi_j(x)|^2. \tag{5.8}$$

In addition, for each $\ell \in \mathcal{V}(\lambda, K)$, we have that $\nu_\ell \approx \lambda$, and so by (5.6) that

$$\sum_{\lambda_j \in \{|\lambda_j - \lambda| \leq K\} \cap \mathcal{E}(\ell,r)} |\varphi_j(x)|^2 \leq C r^{-2} \lambda^{n-1} \tag{5.9}$$

since $\ell \approx \nu_\ell \approx \lambda$. Next, we need the following lemma whose proof we postpone until the end of this section.

Lemma 5.2. *Let (M, g) be any compact smooth manifold of dimension n with Laplace eigenfunctions $\{\varphi_j\}_j$ as in (1.6). Then, for every multi-index $\alpha \in \mathbb{N}$ there exist $K, C, \lambda_0 > 0$ so that*

$$\sum_{|\lambda - \lambda_j| \leq K} |\partial_x^\alpha \varphi_j(x)|^2 \geq C \lambda^{n-1+2|\alpha|}$$

for all $\lambda \geq \lambda_0$.

Returning to the proof of Proposition 5.1, we can combine Lemma 5.2 with (5.9) to obtain that for K sufficiently large,

$$\sum_{\lambda_j \in \{|\lambda_j - \lambda| \leq K\} \cap \mathcal{E}(\ell, r)} |\varphi_j(x)|^2 \leq C r^{-2} \sum_{|\lambda_j - \lambda| \leq K} |\varphi_j(x)|^2. \tag{5.10}$$

Furthermore, since the cardinality of $\mathcal{V}(\lambda, K)$ is proportional to K, we can combine (5.10) with (5.8) to obtain

$$\sum_{\lambda_j \in \mathcal{A}(K,r,\lambda)} |\varphi_j(x)|^2 \geq \left(1 - \frac{C}{r^2}\right) \sum_{|\lambda_j - \lambda| \leq K} |\varphi_j(x)|^2,$$

which completes the proof in case where $|\alpha| = 0$.

In order to prove the statement for higher order derivatives ∂_x^α, one need only show the appropriate analog of (5.4). In particular, this will follow from

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \hat{\chi}(t) \partial_x^\alpha \partial_y^\alpha (U_t(x,y) - e^{i\mathfrak{b}T} U_{t+T}(x,y))|_{y=x} \, dt = \mathcal{O}(\lambda^{n-2+2|\alpha|}). \tag{5.11}$$

This follows directly from the off-diagonal analog of (5.3), which is given by

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \hat{\chi}(t)(U_t(x,y) - e^{i\mathfrak{b}T} U_{t+T}(x,y)) \, dt \\ &\quad = \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{T_y^* M} e^{i\lambda(\langle \exp_y^{-1}(x), \xi \rangle + t(1-|\xi|))} \hat{\chi}(t) \widehat{B}(t,x,y,\lambda\xi) \, d\xi \, dt. \end{aligned}$$

Thus, each derivative in x or y yields at most one additional power of λ, and so by previous arguments we obtain (5.11). The rest of the argument proceeds identically to the $|\alpha| = 0$ case. ∎

Proof of Lemma 5.2. The proof of this lower bound relies on the generalized local Weyl law, which states that if A is a classical polyhomogeneous pseudodifferential operator of order zero, then

$$A\Pi_{[0,\lambda]} A^*(x,x) = \sum_{\lambda_j \leq \lambda} |A\varphi_j(x)|^2 = L_A(x,\lambda)\lambda^n + R_A(\lambda, x), \tag{5.12}$$

where

$$L_A(x) := C \int_{S_x^* M} |\sigma_0(A)(x,\xi)|^2 \, d\xi$$

for some $C > 0$, and $\sup_{x\in M} |R_A(\lambda, x)| \leq C_A \lambda^{n-1}$ for some $C_A > 0$ and all $\lambda \geq 1$ (cf. [32, Theorem 5.2.3]). We note that since A is of order zero, $|L_A(x)| \leq C'_A$ for some $C'_A > 0$. Given these facts, we define for each multi-index α the operator

$$A = \partial_x^\alpha (1 + \Delta_g)^{-|\alpha|/2} \in \Psi^0_{c\ell}(M)$$

whose principal symbol is a homogeneous function in $C^\infty(T^*M \setminus 0)$ which can be written in local coordinates as

$$\sigma_0(A)(x, \xi) = \frac{i^{|\alpha|}\xi^\alpha}{|\xi|_g^{|\alpha|}}.$$

By the local Weyl law, we have

$$\begin{aligned}
&A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x)\\
&\quad = (A\Pi_{\lambda+K}A^*(x, x) - L_A(x)(\lambda + K)^n)\\
&\qquad - (A\Pi_{\lambda-K}A^*(x, x) - L_A(x)(\lambda - K)^n)\\
&\qquad + L_A(x)((\lambda + K)^n - (\lambda - K)^n)\\
&\quad = R_A(\lambda + K, x) - R_A(\lambda - K, x) + L_A(x)(K\lambda^{n-1} + \mathcal{O}_{K,A}(\lambda^{n-2})).
\end{aligned}$$

Since $|R_A(\lambda, x)| \leq C_A\lambda^{n-1}$ and $L_A(x) \geq \delta > 0$ for all $x \in M$ and all $\lambda \geq 1$, we have that

$$A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) \geq (\delta K - C_A)\lambda^{n-1} + \mathcal{O}_{K,A}(\lambda^{n-2}).$$

Thus, if we choose K large enough so that $\delta K - C_A > 0$, there exists a $\lambda_0 > 0$ so that

$$A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) \geq C\lambda^{n-1} \tag{5.13}$$

for some $C > 0$ and all $\lambda \geq \lambda_0$. On the other hand, we can use the functional calculus for Δ_g to write

$$A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) = \sum_{|\lambda-\lambda_j|\leq K} (1 + \lambda_j^2)^{-|\alpha|} |\partial_x^\alpha \varphi_j(x)|^2.$$

Observe that

$$\Big|\frac{1 + \lambda^2}{1 + \lambda_j^2} - 1\Big| = \frac{|\lambda^2 - \lambda_j^2|}{1 + \lambda_j^2} \leq \frac{K(2\lambda + K)}{1 + (\lambda - K)^2},$$

Since $1 + (\lambda - K)^2 \geq \lambda^2/2$ if $\lambda \geq K/4$, we obtain

$$\Big|\frac{1 + \lambda^2}{1 + \lambda_j^2} - 1\Big| \leq CK\lambda^{-1} + \mathcal{O}_K(\lambda^{-2})$$

as $\lambda \to \infty$. Using binomial expansion, we also obtain

$$\left|\frac{(1+\lambda^2)^{|\alpha|}}{(1+\lambda_j^2)^{|\alpha|}} - 1\right| \le C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})$$

for any α. Therefore,

$$\begin{aligned}\Big|(1+\lambda^2)^{|\alpha|} A\Pi_{[\lambda-K,\lambda+K]}A^*(x,x) - \sum_{|\lambda_j-\lambda|\le K} |\partial_x^\alpha \varphi_j(x)|^2\Big| \\ \le (C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})) \sum_{|\lambda_j-\lambda|\le K} |\partial_x^\alpha \varphi_j(x)|^2. \end{aligned} \tag{5.14}$$

Hence, by (5.13),

$$\begin{aligned} C\lambda^{n-1+2|\alpha|} &\le (1+\lambda^2)^{|\alpha|} A\Pi_{[\lambda-K,\lambda+K]}A^*(x,x) \\ &\le (1 + C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})) \sum_{|\lambda_j-\lambda|\le K} |\partial_x^\alpha \varphi_j(x)|^2 \end{aligned}$$

Since $C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})$ tends to zero as $\lambda \to \infty$ for any fixed $K > 0$, this proves the claim. ■

6. Proof of the main results

In this section we complete the proof of Theorem 2.

6.1. Proof of Theorem 2

Let us recall that our goal is to compute the asymptotic behavior of

$$\frac{1}{N}\sum_{j=0}^{N-1} \Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}+\mathrm{w}]}(x,y). \tag{6.1}$$

Recalling our pseudodifferential cutoffs $B_{\varepsilon,\tau}$ and C_ε as well as the definitions of $\mathcal{A}_{\varepsilon,\tau}$ and $\mathcal{B}_{\varepsilon,\tau}$ in (3.6) and (3.7), respectively, we have shown previously that for any $\sigma > 0$, the smoothed projector satisfies

$$\begin{aligned} &\rho_\sigma * \frac{1}{N}\sum_{j=0}^{N-1} \Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}+\mathrm{w}]}(x,y) \\ &= \frac{1}{N}\rho_\sigma * \sum_{j=0}^{N-1} \Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}+\mathrm{w}]}(C_{\varepsilon,\tau}^* + B_{\varepsilon,\tau}^*)(x,y) \end{aligned}$$

$$= \frac{1}{N}\mathcal{A}_{\varepsilon,\tau}(\nu_\ell,\sigma,x,y) + \frac{1}{N}\mathcal{B}_{\varepsilon,\tau}(\lambda,\sigma,x,y) + \frac{1}{N}\sum_{j=0}^{N-1}\Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}+\mathrm{w}]}B^*_{\varepsilon,\tau}(x,y).$$

By Proposition 3.4, we have that for any multi-indices α, β,

$$\lim_{\sigma\to 0+}\lim_{\tau\to 0+}\lim_{\varepsilon\to 0+}\lim_{\delta\to 0+}\limsup_{\ell\to\infty}\sup_{d_g(x,y)\le\delta}\left|\frac{1}{\nu_\ell^{n-1+|\alpha|+|\beta|}}\partial_x^\alpha\partial_y^\beta R_{\varepsilon,\tau}(\ell,\sigma;x,y)\right| = 0, \tag{6.2}$$

where we recall that $R_{\varepsilon,\tau}$ is given by

$$R_{\varepsilon,\tau}(\ell,\sigma;x,y) := \mathcal{A}_{\varepsilon,\tau}(\nu_\ell,\sigma;x,y) - \frac{2\pi N}{T}\cdot\frac{\nu_\ell^{n-1}}{(2\pi)^n}\int_{S_y^*M} e^{i\nu_\ell\langle \exp_y^{-1}(x),\omega\rangle_g}\frac{d\omega}{\sqrt{|g_y|}}.$$

Additionally, we know by Proposition 4.2 that

$$\lim_{\tau\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0+}\limsup_{\ell\to\infty}\sup_{d_g(x,y)<\delta}\nu_\ell^{1-n-|\alpha|-|\beta|}|\partial_x^\alpha\partial_y^\beta\mathcal{B}_{\varepsilon,\tau}(\nu_\ell,\sigma;x,y)| = 0. \tag{6.3}$$

Finally, we have that

$$\lim_{\varepsilon\to 0}\sup_{x,y\in M}\left|\nu_\ell^{1-n-|\alpha|-|\beta|}\partial_x^\alpha\partial_y^\beta\sum_{j=0}^{N-1}\Pi_{[\nu_{\ell+j}-\mathrm{w},\nu_{\ell+j}+\mathrm{w}]}B^*_{\varepsilon,\tau}(x,y)\right| = 0 \tag{6.4}$$

by Proposition 4.3. Therefore, if we combine (6.2), (6.3), and (6.4), we have that the proof of Theorem 2 reduces to the following lemma.

Lemma 6.1. *Suppose that (M, g) is smooth, compact, Zoll manifold with minimal period T. Then, for any $w < \pi/(2T)$ and each pair of multi-indices α, β, we have*

$$\lim_{\sigma\to 0+}\limsup_{\ell\to\infty}\nu_\ell^{1-n-|\alpha|-|\beta|}\sup_{x,y\in M}|\partial_x^\alpha\partial_y^\beta(\Pi_{[\nu_\ell-\mathrm{w},\nu_\ell+\mathrm{w}]}(x,y) - \rho_\sigma * \Pi_{[\nu_\ell-\mathrm{w},\nu_\ell+\mathrm{w}]}(x,y))| = 0. \tag{6.5}$$

Note Lemma 6.1 this is sufficient to complete the proof because the summation over j in (6.1) is finite. Thus, we proceed to prove (6.5).

Proof. Noting that

$$\mathcal{F}_{\tau\mapsto t}(\mathbb{1}_{[-\mathrm{w},\mathrm{w}]}(\tau)) = \int_{-\mathrm{w}}^{\mathrm{w}} e^{-it\tau}\,d\tau = \frac{2\sin(t\mathrm{w})}{t},$$

we can rewrite

$$\Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}(x,y) - \rho_\sigma * \Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}(x,y) = \sum_{j=0}^{\infty} h_{\mathrm{w},\sigma}(\lambda-\lambda_j)\varphi_j(x)\overline{\varphi_j(y)}, \tag{6.6}$$

for any $\lambda > 0$, where

$$h_{\mathrm{w},\sigma}(\tau) = \mathbb{1}_{[-\mathrm{w},\mathrm{w}]}(\tau) - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{it\tau}\hat{\rho}_\sigma(t)\frac{\sin(t\mathrm{w})}{t}\,dt. \tag{6.7}$$

We claim that $h_{\mathrm{w},\sigma}$ satisfies a bound of the form

$$|h_{\mathrm{w},\sigma}(\tau)| \le C_N\Big(1+\frac{||\tau|-\mathrm{w}|}{\sigma}\Big)^{-N} \quad \text{for any } N\in\mathbb{N}. \tag{6.8}$$

To see this, recall that ρ is a Schwartz-class function with $\int_{\mathbb{R}}\rho\,dt = \hat{\rho}(0) = 1$ and $\rho_\sigma(\tau) = (1/\sigma)\rho(\tau/\sigma)$. Thus,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{it\tau}\hat{\rho}_\sigma(t)\frac{\sin(t\mathrm{w})}{t}\,dt = \int_{-\mathrm{w}}^{\mathrm{w}} \frac{1}{\sigma}\rho\Big(\frac{\tau-\mu}{\sigma}\Big)d\mu = \int_{\tau-\mathrm{w}/\sigma}^{(\tau+\mathrm{w})/\sigma}\rho(\mu)\,d\mu.$$

Suppose $\tau > \mathrm{w}$. Then,

$$\Bigg|\int_{\tau-\mathrm{w}/\sigma}^{(\tau+\mathrm{w})/\sigma}\rho(\mu)\,d\mu\Bigg| \le \int_{\tau-\mathrm{w}/\sigma}^{\infty}|\rho(\mu)|\,d\mu \le C_N\Big(1+\frac{\tau-\mathrm{w}}{\sigma}\Big)^{-N}$$

for any N since ρ is Schwartz. The analogous estimate clearly holds in the case where $\tau < -\mathrm{w}$. If instead $|\tau| < \mathrm{w}$, then since ρ integrates to 1 and is rapidly decaying, along with the fact that $\mathbb{1}_{[-\mathrm{w},\mathrm{w}]}$ is identically one on $[-\mathrm{w},\mathrm{w}]$, we have that

$$\begin{aligned}|h_{\mathrm{w},\sigma}(\tau)| &= \Bigg|\mathbb{1}_{[-\mathrm{w},\mathrm{w}]}(\tau) - \int_{\tau-\mathrm{w}/\sigma}^{(\tau+\mathrm{w})/\sigma}\rho(\mu)\,d\mu\Bigg| \\ &\le \int_{-\infty}^{(\tau-\mathrm{w})/\sigma}|\rho(\mu)|\,d\mu + \int_{\tau+\mathrm{w}/\sigma}^{\infty}|\rho(\mu)|\,d\mu \le C_N\Big(1+\frac{||\tau|-\mathrm{w}|}{\sigma}\Big)^{-N}\end{aligned}$$

for any N. Finally, in the case where $|\tau| = \mathrm{w}$, (6.8) only claims that $h_{\mathrm{w},\sigma}(\tau)$ is uniformly bounded in w, σ, which follows immediately from the fact that

$$|h_{\mathrm{w},\sigma}(\mathrm{w})| = \Bigg|1 - \int_0^{2\mathrm{w}/\sigma}\rho(\mu)\,d\mu\Bigg| \le 1$$

along with the analogous statement for $\tau = -\mathrm{w}$. Therefore, we have proved (6.8).

Observe that by (6.6) and (6.7) we have

$$|\partial_x^\alpha \partial_y^\beta (\Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}(x,y) - \rho_\sigma * \Pi_{[\lambda-\mathrm{w},\lambda+\mathrm{w}]}(x,y))| \le \Big(\sum_{j=0}^\infty |h_{\mathrm{w},\sigma}(\lambda-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2\Big)^{1/2}\Big(\sum_{j=0}^\infty |h_{\mathrm{w},\sigma}(\lambda-\lambda_j)||\partial_y^\beta \varphi_j(y)|^2\Big)^{1/2}.$$

Thus, the claim in (6.5) would follow once we prove that given $\alpha \in \mathbb{N}$, setting $\lambda = \nu_\ell$ gives

$$\lim_{\sigma\to 0^+} \limsup_{\ell\to\infty} \frac{1}{\nu_\ell^{n-1+2|\alpha|}} \sum_{j=0}^\infty |h_{\mathrm{w},\sigma}(\nu_\ell-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 = 0. \tag{6.9}$$

For each ℓ, decompose $\mathbb{N} = J_1(\ell) \cup J_2(\ell) \cup J_3(\ell)$ with

$$J_1(\ell) := \Big\{j : |\lambda_j - \nu_\ell| > \frac{\pi}{T}\Big\},$$
$$J_2(\ell) := \{j : |\lambda_j - \nu_\ell| < r\ell^{-1/2}\},$$
$$J_3(\ell) := \Big\{j : r\ell^{-1/2} < |\lambda_j - \nu_\ell| \le \frac{\pi}{T}\Big\}.$$

Note that

$$\sum_{j\in J_1(\ell)} |h_{\mathrm{w},\sigma}(\nu_\ell-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 = \sum_{m=1}^\infty \sum_{|\lambda_j-\nu_\ell|\in[m\pi/T,(m+1)\pi/T]} |h_{\mathrm{w},\sigma}(\nu_\ell-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2. \tag{6.10}$$

Whenever $|\lambda_j - \nu_\ell| \in [m\pi/T, (m+1)\pi/T]$ with $m \ge 1$ and $\mathrm{w} < \pi/(2T)$, we have that

$$|h_{\mathrm{w},\sigma}(\nu_\ell-\lambda_j)| \le C_N\Big(1+\frac{1}{\sigma}\Big|\frac{m\pi}{T}-\mathrm{w}\Big|\Big)^{-N} \le C_N'\Big(\frac{m}{\sigma}\Big)^{-N}$$

for some $C_N' > 0$ by (6.8). For the same range of λ_j, we also have that

$$\sum_{|\lambda_j-\nu_\ell|\in[m\pi/T,(m+1)\pi/T]} |\partial_x^\alpha \varphi_j(x)|^2 \le C\Big(1+\nu_\ell+\frac{m\pi}{T}\Big)^{n-1+2|\alpha|}$$

for some $C, C' > 0$ by the local Weyl law (5.12). Therefore, by (6.10),

$$\sum_{j\in J_1(\ell)} |h_{\mathrm{w},\sigma}(\nu_\ell-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \le \widetilde{C}_N \sigma^N \sum_{m=1}^\infty \Big(1+\nu_\ell+\frac{m\pi}{T}\Big)^{n-1+2|\alpha|} m^{-N}$$

for some $\widetilde{C}_N > 0$. Taking any $N \ge n+1+2\alpha$, we thus obtain

$$\sum_{j\in J_1(\ell)} |h_{\mathrm{w},\sigma}(\nu_\ell-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \le C_1 \sigma^N \nu_\ell^{n-1+2|\alpha|} \tag{6.11}$$

for some $C_1 > 0$ and any $\sigma > 0$ small.

Next, to estimate the sum over $J_2(\ell)$, we note that for each fixed $r, \mathrm{w} > 0$, one can take ℓ sufficiently large so that $|r\ell^{-1/2} - \mathrm{w}| \geq \mathrm{w}/2$, in which case by (6.8) that

$$|h_{\mathrm{w},\sigma}(\nu_\ell - \lambda_j)| \leq C_N\Big(1 + \frac{\mathrm{w}}{\sigma}\Big)^{-N} \leq C_N\Big(\frac{\sigma}{\mathrm{w}}\Big)^N$$

for $|\nu_\ell - \lambda_j| \leq r\ell^{-1/2}$. By the local Weyl law, we have

$$\sum_{j\in J_2(\ell)} |h_{\mathrm{w},\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leq C_2\Big(\frac{\sigma}{\mathrm{w}}\Big)^N \nu_\ell^{n-1+2|\alpha|} \tag{6.12}$$

for some $C_2 > 0$ and all ℓ sufficiently large.

Finally, to estimate the sum over $J_3(\ell)$ we apply Proposition 5.1, which implies that there exist $K > 0$ and $\ell_0 > 0$ such that for all $\ell \geq \ell_0$

$$\sum_{j\in J_3(\ell)} |\partial_x^\alpha \varphi_j(x)|^2 \leq Cr^{-2} \sum_{|\lambda_j - \nu_\ell| \leq K} |\partial_x^\alpha \varphi_j(x)|^2 \leq C' r^{-2} \nu_\ell^{n-1+2|\alpha|},$$

where the final inequality follows from the local Weyl law (5.12). Therefore, since $h_{\mathrm{w},\sigma}$ is bounded by a uniform constant for all $\mathrm{w}, \sigma > 0$, we have

$$\sum_{j\in J_3(\ell)} |h_{\mathrm{w},\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leq C_3 r^{-2} \nu_\ell^{n-1+2|\alpha|} \tag{6.13}$$

for some $C_3 > 0$, all $r > 0$, and all ℓ sufficiently large. Combining (6.11), (6.12), and (6.13),

$$\lim_{\ell\to\infty} \frac{1}{\nu_\ell^{n-1+2|\alpha|}} \sum_{j=0}^{\infty} |h_{\mathrm{w},\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2$$
$$\leq C_1 \sigma^N + C_2\Big(\frac{\sigma}{\mathrm{w}}\Big)^N + C_3 r^{-2}$$

for all $\mathrm{w} < \pi/(2T)$ and all $\sigma, r > 0$. Recalling that $\mathrm{w} > 0$ was fixed in the statement of the proposition, we may send $\sigma \to 0$ and $r \to \infty$ to obtain (6.9), which completes the proof. ∎

7. Proof of Theorem 1

The proof of Theorem 1 follows the same steps as Theorem 2, but is somewhat simpler since the structure of trajectories with period smaller than T is simpler than that of subperiodic loops. In fact, if $\rho \in S^*M$ is periodic with some minimal period $t < T$. Then, $t = T/N$. Since $t > \mathrm{inj}(M)$, this implies $N < T/N$.

We start by integrating (3.5) with respect to x with $C_{\varepsilon,\tau}$ replaced by the identity

$$\begin{aligned}\sum_{j=0}^{N-1}\int \rho_\sigma * \Pi_{[\lambda+2\pi j/T-\mathrm{w},\lambda+2\pi j/T+\mathrm{w}]}(x,x)dx\\ &= \frac{1}{2\pi}\int\int_{-\infty}^{\infty} e^{it(\lambda+(N-1)\pi/T)}\frac{\sin\left(\frac{\pi N t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}\hat{\psi}_\sigma(t)U_t(x,x)\,dt\,dx\\ &= \mathcal{A}(\lambda,\sigma)+\mathcal{B}(\lambda,\sigma),\end{aligned}$$

where (similar to (3.11))

$$\mathcal{A}(\lambda,\sigma)=\sum_{k\in\mathbb{Z}} e^{ikT(\lambda-\mathfrak{b})}\int \partial_\lambda(f_k * \Pi_{[0,\lambda]}(I+Q_k))(x,x)\,dx$$

and (similar to (4.4))

$$\mathcal{B}(\lambda,\sigma)=\sum_{k\in\mathbb{Z}}\frac{e^{ikT(\lambda-\mathfrak{b})}}{2\pi}\int \mathcal{F}^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}(U_t+Q_k(t))(x,x)]\,dx,$$

where $g_{k,N}$ is given in (4.3).

The term $\mathcal{A}(\lambda,\sigma)$ is analyzed by integrating (3.24) and using the estimate from Proposition 3.4 with $x=y$ and $C_\varepsilon=I$ to obtain

$$\lim_{\sigma\to 0^+}\limsup_{\ell\to\infty}\Big(\mathcal{A}(\nu_\ell,\sigma)-\frac{2\pi N}{T}\cdot\frac{\nu_\ell^{n-1}}{(2\pi)^n}\operatorname{vol}(\mathbb{S}^{n-1})\operatorname{vol}_g(M)\Big)=0.$$

To handle $\mathcal{B}(\lambda,\sigma)$, we proceed as in Proposition 4.2, but the analysis is considerably simpler. Let $\chi_{\delta'}\in C^\infty(M\times\omega)$ be such that $|\nabla_{x,\omega}\varphi(x,x,t,\omega)|<\delta'$ for all $(x,\omega)\in\operatorname{supp}\chi_{\delta'}$. We then arrive at the next formula by following the analysis that led to (4.7), we obtain that $\int\mathcal{F}^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}U_t(x,x)]\,dx$ can be expressed as a locally finite sum of terms of the form

$$\begin{aligned}\frac{\lambda^n}{(2\pi)^n}\int\int_{-\infty}^{\infty}\int_0^{\infty}\int_{S^{n-1}} e^{i\lambda(t+r\varphi(x,y,t,\omega))}\hat{g}_{k,N}(t)a^0(x,y,t,\omega)r^{n-1}\,dr\,dt\,d\omega\,dx\\ &= \frac{\lambda^n}{(2\pi)^n}\int\int_{-\infty}^{\infty}\int_0^{\infty}\int_{S^{n-1}} e^{i\lambda(t+r\varphi(x,x,t,\omega))}\chi_{\delta'}(x,\omega)\hat{g}_{k,N}(t)\\ &\qquad\times a^0(x,x,t,\omega)r^{n-1}\,dr\,dt\,d\omega\,dx\\ &\quad+\mathcal{O}(\lambda^{-\infty}).\end{aligned}\tag{7.1}$$

Then, performing stationary phase in (r,t), we arrive at the analog of (4.8). Then, the expression in (7.1) equals

$$\frac{\lambda^{n-1}}{(2\pi)^n}\int_{S^{n-1}}\sum \frac{1}{|\partial_t\varphi|}e^{i\lambda t_c+i\frac{\pi}{4}|\operatorname{sgn}\operatorname{Hess}\tilde{\varphi}|}\chi_{\delta'}(x,\omega)\hat{g}_{k,N}(t_c)a^0(x,x,t_c,\omega)r_c^{n-1}\,d\omega\,dx$$
$$+\,\mathcal{O}_{\delta'}(\lambda^{n-2}),$$

where the sum is taken over all critical points $(t_c(x,\omega),r_c(x,\omega))$. Since

$$\left|t_c-\left(\frac{p}{q}\right)T\right|<\mathrm{w}$$

for some $0<p<q\leq N$, we obtain

$$\int \mathcal{F}^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}U_t(x,x)]dx=o(\lambda^{n-1}).$$

The identical argument, but with U_t replaced by U_tQ_k shows that

$$\int \mathcal{F}^{-1}_{t\mapsto\lambda}[\hat{g}_{k,N}U_tQ_k(x,x)]\,dx=O_k(\lambda^{n-2}).$$

The proof of Theorem 1 is now completed by Lemma 6.1 which implies

$$\lim_{\sigma\to0+}\limsup_{\ell\to\infty}\nu_\ell^{1-n}|\Pi_{[\nu_\ell-\mathrm{w},\nu_\ell+\mathrm{w}]}(x,x)-\rho_\sigma*\Pi_{[\nu_\ell-\mathrm{w},\nu_\ell+\mathrm{w}]}(x,x)|=0.$$

8. Assumption (1.10) on real analytic manifolds

We now assume that (M,g) is a *real analytic* Zoll manifold and show that (1.10) holds for all $x_0\in(M,g)$. The goal of this section is to prove the following result.

Proposition 8.1. *Suppose that (M,g) is a real analytic Zoll manifold with minimal common period T. Then, there is $N>0$ such that for all $x_0\in M$ and $\tau>0$,*

$$\mu_{S^*M}(\mathcal{L}_{N,\tau}(x_0))=0.$$

We start by showing that on a real analytic manifold (Zoll or not) having a positive measure of set of loops at x_0 implies that all geodesics through x_0 loop at some fixed time. We follow the analogous proof in [34, Theorem 5.1].

Lemma 8.2. *Let (M,g) be real analytic. Then, for all $0\leq t_0<t_1$, if*

$$\mu_{S^*_{x_0}M}(\Gamma_{t_1})>0,$$
$$\Gamma_{t_0,t_2}:=\{\rho\in S^*_{x_0}M : \text{there exists } t\in(t_0,t_1) \text{ such that } \pi_M(\varphi_t(\rho))=x_0\},$$

then there is $t_0 < s < t_1$ such that

$$\pi_M(\varphi_s(\rho)) = x_0, \quad \textit{for all } \rho \in S^*_{x_0}M. \tag{8.1}$$

Proof. First, observe that

$$\widetilde{\Gamma}_{t_0,t_1} := \{\xi \in T^*_{x_0}M : t_0 < |\xi|_g < t_1 \text{ such that } \pi_M(\varphi_1(x_0,\xi)) = x_0\},$$

is an intersection of two subanalytic sets, in fact, it is an intersection of the zero set of an analytic function with the subanalytic set $\{\xi \in T^*_{x_0}M : t_0 < |\xi|_g < t_1\}$. In particular, $\widetilde{\Gamma}_{t_0,t_1}$ is stratified and hence contains an embedded open submanifold $Y_{t_0,t_1} \subset \widetilde{\Gamma}_{t_0,t_1}$ of maximal dimension. The arguments in [34] show that $\dim(Y_{t_1}) \leq n-1$ and that $\widetilde{\Gamma}_{t_0,t_1}$ has the same dimension as its radial projection to $S^*_{x_0}M$ which we identify with Γ_{t_0,t_1}. Therefore, if Γ_{t_0,t_1} has positive measure, we conclude $\dim(Y_{t_0,t_1}) = n-1$.

Next, consider the collection of rays

$$C_{t_0,t_1} := \bigcup_{\xi \in Y_{t_0,t_1}} \{t\xi : 0 \leq t \leq 1\}.$$

Then, each ray in C_{t_0,t_1} exponentiates to a loop that returns at $t = 1$ and hence since return times must be constant on open sets (see e.g. [34, Proposition 4.2]), we have $|\xi| = s$ on Y_{t_0,t_1}. Using again that (M, g) is real analytic, we have $\pi_M(\varphi_1(x_0,\xi))$ is constant on $|\xi|_{g(x_0)} = s$ from which the claim (8.1) follows. ■

Next, we show that any common looping time at a point x_0 on a Zoll manifold must be a rational multiple of the minimal common period.

Lemma 8.3. *Suppose that (M, g) is a Zoll manifold with minimal common period T. Then for all $x_0 \in M$ and $0 < t_0 < T$ such that*

$$\pi_M(\varphi_{t_0}(\rho)) = x_0, \quad \textit{for all } \rho \in S^*_{x_0}M.$$

there are $0 < p < q \in \mathbb{Z}_+$ such that $t_0 = (p/q)T$ and $q < T/\operatorname{inj}(M)$.

Proof. We first show that t_0 is a rational multiple of T.

Suppose by contradiction t_0 is not a rational multiple of T. Then, there are $p_n, q_n \in \mathbb{Z}_+$ such that $q_n \to \infty$ and

$$0 < \left|\frac{p_n T}{q_n} - t_0\right| < \frac{T}{q_n^2} \tag{8.2}$$

Then, observe that

$$\pi_M(\varphi_{q_n t_0}(\rho)) = x_0, \quad \pi_M(\varphi_{p_n T}(\rho)) = x_0, \quad \text{for all } \rho \in S^*_{x_0}M.$$

In particular,

$$\pi_M(\varphi_{q_n t_0 - p_n T}(\rho)) = x_0, \quad \text{for all } \rho \in S^*_{x_0}M$$

and hence $|q_n t_0 - p_n T| > \text{inj}(M)$. This contradicts (8.2).

Now, suppose $t_0 = pT/q$ with $\gcd(p,q) = 1$. Then, there are $n, k \in \mathbb{Z}$ such that $np = kq + 1$. Hence,

$$\varphi_{nt_0}(\rho) = \varphi_{nt_0} \circ \varphi_{-kT}(\rho) = \varphi_{nt_0 - kT}(\rho) = \varphi_{T/q}(\rho).$$

In particular, since $\pi_M(\varphi_{nt_0}(\rho)) = x_0$, we have $T/q > \text{inj}(M)$ which completes the proof. ■

Next, we show that, apart from a finite set of geodesics lengths, the geodesic loops through a point x_0 with length strictly between 0 and T have zero measure.

Lemma 8.4. *There exist* $0 < N$ *and* $0 = r_0 < r_1 < \cdots < r_N = T$ *such that*

$$\pi_M(\varphi_{r_n}(\rho)) = x_0, \quad \textit{for all } \rho \in S^*_{x_0}M \tag{8.3}$$

and

$$\mu_{S^*_{x_0}M}(\{\rho \in S^*_{x_0}M : \text{there exist } 0 < t < T \text{ such that } t \notin \{r_0, \dots, r_N\}, \pi_M(\varphi_t(\rho)) = x_0\}) = 0. \tag{8.4}$$

Proof. Let $s_1 := \inf\{0 < t < T : \mu_{S^*_{x_0}M}(\Gamma_{0,t}) > 0\}$. Note that either $s_1 = \infty$ in which case we set $N = 1$ and observe that (8.4) holds.

If $s_1 < \infty$, set $r_1 = s_1$. Then, $\text{inj}(M) \le s_1 < T$ and we claim that

$$\pi_M(\varphi_{s_1}(\rho)) = x_0, \quad \text{for all } \rho \in S^*_{x_0}M.$$

Suppose by contradiction that

$$\text{there exists } \rho \in S^*_{x_0}M \text{ such that } \pi_M(\varphi_{s_1}(\rho)) \neq x_0. \tag{8.5}$$

Then there are $t_n \downarrow s_1$ such that $\mu_{S^*_{x_0}M}(\Gamma_{t_n}) > 0$ and hence, by Lemma 8.2, there are $\text{inj}(M) < t_n' < t_n$ such that

$$\pi_M(\varphi_{t_n'}(\rho)) = x_0, \quad \text{for all } \rho \in S^*_{x_0}M.$$

In particular, $t_n' \to t \in [\text{inj}(M), s_1]$ and hence, by continuity of φ,

$$\pi_M(\varphi_t(\rho)) = x_0, \quad \text{for all } \rho \in S^*_{x_0}M.$$

By (8.5), this implies $\text{inj}(M) < t < s_1$ which contradicts the definition of s_1. Therefore, (8.3) holds for r_1.

Suppose by induction that we have found $r_1 < \cdots < r_{J-1}$ such that (8.3) holds for $n = 1, \dots, J-1$ and

$$\mu_{S^*_{x_0}M}(\{\rho \in S^*_{x_0}M : \text{there exist } 0 < t < r_{J-1} \\ \text{such that } t \notin \{r_0, \dots, r_{J-2}\}, \pi_M(\varphi_t(\rho)) = x_0\}) = 0.$$

Now, define for

$$s_J := \inf\{t > r_{J-1} : \mu_{S^*_{x_0}M}(\Gamma_{r_{J-1},t}) > 0\}.$$

If $s_J = \infty$ then (8.4), holds with $N = J-1$. If not, then $T > s_J > r_{J-1} + \text{inj}(M)$ and we set $r_J = s_J$. The same argument as above then yields (8.1) for $n = J$.

Since $J \,\text{inj}(M) < r_J < T$, this process terminates after finitely many steps and the proof is complete. ∎

Finally, we combine all the lemmas above to prove Proposition 8.1.

Proof of Proposition 8.1. Combining Lemmas 8.3 and 8.4, there is $L > 0$ and $0 = r_0 < r_1 < \cdots < r_L = T$ such that

$$r_j = \frac{p_j}{q_j} T$$

for some $0 < p_j < q_j < T/\text{inj}(M)$, $p_j, q_j \in \mathbb{Z}_+$ and (8.4) holds. Letting N be the least common multiple of $1, 2, \dots, T/\text{inj}(M)$, the proposition follows. ∎

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Yaiza Canzani
Department of Mathematics, University of North Carolina at Chapel Hill, 250 Phillips Hall #3, Chapel Hill, NC 27599, USA; canzani@email.unc.edu

Jeffrey Galkowski
Department of Mathematics, University College London, 25 Gordon Street, London WC1H 0AY, UK; j.galkowski@ucl.ac.uk

Blake Keeler
Department of Mathematical Sciences, Montana Technological University, 1300 West Park Street, Butte, MT 59701, USA; bkeeler@mtech.edu