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Several isoperimetric inequalities of Dirichlet and Neumann eigenvalues of the Witten Laplacian

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Abstract. In this paper, by mainly using the rearrangement technique and suitably constructing trial functions, under the constraint of fixed weighted volume, we successfully obtain several isoperimetric inequalities for the first and the second Dirichlet eigenvalues, the first non-zero Neumann eigenvalue of the Witten Laplacian on bounded domains in space forms. These spectral isoperimetric inequalities extend the classical ones (i.e., the Faber–Krahn inequality, the Hong–Krahn–Szegő inequality, and the Szegő–Weinberger inequality) of the Laplacian.

1. Introduction

The study of extremum problems for prescribed functionals is of great significance in mathematics. For instance, a well-known isoperimetric problem, familiar to nearly all mathematicians, in the n-dimensional ($n \geq 2$) Euclidean space $\mathbb{R}^n$ involves the study of the following extremum problem:

$$\min\{|\partial\Omega|_{n-1} \mid |\Omega|_n = \text{const.}\} \tag{1.1}$$

for bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Here, with a slight abuse of notation, $|\cdot|$ denotes the Hausdorff measure of a given geometric object, with dimension information included as a subscript when necessary. This extremum problem can also be formulated in an alternative way.

> *Among all bounded domains in $\mathbb{R}^n$ with fixed volume, which one minimises the area functional of the boundary?*

This classical problem has been answered completely and one knows that the unique minimiser of the area functional should be a ball with the volume equal to $|\Omega|_n =$ const. – see, e.g., [35, Chapter 1] for an interesting derivation of classical isoperimetric

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inequalities in Euclidean space by using the Schwarz symmetrization. In fact, for any bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, one has

$$|\partial\Omega|^n/|\Omega|^{n-1} \geq |\mathbb{S}^{n-1}|^n/|\mathbb{B}^n|^{n-1}, \tag{1.2}$$

with equality holding if and only if Ω is a Euclidean ball. Obviously, the right-hand side of (1.2) is independent of the choice of radius for the Euclidean n-ball $\mathbb{B}^n$ and the corresponding Euclidean $(n-1)$-sphere $\mathbb{S}^{n-1}$. That is to say, the quantity $|\mathbb{S}^{n-1}|^n/|\mathbb{B}^n|^{n-1}$ is scale invariant. So, for convenience and simplification, we denote by $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ the unit Euclidean n-ball and the unit Euclidean $(n-1)$-sphere, respectively. By (1.2), one easily sees that

- among all bounded domains in $\mathbb{R}^n$ having the same volume, Euclidean balls minimise the boundary area;
- among all bounded domains in $\mathbb{R}^n$ having the same boundary area, Euclidean balls maximise the volume.

Clearly, (1.2) gives the answer to the problem (1.1) completely – for a ball B_Ω with $|B_\Omega|_n = |\Omega|_n =$ const., it follows that[1]

$$|\partial\Omega|_{n-1} \geq |\partial B_\Omega|_{n-1}, \tag{1.3}$$

with equality holding if and only if Ω is a ball in $\mathbb{R}^n$ (which is congruent with B_Ω). Following the convention in [12], we wish to call (1.2)–(1.3) the *geometric isoperimetric inequalities*.

The purpose of this paper is to investigate isoperimetric inequalities from the viewpoint of spectral quantities of the Witten Laplacian. In order to state our conclusions clearly, we wish to first recall several classical results on the Laplacian.

Let $(M^n, \langle\cdot,\cdot\rangle)$ be an n-dimensional $(n \geq 2)$ complete Riemannian manifold with the metric $g := \langle\cdot,\cdot\rangle$. Let $\Omega \subset M^n$ be a bounded domain in M^n with smooth[2] boundary $\partial\Omega$. Denote by Δ and ∇ the Laplace and the gradient operators on M^n associated

[1] Clearly, ∂B_Ω stands for the boundary sphere of the ball B_Ω.

[2] The smoothness assumption for the regularity of the boundary $\partial\Omega$ is strong enough to consider the eigenvalue problems (1.4) and (1.8). For instance, a weaker regularity assumption that $\partial\Omega$ is Lipschitz continuous can also assure the validity about the description of discrete spectrum of the Neumann eigenvalue problem (1.8) of the Laplacian on the fourth page of this paper. However, the Lipschitz continuous assumption might not be enough to consider some other geometric problems involved Neumann eigenvalues of (1.8). Therefore, to avoid excessive focus on the regularity of the boundary $\partial\Omega$ – which is not central to the topic of this paper – we assume, unless otherwise stated, that $\partial\Omega$ is smooth. This framework suggests that certain conclusions of this paper may remain valid even under a weaker regularity assumption for the boundary $\partial\Omega$. Readers who are interested in this situation could try to seek the weakest regularity.

with the metric g, respectively. On Ω, one can consider the following Dirichlet eigenvalue problem of the Laplacian:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \subset M^n, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

which is also known as the *fixed membrane problem* of the Laplacian. In fact, for the eigenvalue problem (1.4), when M^n is chosen to be $\mathbb{R}^3$, this system can be used to describe the vibration of a membrane with boundary fixed, and this is the reason why it is called *fixed membrane problem.* Because of this physical background, eigenvalues of a prescribed eigenvalue problem of some self-adjoint differentiable elliptic operator are called *frequencies*. It is well known that the operator $-\Delta$ in (1.4) only has a discrete spectrum and all the elements (i.e., eigenvalues) can be listed non-decreasingly as follows:

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \uparrow \infty. \tag{1.5}$$

For each eigenvalue $\lambda_i(\Omega)$, $i = 1, 2, \ldots$, all the non-trivial functions satisfying (1.4) form a vector space, which has finite dimension and is called *eigenspace* of $\lambda_i(\Omega)$. Moreover, all the elements in this eigenspace are called *eigenfunctions belonging to* $\lambda_i(\Omega)$. The dimension of this eigenspace is called *multiplicity* of the eigenvalue $\lambda_i(\Omega)$. Each eigenvalue $\lambda_i(\Omega)$ in the sequence (1.5) is repeated according to its multiplicity. By variational principle, the k-th Dirichlet eigenvalue $\lambda_k(\Omega)$ is characterised as follows:

$$\lambda_k(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 dv}{\int_\Omega f^2 dv} \;\middle|\; f \in W_0^{1,2}(\Omega),\, f \neq 0, \int_\Omega f f_i dv = 0 \right\},$$

where dv denotes the Riemannian volume element of M^n, and $f_i, i = 1, 2, \ldots, k-1$, denotes an eigenfunction of $\lambda_i(\Omega)$. Here, as usual, $W_0^{1,2}(\Omega)$ stands for a Sobolev space, which is the completion of the set of smooth functions (with compact support) $C_0^\infty(\Omega)$ under the following Sobolev norm:

$$\|f\|_{1,2} := \left(\int_\Omega f^2 dv + \int_\Omega |\nabla f|^2 dv \right)^{1/2}. \tag{1.6}$$

See, e.g., [12] for the above fundamental facts of the eigenvalue problem (1.4). Moreover, for simplicity and without causing confusion, we will henceforth write λ_i instead of $\lambda_i(\Omega)$, except where otherwise specified. This convention will also apply when encountering other potential eigenvalue problems.

Similar to (1.1), for bounded domains $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $n \geq 2$, it is interesting and important to consider the following extremum problem:

$$\min\{\lambda_k(\Omega) \mid |\Omega|_n = \text{const.}\} \tag{1.7}$$

for each $k = 1, 2, 3, \ldots$. In fact, (1.7) is a natural and classical isoperimetric problem in the study of spectral geometry. To the best of our knowledge, for $k = 1, 2$, one has the following affirmative answers to problem (1.7).

- (Faber–Krahn inequality, [17, 23]) $\lambda_1(\Omega) \geq \lambda_1(B_\Omega)$, and the equality holds if and only if Ω is a ball in $\mathbb{R}^n$ (which is congruent with B_Ω, $|B_\Omega|_n = |\Omega|_n =$ const.). That is to say, among all bounded domains in $\mathbb{R}^n$ having the same volume, Euclidean balls minimise the first Dirichlet eigenvalue of the Laplacian.
- (Hong–Krahn–Szegő inequality, [21,24]) $\lambda_2(\Omega) \geq \lambda_1(\widetilde{B}_\Omega)$, where $\widetilde{B}_\Omega$ is a ball in $\mathbb{R}^n$ such that $2|\widetilde{B}_\Omega|_n = \text{const.} = |\Omega|_n$. Moreover, the minimum of the second Dirichlet eigenvalue of the Laplacian on bounded domains Ω (whose volume equals some prescribed positive constant) should be equal to the first Dirichlet eigenvalue of the Laplacian on a ball $\widetilde{B}_\Omega$ with $|\widetilde{B}_\Omega|_n = |\Omega|_n/2$.

The Hong–Krahn–Szegő inequality implies that under the constraint that the volume of bounded domains is fixed, the second Dirichlet eigenvalue (of the Laplacian) is minimised by two balls of the same volume. However, if one additionally requires that Ω is connected, then under the constraint of volume fixed ($|\Omega|_n = \text{const.}$), this minimiser of $\lambda_2(\Omega)$ cannot be attained but can be approximated by the domain Ω_ε, obtained by joining the union of the two congruent balls (whose volumes equal $|\Omega|_n/2$) by a thin pipe of width ε (sufficiently small) – see [20] for the precise description of this interesting example and see, e.g., [9, 10] for the strict proof of this approximation (as $\varepsilon \to 0$). In two-dimensional case, it has long been conjectured that the ball minimises $\lambda_3(\Omega)$, but there did not have much progress in this direction. For higher order Dirichlet eigenvalues, not much is known. However, there is an interesting result we wish to mention, that is, Berger [3] proved that for planar bounded domain $\Omega \subset \mathbb{R}^2$, the i-th ($i > 4$) Dirichlet eigenvalue $\lambda_i(\Omega)$ is not minimised by any union of disks.

For a bounded domain Ω (with smooth boundary) on a given complete Riemannian n-manifold M^n, one can also consider the Neumann eigenvalue problem of the Laplacian as follows:

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega \subset M^n, \\ \dfrac{\partial u}{\partial \vec{v}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

which is also known as the *free membrane problem* of the Laplacian. Here, $\vec{v}$ stands for the outward unit normal vector of the boundary $\partial\Omega$. In fact, for the eigenvalue problem (1.8), when M^n is chosen to be $\mathbb{R}^3$, this system can be used to describe the vibration of a membrane with free boundary, and this is the reason why it is called *free membrane problem.* It is well known that the operator $-\Delta$ in (1.8) only has a discrete spectrum and all the eigenvalues can be listed non-decreasingly as follows:

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \uparrow \infty. \tag{1.9}$$

The eigenvalue $\mu_0(\Omega) = 0$ has non-zero constant functions as its eigenfunctions. Each eigenvalue $\mu_i(\Omega)$ in the sequence (1.9) is repeated according to its multiplicity (which is finite and actually equals the dimension of $\mu_i(\Omega)$'s eigenspace). By variational principle, the k-th non-zero Neumann eigenvalue $\mu_k(\Omega)$ is characterised as follows:

$$\mu_k(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 dv}{\int_\Omega f^2 dv} \,\middle|\, f \in W^{1,2}(\Omega),\, f \neq 0,\, \int\limits_\Omega f f_i dv = 0 \right\},$$

where f_i, $i = 0, 1, \ldots, k-1$, denotes an eigenfunction of $\mu_i(\Omega)$. Here, as usual, $W^{1,2}(\Omega)$ denotes a Sobolev space which is the completion of the set of smooth functions $C^\infty(\Omega)$ under the Sobolev norm $\|\cdot\|_{1,2}$ defined by (1.6).

Similar to (1.7), for bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geq 2$, the extremum problem

$$\max\{\mu_k(\Omega) \mid |\Omega|_n = \text{const.}\} \tag{1.10}$$

can be asked for each $k = 1, 2, 3, \ldots$. To the best of our knowledge, for $k = 1, 2$, one has the following affirmative answers to the problem (1.10).

- (Szegő–Weinberger inequality, [38,39]) $\mu_1(\Omega) \leq \mu_1(B_\Omega)$, and the equality holds if and only if Ω is a ball in $\mathbb{R}^n$ (which is congruent with B_Ω, $|B_\Omega|_n = |\Omega|_n =$ const.). That is to say, among all bounded domains in $\mathbb{R}^n$ having the same volume, Euclidean balls maximise the first non-zero Neumann eigenvalue of the Laplacian.
- (Bucur and Henrot [8]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that the Sobolev space $W^{1,2}(\Omega)$ is compactly embedded[3] in $L^2(\Omega)$. Then,

$$|\Omega|_n^{2/n} \mu_2(\Omega) \leq 2^{2/n} |B|_n^{2/n} \mu_1(B), \tag{1.11}$$

  where B is any ball in $\mathbb{R}^n$. If equality in (1.11) occurs, then Ω coincides a.e. with the union of two disjoint, equal balls. Clearly, the quantity $2^{2/n}|B|^{2/n}\mu_1(B)$ is scale invariant. Using (1.11) directly, one has $\mu_2(\Omega) \leq 2^{2/n}\mu_1(B_\Omega)$, with a ball B_Ω satisfying $|B_\Omega|_n = |\Omega|_n =$ const., which gives an affirmative answer to the problem (1.10) for $k = 2$.

[3] In fact, the regularity that $\partial\Omega$ is Lipschitz continuous is sufficient such that $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Therefore, the smoothness assumption for the boundary $\partial\Omega$ is much enough to investigate the maximum of $\mu_2(\Omega)$ under the constraint of fixed volume.

For higher order ($k \geq 3$) Neumann eigenvalues, not much is known. However, recent years, some works have shown numerical approaches which propose candidates for the optimisers for Dirichlet/Neumann eigenvalues of the Laplacian and related spectral problems, and which also suggest conjectures about their qualitative properties – see, e.g., [1, 5, 34] for details.

As mentioned above, in some situation, the eigenvalue problems (1.4) and (1.8) have physical backgrounds, and hence eigenvalues in discrete spectrum are called *frequencies*. So, sometimes, spectral isoperimetric inequalities introduced above are also called *physical isoperimetric inequalities*. There is also one more thing we wish to say here, that is, spectral isoperimetric inequalities mentioned above hold may not only in Euclidean spaces but also some curved spaces – for instance, at least one also has the Faber–Krahn inequality in hyperbolic spaces and spheres. In fact, a more general version of Faber–Krahn inequality says the following (see, e.g., [12, Chapter IV]).

Let $\mathbb{M}^n(\kappa)$ be the complete, simply connected, n-dimensional ($n \geq 2$) space form of constant sectional curvature κ, and let $\mathbb{D}$ denote a geodesic disk in $\mathbb{M}^n(\kappa)$. For a complete Riemannian n-manifold M^n, $n \geq 2$, and each open set Ω, consisting of a finite disjoint union of regular[4] domains in M^n, and satisfying

$$|\Omega|_n = |\mathbb{D}|_n. \tag{1.12}$$

(If $\kappa > 0$, then only consider those Ω for which $|\Omega|_n < |\mathbb{M}^n(\kappa)|_n$.) If, for all such Ω in M^n, equality (1.12) implies the geometric isoperimetric inequality

$$|\partial\Omega|_{n-1} \geq |\partial\mathbb{D}|_{n-1}, \tag{1.13}$$

with equality in (1.13) if and only if Ω is isometric to $\mathbb{D}$, then we also have, for every normal domain Ω in M^n, that equality (1.12) implies the inequality

$$\lambda_1(\Omega) \geq \lambda_1(\mathbb{D}), \tag{1.14}$$

with equality in (1.14) if and only if Ω is isometric to $\mathbb{D}$.

This fact can be simply summarised as *under the constraint of volume fixed, the geometric isoperimetric inequality* (1.13) *would imply the physical isoperimetric inequality* (1.14). It is known that, in space forms, (1.13) holds once $|\Omega|_n = |\mathbb{D}|_n$. Hence, in space forms, one has the physical isoperimetric inequality (1.14) under the volume constraint (1.12). From this example, one might have a recognition that geometric

[4]Here, following the convention in [12], *regular* means that the domain considered has compact closure and smooth boundary, while the word *normal* also in this statement means that the domain considered has compact closure and piecewise smooth boundary.

isoperimetric inequalities have a close relation with physical isoperimetric inequalities (of differential operators). A natural question is: *except space forms, whether one could find other spaces on which the geometric isoperimetric inequality* (1.13) *holds under the volume constraint* (1.12)*?* One might refer to [12, Chapter IV] for some interesting progresses on this question.

In the sequel, we will show a way to extend the Faber–Krahn inequality, the Hong–Krahn–Szegő inequality and the Szegő–Weinberger inequality of the Laplacian to the case of the Witten Laplacian.

For a given complete Riemannian n-manifold ($n \geq 2$) with the metric g, let $\Omega \subset M^n$ be a bounded domain (with boundary $\partial\Omega$) in M^n, and $\phi \in C^\infty(M^n)$ be a smooth[5] real-valued function defined on Ω. In this setting, one can define the following elliptic operator

$$\Delta_\phi := \Delta - \langle \nabla\phi, \nabla \cdot \rangle$$

on Ω, which is called the *Witten Laplacian* (also called the *drifting Laplacian* or the *weighted Laplacian*) with respect to the metric g. Consider the Dirichlet eigenvalue problem of the Witten Laplacian as follows:

$$\begin{cases} \Delta_\phi u + \lambda u = 0 & \text{in } \Omega \subset M^n, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{1.15}$$

it is not hard to check that the operator Δ_ϕ in (1.15) is *self-adjoint* with respect to the inner product

$$(h_1, h_2)_\phi := \int_\Omega h_1 h_2 d\eta = \int_\Omega h_1 h_2 e^{-\phi} dv, \tag{1.16}$$

with $h_1, h_2 \in W^{1,2}_{0,\phi}(\Omega)$, where $d\eta := e^{-\phi} dv$ is the weighted measure, and $W^{1,2}_{0,\phi}(\Omega)$ stands for a Sobolev space, which is the completion of the set of smooth functions (with compact support) $C_0^\infty(\Omega)$ under the following Sobolev norm:

$$\|f\|^\phi_{1,2} := \left(\int_\Omega f^2 e^{-\phi} dv + \int_\Omega |\nabla f|^2 e^{-\phi} dv \right)^{1/2} = \left(\int_\Omega f^2 d\eta + \int_\Omega |\nabla f|^2 d\eta \right)^{1/2}. \tag{1.17}$$

[5] In fact, one might see that $\phi \in C^2(\Omega)$ is suitable to derive our main conclusions in this paper. However, in order to avoid putting too much attention on discussion for the regularity of the boundary $\partial\Omega$, and following the assumption on conformal factor $e^{-\phi}$ for the notion of *smooth metric measure spaces* in many literatures (including of course those cited in this paper), without specification, we wish to assume that ϕ is smooth on the domain Ω.

Then, using similar arguments to those of the classical fixed membrane problem of the Laplacian (i.e., the discussions about the existence of discrete spectrum, Rayleigh's theorem, Max-min theorem, etc. Those discussions are standard, and for details, please see for instance [12]), it is not hard to see the following.

- The self-adjoint elliptic operator $-\Delta_{\phi}$ in (1.15) *only* has discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows:

$$0 < \lambda_{1,\phi}(\Omega) < \lambda_{2,\phi}(\Omega) \le \lambda_{3,\phi}(\Omega) \le \cdots \uparrow +\infty. \tag{1.18}$$

Each eigenvalue $\lambda_{i,\phi}$, $i = 1, 2, \ldots$, in the sequence (1.18) was repeated according to its multiplicity (which is finite and equals the dimension of the eigenspace of $\lambda_{i,\phi}$). By applying the standard variational principles, one can obtain that the k-th Dirichlet eigenvalue $\lambda_{k,\phi}(\Omega)$ can be characterised as follows:

$$\lambda_{k,\phi}(\Omega) = \inf\left\{ \frac{\int_{\Omega} |\nabla f|^2 e^{-\phi} dv}{\int_{\Omega} f^2 e^{-\phi} dv} \,\middle|\, f \in W^{1,2}_{0,\phi}(\Omega), f \neq 0, \int_{\Omega} f f_i e^{-\phi} dv = 0 \right\},$$

where f_i, $i = 1, 2, \ldots, k-1$, denotes an eigenfunction of $\lambda_{i,\phi}(\Omega)$. Moreover, the first Dirichlet eigenvalue $\lambda_{1,\phi}(\Omega)$ of the eigenvalue problem (1.15) satisfies

$$\lambda_{1,\phi}(\Omega) = \inf\left\{ \frac{\int_{\Omega} |\nabla f|^2 d\eta}{\int_{\Omega} f^2 d\eta} \,\middle|\, f \in W^{1,2}_{0,\phi}(\Omega), f \neq 0 \right\}.$$

It is interesting and important to study spectral geometric problems related to the Witten Laplacian – we refer to [13, Introduction] for a detailed explanation. We already have some interesting works about spectral estimates and geometric functional inequalities related to the Witten Laplacian – see, e.g., [16, 26, 29, 30, 33, 40].

On Ω, one can also define *weighted volume* (or *ϕ-volume*) as follows:

$$|\Omega|_{n,\phi} := \int_{\Omega} d\eta = \int_{\Omega} e^{-\phi} dv.$$

Using the constraint of fixed weighted volume, we can obtain several spectral isoperimetric inequalities for the first and the second Dirichlet eigenvalues of the Witten Laplacian. However, in order to state our conclusions clearly, we need to impose an assumption on the function ϕ as follows.

Property 1. ϕ is a function of the Riemannian distance parameter $t := d(o, \cdot)$ for some point $o \in M^n$.

Clearly, if a given open Riemannian n-manifold (M^n, g) was endowed with the weighted density $e^{-\phi} dv$, where ϕ satisfies Property 1, then ϕ would be a *radial* function defined on M^n with respect to the radial distance t, $t \in [0, \infty)$. In particular, when the given open n-manifold is chosen to be $\mathbb{R}^n$ or $\mathbb{H}^n$ (i.e., the n-dimensional hyperbolic space of sectional curvature -1), we additionally require that o is the origin of $\mathbb{R}^n$ or $\mathbb{H}^n$.

First, we have the following Faber–Krahn-type inequality for the Witten Laplacian in the Euclidean space.

Theorem 1.1. *Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{R}^n$) and is concave. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $B_R(o)$ be a ball of radius R and centred at the origin o of $\mathbb{R}^n$ such that $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$, i.e., $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Then,*

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)),$$

and the equality holds if and only if (up to measure zero) Ω is the ball $B_R(o)$, which lies entirely in the region $B_{\mathfrak{R}(h)}$ defined by (1.19).

Remark 1.2. (1) Unlike the Neumann case described in Theorems 1.11 and 1.12 below, for the Dirichlet case we do not need to require that the point o locates in the convex hull of the domain Ω in Theorem 1.1. The same situation also happens in Theorem 1.3.

(2) From the previous introduction on the Faber–Krahn inequality of the Laplacian, one knows that under the volume constraint (1.12), the geometric isoperimetric inequality (1.13) makes an important role in the derivation process. What about the Witten Laplacian case? Does some weighted geometric isoperimetric inequality play an important role also? The answer is affirmative. We would like to recall a recent breakthrough of Chambers [11] to the log-convex density conjecture. Given a positive function h in $\mathbb{R}^n$, $n \geq 2$, one can define the weighted perimeter and weighted volume of a set $A \subset \mathbb{R}^n$ of locally finite perimeter as

$$\mathrm{Per}(A) = \int_{\partial A} h \, d\mathcal{H}^{n-1}, \quad \mathrm{Vol}(A) = \int_A h \, d\mathcal{H}^n,$$

where following the usage of notations in [11], $\mathcal{H}^m$ indicates the m-dimensional Hausdorff measure, and ∂A denotes the essential boundary of A. Such positive function h is called a *density* on $\mathbb{R}^n$. If one fixes a positive weighted volume $m > 0$, does there exist a set $A \subset \mathbb{R}^n$ such that $\mathrm{Vol}(A) = m$ and

$$\mathrm{Per}(A) = \inf_{\substack{Q \subset \mathbb{R}^n \\ \mathrm{Vol}(Q) = m}} \mathrm{Per}(Q)?$$

Rosales, Cañete, Bayle, and Morgan considered this problem and gave a partial answer that in $\mathbb{R}^n$ with the density $e^{c|x|^2}$, $c > 0$, round balls about the origin uniquely minimise perimeter for a given volume (see [36, Theorem 5.2]). Moreover, they showed that for any radial, smooth density $h = e^{f(|x|)}$, balls around the origin are stable[6] if and only if f is convex ([36, Theorem 3.10]). This fact motivates the following conjecture (Conjecture 3.12 in their article), first stated by Kenneth Brakke.

Conjecture (Log-convex density conjecture). *In $\mathbb{R}^n$ with a smooth, radial, log-convex[7] density, balls around the origin provide isoperimetric regions of any given volume.*

Chambers [11, Theorem 1.1] gave an answer to the above conjecture as follows.

Fact A. *Given a density $h(x) = e^{f(|x|)}$ on $\mathbb{R}^n$ with f smooth, convex and even, balls around the origin are isoperimetric regions with respect to weighted perimeter and volume.*

Moreover, Chambers [11, Theorem 1.2] characterised the uniqueness of isoperimetric regions as follows.

Fact B. *Up to sets of measure 0, the only isoperimetric regions are balls centred at the origin, and balls that lie entirely in*

$$B_{\mathcal{R}(h)} = \{x \mid |x| \leq \mathcal{R}(h)\}, \tag{1.19}$$

where $\mathcal{R}(h) = \sup\{|x| \mid h(x) = h(0)\}$.

Fact A and Fact B would make an important role in the proof of Theorem 1.1 – see Section 2.1 for details.

(3) Since Chambers' weighted geometric isoperimetric inequality in $\mathbb{R}^n$ (i.e., Fact A) plays an important role in the proof of Theorem 1.1, it implies that, similar to the potential precondition of [11, Theorem 1.1], we also need to require that the boundary $\partial\Omega$ has finite area (or, following the convention in [11], *perimeter*). However, we believe this assumption is so natural when considering isoperimetric problems that we prefer not to list it explicitly in every statement of our main conclusions in this paper. Nevertheless, $\partial\Omega$ should maintain this natural assumption throughout the paper, which we will not mention again.

We can prove the following.

Theorem 1.3. *Let $\mathbb{S}^n_+$ be an n-dimensional hemisphere of radius 1, and let $\Omega \subset \mathbb{S}^n_+$ be a bounded domain whose boundary $\partial\Omega$ has positive constant mean curvature. Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{S}^n_+$) and moreover*

[6] Here *stable* means that $\mathrm{Per}''(0) \geq 0$ under smooth, volume-conserving variations.

[7] Clearly, for a density h here, the log-convex assumption means $(\log h)'' \geq 0$.

$\phi = -\log\cos t$*, where the point* o *mentioned in Property* 1 *should additionally be required to be the base point of* $\mathbb{S}^n_+$*. Then,*

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)),$$

where $B_R(o)$ *denotes a geodesic ball of radius* R *and centred at the base point* o *of* $\mathbb{S}^n_+$ *such that* $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$*. The equality holds if and only if* Ω *is isometric to the geodesic ball* $B_R(o)$*.*

Remark 1.4. (1) When investigating the above Faber–Krahn-type isoperimetric inequality, there is no essential difference between $\mathbb{S}^n_+$ and a hemisphere with radius not equal to 1.

(2) To help readers who may be unfamiliar with the concept of *the base point*, we provide an explanation here. A natural starting point is spherically symmetric manifolds, also known as generalised space forms, as suggested in the work of Katz and Kondo [22]. For a detailed definition, fundamental properties, and notable applications of spherically symmetric manifolds, we refer readers to [18, 27, 32]. The corresponding author has utilised spherically symmetric manifolds as model spaces to establish various interesting comparison theorems – covering volume, eigenvalues of different types, the heat kernel, and other geometric quantities (see, e.g., [18, 28, 31, 40]).

Definition ([18, Definition 2.1]). For a given complete n-manifold M^n, a domain

$$\mathscr{D} = \exp_p([0, l) \times S^{n-1}_p) \subset M^n \setminus \mathrm{Cut}(p),$$

with $l < \mathrm{inj}(p)$, is said to be *spherically symmetric* with respect to a point $p \in \mathscr{D}$ if the matrix $\mathbb{A}(t, \xi)$ satisfies $\mathbb{A}(t, \xi) = f(t)I$, for a function $f \in C^2([0, l))$ with $f(0) = 0$, $f'(0) = 1$ and $f|_{(0,l)} > 0$.

Here S^{n-1}_p denotes the unit sphere of the tangent space T_pM^n, $\mathrm{Cut}(p)$ stands for the cut-locus of the point p, $\mathrm{inj}(p)$ denotes the injectivity radius at p, $\xi \in S^{n-1}_p$, and $\mathbb{A}(t, \xi)\colon \xi^\perp \to \xi^\perp$ is the path of linear transformations well defined in [18, Section 2]. A standard model for spherically symmetric manifolds is given by the quotient of the warped product $[0, l) \times_f \mathbb{S}^{n-1}$ with the metric

$$ds^2 = dt^2 + f^2(t)|d\xi|^2, \quad \text{for all } \xi \in S^{n-1}_p,\ 0 < t < l,$$

where as usual $|d\xi|^2$ denotes the round metric of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$. In this model, all pairs $(0, \xi)$ are identified with the single point p, which is called *the base point* of the spherically symmetric domain $\mathscr{D} = [0, l) \times_f \mathbb{S}^{n-1}$. Clearly, as already shown in [18, (2.12)], a space form with constant sectional curvature κ is also a spherically symmetric manifold, and in this particular situation, the warping function f

satisfies

$$f(t) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}, & l = \dfrac{\pi}{\sqrt{\kappa}}, \kappa > 0, \\ t, & l = +\infty, \kappa = 0, \\ \dfrac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}}, & l = +\infty, \kappa < 0. \end{cases}$$

(3) Since o is required to be the base point of $\mathbb{S}^n_+$, then for the domain $\Omega \subset \mathbb{S}^n_+$ in Theorem 1.3, the range of the Riemannian distance parameter $t = d(o, \cdot)$ should be $(0, \pi/2)$, which implies that the choice of the function $\phi = -\log\cos t$ makes sense. Besides, in fact, $\mathbb{S}^n_+$ can be modelled as $[0, \pi/2] \times_{\sin t} \mathbb{S}^{n-1}$ with the metric $dt^2 + (\sin t)^2|d\xi|^2$, and its base point o should be the vertex of $\mathbb{S}^n_+$.

We also get the following.

Theorem 1.5. *Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{H}^n$) and is strictly concave, where the point o mentioned in Property 1 should additionally be required to be the origin of $\mathbb{H}^n$. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with boundary. Then,*

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)),$$

where $B_R(o)$ denotes a geodesic ball of radius R and centred at the origin o of $\mathbb{H}^n$ such that $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$. The equality holds if and only if Ω is isometric to the geodesic ball $B_R(o)$.

Remark 1.6. (1) The hyperbolic space $\mathbb{H}^n$ can be modelled as $[0, \infty) \times_{\sinh t} \mathbb{S}^{n-1}$ with the metric

$$dt^2 + (\sinh t)^2|d\xi|^2.$$

Since hyperbolic spaces are two-point homogenous, the base point of $\mathbb{H}^n$ is not unique and any point of $\mathbb{H}^n$ can be chosen as the base point, which is different with the case of hemisphere $\mathbb{S}^n_+$. However, for $\mathbb{H}^n$ once its globally defined coordinate system was set up, the origin o would be determined uniquely with respect to this system. As shown above, in order to get the main conclusion in Theorem 1.5, we need to assume that ϕ is radial with respect to some fixed point and is also concave, which leads to the situation that in the statement of Theorem 1.5, it is better to choose the point o to be the origin of $\mathbb{H}^n$ (might not the base point), and correspondingly ϕ is concave with respect to the radial Riemannian distance parameter $t = d(o, \cdot)$.

(2) As mentioned before, one knows two facts: (a) under the constraint of fixed volume, the Faber–Krahn inequality for the first Dirichlet eigenvalue of the Laplacian

also holds in hyperbolic spaces; (b) under the constraint of fixed weighted volume, Fact A (i.e., a weighted geometric isoperimetric inequality in $\mathbb{R}^n$) makes an important role in the proof of the Faber–Krahn-type inequality for the Witten Laplacian in $\mathbb{R}^n$ (i.e., Theorem 1.1). So, it is natural to ask the following.

Could one expect to get a hyperbolic version of Fact A which makes a contribution in the proof of Theorem 1.5?

The answer is affirmative. In fact, Li and Xu [25, Theorem 1.1] obtained a partial result to the hyperbolic version of Fact A for specified density through suitably applying Chambers' result [11] by projecting the hyperbolic space onto $\mathbb{R}^n$ and employing a comparison argument. Very recently, L. Silini [37] solved the above question completely. For an arbitrary base point $o \in \mathbb{H}^n$, and a density h given by $h := e^{f(d(o,\cdot))}$, where $h: \mathbb{R} \to \mathbb{R}$ is a smooth, (strictly) convex, even function, and, similar to before, $d(o,\cdot)$ denotes the Riemannian distance to the point o on $\mathbb{H}^n$, one can define the weighted perimeter and weighted volume of a set with finite perimeter $E \subset \mathbb{H}^n$ as

$$\mathrm{P}_h(E) = \int_{\partial^* E} h \, d\mathcal{H}^{n-1}, \quad \mathrm{V}_h(E) = \int_E h \, d\mathcal{H}^n,$$

where following the usage of notations in [37], $\partial^* E$ denotes the reduced boundary of E, and $\mathcal{H}^m$ indicates the m-dimensional Hausdorff measure. Silini [37, Theorem 1.1] proved the following.

Fact C. *For any strictly radially log-convex density h, geodesic balls centred at $o \in \mathbb{H}^n$ uniquely minimise the weighted perimeter for any given weighted volume with respect to P_h and V_h.*

Fact C would make an important role in the proof of Theorem 1.5 – see Section 3 for details. Using a comparison argument between $H^n_{\mathbb{C}} = U(n,1)/U(n)$ (i.e., the n-dimensional complex hyperbolic space of constant curvature -1) and $\mathbb{H}^{2n}$, together with Fact C, Silini [37] can get further.

In $H^n_{\mathbb{C}}$, geodesic balls are uniquely isoperimetric in the class of Hopf-symmetric sets for all volumes.

This conclusion gives a partial answer to an open conjecture proposed by Gromov and Ros in [19] as follows.

Conjecture. *Geodesic balls are isoperimetric for all volumes in the complex hyperbolic space $H^n_{\mathbb{C}}$.*

Silini's above result on the isoperimetric problem for the class of Hopf-symmetric sets in $H^n_{\mathbb{C}}$ might inspire readers to try to extend the spectral isoperimetric inequality in Theorem 1.5 to a more general space, which we think it is possible. However, due to the structure of this paper, here we just focus on investigating spectral isoperimetric inequalities for the Witten Laplacian on bounded domains in space forms.

(3) As explained in [37, Remark 1.7], since technical difficulties arise from the presence of regions with constant weight, for simplicity it was decided to assume the weight to be strictly log-convex rather than simply log-convex in extending the proof of Brakke's conjecture from the Euclidean space to the hyperbolic space. This is the reason why in Theorem 1.5 we assume that the radial function ϕ is strictly concave (i.e., $-(\log\phi)'' > 0$). Besides, if the domain Ω has a constant weight (i.e., a constant density), then the Witten Laplacian degenerates into the classical Laplacian, and correspondingly, in $\mathbb{H}^n$ one naturally has the Faber–Krahn inequality for the first Dirichlet eigenvalue. In this situation, it is no need to write down Theorem 1.5 any more. Based on this truth, in Theorem 1.5 it is acceptable to assume that the radial function ϕ is strictly concave.

Inspired by the technique used in [4], under other assumptions on ϕ and the constraint of weighted volume fixed, we can also get the following Faber–Krahn-type inequality for the Witten Laplacian in the Euclidean space, which can be seen as a complement to Theorem 1.1.

Theorem 1.7. *Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{R}^n$), ϕ is monotone non-increasing, and for $z \geq 0$, the function*

$$(e^{-\phi(z^{1/n})} - e^{-\phi(0)})z^{1-1/n}$$

is convex. Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, and let $B_R(o)$ be a ball of radius R and centred at the origin o of $\mathbb{R}^n$ such that $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$. Then,

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)).$$

Remark 1.8. Since ϕ satisfies Property 1 and moreover when M^n is chosen to be $\mathbb{R}^n$, we additionally require that o is the origin of $\mathbb{R}^n$, so o corresponds to $z = 0$, and then $\phi(0)$ is actually the value of the function ϕ at the origin o.

For the second Dirichlet eigenvalue of the Witten Laplacian, we can obtain the following Hong–Krahn–Szegő-type inequalities.

Theorem 1.9. *Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{R}^n$) and is concave. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $B_{\widetilde{R}}(o)$ be a ball of radius $\widetilde{R}$ and centred at the origin o of $\mathbb{R}^n$ such that $|\Omega|_{n,\phi}/2 = |B_{\widetilde{R}}(o)|_{n,\phi}$, i.e., $(1/2)\int_\Omega d\eta = \int_{B_{\widetilde{R}}(o)} d\eta$. Then,*

$$\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{\widetilde{R}}(o)).$$

That is to say, under the assumptions for ϕ described above, the minimum of the second Dirichlet eigenvalue of the Witten Laplacian on bounded domains Ω in $\mathbb{R}^n$, whose weighted volume equals some prescribed positive constant, should be equal to the first Dirichlet eigenvalue of the Witten Laplacian on a ball $B_{\widetilde{R}}(o)$ (of radius $\widetilde{R}$ and centred at the origin $o \in \mathbb{R}^n$) such that $|\Omega|_{n,\phi}/2 = |B_{\widetilde{R}}(o)|_{n,\phi}$.

Theorem 1.10. *Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{H}^n$) and is strictly concave, where the point o mentioned in Property 1 should additionally be required to be the origin of $\mathbb{H}^n$. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with boundary. Then,*

$$\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{\widetilde{R}}(o)),$$

where $B_{\widetilde{R}}(o)$ denotes a geodesic ball of radius $\widetilde{R}$ and centred at the origin o of $\mathbb{H}^n$ such that $|\Omega|_{n,\phi}/2 = |B_{\widetilde{R}}(o)|_{n,\phi}$. That is to say, under the assumptions for ϕ described above, the minimum of the second Dirichlet eigenvalue of the Witten Laplacian on bounded domains Ω in $\mathbb{H}^n$, whose weighted volume equals some prescribed positive constant, should be equal to the first Dirichlet eigenvalue of the Witten Laplacian on a geodesic ball $B_{\widetilde{R}}(o)$ (of radius $\widetilde{R}$ and centred at the origin $o \in \mathbb{H}^n$) such that $|\Omega|_{n,\phi}/2 = |B_{\widetilde{R}}(o)|_{n,\phi}$.

For a bounded domain Ω (with boundary $\partial\Omega$) on a given n-dimensional ($n \geq 2$) complete Riemannian manifold M^n, we can also consider the following Neumann eigenvalue problem of the Witten Laplacian:

$$\begin{cases} \Delta_\phi u + \mu u = 0 & \text{in } \Omega \subset M^n, \\ \dfrac{\partial u}{\partial \vec{v}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.20}$$

It is easy to check that the operator Δ_ϕ in (1.20) is *self-adjoint* with respect to the inner product (1.16) with $h_1, h_2 \in W^{1,2}_\phi(\Omega)$. Here $W^{1,2}_\phi(\Omega)$ stands for a Sobolev space, which is the completion of the set of smooth functions $C^\infty(\Omega)$ under the Sobolev norm $\|\cdot\|^\phi_{1,2}$ defined by (1.17). Then, using similar arguments to those of the classical free membrane problem of the Laplacian (see, e.g., [12]), it is not difficult to see the following.

The operator $-\Delta_\phi$ in (1.20) *only* has discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows:

$$0 = \mu_{0,\phi}(\Omega) < \mu_{1,\phi}(\Omega) \leq \mu_{2,\phi}(\Omega) \leq \mu_{3,\phi}(\Omega) \leq \cdots \uparrow +\infty. \tag{1.21}$$

Each eigenvalue $\mu_{i,\phi}$, $i = 0, 1, 2, \ldots$, in the sequence (1.21) is repeated according to its multiplicity (i.e., the dimension of the eigenspace of $\mu_{i,\phi}$). In particular, the zero eigenvalue $\mu_{0,\phi}$ has multiplicity 1 and has non-zero constant function as its eigenfunction. By applying the standard variational principles, one can obtain that the k-th Neumann eigenvalue $\mu_{k,\phi}(\Omega)$ can be characterised as follows:

$$\mu_{k,\phi}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 e^{-\phi} dv}{\int_\Omega f^2 e^{-\phi} dv} \;\middle|\; f \in W^{1,2}_\phi(\Omega), f \neq 0, \int_\Omega f f_i e^{-\phi} dv = 0 \right\},$$

where f_i, $i = 1, 2, \ldots, k-1$, denotes an eigenfunction of $\mu_{i,\phi}(\Omega)$. Moreover, the first non-zero Neumann eigenvalue $\mu_{1,\phi}(\Omega)$ of the eigenvalue problem (1.20) satisfies

$$\mu_{1,\phi}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \;\middle|\; f \in W^{1,2}_\phi(\Omega), f \neq 0, \int_\Omega f d\eta = 0 \right\}. \tag{1.22}$$

In fact, the above facts have been explained more clearly in [13, Section 1]. However, we wish to restate this content for two reasons: the first is to complete the brief introduction to the eigenvalue problem (1.20) presented here; the second is that the characterization (1.22) will be used to derive spectral isoperimetric inequalities for the first non-zero Neumann eigenvalue $\mu_{1,\phi}(\cdot)$ below.

We can prove the following Szegő–Weinberger-type inequalities for the Witten Laplacian.

Theorem 1.11. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{R}^n$ and additionally the point o required to be in the convex hull of Ω, i.e., $o \in \mathrm{hull}(\Omega)$), and ϕ is also a non-increasing convex function defined on $[0, \infty)$. Let $B_R(o)$ be a ball of radius R and centred at the origin o of $\mathbb{R}^n$ such that $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$, i.e., $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Then,*

$$\mu_{1,\phi}(\Omega) \leq \mu_{1,\phi}(B_R(o)),$$

with equality holding if and only if Ω is the ball $B_R(o)$.

Theorem 1.12. *Let Ω be a bounded domain with smooth boundary in $\mathbb{H}^n$. Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{H}^n$ and additionally $o \in \mathrm{hull}(\Omega)$), and ϕ is also a non-increasing convex function defined on $[0, \infty)$. Let $B_R(o)$ be a geodesic ball of radius R and centred at the origin o of $\mathbb{H}^n$ such that $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$. Then,*

$$\mu_{1,\phi}(\Omega) \leq \mu_{1,\phi}(B_R(o)),$$

with equality holding if and only if Ω is isometric to the geodesic ball $B_R(o)$.

Remark 1.13. (1) In fact, in our very recent work [13, Theorems 1.1 and 1.5], we can prove an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$

non-zero Neumann eigenvalues of the Witten Laplacian on bounded domains in $\mathbb{R}^n$ or $\mathbb{H}^n$, which, together with the monotonicity of the sequence (1.21) of Neumann eigenvalues, yields directly our Theorem 1.11 and Theorem 1.12 here. This fact has been already pointed out in [13, Corollaries 1.2 and 1.6], and readers can check there for details.

(2) For two reasons, we insist on writing down Theorem 1.11 and Theorem 1.12 here. The first is to complete the overall structure of this paper, and the second is that our approach to proving Theorem 1.11 and Theorem 1.12 differs somewhat from the one used in [13].

(3) Unlike the Dirichlet case, we need to require that $o \in \mathrm{hull}(\Omega)$ in Theorem 1.11 and Theorem 1.12. This is because we must use the Brouwer fixed point theorem to ensure the existence of an orthonormal frame field such that the origin of the coordinate system (corresponding to the orthonormal frame field) is located in the convex hull of Ω. Then, all the computations involving trial functions constructed are valid. See the proofs of Theorem 1.11 and Theorem 1.12 in Section 3 for details.

The paper is organised as follows. The proofs of the Faber–Krahn-type inequalities, the Hong–Krahn–Szegő-type inequalities, and the Szegő–Weinberger-type inequalities for the Witten Laplacian will be given in Sections 2, 3, and 4, respectively. Besides, in Section A, we will give the detailed information about the first non-zero Neumann eigenvalue and its eigenfunctions of the Witten Laplacian on prescribed (geodesic) balls in space forms.

2. The Faber–Krahn-type inequalities for the Witten Laplacian

2.1. The Euclidean case

Assume that f is an eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_{1,\phi}(\Omega)$. Since f does not change sign on Ω, without loss of generality, we can assume $f > 0$ on Ω (see Lemma 3.1 below for the explanation). Consider the sets $\Omega_s := \{x \in \Omega | f(x) > s\}$, and let Ω_s^* be balls in $\mathbb{R}^n$ with centre at the origin o and satisfying $|\Omega_s|_{n,\phi} = |\Omega_s^*|_{n,\phi}$. Let $B_R(o)$ be a ball of radius R and centred at o of $\mathbb{R}^n$ such that $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$, i.e., $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Define a function f^* on $B_R(o)$ having the following properties:

- f^* is a radial decreasing function;
- f^* takes the value s on the boundary sphere $\partial\Omega_s^*$ of the ball Ω_s^* (for a fixed s).

It is not hard to see that $\Omega_0 = \Omega$ and correspondingly $\Omega_0^* = B_R(o)$. The existence of the balls Ω_s^* can be assured by using the Schwarz symmetrization. Readers can check e.g., [4, 20] for details on how to use symmetrization to get balls Ω_s^* under the constraint of having the same weighted volume.

Now, we agree on the notations used below. Denote by $\widehat{dv}$ the $(n-1)$-dimensional Hausdorff measure of the boundary associated to the Riemannian volume element[8] dv. This convention will be used throughout the paper. Similarly, $\widehat{d\eta} = e^{-\phi}\widehat{dv}$ would be the weighted volume element of the boundary. Besides, for convenience, set $G(s) := \partial\Omega_s$, $S_{t(s)} := (G(s))^* = G^*(s) = \partial\Omega_s^*$, which denotes the sphere with centre at the origin and radius $t(s)$. The following formula is known as the *co-area formula* (see, e.g., [7, 12]).

For any continuous function h defined on Ω, one has

$$\int\limits_{\Omega} h dv = \int\limits_{0}^{\sup f} \int\limits_{G(s)} h|\nabla f|^{-1}\widehat{dv_s} ds,$$

where following the above agreement, $\widehat{dv_s}$ denotes the volume element of the hypersurface $G(s) = f^{-1}(s)$.

Clearly, taking $h = |\nabla f|^2$ and then applying the co-area formula, one has

$$\int\limits_{\Omega} |\nabla f|^2 dv = \int\limits_{0}^{\sup f} \int\limits_{G(s)} |\nabla f|\widehat{dv_s} ds.$$

Denote by the Schwarz symmetric rearrangement mapping $t : [0, \sup f] \to [0, R]$, with R the radius of $B_R(o)$, and ψ the inverse transformation of t, where t additionally satisfies $t(0) = R$, $t(\sup f) = 0$.

Lemma 2.1. *If Ω is a bounded region in $\mathbb{R}^n$, and ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{R}^n$), then*

$$\int\limits_{\Omega} f^2 d\eta = \int\limits_{B_R(o)} (f^*)^2 d\eta, \tag{2.1}$$

where $B_R(o) \subset \mathbb{R}^n$ is the ball defined as in Theorem 1.1.

Proof. By a direct calculation, one can obtain

$$\int\limits_{B_R(o)} (f^*)^2 d\eta = \int\limits_{0}^{R} \int\limits_{\partial B_t(o)} (f^*)^2 e^{-\phi(t)}\widehat{dv_t} dt = \int\limits_{0}^{R} \psi^2(t) \int\limits_{\partial B_t(o)} e^{-\phi(t)}\widehat{dv_t} dt$$

[8] In fact, for domains Ω_s and $\Omega = \Omega_0$, they should have the same volume element dv. However, in order to emphasise that the domain Ω_s depends on s, we wish to additionally write the volume element of Ω_s as dv_s (except $s = 0$).

$$= -\int_0^{\sup f} \psi^2(t(s))t'(s)\left(\int_{\partial B_{t(s)}(o)} e^{-\phi(t(s))}\widehat{dv_t}\right)ds$$

$$= -\int_0^{\sup f} s^2\left(-\int_{G(s)} |\nabla f|^{-1}e^{-\phi|_{G(s)}}\widehat{dv_s}\right)ds = \int_\Omega f^2 d\eta,$$

which implies (2.1) directly. ■

Now, together with Fact A and Fact B, we can prove Theorem 1.1.

Proof of Theorem 1.1. Applying the co-area formula, we have

$$\int_\Omega |\nabla f|^2 e^{-\phi} dv = \int_0^{\sup f}\int_{G(s)} |\nabla f| e^{-\phi}\widehat{dv_s} ds. \tag{2.2}$$

We can obtain, by using the Cauchy–Schwarz inequality, that

$$\int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}}\widehat{dv_s} \geq \frac{\left(\int_{G(s)} e^{-\phi|_{G(s)}}\widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}}\widehat{dv_s}}. \tag{2.3}$$

By Fact A and Fact B, we have

$$\int_{G(s)} e^{-\phi|_{G(s)}}\widehat{dv_s} \geq \int_{G^*(s)} e^{-\phi(t(s))}\widehat{dv_s},$$

with equality holding if and only if $G(s)\setminus E(s) = G^*(s)$, where the set $E(s)$ denotes a set of measure zero. Substituting this into (2.3) yields

$$\int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}}\widehat{dv_s} \geq \frac{\left(\int_{G^*(s)} e^{-\phi(t(s))}\widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}}\widehat{dv_s}}. \tag{2.4}$$

On the other hand, one has

$$\int_{G^*(s)} |\nabla f^*| e^{-\phi(t(s))}\widehat{dv_s} = \frac{\left(\int_{G^*(s)} e^{-\phi(t(s))}\widehat{dv_s}\right)^2}{\int_{G^*(s)} |\nabla f^*|^{-1} e^{-\phi(t(s))}\widehat{dv_s}},$$

since $|\nabla f^*|$ and $e^{-\phi(s)}$ are constant on the sphere $G^*(s)$. We notice that

$$|\Omega_r|_{n,\phi} = \int_{\Omega_r} e^{-\phi} dv = \int_r^{\sup f}\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}}\widehat{dv_s} ds,$$

and so it follows that

$$(|\Omega_r|_{n,\phi})'(s) = -\int\limits_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s},$$

which implies

$$-\int\limits_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} = \frac{d}{ds}|\Omega_s|_{n,\phi} = \frac{d}{ds}|\Omega_s^*|_{n,\phi}. \tag{2.5}$$

Since

$$|\Omega_s^*|_{n,\phi} = \int\limits_0^{t(s)} \int\limits_{\partial B_z(o)} e^{-\phi(z)} \widehat{dv_z} dz,$$

one has

$$\frac{d}{ds}|\Omega_s^*|_{n,\phi} = t'(s) \int\limits_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s}. \tag{2.6}$$

We wish to point out the following fact.

Lemma 2.2. *For the function $t(s)$ in (2.6), one has $t'(s) \neq 0$.*

Proof. Denote by $\mathcal{T}$ the set consisting of points, where the function f attains its critical values. By Sard's theorem (i.e., the set of critical points of a smooth function has measure zero), we can conclude that $\mathcal{T}$ has measure zero. Therefore, one knows

$$\int\limits_{\mathcal{T}} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} = 0,$$

and then

$$\begin{aligned}\int\limits_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} &= \int\limits_{G(s)\setminus\mathcal{T}} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} + \int\limits_{\mathcal{T}} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} \\ &= \int\limits_{G(s)\setminus\mathcal{T}} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s}. \end{aligned} \tag{2.7}$$

This implies that there is no essential difference when doing integrations over $G(s) \setminus \mathcal{T}$ or over $G(s)$. Based on this reason, in the sequel, for convenience and simplicity, we wish to integrate over $G(s)$ directly.

Therefore, combining (2.7) with (2.5)–(2.6), one has

$$\int_{G(s)\setminus\mathcal{T}} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} = t'(s) \int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s},$$

which implies $t'(s) \neq 0$ since the LHS of the above equality cannot be zero. ■

Now, let us go back to our discussion. Putting (2.5)–(2.6) into (2.4) results in[9]

$$\begin{aligned}\int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}} \widehat{dv_s} &\geq \frac{\left(\int_{G^*(s)} e^{-\phi(t(s))} \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s}} \\ &= \frac{\int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s}}{-t'(s)}.\end{aligned}$$

The above expression makes sense since $t'(s) \neq 0$ by Lemma 2.2. Therefore, by substituting the above inequality into (2.2), one obtains

$$\begin{aligned}\int_{\Omega} |\nabla f|^2 d\eta &= \int_0^{\sup f} \int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}} \widehat{dv_s} ds \geq -\int_0^{\sup f} \frac{\int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s}}{t'(s)} ds \\ &= -\int_0^{\sup f} (\psi'(t(s)))^2 t'(s) \int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s} ds \\ &= \int_0^R (\psi'(t))^2 \int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s} dt = \int_{B_R(o)} |\nabla f^*|^2 d\eta. \qquad (2.8)\end{aligned}$$

The equality case in (2.8) implies that

$$\int_{G(0)} e^{-\phi|_{G(0)}} \widehat{dv} = \int_{G^*(0)} e^{-\phi(t)} \widehat{dv}$$

holds. So, one has $G(0) \setminus E(0) = G^*(0)$, that is, $\Omega \setminus E(0) = B_R(o)$. Moreover, this domain should lie entirely in the region $B_{\mathcal{R}(h)}$ defined by (1.19). Furthermore, by Lemma 2.1, we have

$$\lambda_{1,\phi}(\Omega) = \frac{\int_{\Omega} |\nabla f|^2 d\eta}{\int_{\Omega} f^2 d\eta} \geq \frac{\int_{B_R(o)} |\nabla f^*|^2 d\eta}{\int_{B_R(o)} (f^*)^2 d\eta} \geq \lambda_{1,\phi}(B_R(o)),$$

which completes the proof of Theorem 1.1. ■

[9] One would see that similar conclusions can be obtained in the hemisphere case and also the hyperbolic case.

Proof of Theorem 1.7. Use an almost the same argument as that in the above proof of Theorem 1.1 except replacing the usage of Fact A and Fact B by the following fact.

([4]) Assume that the function $a\colon [0, +\infty) \to [0, +\infty)$ satisfies preconditions $a(t)$ is non-decreasing for $t \geq 0$, $(a(z^{1/n}) - a(0))z^{1-1/n}$ is convex, $z \geq 0$, and moreover, assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$. Then,

$$\int_{\partial\Omega} a(|x|)dx \geq \int_{\partial\Omega^*} a(|x|)dx,$$

where $\partial\Omega^*$ is a sphere with centre at the origin and enclosing the weighted volume equal to that of Ω.

Then the conclusion in Theorem 1.7 follows naturally by choosing $a(t) = e^{-\phi(t)}$. ∎

2.2. The hemisphere case

As we know, Schwarz symmetrization can also be carried out on hemispheres and hyperbolic spaces. For convenience, we will continue to use the notions and notations introduced at the beginning of Section 2.1 to investigate Faber–Krahn-type inequalities for the Witten Laplacian in the hemisphere and hyperbolic cases.

Lemma 2.3. *Assume that the function ϕ satisfies Property 1 (with M^n chosen to be $\mathbb{S}^n_+$), where the point o mentioned in Property 1 should additionally be required to be the base point of $\mathbb{S}^n_+$. Then we have*

$$\int_{\Omega} f^2 d\eta = \int_{B_R(o)} (f^*)^2 d\eta,$$

where $B_R(o) \subset \mathbb{S}^n_+$ is the geodesic ball defined as in Theorem 1.3.

Proof. Formally, the computation for the assertion in Lemma 2.3 is almost the same as that for (2.1), and so we omit the details here. ∎

We also need the following fact.

Lemma 2.4 ([6]). *Let $\Omega \subset \mathbb{S}^n_+$ be a compact n-dimensional domain with smooth boundary $\partial\Omega$. Let H be the normalised mean curvature of $\partial\Omega$. Let*

$$V(x) = \cos \mathrm{dist}_{\mathbb{S}^n}(x, o).$$

If H is positive everywhere, then[10]

$$\int_{\partial\Omega} \frac{V}{H} dA \geq n \int_{\Omega} V d\Omega. \tag{2.9}$$

The equality in (2.9) holds if and only if Ω is isometric to a geodesic ball.

Proof of Theorem 1.3. Applying the co-area formula, we have

$$\int_{\Omega} |\nabla f|^2 \cos t \, dv = \int_0^{\sup f} \int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} ds. \tag{2.10}$$

By using the Cauchy–Schwarz inequality, we obtain that

$$\int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} \geq \frac{\left(\int_{G(s)} \cos(t|_{G(s)}) \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s}}. \tag{2.11}$$

By Lemma 2.4 and the assumption that H is a positive constant, one has

$$\int_{G(s)} \cos(t|_{G(s)}) \widehat{dv_s} \geq \int_{G^*(s)} \cos t(s) \widehat{dv_s},$$

and then (2.11) becomes

$$\int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} \geq \frac{\left(\int_{G^*(s)} \cos t(s) \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s}}. \tag{2.12}$$

On the other hand, one has

$$\int_{G^*(s)} |\nabla f^*| \cos t(s) \widehat{dv_s} = \frac{\left(\int_{G^*(s)} \cos t(s) \widehat{dv_s}\right)^2}{\int_{G^*(s)} |\nabla f^*|^{-1} \cos t(s) \widehat{dv_s}},$$

since $|\nabla f^*|$ and $\cos t(s)$ are constant on the sphere $G^*(s)$. Notice that

$$|\Omega_r|_{n,\phi} = \int_{\Omega_r} \cos t(r) dv = \int_r^{\sup f} \int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s} ds,$$

[10] In (2.9), the Hausdorff measures of the domain Ω and its boundary $\partial\Omega$ are given by $d\Omega$ and dA respectively. This usage of notations does not match the convention made at the beginning of Section 2.1, and the reason is that we wish to list here the original statement of the conclusion in Lemma 2.4 proven firstly in the reference [6].

and so it follows that

$$(|\Omega_r|_{n,\phi})'(s) = -\int\limits_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)})\widehat{dv_s},$$

which implies

$$-\int\limits_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)})\widehat{dv_s} = \frac{d}{ds}|\Omega_s|_{n,\phi} = \frac{d}{ds}|\Omega_s^*|_{n,\phi}. \tag{2.13}$$

Since

$$|\Omega_s^*|_{n,\phi} = \int\limits_0^{t(s)} \int\limits_{\partial B_z(o)} \cos z\widehat{dv_z}dz,$$

one has

$$\frac{d}{ds}|\Omega_s^*|_{n,\phi} = t'(s)\int\limits_{S_{t(s)}} \cos t(s)\widehat{dv_s}. \tag{2.14}$$

Putting (2.13)–(2.14) into (2.12) results in

$$\int\limits_{G(s)} |\nabla f| \cos(t|_{G(s)})\widehat{dv_s} \geq \frac{\left(\int_{G^*(s)} \cos t(s)\widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)})\widehat{dv_s}} = \frac{\int_{S_{t(s)}} \cos t(s)\widehat{dv_s}}{-t'(s)}.$$

Therefore, by substituting the above inequality into (2.10), one has

$$\begin{aligned}
\int\limits_{\Omega} |\nabla f|^2 d\eta &= \int\limits_0^{\sup f} \int\limits_{G(s)} |\nabla f| \cos(t|_{G(s)})\widehat{dv_s}ds \geq -\int\limits_0^{\sup f} \frac{\int_{S_{t(s)}} \cos t(s)\widehat{dv_s}}{t'(s)}ds \\
&= -\int\limits_0^{\sup f} (\psi'(t(s)))^2 t'(s) \int\limits_{S_{t(s)}} \cos t(s)\widehat{dv_s}ds \\
&= \int\limits_0^{R} (\psi'(t))^2 \int\limits_{S_{t(s)}} \cos t(s)\widehat{dv_s}dt = \int\limits_{B_R(o)} |\nabla f^*|^2 d\eta.
\end{aligned}$$

Together with Lemma 2.3, it follows that

$$\lambda_{1,\phi}(\Omega) = \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \geq \frac{\int_{B_R(o)} |\nabla f^*|^2 d\eta}{\int_{B_R(o)} (f^*)^2 d\eta} \geq \lambda_{1,\phi}(B_R(o)). \tag{2.15}$$

In particular, if the equality in (2.15) is achieved, then the equality in (2.11) and (2.12) can be attained simultaneously, and the rigidity assertion in Theorem 1.3 follows directly from Lemma 2.4. This completes the proof of Theorem 1.3. ■

2.3. The hyperbolic case

Proof of Theorem 1.5. It is not hard to see that similar to Lemma 2.1, in the hyperbolic case one also has the L^2 integral (with respect to the weighted density $d\eta$) unchanged after the Schwarz symmetrization under the constraint of fixed weighted volume. Besides, if one looks at the proofs of Theorems 1.1 and 1.3, one finds that in the two different cases (i.e., the case of Euclidean spaces and the case of hemispheres), the co-area formula, and most subsequent calculations look similarly in form. The key difference for those two cases is the usage of weighted isoperimetric inequalities (i.e., the way of dealing with (2.4) and (2.11)) *properly*. Based on these facts, using almost the same argument as in the proof of Theorem 1.1, together with the help of Fact C (i.e., the geometric isoperimetric inequality in $\mathbb{H}^n$ under the constraint of fixed weighted volume), we obtain the spectral isoperimetric inequality and the rigidity in Theorem 1.5. ∎

3. The Hong–Krahn–Szegő-type inequalities for the Witten Laplacian

For the Dirichlet eigenvalue problem (1.15), we know from Section 1 that its admissible space is the Sobolev space $W^{1,2}_{0,\phi}(\Omega)$. Using the inner product (1.16), one can define the L^2 space $\widehat{L}^2(\Omega)$ with respect to the weighted density as follows: we say that $u \in \widehat{L}^2(\Omega)$ if

$$\int_{\Omega} u^2 e^{-\phi} dv < \infty.$$

Before giving the proof of the Hong–Krahn–Szegő-type inequalities for the second Dirichlet eigenvalue of the Witten Laplacian, we need the following facts.

Lemma 3.1 (Nodal domain theorem for the Witten Laplacian, [14]). *For the Dirichlet eigenvalue problem* (1.15)*, the eigenvalues consist of a non-decreasing sequence* (1.18)*. Denote by f_i an eigenfunction of the i-th eigenvalue $\lambda_{i,\phi}$, $i = 1, 2, 3, \ldots$, and $\{f_1, f_2, f_3, \ldots\}$ forms a complete orthogonal basis of $\widehat{L}^2(\Omega)$. Then, for each $k = 1, 2, 3, \ldots$, the number of nodal domains of f_k is less than or equal to k.*

Remark 3.2. (1) By Lemma 3.1, one easily sees that the eigenfunction f_1 does not change sign on Ω, and $\lambda_{1,\phi}$ has multiplicity 1. Without loss of generality, we can assume $f_1 > 0$ on Ω. Besides, in Ω, the complement of the nodal set of eigenfunction f_2 of the second Dirichlet eigenvalue $\lambda_{2,\phi}$ has precisely two components. That is to say, f_2 has two nodal domains.

(2) By the way, we have pointed out in of [14, Remark 1.3] that maybe spectral geometers have already known the conclusion of Lemma 3.1, and we still formally

write it down therein for the completion of the structure of [14]. In fact, by making necessary changes to the proof of Courant-type theorem for the characterization of nodal domains to eigenfunctions of the Laplacian in the Riemannian case given by Bérard and Meyer [2], one might get our proof for the conclusion of Lemma 3.1 shown in [14].

Lemma 3.3 ([15]). *Domain monotonicity of eigenvalues with vanishing Dirichlet data also holds for the Dirichlet eigenvalues of the weighted Laplacian.*

A proof of Theorem 1.9 or 1.10. By Lemma 3.1, one knows that the eigenfunction f_2 has two nodal domains and its nodal set lies inside Ω. Denote by Γ the nodal set of f_2. Γ divides the domain Ω into two parts D_1 and D_2. Without loss of generality, assume that $f_2|_{D_1} > 0$ and $f_2|_{D_2} < 0$. Then, it is easy to see that

$$\begin{cases} \Delta_\phi f_2 + \lambda_{2,\phi}(\Omega) f_2 = 0 & \text{in } D_1, \\ f_2 = 0 & \text{on } \partial D_1, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \Delta_\phi f_2 + \lambda_{2,\phi}(\Omega) f_2 = 0 & \text{in } D_2, \\ f_2 = 0 & \text{on } \partial D_2. \end{cases} \tag{3.2}$$

In fact, the nodal set Γ also divides the boundary $\partial\Omega$ into two parts; let us call them $\mathscr{C}_1$ and $\mathscr{C}_2$. It is not hard to see that $\mathscr{C}_1$ and Γ surround one of D_1 and D_2, and without loss of generality, let us say D_1. This implies that the boundary ∂D_1 of D_1 satisfies $\partial D_1 = \mathscr{C}_1 \cup \Gamma$. Correspondingly, one has $\partial D_2 = \mathscr{C}_2 \cup \Gamma$. From (3.1) and (3.2), one knows that f_2 satisfies the eigenvalue problem (1.15) with $\Omega = D_1$ or $\Omega = D_2$, and moreover, f_2 does not change sign on D_i, $i = 1, 2$. Hence, we have $\lambda_{1,\phi}(D_1) = \lambda_{2,\phi}(\Omega) = \lambda_{1,\phi}(D_2)$, and f_2 can be treated as an eigenfunction of $\lambda_{1,\phi}(D_i)$, $i = 1, 2$. Denote by $B_{R_i}(o)$ the (geodesic) ball in $\mathbb{R}^n$ (or $\mathbb{H}^n$) centred at the origin o and radius R_i such that its weighted volume equals that of D_i, $i = 1, 2$, that is, $|B_{R_i}(o)|_{n,\phi} = |D_i|_{n,\phi}$. Then, by Theorem 1.1 (or Theorem 1.5), we know that

$$\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{R_1}(o)), \quad \lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{R_2}(o))$$

hold simultaneously. Hence, one has

$$\lambda_{2,\phi}(\Omega) \geq \max\{\lambda_{1,\phi}(B_{R_1}(o)), \lambda_{1,\phi}(B_{R_2}(o))\}.$$

We may suppose that $|D_1|_{n,\phi} \leq |D_2|_{n,\phi}$. So, $R_1 \leq R_2$, and by Lemma 3.3 we have $\lambda_{1,\phi}(B_{R_1}(o)) \geq \lambda_{1,\phi}(B_{R_2}(o))$. Therefore, in this setting, finding the greatest lower bound for the second eigenvalue $\lambda_2(\Omega)$ among domains with the fixed weighted volume $|\Omega|_{n,\phi} = \text{const.}$, it is sufficient to minimise $\lambda_{1,\phi}(B_{R_1}(o))$. Since one has $|D_1|_{n,\phi} \le |D_2|_{n,\phi}$ and $|D_1|_{n,\phi} + |D_2|_{n,\phi} = |\Omega|_{n,\phi}$, the maximal possibility for the weighted volume of D_1 is that $|D_1|_\phi = |\Omega|_{n,\phi}/2$. Hence, there exists $\widetilde{R} > 0$ such that $|B_{\widetilde{R}}(o)|_\phi = |\Omega|_{n,\phi}/2$, and by Lemma 3.3, in this situation, the eigenvalue $\lambda_{1,\phi}(B_{\widetilde{R}}(o))$ minimises the eigenvalue functional $\lambda_{1,\phi}(B_{R_1}(o))$ as R_1 changes. Hence, one has $\lambda_{2,\phi}(\Omega) \ge \lambda_{1,\phi}(B_{\widetilde{R}}(o))$, and the eigenvalue $\lambda_{1,\phi}(B_{\widetilde{R}}(o))$ equals the minimum value of the eigenvalue functional $\lambda_{2,\phi}(\Omega)$ under the constraint of weighted volume $|\Omega|_{n,\phi} = \text{const.}$ fixed. This completes the proof. ∎

4. The Szegő–Weinberger-type inequalities for the Witten Laplacian

This section is devoted to presenting isoperimetric inequalities for the first non-zero Neumann eigenvalue of the Witten Laplacian under the constraint of fixed weighted volume. Before that, we need the following fact.

Theorem 4.1. *Assume that $B_R(o)$ is a geodesic ball of radius R and centred at some point o in the n-dimensional complete simply connected Riemannian manifold $\mathbb{M}^n(\kappa)$ with constant sectional curvature $\kappa \in \{-1, 0, 1\}$, and that ϕ is a radial function with respect to the distance parameter $t := d(o, \cdot)$, which is also a non-increasing convex function. Then, the eigenfunctions of the first non-zero Neumann eigenvalue $\mu_{1,\phi}(B_R(o))$ of the Witten Laplacian on $B_R(o)$ have the form $T(t)x_i/t$, $i = 1, 2, \ldots, n$, where $T(t)$ satisfies*

$$\begin{cases} T'' + \Big(\dfrac{(n-1)C_\kappa}{S_\kappa} - \phi'\Big)T' + (\mu_{1,\phi}(B_R(o)) - (n-1)S_\kappa^{-2})T = 0, \\ T(0) = 0, \ T'(R) = 0, \ T'|_{[0,R)} \neq 0. \end{cases} \tag{4.1}$$

Here $C_\kappa(t) = (S_\kappa(t))'$ and

$$S_\kappa(t) = \begin{cases} \sin t & \text{if } \mathbb{M}^n(\kappa) = \mathbb{S}^n_+, \\ t & \text{if } \mathbb{M}^n(\kappa) = \mathbb{R}^n, \\ \sinh t & \text{if } \mathbb{M}^n(\kappa) = \mathbb{H}^n, \end{cases}$$

with $\mathbb{S}^n_+$ the n-dimensional hemisphere of radius 1.

The proof of the above fact is somewhat long and does not appear to have a close relation to the main content of this section. Hence, we choose to leave the proof in Appendix A.

Remark 4.2. It is not hard to see (cf. Section A) that x_i, $i = 1, 2, \ldots, n$, are coordinate functions of the globally defined orthonormal coordinate system set up in $\mathbb{M}^n(\kappa)$.

We construct an auxiliary function $h(t)$ such that

$$h(t) = \begin{cases} T(t), & 0 \le t \le R, \\ T(R), & t > R. \end{cases} \tag{4.2}$$

Lemma 4.3. *Assume that the function ϕ satisfies Property 1 (with M^n chosen as $\mathbb{R}^n$ and the point o additionally required to be in the convex hull of Ω, i.e., $o \in \mathrm{hull}(\Omega)$). Assume that $T(t)$ is a monotonically non-decreasing function determined by the system (4.1). Then, $h(t)$ is monotonically non-decreasing, and $(h')^2 + (n-1)h^2/t^2$ is monotonically non-increasing.*

Proof. First, it is easy to check that $h(t)$ defined by (4.2) is non-decreasing. Besides, by a direct calculation, one has

$$\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] = 2h'h'' + 2(n-1)\frac{thh' - h^2}{t^3}.$$

Together with (4.1), we have

$$\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] = -2\mu_{1,\phi}(B_R(o))hh' - (n-1)\frac{(th' - h)^2}{t^3} + 2(h')^2\phi' \le 0,$$

which implies the second assertion of the lemma directly. ∎

Lemma 4.4. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ (or $\mathbb{H}^n$) with smooth boundary. If $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$, with $B_R(o)$ be the (geodesic) ball defined as in Theorem 1.11 (or Theorem 1.12), and the non-constant functions $u(t)$ and $v(t)$ defined on $[0, +\infty)$ are monotonically non-increasing and non-decreasing respectively, then*

$$\int_{\Omega} v(|x|)d\eta \ge \int_{B_R(o)} v(|x|)d\eta, \quad \int_{\Omega} u(|x|)d\eta \le \int_{B_R(o)} u(|x|)d\eta.$$

The equality holds if and only if $\Omega = B_R(o)$ (or Ω is isometric to $B_R(o)$).

Proof. Assume that $Q = \Omega \cap B_R(o)$. Then we have

$$\int_{\Omega} v(|x|)d\eta = \int_{Q} v(|x|)d\eta + \int_{\Omega\setminus Q} v(|x|)d\eta \ge \int_{Q} v(|x|)d\eta + v(R)\int_{\Omega\setminus Q} d\eta.$$

Similarly, one has

$$\int_{B_R(o)} v(|x|)d\eta = \int_{Q} v(|x|)d\eta + \int_{B_R(o)\setminus Q} v(|x|)d\eta \le \int_{Q} v(|x|)d\eta + v(R)\int_{B_R(o)\setminus Q} d\eta.$$

Since $|\Omega|_{n,\phi} = |B_R(o)|_{n,\phi}$, then $\int_{\Omega\backslash Q} d\eta = \int_{B_R(o)\backslash Q} d\eta$, and substituting this into the above two inequalities yields

$$\int_{\Omega} v(|x|)d\eta \geq \int_{B_R(o)} v(|x|)d\eta.$$

In particular, when the equality holds, one has

$$\int_{\Omega\backslash Q} v(|x|)d\eta = v(R)\int_{\Omega\backslash Q} d\eta, \qquad \int_{B_R(o)\backslash Q} v(|x|)d\eta = v(R)\int_{B_R(o)\backslash Q} d\eta$$

simultaneously. Since the non-constant function v is non-increasing, Ω is the ball $B_R(o)$ (or Ω is isometric to $B_R(o)$). The situation for the non-constant function u can be dealt with similarly. ∎

Proof of Theorem 1.11. Define $f(t) := h(t)x_i/t$, where i is chosen to be an integer of the set $\{1, 2, \ldots, n\}$. Then, applying the Brouwer's fixed point theorem and choosing a suitable coordinate origin $o \in \text{hull}(\Omega)$, we can assure $\int_\Omega f d\eta = 0$. This can be seen by using a very similar argument to that on [39, pp. 634–635]. In fact, one can also check our another work [13], where we have given a detailed explanation on how to get the suitable coordinate system such that $\int_\Omega f d\eta = 0$. By the characterization (1.22), and by using a similar calculation to [39, (2.9)–(2.10), p. 635], one has

$$\mu_{1,\phi}(\Omega) \leq \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta}{\int_\Omega h^2 d\eta}.$$

On the other hand, by Lemma 4.3 and Lemma 4.4, we have

$$\int_{\Omega} \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta \leq \int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta$$

and

$$\int_{\Omega} h^2 d\eta \geq \int_{B_R(o)} h^2 d\eta.$$

Therefore, we have

$$\mu_{1,\phi}(\Omega) \leq \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta}{\int_\Omega h^2 d\eta} \leq \frac{\int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta}{\int_{B_R(o)} h^2 d\eta} = \mu_{1,\phi}(B_R(o)),$$

which, together with the description of the equality case in Lemma 4.4, implies the assertion of Theorem 1.11. ∎

Proof of Theorem 1.12. We still use $f(t)$ as the trail function, but now the distance should be the Riemannian distance in the hyperbolic space $\mathbb{H}^n$. In the hyperbolic case, using a similar argument to that in the proof of Theorem 1.11, we have

$$\mu_{1,\phi}(\Omega) \le \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{(\sinh t)^2}\right] d\eta}{\int_\Omega h^2 d\eta}. \tag{4.3}$$

On the other hand,

$$\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{(\sinh t)^2}\right] = 2h'h'' + 2(n-1)\frac{hh' \sinh t - h^2 \cosh t}{(\sinh t)^3}.$$

Putting (4.1) into the above equality and using the facts $\sinh t \ge 0$, $\cosh t \ge 1$ for $t \ge 0$, one has

$$\begin{aligned}
&\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{(\sinh t)^2}\right]\\
&= -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - \frac{2(n-1)\cosh t}{\sinh t}(h')^2\\
&\quad - \frac{2(n-1)\cosh t}{\sinh^3 t}h^2 + \frac{4(n-1)}{\sinh^2 t}hh'\\
&\le -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - \frac{2(n-1)}{\sinh t}(h')^2\\
&\quad - \frac{2(n-1)}{\sinh^3 t}h^2 + \frac{4(n-1)}{\sinh^2 t}hh'\\
&= -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - 2(n-1)\frac{(h')^2\sinh^2 t + h^2 - 2hh'\sinh t}{\sinh^3 t}\\
&= -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - 2(n-1)\frac{(h'\sinh t - h)^2}{\sinh^3 t} \le 0.
\end{aligned}$$

Then, by applying Lemma 4.4, we have

$$\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta \le \int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta$$

and

$$\int_\Omega h^2 d\eta \ge \int_{B_R(o)} h^2 d\eta.$$

Therefore, from (4.3), we obtain

$$\mu_{1,\phi}(\Omega) \le \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta}{\int_\Omega h^2 d\eta} \le \frac{\int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta}{\int_{B_R(o)} h^2 d\eta} = \mu_{1,\phi}(B_R(o)),$$

which together with the description of the equality case in Lemma 4.4 implies the assertion of Theorem 1.12 directly. ∎

A. Appendix

We give a proof of Theorem 4.1 in details. Assume that f is an eigenfunction of the Witten Laplace operator Δ_ϕ, and f can be decomposed into $T(t)G(\xi)$, where $t := d(o,\cdot)$ stands for the Riemannian distance to the point o, and $\xi \in S_o^{n-1} \subset T_o\mathbb{M}^n(\kappa)$. A simple calculation gives us that

$$0 = \Delta_\phi f + \mu f = S_\kappa^{1-n}(S_\kappa^{n-1}T')'G - S_\kappa^2 T v_l G - \phi' T' G + \mu T G,$$

where v_l denotes the closed eigenvalue of the Laplacian on the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$, i.e., $v_l = l(l+n-2)$, $l = 0, 1, 2, \ldots$. Simplifying the above equation gives us a second-order ODE as follows:

$$T'' + \Big[\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\Big]T' + \Big(\mu - \frac{v_l}{S_\kappa^2}\Big)T = 0, \tag{A.1}$$

where $C_\kappa(t) = S_\kappa'(t)$. For the Neumann eigenvalue problem of the Witten Laplacian Δ_ϕ, in order to ensure the smoothness of the function T, we have the following:

- when $l = 0$, $T'(0) = 0$;
- $T(t) \sim t^l$, $l = 1, 2, \ldots$;
- T satisfies the Neumann boundary condition $T'(R) = 0$.

Choosing a sufficiently small positive number ε and letting

$$p(t) = e^{\int_\varepsilon^t (((n-1)C_\kappa)/S_\kappa - \phi')ds},$$

we can simplify (A.1) into a Sturm–Liouville equation

$$(pT')' + (\mu - v_l S_\kappa^{-2})pT = 0. \tag{A.2}$$

Assume that for a fixed v_l, $\mu_{l,j,\phi}$, $j = 1, 2, \ldots$, is the j-th eigenvalue related to v_l, and $T_{l,j,\phi}$ denotes an eigenfunction belonging to $\mu_{l,j,\phi}$. Here, the purpose of placing the symbol ϕ in the subscript of $\mu_{l,j,\phi}$ is to emphasise that, theoretically, $\mu_{l,j,\phi}$ and $T_{l,j,\phi}$ have a close relation to the function ϕ, since the function $p(t)$ in equation (A.2) depends on $\phi'(t)$. In this setting, equation (A.2) can be rewritten as

$$(pT'_{l,j,\phi})' + (\mu_{l,j,\phi} - v_l S_\kappa^{-2})pT_{l,j,\phi} = 0, \tag{A.3}$$

which implies

$$\int_0^R T_{l,j,\phi}\, T_{l,k,\phi}\, p\,dt = 0, \quad \text{when } \mu_{l,j,\phi} \neq \mu_{l,k,\phi}. \tag{A.4}$$

Moreover, one can normalise T so that

$$\int_0^R T_{l,j,\phi} T_{l,j,\phi}\, p dt = 1.$$

For an equation of the form similar to (A.3), we have the following fact.

Lemma A.1. *Assume that the functions f and g satisfy separately the equations*

$$(pf')' + (\alpha - \sigma(t))pf = 0, \tag{A.5}$$

$$(pg')' + (\beta - \tau(t))pg = 0, \tag{A.6}$$

and also the boundary conditions given as in the system (4.1). Then we have

$$p(fg' - f'g)(t) = \int_0^t [\alpha - \beta + (\tau - \sigma)] pfg ds.$$

Proof. Multiplying both sides of equation (A.5) by g, multiplying both sides of equation (A.6) by f, and then taking their difference yields

$$(pf')'g - (pg')'f + [\alpha - \beta + (\tau(t) - \sigma(t))]pfg = 0.$$

Integrating both sides of the above equality from 0 to t, and using the boundary conditions given as in the system (4.1), one can get the assertion of Lemma A.1 directly. ∎

By the standard Sturm–Liouville theory for second-order ODEs, we know that $T_{l,j,\phi}$ has exactly $j-1$ zeros on the interval $(0, R)$. So, $T_{l,1,\phi}$ keeps its sign unchanged on $(0, R)$. Without loss of generality, we may assume that $T_{l,1,\phi}$ and $T_{k,1,\phi}$ are both greater than 0, where $l < k$. Then, by Lemma A.1, when $t = R$, we have $\mu_{l,1,\phi}(R) < \mu_{k,1,\phi}(R)$, $l < k$. Since for the eigenvalue problem (1.20), we know from its sequence (1.21) that $\mu_{1,\phi} = \mu_{0,1,\phi} = 0$. Hence, if one wants to get the first non-zero Neumann eigenvalue $\mu_{1,\phi}$ of the Witten Laplacian on $B_R(o)$, one only needs to know exactly which one is smaller between $\mu_{0,2,\phi}$ and $\mu_{1,1,\phi}$.

The following lemma is fundamental.

Lemma A.2. *When $l \geq 1$, $T'_{l,j,\phi}$ has only $j-1$ zeros in the interval $(0, R)$.*

Proof. From (A.3), one has

$$pT''_{l,j,\phi} + p'T'_{l,j,\phi} + \mu_{l,j,\phi}pT_{l,j,\phi} - v_l S_\kappa^{-2} pT_{l,j,\phi} = 0. \tag{A.7}$$

Since $T_{l,1,\phi}$ has no zero points on the interval $(0, R)$, we can assume that $T_{l,1,\phi}$ is greater than 0. According to the boundary conditions, if $T'_{l,1,\phi}$ is not constantly greater than 0 on the interval $(0, R)$, then there exists a $t_0 < t_1$ such that $T''_{l,1,\phi}(t_0) \leq 0$, $T'_{l,1,\phi}(t_0) = 0$, and $T''_{l,1,\phi}(t_1) \geq 0$, $T'_{l,1,\phi}(t_1) = 0$ hold true. Together with (A.7), we obtain

$$S_\kappa^2(t_0) \geq \frac{v_l}{\mu_{l,1,\phi}} \geq S_\kappa^2(t_1).$$

Due to the increasing property of $S_\kappa(t)$, this contradicts $t_0 < t_1$. Thus, $T'_{l,1,\phi}$ has no zero points in the interval $(0, R)$. For the case $T'_{l,j,\phi}$, $j > 1$, one only needs to repeat the above argument in each nodal domain. ■

It is not hard to see that the function $T_{0,2}$ satisfies

$$\begin{cases} (pT'_{0,2,\phi})' + \mu_{0,2,\phi}\, pT_{0,2,\phi} = 0, \\ T'_{0,2,\phi}(0) = T'_{0,2,\phi}(R) = 0. \end{cases} \tag{A.8}$$

Since $T_{0,1,\phi}$ is a non-zero constant function, and $T_{0,2,\phi}$ is orthogonal to $T_{0,1,\phi}$ in the sense of (A.4), we know that $T_{0,2,\phi}$ changes sign on the interval $(0, R)$. Therefore, we may assume that $T_{0,2,\phi}$ is positive on some interval $(0, r_0)$ and $T_{0,2,\phi}(r_0) = 0$, $0 < r_0 < R$. If there exists $r^* \in [0, r_0)$ such that $T''_{0,2,\phi}(r^*) \geq 0$ and $T'_{0,2,\phi}(r^*) = 0$, then substituting this into (A.8) yields $((pT'_{0,2,\phi})' + \mu_{0,2,\phi}\, pT_{0,2,\phi})(r^*) > 0$, which contradicts with the first equation in the system (A.8). Hence, we conclude that $T'_{0,2,\phi}$ is negative on the interval $(0, r_0)$. Since ϕ is non-increasing, $p' \geq 0$ can be obtained, and then from (A.8) again, we have $T''_{0,2,\phi}(r_0) \geq 0$ at r_0.

We notice that the function $T_{1,1,j}$ satisfies the following equation

$$(pT'_{1,1,\phi})' + (\mu_{1,1,\phi} - (n-1)S_\kappa^{-2})pT_{1,1,\phi} = 0. \tag{A.9}$$

Differentiating both sides of the first equation in the system (A.8) results in

$$(pT''_{0,2,\phi})' + \Big(\mu_{0,2,\phi} + \Big(\frac{p'}{p}\Big)'\Big)pT'_{0,2,\phi} = 0. \tag{A.10}$$

Combining (A.9)–(A.10), and applying Lemma A.1, we can obtain at r_0 that

$$\begin{aligned} &p(T_{1,1,\phi}T''_{0,2,\phi} - T'_{1,1,\phi}T'_{0,2,\phi})(r_0) \\ &= \int_0^{r_0} \Big[\mu_{1,1,\phi} - \mu_{0,2,\phi} + \Big(\Big(-\frac{p'}{p}\Big)' - (n-1)S_\kappa^{-2}\Big)\Big]pT_{1,1,\phi}T'_{0,2,\phi}\,dt. \end{aligned} \tag{A.11}$$

Since ϕ is a convex function, $\phi'' \geq 0$, and so we have

$$-\Big(\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\Big)' - (n-1)S_\kappa^{-2} \geq 0.$$

Substituting

$$p(t) = e^{\int_{\varepsilon}^{t}(((n-1)C_{\kappa})/S_{\kappa}-\phi')ds}$$

into the above inequality, one has

$$\Big(-\frac{p'}{p}\Big)' - (n-1)S_{\kappa}^{-2} \geq 0.$$

Together with the fact that at r_0, $T_{1,1,\phi} > 0$, $T'_{1,1,\phi} \geq 0$, $T_{0,2,\phi} > 0$, $T'_{0,2,\phi} \leq 0$ and $T''_{0,2,\phi} \geq 0$, it follows from (A.11) that $\mu_{1,1,\phi} < \mu_{0,2,\phi}$. That is to say, the first nonzero Neumann eigenvalue $\mu_{1,\phi}(B_R(o))$ of the Witten Laplacian on $B_R(o)$ should be $\mu_{1,\phi} = \mu_{1,1,\phi}$. Substituting into (A.1) results in

$$T'' + \Big[\frac{(n-1)C_{\kappa}}{S_{\kappa}} - \phi'\Big]T' + (\mu_1(B_R(o)) - v_1 S_{\kappa}^{-2})T = 0,$$

which is exactly the first equation in the system (4.1). This completes the proof of Theorem 4.1.

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