

A hot spots theorem for the mixed eigenvalue problem with small Dirichlet region

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Abstract. We prove that on convex domains, first mixed Laplace eigenfunctions have no interior critical points if the Dirichlet region is connected and sufficiently small. We also find two seemingly new estimates on the first mixed eigenvalue to give explicit examples of when the Dirichlet region is sufficiently small.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Let $D \subseteq \bar{\Omega}$ (where $\bar{\Omega}$ is the closure of Ω in \mathbb{R}^2) be a compact set such that $\Omega \setminus D$ is connected. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda & \text{in } \Omega \setminus D, \\ u \equiv 0 & \text{in } D, \\ \partial_\nu u \equiv 0 & \text{in } \partial\Omega \setminus D. \end{cases} \quad (1.1)$$

When D has non-empty interior relative to $\bar{\Omega}$, we interpret this problem as deleting the relative interior of D from Ω and enforcing Dirichlet boundary conditions on ∂D . We will suppose that D is chosen such that this eigenvalue problem admits a discrete, increasing sequence of non-negative eigenvalues $\{\lambda_n^D\}$ and a corresponding orthonormal basis of eigenfunctions $\{u_n^D\}$ for $L^2(\Omega \setminus \text{int}(D))$ that (weakly) solve (1.1). When we wish to emphasize the dependence on the domain, we may write $\lambda_n^D(\Omega) := \lambda_n^D$. Note that because we did not place strong restrictions on D , the (weak) eigenfunctions may not vanish pointwise on D (see [7]). We will assume throughout the paper that D is chosen such that eigenfunctions of (1.1) vanish everywhere on D . For example, this holds if D is the closure of a relatively open subset of $\partial\Omega$ or if $D \subseteq \Omega$ and $\Omega \setminus D$ is a Lipschitz domain. However, one can show that D may be a somewhat more pathological set, such as a line segment in the interior of Ω .

Mathematics Subject Classification 2020: 35P05 (primary); 58J50 (secondary).

Keywords: eigenfunctions, eigenvalues, Laplacian, hot spots.

For some subset $E \subseteq \mathbb{R}^2$, define the *diameter* of E to be the quantity

$$\sup\{|x - y| : x, y \in E\}.$$

Theorem 1.1. *Suppose that Ω is convex. There exists $\varepsilon > 0$ such that if D is connected with diameter at most ε , then u_1^D has no critical points in $\Omega \setminus D$.¹ Moreover, if d denotes the diameter of Ω and $j_0 \approx 2.40483$ is the first zero of the Bessel function of order zero, then it suffices to take*

$$\varepsilon = \sqrt{\frac{1}{\pi}|\Omega|} \cdot \exp\left(-\frac{4\pi}{j_0^2} \cdot \frac{d^2}{|\Omega|}\right).$$

Remark 1.2. If D is compactly contained in Ω , then the hypothesis that D be connected is necessary. We demonstrate this in Example 4.1 below. If D is contained in $\partial\Omega$, then it is unclear to us whether connectivity is a necessary condition or if its requirement is merely an artifact of the proof.

Remark 1.3. Our proofs of Theorems 1.1, 1.6, and 2.3 are heavily inspired by the proof of Miyamoto's [15, Theorem A]. By extension, Theorems 1.6 and 2.3 are modifications of Weinberger's proof of the maximality of the second Neumann eigenvalue of a Euclidean ball [21].

The well-known hot spots conjecture of J. Rauch (see [20]) states that extrema of solutions to the heat equation (with generic initial conditions) on a free membrane tend toward the boundary of the membrane as time tends toward infinity. It is not hard to see (see [3]) that this statement is equivalent to second Neumann eigenfunctions of the Laplace operator (i.e., second eigenfunctions of (1.1) with $D = \emptyset$) having extrema only on the boundary. A natural generalization of this problem is to ask for which sets D does the heat equation with perfect refrigeration on D and perfect insulation on $\partial\Omega \setminus D$ have solutions with extrema tending toward the boundary over time. Theorem 1.1 shows that if Ω is convex and D is connected and sufficiently small, then this is the case.

There has been recent interest by researchers in variants of the hot spots conjecture for the mixed problem. In 2024, the following pre-prints appeared: [1, 8, 9, 14]. To our knowledge, these are the first papers on the topic since a handful of results in the early 2000s: [4, 5, 18].

Theorem 1.1 is the combination of Propositions 1.5 and 2.4 below.

¹Recall that a domain is, by definition, open, so the theorem does not prohibit critical points in $\partial\Omega$.

Definition 1.4. Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain. Let $D \subseteq \bar{\Omega}$ be as above. For each y in the smooth part of $\partial\Omega$, let $\nu(y)$ be the outward normal vector to the boundary at y . We say that the pair (Ω, D) is *Neumann convex* if for any $x \in \Omega \setminus D$ and for any $y \in \partial\Omega \setminus D$ for which $\nu(y)$ is defined, the dot product $(y - x) \cdot \nu(y)$ is non-negative. Equivalently, (Ω, D) is Neumann convex if and only if $\partial\Omega \setminus D$ is contained in the boundary of the convex hull of $\Omega \setminus D$.

For instance, if Ω is convex, then (Ω, D) is Neumann convex for any choice of $D \subseteq \bar{\Omega}$. The converse, however, is not true. For example, if Ω is a nodal domain of a second Neumann eigenfunction of a convex domain $\tilde{\Omega}$ (which one often expects to be non-convex) and D equals the nodal set of the eigenfunction, then (Ω, D) is Neumann convex. For other examples of non-convex but Neumann convex domains, see Examples 4.1, 4.2, and 4.4 in Section 4 below.

Proposition 1.5. Suppose that (Ω, D) is Neumann convex and that D is connected. Let d be the diameter of Ω , and let $j_0 \approx 2.4048$ be the first zero of the Bessel function $J_0(x)$. If

$$\lambda_1^D \leq \left(\frac{j_0}{d}\right)^2,$$

then u_1^D has no critical points in $\Omega \setminus D$.

In Section 2, we solve two shape optimization questions for the mixed problem that appear to be new. Using these estimates, we show some explicit examples where this inequality holds, and we use these results to prove some new results on the hot spots conjecture. Roughly speaking, Theorem 1.6 applies when Ω is long and narrow with D contained near an end of Ω , while Proposition 2.4 applies when the diameter of D is small.

Theorem 1.6. Let (Ω, D) be a Neumann convex pair embedded in the upper half plane $\{(x, y) \mid y \geq 0\}$ such that D is contained in the closed second quadrant $\{(x, y) \mid x \leq 0, y \geq 0\}$. Let ℓ be the length of the projection of Ω onto the y -axis. Let A be the area of the intersection of Ω with the first quadrant $\{(x, y) \mid x, y \geq 0\}$. The first mixed eigenvalue satisfies

$$\lambda_1^D \leq \left(\frac{\pi\ell}{2A}\right)^2.$$

This inequality is sharp, and equality is uniquely achieved when Ω is the rectangle $(0, A/\ell) \times (0, \ell)$ with $D = \{0\} \times [0, \ell]$. Moreover, if

$$\frac{d\ell}{A} \leq \frac{2j_0}{\pi} \approx 1.531,$$

then $\lambda_1^D \leq (j_0/d)^2$, so first mixed eigenfunctions have no critical points in $\Omega \setminus D$.

Let P_n denote a regular polygon with n edges for $n \geq 3$. Let e be the relative interior of an edge of P_n , and let Q_n be the reflection of P_n over e . Let $K_n = P_n \cup e \cup Q_n$. An application of Theorem 1.6 above and [8, Theorem 1.1] is the following, which we prove in Section 4.

Theorem 1.7. *Second Neumann eigenfunctions of the Laplace operator on K_n have no interior critical points. In particular, the hot spots conjecture holds for these domains. Moreover, the second Neumann eigenvalue of these domains is simple.*

We now give a brief outline of the paper. In Section 2, we make estimates on the first mixed eigenvalues, showing in particular that this eigenvalue tends to zero with the diameter of the Dirichlet region. In Section 3, we provide proofs of Theorem 1.1 and Proposition 1.5. To end the paper in Section 4, we provide a number of examples and counterexamples illustrating our results.

2. First mixed eigenvalue estimates

In this section, we prove an eigenvalue estimate that, together with Proposition 1.5, proves Theorem 1.1. We begin by recalling the variational characterization of the first mixed eigenvalue and eigenfunction. Let $H_D^1(\Omega)$ be the completion of the set of C^∞ functions in $\bar{\Omega}$ that vanish on D with respect to the norm

$$\|u\|_{H_D^1}^2 = \int_{\Omega} (|u|^2 + |\nabla u|^2).$$

Then we have

$$\lambda_1^D = \inf_{u \in H_D^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2}, \quad (2.1)$$

and the minimizers of the functional above are exactly the first mixed eigenfunctions of the Laplacian. Constructing appropriate test functions for this functional allows us to give upper bounds on the first mixed eigenvalue.

Let $B(0, R_1)$ denote the ball in \mathbb{R}^2 centered at the origin with radius $R_1 > 0$. Consider the class of all pairs (Ω, D) as in Section 1 such that $D \subseteq B(0, R_1)$ and such that the area of $\Omega \setminus B(0, R_1)$ is some fixed positive number V . Theorem 2.3 below shows that the annulus of inner radius R_1 and area V maximizes the first mixed eigenvalue over all such pairs. This estimate is particularly useful when Ω is a doubly connected domain with D equal to its inner boundary.

The existing literature on eigenvalue optimization for the mixed problem is somewhat limited, but there do exist previous results in this area; see [2, 10, 16, 19]. Hersch's result in [10] solves a similar problem to Theorem 2.3 in which the area of Ω itself

as well as the length L of the inner boundary are both fixed quantities. Using simple estimates on the eigenvalues of annuli (see equation (2.2) and Remark 2.5 below), Theorem 2.3 gives a smaller upper bound on the first mixed eigenvalue than Hersch's theorem does when R_1 is small and L is large.

Notation 2.1. For $0 < R_1 < R_2$, let $A(R_1, R_2) = \{R_1 < |(x, y)| < R_2\}$ be the annulus with inner radius R_1 and outer radius R_2 centered at the origin.

Hypothesis 2.2. Let Ω be a bounded Lipschitz domain, and let $D \subseteq \bar{\Omega}$ be as in Section 1. We may translate Ω and D such that D is contained in the closed ball $\bar{B}(0, R_1)$, where R_1 equals the diameter of D . Let $\Omega_+ = \Omega \setminus B(0, R_1)$, and let

$$R_2 = \sqrt{R_1^2 + |\Omega_+|/\pi}.$$

Theorem 2.3. Let Ω , R_1 , and R_2 be as in Hypothesis 2.2. Then

$$\lambda_1^D(\Omega) \leq \lambda_1^{\partial B(0, R_1)}(A(R_1, R_2)).$$

Equality holds if and only if $\Omega = A(R_1, R_2)$ with $D = \partial B(0, R_1)$.

Proof. Let ϕ be a non-negative first mixed eigenfunction for $A(R_1, R_2)$ with $D = \partial B(0, R_1)$. Since the first mixed eigenvalue is simple and ϕ does not change signs, ϕ must be a radial function. We claim that ϕ is strictly monotonic in the radial direction. Indeed, if not, then the set of critical points of ϕ contains a circle of radius $R_3 \in (R_1, R_2)$. The restriction of ϕ to the annulus $A(R_3, R_2)$ is a non-constant, nowhere vanishing Neumann eigenfunction of $A(R_3, R_2)$. Such a function cannot exist, so the claim must hold.

Extend ϕ to $\{|(x, y)| > R_2\}$ by setting it equal to its (constant) value on the outer boundary of $A(R_1, R_2)$. Extend ϕ to vanish in the disk $\{|(x, y)| < R_1\}$. Note that $A(R_1, R_2)$ has area $|\Omega_+|$, so

$$|A(R_1, R_2) \setminus \Omega_+| = |\Omega_+ \setminus A(R_1, R_2)|.$$

Then the estimates

$$\int_{\Omega} |\nabla \phi|^2 = \int_{\Omega \cap A(R_1, R_2)} |\nabla \phi|^2 \leq \int_{A(R_1, R_2)} |\nabla \phi|^2$$

and

$$\begin{aligned} \int_{\Omega} |\phi|^2 &= \int_{\Omega \cap A(R_1, R_2)} |\phi|^2 + \int_{\Omega \setminus A(R_1, R_2)} |\phi|^2 \\ &\geq \int_{\Omega \cap A(R_1, R_2)} |\phi|^2 + \int_{A(R_1, R_2) \setminus \Omega} |\phi|^2 = \int_{A(R_1, R_2)} |\phi|^2 \end{aligned}$$

combine to give

$$\lambda_1^D(\Omega) \leq \frac{\int_{\Omega} |\nabla \phi|^2}{\int_{\Omega} |\phi|^2} \leq \frac{\int_{A(R_1, R_2)} |\nabla \phi|^2}{\int_{A(R_1, R_2)} |\phi|^2} = \lambda_1^{\partial B(0, R_1)}(A(R_1, R_2)).$$

If each of the above inequalities is actually equality, then either $\Omega = A(R_1, R_2)$ or Ω equals the union of $A(R_1, R_2)$ with a subset of $B(0, R_1)$ with non-empty interior. In the latter case, unique continuation implies that the eigenvalue inequality is strict. ■

Proposition 2.4. *Let Ω , R_1 , and R_2 be as in Hypothesis 2.2. Suppose that $R_1 \leq e^{-2} \cdot R_2$. Then*

$$\lambda_1^D(\Omega) \leq \frac{4}{R_2^2 \ln(R_2/R_1)}.$$

In particular, for fixed Ω , $\lambda_1^D(\Omega)$ approaches 0 as the diameter of D approaches 0.

Proof. We begin by bounding $\lambda_1^{\partial B(0, \varepsilon)}(A(\varepsilon, 1))$ from above when $\varepsilon < e^{-2}$. Let (r, θ) denote polar coordinates on \mathbb{R}^2 . Let $\phi(r, \theta) = 1 - \ln(r)/\ln(\varepsilon)$. Then ϕ vanishes on $\{r = \varepsilon\}$, so its restriction to the annulus is a valid test function. We have

$$\int_{A(\varepsilon, 1)} |\nabla \phi|^2 = \frac{-2\pi}{\ln \varepsilon}.$$

Note that for each fixed r , ϕ is increasing in ε . Moreover, $r \mapsto r\phi(r)^2$ is a convex function whose derivative at $r = 1$ equals $1 - 2/\ln(\varepsilon)$. Since $\varepsilon < e^{-2}$, this implies that $r\phi^2 > 2r - 1 \geq 0$ for $r \in (1/2, 1)$. Hence,

$$\int_{A(\varepsilon, 1)} \phi^2 \geq 2\pi \int_{1/2}^1 (2r - 1) dr = \frac{\pi}{2}.$$

We then have

$$\lambda_1^{\partial B(0, \varepsilon)}(A(\varepsilon, 1)) \leq \frac{-4}{\ln \varepsilon}.$$

Scaling a domain by a positive value C rescales its eigenvalues by $1/C^2$. Thus, the more general annulus $A(R_1, R_2)$ satisfies the inequality

$$\lambda_1^{\partial B(0, R_1)}(A(R_1, R_2)) = \frac{1}{R_2^2} \lambda_1^{\partial B(0, R_1/R_2)}(A(R_1/R_2, 1)) \leq \frac{4}{R_2^2 \ln(R_2/R_1)}. \quad (2.2)$$

Theorem 2.3 then gives the desired estimate. ■

Remark 2.5. Using a Poincaré-type inequality, one can show that the first mixed eigenvalue of the annulus is bounded below by

$$\lambda_1^{B(0,R_1)}(A(R_1, R_2)) \geq \frac{2}{R_2^2 \ln(R_2/R_1)}.$$

Hence, the estimate on $\lambda_1^{B(0,R_1)}(A(R_1, R_2))$ obtained in the proof of Proposition 2.4 is sharp up to a constant multiple. This should be compared with the more general result in [7], which is stated in terms of the Sobolev capacity of D .

3. Proofs of main results

By Proposition 2.4, Theorem 1.1 follows from Proposition 1.5, which we prove in this section. We begin by reviewing some preparatory results on nodal sets (i.e., zero-level sets) of Laplace eigenfunctions.

Nodal sets of Laplace eigenfunctions have been studied for many years. It is well known that non-constant functions satisfying $-\Delta u = \lambda u$ on an open set have nodal sets that are unions of real-analytic arcs (see, e.g., [6]) and, in particular, have no isolated points. If such u vanishes at a critical point p , then there exists a disk neighborhood U of p such that $(U \cap u^{-1}(\{0\})) \setminus \{p\}$ is the union of at least four arcs. Moreover, U can be chosen such that u is positive in half of the connected components of $U \setminus u^{-1}(\{0\})$ and negative in the other components. The following lemma places an additional restriction on the nodal sets of eigenfunctions whose eigenvalue is λ_1^D .

Lemma 3.1. *Let u be a pointwise solution of $-\Delta u = \lambda_1^D u$ (u is not necessarily equal to u_1^D) in Ω that extends continuously to $\bar{\Omega}$. Identify u with its extension. If $u^{-1}(\{0\})$ contains a loop intersecting Ω , then the region bounded by this loop intersects D .*

Proof. Recall the variational formulation (2.1) for λ_1^D and the fact that the only minimizers of the functional are scalar multiples of u_1^D . Suppose for contradiction that $u^{-1}(\{0\})$ does contain a loop that bounds a topological disk B that does not intersect D . Then $\phi := u\chi_B$ (where χ_B is the indicator function for B) is an element of $H_D^1(\Omega)$, and integration by parts gives

$$\int_{\Omega} |\nabla \phi|^2 = \lambda_1^D \int_{\Omega} |\phi|^2$$

since $\phi = 0$ on $\partial\Omega$. This implies that ϕ is a scalar multiple of u_1^D . Since the loop intersects Ω , the set $\Omega \setminus B$ is non-empty and open, so we obtain a contradiction to unique continuation. ■

Let J_0 be the order zero Bessel function of the first kind (see, for instance, [17, Chapter 10]), and let $j_0 \approx 2.4048$ be the first positive zero of $J_0(x)$. Note that $J_0(0) = 1$, so J_0 is positive on $[0, j_0)$. Note also that $J'_0(0) = 0$ and that $J'_0(x) < 0$ on $(0, j_0]$. Let d be the diameter of Ω , and suppose that D is chosen such that $\lambda_1^D \leq \left(\frac{j_0}{d}\right)^2$.

Proof of Proposition 1.5. Suppose toward a contradiction that u_1^D has a critical point $p \in \Omega$. Since u does not vanish anywhere in $\Omega \setminus D$, we may suppose without loss of generality that $u_1^D \geq 0$. By translating Ω , we may suppose without loss of generality that p equals the origin. Using polar coordinates, let $w(x, y) = J_0(\sqrt{\lambda_1^D} r)$. For $(x, y) \in \bar{\Omega}$, we have $r < d$ (with strict inequality since p is an interior point), from which it follows that $w > 0$ in $\bar{\Omega}$. By definition, w satisfies $-\Delta w = \lambda_1^D w$ in Ω . Define $f(x, y) = u_1^D(0)w(x, y) - u_1^D(x, y)$. Then $-\Delta f = \lambda_1^D f$. Moreover, f is non-constant since $\lambda_1 > 0$ and $f > 0$ on D . We will use f to construct a test function that contradicts the variational formulation (2.1) for λ_1^D .

We have $f(0) = 0$ and $\nabla f(0) = 0$ since 0 is a critical point for both u_1^D and w . Thus, there exists a neighborhood U of 0 such that $U \setminus f^{-1}(\{0\})$ has at least four connected components, with f taking alternating signs on each connected component. By Lemma 3.1 and using that $f > 0$ on D , $\Omega \setminus f^{-1}(\{0\})$ has at least four connected components, and f is positive in at least two of these components. Since D is connected, there exists a unique connected component of $\Omega \setminus f^{-1}(\{0\})$ whose closure contains D . Thus, there exists a connected component B of $\Omega \setminus f^{-1}(\{0\})$ on which f is positive and whose closure does not intersect D . Let $\phi = f\chi_B \in H_D^1(\Omega)$.

We claim that $\partial_\nu \phi < 0$ almost everywhere on $\partial\Omega \cap \partial B$. Indeed, since $|(x, y)| < d$ on $\partial\Omega$ and since u_1^D is Neumann on $\partial\Omega \setminus D$, the following holds for almost every $(x, y) \in \partial\Omega \setminus D$:

$$\partial_\nu \phi(x, y) = u_1^D(0)\partial_\nu w(x, y) = u_1^D(0)\sqrt{\lambda_1^D}J'_0\left(\sqrt{\lambda_1^D}|(x, y)|\right)\frac{(x, y)}{|(x, y)|} \cdot \nu.$$

Because (Ω, D) is Neumann convex, we have $(x, y) \cdot \nu \geq 0$ for all $(x, y) \in \partial\Omega \setminus D$, so $\partial_\nu \phi \leq 0$ on $\partial\Omega \cap \partial B$. Using integration by parts and our knowledge of where ϕ vanishes, we have

$$\int_{\Omega} |\nabla \phi|^2 = \lambda_1^D \int_{\Omega} \phi^2 + \int_{\partial\Omega \cap \partial B} \phi \partial_\nu \phi \leq \lambda_1^D \int_{\Omega} \phi^2.$$

Dividing both sides of this inequality by $\int_{\Omega} \phi^2$ gives a contradiction to the variational characterization (2.1) of λ_1^D since ϕ vanishes on a non-empty open set. ■

Proof of Theorem 1.1. The first statement follows from Propositions 1.5 and 2.4. We now quantify which values of ε are sufficiently small for the first statement of the

theorem to hold. Let R_1 and R_2 be as in Hypothesis 2.2, and let d denote the diameter of Ω . Then the inequality $|\Omega_+| \geq |\Omega| - \pi R_1^2$ implies that $R_2 \geq \sqrt{|\Omega|/\pi}$. Recall that the ratio $d^2/|\Omega|$ is minimized by a Euclidean disk, so $d^2/|\Omega| \geq 4/\pi$. Since $4\pi/j_0^2 > \pi/2$, we get

$$R_1 \leq \sqrt{|\Omega|/\pi} \exp\left(-\frac{4\pi}{j_0^2} \cdot \frac{d^2}{|\Omega|}\right) \leq \sqrt{|\Omega|/\pi} \exp\left(-\frac{\pi}{2} \cdot \frac{4}{\pi}\right) = e^{-2} \sqrt{|\Omega|/\pi} \leq e^{-2} R_2.$$

Thus, by Proposition 2.4, we have

$$\lambda_1^D(\Omega) \leq \frac{4}{R_2^2 \ln(R_2/R_1)}. \quad (3.1)$$

By hypothesis, we have

$$R_1 \leq \sqrt{|\Omega|/\pi} \cdot \exp\left(-\frac{4\pi}{j_0^2} \cdot \frac{d^2}{|\Omega|}\right) \leq R_2 \exp\left(-\frac{4\pi}{j_0^2} \cdot \frac{d^2}{|\Omega|}\right),$$

which may be rearranged to give

$$\frac{4}{(|\Omega|/\pi) \ln(R_2/R_1)} \leq \left(\frac{j_0}{d}\right)^2.$$

Concatenating this with equation (3.1) gives

$$\lambda_1^D(\Omega) \leq \frac{4}{(|\Omega|/\pi) \ln(R_2/R_1)} \leq \left(\frac{j_0}{d}\right)^2,$$

and the result follows from Proposition 1.5. ■

We next prove Theorem 1.6, which in some cases gives a more relaxed geometric condition than Theorem 1.1 under which a Neumann convex pair (Ω, D) has first mixed eigenfunctions with no interior critical points.

Proof of Theorem 1.6. Orient (Ω, D) in the plane as in the paragraph preceding the theorem statement. Let R denote the rectangle $(0, A/\ell) \times (0, \ell)$. Let $\Omega_+ = \Omega \cap \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. Then R and Ω_+ have the same area, and $|R \setminus \Omega_+| = |\Omega_+ \setminus R|$. Define a test function

$$\phi(x, y) = \begin{cases} 0 & \text{if } x \leq 0, \\ \sin\left(\frac{\pi \ell x}{2A}\right) & \text{if } 0 < x < A/\ell, \\ 1 & \text{if } A/\ell \leq x. \end{cases}$$

Then, $\phi|_{\Omega} \in H_D^1(\Omega)$. The two estimates

$$\int_{\Omega} |\nabla \phi|^2 = \int_{\Omega \cap R} |\nabla \phi|^2 \leq \int_R |\nabla \phi|^2 \quad \text{and}$$

$$\int_{\Omega} |\phi|^2 = \int_{\Omega_+ \cap R} |\phi|^2 + \int_{\Omega_+ \setminus R} |\phi|^2 \geq \int_{\Omega_+ \cap R} |\phi|^2 + \int_{R \setminus \Omega_+} |\phi|^2 = \int_R |\phi|^2$$

combine to give

$$\lambda_1^D \leq \frac{\int_{\Omega} |\nabla \phi|^2}{\int_{\Omega} |\phi|^2} \leq \frac{\int_R |\nabla \phi|^2}{\int_R |\phi|^2} = \left(\frac{\pi \ell}{2A} \right)^2.$$

The sharpness of the estimate and uniqueness of the extremizer follow from a similar argument to that given in the proof of Theorem 2.3.

Now, suppose that, $d\ell/A \leq 2j_0/\pi$. Then we have

$$\lambda_1^D \leq \left(\frac{\pi \ell}{2A} \right)^2 \leq \left(\frac{j_0}{d} \right)^2.$$

The last statement follows from this estimate combined with Proposition 1.5. ■

4. Examples

In this section, we construct several examples that satisfy the hypotheses of Proposition 1.5 and Theorem 1.1 as well as examples illustrating the necessity of the connectivity hypothesis in these results. We begin with the latter.

Example 4.1. Let Ω be the square given by the product of intervals $(-1, 1) \times (-1, 1)$. We first show that if we allow D to have two connected components, then there exist arbitrarily small sets D in Ω such that u_1^D has an interior critical point. However, we do not know whether this critical point is a local extremum.

Let $0 < \varepsilon < 1/2$. Let D_ε equal the union of the disks $B((\pm\varepsilon, 0), \varepsilon/2)$. See Figure 1. Then the first eigenfunction $u_1^{D_\varepsilon}$ is even about both the x - and y -axis since it is non-negative. Hence, $\partial_x u_1^{D_\varepsilon} \equiv 0$ on the y -axis and $\partial_y u_1^{D_\varepsilon} \equiv 0$ on the x -axis. In particular, the origin is a critical point of u for all such ε .

Example 4.2. We now show that if D is allowed to have three connected components, then there exists a convex domain with the diameter of D arbitrarily small and such that the first mixed eigenfunction has an interior local extremum. Let Ω be an equilateral triangle, and let $\varepsilon > 0$. Number the vertices v_1, v_2, v_3 . For each $i = 1, 2, 3$, let B_i denote the angle bisector at vertex v_i . Let p_i be the point on B_i between v_i and the incenter of Ω such that p_i is distance $2\varepsilon/\sqrt{3}$ from the incenter. Let D_ε equal the union of the closed disks of radius $\varepsilon/2$ centered at the p_i . For ε sufficiently small, D_ε is contained in Ω and has three connected components. See Figure 1. Since D is invariant under the isometry group of Ω and a first mixed eigenfunction u_1 does not change signs in Ω , it follows that u_1 is invariant under the isometry group of Ω . Thus, u_1 restricts to a first mixed eigenfunction of each fundamental domain Ω' for this group action with Dirichlet conditions on $\Omega' \cap D_\varepsilon$ and Neumann conditions on

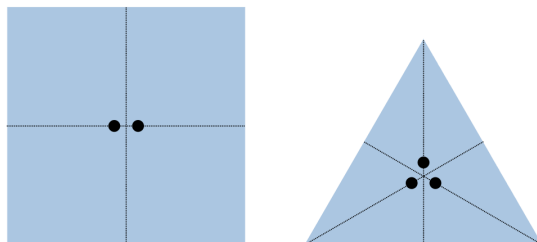


Figure 1. The pairs (Ω, D) constructed in Examples 4.1 and 4.2 – each with $\varepsilon = 0.1$. The dotted lines show the lines of symmetry for each figure. These graphics were created with Desmos (2024).

$\partial\Omega' \setminus D_\varepsilon$. By [12, Proposition 2.1], each acute vertex of Ω' is a local extremum of u_1 . It follows that the incenter of Ω is a local extremum of u_1 .

Example 4.3. Let Ω denote the unit disk, and let D be a subarc of $\partial\Omega$ of length $\alpha \leq \pi$. Then, using the notation of Theorem 1.6, we have

$$\frac{d\ell}{A} = \frac{4}{\pi - \alpha/2 + \sin(\alpha/2) \cos(\alpha/2)}.$$

Numerical computation shows that $d\ell/A \leq 2j_0/d$ as long as $\alpha \leq 1.976$. In particular, if D is a subarc of $\partial\Omega$ of length at most $\pi/2$, then u_1^D has no interior critical points.

Example 4.4. The definition of Neumann convexity allows Theorem 1.6 to apply to quite wild domains. Let $1 < a$, and let R be the rectangle $(0, a-1) \times (0, 1)$. Let R_- be the closed square $[-1, 0] \times [0, 1]$. Let γ be any simple curve contained in R_- joining the points $(0, 0)$ and $(0, 1)$ that can be locally expressed as the graph of a Lipschitz function. Let Ω be the region bounded by the edges of R not contained in the y -axis and by γ . See Figure 2. Then

$$\frac{d\ell}{A} \leq \frac{\sqrt{a^2 + 1}}{a - 1} \rightarrow 1 \text{ as } a \rightarrow \infty.$$

Thus, for a sufficiently large, $D = \gamma$ defines an eigenvalue problem for which u_1^D has no critical points.

Theorem 4.5. Let P_n denote a regular n -gon with $n \geq 4$, and let D be a line segment contained in an edge of P_n (D may equal the entire edge). Then

$$\lambda_1^D \leq \left(\frac{j_0}{d_n} \right)^2,$$

where d_n is the diameter of P_n . In particular, corresponding first mixed eigenfunctions have no interior critical points by Proposition 1.5.



Figure 2. An example of a Neumann convex pair (Ω, D) discussed in Example 4.4. The dashed line segments show the square R_- , and the solid black curve denotes $D = \gamma$. Created with Adobe Illustrator (2024).

Proof. Normalize P_n so that the distance from the center of P_n to each vertex equals 1. Suppose that P_n is embedded in the first quadrant of \mathbb{R}^2 with D contained in the y -axis as in Theorem 1.6. Using the notation of Theorem 1.6, we have

$$\frac{d\ell}{A} \leq \frac{4}{A} = \frac{4}{n \sin(\pi/n) \cos(\pi/n)}$$

for all $n \geq 4$. The right-hand side of this inequality is decreasing in n . One can numerically check that we then have

$$\frac{d\ell}{A} \leq \frac{4}{A} < \frac{2j_0}{\pi}$$

for $n = 7$ and therefore for all $n \geq 7$. However, the latter inequality fails to hold for $n \leq 6$, so the cases $n = 4, 5, 6$ require other arguments.

One can check the $n = 4$ case directly using the first mixed eigenfunction with D equal to an edge of P_4 as a test function when D is a subset of an edge.

For $n = 5$ with P_5 normalized as above, we have

$$\frac{d\ell}{A} = \frac{2 \sin(2\pi/5)(1 + \cos(\pi/5))}{5 \sin(\pi/5) \cos(\pi/5)} < \frac{2j_0}{\pi}$$

by numerical computation.

The case $n = 6$ is the most involved. We will show directly that $\lambda_1^D \leq (j_0/d)^2$ without reference to Theorem 1.6. Rescale, translate, and rotate P_6 such that D is contained in the y -axis, P_6 has a vertex at the origin, and such that the length ℓ of the projection of P_n onto the x -axis equals 1. So P_6 has diameter $2/\sqrt{3}$. Let R be the rectangle $(0, 1) \times (0, 1/\sqrt{3})$. Let \tilde{R} be the rectangle $(0, 1/2) \times (0, 1/2\sqrt{3})$. Note that $R \subseteq P_6$ (perhaps after reflecting P_6 about the y -axis) and that $P_6 \setminus R$ has twice the area of \tilde{R} . Define a test function $\phi(x, y) = \sin(\pi x/2) \in H_D^1(P_6)$. Then using the monotonicity of $\phi|_{P_6}$ and its derivative in x , we have

$$\int_{P_6} |\nabla \phi|^2 \leq \int_R |\nabla \phi|^2 + 2 \int_{\tilde{R}} |\nabla \phi|^2 = \frac{1}{2\sqrt{3}} \left(\frac{\pi}{2}\right)^2 \left(\frac{3}{2} + \frac{1}{\pi}\right)$$

and

$$\int_{P_6} \phi^2 \geq \int_R \phi^2 + 2 \int_{\tilde{R}} \phi^2 = \frac{1}{2\sqrt{3}} \left(\frac{3}{2} - \frac{1}{\pi} \right),$$

so we have

$$\lambda_1^D \leq \frac{\int_{P_6} |\nabla \phi|^2}{\int_{P_6} \phi^2} \leq \left(\frac{\pi}{2} \right)^2 \cdot \frac{\frac{3}{2} + \frac{1}{\pi}}{\frac{3}{2} - \frac{1}{\pi}} < \left(\frac{j_0}{d} \right)^2,$$

where the last inequality holds by direct numerical computation. ■

Proof of Theorem 1.7. Let u denote a second Neumann eigenfunction of K_n . Let e be the line of reflection of K_n as in the definition given in Section 1, and suppose that e is contained in the x -axis. Then u decomposes into a sum of even and odd eigenfunctions about this line:

$$\begin{aligned} u_{\text{even}}(x, y) &= \frac{1}{2}(u(x, y) + u(x, -y)), \\ u_{\text{odd}}(x, y) &= \frac{1}{2}(u(x, y) - u(x, -y)). \end{aligned}$$

We claim that u_{even} is identically equal to zero. If not, then it restricts to a second Neumann eigenfunction of P_n . In this case, every second Neumann eigenfunction of P_n extends via reflection to a second Neumann eigenfunction of K_n . By [11, Lemma 3.3], the nodal set of u_{even} cannot have distinct endpoints in e . Thus, the nodal set of u_{even} has end points in two disjoint edges of P_n . By precomposing u_{even} by an appropriate rotational symmetry, there exists another second Neumann eigenfunction v of P_n whose nodal set does not intersect e . Since the extension of v to K_n by reflection has three nodal domains, we get a contradiction to Courant's nodal domains theorem, and the claim holds.

Thus, the second Neumann eigenspace of K_n is spanned by extensions of first mixed eigenfunctions of P_n with Dirichlet conditions on e . By the simplicity of the first mixed eigenvalue, the second Neumann eigenvalue of K_n is simple. By the Hopf lemma, these eigenfunctions have no critical points in e . By Theorem 4.5, the theorem holds for $n \geq 4$. For $n = 3$, the theorem is a special case of [8, Theorem 1.1] or of [14, Corollary 1.6]. ■

We end the paper by using Proposition 1.5 to prove a more general result pertaining to the hot spots conjecture.

Theorem 4.6. *Let Ω be a convex domain, and let u be a second Neumann eigenfunction of Ω . Let $\Omega_+ = \{x \in \Omega \mid u(x) > 0\}$. If the diameter of Ω_+ is at most half of the diameter of Ω , then u has no interior critical points in Ω_+ . The same statement applies to the set on which u is negative.*

Remark 4.7. We do not know of any examples of a domain with a second Neumann eigenfunction satisfying the hypothesis of Theorem 4.6. It would be interesting to find out whether or not such a pair exists.

Before proving Theorem 4.6, we remind the reader of an upper bound on the second Neumann eigenvalue on convex domains that appears as [3, Corollary 2.1] and as Theorem 1 of Kröger’s paper [13].

Lemma 4.8. *Let Ω be a convex domain with diameter d . The second Neumann eigenvalue μ_2 of the Laplacian on Ω is at most $4(j_0/d)^2$.*

Proof of Theorem 4.6. Let Ω , u , and Ω_+ be as in the theorem statement. Let μ_2 be the eigenvalue associated to u . Because the restriction of u to Ω_+ is a first mixed eigenfunction with Dirichlet conditions on $u^{-1}(\{0\})$, we have $\lambda_1^{u^{-1}(\{0\})}(\Omega_+) = \mu_2$. Let d be the diameter of Ω , and let d_+ be the diameter of Ω_+ . Since $d_+ \leq d/2$, we have

$$\lambda_1^{u^{-1}(\{0\})} = \mu_2 \leq 4\left(\frac{j_0}{d}\right)^2 \leq 4\left(\frac{j_0}{2d_+}\right)^2 = \left(\frac{j_0}{d_+}\right)^2$$

by Lemma 4.8. By the discussion in Section 1, $(\Omega_+, u^{-1}(\{0\}))$ is Neumann convex, so we may apply Proposition 1.5 to see that $u|_{\Omega_+}$ has no critical points in Ω_+ . To get the second statement of the theorem, we simply apply the same argument to the eigenfunction $-u$. ■

Acknowledgments. The author would like to thank Matthew Lowe for illustrating Figure 2. He would also like to thank Mark Ashbaugh for providing several helpful references on the mixed shape optimization problem.

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