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Spectral analysis of Dirac operators for dislocated potentials with a purely imaginary jump

Lyonell Boulton, David Krejčiřík, and Tho Nguyen Duc

Abstract. In this paper we present a complete spectral analysis of Dirac operators with non-Hermitian matrix potentials of the form $i\,\mathrm{sgn}(x) + V(x)$ where $V \in L^1$. For $V = 0$, we compute explicitly the matrix Green function. This allows us to determine the spectrum, which is purely essential, and its different types. It also allows us to find sharp enclosures for the pseudospectrum and its complement, in all parts of the complex plane. Notably, this includes the instability region, corresponding to the interior of the band that forms the numerical range. Then, with the help of a Birman–Schwinger principle, we establish in precise manner how the spectrum and pseudospectrum change when $V \neq 0$, assuming the hypotheses $\|V\|_{L^1} < 1$ or $V \in L^1 \cap L^p$ where $p > 1$. We show that the essential spectra remain unchanged and that the ε-pseudospectrum stays close to the instability region for small ε. We determine sharp asymptotics for the discrete spectrum, whenever V satisfies further conditions of decay at infinity. Finally, in one of our main findings, we give a complete description of the weakly-coupled model.

*Dedicated to the memory of Professor E. Brian Davies FRS,
a mentor and a friend whose generosity and mathematical talent
will be greatly missed*

1. Introduction

1.1. Context and motivations

The significance of the Dirac equation lies on the fact that it describes, not only relativistic particles in quantum mechanics, but also advanced materials such as graphene. Mathematically, the study of this equation is notoriously difficult, partly because of the spinorial structure of the underlying Hilbert space and partly because of the lack

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of semi-boundedness and positivity-preserving properties. In this respect, the spectral analysis of the underlying matrix differential operator is considerably more challenging than that of its non-relativistic counterpart, the scalar Schrödinger operator.

Recently, we have seen unprecedented interest in the study of non-selfadjoint electromagnetic perturbations of the Dirac equation [3, 5–10, 13, 15, 17, 18, 24]. This is justified, on the one hand, by the availability of new proper connections with quantum mechanics [21, 26], and, on the other hand, by the new conceptual paradigms posed by the notions of pseudospectra of linear operators.

This paper is motivated by the current lack of understanding of the structure of the pseudospectra of non-selfadjoint Dirac operators. Indeed, the only available work in this direction seems to be the recent non-semiclassical construction of pseudomodes given in [18].

In order to provide an insight into the pseudospectral properties of the Dirac equation in general, and at the same time set a benchmark that differs from the available scalar models given by the classical non-selfadjoint Schrödinger operator, the present paper is devoted to introducing and examining in close detail the non-selfadjoint Dirac operator

$$\mathcal{L}_{m,V} = \begin{pmatrix} m & -\partial_x \\ \partial_x & -m \end{pmatrix} + \begin{pmatrix} i\,\mathrm{sgn}(x) & 0 \\ 0 & i\,\mathrm{sgn}(x) \end{pmatrix} + V(x), \tag{1.1}$$

on a suitable domain of $L^2(\mathbb{R}, \mathbb{C}^2)$ where $V\colon \mathbb{R} \to \mathbb{C}^{2\times 2}$ is a long-range possibly non-Hermitian matrix potential. The unperturbed operator $\mathcal{L}_m = \mathcal{L}_{m,0}$ is a relativistic non-selfadjoint version of the quantum mechanical infinite square well.

The choice is motivated by the scalar Schrödinger case on $L^2(\mathbb{R}, \mathbb{C})$ considered in [14],

$$-\partial_x^2 + i\,\mathrm{sgn}(x) \tag{1.2}$$

with potential perturbations, and it serves as a link with the analysis conducted in [7]. The scalar model (1.2) was instrumental for the non-semiclassical construction of pseudomodes established in [20], which eventually solved a notorious open problem raised during a workshop at the American Institute of Mathematics in 2015, cf. [23, Open Problem 10.1]. The relativistic variant of this construction has now been reported in [18].

The simplicity of the linear operator (1.2) is deceiving, as it hides a non-trivial structure on its pseudospectrum. Our findings reveal two main distinctions between (1.1) and this scalar model. One is about this structure and one is about the available tools to analyse it. The latter is to be expected, given the higher degree of complexity of a matrix versus a scalar operator. But the former is rather surprising. As we shall see in the next section, the linear operator (1.1) has a resolvent norm that grows quadratically, rather than linearly, at infinity inside the band forming the spectral instability region.

1.2. Structure of the paper

The organisation of the paper is as follows. Section 2 is devoted to a proper mathematical description of our main contributions. The body of the paper is Sections 3–5, where we establish the proofs of these results. After this, Appendix A includes complete details of the Birman–Schwinger-type principle formulated in [12] which is crucial to our analysis.

The core Sections 3–5 consist of two parts.

The unperturbed operator (comprising Sections 3 and 4). We begin our study of (1.1) in Section 3 by finding the explicit expression of the matrix Green function of the unperturbed operator $\mathcal{L}_m$. This allows to determine the spectrum of $\mathcal{L}_m$. As it turns out, the latter is purely essential and it comprises four symmetric segments on the boundary of the band $\Sigma = \{|\operatorname{Im} z| < 1\}$ for $m > 0$, while by contrast it is equal to $\overline{\Sigma}$ for $m = 0$. In Section 4 we find enclosures for the pseudospectrum and its complement, in all parts of the complex plane. In one of our main contributions, we compute the explicit asymptotic constants, up to order 0, of the resolvent norm of $\mathcal{L}_m$ inside the instability region Σ. The latter coincides with the interior of the numerical range.

Perturbations (comprising Section 5). The second part of the paper is devoted to the case $V \neq 0$. By applying a non-selfadjoint version of the classical Birman–Schwinger principle, we establish in precise manner how spectrum and pseudospectrum change, under two general hypotheses, $\|V\|_{L^1} < 1$ or $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}) \cap L^p(\mathbb{R}, \mathbb{C}^{2\times 2})$ for some $p > 1$. We show that the essential spectra remain unchanged in both cases and that the ε-pseudospectrum stays close to the instability region for small $\varepsilon > 0$ whenever $\|V\|_{L^1} < 1$. Then, we formulate our two other main contributions of this part. On the one hand, we determine sharp asymptotics for the discrete spectrum, whenever V satisfies further conditions of decay at infinity. On the other hand, we give a complete description of the weakly-coupled model, corresponding to potential ϵV in the regime $\epsilon \to 0$.

1.3. Notation used throughout the work

The following specific conventions will be used throughout this paper.

- $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.
- $[[m, n]] = \{k \in \mathbb{N} : m \leq k \leq n\}$ for $m, n \in \mathbb{R}$.
- For a closed operator $\mathcal{L}\colon \operatorname{Dom}(\mathcal{L}) \to \mathscr{H}$, the *spectrum* is denoted by

$$\operatorname{Spec}(\mathcal{L}) = \operatorname{Spec}_{\mathrm{p}}(\mathcal{L}) \cup \operatorname{Spec}_{\mathrm{r}}(\mathcal{L}) \cup \operatorname{Spec}_{\mathrm{c}}(\mathcal{L}),$$

where the *point*, *residual* and *continuous* spectrum denote, as usual,

$$\begin{aligned}\mathrm{Spec_p}(\mathcal{L}) &= \{z \in \mathbb{C} : \mathcal{L} - z \text{ is not injective}\},\\ \mathrm{Spec_r}(\mathcal{L}) &= \{z \in \mathbb{C} : \mathcal{L} - z \text{ is injective and } \overline{\mathrm{Ran}(\mathcal{L} - z)} \subsetneq \mathcal{H}\},\\ \mathrm{Spec_c}(\mathcal{L}) &= \{z \in \mathbb{C} : \mathcal{L} - z \text{ is injective and } \overline{\mathrm{Ran}(\mathcal{L} - z)} = \mathcal{H}\\ &\qquad \text{and } \mathrm{Ran}(\mathcal{L} - z) \subsetneq \mathcal{H}\}.\end{aligned}$$

The *resolvent set* is denoted by $\rho(\mathcal{L})$.

- We use the classical definitions of the *five types of essential spectrum*, [11, Section IX] or [19, Section 5.4]. We write $\mathrm{Spec_{ej}}(\mathcal{L})$ for $\mathrm{j} \in [[1, 5]]$, where $\mathcal{L} - z$
  - is not semi-Fredholm for e1,
  - possess a singular Weyl sequence for e2,
  - is not Fredholm for e3,
  - is not Fredholm of index 0 for e4,
  - is such that either $z \in \mathrm{Spec_{e1}}(\mathcal{L})$ or the resolvent set does not intersect the connected component of the complement of $\mathrm{Spec_{e1}}(\mathcal{L})$ where z lies, for e5.
- The *discrete spectrum* is denoted as

$$\mathrm{Spec_{dis}}(\mathcal{L}) = \mathbb{C} \setminus \mathrm{Spec_{e5}}(\mathcal{L}),$$

  and it is the set of isolated eigenvalues such that the Riesz projector has a finite-dimensional range.
- For $\varepsilon > 0$, the ε*-pseudospectrum* is denoted by

$$\begin{aligned}\mathrm{Spec}_\varepsilon(\mathcal{L}) &= \mathrm{Spec}(\mathcal{L}) \cup \{z \in \rho(\mathcal{L}) : \|(\mathcal{L} - z)^{-1}\| > \varepsilon^{-1}\}\\ &= \bigcup_{\|V\| < \varepsilon} \mathrm{Spec}(\mathcal{L} + V).\end{aligned} \tag{1.3}$$

  By convention, we set $\mathrm{Spec}_0(\mathcal{L}) = \mathrm{Spec}(\mathcal{L})$.
- $|\cdot|_{\mathbb{C}^2}$ is the Euclidean norm for vectors.
- $|\cdot|_{\mathbb{C}^{2\times 2}}$ is the operator norm induced by $|\cdot|_{\mathbb{C}^2}$ for matrices.
- $|\cdot|_{\mathrm{F}}$ is the Frobenius norm for matrices.
- For a measurable matrix-valued function $V : \mathbb{R} \to \mathbb{C}^{2\times 2}$ and for $p \in [1, \infty]$, we write $V \in L^p(\mathbb{R}, \mathbb{C}^{2\times 2})$ to indicate that $|V|_{\mathbb{C}^{2\times 2}} \in L^p(\mathbb{R}, \mathbb{C})$. We denote the L^p norm of V by

$$\|V\|_{L^p} = \||V|_{\mathbb{C}^{2\times 2}}\|_{L^p}.$$

- For a positive function $\nu'(x)$, we define the measure $\mathrm{d}\nu(x) = \nu'(x)\mathrm{d}x$. We say that $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}; \mathrm{d}\nu)$ if

$$\|V\|_{L^1(\mathrm{d}\nu)} = \int_{\mathbb{R}} |V(x)|_{\mathbb{C}^{2\times 2}} \nu'(x)\,\mathrm{d}x < \infty.$$

2. Mathematical framework and summary of results

Below, we set

$$S_m = \begin{cases} \{z \in \mathbb{C} : |\operatorname{Re} z| \geq m, |\operatorname{Im} z| = 1\} & \text{if } m > 0, \\ \{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1\} & \text{if } m = 0, \end{cases} \tag{2.1}$$

and we write $L^2(\mathbb{R}, \mathbb{C}^2) = L^2(\mathbb{R}, \mathbb{C}) \oplus L^2(\mathbb{R}, \mathbb{C})$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)}\,\mathrm{d}x \quad \text{for components } f_j,\ g_j \in L^2(\mathbb{R}, \mathbb{C}).$$

As usual,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli matrices. For mass $m \geq 0$, let $\mathcal{D}_m \colon H^1(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ be the one-dimensional free particle Dirac operator, given by

$$(\mathcal{D}_m f)(x) = (-i\partial_x)\sigma_2 f(x) + m\sigma_3 f(x). \tag{2.2}$$

It is well known that $\mathcal{D}_m$ is a selfadjoint linear operator and that its spectrum is

$$\operatorname{Spec}(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty).$$

Moreover, $\mathcal{D}_m$ is unitarily equivalent to the selfadjoint operator $\widetilde{\mathcal{D}}_m = (-i\partial_x)\sigma_1 + m\sigma_3 \colon H^1(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ via transformation by $\left(\begin{smallmatrix} i & 0 \\ 0 & -1 \end{smallmatrix}\right)$. This other realisation of the free particle Dirac operator is the one considered in [7, 8, 18]. In the present paper, we prefer the formulation (2.2), noting that the results we report below, map in a straightforward manner onto $\widetilde{\mathcal{D}}_m$.

2.1. The Dirac operator with a dislocation

In the present paper, we examine the linear operator $\mathcal{L}_m \colon H^1(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ given by

$$(\mathcal{L}_m f)(x) = \mathcal{D}_m f(x) + i \operatorname{sgn}(x) f(x) \tag{2.3}$$

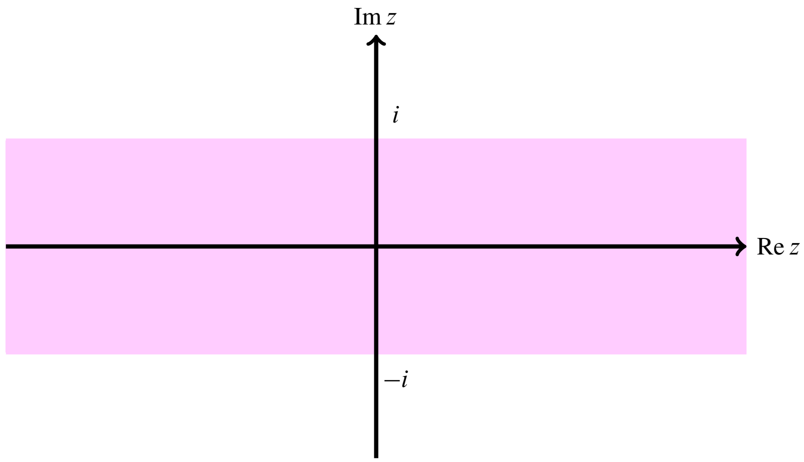

Figure 1. The numerical range of $\mathcal{L}_m$.

and its perturbations by a potential. Since multiplication by $i\,\mathrm{sgn}(x)I$ is a bounded operator, $\mathcal{L}_m$ is also closed. From the explicit expression of the adjoint,

$$(\mathcal{L}_m^* f)(x) = \mathcal{D}_m f(x) - i\,\mathrm{sgn}(x) f(x),$$

it follows that $\mathcal{L}_m$ is not a normal operator. As we shall see below, the spectral and pseudospectral properties of $\mathcal{L}_m$ and its perturbations, are rather unusual and very interesting.

The numerical range of $\mathcal{L}_m$ is the infinite closed strip

$$\mathrm{Num}(\mathcal{L}_m) = \mathbb{R} + i[-1, 1].$$

See Figure 1 and the beginning of Section 3. Unlike the Schrödinger operator with dislocated potential analysed in [14], $\mathcal{L}_m$ is neither sectorial nor $\mathcal{PT}$-symmetric. However, $\mathcal{L}_m$ is $\mathcal{T}$-selfadjoint and this implies that the spectrum respects some of the symmetries that the Schrödinger model possesses.

Here and everywhere below, a distinction of the case $m = 0$ is evident, unavoidable, and it is typical of families of non-selfadjoint operators dependent on a parameter, cf. the examples in [1] and references therein. Moreover, setting a unitary operator $Sf = g$ such that $g(x) = af(a^2 x)$ for fixed non-zero $a \in \mathbb{R} \cup i\mathbb{R}$, it is readily seen that S preserves the domains and

$$(S^* \mathcal{L}_{a^2 m} S f)(x) = a^2 \Big(\mathcal{D}_m f(x) + \frac{i}{a^2} \mathrm{sgn}\Big(\frac{x}{a^2}\Big) f(x) \Big).$$

This shows that both the case $m < 0$ and the case of a dislocated potential of the form $c\,\mathrm{sgn}(x)I$, where $c > 0$ is a constant, are covered by our results below.

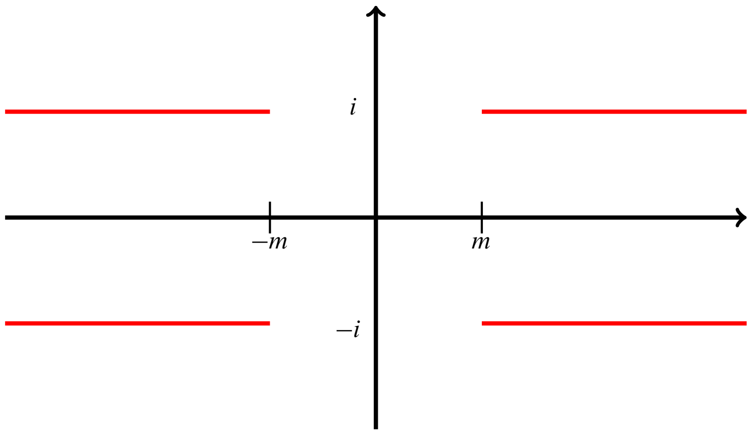

Figure 2. The red lines represent the spectrum of the operator $\mathcal{L}_m$ when $m > 0$.

As we shall see in Section 3, the spectrum of $\mathcal{L}_m$ can be determined analytically using arguments based on resolvent construction and Weyl sequences. For all $m > 0$, the spectrum is purely continuous

$$\mathrm{Spec}(\mathcal{L}_m) = \mathrm{Spec}_c(\mathcal{L}_m) = S_m.$$

Furthermore, all essential spectra coincide,

$$\mathrm{Spec}_{ej}(\mathcal{L}_m) = \mathrm{Spec}(\mathcal{L}_m), \quad \text{for all j} \in [[1, 5]].$$

Note that $\mathcal{T}$-selfadjointness implies equality of the essential spectra for $\mathrm{j} \in [[1, 4]]$ only, so the proof of equality of the full set requires extra arguments given below. An illustration of the spectrum is included in Figure 2.

In contrast, for $m = 0$, the spectrum is characterised by

$$\mathrm{Spec}_p(\mathcal{L}_0) = \{z \in \mathbb{C} : |\,\mathrm{Im}\, z| < 1\} \quad \text{and} \quad \mathrm{Spec}_c(\mathcal{L}_0) = \{z \in \mathbb{C} : |\,\mathrm{Im}\, z| = 1\}.$$

Unlike the case $m > 0$, the essential spectra are not identical:

$$\mathrm{Spec}_{ej}(\mathcal{L}_0) = \{z \in \mathbb{C} : |\,\mathrm{Im}\, z| = 1\},$$

for all $\mathrm{j} \in [[1, 4]]$ and $\mathrm{Spec}_{e5}(\mathcal{L}_0) = \mathrm{Spec}(\mathcal{L}_0)$. See the illustration in Figure 3.

For all $m \geq 0$, the comparison with the spectrum of $\mathcal{D}_m$ is obvious and striking. Despite the seemingly simple structure of the dislocation $i\,\mathrm{sgn}(x)I$, the perturbation splits the spectrum into two halves. In the case $m > 0$, this phenomenon mimics the findings of [14, Theorem 2.1] for the Schrödinger case. For $m = 0$, rather than a clear-cut split, the spectrum smears throughout the full band that forms the numerical range. In Lemma 3.4 we show that, at any z in the point spectrum, $(\mathcal{L}_0 - z)$ is

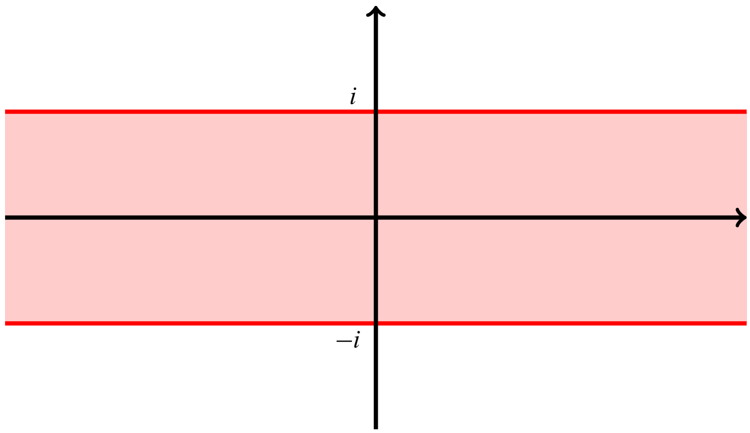

Figure 3. The red lines show the continuous spectrum of $\mathcal{L}_0$. These lines coincide with the essential spectra e1, . . . , e4. The inner part in light colour shows the point spectrum. The union of both these regions form the essential spectrum e5 and also the full $\operatorname{Spec}(\mathcal{L}_0)$.

Fredholm with index zero. Moreover, the geometric multiplicity of z is one while its algebraic multiplicity is infinite. This is neither obvious, nor a consequence of general principles.

Let us now describe the pseudospectrum of $\mathcal{L}_m$. We adopt the following terminology, which is convenient in order to articulate our findings. We say that a closed operator $\mathcal{L}$ has a *trivial pseudospectrum* if there exists a constant $C \geq 1$ such that

$$\operatorname{Spec}_\varepsilon(\mathcal{L}) \subseteq \{z \in \mathbb{C} : \operatorname{dist}(z, \operatorname{Spec} \mathcal{L}) \leq C\varepsilon\}$$

for all $\varepsilon > 0$. Any normal operator has a trivial pseudospectrum with $C = 1$, since in this case the ε-pseudospectrum coincides with the ε-neighbourhood of the spectrum. Any closed operator which is similar to a normal operator, via a bounded and boundedly invertible similarity transformation S, has also trivial pseudospectrum with $C = \|S\|\|S^{-1}\|$, the condition number of S.

According to the well-known inequalities

$$\frac{1}{\operatorname{dist}(z, \operatorname{Spec} \mathcal{L}_m)} \leq \|(\mathcal{L}_m - z)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, \operatorname{Num} \mathcal{L}_m)}$$

for all $z \notin \operatorname{Num}(\mathcal{L}_m)$, it is readily seen that for $|\operatorname{Im} z| > 1$,

- for $|\operatorname{Re} z| \geq m$,

$$\|(\mathcal{L}_m - z)^{-1}\| = \frac{1}{|\operatorname{Im} z| - 1}; \tag{2.4}$$

- for $|\operatorname{Re} z| < m$,

$$\frac{1}{\sqrt{(|\operatorname{Re} z| - m)^2 + (|\operatorname{Im} z| - 1)^2}} \leq \|(\mathcal{L}_m - z)^{-1}\| \leq \frac{1}{|\operatorname{Im} z| - 1}.$$

In the next theorem we see that, in stark contrast, the resolvent norm increases at a quadratic rate as $z \to \infty$ inside the numerical range. This statement is our first main contribution.

Theorem 2.1. *Let $m > 0$ and let $\mathcal{L}_m$ be the operator given by* (2.3)*. Assume that $z \in \mathbb{C}$ is such that $|\operatorname{Im} z| < 1$. Then,*

$$\|(\mathcal{L}_m - z)^{-1}\| = \frac{|\operatorname{Re} z|^2}{m(1 - |\operatorname{Im} z|^2)}(1 + \mathcal{O}(|\operatorname{Re} z|^{-2}))$$

as $|\operatorname{Re} z| \to \infty$. The $\mathcal{O}$ term is uniform in $|\operatorname{Im} z|$ and locally uniform in m (i.e., the constants involved can be chosen independent of $z \in \mathbb{C}$ for all $|\operatorname{Im} z| < 1$ and m on any fixed compact subset of $(0, \infty)$).

According to this theorem, when we consider the level set

$$\Big\{z \in \rho(\mathcal{L}_m) : \|(\mathcal{L}_m - z)^{-1}\| = \frac{1}{\varepsilon}\Big\}$$

for some $\varepsilon > 0$, we obtain curves whose points $z \in \mathbb{C}$ satisfy

$$|\operatorname{Im} z|^2 = 1 - \frac{\varepsilon|\operatorname{Re} z|^2}{m}\Big(1 + \mathcal{O}\Big(\frac{1}{|\operatorname{Re} z|^2}\Big)\Big)$$

in the asymptotic regime $|\operatorname{Re} z| \to +\infty$. Motivated by this, for $\alpha \in (0, 1)$ and $\varepsilon > 0$, we set regions

$$\Lambda_{\pm}^{\alpha}(\varepsilon) = \Big\{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1 + \varepsilon \text{ and } |\operatorname{Re} z|^2 \geq \frac{m(1 - |\operatorname{Im} z|^2)}{(1 \pm \alpha)\varepsilon}\Big\}.$$

These are illustrated in Figure 4 for two different values of ε.

In the next corollary, we establish sharp inclusions and exclusion zones for the pseudospectra of $\mathcal{L}_m$ in terms of these regions. Note that $\mathcal{L}_0$ has trivial pseudospectra although it is a non-normal operator.

Corollary 2.2. *Let $m \geq 0$ and let $\mathcal{L}_m$ be the operator given by* (2.3)*.*

(1) *If $m = 0$, then the pseudospectrum of $\mathcal{L}_0$ is trivial and for all $\varepsilon > 0$,*

$$\operatorname{Spec}_{\varepsilon}(\mathcal{L}_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1 + \varepsilon\}.$$

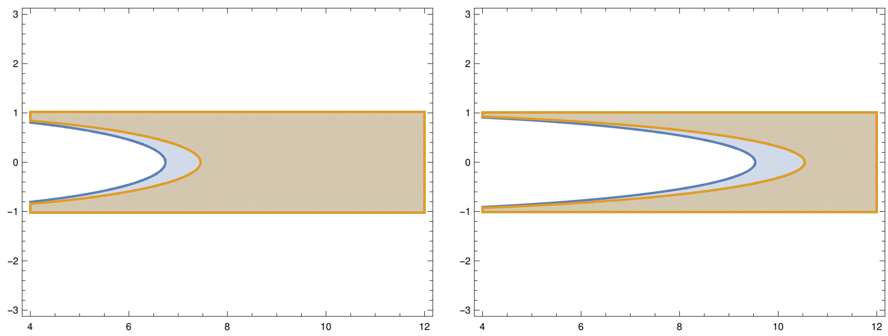

Figure 4. Regions $\Lambda_+^{0.1}(\varepsilon)$ and $\Lambda_-^{0.1}(\varepsilon)$ in the box as shown, for $m = 1$, $\varepsilon = 0.02$ (left) and $\varepsilon = 0.01$ (right). The part in blue corresponds to $\Lambda_+^{0.1}(\varepsilon) \setminus \Lambda_-^{0.1}(\varepsilon)$ and $\Lambda_-^{0.1}(\varepsilon)$ is shows in brick.

(2) *If* $m > 0$, *then the pseudospectrum of* $\mathcal{L}_m$ *is non-trivial. Namely, for every fixed* $\alpha \in (0, 1)$, *there exists* $M > 0$ *such that*

$$\Lambda_-^{\alpha}(\varepsilon) \cap \{|\operatorname{Re} z| > M\} \subset (\operatorname{Spec}_\varepsilon \mathcal{L}_m) \cap \{|\operatorname{Re} z| > M\}, \tag{2.5}$$

$$(\operatorname{Spec}_\varepsilon \mathcal{L}_m) \cap \{|\operatorname{Re} z| > M\} \subset \Lambda_+^{\alpha}(\varepsilon) \cap \{|\operatorname{Re} z| > M\} \tag{2.6}$$

for all $\varepsilon > 0$.

Proof. For $m = 0$, the statement follows directly from the spectrum of $\mathcal{L}_0$ and (2.4). Here is the proof for $m > 0$. According to Theorem 2.1, there exists a constant $C > 0$ such that

$$\left(1 - \frac{C}{|\operatorname{Re} z|^2}\right)\frac{|\operatorname{Re} z|^2}{m(1 - |\operatorname{Im} z|^2)} < \|(\mathcal{L}_m - z)^{-1}\| < \left(1 + \frac{C}{|\operatorname{Re} z|^2}\right)\frac{|\operatorname{Re} z|^2}{m(1 - |\operatorname{Im} z|^2)}$$

for large $|\operatorname{Re}(z)|$ and $|\operatorname{Im} z| < 1$. Thus, for fixed $\alpha \in (0, 1)$ and $|\operatorname{Re} z| \geq \sqrt{C/\alpha}$, we have the following. For $\varepsilon > 0$, (2.5) is ensured whenever $|\operatorname{Re} z|$ is such that

$$(1 - \alpha)\frac{|\operatorname{Re} z|^2}{m(1 - |\operatorname{Im} z|^2)} \geq \frac{1}{\varepsilon}.$$

Similarly, if

$$\frac{1}{\varepsilon} \leq \|(\mathcal{L}_m - z)^{-1}\|,$$

then

$$\frac{1}{\varepsilon} < (1 + \alpha)\frac{|\operatorname{Re} z|^2}{m(1 - |\operatorname{Im} z|^2)}.$$

Therefore, (2.6) is valid. For $|\operatorname{Im} z| > 1$, we recall (2.4). ∎

2.2. Long-range potential perturbations

In Section 5 we examine perturbations of $\mathcal{L}_m$ by an integrable matrix-valued function V. We establish results about the localisation of eigenvalues and pseudospectrum, under general assumptions on the decay of V at infinity.

Our starting point is the construction of a specific closed densely defined extension of the differential operator

$$\mathcal{L}_m + V : H^1(\mathbb{R}, \mathbb{C}^2) \cap \mathrm{Dom}(V) \to L^2(\mathbb{R}, \mathbb{C}^2),$$

via a version of the classical selfadjoint framework of Kato [16], obtained in [12]. Concretely, for almost[1] all $x \in \mathbb{R}$, let

$$V(x) = B(x)A(x) \quad \text{where } B(x) = U(x)|V(x)|^{1/2} \text{ and } A(x) = |V(x)|^{1/2},$$

in the polar decomposition $V(x) = U(x)|V(x)|$, where $U(x)$ are partial isometries and $|V(x)| = (V^*(x)V(x))^{1/2}$. Consider the family of closed operators

$$Q(z) = \overline{AR_0(z)B}, \quad \text{where } R_0(z) = (\mathcal{L}_m - z)^{-1},$$

for $z \in \rho(\mathcal{L}_m)$. According to Theorem A.1 quoted in Appendix A, if

(H1) there exists $z_0 \in \rho(\mathcal{L}_m)$ such that $-1 \in \rho(Q(z_0))$,

then there exists a closed extension $\mathcal{L}_{m,V} \supseteq \mathcal{L}_m + V$, whose resolvent coincides with the family of bounded operators

$$R_0(z) - \overline{R_0(z)B}(I + Q(z))^{-1}AR_0(z).$$

Our analysis below refers to the closed operator $\mathcal{L}_{m,V}$.

We begin Section 5 by showing that the abstract condition (H1) holds, for a measurable function V satisfying either of the following hypotheses:

(H2) $\|V\|_{L^1} < 1$;

(H3) $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}) \cap L^p(\mathbb{R}, \mathbb{C}^{2\times 2})$ for some $p \in (1, \infty]$.

Once we settle that, we formulate the next spectral stability result.

Proposition 2.3. *Let $m \geq 0$ and let $\mathcal{L}_m$ be as in (2.3). Let $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$ and assume that either (H2) or (H3) holds true. Then, there exists a closed densely defined extension $\mathcal{L}_{m,V} \supseteq \mathcal{L}_m + V$ with a non-empty resolvent set and its spectrum is as follows:*

[1] Here and elsewhere below, we assume without further mentioning, that all the identities involving point-wise evaluation are valid almost everywhere with respect to the Borel σ-algebra on $\mathbb{R}$.

(1) *if* $m > 0$, *all the essential spectra of* $\mathcal{L}_{m,V}$ *are stable and coincide; namely,*

$$\mathrm{Spec}_{\mathrm{ej}}(\mathcal{L}_{m,V}) = \mathrm{Spec}_{\mathrm{ej}}(\mathcal{L}_m) = S_m \quad \textit{for all } \mathrm{j} \in [[1,5]];$$

(2) *if* $m = 0$, *all the first four notions of essential spectra are stable and coincide; namely,*

$$\mathrm{Spec}_{\mathrm{ej}}(\mathcal{L}_{0,V}) = \mathrm{Spec}_{\mathrm{ej}}(\mathcal{L}_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| = 1\} \quad \textit{for all } \mathrm{j} \in [[1,4]].$$

Moreover,

(3) *if* V *satisfies the assumption* (H2), *we have*

$$\mathrm{Spec}_{\mathrm{p}}(\mathcal{L}_{0,V}) \subset S_0;$$

(4) *if* V *satisfies the assumption* (H3) *for* $p < \infty$, *we have*

$$\mathrm{Spec}_{\mathrm{p}}(\mathcal{L}_{0,V}) \subset \left\{z \in \mathbb{C} : |\operatorname{Im} z| \le 1 + \frac{2(p-1)}{p} \|V\|_{L^p}^{p/(p-1)}\right\};$$

if V *satisfies the assumption* (H3) *for* $p = \infty$, *then*

$$\mathrm{Spec}_{\mathrm{p}}(\mathcal{L}_{0,V}) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \le 1 + \|V\|_{L^\infty}\}.$$

In Remark 5.1, we expand further on the justification for the condition (H2) and compare with the case of the dislocated Schrödinger operator. This assumption has already been documented in [7] for $\mathcal{D}_m$ in place of $\mathcal{L}_m$. Noticeably, it gives regions of inclusion for the perturbed eigenvalues. Our next result shows that a control on $\|V\|_{L^1}$ also gives inclusions for the spectrum and pseudospectrum of $\mathcal{L}_{m,V}$.

Theorem 2.4. *For* $m \ge 0$, *let*

$$D = \left\{z \in \mathbb{C} : |\operatorname{Im} z| < \frac{3}{2}, |\operatorname{Re} z| < \frac{5}{2} m\right\}.$$

There exists a constant $0 < C(m) \le 1$ *such that, if* $\|V\|_{L^1} < C(m)$, *then*

$$\mathrm{Spec}_\varepsilon(\mathcal{L}_{m,V}) \subseteq \bar{D} \cup \left\{z \in \mathbb{C} : |\operatorname{Im} z| \le 1 + \varepsilon\left(1 + \frac{\|V\|_{L^1}}{4C(m)(C(m) - \|V\|_{L^1})}\right)\right\},$$

for all $\varepsilon \ge 0$. *Moreover, for* $C(0) = 1$ *this statement is valid.*

According to [7, Corollary 4.6], $\mathcal{D}_m + V$ has point spectrum lying on a strip covering the real axis for $m = 0$. According to [7, Theorem 2.1], for $m > 0$, it lies inside two disks centred near the endpoints of the essential spectrum. The radius of these disks decreases as $\|V\|_{L^1} \to 0$. Our next result shows that, rather unexpectedly, the point spectrum of $\mathcal{L}_{m,V}$ behaves differently from that of $\mathcal{D}_m + V$, under fast rates of decay of the perturbation at infinity. Namely, it escapes to infinity through the channel created by the numerical range, rather than concentrating at the end-points of the essential spectrum, as $\|V\|_{L^1} \to 0$. The condition we impose is analogous to the one considered in [14, Theorem 2.3] and in turn allows refined versions of the conclusions (3)–(4) of Proposition 2.3 for $m > 0$.

Here and everywhere below, we write

$$\begin{aligned} \nu_1'(x) &= \sqrt{1+x^2}, & \mathrm{d}\nu_1(x) &= \nu_1'(x)\mathrm{d}x, \\ \nu_2'(x) &= \nu_1'(x)^2 = 1+x^2, & \mathrm{d}\nu_2(x) &= \nu_2'(x)\mathrm{d}x. \end{aligned}$$

Theorem 2.5. *Let $m > 0$ be fixed. Then, there exists a constant $C(m) > 0$, ensuring the following. For all $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}; \mathrm{d}\nu_1)$ such that*

$$\|V\|_{L^1(\mathrm{d}\nu_1)} < C(m),$$

we have

$$\mathrm{Spec}_{\mathrm{p}}(\mathcal{L}_{m,V}) \subseteq \left\{ z \in \mathbb{C} : |\operatorname{Im} z| \le 1,\ |\operatorname{Re}(z)| > \frac{C(m)}{\|V\|_{L^1}^{1/2}} \right\}.$$

Succinctly, this theorem says that for potentials V decaying sufficiently rapidly at ∞, the discrete spectrum escapes to ∞ inside the instability band, $\{|\operatorname{Im} z| < 1\}$, at a rate proportional to $\|V\|_{L^1}^{-1/2}$ in the regime $\|V\|_{L^1} \to 0$. In the conclusion, the norm is taken with respect to the Lebesgue measure.

For Schrödinger operators, a similar result was established in [14, Theorem 2.3]. In the latter, the density considered was ν_2' instead of ν_1', and upper bounds for the norm of $Q(z)$ were obtained via the Hilbert–Schmidt norm. Here we employ the Schur test instead, which allows a better control of the upper bounds for the norm of the integral operator $Q(z)$. Therefore, we manage to weaken the condition on the decay of V.

Theorem 2.5 opens the question of whether there exists $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$ with non-empty discrete spectrum for $m > 0$. We give an answer to this question in Section 5.3. Our concrete finding in this respect is the following. Consider the step potential

$$V(x) = V_{a,b}(x) = (-i\,\mathrm{sgn}(x) - b)\chi_{[-a,a]}(x) I,$$

where $a > 0$ and $b \in \mathbb{R}$. Set $\mathrm{Dom}(\mathcal{L}_{m,V_{a,b}}) = H^1(\mathbb{R}, \mathbb{C}^2)$. According to Proposition 2.3, the essential spectra of $\mathcal{L}_{m,V_{a,b}}$ are

$$\mathrm{Spec}_{\mathrm{ej}}(\mathcal{L}_{m,V_{a,b}}) = \{z \in \mathbb{C} : |\operatorname{Im} z| = 1,\ |\operatorname{Re} z| \ge m\},$$

for all $\mathrm{j} \in [[1,4]]$, including $\mathrm{j} = 5$ if $m > 0$.

Proposition 2.6. *For $m \geq 0$, let $\mathcal{L}_{m,V_{a,b}}$ be as in the previous paragraph.*

(1) *If $m = 0$, then the point spectrum of $\mathcal{L}_{0,V_{a,b}}$ is the band*

$$\mathrm{Spec}_{\mathrm{p}}(\mathcal{L}_{0,V_{a,b}}) = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}.$$

(2) *If $m > 0$, then the operator $\mathcal{L}_{m,V_{a,b}}$ has infinitely many isolated real eigenvalues accumulating at $\pm\infty$.*

In the context of Theorem 2.5, it is also natural to examine the behaviour of the family of operators $\mathcal{L}_{m,\epsilon V}$ for fixed $m > 0$, fixed potential V, and a small moving complex parameter $\epsilon \in \mathbb{C}$, in the regime $|\epsilon| \to 0$. This is the so-called *weakly coupled model*. For this model, we establish the following in Section 5.4.

Let $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}; \mathrm{d}\nu_1)$. From Proposition 2.3 and Theorem 2.5, it follows that

$$\mathrm{Spec}_{\mathrm{ej}}(\mathcal{L}_{m,\epsilon V}) = \{z \in \mathbb{C} : |\operatorname{Im} z| = 1,\ |\operatorname{Re} z| \geq m\}, \quad \text{for all } j \in [[1,5]]$$

and, for a suitable constant $C(m) > 0$,

$$\mathrm{Spec}_{\mathrm{dis}}(\mathcal{L}_{m,\epsilon V}) \subseteq \left\{z \in \mathbb{C} : |\operatorname{Im} z| < 1,\ |\operatorname{Re} z| > \frac{C(m)}{|\epsilon|^{1/2}\|V\|_{L^1}^{1/2}}\right\}, \tag{2.7}$$

for all $|\epsilon|$ small enough. If V satisfies additional conditions of regularity and decay at infinity, then Theorem 2.5 refines as follows.

Theorem 2.7. *Let $m > 0$. For any $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}; \mathrm{d}\nu_2) \cap W^{2,1}(\mathbb{R}, \mathbb{C}^{2\times 2})$, there exists a constant $C(m) > 0$ independent of ϵ, such that*

$$\mathrm{Spec}_{\mathrm{dis}}(\mathcal{L}_{m,\epsilon V}) \subset \left\{z \in \mathbb{C} : |\operatorname{Im} z| < 1,\ |\operatorname{Re} z| > \frac{C(m)}{|\epsilon|}\right\},$$

for all $|\epsilon|$ small enough.

3. Resolvent and spectrum of $\mathcal{L}_m$

In this section, we settle the framework of the differential expression (2.3) and the associated linear operator. We begin by determining $\mathrm{Num}(\mathcal{L}_m)$. We then compute the Green matrix associated with the resolvent $(\mathcal{L}_m - z)^{-1}$ and find $\mathrm{Spec}(\mathcal{L}_m)$.

Since $\mathcal{D}_m^* = \mathcal{D}_m$ and multiplication by $i\,\mathrm{sgn}(x)I$ is bounded, it follows that the domain of $\mathcal{L}_m^*$ is also $H^1(\mathbb{R}, \mathbb{C}^2)$. Let $\mathcal{T}$ be the antilinear operator of complex conjugation, $\mathcal{T} g = \bar{g}$ and $\mathcal{P}$ be the parity operator, $(\mathcal{P} g)(x) = g(-x)$. For $f \in H^1(\mathbb{R}, \mathbb{C}^2)$, direct substitution of the operators involved, gives

$$\begin{aligned}(\mathcal{T}\mathcal{L}_m\mathcal{T} f)(x) &= \overline{(-i\,\partial_x)\sigma_2\overline{f(x)} + m\sigma_3\overline{f(x)} + i\,\mathrm{sgn}(x)\overline{f(x)}} \\ &= (-i\,\partial_x)\sigma_2 f(x) + m\sigma_3 f(x) - i\,\mathrm{sgn}(x) f(x) = (\mathcal{L}_m^* f)(x)\end{aligned}$$

however,

$$\mathcal{P}\mathcal{T}\mathcal{L}_m f \neq \mathcal{L}_m \mathcal{P}\mathcal{T} f.$$

Therefore, $\mathcal{L}_m$ is $\mathcal{T}$-selfadjoint but it is not $\mathcal{P}\mathcal{T}$-symmetric.

Now, we show that

$$\operatorname{Num}(\mathcal{L}_m) = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1\}. \tag{3.1}$$

Indeed, let $f = \binom{f_1}{f_2} \in \operatorname{Dom}(\mathcal{L}_m)$ be such that $\|f\| = 1$. Integration by parts gives

$$\langle \mathcal{L}_m f, f\rangle = m(\|f_1\|^2 - \|f_2\|^2) + 2\operatorname{Re}\langle f_1', f_2\rangle + i(\|f\|^2_{L^2(\mathbb{R}_+)} - \|f\|^2_{L^2(\mathbb{R}_-)}).$$

Since $\|f\| = 1$, it follows immediately that $|\operatorname{Im}\langle \mathcal{L}_m f, f\rangle| \leq 1$, which implies that

$$\operatorname{Num}(\mathcal{L}_m) \subseteq \{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1\}.$$

Conversely, for the reverse inclusion, we construct appropriate test functions as follows. Choose smooth g_1 and g_2 with supports in $\mathbb{R}_+$ such that $\|g_1\|^2 + \|g_2\|^2 = 1$ and $\operatorname{Re}\langle g_1', g_2\rangle > 0$. For $\lambda > 0$, let

$$f_j(x) = \lambda^{1/2} g_j(\lambda x).$$

Then, $\|f\| = 1$ and

$$\langle \mathcal{L}_m f, f\rangle = m(\|g_1\|^2 - \|g_2\|^2) + 2\lambda \operatorname{Re}\langle g_1', g_2\rangle + i.$$

Hence, by increasing λ, it follows that $\mathbb{R}_+ + i \subseteq \operatorname{Num}(\mathcal{L}_m)$. Similar arguments, choosing g_1 and g_2 with support in $\mathbb{R}_-$ or changing the sign of $\operatorname{Re}\langle g_1', g_2\rangle$, give $\mathbb{R} \pm i \subseteq \operatorname{Num}(\mathcal{L}_m)$. The convexity of the numerical range completes the proof of (3.1).

Below, we often write $\Sigma = \operatorname{Int}(\operatorname{Num} \mathcal{L}_m)$ and call this set the *instability band*.

3.1. Integral form of the resolvent

We compute explicitly the integral kernel associated to the resolvent of $\mathcal{L}_m$. We also establish preliminary estimates on its norm. We begin by fixing a notation that will prove crucial for simplifying our expressions and subsequent analysis.

For $z \in \mathbb{C} \setminus \{\pm m \pm i\}$, we write

$$\begin{aligned} \mu_z^- &= \sqrt{(m+i+z)(m-i-z)}, & \mu_z^+ &= \sqrt{(m-i+z)(m+i-z)}, \\ w_z^- &= \frac{m-i-z}{\mu_z^-} = \frac{\sqrt{m-i-z}}{\sqrt{m+i+z}}, & w_z^+ &= \frac{m+i-z}{\mu_z^+} = \frac{\sqrt{m+i-z}}{\sqrt{m-i+z}}. \end{aligned} \tag{3.2}$$

Here and throughout the paper, we pick the principal branch $z \mapsto \sqrt{z} = z^{1/2}$ defined on $\mathbb{C}$, holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and having a positive imaginary part on $(-\infty, 0)$. This choice ensures that $w_z^\pm$ have the stated expressions in (3.2).

Since

$$(m \pm i + z)(m \mp i - z) \in (-\infty, 0] \iff |\operatorname{Re} z| \geq m \text{ and } \operatorname{Im} z = \mp 1,$$

then

$$\operatorname{Re} \mu_z^{\pm} > 0 \iff z \notin \{\tau \pm i : \tau \in \mathbb{R},\ |\tau| \geq m\}. \tag{3.3}$$

Moreover, $\overline{\mu_z^{\pm}} = \mu_{\bar{z}}^{\mp}$ and $\overline{w_z^{\pm}} = w_{\bar{z}}^{\mp}$ for all z satisfying (3.3). Below we will use these facts repeatedly, often without further explicit mention.

Recall S_m given by (2.1). In Section 3.2, we will establish that $\operatorname{Spec}(\mathcal{L}_m) = S_m$. For $z \in \mathbb{C} \setminus S_m$, define the matrices

$$N_{1,z} = \frac{1}{2}\frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}\begin{pmatrix} 1/w_z^- & 1 \\ 1 & w_z^- \end{pmatrix}, \qquad N_{6,z} = \frac{1}{2}\begin{pmatrix} -1/w_z^+ & 1 \\ -1 & w_z^+ \end{pmatrix},$$

$$N_{2,z} = \frac{1}{2}\begin{pmatrix} -1/w_z^- & -1 \\ 1 & w_z^- \end{pmatrix}, \qquad N_{7,z} = N_{4,z},$$

$$N_{3,z} = \frac{1}{w_z^+ + w_z^-}\begin{pmatrix} -1 & -w_z^- \\ w_z^+ & w_z^+ w_z^- \end{pmatrix}, \qquad N_{8,z} = \frac{1}{w_z^+ + w_z^-}\begin{pmatrix} -1 & w_z^+ \\ -w_z^- & w_z^+ w_z^- \end{pmatrix},$$

$$N_{4,z} = \frac{1}{2}\frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}\begin{pmatrix} -1/w_z^+ & 1 \\ 1 & -w_z^+ \end{pmatrix}, \quad N_{9,z} = \frac{1}{2}\begin{pmatrix} -1/w_z^- & 1 \\ -1 & w_z^- \end{pmatrix},$$

$$N_{5,z} = \frac{1}{2}\begin{pmatrix} -1/w_z^+ & -1 \\ 1 & w_z^+ \end{pmatrix}, \qquad N_{10,z} = N_{1,z}.$$

For $x, y \in \mathbb{R}$, $x \neq y$, let the matrix kernel $\mathcal{R}_z(x, y)$ be given in regions of the plane $\mathbb{R}^2$ by

$$\mathcal{R}_z(x,y) = \begin{cases} N_{1,z}\, e^{\mu_z^-(x+y)} + N_{2,z}\, e^{-\mu_z^-(x-y)} & \text{for } y < x < 0, \\ N_{3,z}\, e^{-\mu_z^+ x + \mu_z^- y} & \text{for } y < 0 < x, \\ N_{4,z}\, e^{-\mu_z^+(x+y)} + N_{5,z}\, e^{-\mu_z^+(x-y)} & \text{for } 0 < y < x, \\ N_{6,z}\, e^{\mu_z^+(x-y)} + N_{7,z}\, e^{-\mu_z^+(x+y)} & \text{for } 0 < x < y, \\ N_{8,z}\, e^{\mu_z^- x - \mu_z^+ y} & \text{for } x < 0 < y, \\ N_{9,z}\, e^{\mu_z^-(x-y)} + N_{10,z}\, e^{\mu_z^-(x+y)} & \text{for } x < y < 0. \end{cases} \tag{3.4}$$

Note that $\mathcal{R}_z$ is continuous across the lines $x = 0$ and $y = 0$, but discontinuous across the line $x = y$. Here it is not important what values we give to this matrix function on these lines, so we leave them unassigned.

We mention that, in contrast to the case of the resolvent kernel of the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + i\operatorname{sgn}(x)$, considered in [14, Proposition 3.1], $\mathcal{R}_z$ is not symmetric with respect to the line $x = y$. We will see in Lemma 3.2 that the induced operator norm of $\mathcal{R}_z$ is nonetheless symmetric. That is, $|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} = |\mathcal{R}_z(y,x)|_{\mathbb{C}^{2\times 2}}$ for all $(x,y) \in \mathbb{R}^2$.

As we shall see from the next statements, $\mathcal{R}_z$ is the (matrix) Green function associated to the operator $\mathcal{L}_m - z$. This explicit formula has a far-reaching role in the analysis that we conduct below. Remarkably, we will see that it enables the systematic study of perturbation of $\mathcal{L}_m$ via a Birman–Schwinger principle, as described in Section 5.

Proposition 3.1. *For $m \geq 0$, let $\mathcal{L}_m$ be the operator given by* (2.3)*. Then, $\mathbb{C} \setminus S_m \subset \rho(\mathcal{L}_m)$. Moreover,*

$$[(\mathcal{L}_m - z)^{-1} f](x) = \int_{\mathbb{R}} \mathcal{R}_z(x,y) f(y)\,\mathrm{d}y,$$

for all $z \in \mathbb{C} \setminus S_m$ and $f \in L^2(\mathbb{R}, \mathbb{C}^2)$,

Before proceeding to the proof, we highlight that the resolvent kernel of the free Dirac operator [7, Proof of Theorem 2.1], is a convolution kernel and it contains only exponential terms involving the difference $x - y$. By contrast, for $m > 0$, $\mathcal{R}_z$ also includes mixed exponential terms and it cannot be expressed as a pure tensor product of two matrix functions. We will see below that this more complex structure of the Green function is responsible for the asymptotic growth of the resolvent norm inside the numerical range.

The remainder of this section is devoted to the proof of Proposition 3.1. Although this proof follows a routine argumentation, we include crucial details that will aid in the computation of sharp estimates for the resolvent norm. We split it into four steps.

Step 1. Fix $z \in \mathbb{C} \setminus S_m$ and $f \in L^2(\mathbb{R}, \mathbb{C}^2)$. Consider the eigenvalue equation

$$(\mathcal{L}_m - z)u = f,$$

which in matrix form reads

$$\begin{pmatrix} m + i\,\mathrm{sgn}(x) - z & -\partial_x \\ \partial_x & -m + i\,\mathrm{sgn}(x) - z \end{pmatrix} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}. \tag{3.5}$$

Our final goal in the proof is to determine the explicit integral expression for the solution. Multiplying by the matrix $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ on the left of the two sides of (3.5), gives the equivalent non-homogeneous linear system of first order,

$$\partial_x u(x) = A_z(x) u(x) + g(x), \tag{3.6}$$

where

$$A_z(x) = \begin{pmatrix} 0 & m - i\,\mathrm{sgn}(x) + z \\ m + i\,\mathrm{sgn}(x) - z & 0 \end{pmatrix} \quad \text{and} \quad g(x) = -Jf(x).$$

We now seek for the solution of this system. Consider equation (3.6) for $x > 0$ and $x < 0$, where the corresponding solutions are indicated as u_- and u_+. Let

$$A_z^- = \begin{pmatrix} 0 & m+i+z \\ m-i-z & 0 \end{pmatrix} \quad \text{and} \quad A_z^+ = \begin{pmatrix} 0 & m-i+z \\ m+i-z & 0 \end{pmatrix}$$

be the expressions for $A_z(x)$ for $x > 0$ and $x < 0$, respectively. Let $\Phi_z^\pm(x)$ be fundamental matrices for $\partial_x u_\pm = A_z^\pm u^\pm$. The columns of $\Phi_z^\pm(x)$ ought to be linear independent for all $x \in \mathbb{R}$, thus, we can pick the choice

$$\Phi_z^\pm(x) = \exp(A_z^\pm x) = \begin{pmatrix} \cosh(\mu_z^\pm x) & \sinh(\mu_z^\pm x)/w_z^\pm \\ w_z^\pm \sinh(\mu_z^\pm x) & \cosh(\mu_z^\pm x) \end{pmatrix}. \tag{3.7}$$

Note that

$$(\Phi_z^\pm(x))^{-1} = \Phi_z^\pm(-x) \quad \text{and} \quad \Phi_z^\pm(x)\Phi_z^\pm(y) = \Phi_z^\pm(x+y). \tag{3.8}$$

We now determine the solutions $u_\pm$ of the inhomogeneous system

$$\partial_x u_\pm(x) = A_z^\pm u_\pm(x) + g(x) \quad \text{on } \mathbb{R}_\pm \tag{3.9}$$

in the form $u_\pm(x) = \Phi_z^\pm(x) v_\pm(x)$, where $v_\pm : \mathbb{R}_\pm \to \mathbb{C}^2$ are to be determined. Taking the derivative of $u_\pm$ and using (3.7), we obtain

$$\begin{aligned} \partial_x u_\pm(x) &= A_z^\pm \Phi_z^\pm(x) v_\pm(x) + \Phi_z^\pm(x)\, \partial_x v_\pm(x) \\ &= A_z^\pm u_\pm(x) + \Phi_z^\pm(x)\, \partial_x v_\pm(x) \quad \text{on } \mathbb{R}_\pm. \end{aligned}$$

Therefore, $u_\pm$ are solutions of (3.9) if and only if $\Phi_z^\pm(x)\, \partial_x v_\pm(x) = g(x)$ on $\mathbb{R}_\pm$. Hence, we choose

$$v_\pm(x) = \int_0^x (\Phi_z^\pm(y))^{-1} g(y) \mathrm{d}y + \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix},$$

where $\alpha_\pm, \beta_\pm \in \mathbb{C}$ are constants to be determined. Then, the general solutions of (3.9) are given by

$$u_\pm(x) = \Phi_z^\pm(x) \int_0^x (\Phi_z^\pm(y))^{-1} g(y) \mathrm{d}y + \Phi_z^\pm(x) \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix}.$$

As the condition

$$\lim_{x \to 0^-} u_-(x) = \lim_{x \to 0^+} u_+(x)$$

is needed for a solution to be in the domain of the operator, then necessarily

$$\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} = \begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} =: \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This gives the appropriate regularity to the solution for all $x \in \mathbb{R}$.

Step 2. Our next objective is to determine α and β such that $u_\pm$ decay to zero at infinity. According to (3.8), we have

$$\begin{aligned} u_\pm(x) &= \int_0^x \Phi_z^\pm(x-y)g(y)\,\mathrm{d}y + \Phi_z^\pm(x)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\ &= \int_0^x \begin{pmatrix} \cosh(\mu_z^\pm(x-y)) & \sinh(\mu_z^\pm(x-y))/w_z^\pm \\ w_z^\pm \sinh(\mu_z^\pm(x-y)) & \cosh(\mu_z^\pm(x-y)) \end{pmatrix} g(y)\,\mathrm{d}y \\ &\quad + \begin{pmatrix} \cosh(\mu_z^\pm x) & \sinh(\mu_z^\pm x)/w_z^\pm \\ w_z^\pm \sinh(\mu_z^\pm x) & \cosh(\mu_z^\pm x) \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \end{aligned}$$

When the right-hand side is written in exponential form, it reduces to

$$\begin{aligned} u_\pm(x) &= \int_0^x e^{\mu_z^\pm(x-y)} S_z^\pm g(y)\mathrm{d}y + e^{\mu_z^\pm x} S_z^\pm \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\ &\quad + \int_0^x e^{-\mu_z^\pm(x-y)} T_z^\pm g(y)\mathrm{d}y + e^{-\mu_z^\pm x} T_z^\pm \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \end{aligned} \tag{3.10}$$

where

$$S_z^\pm = \frac{1}{2}\begin{pmatrix} 1 & 1/w_z^\pm \\ w_z^\pm & 1 \end{pmatrix} \quad \text{and} \quad T_z^\pm = \frac{1}{2}\begin{pmatrix} 1 & -1/w_z^\pm \\ -w_z^\pm & 1 \end{pmatrix}.$$

We seek for conditions ensuring $\lim_{x\to\pm\infty} u_\pm(x) = 0$. Since the function g belongs to L^2 but not necessarily to L^1, we cannot directly use the dominated convergence theorem. Instead, we appeal to the density of $C_c^\infty(\mathbb{R},\mathbb{C}^2)$ in $L^2(\mathbb{R},\mathbb{C}^2)$ as follows. For $\epsilon > 0$ small, let $g_\epsilon \in C_c^\infty(\mathbb{R},\mathbb{C}^2)$ be such that $\|g - g_\epsilon\| < \epsilon$. Then,

$$\begin{aligned} \left| \int_0^x e^{-\mu_z^+(x-y)} T_z^+ g(y)\,\mathrm{d}y \right|_{\mathbb{C}^2} &\le \int_0^x e^{-\operatorname{Re}\mu_z^+(x-y)} \le T_z^+|_{\mathbb{C}^{2\times 2}} |g(y) - g_\epsilon(y)|_{\mathbb{C}^2}\,\mathrm{d}y \\ &\quad + \int_0^x e^{-\operatorname{Re}\mu_z^+(x-y)} t|T_z^+|_{\mathbb{C}^{2\times 2}} |g_\epsilon(y)|_{\mathbb{C}^2}\,\mathrm{d}y \\ &\le |T_z^+|_{\mathbb{C}^{2\times 2}} \sqrt{\frac{1 - e^{-2\operatorname{Re}\mu_z^+ x}}{2\operatorname{Re}\mu_z^+}} \|g - g_\epsilon\| \\ &\quad + |T_z^+|_{\mathbb{C}^{2\times 2}} \int_0^x e^{-\operatorname{Re}\mu_z^+(x-y)} |g_\epsilon(y)|_{\mathbb{C}^2}\,\mathrm{d}y. \end{aligned}$$

Recall (3.3). Since $\operatorname{Re}\mu_z^+ > 0$, the integrand $e^{-\operatorname{Re}\mu_z^+(x-y)}|g_\epsilon(y)|_{\mathbb{C}^2}$ is bounded by the L^1 function $|g_\epsilon|_{\mathbb{C}^2}$, then, according to the dominated convergence theorem, we have

$$\lim_{x\to+\infty}\left|\int_0^x e^{-\mu_z^+(x-y)}T_z^+ g(y)\,\mathrm{d}y\right|_{\mathbb{C}^2} \le |T_z^+|_{\mathbb{C}^{2\times 2}}\frac{\epsilon}{\sqrt{\operatorname{Re}\mu_z^+}}.$$

Because ϵ is arbitrary, the limit on the left-hand side is zero. In turn, we gather that

$$\begin{aligned}\lim_{x\to+\infty} u_+(x) = 0 \iff& \lim_{x\to+\infty} e^{\mu_z^+ x}\left(\int_0^x e^{-\mu_z^+ y} S_z^+ g(y)\mathrm{d}y + S_z^+\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) = 0\\ \iff& S_z^+\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -\int_0^{+\infty} e^{-\mu_z^+ y} S_z^+ g(y)\,\mathrm{d}y\\ \iff& \alpha + \frac{1}{w_z^+}\beta = -\int_0^{+\infty}\left(g_1(y) + \frac{1}{w_z^+}g_2(y)\right)e^{-\mu_z^+ y}\,\mathrm{d}y.\end{aligned} \tag{3.11}$$

For the latter, note that $\det S_z^+ = 0$.

Similarly, by considering the limit to $-\infty$, we conclude that

$$\lim_{x\to-\infty} u_-(x) = 0 \iff \alpha - \frac{1}{w_z^-}\beta = \int_{-\infty}^0\left(g_1(y) - \frac{1}{w_z^-}g_2(y)\right)e^{\mu_z^- y}\,\mathrm{d}y. \tag{3.12}$$

Now, (3.11)–(3.12) can be re-written as a single linear system of equations,

$$\begin{aligned}\begin{pmatrix}1 & 1/w_z^+\\ 1 & -1/w_z^-\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} =& \int_0^{+\infty} e^{-\mu_z^+ y}\begin{pmatrix}-1 & -1/w_z^+\\ 0 & 0\end{pmatrix}g(y)\,\mathrm{d}y\\ &+\int_{-\infty}^0 e^{\mu_z^- y}\begin{pmatrix}0 & 0\\ 1 & -1/w_z^-\end{pmatrix}g(y)\,\mathrm{d}y.\end{aligned}$$

Since

$$\det\begin{pmatrix}1 & 1/w_z^+\\ 1 & -1/w_z^-\end{pmatrix} \neq 0$$

for all $z \in \mathbb{C}\setminus S_m$, then

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \int_0^{+\infty} e^{-\mu_z^+ y} M_z^+ g(y)\,\mathrm{d}y + \int_{-\infty}^0 e^{\mu_z^- y} M_z^- g(y)\,\mathrm{d}y,$$

where

$$M_z^+ = \frac{-1}{w_z^+ + w_z^-}\begin{pmatrix} w_z^+ & 1 \\ w_z^+ w_z^- & w_z^- \end{pmatrix}, \quad M_z^- = \frac{1}{w_z^+ + w_z^-}\begin{pmatrix} w_z^- & -1 \\ -w_z^+ w_z^- & w_z^+ \end{pmatrix}.$$

Step 3. We now verify the expression for the matrix integral kernel $\mathcal{R}_z$. Replacing the constant vector $\binom{\alpha}{\beta}$ into the formula for $u_\pm$, gives

$$\begin{aligned} u_\pm(x) &= \int_0^x e^{\mu_z^\pm(x-y)} S_z^\pm g(y)\mathrm{d}y + \int_0^{+\infty} e^{\mu_z^\pm x - \mu_z^+ y} S_z^\pm M_z^+ g(y)\,\mathrm{d}y \\ &\quad + \int_{-\infty}^0 e^{\mu_z^\pm x + \mu_z^- y} S_z^\pm M_z^- g(y)\,\mathrm{d}y + \int_0^x e^{-\mu_z^\pm(x-y)} T_z^\pm g(y)\mathrm{d}y \\ &\quad + \int_0^{+\infty} e^{-\mu_z^\pm x - \mu_z^+ y} T_z^\pm M_z^+ g(y)\,\mathrm{d}y + \int_{-\infty}^0 e^{-\mu_z^\pm x + \mu_z^- y} T_z^\pm M_z^- g(y)\,\mathrm{d}y. \end{aligned}$$

Since $S_z^+ M_z^+ = -S_z^+$, $S_z^+ M_z^- = 0$ and $g = -Jf$, then

$$\begin{aligned} u_+(x) &= \int_x^{+\infty} N_{6,z} e^{\mu_z^+(x-y)} f(y)\,\mathrm{d}y + \int_0^x N_{5,z} e^{-\mu_z^+(x-y)} f(y)\mathrm{d}y \\ &\quad + \int_0^{+\infty} N_{4,z} e^{-\mu_z^+(x+y)} f(y)\,\mathrm{d}y + \int_{-\infty}^0 N_{3,z} e^{-\mu_z^+ x + \mu_z^- y} f(y)\,\mathrm{d}y. \end{aligned}$$

Similarly, since $T_z^- M_z^- = T_z^-$ and $T_z^- M_z^+ = 0$, then

$$\begin{aligned} u_-(x) &= \int_{-\infty}^x N_{2,z} e^{-\mu_z^-(x-y)} f(y)\,\mathrm{d}y + \int_x^0 N_{9,z} e^{\mu_z^-(x-y)} f(y)\mathrm{d}y \\ &\quad + \int_{-\infty}^0 N_{1,z} e^{\mu_z^-(x+y)} f(y)\,\mathrm{d}y + \int_0^{+\infty} N_{8,z} e^{\mu_z^- x - \mu_z^+ y} f(y)\,\mathrm{d}y. \end{aligned}$$

Step 4. So far, we have obtained a solution to (3.5) on $\mathbb{R}$, which is continuous and decaying at infinity. This solution has the form

$$u(x) = \begin{cases} u_+(x) & \text{if } x \geq 0, \\ u_-(x) & \text{if } x \leq 0, \end{cases}$$

and has the integral representation

$$u(x) = \int_{\mathbb{R}} \mathcal{R}_z(x,y) f(y)\, \mathrm{d}y \tag{3.13}$$

as in the statement of Proposition 3.1. To complete the proof, it remains to check that $u \in L^2(\mathbb{R}, \mathbb{C}^2)$ and thus $\mathcal{L}_m u = zu + f \in L^2(\mathbb{R}, \mathbb{C}^2)$, so that $u \in H^1(\mathbb{R}, \mathbb{C}^2)$. This will be the consequence of the next two lemmas, which will be useful in their own right later on.

Firstly, we explicitly compute the matrix norm of the kernel.

Lemma 3.2. *Let $\mathcal{R}_z$ be as defined in (3.4). Assume that $z \in \mathbb{C} \setminus S_m$ is fixed. Let,*

- *for $t \le 0$,*

$$\varphi_z(t) = \frac{\sqrt{1+|w_z^-|^2}}{2}\Big(\frac{1}{|w_z^-|^2}|k_z e^{2\mu_z^- t} - 1|^2 + |k_z e^{2\mu_z^- t} + 1|^2\Big)^{1/2}, \tag{3.14a}$$

- *for $t \ge 0$,*

$$\varphi_z(t) = \frac{\sqrt{1+|w_z^+|^2}}{2}\Big(\frac{1}{|w_z^+|^2}|k_z e^{-2\mu_z^+ t} + 1|^2 + |k_z e^{-2\mu_z^+ t} - 1|^2\Big)^{1/2}, \tag{3.14b}$$

where

$$k_z = \frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}. \tag{3.15}$$

Then, $\varphi_z\colon \mathbb{R} \to \mathbb{R}_+$ is bounded and continuous, and

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} = \begin{cases} \varphi_z(\max\{x,y\})\, e^{-\operatorname{Re}\mu_z^-|x-y|} & \text{for } x<0,\ y<0,\\ \varphi_z(\min\{x,y\})\, e^{-\operatorname{Re}\mu_z^+|x-y|} & \text{for } x>0,\ y>0,\\ \varphi_z(0)\, e^{-\operatorname{Re}\mu_z^+|x|-\operatorname{Re}\mu_z^-|y|} & \text{for } x>0,\ y<0,\\ \varphi_z(0)\, e^{-\operatorname{Re}\mu_z^-|x|-\operatorname{Re}\mu_z^+|y|} & \text{for } x<0,\ y>0. \end{cases} \tag{3.16}$$

Proof. The function φ_z is continuous, since

$$\lim_{t\to 0_-} \varphi_z(t) = \frac{\sqrt{(1+|w_z^+|^2)(1+|w_z^-|^2)}}{|w_z^+ + w_z^-|} = \lim_{t\to 0+} \varphi_z(t).$$

It is bounded, since

$$\begin{aligned} \sup_{t\in\mathbb{R}_\pm} \varphi_z(t) &\le (|k_z|+1)\frac{\sqrt{1+|w_z^\pm|^2}}{2}\sqrt{1+\frac{1}{|w_z^\pm|^2}} \\ &= \frac{1}{2}(|k_z|+1)\Big(|w_z^\pm| + \frac{1}{|w_z^\pm|}\Big). \end{aligned} \tag{3.17}$$

Now, recall that, for a singular matrix $A \in \mathbb{C}^{2\times 2}$, the operator norm and the Frobenius norm coincide

$$|A|_{C^{2\times 2}} = \sqrt{\sum_{1\le i,j\le 2} |A_{ij}|^2}.$$

Indeed, $\det(A^*A) = 0$, therefore the two non-negative eigenvalues of A^*A are 0 and $\operatorname{Tr}(A^*A)$. Applying this, alongside a straightforward computation, gives

- for $\pm x > 0$ and $\pm y > 0$,

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} = \frac{\sqrt{1+|w_z^\pm|^2}}{2}\Big(\frac{1}{|w_z^\pm|^2}|k_z e^{-\mu_z^\pm|x+y|} \pm e^{-\mu_z^\pm|x-y|}|^2 + |k_z e^{-\mu_z^\pm|x+y|} \mp e^{-\mu_z^\pm|x-y|}|^2\Big)^{1/2};$$

- for $\pm x > 0$ and $\pm y < 0$,

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} = \frac{\sqrt{(1+|w_z^+|^2)(1+|w_z^-|^2)}}{|w_z^+ + w_z^-|} e^{-\operatorname{Re}\mu_z^\pm|x| - \operatorname{Re}\mu_z^\mp|y|}.$$

Factoring out suitable exponential terms and substitution of the expression (3.14) yields (3.16). ∎

The next lemma directly implies that the function u given by (3.13) belongs to $L^2(\mathbb{R},\mathbb{C}^2)$, and hence $z \in \rho(\mathcal{L}_m)$. This conclusion completes the proof of Proposition 3.1. Additionally, the lemma provides an initial estimate for the upper bound of the resolvent norm of the operator $\mathcal{L}_m$. This estimate is derived directly from the explicit expression of $|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}$ in terms of the supremum of $\varphi_z(t)$, using the Schur test. We will refine this upper bound in the next section.

Lemma 3.3. *Let $z \in \mathbb{C} \setminus S_m$. Then, the integral operator on the right-hand side of* (3.13) *is bounded on $L^2(\mathbb{R},\mathbb{C}^2)$, and*

$$\|(\mathcal{L}_m - z)^{-1}\| \le \frac{2}{\min\{\operatorname{Re}\mu_z^-, \operatorname{Re}\mu_z^+\}} \sup_{t\in\mathbb{R}} \varphi_z(t).$$

Proof. We will prove this lemma by the Schur test. Our goal is to show that

$$\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}\, dy \le \frac{2}{\min\{\operatorname{Re}\mu_z^-, \operatorname{Re}\mu_z^+\}} \sup_{t\in\mathbb{R}} \varphi_z(t), \quad \text{for a.e. } x \in \mathbb{R}, \tag{3.18a}$$

$$\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}\, dx \le \frac{2}{\min\{\operatorname{Re}\mu_z^-, \operatorname{Re}\mu_z^+\}} \sup_{t\in\mathbb{R}} \varphi_z(t), \quad \text{for a.e. } y \in \mathbb{R}. \tag{3.18b}$$

Since $|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}$ is symmetric with respect to swapping x and y, it suffices to show the first identity in (3.18).

For $x \le 0$, (3.16) yields

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}\, \mathrm{d}y \\
&= \int_{-\infty}^{x} \varphi_z(x) e^{\operatorname{Re}\mu_z^-(y-x)}\, \mathrm{d}y + \int_{x}^{0} \varphi_z(y) e^{\operatorname{Re}\mu_z^-(x-y)}\, \mathrm{d}y \\
&\quad + \int_{0}^{+\infty} \varphi_z(0) e^{\operatorname{Re}\mu_z^- x - \operatorname{Re}\mu_z^+ y}\, \mathrm{d}y \\
&\le \varphi_z(x) \frac{1}{\operatorname{Re}\mu_z^-} + \big(\sup_{t\le 0} \varphi_z(t)\big) \frac{1-e^{\operatorname{Re}\mu_z^- x}}{\operatorname{Re}\mu_z^-} + \varphi_z(0) \frac{e^{\operatorname{Re}\mu_z^- x}}{\operatorname{Re}\mu_z^+} \\
&\le \big(\sup_{x\le 0} \varphi_z(x)\big) \left[\frac{2}{\operatorname{Re}\mu_z^-} + \left(\frac{1}{\operatorname{Re}\mu_z^+} - \frac{1}{\operatorname{Re}\mu_z^-} \right) e^{\operatorname{Re}\mu_z^- x} \right] \\
&\le \big(\sup_{x\le 0} \varphi_z(x)\big) \frac{2}{\min\{\operatorname{Re}\mu_z^-, \operatorname{Re}\mu_z^+\}}.
\end{aligned}$$

A similar estimate is obtained for $x \ge 0$, where now the supremum is taken for $x \ge 0$. From this, (3.18) follows. ■

3.2. The spectrum of $\mathcal{L}_m$

According to Proposition 3.1,

$$\operatorname{Spec}(\mathcal{L}_m) \subseteq S_m.$$

Since $\mathcal{L}_m$ is $\mathcal{T}$-selfadjoint, the residual spectrum is empty [19, Section 5.2.5.4] and the first four essential spectra coincide [11, Theorem IX.1.6 (ii)]. We show that the spectrum coincides exactly with S_m and find $\operatorname{Spec}_{e5}(\mathcal{L}_m)$. We treat the case $m = 0$ separately.

Case $m > 0$. Let $z^\pm = \tau \pm i$ for fixed $\tau \in \mathbb{R}$ such that $|\tau| \ge m$, so $z^\pm \in S_m$. Our goal is to construct singular Weyl sequences $(\Psi_n^\pm)_{n\in\mathbb{N}}$ for $z^\pm$, to conclude that they are in the appropriate part of the spectrum. With this purpose in mind, let

$$u(x) = \begin{pmatrix} \sqrt{|m+\tau|} \\ i\,\operatorname{sgn}(\tau)\sqrt{|m-\tau|} \end{pmatrix} e^{i\sqrt{\tau^2-m^2}x}, \tag{3.19}$$

which is one of the solutions of

$$(\mathcal{D}_m - \tau I)u = 0, \tag{3.20}$$

for the free Dirac operator (2.2). Let

$$\Psi_n^{\pm}(x) = c_n \varphi_n(\pm x) u(x),$$

where

$$c_n = \frac{1}{\sqrt{2|\tau|\left(n + \frac{2}{3}\right)}}$$

and φ_n is the continuous compact support cut-off function given by

$$\varphi_n(x) = \begin{cases} x - n & \text{for } x \in [n, n+1], \\ 1 & \text{for } x \in [n+1, 2n+1], \\ 2n + 2 - x & \text{for } x \in [2n+1, 2n+2], \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\|\Psi_n^{\pm}\| = 1$.

By considering the location of the supports and from (3.20), it follows that

$$\begin{aligned}(\mathcal{L}_m - z^{\pm} I)\Psi_n^{\pm}(x) &= (\mathcal{D}_m - \tau I)\Psi_n^{\pm}(x) \\ &= c_n \varphi_n(\pm x)(\mathcal{D}_m - \tau I)u(x) + c_n[\mathcal{D}_m - \tau I, \varphi_n(\pm x) I]u(x) \\ &= \pm c_n \varphi_n'(\pm x) J u(x).\end{aligned}$$

Then,

$$\|(\mathcal{L}_m - z^{\pm} I)\Psi_n^{\pm}(x)\| = \|c_n \varphi_n'(\pm x) J u(x)\| = \sqrt{\frac{2}{n + \frac{2}{3}}} \to 0$$

as $n \to \infty$. Therefore, indeed, $(\Psi_n^{\pm})_{n \in \mathbb{N}}$ is a Weyl sequence for $z^{\pm}$. Thus, $z^{\pm} \in \operatorname{Spec}(\mathcal{L}_m)$. This completes the proof that $\operatorname{Spec}(\mathcal{L}_m) = S_m$ for all $m > 0$, which we will use without further explicit mention.

Since the support of the Weyl sequence moves to infinity, $S_m \subset \operatorname{Spec}_{\mathrm{e2}}(\mathcal{L}_m)$. Consequently, $\operatorname{Spec}_{\mathrm{ej}}(\mathcal{L}_m) = S_m$ for $\mathrm{j} \in [[1, 4]]$. To further deduce that

$$\operatorname{Spec}_{\mathrm{e5}}(\mathcal{L}_m) = S_m,$$

it suffices to note that $\mathbb{C} \setminus \operatorname{Spec}_{\mathrm{e1}}(\mathcal{L}_m) = \rho(\mathcal{L}_m)$ is connected [19, Proposition 5.4.4].

Finally, observe that $\operatorname{Spec}_{\mathrm{p}}(\mathcal{L}_m) = \varnothing$. Indeed, the expression of the fundamental matrices (3.7) is also valid for $z \in S_m$. Applying (3.10) with $g = 0$ and noting that $\operatorname{Re} \mu_z^{\pm} = 0$ for $z \in \{\tau \pm i : |\tau| \geq m\}$, we conclude that there are no non-trivial solutions to the eigenvalue equation $(\mathcal{L}_m - z)u = 0$ that belong to $L^2(\mathbb{R}, \mathbb{C}^2)$. Therefore, no point in S_m can be an eigenvalue. This completes the description of the spectrum of $\mathcal{L}_m$ for $m > 0$.

Case $m = 0$. For $\mathcal{L}_0$ and $z^\pm = \tau \pm i$ where $\tau \neq 0$, the previous argumentation involving $(\Psi_n^\pm)_{n\in\mathbb{N}}$ is still applicable, so we know that there exist singular Weyl sequences for $z^\pm \neq \pm i$. On the other hand, for $\tau = 0$, we can choose constant $u(x) = \binom{1}{0}$ instead of (3.19), which is a solution to the eigenvalue equation (3.20), and run a similar proof as above. Thus, we know up front that

$$\partial S_0 = \{z \in \mathbb{C} : |\operatorname{Im} z| = 1\} \subset \operatorname{Spec}_{e2}(\mathcal{L}_0) = \operatorname{Spec}_{ej}(\mathcal{L}_0) \tag{3.21}$$

for j = 1, 3, 4.

Our focus now is to show that all the points z in the interior of the strip S_0 are eigenvalues with $\mathcal{L}_0 - z$ being a Fredholm operator of index zero.

Lemma 3.4. *For $m = 0$, let $\mathcal{L}_0$ be defined as in (2.3). Let $z \in \mathbb{C}$ be such that $|\operatorname{Im} z| < 1$. Then, the following holds true.*

(a) $\operatorname{Ker}(\mathcal{L}_0 - z) = \operatorname{span}\{v_z\}$, *where*

$$v_z(x) = \begin{cases} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1-iz)x} & \text{for } x \leq 0, \\ \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-(1+iz)x} & \text{for } x \geq 0. \end{cases}$$

(b) $\operatorname{Ker}(\mathcal{L}_0^* - z) = \operatorname{span}\{\tilde{v}_z\}$, *where*

$$\tilde{v}_z(x) = \begin{cases} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+iz)x} & \text{for } x \leq 0, \\ \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-(1-iz)x} & \text{for } x \geq 0. \end{cases}$$

(c) $\operatorname{Ran}(\mathcal{L}_0 - z)$ *is closed.*

(d) *The algebraic multiplicity of z is infinite.*

Proof. Consider once again the equation $(\mathcal{L}_0 - z)u = f$ as in Section 3.1. When $m = 0$ and $|\operatorname{Im} z| < 1$, the quantities $\mu_z^\pm$ and $w_z^\pm$ in (3.2) reduce to

$$\mu_z^+ = 1 + iz, \quad \mu_z^- = 1 - iz, \quad w_z^+ = i, \quad w_z^- = -i.$$

We start with the proof of (a). Let $u \in H^1(\mathbb{R}, \mathbb{C}^2)$ be a solution to this eigenvalue equation. We know that the restrictions of u to $\mathbb{R}_\pm$, match (3.10) with $g = 0$, namely

$$u_\pm(x) = e^{\mu_z^\pm x} S_z^\pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + e^{-\mu_z^\pm x} T_z^\pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

with $(\alpha, \beta) \in \mathbb{C}^2$ and

$$S_z^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}, \quad T_z^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \mp i & 1 \end{pmatrix}.$$

Since $\operatorname{Re} \mu_z^{\pm} > 0$, the decay of u at $\pm\infty$ should match the conditions (3.11) and (3.12) for α and β. Therefore, $\alpha = i\beta$. Hence,

$$u_+(x) = \alpha e^{-\mu_z^+ x}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad u_-(x) = \alpha e^{\mu_z^- x}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Thus, indeed (a) is valid. The proof of (b) is identical, so we omit it.

To confirm (c), we show that $\overline{\operatorname{Ran}(\mathcal{L}_0 - z)} \subseteq \operatorname{Ran}(\mathcal{L}_0 - z)$. Take $f \in \overline{\operatorname{Ran}(\mathcal{L}_0 - z)}$ and seek for $u \in H^1(\mathbb{R}, \mathbb{C}^2)$ such that $(\mathcal{L}_0 - z)u = f$. Now, $u_\pm$ are given in the form (3.10) with $g = -Jf$. Since

$$\overline{\operatorname{Ran}(\mathcal{L}_0 - z)} = (\operatorname{Ker}(\mathcal{L}_0^* - \bar{z}))^\perp,$$

according to (b), it follows that

$$\int_{-\infty}^{0} (f_1(y) - i f_2(y)) e^{(1-iz)y}\, \mathrm{d}y + \int_{0}^{\infty} (f_1(y) - i f_2(y)) e^{-(1+iz)y}\, \mathrm{d}y = 0.$$

By proceeding as in the proof of Proposition 3.1, this is equivalent to the fact that both right-hand sides of the decaying conditions (3.11) and (3.12) are equal. That is,

$$-\int_{0}^{\infty} (g_1(y) - i g_2(y)) e^{-\mu_z^+ y}\, \mathrm{d}y = \alpha - i\beta = \int_{-\infty}^{0} (g_1(y) - i g_2(y)) e^{\mu_z^- y}\, \mathrm{d}y.$$

Set $\beta = 0$. Then,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 & i \\ 0 & 0 \end{pmatrix} \int_{0}^{\infty} g(y) e^{-\mu_z^+ y}\, \mathrm{d}y = \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \int_{-\infty}^{0} g(y) e^{\mu_z^- y}\, \mathrm{d}y.$$

Replacing these constants in (3.10) and writing the expression back in terms of f_1 and f_2, using $g = -Jf$, gives

$$u_+(x) = C_1^+ \int_{x}^{+\infty} e^{\mu_z^+ (x-y)} f(y)\, \mathrm{d}y + C_2^+ \int_{0}^{x} e^{-\mu_z^+ (x-y)} f(y)\, \mathrm{d}y$$
$$+ C_3^+ \int_{0}^{\infty} e^{-\mu_z^+ (x+y)} f(y)\, \mathrm{d}y,$$

and

$$u_-(x) = C_1^- \int_{-\infty}^{x} e^{-\mu_z^-(x-y)} f(y)\,\mathrm{d}y + C_2^- \int_{x}^{0} e^{\mu_z^-(x-y)} f(y)\,\mathrm{d}y + C_3^- \int_{-\infty}^{0} e^{\mu_z^-(x+y)} f(y)\,\mathrm{d}y,$$

for suitable constant matrices $C_j^{\pm}$. Now, by invoking the Schur test in identical manner as in the proof of Proposition 3.1 – we omit the details – we find that $u \in L^2(\mathbb{R}, \mathbb{C}^2)$ and thus $u \in H^1(\mathbb{R}, \mathbb{C}^2)$. Hence, indeed, $f \in \operatorname{Ran}(\mathcal{L}_m - z)$. This completes the proof of (c).

For the final statement, we invoke [22, Theorem 2.16 in Chapter 2], which establishes that only one of the following cases can occur. Either

$$\Sigma = \{w \in \mathbb{C} : |\operatorname{Im} w| < 1 \text{ and the algebraic multiplicity of } w \text{ is finite}\}$$

or

$$\Sigma = \{w \in \mathbb{C} : |\operatorname{Im} w| < 1 \text{ and the algebraic multiplicity of } w \text{ is infinite}\}.$$

From the above, the geometric multiplicity of all $z \in \Sigma$ is one. Therefore, no point in Σ has algebraic multiplicity zero. Thus, according to [22, Theorem 2.15 in Chapter 2], the set on the right-hand side of the first possibility is countable. But this is not the case for Σ, hence, the second possibility is the one that holds. This confirms (d) and completes the proof of the lemma. ∎

According to Lemma 3.4, we now know that

$$\operatorname{int} S_0 = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\} \subseteq \operatorname{Spec}_{\mathrm{p}}(\mathcal{L}_0) \setminus \operatorname{Spec}_{\mathrm{e3}}(\mathcal{L}_0). \tag{3.22}$$

Combining this with (3.21), we conclude that indeed

$$\operatorname{Spec}(\mathcal{L}_0) = S_0 \quad \text{and} \quad \operatorname{Spec}_{\mathrm{ej}}(\mathcal{L}_0) = \operatorname{Spec}_{\mathrm{e2}}(\mathcal{L}_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| = 1\}$$

for $\mathrm{j} = 1, 3, 4$. Since no eigenvalue on the left-hand side of (3.22) is isolated, and S_0 is the closure of its interior, then $\operatorname{Spec}_{\mathrm{e5}}(\mathcal{L}_0) = S_0$. Finally, since no point on the lines $\{\tau \pm i : \tau \in \mathbb{R}\}$ is an eigenvalue, these lines exactly form the continuous spectrum of $\mathcal{L}_0$. In summary,

$$\operatorname{Spec}_{\mathrm{p}}(\mathcal{L}_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}.$$

We have therefore completed the analysis of the spectrum of $\mathcal{L}_m$ for all $m \geq 0$.

4. The resolvent norm of $\mathcal{L}_m$ inside the numerical range

Let $m > 0$. In the second part of this section we give the proof of the sharp estimate for the resolvent norm of $\mathcal{L}_m$ claimed in Theorem 2.1. Before that, and in order to visualise the asymptotic behaviour of the different parameters involved in this proof, we establish a sub-optimal version of the statement. Later on, this preliminary estimate will provide a link with the analysis of the pseudospectrum of the perturbations of $\mathcal{L}_m$.

From now on, we write $\tau = \operatorname{Re} z$ and $\delta = \operatorname{Im} z$, often without explicit mention.

4.1. Preliminary estimate

A direct application of the Schur test gives the next preliminary version of Theorem 2.1.

Theorem 4.1. *For $m > 0$, let $\mathcal{L}_m$ be given by (2.3). Let $z = \tau + i\delta$ for $|\delta| < 1$. Then, we get*

$$\|(\mathcal{L}_m - z)^{-1}\| \geq \frac{1}{2\sqrt{2}m\sqrt{1-\delta^2}}\Big[\tau^2 + \frac{m}{2}\tau + \frac{8(1+\delta^2) + 4m(1+\delta) - m^2}{8} + \mathcal{O}(\tau^{-1})\Big] \tag{4.1a}$$

and

$$\|(\mathcal{L}_m - z)^{-1}\| \leq \frac{4}{m(1-\delta^2)}\Big[\tau^2 + \frac{1+\delta^2}{4} + m + \mathcal{O}(\tau^{-2})\Big], \tag{4.1b}$$

as $|\tau| \to +\infty$. Both limits inside the brackets converge uniformly[2] *for all $|\delta| < 1$ and m on compact set.*

According to this statement, the resolvent norm of $\mathcal{L}_m$ has leading asymptotic behaviour $|\operatorname{Re} z|^2$, as $z \to \infty$ inside the band forming the numerical range. If we assume that Theorem 2.1 has already been proved, then we see that the leading order coefficient of the upper bound in (4.1) is optimal, up to a factor 4. However, the leading coefficient of the lower bound is sub-optimal. The latter is a consequence of choosing below a sub-optimal pseudomode. We have included this calculation, in

[2] Here and elsewhere below, we adopt the common say that two function are related by $f(z) = g(z) + \mathcal{O}(\tau^{-p})$ as $|\tau| \to \infty$ and that the limit converges uniformly for parameters satisfying a condition, if and only if there exist a constant $K > 0$ independent of τ and these parameters, such that $|f(z) - g(z)| \leq K|\tau|^{-p}$ for all $|\tau| \geq 1$ and all such parameters satisfying the condition.

order to illustrate the construction of the optimal pseudomode for the full proof of the main Theorem 2.1 in Section 4.2, without too many technical details.

Before proceeding to the asymptotic analysis of the coefficients involved, we highlight the connection with recent results of the same nature.

According to [14, Theorem 2.2], inside the numerical range the Schrödinger operator with a dislocated potential, $-\mathrm{d}^2/\mathrm{d}x^2 + i\,\mathrm{sgn}(x)$, has a resolvent norm with linear leading order of growth at infinity in $|\operatorname{Re} z|$. This discrepancy is explained by the fact that $\mathcal{L}_m$ is an operator of order 1, therefore the perturbation affects the resolvent by a higher order of magnitude.

When the potential is smooth and asymptotic to a dislocation, the effect of the perturbation on the Dirac operator appears to become substantially stronger. For instance, in the case of the operator $\widetilde{\mathcal{L}} = \mathcal{D}_m + i\,2\arctan(x)I/\pi$, by using the recent results of [18, Theorem 3.11], it might be possible to show that there exists a WKB-pseudomode $\Psi_{\tau,n}$, such that

$$\frac{\|(\widetilde{\mathcal{L}} - \tau I)\Psi_{\tau,n}\|}{\|\Psi_{\tau,n}\|} = \mathcal{O}(\tau^{-n})$$

as $|\tau| \to \infty$ for all $n \in \mathbb{N}$. This may be closely linked to the fact that interior singularities of a differential expression would create an exponential growth of the resolvent norm away from the spectrum. The phenomenon could be similar to the one reported recently in [2] for the case of a singular Sturm–Liouville operator and is worthy of further exploration.

We now give the proof of Theorem 4.1. We split the proof into three steps.

Step 1. The first step is an asymptotic analysis of the parameters involved in the resolvent estimate. This result is useful on its own, and will be invoked repeatedly in later sections.

Lemma 4.2. *For $m > 0$, let $\mu_z^\pm$ and $w_z^\pm$ be as in (3.2), and let k_z be as in (3.15). For indices $p \in [[-4, 2]]$ and parameters*

$$\mathsf{P} \in \left\{\mu_z^\pm, \frac{\operatorname{Re}\mu_z^\pm}{1 \mp \delta}, w_z^\pm, |w_z^\pm|, k_z\right\},$$

let the coefficients $C_p(\mathsf{P}) \in \mathbb{C}$ be prescribed in Table 1. There exists a constant $K_m > 0$, independent of τ and δ, such that

$$\left|\mathsf{P} - \sum_{p=-4}^{2} C_p(\mathsf{P})\tau^p\right| \le \frac{K_m}{\tau^5}$$

for all $|\tau| \ge 1$ and $|\delta| < 1$. This constant can be replaced by a uniform constant for all m on a compact set.

P	2	1	0	−1	−2	−3	−4
$\mu_z^{\pm}$	0	$\pm i$	$1 \mp \delta$	$\mp \frac{im^2}{2}$	$\frac{(1\mp\delta)m^2}{2}$	$\frac{\mp im^2}{8}[m^2 - 4(1 \mp \delta)^2]$	$\mp \frac{m^2}{8}(1 \mp \delta)[3m^2 - 4(1 \mp \delta)^2]$
$w_z^{\pm}$	0	0	$\pm i$	$\mp im$	$\frac{m}{2}[2(1 \mp \delta) \pm im]$	$\frac{m}{2}[im \mp (1+i)(1 \mp \delta)]$ $\times[(1+i)(1 \mp \delta) \mp m]$	$\frac{m}{8}\{4(1 \mp \delta)[3m^2 + 3i(1 \mp \delta)m$ $- 2(1 \mp \delta)^2] \pm 3im^3\}$
$\lvert w_z^{\pm} \rvert$	0	0	1	$-m$	$\frac{m^2}{2}$	$\frac{m}{2}[2(1 \mp \delta)^2 - m^2]$	$m^2\left[\frac{3}{8}m^2 - (1 \mp \delta)^2\right]$
k_z	$\frac{i}{m}$	$-\frac{2\delta}{m}$	$i\left(\frac{\delta^2 - 1}{m} - m\right)$	0	im	$2\delta m$	$im(1 + 3\delta^2 - m^2)$
$\frac{\mathrm{Re}\,\mu_z^{\pm}}{1 \mp \delta}$	0	0	1	0	$\frac{m^2}{2}$	0	$\mp \frac{m^2}{8}[3m^2 - 4(1 \mp \delta)^2]$

Table 1. Coefficients $C_p(\mathsf{P})$ for the parameters in Lemma 4.2.

Proof. We only provide a proof sketch, because the details are lengthy routine calculations.

The strategy for each of the parameters P, is to find a suitable Taylor expansion from which the coefficients $C_p \in \mathbb{C}$ are obtained from those of the different powers of $1/\tau$. The remainder at power 5 in this expansion gives the uniform bound in terms of δ and m. The specific Taylor series and coefficients are determined from the following reductions:

$$\mu_z^{\pm} = i\tau\sqrt{1 - \frac{m^2 + (1 \mp \delta)^2}{\tau^2} \mp i\frac{2(1 \mp \delta)}{\tau}}$$

and

$$w_z^{\pm} = \sqrt{\frac{1 - \frac{m \pm i(1 \mp \delta)}{\tau}}{1 + \frac{m \mp i(1 \mp \delta)}{\tau}}}.$$

For $\mathsf{P} = |w_z^{\pm}|$, we use

$$|w_z^{\pm}| = \sqrt{w_z^{\pm} w_z^{\mp}} = \sqrt{1 + g(\tau)}$$

where $g(\tau) = \mathcal{O}\left(\frac{1}{\tau}\right)$ as $|\tau| \to \infty$. For $\mathsf{P} = k_z$, we use (3.15) and the expansions of $w_{\pm}$.

Concretely, for $\mathsf{P} = \mu_z^+$, we write $\mu_z^+ = i\tau f\left(\frac{1}{\tau}\right)$ where f is analytic near 0. Expanding in power series at that point gives

$$i\frac{f(x)}{x} = \sum_{k=-1}^{4} C_{-k}(\mu_z^+)x^p + \frac{x^5}{6!}\int_0^1 (1-s)^5 f^{(6)}(xs)\,\mathrm{d}s,$$

where the coefficients $C_p(\mu_z^+)$ coincide with those in Table 1. Moreover, $f^{(6)}$ has a denominator involving a term of the form $(a(\delta, m)x^2 + b(\delta, m)x + 1)^{11/2}$, where a and b are bounded uniformly in both variables. Therefore, taking $|x|$ small enough, ensures that $|f^{(6)}(x)|$ is bounded uniformly in x, $|\delta| < 1$ and $m > 0$ in compact sets.

For the expansion of $\operatorname{Re}\mu_z^{\pm}/1 \mp \delta$, recall that $\operatorname{Re}\sqrt{\gamma} = |\operatorname{Im}\gamma|/\sqrt{2(|\gamma| - \operatorname{Re}\gamma)}$ for $\gamma \in \mathbb{C} \setminus \mathbb{R}$, and so

$$\frac{\operatorname{Re}\mu_z^{\pm}}{1 \mp \delta} = \sqrt{\frac{2}{1 - \frac{m^2+(1\mp\delta)^2}{\tau^2} + \sqrt{\left(1 - \frac{m^2+(1\mp\delta)^2}{\tau^2}\right)^2 + \frac{4(1\mp\delta)^2}{\tau^2}}}}. \qquad \blacksquare$$

Step 2. Consider the upper bound first. According to Lemma 3.3, we have that

$$\begin{aligned}\|(\mathcal{L}_m - z)^{-1}\| &\le \frac{(|k_z|+1)\max\left\{|w_z^{\pm}| + \frac{1}{|w_z^{\pm}|}, |w_z^+| + \frac{1}{|w_z^+|}\right\}}{\min\{\operatorname{Re}\mu_z^-, \operatorname{Re}\mu_z^+\}} \\ &\le (|k_z|+1)\left(|w_z^+| + \frac{1}{|w_z^+|} + |w_z^-| + \frac{1}{|w_z^-|}\right)\left(\frac{1}{\operatorname{Re}\mu_z^-} + \frac{1}{\operatorname{Re}\mu_z^+}\right),\end{aligned}$$

for all $z \in \rho(\mathcal{L}_m)$. By Lemma 4.2, the asymptotic behaviour of each of the multiplying terms above is

$$
\begin{aligned}
|k_z| + 1 &= \frac{1}{m}\tau^2 + \Big(\frac{1+\delta^2}{m} - m + 1\Big) + \mathcal{O}(\tau^{-4}),\\
|w_z^+| + \frac{1}{|w_z^+|} + |w_z^-| + \frac{1}{|w_z^-|} &= 2 + m^2\tau^{-2} + \mathcal{O}(\tau^{-4}), \qquad (4.2)\\
\frac{1}{\operatorname{Re}\mu_z^-} + \frac{1}{\operatorname{Re}\mu_z^+} &= \frac{1}{1-\delta^2}[2 - m^2\tau^{-2} + \mathcal{O}(\tau^{-4})], \qquad (4.3)
\end{aligned}
$$

as $|\tau| \to \infty$. Note that the first two coefficients of the terms involving the maximum above are identical. Multiplying and collecting terms gives the upper bound in (4.1) as required. Observe that the fact that the limit inside the bracket is uniform in the parameters, carries over from the conclusion of Lemma 4.2.

Step 3. Now, consider the lower estimate of the resolvent in (4.1). Set a pseudomode

$$
\psi_0(x) = \begin{pmatrix} 1 \\ w_z^+ \end{pmatrix} e^{\mu_z^+ x} \chi_{(-\infty,0)}(x),
$$

which belongs to $L^2(\mathbb{R}, \mathbb{C}^2)$ and has a norm $\|\psi_0\| = \sqrt{1 + |w_z^-|^2}/\sqrt{2\operatorname{Re}\mu_z^-}$. From the formula of the kernel in (3.4) and considering the location of the support of ψ_0, we have

$$
\begin{aligned}
\|(\mathcal{L}_m - z)^{-1}\psi_0\|^2 &\geq \int_0^{+\infty} \Big| \int_{-\infty}^0 N_{3,z} \begin{pmatrix} 1 \\ w_z^+ \end{pmatrix} e^{-\mu_z^+ x + 2\operatorname{Re}\mu_z^- y}\, \mathrm{d}y \Big|_{\mathbb{C}^2}^2 \,\mathrm{d}x\\
&= \Big| N_{3,z} \begin{pmatrix} 1 \\ w_z^+ \end{pmatrix} \Big|_{\mathbb{C}^2}^2 \frac{1}{8(\operatorname{Re}\mu_z^-)^2 \operatorname{Re}\mu_z^+}\\
&= \bigg(\frac{(1 + |w_z^-|^2)\sqrt{1 + |w_z^+|^2}}{|w_z^+ + w_z^-|} \bigg)^2 \frac{1}{8(\operatorname{Re}\mu_z^-)^2 \operatorname{Re}\mu_z^+}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|(\mathcal{L}_m - z)^{-1}\| &\geq \frac{\|(\mathcal{L}_m - z)^{-1}\psi_0\|}{\|\psi_0\|}\\
&= \frac{1}{2}\Big[\frac{(1 + |w_z^-|^2)(1 + |w_z^+|^2)}{\operatorname{Re}(\mu_z^-)\operatorname{Re}(\mu_z^+)} \Big]^{1/2} |w_z^+ + w_z^-|^{-1}.
\end{aligned}
$$

According to Lemma 4.2, the first bracket is $\mathcal{O}(1)$ and the second term is $\mathcal{O}(\tau^2)$. Multiplying and collecting coefficients, gives the lower estimate of the resolvent in (4.1). We omit further details, but this confirms the validity of Theorem 4.1.

4.2. Proof of Theorem 2.1

We now establish improved estimates, which enable the identification of the sharp constants in the asymptotics for the resolvent norm of $\mathcal{L}_m$ in the instability band. These will render Theorem 2.1 as a corollary. The strategy for the computation of the different constants has two parts. We first split the resolvent as the sum of two integral operators,

$$(\mathcal{L}_m - zI)^{-1} = T_1(z) + T_2(z),$$

where the norm of the operator $T_1(z)$ carries the leading order behaviour and $\|T_2(z)\|$ is $\mathcal{O}(1)$ at infinity. We then proceed as in the proof of Theorem 4.1, and compute the asymptotic coefficients of $\|T_1(z)\|$ and $\|T_2(z)\|$. In the case of the lower bound, we find a pseudomode for $T_1(z)$ that is sharp for the leading order, by optimising the contribution of the two components of the wave function in an adaptive manner as z moves towards infinity. For a further discussion on the computation of other coefficients, see Remark 4.6 at the end of this section.

The explicit formulas of the two integral operators is the following. For $j = 1, 2$,

$$T_j(z) f(x) = \int_{\mathbb{R}} \mathcal{R}_{j,z}(x, y) f(y)\, \mathrm{d}y, \tag{4.4}$$

where

$$\mathcal{R}_{1,z}(x, y) = \begin{cases} N_{1,z}\, e^{\mu_z^-(x+y)} & \text{for } \{y < x < 0\}, \\ N_{3,z}\, e^{-\mu_z^+ x + \mu_z^- y} & \text{for } \{y < 0 < x\}, \\ N_{4,z}\, e^{-\mu_z^+(x+y)} & \text{for } \{0 < y < x\}, \\ N_{7,z}\, e^{-\mu_z^+(x+y)} & \text{for } \{0 < x < y\}, \\ N_{8,z}\, e^{\mu_z^- x - \mu_z^+ y} & \text{for } \{x < 0 < y\}, \\ N_{10,z}\, e^{\mu_z^-(x+y)} & \text{for } \{x < y < 0\}, \end{cases} \tag{4.5}$$

and

$$\mathcal{R}_{2,z}(x, y) = \begin{cases} N_{2,z}\, e^{-\mu_z^-(x-y)} & \text{for } \{y < x < 0\}, \\ 0 & \text{for } \{y < 0 < x\}; \\ N_{5,z}\, e^{-\mu_z^+(x-y)} & \text{for } \{0 < y < x\}, \\ N_{6,z}\, e^{\mu_z^+(x-y)} & \text{for } \{0 < x < y\}, \\ 0 & \text{for } \{x < 0 < y\}, \\ N_{9,z}\, e^{\mu_z^-(x-y)} & \text{for } \{x < y < 0\}. \end{cases} \tag{4.6}$$

Recall the definition of the matrices $N_{k,z}$ at the beginning of Section 3.

A crucial reason for the split of the resolvent in this fashion is to observe that the term $1/w_z^+ + w_z^-$ is the only one responsible for the quadratic growth of the norm and that this term appears only in $N_{k,z}$ for $k \in \{1, 3, 4, 7, 8, 10\}$. Moreover, now the kernel $\mathcal{R}_{1,z}$ is separable, therefore a sharp upper bound on its norm is obtained by means of Hölder's inequality.

For $j \in [[1, 10]]$, we denote $n_j = n_j(z) = |N_{j,z}|_{\mathbb{C}^{2\times 2}}$. The fact that $\det N_{j,z} = 0$ and substitutions give

$$n_1 = n_{10} = \frac{1}{2}\left|\frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}\right|\Big(|w_z^-| + \frac{1}{|w_z^-|}\Big), \tag{4.7a}$$

$$n_2 = n_9 = \frac{1}{2}\Big(|w_z^-| + \frac{1}{|w_z^-|}\Big), \tag{4.7b}$$

$$n_3 = n_8 = \frac{\sqrt{(1 + |w_z^+|^2)(1 + |w_z^-|^2)}}{|w_z^+ + w_z^-|}, \tag{4.7c}$$

$$n_4 = n_7 = \frac{1}{2}\left|\frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}\right|\Big(|w_z^+| + \frac{1}{|w_z^+|}\Big), \tag{4.7d}$$

$$n_5 = n_6 = \frac{1}{2}\Big(|w_z^+| + \frac{1}{|w_z^+|}\Big). \tag{4.7e}$$

Note that the asymptotics of all the terms in the formulas for n_j are available from the calculations performed in the previous subsection. Also, note that the operator norms of $\mathcal{R}_{j,z}$ are symmetric with respect to the axis $y = x$, as was the case for $\mathcal{R}_z$.

The next proposition gives explicit bounds for the norm of the leading operator $T_1(z)$. The match of all terms in these bounds except $\widetilde{B}_z$ and B_z is the main ingredient that enables the sharp constants below. For $z \neq 0$, we write

$$\operatorname{csgn}(z) = \frac{z}{|z|}.$$

Proposition 4.3. *Let $m > 0$. Let the operator $T_1(z)$ be as in (4.4)–(4.5). For all $z \in \rho(\mathcal{L}_m)$, let*

$$A_z = \frac{n_1^2}{4(\operatorname{Re}\mu_z^-)^2},$$

$$C_z = \frac{n_4^2}{4(\operatorname{Re}\mu_z^+)^2},$$

$$D_z = \frac{n_3^2}{4(\operatorname{Re}\mu_z^-)(\operatorname{Re}\mu_z^+)},$$

$$B_z = \frac{n_3}{2\sqrt{(\operatorname{Re}\mu_z^+)(\operatorname{Re}\mu_z^-)}}\Big(\frac{n_1}{\operatorname{Re}\mu_z^-} + \frac{n_4}{\operatorname{Re}\mu_z^+}\Big),$$

$$\widetilde{B}_z = \frac{n_3}{2\sqrt{(\operatorname{Re}\mu_z^+)(\operatorname{Re}\mu_z^-)}}\Bigg(\frac{n_1 \operatorname{Re}\big(\operatorname{csgn}\big(\frac{w_z^+ - w_z^-}{w_z^-}\big)\big)}{\operatorname{Re}\mu_z^-} - \frac{n_4 \operatorname{Re}\big(\operatorname{csgn}\big(\frac{w_z^+ - w_z^-}{w_z^+}\big)\big)}{\operatorname{Re}\mu_z^+}\Bigg).$$

Then, $T_1(z)$ is bounded for all $z \in \rho(\mathcal{L}_m)$. Moreover,

$$\|T_1(z)\| \leq \frac{1}{\sqrt{2}}\sqrt{\sqrt{(A_z - C_z)^2 + B_z^2} + A_z + C_z + 2D_z} \tag{4.8a}$$

and

$$\|T_1(z)\| \geq \frac{1}{\sqrt{2}}\sqrt{\sqrt{(A_z - C_z)^2 + \widetilde{B}_z^2} + A_z + C_z + 2D_z}. \tag{4.8b}$$

Proof. We first prove the upper bound, then the lower bound.

Upper bound. Let $\Psi \in L^2(\mathbb{R}, \mathbb{C}^2)$ be such that $\|\Psi\|^2 = 1$. Then,

$$\|T_1(z)\Psi\|^2 \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathcal{R}_{1,z}(x,y)|_{\mathbb{C}^{2\times 2}} |\Psi(y)|_{\mathbb{C}^2} \, \mathrm{d}y \right)^2 \mathrm{d}x.$$

The right-hand side of this expression is given explicitly as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathcal{R}_{1,z}(x,y)|_{\mathbb{C}^{2\times 2}} |\Psi(y)|_{\mathbb{C}^2} \, \mathrm{d}y \right)^2 \mathrm{d}x \\
&= \frac{1}{2(\operatorname{Re}\mu_z^-)} \left(n_1 \int_{-\infty}^{0} e^{(\operatorname{Re}\mu_z^-)y} |\Psi(y)|_{\mathbb{C}^2} \, \mathrm{d}y + n_8 \int_{0}^{+\infty} e^{-(\operatorname{Re}\mu_z^+)y} |\Psi(y)|_{\mathbb{C}^2} \, \mathrm{d}y \right)^2 \\
&\quad + \frac{1}{2(\operatorname{Re}\mu_z^+)} \left(n_3 \int_{-\infty}^{0} e^{(\operatorname{Re}\mu_z^-)y} |\Psi(y)|_{\mathbb{C}^2} \, \mathrm{d}y + n_4 \int_{0}^{+\infty} e^{-(\operatorname{Re}\mu_z^+)y} |\Psi(y)|_{\mathbb{C}^2} \, \mathrm{d}y \right)^2.
\end{aligned}$$

By applying Holder's inequality to the inner integrals, we get

$$\begin{aligned}
\|T_1(z)\Psi\|^2 &\leq \frac{1}{2\operatorname{Re}\mu_z^-} \Big(\frac{n_1}{\sqrt{2\operatorname{Re}\mu_z^-}} \|\Psi\|_{L^2(\mathbb{R}_-)} + \frac{n_8}{\sqrt{2\operatorname{Re}\mu_z^+}} \|\Psi\|_{L^2(\mathbb{R}_+)} \Big)^2 \\
&\quad + \frac{1}{2\operatorname{Re}\mu_z^+} \Big(\frac{n_3}{\sqrt{2\operatorname{Re}\mu_z^-}} \|\Psi\|_{L^2(\mathbb{R}_-)} + \frac{n_4}{\sqrt{2\operatorname{Re}\mu_z^+}} \|\Psi\|_{L^2(\mathbb{R}_+)} \Big)^2 \\
&\leq \frac{n_1^2}{4(\operatorname{Re}\mu_z^-)^2} \|\Psi\|^2_{L^2(\mathbb{R}_-)} + \frac{n_4^2}{4(\operatorname{Re}\mu_z^+)^2} \|\Psi\|^2_{L^2(\mathbb{R}_+)} + \frac{n_3^2}{4(\operatorname{Re}\mu_z^-)(\operatorname{Re}\mu_z^+)} \\
&\quad + \frac{n_3}{2\sqrt{(\operatorname{Re}\mu_z^+)(\operatorname{Re}\mu_z^-)}} \Big(\frac{n_1}{\operatorname{Re}\mu_z^-} + \frac{n_4}{\operatorname{Re}\mu_z^+} \Big) \|\Psi\|_{L^2(\mathbb{R}_-)} \|\Psi\|_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

Here we have used that $n_1 = n_{10}$, $n_3 = n_8$ and that $\|\Psi\| = 1$.

Now, for constants $A, B, C \in \mathbb{R}$,

$$\max_{x^2+y^2=1} Ax^2 + Bxy + Cy^2 = \frac{A + C + \sqrt{(A-C)^2 + B^2}}{2}. \tag{4.9}$$

Then, taking $x = \|\Psi\|_{L^2(\mathbb{R}_-)}$ and $y = \|\Psi\|_{L^2(\mathbb{R}_+)}$, we obtain for all $\Psi \in L^2(\mathbb{R}, \mathbb{C}^2)$ such that $\|\Psi\|^2 = 1$,

$$\|T_1(z)\Psi\|^2 \leq \frac{1}{2}\Big(A_z + C_z + \sqrt{(A_z - C_z)^2 + B_z^2} + 2D_z\Big),$$

where A_z, B_z, C_z, D_z are as above. Moreover, since all these 4 terms are finite for all $z \in \rho(\mathcal{L}_m)$, then indeed $T_1(z)$ is a bounded operator.

Lower bound. In order to obtain the lower bound estimate on the norm of $T_1(z)$, consider a pseudomode of the form

$$\Psi_0(y) = \begin{cases} \beta \begin{pmatrix} -1 \\ w_{\bar{z}}^- \end{pmatrix} e^{-\overline{\mu_z^+}\, y} & \text{if } y \geq 0, \\ \alpha \begin{pmatrix} 1 \\ w_{\bar{z}}^+ \end{pmatrix} e^{\overline{\mu_z^-}\, y} & \text{if } y \leq 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ will be chosen later to normalise $\|\Psi_0\|$ and maximise $\|T_1(z)\Psi_0\|$.

We first compute the square of $\|T_1(z)\Psi_0\|$. Indeed,

$$\begin{aligned} \|T_1(z)\Psi_0\|^2 &= \int_{-\infty}^{0} \left| \int_{-\infty}^{+\infty} \mathcal{R}_{1,z}(x,y)\Psi_0(y)\,\mathrm{d}y \right|_{\mathbb{C}^2}^2 \mathrm{d}x \\ &\quad + \int_{0}^{+\infty} \left| \int_{-\infty}^{+\infty} \mathcal{R}_{1,z}(x,y)\Psi_0(y)\,\mathrm{d}y \right|_{\mathbb{C}^2}^2 \mathrm{d}x \\ &= K_1 + K_2. \end{aligned}$$

By the definition of Ψ_0, we have, for $x < 0$,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathcal{R}_{1,z}(x,y)\Psi_0(y)\,\mathrm{d}y \\ &\quad = e^{\mu_z^- x}\left[\frac{\alpha \frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}(1 + |w_z^-|^2)\frac{1}{w_z^-}}{4\,\mathrm{Re}\,\mu_z^-} + \frac{\frac{\beta(1+|w_z^+|^2)}{w_z^+ + w_z^-}}{2\,\mathrm{Re}\,\mu_z^+} \right] \begin{pmatrix} 1 \\ w_z^- \end{pmatrix}, \end{aligned}$$

and for $x > 0$,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathcal{R}_{1,z}(x,y)\Psi_0(y)\,\mathrm{d}y \\ &\quad = e^{-\mu_z^+ x}\left[\frac{\frac{\alpha(1+|w_z^-|^2)}{w_z^+ + w_z^-}}{2\,\mathrm{Re}\,\mu_z^-} + \frac{\beta \frac{w_z^+ - w_z^-}{w_z^+ + w_z^-}(1 + |w_z^+|^2)\frac{-1}{w_z^+}}{4\,\mathrm{Re}\,\mu_z^+} \right] \begin{pmatrix} -1 \\ w_z^+ \end{pmatrix}. \end{aligned}$$

Now,

$$K_1 = \frac{1}{2\operatorname{Re}\mu_z^-}\Big[\frac{|\alpha|^2 n_1^2(1+|w_z^-|^2)}{4(\operatorname{Re}\mu_z^-)^2} + \frac{|\beta|^2 n_8^2(1+|w_z^+|^2)}{4(\operatorname{Re}\mu_z^+)^2} + \frac{n_1 n_8 \alpha\beta\sqrt{(1+|w_z^-|^2)(1+|w_z^+|^2)}}{2(\operatorname{Re}\mu_z^+)(\operatorname{Re}\mu_z^-)}\operatorname{Re}\Big(\operatorname{csgn}\Big(\frac{w_z^+ - w_z^-}{w_z^-}\Big)\Big)\Big],$$

and

$$K_2 = \frac{1}{2\operatorname{Re}\mu_z^+}\Big[\frac{|\alpha|^2 n_3^2(1+|w_z^-|^2)}{4(\operatorname{Re}\mu_z^-)^2} + \frac{|\beta|^2 n_4^2(1+|w_z^+|^2)}{4(\operatorname{Re}\mu_z^+)^2} - \frac{n_3 n_4 \alpha\beta\sqrt{(1+|w_z^-|^2)(1+|w_z^+|^2)}}{2(\operatorname{Re}\mu_z^+)(\operatorname{Re}\mu_z^-)}\operatorname{Re}\Big(\operatorname{csgn}\Big(\frac{w_z^+ - w_z^-}{w_z^+}\Big)\Big)\Big].$$

Moreover,

$$\|\Psi_0\|^2 = |\alpha|^2\frac{1+|w_z^-|^2}{2\operatorname{Re}\mu_z^-} + |\beta|^2\frac{1+|w_z^+|^2}{2\operatorname{Re}\mu_z^+}.$$

To normalise Ψ_0 properly, we re-write

$$\alpha = \frac{\sqrt{2\operatorname{Re}\mu_z^-}}{\sqrt{1+|w_z^-|^2}}x$$

and

$$\beta = \frac{\sqrt{2\operatorname{Re}\mu_z^+}}{\sqrt{1+|w_z^+|^2}}y,$$

for $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$. We seek for pairs (x, y), such that the expression

$$\|T_1(z)\Psi_0\|^2 = \frac{n_1^2}{4(\operatorname{Re}\mu_z^-)^2}x^2 + \frac{n_4^2}{4(\operatorname{Re}\mu_z^+)^2}y^2 + \frac{n_3^2}{4(\operatorname{Re}\mu_z^-)(\operatorname{Re}\mu_z^+)} + \frac{n_3}{2\sqrt{(\operatorname{Re}\mu_z^+)(\operatorname{Re}\mu_z^-)}}\Bigg(\frac{n_1\operatorname{Re}\big(\operatorname{csgn}\big(\frac{w_z^+-w_z^-}{w_z^-}\big)\big)}{\operatorname{Re}\mu_z^-} - \frac{n_4\operatorname{Re}\big(\operatorname{csgn}\big(\frac{w_z^+-w_z^-}{w_z^+}\big)\big)}{\operatorname{Re}\mu_z^+}\Bigg)xy$$

attains its maximum. Therefore, according to (4.9) once again, we have

$$\|T_1(z)\|^2 \geq \|T_1(z)\Psi_0\|^2 = \frac{1}{2}(A_z + C_z + \sqrt{(A_z - C_z)^2 + \widetilde{B}_z^2} + 2D_z).$$

Thus, the lower bound of $T_1(z)$ is indeed confirmed. ∎

Our next goal is to determine an upper bound for the norm of $T_2(z)$.

Proposition 4.4. *Let $m > 0$. Let the operator $T_2(z)$ be as in (4.4)–(4.6). Then, $T_2(z)$ is bounded for all $z \in \rho(\mathcal{L}_m)$. Moreover,*

$$\|T_2(z)\| \leq \max\Big\{\frac{n_2}{\operatorname{Re}\mu_z^-}, \frac{n_5}{\operatorname{Re}\mu_z^+}\Big\}.$$

Proof. We use the Schur test to prove this proposition. For $x < 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{R}_{2,z}(x,y)|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}y &= \int_{-\infty}^{x} n_2 e^{-\operatorname{Re}\mu_z^-(x-y)}\,\mathrm{d}y + \int_{x}^{0} n_9 e^{\operatorname{Re}\mu_z^-(x-y)}\,\mathrm{d}y \\ &= \frac{n_2}{\operatorname{Re}\mu_z^-}(2 - e^{\operatorname{Re}\mu_z^- x}) \\ &\leq \frac{n_2}{\operatorname{Re}\mu_z^-}.\end{aligned}$$

Note that $n_2 = n_9$. Similarly, for $x \geq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{R}_{2,z}(x,y)|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}y &= \int_{0}^{x} n_5 e^{-\operatorname{Re}\mu_z^+(x-y)}\,\mathrm{d}y + \int_{x}^{+\infty} n_6 e^{\operatorname{Re}\mu_z^+(x-y)}\,\mathrm{d}y \\ &= \frac{n_5}{\operatorname{Re}\mu_z^-}(2 - e^{-\operatorname{Re}\mu_z^+ x}) \\ &\leq \frac{n_5}{\operatorname{Re}\mu_z^+}.\end{aligned}$$

Note that $n_5 = n_6$. Now, recall that $|\mathcal{R}_{2,z}(x,y)|_{\mathbb{C}^{2\times 2}} = |\mathcal{R}_{2,z}(y,x)|_{\mathbb{C}^{2\times 2}}$. Hence, by virtue of the Schur test, the claimed conclusion follows. ∎

With the bounds for the norms of the components $T_j(z)$ at hand from the previous two propositions, we are now in the position to tackle Theorem 2.1. This will be a direct consequence of the following theorem, which gives slightly more refined asymptotic estimates. This theorem is one of the main contributions of this paper and it directly implies Theorem 2.1.

Theorem 4.5. *For $m > 0$, let $\mathcal{L}_m$ be the dislocated Dirac operator (2.3). Let $z = \tau + i\delta$ for $|\delta| < 1$. Then, as $|\tau| \to \infty$,*

$$\|(\mathcal{L}_m - z)^{-1}\| \leq \frac{1}{m(1-\delta^2)}[\tau^2 + \mathsf{P}_0^+ + \mathsf{P}_{-2}^+\tau^{-2} + \mathcal{O}(\tau^{-4})]$$

and

$$\|(\mathcal{L}_m - z)^{-1}\| \geq \frac{1}{m(1-\delta^2)}[\tau^2 + \mathsf{P}_0^- + \mathsf{P}_{-2}^-\tau^{-2} + \mathcal{O}(\tau^{-4})],$$

where

$$\mathsf{P}_0^{\pm} = \frac{1}{4}(3 + 2\delta^2 - 2m^2) \pm 2m,$$
$$\mathsf{P}_{-2}^{+} = \frac{1}{4}(3m^2 + \delta^2(9m^2 - 8)),$$
$$\mathsf{P}_{-2}^{-} = \frac{1}{8}(5m^2 + \delta^2(19m^2 - 16)).$$

The limits inside the brackets converge uniformly for all $|\delta| < 1$ *and* m *on compact sets.*

The remainder of this section is devoted to sketching the proof of this theorem. That involves computing the coefficients in the asymptotic expansions of the various terms in the expressions (4.8). We only give the intermediate results in the computation of these coefficients.

Coefficients for $(A_z - C_z)^2$. We take the square of the following:

$$A_z - C_z = \frac{-\delta}{m^2(1-\delta^2)^2}\Big[\sum_{k=-4}^{4} \mathsf{P}_k \tau^k + \mathcal{O}(\tau^{-6})\Big]$$

where[3]

$$\mathsf{P}_4 = 1,$$
$$\mathsf{P}_2 = 2(\delta^2 - m^2 + 1),$$
$$\mathsf{P}_0 = \delta^4 + 2\delta^2(m^2 - 1) + m^4 + 1,$$
$$\mathsf{P}_{-2} = -2(\delta^4 + 6\delta^2 + 1)m^2,$$
$$\mathsf{P}_{-4} = m^2(2\delta^6 - 5\delta^4(m^2 - 6) - \delta^2(22m^2 - 30) - 6m^2 + 2).$$

Coefficients for B_z^2. We use

$$B_z^2 = D_z\Big(\frac{n_3}{\operatorname{Re}\mu_z^-} + \frac{n_4}{\operatorname{Re}\mu_z^+}\Big)^2$$

together with the following. On the one hand,

$$D_z = \frac{1}{m^2(1-\delta^2)}\Big[\sum_{k=-4}^{4} \mathsf{P}_k \tau^k + \mathcal{O}(\tau^{-6})\Big]$$

[3]Here and henceforth, $\mathsf{P}_k = 0$ for k odd.

where

$$\mathsf{P}_4 = \frac{1}{4},$$
$$\mathsf{P}_2 = \frac{1}{2}(\delta^2 - m^2 + 1),$$
$$\mathsf{P}_0 = \frac{1}{4}(\delta^4 + \delta^2(m^2 - 2) + m^4 + 1),$$
$$\mathsf{P}_{-2} = -4\delta^2 m^2,$$
$$\mathsf{P}_{-4} = -\frac{1}{4}(m^2(\delta^4(m^2 - 32) + \delta^2(31m^2 - 32) + m^2)).$$

On the other hand,

$$\Big(\frac{n_3}{\operatorname{Re}\mu_z^-} + \frac{n_4}{\operatorname{Re}\mu_z^+}\Big)^2 = \frac{1}{m^2(1-\delta^2)^2}\Big[\sum_{k=-4}^{4} \mathsf{P}_k \tau^k + \mathcal{O}(\tau^{-6})\Big]$$

where

$$\mathsf{P}_4 = 4,$$
$$\mathsf{P}_2 = 8(\delta^2 - m^2 + 1),$$
$$\mathsf{P}_0 = 4(\delta^4 + \delta^2(3m^2 - 2) + m^4 - m^2 + 1),$$
$$\mathsf{P}_{-2} = -16\delta^2(\delta^2 + 3)m^2,$$
$$\mathsf{P}_{-4} = -4m^2(-4\delta^6 + 8\delta^4(m^2 - 5) + 5\delta^2(5m^2 - 4)).$$

Coefficients for $\sqrt{\sqrt{(A_z - C_z)^2 + B_z^2} + A_z + C_z + 2D_z}$. We use the following:

$$A_z + C_z + 2D_z = \frac{1}{m^2(1-\delta^2)^2}\Big[\sum_{k=-2}^{4} \mathsf{P}_k \tau^k + \mathcal{O}(\tau^{-4})\Big]$$

where

$$\mathsf{P}_4 = 1,$$
$$\mathsf{P}_2 = 2(\delta^2 - m^2 + 1),$$
$$\mathsf{P}_0 = \frac{1}{2}(2\delta^4 + \delta^2(5m^2 - 4) + 2m^4 - m^2 + 2),$$
$$\mathsf{P}_{-2} = -16\delta^2 m^2,$$

and

$$\sqrt{(A_z - C_z)^2 + B_z^2} = \frac{1}{m^2(1-\delta^2)^2}\Big[\sum_{k=-2}^{4} \mathsf{P}_k \tau_k + \mathcal{O}(\tau^{-4})\Big]$$

where

$$\begin{aligned}
\mathsf{P}_4 &= 1,\\
\mathsf{P}_2 &= 2(\delta^2 - m^2 + 1),\\
\mathsf{P}_0 &= \frac{1}{2}(2\delta^4 + \delta^2(5m^2 - 4) + 2m^4 - m^2 + 2),\\
\mathsf{P}_{-2} &= -256\delta^2 m^2.
\end{aligned}$$

Note that the first three coefficients of these two terms match exactly. This gives,

$$\|T_1(z)\| \le \frac{1}{m(1-\delta^2)}[\tau^2 + \mathsf{P}_0 + \mathsf{P}_{-2}\tau^{-2} + \mathcal{O}(\tau^{-4})], \tag{4.10}$$

where

$$\begin{aligned}
\mathsf{P}_0 &= \frac{1}{4}(3 + 2\delta^2 - 2m^2),\\
\mathsf{P}_{-2} &= \frac{1}{4}(3m^2 + \delta^2(9m^2 - 8)).
\end{aligned}$$

Coefficients for $\sqrt{\sqrt{(A_z - C_z)^2 + \widetilde{B}_z^2} + A_z + C_z + 2D_z}$. We use the fact that

$$\operatorname{Re}(\operatorname{csgn}(z)) = \frac{\operatorname{Re} z}{|z|}$$

to obtain the asymptotics

$$\operatorname{Re}\Big(\operatorname{csgn}\frac{w_z^+ - w_z^-}{w_z^\mp}\Big) = \mp 1 \pm \frac{m^2}{2\tau^4} + \mathcal{O}(\tau^{-6}).$$

Hence, we get

$$\sqrt{(A_z - C_z)^2 + \widetilde{B}_z^2} = \frac{1}{m^2(1-\delta^2)^2}\Big[\sum_{k=-2}^{4} \mathsf{P}_k \tau_k + \mathcal{O}(\tau^{-4})\Big]$$

where

$$\begin{aligned}
\mathsf{P}_4 &= 1,\\
\mathsf{P}_2 &= 2(\delta^2 - m^2 + 1),\\
\mathsf{P}_0 &= \delta^4 + \delta^2(3m^2 - 2) + m^4 - m^2 + 1,\\
\mathsf{P}_{-2} &= -32\delta^2(\delta^2 + 7).
\end{aligned}$$

Note that only the two final coefficients are different from the ones of B_z^2. Hence,

$$\|T_1(z)\| \ge \frac{1}{m(1-\delta^2)}[\tau^2 + \mathsf{P}_0 + \widetilde{\mathsf{P}}_{-2}\tau^{-2} + \mathcal{O}(\tau^{-4})] \tag{4.11}$$

where

$$\mathsf{P}_0 = \frac{1}{4}(3 + 2\delta^2 - 2m^2),$$
$$\widetilde{\mathsf{P}}_{-2} = \frac{1}{8}(5m^2 + \delta^2(19m^2 - 16)).$$

Completion of the proof. According to (4.2)–(4.3), we have

$$\|T_2(z)\| \leq (n_2 + n_5)\Big(\frac{1}{\operatorname{Re}\mu_z^-} + \frac{1}{\operatorname{Re}\mu_z^+}\Big) = \frac{1}{1-\delta^2}(2 + \mathcal{O}(\tau^{-4})),$$

as $|\tau| \to \infty$. Thus, we complete the proof of Theorem 4.5, by combining the asymptotic formulas (4.10)–(4.11) with this and invoking the triangle inequality.

Remark 4.6. We suspect that with some more effort, it is possible to show that there exists P_0 such that

$$\|(\mathcal{L}_m - z)^{-1}\| = \frac{1}{1 - |\operatorname{Im} z|^2}\Big(\frac{1}{m}|\operatorname{Re} z|^2 + \frac{1}{2m}[|\operatorname{Im} z|^2 + \mathsf{P}_0] + \mathcal{O}(|\operatorname{Re} z|^{-2})\Big),$$

where P_0 is a constant independent of z such that

$$|\mathsf{P}_0| \leq \frac{3}{2} + 4m + m^2.$$

Note that the only point left to show here is that the coefficient P_0 actually exists. The bound on its magnitude follows directly from our Theorem 4.5. Two comments on this are in place.

• To prove this claim, one possibility is to find sharp lower bounds for $\|T_2(z)\|$, then feed into the final step above. This might not be easy and a more refined construction of pseudomodes seems to be needed.

• To confirm existence of P_0 without control on $\|T_2(z)\|$, perhaps it is possible to begin by proving that the function

$$s_1(\tau, \delta) = \frac{1}{\|(\mathcal{L}_m - \tau - i\delta)^{-1}\|},$$

being the first singular value of the operator $\mathcal{L}_m - \tau - i\delta$, is real analytic in the variables $\tau \in \mathbb{R}$ and $\delta \in (-1, 1)$. This might follow from the fact that $\mathcal{L}_m - z$ is an integral operator with coefficients analytic in z, but it is not automatic and an analysis of its multiplicity is required.

5. Perturbations of $\mathcal{L}_m$ by a long-range potential

We now consider perturbations of the form $\mathcal{L}_m + V$ where $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$. To aid the reader through this section, we recall the hypotheses

- (H2) $\|V\|_{L^1} < 1$;
- (H3) $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}) \cap L^p(\mathbb{R}, \mathbb{C}^{2\times 2})$ for some $p \in (1, \infty]$.

Either will be assumed throughout.

Our main goal in the first part of the section is to identify a closed extension, $\mathcal{L}_{m,V} \supseteq \mathcal{L}_m + V$, which is well defined under either assumptions and preserves the essential spectra. Then, in the second part, we examine the spectrum and pseudospectrum of $\mathcal{L}_{m,V}$ inside the instability band, $\Sigma = \{|\operatorname{Im} z| < 1\}$.

Section 5.2 is devoted to the proof of the main Theorem 2.5 about the localisation of the point spectrum for $m > 0$. Section 5.3 addresses confirmation that there can be potentials with infinitely many eigenvalues in the instability band. Section 5.4 discusses the effect of weak coupling, when imposing further concrete hypotheses on the decay and regularity of V, and comprises the proof of Theorem 2.7. Finally, the proof of Theorem 2.4 about the pseudospectrum is given in Section 5.5.

5.1. Proof of Proposition 2.3

In order to show the existence of the closed extension $\mathcal{L}_{m,V}$, and determine its spectrum and pseudospectrum, our main tool is the Birman–Schwinger-type principle formulated in [12]. We quote that result in Theorem A.1 and apply it as follows.

Set the Hilbert spaces $\mathcal{H} = \mathcal{K} = L^2(\mathbb{R}, \mathbb{C}^2)$. Set the unperturbed operator $H_0 = \mathcal{L}_m$. The auxiliary factors A and B take the following form. Write

$$V(x) = U(x)|V(x)|$$

in polar form, where $U(x)$ is the partial isometry and $|V(x)| = (V^*(x)V(x))^{1/2}$. We set

$$A(x) = |V(x)|^{1/2} \quad \text{and} \quad B(x) = U(x)|V(x)|^{1/2}. \tag{5.1}$$

Then, we have

$$|A(x)|_{\mathbb{C}^2\times\mathbb{C}^2} = |V(x)|^{1/2}_{\mathbb{C}^2\times\mathbb{C}^2} = |B(x)|_{\mathbb{C}^2\times\mathbb{C}^2}. \tag{5.2}$$

The first equality follows from the property that the norm of the square root of a matrix equals the square root of the norm of the matrix itself. To establish the second equality in (5.2), observe that $\operatorname{Ker}(|V(x)|) \subset \operatorname{Ker}(|V(x)|^{1/2})$. Consequently, we have

$\overline{\operatorname{Im}(|V(x)|^{1/2})} \subset [\operatorname{Ker}(U(x))]^{\perp}$, because $\operatorname{Ker}(|V(x)|) = \operatorname{Ker}(U(x))$, cf. [4, Proposition 2.85]. Hence, as $U(x)$ is a partial isometry, it follows that

$$\begin{aligned} |B(x)|_{\mathbb{C}^2\times\mathbb{C}^2} &= \sup_{|v|_{\mathbb{C}^2}=1} |U(x)|V(x)|^{1/2}v|_{\mathbb{C}^2} \\ &= \sup_{|v|_{\mathbb{C}^2}=1} ||V(x)|^{1/2}v|_{\mathbb{C}^2} \\ &= |A(x)|_{\mathbb{C}^2\times\mathbb{C}^2} = |V(x)|^{1/2}_{\mathbb{C}^2\times\mathbb{C}^2}. \end{aligned}$$

The operators A and B denote the multiplication operators

$$Af(x) = A(x)f(x) \quad \text{and} \quad Bf(x) = B(x)f(x)$$

with their maximal domains. By construction, both A and B are closed and densely defined. For each $z \in \rho(\mathcal{L}_m)$, let

$$R_0(z) = (\mathcal{L}_m - z)^{-1} \quad \text{and} \quad Q(z) = \overline{AR_0(z)B}.$$

This settles the framework of Theorem A.1.

Now, we give the proof of Proposition 2.3. We split it into three steps. In the first step, we verify that the hypotheses (B1) and (B2) of Theorem A.1 hold, for all $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$. In the second step, we verify that (B3) is implied by either of the two different assumptions, (H2) or (H3). In the final step we derive the claims made about the spectrum.

Step 1. Conditions (B1) and (B2). Let $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$. Our goal is to show that, for suitable $z \in \rho(\mathcal{L}_m)$,

(B1) the operator $R_0(z)B$ is closable, and both $AR_0(z)$ and $\overline{R_0(z)B}$ are bounded;

(B2) the operator $AR_0(z)B$ has a bounded closure $Q(z) = \overline{AR_0(z)B}$.

Firstly, note that the Schur test conditions,

$$\begin{aligned} &\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|^n_{\mathbb{C}^{2\times 2}}\,\mathrm{d}y \le K_n(z), \qquad \text{for a.e. } x \in \mathbb{R}, \\ &\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|^n_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x \le K_n(z), \qquad \text{for a.e. } y \in \mathbb{R}, \end{aligned} \tag{5.3}$$

hold for all fixed $n \in \mathbb{N}$. The proof of this is the same as for the case of the power $n = 1$, addressed in the verification of (3.18), as the integration only involves exponential functions. Here,

$$K_n(z) = \frac{2}{n \min\{\operatorname{Re}\mu_z^-, \operatorname{Re}\mu_z^+\}} (\sup_{t\in\mathbb{R}} \varphi_z(t))^n,$$

φ_z is as in (3.14), and $K_n(z)$ is finite for each $z \in \rho(\mathcal{L}_m)$.

Since we have $H^1(\mathbb{R},\mathbb{C}^2) \subset L^\infty(\mathbb{R},\mathbb{C}^2)$ and $|V|^{1/2} \in L^2(\mathbb{R},\mathbb{C}^2)$, we obtain that $H^1(\mathbb{R},\mathbb{C}^2) \subset \mathrm{Dom}(A)$. As

$$R_0(z)\colon L^2(\mathbb{R},\mathbb{C}^2) \to H^1(\mathbb{R},\mathbb{C}^2),$$

it follows that $\mathrm{Dom}(AR_0(z)) = L^2(\mathbb{R},\mathbb{C}^2)$. Therefore, $AR_0(z)$ is closed and hence bounded by the closed graph theorem. This confirms the second requirement in (B1). Moreover, $AR_0(z)$ is a Hilbert–Schmidt operator. Indeed, from (5.2) and (5.3) with $n = 2$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^2} |A(x)\mathcal{R}_z(x,y)|^2_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x\,\mathrm{d}y &\le \int_{\mathbb{R}^2} |V(x)|_{\mathbb{C}^{2\times 2}}|\mathcal{R}_z(x,y)|^2_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le K_2(z)\|V\|_{L^1}. \end{aligned} \tag{5.4}$$

Similarly, we also have

$$\int_{\mathbb{R}} |\mathcal{R}_z(x,y)B(y)|^2_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x\,\mathrm{d}y \le K_2(z)\|V\|_{L^1}, \tag{5.5}$$

and thus the integral operator $f(\cdot) \mapsto \int_{\mathbb{R}} \mathcal{R}_z(\cdot,y)B(y)f(y)\,\mathrm{d}y$ is Hilbert–Schmidt. But the latter is an extension of the operator $R_0(z)B$, thus $R_0(z)B$ is closable. Furthermore, since the domain $\mathrm{Dom}(R_0(z)B) = \mathrm{Dom}(B)$ is dense in $L^2(\mathbb{R};\mathbb{C}^2)$ and the integral operator above is continuous, $\overline{R_0(z)B}$ coincides with it. This completes the confirmation of (B1).

Now, we address (B2). Once again, we appeal to the Schur test, in order to show that $AR_0(z)B$ has a bounded closure, but now we consider the weight

$$p(x) = \begin{cases} |V(x)|^{1/2}_{\mathbb{C}^{2\times 2}} & \text{if } |V(x)|_{\mathbb{C}^{2\times 2}} > 0, \\ 1 & \text{if } |V(x)|_{\mathbb{C}^{2\times 2}} = 0. \end{cases}$$

According to (5.2), for almost every $x \in \mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} |A(x)\mathcal{R}_z(x,y)B(y)|_{\mathbb{C}^{2\times 2}}\,p(y)\,\mathrm{d}y &\le p(x)\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}\,p(y)^2\,\mathrm{d}y \\ &= p(x)\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}|V(y)|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}y\end{aligned}$$

and for almost every $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} |A(x)\mathcal{R}_z(x,y)B(y)|_{\mathbb{C}^{2\times 2}}\,p(x)\,\mathrm{d}x \le p(y)\int_{\mathbb{R}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}}|V(x)|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x.$$

Since $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$ and $|\mathcal{R}_z(\cdot,\cdot)|_{\mathbb{C}^{2\times 2}} \in L^\infty(\mathbb{R}^2)$ (recall Lemma 3.2), then by the Schur test, the integral operator $f(\cdot) \mapsto \int_{\mathbb{R}} A(\cdot)\mathcal{R}_z(\cdot, y)B(y)f(y)\,\mathrm{d}y$ is bounded. Thus, since $H^1(\mathbb{R}, \mathbb{C}^2) \subset \mathrm{Dom}(A)$, $\mathrm{Dom}(AR_0(z)B) = \mathrm{Dom}(B)$, and since $\mathrm{Dom}(B)$ is dense, continuity implies that $Q(z)$ is equal to this integral operator. Moreover,

$$\|Q(z)\| \leq \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |V(x)|_{\mathbb{C}^{2\times 2}} |\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} \mathrm{d}x. \tag{5.6}$$

This ensures the validity of (B2).

Step 2. Hypothesis (B3) and existence of $\mathcal{L}_{m,V}$. We now need to verify that

(B3) $\{z \in \rho(H_0) : -1 \in \rho(Q(z))\} \neq \varnothing$.

We split the proof into the two sub-cases depending on the hypothesis on V.

First, assume that (H2) is the one condition that holds. According to (5.6), (3.16), and (3.17), for all $z \in \rho(\mathcal{L}_m)$,

$$\begin{aligned}\|Q(z)\| &\leq \|V\|_{L^1} \sup_{(x,y)\in\mathbb{R}^2} |\mathcal{R}_z(x,y)| \\ &\leq \frac{\|V\|_{L^1}}{2}(|k_z|+1)\max\Big\{|w_z^\pm| + \frac{1}{|w_z^\pm|}\Big\}. \end{aligned} \tag{5.7}$$

From (3.2) and (3.15), it follows that $w_{i\delta}^\pm \to -i$ and $k_{i\delta} \to 0$ as $\delta \to \infty$. Thus, the terms inside the maximum become close to 2 as $\delta \to \infty$. Since $\|V\|_{L^1} < 1$, then indeed (B3) follows taking z on the imaginary axis with sufficiently large modulus.

Now, if instead condition (H3) holds for $p \in (1, \infty]$, by Holder's inequality and (5.3), we get

$$\|Q(z)\| \leq \|V(x)\|_{L^p} (K_q(z))^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{5.8}$$

Therefore, an argument similar to the previous sub-case, yields $K_q(z) \to 0$ for z on the imaginary axis with sufficiently large modulus. Hence, (B3) is also valid under assumption (H3).

As a conclusion to this step, we now have that, by virtue of the first part of Theorem A.1, a closed extension $\mathcal{L}_{m,V} \supseteq \mathcal{L}_m + V$ indeed exists and

$$(\mathcal{L}_{m,V} - z)^{-1} = R_0(z) - \overline{R_0(z)B}(I + Q(z))^{-1}AR_0(z)$$

for all z such that $-1 \in \rho(Q(z))$. Note that, under the hypotheses, the latter is non-empty.

Step 3. Conclusions (1)–(4) of the proposition. The crucial point in the proof is to observe that $AR_0(z)$ and $\overline{R_0(z)B}$ are Hilbert–Schmidt operators, therefore the difference

$$(\mathcal{L}_m - z)^{-1} - (\mathcal{L}_{m,V} - z)^{-1} = \overline{R_0(z)B}(I + Q(z))^{-1}AR_0(z)$$

is compact. The stability of the first four essential spectra follows directly from [11, Theorem IX.2.4]. Moreover, when $m > 0$, $\mathbb{C} \setminus \mathrm{Spec}_{e1}(\mathcal{L}_{m,V})$ has one connected component, then $\mathrm{Spec}_{e5}(\mathcal{L}_{m,V})$ in that case is also identical to the other essential spectra, cf. [19, Proposition 5.5.4]. This confirms the conclusions (1) and (2).

The rest of the proof assumes that $m = 0$. In that case, notice that $k_z = 0$ and $|w_z^{\pm}| = 1$. For (3), observe that by direct substitution into (5.7), we have that, if V satisfies (H2), then $\|Q(z)\| \le \|V\|_{L^1} < 1$ for all

$$z \in \rho(\mathcal{L}_m) = \mathbb{C} \setminus \{z \in \mathbb{C} : |\operatorname{Im} z| \le 1\}.$$

Hence, $-1 \in \rho(Q(z))$ for all $z \in \mathbb{C}$ such that $|\operatorname{Im} z| > 1$. That is, any eigenvalue of $\mathcal{L}_{m,V}$ should all be located in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z \le 1\}$ as claimed.

Finally, for (4), if V satisfies (H3) instead, the estimate (5.8), the fact that

$$\min\{\operatorname{Re}\mu_z^+, \operatorname{Re}\mu_z^-\} = |\operatorname{Im} z| - 1 \quad \text{for } |\operatorname{Im} z| > 1,$$

and that $\varphi_z(t) = 1$ for all $t \in \mathbb{R}$ ensure that

$$\|Q(z)\| \le \|V\|_{L^p}\left(\frac{2}{q(|\operatorname{Im} z| - 1)}\right)^{1/q}, \quad q = \frac{p}{p-1},$$

for all $z \in \rho(\mathcal{L}_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| > 1\}$. Hence, for all $z \in \mathbb{C}$ such that $|\operatorname{Im} z| > 1 + 2\|V\|_{L^p}^q/q$, the right-hand side is strictly smaller than 1. This gives (4) for $p < \infty$. For the case $p = \infty$, note that $\mathrm{Spec}(\mathcal{L}_{0,V}) \subset \mathrm{Spec}_{\|V\|_\infty}(\mathcal{L}_0)$ (from the second equality in (1.3)). This completes the proof of Proposition 2.3.

For later purposes, note that the operator $Q(z)$ is also Hilbert–Schmidt. Indeed, the identities (5.2), (3.16), and (3.17) yield

$$\begin{aligned}
&\int_{\mathbb{R}^2} |A(x)\mathcal{R}_z(x,y)B(y)|^2_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le \|V\|_{L^1}^2 \sup_{(x,y)\in\mathbb{R}^2} |\mathcal{R}_z(x,y)|^2_{\mathbb{C}^{2\times 2}} \\
&\le \frac{\|V\|_{L^1}^2}{4}\max\left\{(|k_z|+1)^2\left(|w_z^-| + \frac{1}{|w_z^-|}\right)^2, (|k_z|+1)^2\left(|w_z^+| + \frac{1}{|w_z^+|}\right)^2\right\},
\end{aligned} \tag{5.9}$$

where the right-hand side is finite for each $z \in \rho(\mathcal{L}_m)$.

We close this subsection with a remark that provides the context of our findings about perturbations of $\mathcal{L}_m$ by long-range potentials.

Remark 5.1. In the case of long-range perturbations of the dislocated Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + i\,\mathrm{sgn}(x)$, considered in [14, Section 5.1], the condition $V \in L^1(\mathbb{R})$ alone gives the existence of a closed extension directly, via the associated quadratic forms. The distinction with the present case and the need for the condition (H2) can be explained through the behaviour of the resolvent kernel, as follows.

The estimates around [14, estimate (5.6)] ensure that the L^∞ norm of the kernel of the unperturbed resolvent for the Schrödinger model decays to zero as $z \to \infty$ along the imaginary axis. By contrast, the kernel $\mathcal{R}_z(x, y)$ in the present case does not have a decaying L^∞ norm. Effectively, the right-hand sides in (3.16) and (3.17) do not decay to zero. Note that this is analogous to the contrast between the free Schrödinger kernel and the free particle Dirac kernel. For details, see [7, (13)–(14)].

5.2. Proof of Theorem 2.5

Proposition 2.3, which has been already proven, provides a general statement about exclusion regions (semi-planes) for the eigenvalues of $\mathcal{L}_{0,V}$. We now proceed to give the proof of the first main result of this section, Theorem 2.5. This theorem, in a similar vein, describes regions of exclusion for the eigenvalues of $\mathcal{L}_{m,V}$ in the case $m > 0$. Thus, for the remainder of this subsection, we assume that $m \neq 0$.

Our main arguments involve deriving upper bound for the norm of the *Birman–Schwinger operator*, $Q(z)$, introduced in (5.6). To achieve this, we give sharp estimates on the matrix norm of the kernel in various regions of the complex z-plane. Specifically, we partition the resolvent set into three main disjoint regions as follows:

$$\rho(\mathcal{L}_m) = D \cup W \cup U,$$

where

$$W = \Big\{ z \in \mathbb{C} : |\operatorname{Re}(z)| \geq \frac{5}{2}m,\ |\operatorname{Im} z| < 1 \Big\},$$

$$D = D(m+i) \cup D(-m+i) \cup D(-m-i) \cup D(m-i),$$

for

$$\begin{aligned} D(m \pm i) = \Big\{ z \in \mathbb{C} \setminus \rho(\mathcal{L}_m) : |\operatorname{Re}(z - (m \pm i))| \leq \frac{3}{2}m, \\ |\operatorname{Im}(z - (m \pm i))| \leq \frac{3}{2} \Big\}, \\ D(-m \pm i) = \Big\{ z \in \mathbb{C} \setminus \rho(\mathcal{L}_m) : |\operatorname{Re}(z - (-m \pm i))| \leq \frac{3}{2}m, \\ |\operatorname{Im}(z - (-m \pm i))| \leq \frac{3}{2} \Big\}, \end{aligned}$$

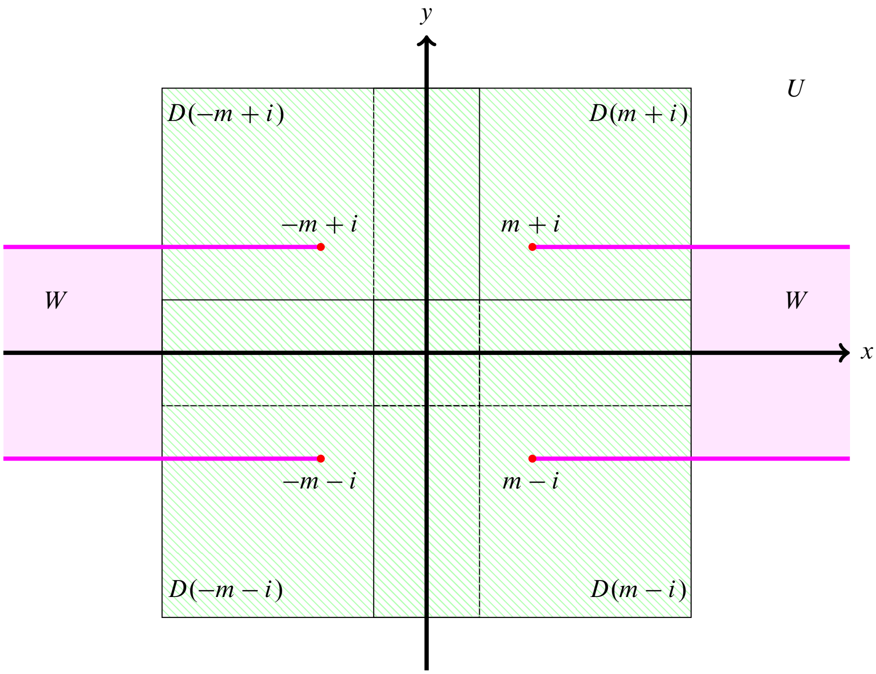

Figure 5. Partition of the resolvent set of $\mathcal{L}_m$ in subdomains for the proof of Theorem 2.5.

and

$$U = \rho(\mathcal{L}_m) \setminus (D \cup W).$$

See Figure 5.

We split the proof of Theorem 2.5 according to this partition, deriving suitable estimates for the norm of $Q(z)$ where z is in each of the sub-regions. In this proof, the parameters $c_j \equiv c_j(m) > 0$ only depend on m.

Estimates on *D*. The region $D \subset \mathbb{C}$ is the union of four neighbourhoods of singularities of the matrix kernel $\mathcal{R}_z$. We will see that these singularities are removable. That is, the matrix norm $|\mathcal{R}_z(x, y)|_{\mathbb{C}^{2\times 2}}$ is bounded uniformly for all $(x, y) \in \mathbb{R}^2$, whenever $z \in D(\pm m \pm i)$. We treat each different component separately.

Consider first that $z \in D(m + i)$. From the formulas of μ_z^+, $w_z^\pm$ in (3.2), and k_z in (3.15), it follows that

$$\begin{aligned} &\lim_{z\to m+i} w_z^+ = 0, \qquad \lim_{z\to m+i} w_z^- = \frac{\sqrt{-1-im}}{\sqrt{m^2+1}}, \\ &\lim_{z\to m+i} \frac{\mu_z^+}{w_z^+} = 2m, \qquad \lim_{z\to m+i} k_z = -1. \end{aligned} \tag{5.10}$$

Then, we gather the following. According to (3.16), for all $(x,y) \in \mathbb{R}^2$ such that $xy \le 0$,

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} \le \varphi_z(0) = \frac{\sqrt{(1+|w_z^+|^2)(1+|w_z^-|^2)}}{|w_z^+ + w_z^-|} \le c_1. \tag{5.11}$$

According to (3.17), for all $(x,y) \in \mathbb{R}^2$ such that $x \le 0$, $y \le 0$,

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} \le \sup_{t\le 0} \varphi_z(t) \le \frac{1}{2}(|k_z|+1)\Big(|w_z^-| + \frac{1}{|w_z^-|}\Big) \le c_2. \tag{5.12}$$

According to (3.16), for all $(x,y) \in \mathbb{R}^2$ such that $x \ge 0$, $y \ge 0$,

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} \le \varphi_z(\min\{x,y\}).$$

Now, for all $t \ge 0$, we know that

$$\varphi_z(t) = \frac{\sqrt{1+|w_z^+|^2}}{2}\Big(\frac{1}{|w_z^+|^2}|k_z e^{-2\mu_z^+ t}+1|^2 + |k_z e^{-2\mu_z^+ t}-1|^2\Big)^{1/2}.$$

Because of (5.10), the term that we should take care of is $|k_z e^{-2\mu_z^+ t}+1|/|w_z^+|$, as it might potentially exhibit a singularity. We show that this is not the case. Indeed, from (3.15) and (5.10), for $z \in D(m+i)$ and $x \ge 0$, $y \ge 0$, we have

$$\begin{aligned}\frac{1}{|w_z^+|}|k_z e^{-2\mu_z^+ t}+1| &= \frac{1}{|w_z^+|}|k_z(e^{-2\mu_z^+ t}-1)+1+k_z| \\ &= \frac{1}{|w_z^+|}\Big|k_z(e^{-2\mu_z^+ t}-1)+\frac{2w_z^+}{w_z^+ + w_z^-}\Big| \\ &\le \frac{|\mu_z^+|}{|w_z^+|}|k_z||t| + \frac{2}{|w_z^+ + w_z^-|} \\ &\le c_3(|t|+1).\end{aligned}$$

Here we have used the inequality $|e^{-z}-1| \le |z|$ for $\operatorname{Re} z \ge 0$. Thus, for all $z \in D(m+i)$ and $x \ge 0$, $y \ge 0$, we have

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} \le \varphi_z(\min\{x,y\}) \le c_4(1+|\min\{x,y\}|) \le c_5 \nu_1'(x). \tag{5.13}$$

Finally, by virtue of (5.6), combining the above estimates we obtain

$$\|Q(z)\| \le c_6 \int_{\mathbb{R}} |V(x)|_{\mathbb{C}^{2\times 2}} \nu_1'(x)\,\mathrm{d}x \tag{5.14}$$

for all $z \in D(m+i)$.

Now, let $z \in D(m-i)$. We have

$$\lim_{z\to m-i} w_z^- = 0, \qquad \lim_{z\to m-i} w_z^+ = \frac{\sqrt{-1+im}}{\sqrt{m^2+1}},$$
$$\lim_{z\to m-i} \frac{\mu_z^-}{w_z^-} = 2m, \qquad \lim_{z\to m-i} k_z = 1.$$

By means of an identical proof to the one given in the previous region, we can show that if $xy \le 0$ then (5.11) holds true and if $x \ge 0$, $y \ge 0$ then (5.12) holds true. We omit the details of that. Moreover, in the case $x \le 0$ and $y \le 0$, the term $|k_z e^{2\mu_z^- t} - 1|/|w_z^-|$ is bounded by $c_7(|t|+1)$. Thus, we gather that

$$\begin{aligned}|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} &\le \varphi_z(\max\{x,y\}) \\ &\le c_8(1+|\max\{x,y\}|) \\ &\le c_9(1+\min\{|x|,|y|\}) \le c_{10}\nu_1'(x).\end{aligned}$$

Hence, by virtue of (5.6), we also obtain the estimate (5.14) for all $z \in D(m-i)$, perhaps with a larger constant c_6.

Now, let $z \in D(-m+i)$. Then,

$$\lim_{z\to -m+i} w_z^+ = \infty, \qquad \lim_{z\to -m+i} w_z^- = \sqrt{-1-im},$$
$$\lim_{z\to -m+i} \mu_z^+ w_z^+ = 2m, \qquad \lim_{z\to -m+i} k_z = 1.$$

We obtain the same estimates as (5.11) for all $(x,y) \in \mathbb{R}^2$ satisfying $xy \le 0$ and (5.12) for all $(x,y) \in \mathbb{R}^2$ satisfying $x \le 0$, $y \le 0$, by writing

$$\frac{\sqrt{(1+|w_z^+|^2)(1+|w_z^-|^2)}}{|w_z^+ + w_z^-|} = \frac{\sqrt{\big(\frac{1}{|w_z^+|^2}+1\big)(1+|w_z^-|^2)}}{\big|1+\frac{w_z^-}{w_z^+}\big|}.$$

On the other hand, whenever $x \ge 0$, $y \ge 0$, we invoke the same estimate

$$|\mathcal{R}_z(x,y)|_{\mathbb{C}^{2\times 2}} \le \varphi_z(\min\{x,y\})$$

as valid for $z \in D(m+i)$, in which we rewrite

$$\varphi_z(t) = \frac{1}{2}\Big(\frac{1+|w_z^+|^2}{|w_z^+|^2}|k_z e^{-2\mu_z^+ t}+1|^2 + (1+|w_z^+|^2)|k_z e^{-2\mu_z^+ t}-1|^2\Big)^{1/2}.$$

It is readily seen that, for all $t \ge 0$ and $z \in D(-m+i)$,

$$\frac{1+|w_z^+|^2}{|w_z^+|^2}|k_z e^{-2\mu_z^+ t}+1|^2 + |k_z e^{-2\mu_z^+ t}-1|^2 \le c_{11}.$$

For the remaining term $|w_z^+|^2|k_z e^{-2\mu_z^+ t} - 1|^2$, note that

$$\begin{aligned} |w_z^+||k_z e^{-2\mu_z^+ t} - 1| &= |w_z^+||k_z(e^{-2\mu_z^+ t} - 1) + k_z - 1| \\ &= |w_z^+|\Big|k_z(e^{-2\mu_z^+ t} - 1) - \frac{2w_z^-}{w_z^+ + w_z^-}\Big| \\ &\le |\mu_z^+ w_z^+||k_z||t| + \frac{2|w_z^-|}{\big|1 + \frac{w_z^-}{w_z^+}\big|} \\ &\le c_{12}(|t| + 1). \end{aligned}$$

Thus, the estimate (5.13) is valid for all $(x, y) \in \mathbb{R}^2$ such that $x \ge 0$, $y \ge 0$, also whenever $z \in D(-m + i)$. Hence, combining the estimates, once again we gather that (5.14) also holds true for all $z \in D(-m + i)$.

Finally, the case $z \in D(-m - i)$ is similar to the case $z \in D(-m + i)$, so we omit further details. Hence, we then gather that the upper bound

$$\|Q(z)\| \le c_{13}(m)\|V\|_{L^1(\mathrm{d}\nu_1)} \tag{5.15}$$

is valid for all $z \in D$.

Estimate on $U = \mathbb{C} \setminus \overline{D \cup W}$. Split $\bar{U} = U^+ \cup U^-$ for

$$U^{\pm} = \{z \in \bar{U} : \pm \operatorname{Im} z > 0\}.$$

Then, according to (3.2) and (3.15),

$$\lim_{\substack{z\to\infty \\ z\in U^+}} w_z^{\pm} = -i, \qquad \lim_{\substack{z\to\infty \\ z\in U^-}} w_z^{\pm} = i, \qquad \lim_{\substack{z\to\infty \\ z\in \bar{U}}} k_z = 0.$$

Hence, by continuity of each of the quantities $|w_z^{\pm}|$, $|w_z^{\pm}|^{-1}$, and $|k_z|$, in z, we have that

$$(|k_z| + 1)\max\Big\{|w_z^{\pm}| + \frac{1}{|w_z^{\pm}|}\Big\} \le c_{14}(m) \tag{5.16}$$

for all $z \in U$. According to (5.6), (3.16), and (3.17), we then gather that

$$\|Q(z)\| \le c_{14}(m)\|V\|_{L^1} \le c_{14}(m)\|V\|_{L^1(\mathrm{d}\nu_1)} \tag{5.17}$$

for all $z \in U$.

Estimate on W. By invoking Lemma 4.2, we obtain the following bounds for all $z \in W$:

$$\frac{\sqrt{(1 + |w_z^+|^2)(1 + |w_z^-|^2)}}{|w_z^+ + w_z^-|} \le c_{15}(\operatorname{Re} z)^2, \quad |w_z^{\pm}| \le c_{16},$$
$$|w_z^{\pm}|^{-1} \le c_{17}, \qquad |k_z| \le c_{18}(\operatorname{Re} z)^2.$$

Then, by virtue of (5.6), (3.16), and (3.17), it follows that

$$\|Q(z)\| \le c_{19}(m)(\operatorname{Re} z)^2 \|V\|_{L^1} \tag{5.18}$$

for all $z \in W$.

Completion of the proof of Theorem 2.5. According to the estimates (5.15) on D, (5.17) on U and (5.18) on W, we conclude that there exists a constant $C(m) > 0$ such that

$$\|Q(z)\| \le \frac{1}{C(m)} \|V\|_{L^1(\mathrm{d}\nu_1)} \qquad \text{for all } z \in \rho(\mathcal{L}_m) \setminus W, \tag{5.19}$$

and

$$\|Q(z)\| \le \frac{1}{C(m)^2} (\operatorname{Re} z)^2 \|V\|_{L^1} \quad \text{for all } z \in W. \tag{5.20}$$

Hence, if we assume $\|V\|_{L^1(\mathrm{d}\nu_1)} < C(m)$, it follows that $\|Q(z)\| < 1$ for all $z \in \rho(\mathcal{L}_m) \setminus W$. Hence, -1 is not in the spectrum of $Q(z)$ whenever $z \in \rho(\mathcal{L}_m) \setminus W$. Theorem A.2, thus implies that all points in $\rho(\mathcal{L}_m) \setminus W$ are not eigenvalues of $\mathcal{L}_{m,V}$. Moreover, the points $z \in W$ satisfying $|\operatorname{Re} z| < C(m)\sqrt{1/\|V\|_{L^1}}$ cannot be eigenvalues of $\mathcal{L}_{m,V}$. Therefore, we deduce that the eigenvalues of $\mathcal{L}_{m,V}$ (if there is any), should lie in the subset of W, corresponding to real part larger than or equal to $C(m)\sqrt{1/\|V\|_{L^1}}$. This completes the proof of Theorem 2.5.

5.3. Step potentials

We construct a potential $V \in L^1$ such that $\mathcal{L}_{m,V}$ has infinitely many eigenvalues inside the instability band. We compute explicitly the eigenfunctions by means of an argument similar to the one employed in the proof of Proposition 3.1. We end the subsection by giving the proof of Proposition 2.6.

Set

$$V_{a,b}(x) = (-i\operatorname{sgn}(x) - b)\chi_{[-a,a]}(x) I,$$

for $a > 0$ and $b \in \mathbb{R}$. If $u \neq 0$ is such that $(\mathcal{L}_{m,V_{a,b}} - z)u = 0$, then

$$u(x) = \begin{cases} u_-(x) & \text{for } x \in (-\infty, -a), \\ u_0(x) & \text{for } x \in [-a, a], \\ u_+(x) & \text{for } x \in (a, +\infty), \end{cases}$$

where $u_\dagger$ for $\dagger \in \{-, 0, +\}$, are given in a similar manner to (3.10), by

$$u_\dagger(x) = (e^{\mu_z^\dagger x} S_z^\dagger + e^{-\mu_z^\dagger x} T_z^\dagger) \begin{pmatrix} \alpha_\dagger \\ \beta_\dagger \end{pmatrix}. \tag{5.21}$$

Here, $\mu_z^\pm$ and $w_z^\pm$ are the same parameters from (3.2) that we considered above,

$$\mu_z^0 = \sqrt{(m+b+z)(m-b-z)} \quad \text{and} \quad w_z^0 = \frac{\sqrt{m-b-z}}{\sqrt{m+b+z}}.$$

Also

$$S_z^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1/w_z^\dagger \\ w_z^\dagger & 1 \end{pmatrix} \quad \text{and} \quad T_z^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1/w_z^\dagger \\ -w_z^\dagger & 1 \end{pmatrix}.$$

Note that there is a singularity of $u_0(x)$ for $z = \pm m - b$, but taking the limits $z \to \pm m - b$ in (5.21) for $\dagger = 0$, we get

$$u_0(x) = \begin{cases} \begin{pmatrix} 1 & 2mx \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} & \text{if } z = m - b, \\ \begin{pmatrix} 1 & 0 \\ 2mx & 1 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} & \text{if } z = -m - b. \end{cases}$$

We seek for $z \in \mathbb{C}$, $\alpha_\dagger$ and $\beta_\dagger$ (non-vanishing simultaneously), such that we have $u \in H^1(\mathbb{R}, \mathbb{C}^2)$, and so u becomes a proper eigenfunction.

Since u has to decay at $\pm\infty$ and must be continuous at $\pm a$, then

$$\begin{pmatrix} -w_z^- & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_z^+ & 1 \\ e^{-\mu_z^- a} & e^{-\mu_z^- a}/w_z^- & -(e^{\mu_z^0 a} + e^{-\mu_z^0 a}) & (e^{\mu_z^0 a} - e^{-\mu_z^0 a})/w_z^0 & 0 & 0 \\ w_z^- e^{-\mu_z^- a} & e^{-\mu_z^- a} & w_z^0(e^{\mu_z^0 a} - e^{-\mu_z^0 a}) & -(e^{\mu_z^0 a} + e^{-\mu_z^0 a}) & 0 & 0 \\ 0 & 0 & (e^{\mu_z^0 a} + e^{-\mu_z^0 a}) & (e^{\mu_z^0 a} - e^{-\mu_z^0 a})/w_z^0 & -e^{-\mu_z^+ a} & e^{-\mu_z^+ a}/w_z^+ \\ 0 & 0 & w_z^0(e^{\mu_z^0 a} - e^{-\mu_z^0 a}) & (e^{\mu_z^0 a} + e^{-\mu_z^0 a}) & w_z^+ e^{-\mu_z^+ a} & -e^{-\mu_z^+ a} \end{pmatrix}\begin{pmatrix} \alpha_- \\ \beta_- \\ \alpha_0 \\ \beta_0 \\ \alpha_+ \\ \beta_+ \end{pmatrix} = 0.$$

The determinant of the matrix vanishes if and only if

$$e^{4a\mu_z^0}(w_z^0 + w_z^+)(w_z^0 + w_z^-) = (w_z^0 - w_z^+)(w_z^0 - w_z^-) \tag{5.22}$$

for eigenvalues $z \neq \pm m - b$ such that

$$z \notin \{z \in \mathbb{C} : |\operatorname{Re} z| \geq m, |\operatorname{Im} z| = 1\}.$$

And it vanishes for $z = \pm m - b$ if and only if

$$w^+_{m-b} + w^-_{m-b} + 4am\, w^+_{m-b} w^-_{m-b} = 0 \quad \text{when } z = m - b$$

and

$$w^+_{-m-b} + w^-_{-m-b} + 4am = 0 \qquad \text{when } z = -m - b.$$

Note that, no points in $\{z \in \mathbb{C} : |\operatorname{Re} z| \geq m, |\operatorname{Im} z| = 1\}$ can be an eigenvalue.

Proof of Proposition 2.6. Let $m = 0$. Directly from the definition, it follows that w^0_z takes one of two values, $\pm i$, depending on the location of $z \in \mathbb{C} \setminus \{\pm m - b\}$. Moreover, substitution shows that $|\operatorname{Im} z| < 1$, if and only if $w^+_z = i$ and $w^-_z = -i$. Also, we directly see that $z = m - b$ or $z = -m - b$ are eigenvalues if $|\operatorname{Im} b| < 1$. This gives the first conclusion of Proposition 2.6.

Let $m > 0$. We show the existence of infinitely many large real eigenvalues. Since, $(w^0_z + w^+_z)(w^0_z + w^-_z) \neq 0$ whenever $z \in \mathbb{R}$, then (5.22) reduces to

$$e^{i4a\sqrt{(z+b)^2 - m^2}} = \frac{(w^0_z - w^+_z)(w^0_z - w^-_z)}{(w^0_z + w^+_z)(w^0_z + w^-_z)}.$$

Now, $\cot(w) = i(e^{i2w} - 1)/(e^{i2w} + 1)$ for $w \in \mathbb{R}$ and note that $w^+_z + w^-_z \neq 0$ for all $z \in \mathbb{C}$. Thus, we can rewrite this equation as

$$\cot(2a\sqrt{(z+b)^2 - m^2}) = -i\,\frac{(w^0_z)^2 + w^+_z w^-_z}{w^0_z(w^+_z + w^-_z)}.$$

The right-hand side of this expression is real for real z with sufficiently large modulus. Indeed,

$$-i\,\frac{(w^0_z)^2 + w^+_z w^-_z}{w^0_z(w^+_z + w^-_z)} = \begin{cases} S(z) & \text{if } z > m - b, \\ -S(z) & \text{if } z < -m - b, \end{cases}$$

where

$$S(z) = \sqrt{\frac{z + m + b}{z - m + b}}\,\frac{(z - m + b)\sqrt{(z+m)^2 + 1} - (z + m + b)\sqrt{(z-m)^2 + 1}}{2(z + m + b)\operatorname{Re}(\sqrt{z + m + i}\,\sqrt{m + i - z})}.$$

Since,

$$\lim_{z \to +\infty} S(z) = \lim_{z \to -\infty} S(z) = b,$$

by periodicity, we have infinitely many positive and negative solutions. This is the second statement in Proposition 2.6, and completes its proof. ∎

5.4. The weakly coupled model

At the end of this subsection we give the proof of Theorem 2.7, about localisation of the discrete spectrum for a potential of the form ϵV in the asymptotic regime $|\epsilon| \to 0$, where $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}; \mathrm{d}\nu_1)$ satisfies additional hypotheses.

Recall (2.7),

$$\mathrm{Spec}_{\mathrm{dis}}(\mathcal{L}_{m,\epsilon V}) \subseteq \Big\{ z \in \mathbb{C} : |\operatorname{Im} z| < 1,\ |\operatorname{Re} z| > \frac{C(m)}{|\epsilon|^{1/2} \|V\|_{L^1}^{1/2}} \Big\}.$$

Hence, for small enough $|\epsilon|$, the eigenvalues of $\mathcal{L}_{m,\epsilon V}$ lie in the instability band

$$\Sigma = \{ z \in \mathbb{C} : |\operatorname{Im} z| < 1 \},$$

and they escape to infinity in the regime $|\epsilon| \to 0$ at a rate proportional to $|\epsilon|^{-1/2}$ or faster. We now formulate a more precise statement about this, for V satisfying a faster decay rate at infinity. The next result is a precursor of Theorem 2.7 and it is one of the main contributions of this paper. We recall that the expression of the densities ν_k' are given in Section 2 in the paragraph above Theorem 2.7.

Theorem 5.2. *Let $m > 0$ and let $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}; \mathrm{d}\nu_2)$. Let*

$$\Upsilon_z = \frac{1}{2(4|z|^2 + m^2)} \begin{pmatrix} -(2z+m)^2 & i(4z^2 - m^2) \\ i(4z^2 - m^2) & (2z - m)^2 \end{pmatrix} \in \mathbb{C} \tag{5.23}$$

for all $z \in \mathbb{C}$. Then,

$$\mathrm{Spec}_{\mathrm{dis}}(\mathcal{L}_{m,\epsilon V}) \subset \Big\{ z \in \mathbb{C} : |\operatorname{Im} z| < 1,\ \frac{4m}{4|z|^2 + m^2} = W_z(\epsilon) \Big\}, \tag{5.24}$$

where

$$W_z(\epsilon) = \epsilon \int_{\mathbb{R}} e^{-2(|x| + i(z - m^2/(2\operatorname{Re} z))x)} \langle V(x), \overline{\Upsilon_z} \rangle_{\mathrm{F}} \, \mathrm{d}x + \mathcal{O}(\epsilon^2)$$

in the regime $|\epsilon| \to 0$.

The verification of the validity of this statement occupies most of this subsection. The explicit dependence on z of $W_z(\epsilon)$ and the constant in the limit, will be established below. The proof of (5.24) will invoke Theorem A.2 once again. Note that the Birman–Schwinger operator associated to ϵV is $\epsilon Q(z)$. We test whether $z \in \Sigma$ does not belong to $\mathrm{Spec}_{\mathrm{p}}(\mathcal{L}_{m,\epsilon V})$, by testing whether $-1 \notin \mathrm{Spec}_{\mathrm{p}}(\epsilon Q(z))$. However, for this we will not appeal directly to estimates for the norm $\|\epsilon Q(z)\|$ as we did in Section 5.2. Instead, we will find a criterion, involving $W_z(\epsilon)$, to test whether the operator $\epsilon Q(z) + 1$ is bijective. Since $Q(z)$ is compact (see the paragraph above Remark 5.1), this is a test on whether $-1 \notin \mathrm{Spec}_{\mathrm{p}}(\epsilon Q(z))$.

The position of the discrete spectrum is associated with sub-regions of Σ where the norm of the resolvent is singular. In Section 4.2, we isolated the singular part of the unperturbed kernel through the term $\mathcal{R}_{1,z}$. Since the latter still carries the leading order information about the singularities of the perturbed kernel (under small enough perturbations), we now consider a decomposition of the kernel of the integral operator associated to $Q(z)$ that takes the singular contribution of $\mathcal{R}_{1,z}$ into account. For this, we proceed as follows.

Let

$$Q(z) = L(z) + M(z),$$

where

$$L(z)f(x) = \int_{\mathbb{R}} A(x)\mathcal{L}_z(x,y)B(y)f(y)\,\mathrm{d}y, \quad \mathcal{L}_z(x,y) = e^{-\eta_z(x)-\eta_z(y)}U_z, \tag{5.25a}$$

$$\eta_z(w) = i\Big(z - \frac{m^2}{2\operatorname{Re} z}\Big)w + |w|, \tag{5.25b}$$

$$U_z = \frac{1}{8m}\begin{pmatrix} -(2z+m)^2 & i(4z^2-m^2) \\ i(4z^2-m^2) & (2z-m)^2 \end{pmatrix}. \tag{5.25c}$$

Then, as we shall see below, $L(z)$ is a rank-one operator for all $z \in \Sigma$ and it carries the leading order contribution of the norm of $Q(z)$ in the asymptotic regime $|\operatorname{Re} z| \to \infty$. Notably, the kernel $\mathcal{L}_z$ encodes the contribution of the singular part of $\mathcal{R}_{1,z}$ in this regime, while $\|M(z)\|$ is uniformly bounded with respect to z. Before proving all this, we motivate this decomposition and determine the asymptotic coefficients of the different terms involved in the argument. Recall the notation $z = \tau + i\delta$ for $\tau, \delta \in \mathbb{R}$ with $|\delta| < 1$.

On the one hand, we note that the expression of the powers $\eta_z(w)$, is motivated by the fact that

$$-\eta_z(w) = \begin{cases} \eta_z^- w & \text{if } w \le 0, \\ -\eta_z^+ w & \text{if } w \ge 0, \end{cases}$$

$$\eta_z^- = (1+\delta) - i\Big(\tau - \frac{m^2}{2\tau}\Big),$$

$$\eta_z^+ = (1-\delta) + i\Big(\tau - \frac{m^2}{2\tau}\Big),$$

where $\eta_z^\pm$ are the leading order coefficients in the expansion of the terms $\mu_z^\pm$:

$$\mu_z^\pm = \eta_z^\pm + \frac{m^2(1 \mp \delta)}{2\tau^2} + \mathcal{O}\Big(\frac{1}{|\tau|^3}\Big) \tag{5.26}$$

as $|\tau| \to +\infty$. See Lemma 4.2 and Table 1.

On the other hand, the expression of the matrix U_z, has a more involved motivation which is given after the following lemma.

Lemma 5.3. *Let the matrices appearing in the expression of the kernel $\mathcal{R}_{1,z}$ in (4.5) be $N_{j,z}$, as defined at the beginning of Section 3.1. Let*

$$U^0_{\tau+i\delta} = \begin{pmatrix} -\frac{1}{2m}\tau^2 - \left(\frac{1}{2} + \frac{i\delta}{m}\right)\tau & \frac{i}{2m}\tau^2 - \frac{\delta}{m}\tau \\ \frac{i}{2m}\tau^2 - \frac{\delta}{m}\tau & \frac{1}{2m}\tau^2 - \left(\frac{1}{2} - \frac{i\delta}{m}\right)\tau \end{pmatrix}.$$

Then, $N_{j,z} = U^0_z + \widehat{N}_{j,z}$ where

$$|\widehat{N}_{j,\tau+i\delta}|_{\mathrm{F}} = \mathcal{O}(1)$$

as $|\tau| \to \infty$ for all $j \in \{1,3,4,7,8,10\}$.

Proof. By writing

$$\frac{1}{w_z^+ + w_z^-} = \frac{-i}{4m}[(m+i+z)\mu_z^+ - (m-i+z)\mu_z^-],$$

we get that

$$\frac{1}{w_z^+ + w_z^-} = \frac{1}{2m}\tau^2 + \left(\frac{1}{2} + \frac{i\delta}{m}\right)\tau + \left(\frac{1-\delta^2}{2m} - \frac{m}{4} + \frac{i\delta}{2}\right) + \mathcal{O}\left(\frac{1}{\tau}\right)$$

as $|\tau| \to 0$. The claimed asymptotic formula is achieved by substituting the expansions of $w_z^\pm$, $1/w_z^\pm$, and this expression, into the formulas of the different $N_{j,z}$. See Lemma 4.2. ■

Now, we return to the motivation for the expression of U_z. We pick the latter, so that it approximates all the matrices $N_{j,z}$ up to order τ. Indeed, writing U_z in terms of τ and δ gives

$$U_{\tau+i\delta} = U^0_{\tau+i\delta} + U^1_\delta, \tag{5.27}$$

for U^0_z as in Lemma 5.3 and

$$U^1_\delta = \begin{pmatrix} \frac{\delta^2}{2m} - \frac{m}{8} - \frac{i\delta}{2} & -i\left(\frac{\delta^2}{2m} + \frac{m}{8}\right) \\ -i\left(\frac{\delta^2}{2m} + \frac{m}{8}\right) & -\frac{\delta^2}{2m} + \frac{m}{8} - \frac{i\delta}{2} \end{pmatrix}.$$

Note that for all $z \in \Sigma$, $\det U_z = 0$ and U_z is of rank one. Then, the matrix norm of U_z can be computed explicitly,

$$|U_z|_{\mathbb{C}^{2\times 2}} = |U_z|_{\mathrm{F}} = \frac{|z|^2}{m} + \frac{m}{4}.$$

We are now ready to formulate the crucial step in the proof of Theorem 5.2.

Lemma 5.4. *Let $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}, \mathrm{d}\nu_2)$. For $A(x)$ and $B(y)$ as in (5.1), let $L(z)$ be the integral operator (5.25) and let $M(z) = Q(z) - L(z)$. Let $\Upsilon_z = U_z/|U_z|_{\mathrm{F}}$ be as in (5.23). Then,*

$$L(z)f = \left(\frac{|z|^2}{m} + \frac{m}{4}\right)\langle f, \overline{\psi_z}\rangle\, \phi_z,$$

where

$$\psi_z(w) = \begin{cases} e^{-\eta_z(w)} B(w)^T \begin{pmatrix} (\Upsilon_z)_{11} \\ (\Upsilon_z)_{12} \end{pmatrix} & \text{if } z \neq \dfrac{-m}{2}, \\ e^{-\eta_z(w)} B(w)^T \begin{pmatrix} (\Upsilon_z)_{21} \\ (\Upsilon_z)_{22} \end{pmatrix} & \text{if } z = \dfrac{-m}{2}, \end{cases}$$

$$\phi_z(w) = \begin{cases} e^{-\eta_z(w)} A(w) \begin{pmatrix} 1 \\ (\Upsilon_z)_{21}/(\Upsilon_z)_{11} \end{pmatrix} & \text{if } z \neq \dfrac{-m}{2}, \\ e^{-\eta_z(w)} A(w) \begin{pmatrix} (\Upsilon_z)_{12}/(\Upsilon_z)_{22} \\ 1 \end{pmatrix} & \text{if } z = \dfrac{-m}{2}. \end{cases}$$

Moreover, there exists a constant $C(m) > 0$ such that $\|M_z\| \le C(m)$ for all $z \in \Sigma$.

Proof. Assume first that $z \neq -m/2$. Since U_z is singular and non-zero, then

$$U_z = \begin{pmatrix} 1 \\ (U_z)_{21}/(U_z)_{11} \end{pmatrix} \begin{pmatrix} (U_z)_{11} & (U_z)_{12} \end{pmatrix}.$$

Thus, we can rewrite the action of the operator L_z in the form

$$\begin{aligned} L_z f(x) &= |U_z|_{\mathrm{F}} \int_{\mathbb{R}} e^{-\eta_z(x)-\eta_z(y)} A(x) \begin{pmatrix} 1 \\ \frac{(\Upsilon_z)_{21}}{(\Upsilon_z)_{11}} \end{pmatrix} \begin{pmatrix} (\Upsilon_z)_{11} & (\Upsilon_z)_{12} \end{pmatrix} B(y) f(y)\,\mathrm{d}y \\ &= |U_z|_{\mathrm{F}} \left(\int_{\mathbb{R}} e^{-\eta_z(y)} \begin{pmatrix} (\Upsilon_z)_{11} & (\Upsilon_z)_{12} \end{pmatrix} B(y) f(y)\,\mathrm{d}y \right) \phi_z(x) \\ &= |U_z|_{\mathrm{F}} \left(\int_{\mathbb{R}} \langle f(y), \overline{\psi_z(y)} \rangle_{\mathbb{C}^2}\,\mathrm{d}y \right) \phi_z(x). \end{aligned}$$

This is the claimed expression for the operator L_z.

Both vector-valued functions ψ_z and ϕ_z belong to $L^2(\mathbb{R}, \mathbb{C}^2)$ because $\operatorname{Re} \eta_z^{\pm} = 1 \mp \delta > 0$ and thus, for all $z \in \Sigma \setminus \{-m/2\}$,

$$\|\psi_z\| \le \|V\|_{L^1}^{1/2} \sqrt{(\Upsilon_z)_{11}^2 + (\Upsilon_z)_{12}^2}, \quad \|\phi_z\| \le \|V\|_{L^1}^{1/2} \sqrt{1 + \frac{(\Upsilon_z)_{21}^2}{(\Upsilon_z)_{11}^2}}. \tag{5.28}$$

Therefore, we can conclude that, for all $z \in \Sigma \setminus \{-m/2\}$, L_z is an operator of rank-one and

$$\|L_z\| = |U_z|_{\mathrm{F}} \|\psi_z\| \|\phi_z\| \le |U_z|_{\mathrm{F}} \|V\|_{L^1}. \tag{5.29}$$

Now, to achieve the same conclusion for $z = -m/2$, we can just write the matrix U_z in different form, as

$$U_z = \begin{pmatrix} (U_z)_{12}/(U_z)_{22} \\ 1 \end{pmatrix} \begin{pmatrix} (U_z)_{21} & (U_z)_{22} \end{pmatrix},$$

and follow the same argument as above.

Next, let us complete the proof of the lemma by showing the validity of the claim made about M_z. Let

$$\mathcal{M}_z(x,y) = \mathcal{R}_z(x,y) - \mathcal{L}_z(x,y).$$

Then,

$$\mathcal{M}_z(x,y) = (\mathcal{R}_{1,z}(x,y) - \mathcal{L}_z(x,y)) + \mathcal{R}_{2,z}(x,y). \tag{5.30}$$

By virtue of (4.6), (4.7), and Lemma 4.2, it follows that there exists a constant $c_1 > 0$ such that

$$|\mathcal{R}_{2,z}|_{\mathbb{C}^{2\times 2}} \le c_1, \tag{5.31}$$

for all $|\tau| \ge 1$.

Let us now show that the difference of $\mathcal{R}_{1,z} - \mathcal{L}_z$ is also uniformly bounded. Let

$$\mu_z(x) = \begin{cases} \mu_z^- x & \text{if } x \le 0, \\ -\mu_z^+ x & \text{if } x \ge 0. \end{cases}$$

Then

$$\mathcal{R}_{1,z}(x,y) = N_{j,z} e^{\mu_z(x)+\mu_z(y)}$$

for j chosen depending on the signs of x and y as in (4.5). For all $(x,y) \in \mathbb{R}^2$, write

$$\Lambda_z(x,y) = \mu_z(x) + \eta_z(x) + \mu_z(y) + \eta_z(y).$$

Then,

$$\mathcal{R}_{1,z}(x,y) - \mathcal{L}_z(x,y) = U_z(e^{\Lambda_z(x,y)} - 1)e^{-\eta_z(x)-\eta_z(y)} + (N_{j,z} - U_z)e^{\mu_z(x)+\mu_z(y)}.$$

Since $\operatorname{Re}\Lambda_z(x,y) < 0$ for all $(x,y) \in \mathbb{R}^2$ and by virtue of (5.26), considering $|\tau|$ sufficiently large so that $\operatorname{Re}(\mu_z^+ + \eta_z^+) > 0$ and $\operatorname{Re}(\mu_z^- + \eta_z^-) > 0$, gives

$$|e^{\Lambda_z(x,y)} - 1| \le |\Lambda_z(x,y)| \le \frac{c_2}{\tau^2}(|x| + |y|).$$

Therefore, since $\operatorname{Re}\mu_z(x) < 0$ and $\operatorname{Re}\eta_z(x) > 0$ for all $x \in \mathbb{R}$, according to Lemma 5.3 and to (5.27), we have

$$\begin{aligned} |\mathcal{R}_{1,z}(x,y) - \mathcal{L}_z(x,y)|_{\mathbb{C}^{2\times 2}} &\le |U_z|_{\mathbb{C}^{2\times 2}}|e^{\Lambda_z(x,y)} - 1| + |N_{j,z} - U_z|_{\mathbb{C}^{2\times 2}} \\ &\le c_3(|x| + |y| + 1), \end{aligned} \tag{5.32}$$

for all $|\tau| \ge \tau_0$.

From (5.31) and (5.32), it follows that, by taking the Hilbert–Schmidt norm in the decomposition (5.30),

$$\begin{aligned}\|M(z)\|^2 &\le c_4 \int_{\mathbb{R}^2} |V(x)|_{\mathbb{C}^{2\times 2}} (1+|x|+|y|)^2 |V(y)|_{\mathbb{C}^{2\times 2}} \mathrm{d}x\, \mathrm{d}y \\ &\le c_5 \Big(\int_{\mathbb{R}} |V(x)|_{\mathbb{C}^{2\times 2}} \nu_2'(x)\, \mathrm{d}x \Big)\Big(\int_{\mathbb{R}} |V(y)|_{\mathbb{C}^{2\times 2}} \nu_2'(y)\, \mathrm{d}y \Big),\end{aligned}$$

where the right-hand side is finite and independent of z, for all $|\tau| \ge \tau_0$. On the other hand, according to (5.29) for $L(z)$, and (5.19)–(5.20) for $Q(z)$, the triangle inequality yields,

$$\|M(z)\| \le \|Q(z)\| + \|L(z)\| \le c_6 \int_{\mathbb{R}} |V(x)|_{\mathbb{C}^{2\times 2}} \nu_1'(x)\, \mathrm{d}x < \infty$$

for all $|\tau| \le \tau_0$, where the right-hand side is also independent of z. This completes the proof of the lemma. ■

With this lemma at hand, we complete the proof of Theorem 5.2 as follows.

Proof of Theorem 5.2. Since $\|M(z)\|$ is uniformly bounded for all $z \in \Sigma$, the operator $I + \epsilon M(z)$ is invertible whenever

$$0 < \epsilon < \frac{1}{C(m)}.$$

Assume this from now on. Then, since

$$I + \epsilon Q(z) = I + \epsilon(L(z) + M(z)) = (I + \epsilon M(z))(I + \epsilon (I + \epsilon M(z))^{-1} L(z)),$$

we have

$$z \in \operatorname{Spec}_{\mathrm{p}}(\mathcal{L}_{m,\epsilon V}) \iff -1 \in \operatorname{Spec}(\epsilon (I + \epsilon M(z))^{-1} L(z)). \tag{5.33}$$

By Lemma 5.4, $L(z)$ is a rank-one operator. Therefore, $\epsilon(I + \epsilon M(z))^{-1} L(z)$ is also a rank-one operator, such that

$$\epsilon (I + \epsilon M(z))^{-1} L(z) f = \epsilon |U_z|_{\mathrm{F}} \langle f, \overline{\psi_z} \rangle (I + \epsilon M(z))^{-1} \phi_z.$$

Hence, $\epsilon(I + \epsilon M(z))^{-1} L(z)$, being a compact operator on the infinite-dimensional Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$, has spectrum

$$\operatorname{Spec}(\epsilon (I + \epsilon M(z))^{-1} L(z)) = \{0, \epsilon |U_z|_{\mathrm{F}} \langle (I + \epsilon M(z))^{-1} \phi_z, \overline{\psi_z} \rangle\}.$$

Thus, from (5.33) and by writing $(I+\epsilon M(z))^{-1} = I - \epsilon M(z)(I+\epsilon M(z))^{-1}$, we have that

$$
\begin{aligned}
z \in \mathrm{Spec_p}(\mathcal{L}_{m,\epsilon V}) &\iff -1 = \epsilon |U_z|_{\mathrm{F}} \langle (I+\epsilon M(z))^{-1}\phi_z, \overline{\psi_z}\rangle \\
&\iff \frac{1}{|U_z|_{\mathrm{F}}} = -\epsilon\langle \phi_z, \overline{\psi_z}\rangle + \epsilon^2 \langle M(z)(I+\epsilon M(z))^{-1}\phi_z, \overline{\psi_z}\rangle \\
&\iff \frac{4m}{4|z|^2+m^2} = W_z(\epsilon) \qquad (5.34)
\end{aligned}
$$

where $W_z(\epsilon)$ is the right-hand side of the second line in this expression. This is (5.24).

We complete the proof by confirming the claimed asymptotic formula for $W_z(\epsilon)$. Notice that

$$\langle \phi_z, \overline{\psi_z}\rangle = -\int_{\mathbb{R}} e^{-2\eta_z(x)} \langle V(x), \overline{\Upsilon_z}\rangle_{\mathrm{F}}\,\mathrm{d}x, \qquad (5.35)$$

because $V(x) = B(x)A(x)$ and $(\Upsilon_z)_{12} = (\Upsilon_z)_{21}$. Now,

$$|\langle M(z)(I+\epsilon M(z))^{-1}\phi_z, \overline{\psi_z}\rangle| \le \frac{\|M(z)\|}{1-\epsilon\|M(z)\|}\|\phi_z\|\|\psi_z\| \le \frac{\|M_z\|\|V\|_{L^1}}{1-\epsilon\|M_z\|} \le c_7.$$

This is a consequence of (5.28) and the fact that $\|M(z)\|$ is uniformly bounded on Σ, and it completes the proof of the theorem. ∎

Proof of Theorem 2.7. As setting in (5.34), we write

$$W_z(\epsilon) = \epsilon a_z + \epsilon^2 b_z(\epsilon)$$

where $a_z = -\langle \phi_z, \overline{\psi_z}\rangle$ is independent of ϵ and $|b_z(\epsilon)| \le c_1$ for all $z \in \Sigma$ and sufficiently small $|\epsilon|$.

We aim to show that there exists a constant $c_2 > 0$ such that

$$|a_{\tau+i\delta}| \le \frac{c_2}{|U_{\tau+i\delta}|_{\mathrm{F}}} \qquad (5.36)$$

for all $|\tau| \ge 1$. To establish this, we first use integration by parts twice to obtain the following results:

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-2\eta_z(x)} v(x)\,\mathrm{d}x &= \Big(\frac{1}{4(\eta_z^+)^2} - \frac{1}{4(\eta_z^-)^2}\Big)v'(0) - \Big(\frac{1}{2\eta_z^-} + \frac{1}{2\eta_z^+}\Big)v(0) \\
&\quad + \frac{1}{4(\eta_z^-)^2}\int_{-\infty}^{0} e^{2\eta_z^- x} v''(x)\,\mathrm{d}x \\
&\quad + \frac{1}{4(\eta_z^+)^2}\int_{0}^{+\infty} e^{-2\eta_z^+ x} v''(x)\,\mathrm{d}x,
\end{aligned}
$$

for any $v \in W^{2,1}(\mathbb{R})$. Now, from the expressions for $\eta_z^{\pm}$, it follows that for suitable constants $c_k > 0$,

$$\left|\frac{1}{2\eta_z^-} + \frac{1}{2\eta_z^+}\right| = \frac{1}{|\eta_z^- \eta_z^+|} \leq \frac{c_3}{\tau^2} \leq \frac{c_4}{|U_z|_{\mathrm{F}}} \quad \text{and} \quad \frac{1}{|\eta_z^{\pm}|^2} \leq \frac{c_5}{|U_z|_{\mathrm{F}}}$$

for all $|\tau| \geq 1$. Since the Frobenius norm of Υ_z is one, setting $v(x) = \langle V(x), \overline{\Upsilon_z} \rangle_{\mathrm{F}}$ gives

$$|v^{(j)}(x)| \leq |V^{(j)}(x)|_{\mathrm{F}},$$

for $j \in \{0, 1, 2\}$ and all $x \in \mathbb{R}$. Hence, according to (5.35), we have that (5.36) is indeed valid under the hypothesis $V \in W^{2,1}(\mathbb{R}, \mathbb{C}^{2\times 2})$.

According to Theorem 5.2, z belongs to $\mathrm{Spec}_{\mathrm{dis}}(\mathcal{L}_{m,\epsilon V})$ if and only if

$$\frac{1}{|U_z|_{\mathrm{F}}} = W_z(\epsilon).$$

To satisfy this condition, z must be in Σ such that

$$\left|\frac{1}{|U_z|_{\mathrm{F}}} - \epsilon a_z\right| = \epsilon^2 |b_z(\epsilon)|.$$

Then, z should meet the requirement

$$\frac{1}{|U_z|_{\mathrm{F}}}(1 - \epsilon c_2) \leq \epsilon^2 c_1.$$

Therefore, there must exist a constant $c_6 > 0$ such that

$$\frac{1}{|U_z|_{\mathrm{F}}} \leq c_6 \epsilon^2$$

for all sufficiently small $|\epsilon|$. Consequently, for z to be an eigenvalue of $\mathcal{L}_{m,\epsilon V}$ when $|\epsilon|$ is small, we require

$$|\operatorname{Re} z|^2 \geq c_7 |U_z|_{\mathrm{F}} > c_8 \epsilon^{-2}.$$

This condition ensures the validity of Theorem 2.7. ■

5.5. Proof of Theorem 2.4

Assume that V satisfies the condition (H2). According to Proposition 2.3, we know that $Q(z) = \overline{A R_0(z) B}$ is well defined for all $z \in \rho(\mathcal{L}_m)$. Moreover, by virtue of (5.9),

$$\|Q(z)\| \leq \|V\|_{L^1} \frac{(|k_z| + 1) \max\left\{|w_z^{\pm}| + \frac{1}{|w_z^{\pm}|}\right\}}{2}$$

for all such z. Recall the set U defined in Section 5.2 and recall that (5.16) is valid for all $z \in U$. Then, for $\|V\|_{L^1} < 2/c_{14}(m) =: C(m)$, we get $\|Q(z)\| < 1$, and so

$$\|(1 + Q(z))^{-1}\| \le \frac{C(m)}{C(m) - \|V\|_{L^1}},$$

for all $z \in U$. Now, from (5.4) and (5.5), we have

$$\max\{\|\overline{R_0(z)B}\|, \|AR_0(z)\|\} \le \|V\|_{L^1}^{1/2} \frac{(|k_z| + 1)\max\{|w_z^\pm| + \frac{1}{|w_z^\pm|}\}}{2(\min \operatorname{Re} \mu_z^\pm)^{1/2}},$$

for all $z \in \rho(\mathcal{L}_m)$.

Consider the minimum in the denominator of the right-hand side. Since

$$\operatorname{Re} \sqrt{\gamma} = \frac{1}{\sqrt{2}} \sqrt{|\gamma| + \operatorname{Re} \gamma},$$

we have that

$$\operatorname{Re} \mu_z^\pm = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(\tau^2 - m^2 - (1 \mp \delta)^2)^2 + 4(1 \mp \delta)^2 \tau^2} + m^2 + (1 \mp \delta)^2 - \tau^2}.$$

Hence, since $\tau^2 \ge \tau^2 - m^2$, then

$$\begin{aligned}
&(\tau^2 - m^2 - (1 \mp \delta)^2)^2 + 4(1 \mp \delta)^2 \tau^2 \\
&\quad \ge (\tau^2 - m^2 - (1 \mp \delta)^2)^2 + 4(1 \mp \delta)^2 (\tau^2 - m^2) \\
&\quad = (\tau^2 - m^2 + (1 \mp \delta)^2)^2.
\end{aligned}$$

Thus, $\operatorname{Re} \mu_z^\pm \ge |1 \mp \delta|$ for all $z \in \mathbb{C}$ and so $\min \operatorname{Re} \mu_z^\pm \ge |\delta| - 1$ for all $|\delta| > 1$.

Gathering the estimates above, we then have that, if $\|V\|_{L^1} \le C(m)$,

$$\begin{aligned}
\|(\mathcal{L}_{m,V} - z)^{-1} - (\mathcal{L}_m - z)^{-1}\| &\le \|\overline{R_0(z)B}\| \, \|(I + Q(z))^{-1}\| \, \|AR_0(z)\| \\
&\le \frac{\|V\|_{L^1}}{(|\operatorname{Im} z| - 1)C(m)(C(m) - \|V\|_{L^1})},
\end{aligned}$$

for all $z \in U$. Moreover, since U lies outside the numerical range of $\mathcal{L}_m$, then

$$\|(\mathcal{L}_m - z)^{-1}\| \le \frac{1}{|\operatorname{Im} z| - 1},$$

for all $z \in U$. Therefore, according to the triangle inequality, whenever $\|V\|_{L^1} < C(m)$, we have that

$$\|(\mathcal{L}_{m,V} - z)^{-1}\| \le \left(1 + \frac{\|V\|_{L^1}}{C(m)(C(m) - \|V\|_{L^1})}\right) \frac{1}{|\operatorname{Im} z| - 1},$$

for all $z \in U$. This gives the claim made in Theorem 2.4.

A. Abstract Birman–Schwinger principle

To characterise the domain of the operator $\mathcal{L}_{m,V}$ in terms of the resolvent of $\mathcal{L}_m$, we follow the classical approach of [16], which was extended to the non-selfadjoint regime in [12]. We include precise details of this construction in this appendix, as the abstract framework might be applicable to other families of operators related to perturbations of $\mathcal{D}_m$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Let $\operatorname{Dom}(H_0) \subset \mathcal{H}$, $\operatorname{Dom}(A) \subset \mathcal{H}$ and $\operatorname{Dom}(B) \subset \mathcal{K}$ be three dense subspaces. Let

$$H_0\colon \operatorname{Dom}(H_0) \to \mathcal{H}, \quad A\colon \operatorname{Dom}(A) \to \mathcal{K}, \quad B\colon \operatorname{Dom}(B) \to \mathcal{H},$$

be three closed operators such that H_0 has a non-empty resolvent set. Let

$$R_0(z) = (H_0 - zI_{\mathcal{H}})^{-1}, \quad z \in \rho(H_0).$$

We consider the following hypotheses.

(B1) For some (and hence all) $z \in \rho(H_0)$,

- the operator $R_0(z)B$ is closable and the closure $\overline{R_0(z)B}\colon \mathcal{K} \to \mathcal{H}$ is bounded,
- and the operator $AR_0(z)\colon \mathcal{H} \to \mathcal{K}$ is bounded.

(B2) For some (and hence all) $z \in \rho(H_0)$, the operator $AR_0(z)B$ has a bounded closure

$$Q(z) = \overline{AR_0(z)B}\colon \mathcal{K} \to \mathcal{K}.$$

(B3) The set

$$\mathsf{R} = \{z \in \rho(H_0) : -1 \in \rho(Q(z))\}$$

is non-empty.

For all $z \in \mathsf{R}$, we let

$$R(z) = R_0(z) - \overline{R_0(z)B}(I_{\mathcal{K}} + Q(z))^{-1} AR_0(z)\colon \mathcal{H} \to \mathcal{H}. \tag{A.1}$$

Theorem A.1. *Assume that the closed operators H_0, A, and B are related in the above manner and that (B1)–(B3) hold true. Then, there exists a closed densely defined extension*[4]

$$H \supseteq H_0 + BA\colon \operatorname{Dom}(H_0) \cap \operatorname{Dom}(BA) \to \mathcal{H},$$

whose resolvent is given by

$$(H - zI_{\mathcal{H}})^{-1} = R(z) \quad \textit{for all } z \in \mathsf{R}.$$

[4]Here $\operatorname{Dom}(BA) = \{f \in \operatorname{Dom}(A) : Af \in \operatorname{Dom}(B)\}$.

The conclusion of Theorem A.1 can be found in [12, Theorem 2.3], assuming the slightly more stringent conditions [12, Hypothesis 2.1 (i)],

$$\operatorname{Dom}(H_0) \subset \operatorname{Dom}(A) \quad \text{and} \quad \operatorname{Dom}(H_0^*) \subset \operatorname{Dom}(B^*).$$

In this appendix we show that the assumption (B1) also ensures the claimed conclusion.

Proof. We split the proof into four steps.

Step 1. Let $z \in \rho(H_0)$, then the product $AR_0(z)B$ is always closable. Indeed, we claim that $AR_0(z)B$ is densely defined and its adjoint is also densely defined, see [25, Theorem 1.8 (i)]. The former is an immediate consequence of (B1), so that

$$\operatorname{Dom}(AR_0(z)B) = \operatorname{Dom}(B). \tag{A.2}$$

For the latter, according to [25, Proposition 1.7], we know that

$$(AR_0(z)B)^* \supset (R_0(z)B)^* A^*.$$

Then, by virtue of [25, Theorem 1.8 (ii)], $(R_0(z)B)^* = (\overline{R_0(z)B})^*$, so by (B1) we have

$$\operatorname{Dom}((R_0(z)B)^* A^*) = \operatorname{Dom}(A^*).$$

Since A is closed, $\operatorname{Dom}(A^*)$ and thus $\operatorname{Dom}((AR_0(z)B)^*)$ are dense in $\mathcal{K}$.

Step 2. Let $z \in \mathsf{R}$. Our second task is to construct a closed densely defined operator, $H(z)$, such that

$$R(z) = (H(z) - zI_{\mathcal{H}})^{-1}. \tag{A.3}$$

For this purpose, firstly observe that

$$\overline{R_0(z)B}\,(\mathcal{K}) \subset \operatorname{Dom}(A) \quad \text{and} \quad Q(z) = A\overline{R_0(z)B}. \tag{A.4}$$

Indeed, for $f \in \operatorname{Dom}(B)$ and $g \in \operatorname{Dom}(A^*)$, we have

$$\langle R_0(z)Bf, A^*g\rangle_{\mathcal{H}} = \langle AR_0(z)Bf, g\rangle_{\mathcal{K}} = \langle Q(z)f, g\rangle_{\mathcal{K}}.$$

Here the second equality is valid as a consequence of (A.2). Thus, the continuity of $\overline{R_0(z)B}$ and $Q(z)$, combined with the density of $\operatorname{Dom}(B)$ in $\mathcal{K}$, imply

$$\langle \overline{R_0(z)B}\, f, A^*g\rangle_{\mathcal{H}} = \langle Q(z)f, g\rangle_{\mathcal{K}}$$

for all $f \in \mathcal{K}$ and $g \in \operatorname{Dom}(A^*)$. This gives (A.4).

Now, multiplying $R(z)$ by A on the left, according to (A.1) and (A.4), we get that

$$AR(z) = AR_0(z) - Q(z)(I_{\mathcal{K}} + Q(z))^{-1}AR_0(z) = (I_{\mathcal{K}} + Q(z))^{-1}AR_0(z).$$

This yields the following. If $R(z)f = 0$, where $f \in \mathcal{H}$, then $AR_0(z)f = 0$. Hence, by (A.1), $R_0(z)f = 0$ and so $f = 0$. That is, $R(z)$ is necessarily injective.

The adjoint of $R(z)$ is given by

$$R(z)^* = R_0(z)^* - (AR_0(z))^*(I_{\mathcal{K}} + Q(z)^*)^{-1}(R_0(z)B)^*. \tag{A.5}$$

By virtue of [25, Theorem 1.8 (ii) and Proposition 1.7 (ii)], we gather that $Q(z)^* = B^*(AR_0(z))^*$. In conjunction with (A.5), this yields

$$\begin{aligned} B^*R(z)^* &= (R_0(z)B)^* - Q(z)^*(I_{\mathcal{K}} + Q(z)^*)^{-1}(R_0(z)B)^* \\ &= (I_{\mathcal{K}} + Q(z)^*)^{-1}(R_0(z)B)^*. \end{aligned} \tag{A.6}$$

Reasoning out as proving the injection of $R(z)$ as above by using (A.5) and (A.6), we obtain

$$\operatorname{Ker}(R(z)^*) = \{0\}.$$

Hence, $R(z)(\mathcal{K})$ is dense in $\mathcal{H}$.

For each $z \in \mathsf{R}$, we can then set $H(z) = R(z)^{-1} + zI_{\mathcal{H}}$. By construction, $H(z)$ is a closed densely defined operator satisfying (A.3). Note that $H(z)$ is ensured to be closed as a consequence of [25, Theorem 1.8 (vi)].

Step 3. We now show that the operator $H(z)$ from the previous step, does not depend on $z \in \mathsf{R}$.

Let $z_1, z_2 \in \mathsf{R}$. Then, $H(z_j)$ are such that

$$R(z_j) = (H(z_j) - z_j I_{\mathcal{H}})^{-1}.$$

We claim that $H(z_1) = H(z_2)$. To prove this claim, note that $R(z)$ satisfies the resolvent identity

$$R(z_1) - R(z_2) = (z_1 - z_2)R(z_2)R(z_1). \tag{A.7}$$

Indeed, by (A.1), (A.8), and (A.9), it follows that

$$\begin{aligned} &(z_1 - z_2)R(z_2)R(z_1) \\ &\quad = R_0(z_1) - R_0(z_2) - [\overline{R_0(z_1)B} - \overline{R_0(z_2)B}](I_{\mathcal{K}} + Q(z_1))^{-1}AR_0(z_1) \\ &\qquad - \overline{R_0(z_2)B}(I_{\mathcal{K}} + Q(z_2))^{-1}A[R_0(z_1) - R_0(z_2)] \\ &\qquad + \overline{R_0(z_2)B}(I_{\mathcal{K}} + Q(z_2))^{-1}[Q(z_1) - Q(z_2)](I_{\mathcal{K}} + Q(z_1))^{-1}AR_0(z_1) \\ &\quad = R(z_1) - R(z_2) + \overline{R_0(z_2)B}[(I_{\mathcal{K}} + Q(z_1))^{-1} - (I_{\mathcal{K}} + Q(z_2))^{-1}]AR_0(z_1) \\ &\qquad + \overline{R_0(z_2)B}(I_{\mathcal{K}} + Q(z_2))^{-1}[Q(z_1) - Q(z_2)](I_{\mathcal{K}} + Q(z_1))^{-1}AR_0(z_1) \\ &\quad = R(z_1) - R(z_2). \end{aligned}$$

By multiplying both sides of (A.7) by $H(z_2) - z_2 I_{\mathcal{H}}$ on the left, we get

$$(H(z_2) - z_2 I_{\mathcal{H}})R(z_1) - I_H = (z_1 - z_2)R(z_1).$$

This is equivalent to

$$(H(z_2) - z_1 I_{\mathcal{H}})R(z_1) = I_{\mathcal{H}}.$$

Multiplying this identity by $H(z_1) - z_1 I_{\mathcal{H}}$ on the right, ensures that $H(z_1) = H(z_2)$ as claimed above. We can therefore write $H = H(z)$.

Step 4. We complete the proof by showing that H is an extension of the operator $H_0 + BA$.

Let $z \in \mathrm{R}$ and let $f \in \mathrm{Dom}(H_0 + BA) = \mathrm{Dom}(H_0) \cap \mathrm{Dom}(BA)$. We set $g = (H_0 - zI_{\mathcal{H}})f$. It follows from (A.1) and (A.6) that

$$R(z) = R_0(z) - \overline{R(z)B}AR_0(z),$$

and thus that $R(z)g = f - R(z)BAf$. This completes the proof of the theorem. ■

The next comments about the validity of the hypotheses (B1) and (B2), when moving z, as now in place. The next observations was used in the Step 3 in the proof of Theorem A.1 and will be useful in the proof of Theorem A.2. Here, $z_1, z_2 \in \rho(H_0)$.

If $AR_0(z_1) \in L(\mathcal{H}, \mathcal{K})$, then $AR_0(z_2) \in L(\mathcal{H}, \mathcal{K})$. Indeed, we have

$$\begin{aligned} AR_0(z_2) &= A[R_0(z_1) + (z_2 - z_1)R_0(z_1)R_0(z_2)] \\ &\supseteq \underbrace{AR_0(z_1)}_{\in L(\mathcal{H},\mathcal{K})} + (z_2 - z_1)\underbrace{AR_0(z_1)}_{\in L(\mathcal{H},\mathcal{K})}\underbrace{R_0(z_2)}_{\in L(\mathcal{H})}. \end{aligned}$$

In particular, since the domain of the right-hand side is total $\mathcal{H}$, we obtain

$$AR_0(z_2) = AR_0(z_1) + (z_2 - z_1)AR_0(z_1)R_0(z_2).$$

Thus, $AR_0(z_2) \in L(\mathcal{H}, \mathcal{K})$.

If $R_0(z_1)B$ is closable and $\overline{R_0(z_1)B} \in L(\mathcal{K}, \mathcal{H})$, then $R_0(z_2)B$ is also closable and $\overline{R_0(z_2)B} \in L(\mathcal{K}, \mathcal{H})$. Indeed, since $\mathrm{Dom}(R_0(z_2)B) = \mathrm{Dom}(B)$ which is densely defined, we have

$$\begin{aligned} (R_0(z_2)B)^* &= [R_0(z_1)B + (z_2 - z_1)R_0(z_2)R_0(z_1)B]^* \\ &\supseteq [R_0(z_1)B]^* + (\overline{z_2} - \overline{z_1})[R_0(z_2)R_0(z_1)B]^* \\ &= \underbrace{(\overline{R_0(z_1)B})^*}_{\in L(\mathcal{H},\mathcal{K})} + (\overline{z_2} - \overline{z_1})\underbrace{(\overline{R_0(z_1)B})^*}_{\in L(\mathcal{H},\mathcal{K})}\underbrace{R_0(z_2)^*}_{\in L(\mathcal{H})}, \end{aligned}$$

where we used [25, Proposition 1.6 (vi)] for the inclusion and [25, Proposition 1.7 (ii) and Theorem 1.8 (ii)] for the last equality. Since the domain of the right-hand side is the whole $\mathcal{H}$, we deduce that

$$(R_0(z_2)B)^* = (\overline{R_0(z_1)B})^* + (\overline{z_2} - \overline{z_1})(\overline{R_0(z_1)B})^* R_0(z_2)^* \in L(\mathcal{H}, \mathcal{K}).$$

Therefore, from [25, Theorem 1.8 (i)–(ii)], $R_0(z_2)B$ is closable and $\overline{R_0(z_2)B} = (R_0(z_2)B)^{**}$ belongs to $L(\mathcal{K}, \mathcal{H})$, as claimed. Moreover, the following identity holds true:

$$\overline{R_0(z_1)B} - \overline{R_0(z_2)B} = (z_1 - z_2) R_0(z_2) \overline{R_0(z_1)B}. \tag{A.8}$$

If $\overline{AR_0(z_1)B} \in L(\mathcal{K})$, then $\overline{AR_0(z_2)B} \in L(\mathcal{K})$. This can be proved in the same manner as above:

$$\begin{aligned}(AR_0(z_2)B)^* &= [AR_0(z_1)B + (z_2 - z_1)AR_0(z_2)R_0(z_1)B]^* \\ &\supset [AR_0(z_1)B]^* + (\overline{z_2} - \overline{z_1})[AR_0(z_2)R_0(z_1)B]^* \\ &= \underbrace{(\overline{AR_0(z_1)B})^*}_{\in L(\mathcal{K})} + (\overline{z_2} - \overline{z_1}) \underbrace{(\overline{R_0(z_1)B})^*}_{\in L(\mathcal{H},\mathcal{K})} \underbrace{(AR_0(z_2))^*}_{\in L(\mathcal{K},\mathcal{H})}.\end{aligned}$$

Once again, since the domain of the right-hand side is total $\mathcal{K}$,

$$(AR_0(z_2)B)^* = (\overline{AR_0(z_1)B})^* + (\overline{z_2} - \overline{z_1})(\overline{R_0(z_1)B})^*(AR_0(z_2))^* \in L(\mathcal{K}),$$

and the conclusion follows. Moreover, we get the formula

$$Q(z_1) - Q(z_2) = (z_1 - z_2) AR_0(z_2) \overline{R_0(z_1)B}. \tag{A.9}$$

Theorem A.2. *Assume that the closed operators H_0, A, and B are related in the above manner, and that (B1)–(B3) hold true. Additionally, assume that* R *has at least two elements. Then, for all $z \in \rho(H_0)$,*

$$z \in \operatorname{Spec}_{\mathrm{p}}(H) \iff -1 \in \operatorname{Spec}_{\mathrm{p}}(Q(z)).$$

Proof. Let $f \in \operatorname{Dom}(H) \setminus \{0\}$ be such that $Hf = zf$. Let $z_0 \in \mathsf{R} \setminus \{z\}$. The equation $Hf = zf$ is equivalent to $f = (z - z_0)R(z_0)f$. By virtue of (A.1), the latter happens if and only if

$$(H_0 - zI_{\mathcal{H}})R_0(z_0)f = -(z - z_0)\overline{R_0(z_0)B}(I_{\mathcal{K}} + Q(z_0))^{-1}AR_0(z_0)f.$$

By setting

$$v = (I_{\mathcal{K}} + Q(z_0))^{-1}AR_0(z_0)f \neq 0$$

and applying $(I_{\mathcal{K}} + Q(z_0))^{-1}AR_0(z)$ on both sides, we obtain

$$v = -(z - z_0)(I_{\mathcal{K}} + Q(z_0))^{-1}AR_0(z)\overline{R_0(z_0)B}v.$$

Then, according to (A.9),

$$v = (I_{\mathcal{K}} + Q(z_0))^{-1}(Q(z_0) - Q(z))v \iff Q(z)v = -v.$$

In other words, -1 is the eigenvalue of $Q(z)$ with the eigenfunction v, as the right-hand side of the conclusion states.

We now show the converse. Assume that $Q(z)v = -v$ for some $v \in \mathcal{K} \setminus \{0\}$. Take $z_0 \in \mathsf{R} \setminus \{z\}$ and write

$$\begin{aligned} v &= v - (I_{\mathcal{K}} + Q(z_0))^{-1}(I_{\mathcal{K}} + Q(z))v \\ &= (I_{\mathcal{K}} + Q(z_0))^{-1}(Q(z_0) - Q(z))v \\ &= (z_0 - z)(I_{\mathcal{K}} + Q(z_0))^{-1} A R_0(z_0) \overline{R_0(z) B} v. \end{aligned}$$

Here, in the last step, we applied (A.9). By setting $f = -\overline{R_0(z)B}v \neq 0$, we get

$$v = (z - z_0)(I_{\mathcal{K}} + Q(z_0))^{-1} A R_0(z_0) f.$$

Then, applying $\overline{R_0(z_0)B}$ on both sides, and using (A.8) and (A.1), we gather that

$$-f + (z - z_0) R_0(z_0) f = (z - z_0)(R_0(z_0) - R(z_0)) f.$$

This is equivalent to $(z - z_0)(H - z_0 I_{\mathcal{H}})^{-1} f = f$ and thus, $f \in \operatorname{Dom}(H)$ and $Hf = zf$. ∎

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Lyonell Boulton
Department of Mathematics and Maxwell Institute for Mathematical Sciences,
Heriot–Watt University, Riccarton, Edinburgh EH14 4AS, UK; l.boulton@hw.ac.uk

David Krejčiřík
Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague, Trojanova 13, 12000 Prague 2, Czech Republic;
david.krejcirik@fjfi.cvut.cz

Tho Nguyen Duc
Faculty of Mathematics and Statistics, Ton Duc Thang University,
No. 19 Nguyen Huu Tho Street, Tan Phong Ward, District 7, Ho Chi Minh City, Vietnam;
nguyenductho@tdtu.edu.vn