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Crystalline representations and Wach modules in the imperfect residue field case

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Abstract. For an absolutely unramified extension $L/\mathbb{Q}_p$ with imperfect residue field, we define and study Wach modules in the setting of (φ, Γ)-modules for L. Our main result establishes a direct equivalence between the category of lattices inside crystalline representations of the absolute Galois group of L and the category of integral Wach modules for L. Moreover, we provide a direct relation between a rational Wach module equipped with the Nygaard filtration and the filtered φ-module of its associated crystalline representation.

1. Introduction

In classical p-adic Hodge theory, Fontaine introduced and developed the idea of studying a p-adic representation of the absolute Galois group of $\mathbb{Q}_p$ (and its extensions) via semilinear algebraic objects attached to the representation. More concretely, for an extension $F/\mathbb{Q}_p$ with perfect residue field and absolute Galois group G_F, in [27], Fontaine showed that the category of $\mathbb{Z}_p$-representations of G_F is equivalent to the category of étale (φ, Γ_F)-modules, where Γ_F is an open subgroup of $\mathbb{Z}_p^\times$ (see Section 1.1). On the other hand, to understand p-adic representations coming from geometry, Fontaine defined several classes of representations such as *crystalline*, semistable, etc. in [26]. Putting the two point of views together, Fontaine asked the following natural question: is it possible to describe crystalline representations of G_F in terms of (φ, Γ_F)-modules? For an unramified extension $F/\mathbb{Q}_p$, Fontaine studied this question in [27], and introduced the notion of finite crystalline-height representations (*représentations de cr-hauteur finie*) of G_F, which was further developed by Wach [46, 47], Colmez [20] and Berger [8]. More precisely, [8] showed that the category of G_F-stable $\mathbb{Z}_p$-lattices of p-adic crystalline representations is equivalent to the category of *Wach modules*, where a Wach module is a certain integral lattice inside the étale (φ, Γ_F)-module associated to the representation (see Section 1.1).

The two point of views of Fontaine admit natural generalisations to a relative base, i.e., formally étale algebras over a formal torus. In particular, relative étale (φ, Γ)-modules were studied by Andreatta [5] and relative p-adic crystalline representations were studied by Faltings [24] and Brinon [15]. In [3], we introduced and studied the notion of relative

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Wach modules for an absolutely unramified (at p) relative base. However, compared to the classical case, the results of [3] are restrictive, i.e., we only show that relative Wach modules give rise to lattices inside relative crystalline representations; the converse is the following *difficult open question: can one functorially associate a relative Wach module to a $\mathbb{Z}_p$-lattice inside a relative crystalline representation?*

In this article, we resolve the *open question* for the imperfect residue field case (see Theorem 1.1), and we use the result thus obtained, in a subsequent work [1], to resolve the *open question* in the relative case. More concretely, for a complete discrete valuation field $L/\mathbb{Q}_p$ with imperfect residue field, [5] developed the theory of étale (φ, Γ_L)-modules, where Γ_L is an open subgroup of $\mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$ with d being the transcendence degree of $L/\mathbb{Q}_p$, and [14] developed the theory of p-adic crystalline representations of G_L, the absolute Galois group of L. However, for absolutely unramified $L/\mathbb{Q}_p$, the theory of Wach modules for L was missing from the picture. So, in this article, we define Wach modules for L and prove our *first main result*.

Theorem 1.1 (Corollary 4.2). *The category of G_L-stable $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_L is equivalent to the category of Wach modules for L.*

As mentioned above, the difficult part of Theorem 1.1 is to functorially associate a Wach module to any G_L-stable $\mathbb{Z}_p$-lattice T inside a p-adic crystalline representation of G_L. To resolve this, let us note that using the classical theory of [8] in the perfect residue field case, one can associate to T a φ-module N over the base ring of Wach modules for L. However, equipping N with a natural action of Γ_L is highly non-trivial, where the difficulty arises because Γ_L is quite large compared to Γ_F from the classical case. The heart of this article constitutes a direct construction of the natural action of Γ_L on N (see Section 1.2.3 for details). Let us remark that the analogous theory of Breuil–Kisin modules in the imperfect residue field case was studied by Brinon and Trihan [16]. However, the theory of loc. cit. is different from the theory of Wach modules, in particular, the construction of the action of Γ_L does not feature in [16].

Besides being natural generalisations of classical results to the relative case, the usefulness of relative Wach modules stems from its applications in the computation of p-adic vanishing cycles using syntomic complexes. Indeed, to generalise the computation of p-adic vanishing cycles by Colmez and Nizioł [22] to the case of crystalline coefficients, in [4], crucial inputs were the results on relative Wach modules from [3]. However, as mentioned above, the results of [3], and therefore, of [4] only work for a restrictive class of crystalline coefficients. In order to generalise the results of [22] to all crystalline coefficients, we need the more general result on relative Wach modules from [1, Theorem 1.5], for which Theorem 1.1 is a crucial input. Furthermore, in op. cit. we provide an interesting application of Theorem 1.1, in particular, we give a new criteria for checking the crystallinity of relative p-adic representations (see [1, Theorem 1.7 and Corollary 1.8]).

An additional motivation for considering Wach modules is to construct a *deformation of the functor* $\mathbf{D}_{\mathrm{cris}}$ from classical p-adic Hodge theory (see [27, Section B.2.3]). This construction was carried out in the Fontaine–Laffaille range by Wach [47, Theoreme 3], and more generally, by Berger [8, Théorème III.4.4]. In this article, our *second main result* provides a generalisation of loc. cit. to the imperfect residue field case (see Theorem 1.8). Let us remark that the general idea of deformations of crystalline and de Rham cohomologies has led to exciting new developments in integral p-adic Hodge theory via the introduction and development of prismatic cohomology [9–11,42].

Finally, note that recent developments in the theory of prismatic F-crystals [12,23,30] provide a new approach to the classification of lattices inside crystalline representations. While the prismatic point of view is an exciting development, in the current paper, we employ techniques from the theory of (φ, Γ)-modules to obtain our results. This is due to the fact that, in our approach, the construction of Wach modules for L and the proof of Theorems 1.1 and 1.8, are explicit and direct, which could be advantageous for "arithmetic" applications. In Section 1.2.4, we will provide more details on relations of our results in this article to other works. In the rest of this section, we will describe the results mentioned above in more detail. We begin by recalling the main classical result.

1.1. The classical case

Let p be a fixed prime number and let κ denote a perfect field of characteristic p; set $O_F := W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F := \mathrm{Frac}(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of F, let $\mathbb{C}_p := \widehat{\overline{F}}$ denote the p-adic completion, and $G_F := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Moreover, let $F_\infty := \bigcup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$. Furthermore, let $F_\infty^\flat$ denote the tilt of F_∞ (see Section 1.3) and fix $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$ in $O_{F_\infty}^\flat$, and $\mu := [\varepsilon] - 1$ and $[p]_q := \varphi(\mu)/\mu$ in $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty}) := W(O_{F_\infty}^\flat)$, the ring of p-typical Witt vectors with coefficients in $O_{F_\infty}^\flat$.

In [27], Fontaine established a categorical equivalence between $\mathbb{Z}_p$-representations of G_F and étale (φ, Γ_F)-modules over a certain period ring $\mathbf{A}_F := O_F[\![\mu]\!][1/\mu]^\wedge \subset W(F_\infty^\flat)$, where $^\wedge$ denotes the p-adic completion, and $\mathbf{A}_F$ is stable under the natural (φ, Γ_F)-action on $W(F_\infty^\flat)$. For a fixed finite free $\mathbb{Z}_p$-representation T of G_F, the associated finite free étale (φ, Γ_F)-module over $\mathbf{A}_F$ is given as $\mathbf{D}_F(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$, where $\mathbf{A} \subset W(\mathbb{C}_p^\flat)$ is the maximal unramified extension of $\mathbf{A}_F$ inside $W(\mathbb{C}_p^\flat)$. In loc. cit., Fontaine conjectured that if $V := T[1/p]$ is crystalline then there exists a lattice inside $\mathbf{D}_F(V) := \mathbf{D}_F(T)[1/p]$ over which the action of Γ_F admits a simpler form. Denote by $\mathbf{A}_F^+ := O_F[\![\mu]\!] \subset \mathbf{A}_F$, which is stable under the (φ, Γ_F)-action, and note the following.

Definition 1.2. Let $a, b \in \mathbb{Z}$ with $b \geqslant a$. A *Wach module* over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_F^+$-module N equipped with a continuous and semilinear action of Γ_F such that,

1. The action of Γ_F on $N/\mu N$ is trivial.
2. There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$, commuting with the action of Γ_F, and such that $\varphi(\mu^b N) \subset \mu^b N$ and the cokernel of the injective map $(1 \otimes \varphi)\colon \varphi^*(\mu^b N) := \mathbf{A}_F^+ \otimes_{\varphi, \mathbf{A}_F^+} \mu^b N \to \mu^b N$ is killed by $[p]_q^{b-a}$.

Denote the category of Wach modules over $\mathbf{A}_F^+$ as $(\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$, with morphisms between objects being $\mathbf{A}_F^+$-linear, Γ_F-equivariant and φ-equivariant (after inverting μ). Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F. To any T in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$, using [20, 46], Berger functorially attached a Wach module $\mathbf{N}_F(T)$ over $\mathbf{A}_F^+$ in [8]. The main result in the arithmetic case is as follows (see [8]).

Theorem 1.3. *The Wach module functor induces a natural equivalence of $\otimes$-categories:*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_F(T),$$

with a natural quasi-inverse $\otimes$-functor given as $N \mapsto (W(\mathbb{C}_p^\flat) \otimes_{\mathbf{A}_F^+} N)^{\varphi=1}$.

1.2. The imperfect residue field case

Let $d \in \mathbb{N}$ and let $X_1, X_2, \ldots, X_d$ be some indeterminates. We define $O_{L^\square}$ to be the p-adic completion of the localisation of the algebra $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm d}]$ at the prime ideal (p). It is a complete discrete valuation ring with uniformiser p, imperfect residue field $\kappa(X_1, \ldots, X_d)$ and fraction field $L^\square := O_{L^\square}[1/p]$. Let O_L denote a finite étale extension of $O_{L^\square}$ such that it is a complete discrete valuation ring with uniformiser p, imperfect residue field a finite étale extension of $\kappa(X_1, \ldots, X_d)$ and fraction field $L := O_L[1/p]$. Let G_L denote the absolute Galois group of L for a fixed algebraic closure $\overline{L}/L$; let $\Gamma_L \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$ denote the Galois group of L_∞ over L, where L_∞ is the fraction field of O_{L_∞} obtained by adjoining to O_L all p-power roots of unity and all p-power roots of X_i, for all $1 \leqslant i \leqslant d$ (see Section 2). In this setting, we have the theory of crystalline representations of G_L [14] and étale (φ, Γ)-modules [5]. However, the theory of Wach modules for L, i.e., a description of the p-adic crystalline representations G_L in terms of (φ, Γ_L)-modules, was missing from the picture. The main goal of this article is to complete this picture, which we discuss next.

1.2.1. Wach modules. For $1 \leqslant i \leqslant d$, let us set $X_i^\flat := (X_i, X_i^{1/p}, \ldots)$ in $O_{L_\infty}^\flat$ and take $[X_i^\flat]$ in $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = W(O_{L_\infty}^\flat)$ to be the Teichmüller representative of $X_i^\flat$. Let $\mathbf{A}_L^+$ denote the unique finite étale extension (along the finite étale map $O_{L^\square} \to O_L$) of the (p, μ)-adic completion of the localisation $O_F[\![\mu]\!][[X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]_{(p,\mu)}$. The ring $\mathbf{A}_L^+$ is equipped with a Frobenius endomorphism φ and a continuous action of Γ_L (see Sections 1.3 and 2.1), and note the following.

Definition 1.4. Let $a, b \in \mathbb{Z}$ with $b \geqslant a$. A *Wach module* over $\mathbf{A}_L^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_L^+$-module N equipped with a continuous and semilinear action of Γ_L satisfying the following assumptions:

(1) The action of Γ_L on $N/\mu N$ is trivial.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$, commuting with the action of Γ_L, and such that $\varphi(\mu^b N) \subset \mu^b N$ and the cokernel of the injective map $(1 \otimes \varphi)\colon \varphi^*(\mu^b N) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mu^b N \to \mu^b N$ is killed by $[p]_q^{b-a}$.

Say that N is *effective* if one can take $b = 0$ and $a \leqslant 0$. Denote the category of Wach modules over $\mathbf{A}_L^+$ as $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}$, with morphisms between objects being $\mathbf{A}_L^+$-linear, Γ_L-equivariant and φ-equivariant (after inverting μ).

Set $\mathbf{A}_L := \mathbf{A}_L^+[1/\mu]^\wedge$ as the p-adic completion, equipped with a Frobenius endomorphism φ and a continuous action of Γ_L. Let T be a finite free $\mathbb{Z}_p$-module equipped with a continuous action of G_L, and note that one can functorially attach to T a finite free étale (φ, Γ_L)-module $\mathbf{D}_L(T)$ over $\mathbf{A}_L$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$, equipped with a Frobenius-semilinear operator φ and a semilinear and continuous action of Γ_L. In fact, the preceding functor induces an equivalence between finite free $\mathbb{Z}_p$-representations of G_L and finite free étale (φ, Γ_L)-modules over $\mathbf{A}_L$ (see Section 2.2).

Remark 1.5. The category of Wach modules over $\mathbf{A}_L^+$ can be realised as a full subcategory of étale (φ, Γ)-modules over $\mathbf{A}_L$ (see Proposition 3.3).

1.2.2. Main results. Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_L. The main result of this article, i.e., Theorem 1.1, can be stated more precisely as follows.

Theorem 1.6 (Corollary 4.2). *The Wach module functor induces a natural equivalence of $\otimes$-categories:*

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L) &\xrightarrow{\sim} (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} \\ T &\longmapsto \mathbf{N}_L(T), \end{aligned}$$

with a natural quasi-inverse $\otimes$-functor given as

$$N \longmapsto \mathbf{T}_L(N) := \big(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} N\big)^{\varphi=1},$$

where $\mathbb{C}_L := \widehat{\overline{L}}$.

Our strategy for the proof of Theorem 1.6 will be described in Section 1.2.3.

Remark 1.7. Let us note that in Theorem 1.6, we do not expect the functor $\mathbf{N}_L$ to be exact (see [18, Example 7.1] for an example in the arithmetic case). However, after passing to the associated isogeny categories, the Wach module functor induces an exact equivalence of $\otimes$-categories

$$\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi, \Gamma)\text{-Mod}_{\mathbf{B}_L^+}^{[p]_q},$$

via $V \mapsto \mathbf{N}_L(V)$, with an exact quasi-inverse $\otimes$-functor given as (see Corollary 4.3),

$$M \longmapsto \mathbf{V}_L(M) := \big(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} M\big)^{\varphi=1}.$$

As indicated earlier, the proof of Theorem 1.6 is based on techniques employed in the theory of (φ, Γ)-modules. One of the advantages of using this approach is that it enables us to establish several comparison results between objects appearing in the p-adic Hodge theory over L (see Propositions 3.14, 4.21, Corollaries 4.22 and 3.16). In order to keep the introduction light, we only mention one of the comparison results here and refer the reader to the main body of this article for the rest.

Let N be a Wach module over $\mathbf{A}_L^+$. We equip N with a Nygaard filtration defined as $\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}$. Then, we note that $(N/\mu N)[1/p]$ is a φ-module over L, since $[p]_q = p \bmod \mu\mathbf{A}_L^+$, and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with the induced filtration, in particular, it is a filtered φ-module over L. Moreover, let $V := \mathbf{T}_L(N)[1/p]$ denote the associated crystalline representation of G_L from Theorem 1.6. Then, we can functorially associate to V a filtered (φ, ∂)-module over L denoted $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ (see Section 2.3), and show the following.

Theorem 1.8 (Corollary 3.16). *Let N be a Wach module over $\mathbf{A}_L^+$ and $V = \mathbf{T}_L(N)[1/p]$ the associated crystalline representation from Theorem 1.6. Then, we have a natural isomorphism $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ as filtered φ-modules over L.*

The proof of Theorem 1.8 is obtained by utilising the computations done in the proof of Theorem 3.12, more specifically, using Proposition 3.14.

Remark 1.9. The statement of Theorem 1.8 is motivated by the results [27, Section B.2.3] and [8, Théorème III.4.4] in the perfect residue field case, but our proof is independent of those results. However, note that it is also possible to deduce that the isomorphism in Theorem 1.8 is compatible with filtrations, by using [8, Théorème III.4.4] as an input (see [1]).

Remark 1.10. Based on the expectation put forth in [3, Remark 4.48], it is reasonable to expect that the L-vector space $(N/\mu N)[1/p]$ may be equipped with a connection by defining a q-connection on N using the action of the geometric part of Γ_L, i.e., Γ_L' (see Section 2), and inducing a connection via $N \xrightarrow{q\mapsto 1} N/\mu N$. Moreover, the isomorphism $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ in Theorem 1.8 should be further compatible with connections. These expectations will be verified in [1].

1.2.3. Strategy for the proof of Theorem 1.6. To prove the theorem, starting with a $\mathbb{Z}_p$-lattice T inside a p-adic crystalline representation of G_L, we first use the result in the perfect residue field case (see Theorem 1.3) and its compatibility with the results of [35, 36] (see Section 4.2) to construct a finite free module $\mathbf{N}_{\mathrm{rig},L}(V)$ (associated to $V = T[1/p]$), over the ring of functions of the open unit disk over L (denoted $\mathbf{B}_{\mathrm{rig},L}^+$), such that $\mathbf{N}_{\mathrm{rig},L}(V)$ satisfies a Frobenius finite $[p]_q$-height condition. However, proving the existence of a non-trivial action of Γ_L on $\mathbf{N}_{\mathrm{rig},L}(V)$ is a difficult question and it does not follow from the classical theory because

$$\Gamma_L \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,$$

whereas we have $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$ in the classical case. To resolve this issue, our innovation is to use the Galois action on V and its crystallinity to explicitly show that $\mathbf{N}_{\mathrm{rig},L}(V)$

is equipped with an action of Γ_L (see Proposition 4.19). Furthermore, we show that our construction is compatible with the theory of (overconvergent) étale (φ, Γ_L)-modules from [5,6], establishing the naturality of the action of Γ_L on $\mathbf{N}_{\mathrm{rig},L}(V)$ (see Section 4.4). Next, we set

$$\mathbf{N}_L(V) := \mathbf{N}_{\mathrm{rig},L}(V) \cap \mathbf{D}_L^\dagger(V) \subset \mathbf{D}_{\mathrm{rig},L}^\dagger(V)$$

as a module over $\mathbf{B}_L^+ = \mathbf{A}_L^+[1/p]$, where $\mathbf{D}_L^\dagger(V)$ is the overconvergent étale (φ, Γ_L)-module associated to V and $\mathbf{D}_{\mathrm{rig},L}^\dagger(V)$ is the slope zero (φ, Γ_L)-module associated to V over the Robba ring (see Section 2.2 and Definition 4.24). Finally, we set $\mathbf{N}_L(T) := \mathbf{N}_L(V) \cap \mathbf{D}_L(T) \subset \mathbf{D}_L(V)$ as an $\mathbf{A}_L^+$-module and show that it satisfies the axioms of Definition 1.4 (see the proof of Theorem 4.1 in Section 4.5). In the opposite direction, starting with a Wach module N over $\mathbf{A}_L^+$, we use ideas developed in [3] to show that $\mathbf{T}_L(N)[1/p]$ is crystalline (see Theorem 3.12).

1.2.4. Relation to other works. Our first main result, Theorem 1.6, is a direct generalisation of Theorem 1.3 from [8,20,46]. As indicated in Section 1.2.3, starting with a crystalline $\mathbb{Z}_p$-representation T of G_L, the construction of a finite $[p]_q$-height module $\mathbf{N}_L(T)$ uses classical Wach modules and its compatibility with the results of [35,36]. However, equipping $\mathbf{N}_L(T)$ with a natural action of Γ_L is highly non-trivial, in particular, it does not follow from previous works and constitutes the heart of this article. For the converse, starting with a Wach module N over $\mathbf{A}_L^+$, we use ideas from [3] to show that $\mathbf{T}_L(N)[1/p]$ is crystalline. Moreover, as mentioned earlier, the results on Wach modules in the current paper are different from the theory of Breuil–Kisin modules in the imperfect residue field case studied in [16].

Now, let us note that using the unpublished results of Tsuji in [44] and the use of [16] in [23], it can be seen that the current paper is a crucial input to the construction of relative Wach modules in [1]. Moreover, recent developments in the theory of prismatic F-crystals [12,23,30], would suggest that there is a categorical equivalence between the category of Wach modules over $\mathbf{A}_L^+$ and the category of prismatic F-crystals on the absolute prismatic site of O_L. At this point, let us remark that unlike the case of Breuil–Kisin modules from [23], obtaining the aforementioned equivalence directly is a difficult question, in particular, it is highly non-trivial to directly show that the natural functor from prismatic F-crystals to Wach modules is essentially surjective. This point will be explored in another work [2] and the current article is independent of the results in the prismatic theory.

As indicated previously, the motivation for interpreting a Wach module as a q-de Rham complex and as q-deformation of crystalline cohomology, i.e., $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$, comes from [27, Section B.2.3] and [8, Théorème III.4.4]. Our second main result, Theorem 1.8, is an important step towards verifying such expectations. In addition, we note that our proof of Theorem 1.8 is entirely independent to that of loc. cit., thus providing an alternative proof (as well as a generalisation) of the important classical result in loc. cit. Furthermore, in Proposition 4.19 and Corollary 4.22 (see Remark 4.23), we generalise some results of [7,8] to obtain comparison results between Wach modules, overconvergent étale (φ, Γ_L)-modules and filtered (φ, ∂)-modules associated to p-adic crystalline representations. In particular, for a p-adic crystalline representation V of G_L, we prove a comparison isomorphism between the associated (φ, Γ_L)-module over the Robba ring and the scalar extension of $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ to the Robba ring, where we use the connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ to equip the scalar extension with an action of Γ_L (see Section 4.3 and Remark 4.23).

Finally, let us remark that using the theory of Breuil–Kisin modules in the imperfect residue field case from [16], in [29], Gao studied lattices inside crystalline (more generally, semistable) representations using Breuil–Kisin G_L-modules. However, the objects of loc. cit. are very different from Wach modules considered in this paper. More specifically, Breuil–Kisin G_L-modules are defined using the "Kummer tower" and admit an action of the big Galois group G_L. In contrast, Wach modules are defined using the "cyclotomic tower", as in the theory of étale (φ, Γ)-modules, and admit an action of Γ_L, which is much smaller than G_L. Moreover, [29] only proves a full faithfulness result, whereas Theorem 1.6 proves a categorical equivalence which was a difficult open question.

1.3. Setup and notations

We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers. Let p be a fixed prime number, κ a perfect field of characteristic p, $O_F := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F := O_F[1/p]$, the fraction field of W. In particular, F is an unramified extension of $\mathbb{Q}_p$ with ring of integers O_F. Let $\overline{F}$ be a fixed algebraic closure of F so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Furthermore, we denote by $G_F := \mathrm{Gal}(\overline{F}/F)$, the absolute Galois group of F.

We fix $d \in \mathbb{N}$ and let $X_1, X_2, \ldots, X_d$ be some indeterminates. Set $R^\square$ to be the p-adic completion of $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. Let $\varphi\colon R^\square \to R^\square$ denote a morphism extending the natural Frobenius on O_F by setting $\varphi(X_i) = X_i^p$, for all $1 \leqslant i \leqslant d$. The endomorphism φ of $R^\square$ is flat by [15, Lemma 7.1.5] and faithfully flat since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \subset R^\square$. Moreover, it is finite of degree p^d using Nakayama Lemma and the fact that φ modulo p is evidently of degree p^d. Let $O_{L^\square} := (R^\square_{(p)})^\wedge$, where $^\wedge$ denotes the p-adic completion. It is a complete discrete valuation ring with uniformiser p, imperfect residue field $\kappa(X_1, \ldots, X_d)$ and fraction field $L^\square := O_{L^\square}[1/p]$. The Frobenius on $R^\square$ extends to a unique faithfully flat and finite of degree p^d Frobenius endomorphism $\varphi\colon O_{L^\square} \to O_{L^\square}$, lifting the absolute Frobenius on $O_{L^\square}/pO_{L^\square}$.

Let O_L denote a finite étale extension of $O_{L^\square}$ such that it is a domain. Then O_L is a complete discrete valuation ring with uniformiser p, imperfect residue field a finite étale extension of $\kappa(X_1, \ldots, X_d)$ and fraction field $L := O_L[1/p]$. Fix an algebraic closure $\overline{L}/L$ and let $G_L := \mathrm{Gal}(\overline{L}/L)$ denote the absolute Galois group. The Frobenius on $O_{L^\square}$ extends to a unique faithfully flat and finite of degree p^d Frobenius endomorphism $\varphi\colon O_L \to O_L$ lifting the absolute Frobenius on O_L/pO_L (see [22, Proposition 2.1]). For $k \in \mathbb{N}$, let $\Omega^k_{O_L}$ denote the p-adic completion of the module of k-differentials of O_L relative to $\mathbb{Z}$. Then, we have that

$$\Omega^1_{O_L} = \bigoplus_{i=1}^{d} O_L\, d\log X_i \quad \text{and} \quad \Omega^k_{O_L} = \wedge^k_{O_L} \Omega^1_{O_L}.$$

Next, let K be one of the fields F_∞, L_∞, $\overline{F}$ or $\overline{L}$, where we have $F_\infty := F(\mu_{p^\infty})$ and $L_\infty := \bigcup_{i=1}^{d} L(\mu_{p^\infty}, X_i^{1/p^\infty})$, and set O_K to be the ring of integers of K. Then, the tilt of O_K is defined as $O_K^\flat := \lim_\varphi O_K/pO_K$, and the tilt of K is defined as $K^\flat := \mathrm{Frac}(O_K^\flat)$ (see [25, Chapitre V, Section 1.4]). Finally, let A be a $\mathbb{Z}_p$-algebra equipped with a Frobenius endomorphism φ lifting the absolute Frobenius on A/pA, then for any A-module M we write $\varphi^*(M) := A \otimes_{\varphi,A} M$.

1.4. Outline of the paper

This article consists of three main sections. In Section 2, we collect relevant results on p-adic Hodge theory in the imperfect residue field case. In Section 2.1, we define several period rings, in particular, we recall crystalline period rings, (φ, Γ)-module theory rings, overconvergent rings and Robba rings and prove several important technical results to be used in our main proofs in Section 4. In Section 2.2, we quickly recall the relation between p-adic representations and (φ, Γ)-module theory over the period rings described in the previous section. In Section 2.3, we focus on crystalline representations and prove some results relating the Galois action on a crystalline representation to its associated filtered (φ, ∂)-module. The goal of Section 3 is to define Wach modules in the imperfect residue field case and study the associated $\mathbb{Z}_p$-representations of G_L. In Section 3.1, we give the definition of Wach modules and relate it to étale (φ, Γ)-modules (see Proposition 3.3). Then, given a Wach module, we functorially associate to it a $\mathbb{Z}_p$-representation of G_L and in Section 3.2, we show that these are related to finite $[p]_q$-height representations studied in [3]. Finally, in Section 3.3, we show that the $\mathbb{Z}_p$-representation of G_L, associated to a Wach module, is a lattice inside a p-adic crystalline representation of G_L (see Theorem 3.12) and prove the filtered isomorphism claimed in Theorem 1.8. In Section 4, we prove our main result, i.e., Theorem 1.6. In Section 4.1, we collect important properties of classical Wach modules, i.e., the perfect residue field case. In Section 4.2, we use ideas from [35, 36] to construct a finite $[p]_q$-height module on the open unit disk over L. On the module thus obtained, we use results of Section 2.3 to construct an action of Γ_L and study its properties in Section 4.3. Then, in Section 4.4, we check that our construction is compatible with the theory of étale (φ, Γ_L)-modules. Finally, in Section 4.5, we construct the promised Wach module and prove Theorem 1.6.

2. Period rings and p-adic representations

We will use the setup and notations from Section 1.3. Recall that O_L is a finite étale algebra over $O_{L^\square}$. Set $L_\infty := \bigcup_{i=1}^{d} L(\mu_{p^\infty}, X_i^{1/p^\infty})$ and for $1 \leqslant i \leqslant d$, fix $X_i^\flat := (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots)$ in $O_{L_\infty}^\flat$. Then, we have the following Galois groups (see [32, Section 1.1] for details):

$$G_L := \mathrm{Gal}(\overline{L}/L), \quad H_L := \mathrm{Gal}(\overline{L}/L_\infty),$$
$$\Gamma_L := G_L/H_L = \mathrm{Gal}(L_\infty/L) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,$$
$$\Gamma'_L := \mathrm{Gal}\left(L_\infty/L(\mu_{p^\infty})\right) \xrightarrow{\sim} \mathbb{Z}_p(1)^d, \quad \mathrm{Gal}\left(L(\mu_{p^\infty})/L\right) = \Gamma_L/\Gamma'_L \xrightarrow{\sim} \mathbb{Z}_p^\times.$$

Let $O_{\breve{L}} := (\bigcup_{i=1}^{d} O_L[X_i^{1/p^\infty}])^\wedge$, where $^\wedge$ denotes the p-adic completion. The O_L-algebra $O_{\breve{L}}$ is a complete discrete valuation ring with perfect residue field, uniformiser p and fraction field $\breve{L} := O_{\breve{L}}[1/p]$. The Witt vector Frobenius on $O_{\breve{L}}$ is given by the Frobenius on O_L described in Section 1.3 and setting $\varphi(X_i^{1/p^n}) = X_i^{1/p^{n-1}}$, for all $1 \leqslant i \leqslant d$ and $n \geqslant 1$. Let $\breve{L}_\infty := \breve{L}(\mu_{p^\infty})$ and let $\overline{\breve{L}} \supset \overline{L}$ denote a fixed algebraic closure of $\breve{L}$. Then, we have the following Galois groups:

$$
\begin{aligned}
G_{\breve{L}} &:= \mathrm{Gal}(\overline{\breve{L}}/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}\Big(\overline{L}/\bigcup_{i=1}^{d} L\big(X_i^{1/p^\infty}\big)\Big),\\
H_{\breve{L}} &:= \mathrm{Gal}(\overline{\breve{L}}/\breve{L}_\infty) \xrightarrow{\sim} \mathrm{Gal}(\overline{L}/L_\infty),\\
\Gamma_{\breve{L}} &:= G_{\breve{L}}/H_{\breve{L}} = \mathrm{Gal}(\breve{L}_\infty/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}\Big(L_\infty/\bigcup_{i=1}^{d} L\big(X_i^{1/p^\infty}\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\sim} \mathrm{Gal}\big(L(\mu_{p^\infty})/L\big) \xrightarrow{\sim} \mathbb{Z}_p^\times.
\end{aligned}
$$

From the description above note that $G_{\breve{L}}$ may be identified with a subgroup of G_L, $H_{\breve{L}} \xrightarrow{\sim} H_L$ and $\Gamma_{\breve{L}}$ may be identified with a quotient of Γ_L.

2.1. Period rings

In this section, we will quickly recall and fix notations for all the period rings that will be used in this article. For details on constructions of period rings, please refer to [5, 14, 40].

As we will recall many period rings in this subsection, let us first briefly mention the usefulness of some of those rings in the constructions carried out for our main results (for precise definitions, please refer to Sections 2.1.1–2.1.5).

Remark 2.1. The period rings defined in Section 2.1.1, for example, $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$, $\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$, $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$, etc. will be used to define and study properties of crystalline representations of G_L (see Section 2.3), to show that Wach modules in the imperfect residue field case are crystalline (see Section 3.3), and to study the action of Γ_L on various scalar extensions of a Wach module associated to a crystalline representation (see Section 4.3). Note that Wach modules are certain (φ, Γ_L)-modules and the rings introduced in Section 2.1.2 provide the basic setup for defining these objects and studying their properties. In particular, we remark that an (integral) Wach module $\mathbf{N}_L(T)$, associated to a crystalline $\mathbb{Z}_p$-representation T of G_L, lives over the ring $\mathbf{A}_L^+$, and the étale (φ, Γ_L)-module associated to T lives over $\mathbf{A}_L$ (see Sections 2.2 and 3.1). Next, the overconvergent period rings from Section 2.1.3 will be used to define overconvergent étale (φ, Γ)-modules over $\mathbf{A}_L^\dagger$ (see Section 2.2), which will be a crucial input for the construction of the Wach module associated to a crystalline representation of G_L (see Section 4.5), and will be used to check that our constructions are compatible with the theory of étale (φ, Γ_L)-modules (see Section 4.4). Furthermore, the analytic rings of Section 2.1.4 will be the most important technical input for our constructions. For example, as a first step in our construction of $\mathbf{N}_L(T)$, we construct an intermediate φ-module $\mathbf{N}_{\mathrm{rig},L}(V)$ (where $V = T[1/p]$), over the ring $\mathbf{B}_{\mathrm{rig},L}^+$ using some ideas of Kisin (see Section 4.2). Additionally, to equip $\mathbf{N}_{\mathrm{rig},L}(V)$

with an action of Γ_L, we use the rings $\mathbf{B}^+_{\mathrm{rig},L}$, $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}$, $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$, etc.; this is the main technical innovation of this article (see Proposition 4.19). Moreover, the rings such as $\mathbf{B}^\dagger_{\mathrm{rig},L}$ and $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L}$ are used to study the compatibility of $\mathbf{N}_{\mathrm{rig},L}(V)$ with the theory of (φ, Γ_L)-modules of Andreatta (see Section 4.4). Finally, the corresponding period rings over $\breve{L}$ in Section 2.1.5 are helpful in recollecting the results on classical Wach modules which are crucial inputs to our constructions (see Sections 4.1 and 4.2).

2.1.1. Crystalline period rings. We set $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) := W(O^\flat_{L_\infty})$ and $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) := W(O^\flat_{\overline{L}})$ equipped with the Frobenius on Witt vectors and continuous G_L-action (for the weak topology). We fix $\bar{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$ is in $O^\flat_{F_\infty}$ with ζ_{p^n} being a primitive p^n-th root of unity, for each $n \geqslant 1$. Set $\mu := [\varepsilon] - 1$ and $\xi := \mu/\varphi^{-1}(\mu)$ in $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$. For any g in G_L, we have that $g(1+\mu) = (1+\mu)^{\chi(g)}$, where χ is the p-adic cyclotomic character. Moreover, we have a G_L-equivariant surjection $\theta\colon \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \to O_{\mathbb{C}_L}$, where $\mathbb{C}_L := \widehat{\overline{L}}$ and $O_{\mathbb{C}_L}$ is its ring of integers; note that $\operatorname{Ker}\theta = \xi \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$. The map θ further induces a Γ_L-equivariant surjection $\theta\colon \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \to O_{\hat{L}_\infty}$.

Recall that, for $1 \leqslant i \leqslant d$, we fixed $X_i^\flat = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots)$ in $O^\flat_{L_\infty}$ and we take $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ to be topological generators of Γ_L such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ'_L and γ_0 is a topological generator of Γ_L/Γ'_L and $\gamma_j(X_i^\flat) = \varepsilon X_i^\flat$, if $i = j$, and $X_i^\flat$, otherwise. Let us also fix Teichmüller lifts $[X_i^\flat]$ in $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$. We set $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})\langle \xi^k/k!,\ k \in \mathbb{N}\rangle$. Let $t := \log(1+\mu)$ which converges in $\mathbf{A}_{\mathrm{cris}}(O_{F_\infty})$ and set $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})[1/p]$ and $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})[1/t]$. For any g in G_L, we have that $g(t) = \chi(g)t$. Furthermore, one can define period rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$. These rings are equipped with a Frobenius endomorphism φ and a continuous Γ_L-action, and the former two rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ and $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ are further equipped with an appropriate extension of the map θ. Rings with a subscript "cris" are equipped with a decreasing filtration and rings with a prefix "$\mathcal{O}$" are further equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration (see [3, Section 2.2] for definitions over R with similar notations). One can define variations of these rings over $\overline{L}$ which are further equipped with a continuous G_L-action. Moreover, from [39, Lemma 4.32], note that

$$\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) = \mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})^{H_L} \quad \text{and} \quad \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) = \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})^{H_L}.$$

We have two O_L-algebra structures on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$: a canonical structure coming from the definition of $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ and a non-canonical $(\varphi, \Gamma_{\breve{L}})$-equivariant structure $O_L \to \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ given by the map

$$x \longmapsto \sum_{\mathbf{k} \in \mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d \big([X_i^\flat] - X_i\big)^{[k_i]},$$

where $\partial_i := \frac{\partial}{\partial X_i}$ is a differential operator defined over O_L, for $1 \leqslant i \leqslant d$. In particular, under the preceding map, we have that $X_i \mapsto [X_i^\flat]$.

2.1.2. Rings of (φ, Γ)-modules. For detailed explanations of objects defined in this subsubsection, see [5]. Recall that $O_{L^\square}$ is a complete discrete valuation ring with a uniformiser p and an imperfect residue field, and O_L is a finite étale $O_{L^\square}$-algebra. Let us set $\mathbf{A}_{L^\square}^+$ to be the (p, μ)-adic completion of the localisation $O_F[\![\mu]\!][[X_1^\flat]^{\pm 1}, \dots, [X_d^\flat]^{\pm 1}]_{(p,\mu)}$. We have a natural embedding $\mathbf{A}_{L^\square}^+ \hookrightarrow \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ and $\mathbf{A}_{L^\square}^+$ is stable under the Witt vector Frobenius and Γ_L-action on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$; we equip $\mathbf{A}_{L^\square}^+$ with induced structures. Moreover, we have an injective homomorphism of rings $\iota \colon O_{L^\square} \to \mathbf{A}_{L^\square}^+$, via the map $X_i \mapsto [X_i^\flat]$, and it extends to an isomorphism of rings $O_{L^\square}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_{L^\square}^+$. Equip $O_{L^\square}[\![\mu]\!]$ with a faithfully flat and finite of degree p^{d+1} Frobenius endomorphism using the Frobenius on $O_{L^\square}$ and by setting $\varphi(\mu) = (1+\mu)^p - 1$. Then, the injective homomorphism ι and the isomorphism $O_{L^\square}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_{L^\square}^+$ are Frobenius-equivariant.

Let $\mathbf{A}_L^+$ denote the (p, μ)-adic completion of the unique extension of the embedding $\mathbf{A}_{L^\square}^+ \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ along the finite étale map $O_{L^\square} \to O_L$ (see [22, Proposition 2.1]). We have a natural embedding of O_F-algebras $\mathbf{A}_L^+ \hookrightarrow \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ and $\mathbf{A}_L^+$ is stable under the induced Frobenius and Γ_L-action. Note that the injective homomorphism of rings $\iota \colon O_{L^\square} \to \mathbf{A}_{L^\square}^+ \subset \mathbf{A}_L^+$ and the isomorphism $O_{L^\square}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_{L^\square}^+ \subset \mathbf{A}_L^+$, respectively, extend to a unique injective homomorphism of rings $\iota \colon O_L \to \mathbf{A}_L^+$ and an isomorphism $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_L^+$. Equip $O_L[\![\mu]\!]$ with a faithfully flat and finite of degree p^{d+1} Frobenius endomorphism using the Frobenius on O_L and by setting $\varphi(\mu) = (1+\mu)^p - 1$. Then, the injective homomorphism ι and the isomorphism $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_L^+$ are Frobenius-equivariant. In particular, the Frobenius morphism $\varphi \colon \mathbf{A}_L^+ \to \mathbf{A}_L^+$ is faithfully flat and finite of degree p^{d+1}. Let $u_\alpha := (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$, where $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_d)$ is a d-tuple with α_i in $\{0, 1, \dots, p-1\}$, for $0 \leqslant i \leqslant d$. Then, we have that $\varphi^*(\mathbf{A}_L^+) := \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L^+ \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}_L^+) u_\alpha$.

Recall that $\mathbb{C}_L = \hat{\overline{L}}$ and set $\tilde{\mathbf{A}} := W(\mathbb{C}_L^\flat)$ and $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[1/p]$ admitting the Frobenius on Witt vectors and continuous G_L-action (for the weak topology). Set $\mathbf{A}_L := \mathbf{A}_L^+[1/\mu]^\wedge$, where $^\wedge$ denotes the p-adic completion; equip $\mathbf{A}_L^+$ with the induced Frobenius endomorphism and continuous Γ_L-action. Note that $\mathbf{A}_L$ is a complete discrete valuation ring with maximal ideal $p\mathbf{A}_L$, residue field $(O_L/p)((\mu))$ and fraction field $\mathbf{B}_L := \mathbf{A}_L[1/p]$. Similar to above, $\varphi \colon \mathbf{A}_L \to \mathbf{A}_L$ is faithfully flat and finite of degree p^{d+1} and we have that

$$\begin{aligned}\varphi^*(\mathbf{A}_L) := \mathbf{A}_L \otimes_{\varphi, \mathbf{A}_L} \mathbf{A}_L \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}_L) u_\alpha = \big(\oplus_\alpha\, \varphi(\mathbf{A}_L^+) u_\alpha\big) \otimes_{\varphi(\mathbf{A}_L^+)} \varphi(\mathbf{A}_L) \\ \xleftarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L.\end{aligned}$$

Furthermore, we have a natural Frobenius and Γ_L-equivariant embedding $\mathbf{A}_L \subset \tilde{\mathbf{A}}^{H_L}$. Let $\mathbf{A}$ denote the p-adic completion of the maximal unramified extension of $\mathbf{A}_L$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B} := \mathbf{A}[1/p] \subset \tilde{\mathbf{B}}$, i.e., $\mathbf{A}$ is the ring of integers of $\mathbf{B}$. The rings $\mathbf{A}$ and $\mathbf{B}$ are stable under the induced Frobenius and G_L-action, and we have $\mathbf{A}_L = \mathbf{A}^{H_L}$ and $\mathbf{B}_L = \mathbf{B}^{H_L}$ stable under the induced Frobenius and residual Γ_L-action.

2.1.3. Overconvergent rings. We begin by defining the ring of overconvergent coefficients stable under Frobenius and G_L-action (see [6, 19]). Denote the natural valuation on

$O_{\overline{L}}^{\flat}$ by $\upsilon^{\flat}$ extending the valuation on $O_{\overline{F}}^{\flat}$. Let $r \in \mathbb{Q}_{>0}$ and set,

$$\widetilde{\mathbf{A}}^{\dagger,r} := \Big\{ \sum_{k \in \mathbb{N}} p^k [x_k] \in \widetilde{\mathbf{A}} \text{ such that } \upsilon^{\flat}(x_k) + \tfrac{pr}{p-1} k \to +\infty \text{ as } k \to +\infty \Big\}.$$

The continuous G_L-action and Frobenius φ on $\widetilde{\mathbf{A}}$ induce commuting actions of G_L and φ on $\widetilde{\mathbf{A}}^{\dagger,r}$ such that $\varphi(\widetilde{\mathbf{A}}^{\dagger,r}) = \widetilde{\mathbf{A}}^{\dagger,pr}$. Define the ring of *overconvergent coefficients* as $\widetilde{\mathbf{A}}^{\dagger} := \bigcup_{r \in \mathbb{Q}_{>0}} \widetilde{\mathbf{A}}^{\dagger,r} \subset \widetilde{\mathbf{A}}$ equipped with the induced Frobenius and continuous G_L-action. Moreover, inside $\widetilde{\mathbf{A}}$ we take $\mathbf{A}_L^{\dagger,r} := \mathbf{A}_L \cap \widetilde{\mathbf{A}}^{\dagger,r}$ and $\mathbf{A}^{\dagger,r} := \mathbf{A} \cap \widetilde{\mathbf{A}}^{\dagger,r}$. Define $\mathbf{A}_L^{\dagger} := \mathbf{A}_L \cap \widetilde{\mathbf{A}}^{\dagger} = \bigcup_{r \in \mathbb{Q}_{>0}} \mathbf{A}_L^{\dagger,r}$ and $\mathbf{A}^{\dagger} := \mathbf{A} \cap \widetilde{\mathbf{A}}^{\dagger} = \bigcup_{r \in \mathbb{Q}_{>0}} \mathbf{A}^{\dagger,r}$, equipped with the induced Frobenius endomorphism and continuous G_L-action from the respective actions on $\widetilde{\mathbf{A}}$; we have $\mathbf{A}_L^{\dagger} = (\mathbf{A}^{\dagger})^{H_L}$. Upon inverting p in the definitions above one obtains $\mathbb{Q}_p$-algebras inside $\widetilde{\mathbf{B}}$, i.e., set $\widetilde{\mathbf{B}}^{\dagger,r} := \widetilde{\mathbf{A}}^{\dagger,r}[1/p]$, $\widetilde{\mathbf{B}}^{\dagger} := \widetilde{\mathbf{A}}^{\dagger}[1/p]$, $\mathbf{B}^{\dagger,r} := \mathbf{A}^{\dagger,r}[1/p]$, $\mathbf{B}^{\dagger} := \mathbf{A}^{\dagger}[1/p]$, equipped with the induced Frobenius and G_L-action. Moreover, set $\widetilde{\mathbf{B}}_L^{\dagger,r} := (\widetilde{\mathbf{B}}^{\dagger,r})^{H_L}$, $\widetilde{\mathbf{B}}_L^{\dagger} := (\widetilde{\mathbf{B}}^{\dagger})^{H_L}$, $\mathbf{B}_L^{\dagger,r} := (\mathbf{B}^{\dagger,r})^{H_L} = \mathbf{A}_L^{\dagger,r}[1/p]$ and $\mathbf{B}_L^{\dagger} := (\mathbf{B}^{\dagger})^{H_L} = \mathbf{A}_L^{\dagger}[1/p]$, equipped with the induced Frobenius and residual Γ_L-action.

2.1.4. Analytic rings. In this section, we will define the Robba ring over L following [34, Section 2] and [41, Section 1]. However, we will use the notations of [7, Section 2] in the perfect residue field case (see [41, Section 1.10] for compatibility between different notations). Define

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} := \bigcup_{r \geqslant 0} \bigcap_{s \geqslant r} \big(\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})\langle \tfrac{p}{[\overline{\mu}]^r}, \tfrac{[\overline{\mu}]^s}{p} \rangle\big[\tfrac{1}{p}\big]\big).$$

The ring $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ can also be defined as $\bigcup_{r \in \mathbb{Q}_{>0}} \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$, where $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ denotes the Fréchet completion of $\widetilde{\mathbf{B}}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger,r}[1/p]$ for a certain family of valuations (see [34, Section 2] and [41, Section 1.6]). The Frobenius and G_L-action on $\widetilde{\mathbf{B}}^{\dagger,r}$, respectively, induce Frobenius and G_L-action on $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$, which extend to respective actions on $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$. In particular, we have a Frobenius and G_L-equivariant inclusion $\widetilde{\mathbf{B}}^{\dagger} \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ (see [41, Sections 1.6 and 1.10]). Set

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+} := \bigcap_{n \in \mathbb{N}} \varphi^n \big(\mathbf{B}_{\mathrm{cris}}^{+}(O_{\overline{L}})\big),$$

equipped with an induced Frobenius endomorphism and G_L-action from the respective actions on $\mathbf{B}_{\mathrm{cris}}^{+}(O_{\overline{L}})$. The descriptions of rings in [7, Lemme 2.5, Exemple 2.8, Section 2.3] directly extend to our situation as the aforementioned results do not depend on structure of the residue field of the base ring O_L. Therefore, from loc. cit. it follows that we have a natural inclusion $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+} \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ compatible with Frobenius and G_L-action. Moreover, we set $\widetilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger,r} := (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r})^{H_L}$, $\widetilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger} := (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger})^{H_L}$ and $\widetilde{\mathbf{B}}_{\mathrm{rig},L}^{+} := (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+})^{H_L} \subset \widetilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger}$, equipped with the induced Frobenius endomorphism and residual Γ_L-action.

Remark 2.2. Note that the definition of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ and $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+}$ as rings does not depend on L, in particular, one may define these rings using $\mathbf{A}_{\mathrm{inf}}(O_{\overline{\overline{L}}})$ and equip them with a Frobenius endomorphism compatible with the Frobenius endomorphism defined above.

Lemma 2.3. *We have* $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{\varphi=1} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{\varphi=1} = \mathbb{Q}_p$.

Proof. Using Remark 2.2, note that the Frobenius invariant elements can be computed using the corresponding results in the perfect residue field case. In particular, we have that $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{\varphi=1} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{\varphi=1} = \mathbb{Q}_p$, where the first equality follows from [8, Proposition I.4.1] and the second equality follows from [21, Proposition 9.15]. ∎

Recall that from Section 2.1.2 we have a Frobenius-equivariant injective homomorphism of rings $\iota\colon O_L \to \mathbf{A}_L^+$. Then, from [41, Section 1.6] the ring $\mathbf{A}_L^{\dagger,r}$ admits the following description:

$$\mathbf{A}_L^{\dagger,r} \xrightarrow{\sim} \Big\{\sum_{k\in\mathbb{Z}} \iota(a_k)\mu^k \text{ such that } a_k \in O_L \text{ and for any } \tfrac{1}{p^{1/r}} \leqslant \rho < 1,\ \lim_{k\to-\infty} |a_k|\rho^k = 0\Big\}.$$

Moreover, we have that $\mathbf{B}_L^{\dagger,r} = \mathbf{A}_L^{\dagger,r}[1/p]$ and we set

$$\mathbf{B}^{\dagger,r}_{\mathrm{rig},L} := \Big\{\sum_{k\in\mathbb{Z}} \iota(a_k)\mu^k \text{ such that } a_k \in L \text{ and for any } \tfrac{1}{p^{1/r}} \leqslant \rho < 1,\ \lim_{k\to\pm\infty} |a_k|\rho^k = 0\Big\}.$$

The ring $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ can also be defined as the Fréchet completion of $\mathbf{B}_L^{\dagger,r}$ for a family of valuations induced by the inclusion $\mathbf{B}_L^{\dagger,r} \subset \widetilde{\mathbf{B}}^{\dagger,r}$ (see [34, Section 2], [41, Section 1.6]). Define the *Robba ring* over L as $\mathbf{B}^{\dagger}_{\mathrm{rig},L} := \bigcup_{r\geqslant 0} \mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$. The Frobenius and G_L-action on $\mathbf{B}_L^{\dagger,r}$ induce respective Frobenius and G_L-action on $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$, which extend to respective actions on $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ (also see [41, Section 4.3] where Ohkubo constructs the differential action of $\operatorname{Lie}\Gamma_L$; one may also obtain the action of Γ_L by exponentiating the action of $\operatorname{Lie}\Gamma_L$). From the preceding discussion, we have a Frobenius and Γ_L-equivariant injection $\mathbf{B}_L^{\dagger} \subset \mathbf{B}^{\dagger}_{\mathrm{rig},L}$ and the former ring $\mathbf{B}_L^{\dagger}$ is also known as the *bounded Robba ring*. Furthermore, note that $\mathbf{B}_L^{\dagger,r} \subset \widetilde{\mathbf{B}}_L^{\dagger,r} = (\widetilde{\mathbf{B}}^{\dagger,r})^{H_L} \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},L}$, where the last term can also be described as the Fréchet completion of the middle term for a family of valuations induced by the inclusion $\widetilde{\mathbf{B}}_L^{\dagger,r} \subset \widetilde{\mathbf{B}}^{\dagger,r}$ (see [34, Section 2] and [41, Section 1.6]).

To summarise, for $r \in \mathbb{Q}_{>0}$, we have the following commutative diagram with injective arrows:

$$\begin{tikzcd}
\mathbf{B}^{\dagger,r} \arrow[r] & \widetilde{\mathbf{B}}^{\dagger,r} \arrow[dr] & \\
\mathbf{B}_L^{\dagger,r} \arrow[r] \arrow[u] \arrow[d] & \widetilde{\mathbf{B}}_L^{\dagger,r} \arrow[r] \arrow[u] \arrow[d] & \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}, \\
\mathbf{B}^{\dagger,r}_{\mathrm{rig},L} \arrow[r] & \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},L} \arrow[ur] &
\end{tikzcd}$$

where in the second row, the two rings on the left are obtained by taking H_L-invariants of the corresponding rings in the first row and the rightmost ring in the second row is obtained as the Fréchet completion of the rightmost ring in the first row. The bottom row is obtained as the Fréchet completion of the two rings on the left in the second row. These inclusions are compatible with the respective Frobenii and Γ_L-actions and these compatibilities are preserved after passing to the respective Fréchet completions. In particular, we have a Frobenius and Γ_L-equivariant embedding $\mathbf{B}^{\dagger}_{\mathrm{rig},L} \subset \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L}$.

Definition 2.4. Define $\mathbf{B}^{+}_{\mathrm{rig},L} := \mathbf{B}^{\dagger}_{\mathrm{rig},L} \cap \widetilde{\mathbf{B}}^{+}_{\mathrm{rig},L} \subset \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L}$, equipped with the induced natural Frobenius endomorphism and Γ_L-action.

Lemma 2.5. *The ring $\mathbf{B}^{+}_{\mathrm{rig},L}$ may be identified with the ring of convergent power series over the open unit disk in one variable over L, i.e.,*

$$\mathbf{B}^{+}_{\mathrm{rig},L} \xrightarrow{\sim} \Big\{ \sum_{k \in \mathbb{N}} \iota(a_k)\mu^k \text{ such that } a_k \in L \text{ and for any } 0 \leqslant \rho < 1, \lim_{k \to +\infty} |a_k|\rho^k = 0 \Big\},$$

Proof. Let x be any element of $\mathbf{B}^{+}_{\mathrm{rig},L} \subset \mathbf{B}^{\dagger}_{\mathrm{rig},L}$. Using the explicit description of $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ and $\mathbf{B}^{\dagger,r}_{L}$ for $r \in \mathbb{Q}_{>0}$, we can write $x = x^+ + x^-$, with x^+ convergent on the open unit disk over L and x^- in $\mathbf{B}^{\dagger,r}_{L}$, for some $r \in \mathbb{Q}_{>0}$, in particular, we have that x^+ is in $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$. Moreover, using Remark 2.2 and [7, Lemma 2.18, Corollaire 2.28], we have an exact sequence

$$0 \longrightarrow \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \longrightarrow \widetilde{\mathbf{B}}^{\dagger,r} \oplus \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \longrightarrow \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \longrightarrow 0,$$

where $\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) = \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/p]$. So, x is in $\mathbf{B}^{+}_{\mathrm{rig},L} \subset \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ if and only if $x^- = x - x^+$ is in $\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \cap \mathbf{B}^{\dagger,r}_{L} = \mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \cap \mathbf{B}^{\dagger,r}_{L} = \mathbf{B}^{+}_{L}$, where we have used that $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})^{H_L} = \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ (see [5, Proposition 7.2]). Hence, x converges on the open unit disk over L. The other inclusion is obvious, allowing us to conclude. ∎

Remark 2.6. The topology on $\mathbf{B}^{+}_{\mathrm{rig},L}$ can be described as follows: let $D(L,\rho)$ denote the closed disk of radius $0 < \rho < 1$ over L and let $\mathcal{O}(D(L,\rho))$ denote the ring of analytic functions, i.e., power series converging on the closed disk $D(L,\rho)$. Then, $\mathcal{O}(D(L,\rho))$ is equipped with a topology induced by the supremum norm $\|f\|_\rho := \sup_{x \in D(L,\rho)} |f(x)| < +\infty$. We have that $\mathbf{B}^{+}_{\mathrm{rig},L} = \lim_\rho \mathcal{O}(D(L,\rho)) \subset L[\![\mu]\!]$ and we equip it with the topology induced by the Fréchet limit of the topology on $\mathcal{O}(D(L,\rho))$ induced by the supremum norm, i.e., the topology on $\mathbf{B}^{+}_{\mathrm{rig},L}$ can be described by uniform convergence on $D(L,\rho)$ for $\rho \to 1^-$.

Lemma 2.7. *The natural map $\mathbf{B}^{+}_{L} \to \mathbf{B}^{+}_{\mathrm{rig},L}$ is faithfully flat.*

Proof. Note that $\mathbf{B}^{+}_{L}$ is a principal ideal domain and $\mathbf{B}^{+}_{\mathrm{rig},L}$ is a domain, so the map in the claim is flat. To show that it is faithfully flat, it is enough to show that for any maximal ideal $\mathfrak{m} \subset \mathbf{B}^{+}_{L}$, we have that $\mathfrak{m}\mathbf{B}^{+}_{\mathrm{rig},L} \neq \mathbf{B}^{+}_{\mathrm{rig},L}$. Note that if $\mathfrak{m} \subset \mathbf{B}^{+}_{L}$ is a maximal ideal, then $\mathfrak{m} = (f)$, where f is an irreducible distinguished polynomial in the sense of [37, Chapter 5, Section 2]. Since any f as above admits a zero over the open unit disk, therefore, it follows that f is not a unit in $\mathbf{B}^{+}_{\mathrm{rig},L}$. Hence, $\mathfrak{m}\mathbf{B}^{+}_{\mathrm{rig},L} \neq \mathbf{B}^{+}_{\mathrm{rig},L}$. ∎

Remark 2.8. From Section 1.3 recall that $\varphi\colon L \to L$ is finite of degree p^d and we also have that $\varphi(\mu) = (1+\mu)^p - 1$. Therefore, from the explicit description of $\mathbf{B}^{+}_{\mathrm{rig},L}$ in

Lemma 2.5, it follows that the Frobenius endomorphism $\varphi\colon \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ is faithfully flat and finite of degree p^{d+1}.

2.1.5. Period rings for $\check{L}$. Definitions above may be adopted almost verbatim to define corresponding period rings for $\check{L}$, in particular, one recovers definitions of period rings in [7, 19, 27], i.e., we have period rings $\mathbf{A}^+_{\check{L}}$, $\mathbf{A}_{\check{L}}$, $\mathbf{A}^\dagger_{\check{L}}$, $\mathbf{B}^+_{\mathrm{rig},\check{L}}$ and $\mathbf{B}^\dagger_{\mathrm{rig},\check{L}}$ equipped with a Frobenius endomorphism φ and $\Gamma_{\check{L}}$-action. Note that we have a natural identification $O_{\check{L}}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_{\check{L}}$, where the right hand side is equipped with a faithfully flat and finite of degree p Frobenius endomorphism using the natural Frobenius on $O_{\check{L}}$ and setting $\varphi(\mu) = (1+\mu)^p - 1$ and a $\Gamma_{\check{L}}$-action given as $g(\mu) = (1+\mu)^{\chi(g)} - 1$, for any g in $\Gamma_{\check{L}}$. Moreover, the preceding isomorphism naturally extends to a Frobenius and $\Gamma_{\check{L}}$-equivariant isomorphism $\mathbf{A}_{\check{L}} \xrightarrow{\sim} O_{\check{L}}[\![\mu]\!][1/\mu]^\wedge$, where $\wedge$ denotes the p-adic completion.

Similar to above, we further equip $O_L[\![\mu]\!]$ with an O_L-linear action of $\Gamma_{\check{L}}$, by setting $g(\mu) = (1+\mu)^{\chi(g)} - 1$, for any g in $\Gamma_{\check{L}}$. Then the isomorphism $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_L$ from Section 2.1.2, is Frobenius and $\Gamma_{\check{L}}$-equivariant. Now, recall that the Frobenius-equivariant embedding $O_L \to O_{\check{L}}$ is faithfully flat and it naturally extends to a Frobenius and $\Gamma_{\check{L}}$-equivariant faithfully flat embedding $O_L[\![\mu]\!] \to O_{\check{L}}[\![\mu]\!]$. So, using the preceding embedding and the Frobenius and $\Gamma_{\check{L}}$-equivariant isomorphisms – the inverse of $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_L$ and the isomorphism $O_{\check{L}}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_{\check{L}}$ – we obtain a Frobenius and $\Gamma_{\check{L}}$-equivariant faithfully flat embedding $\mathbf{A}^+_L \to \mathbf{A}^+_{\check{L}}$, sending $[X_i^\flat] \mapsto X_i$. This further extends to a Frobenius and $\Gamma_{\check{L}}$-equivariant faithfully flat embedding $\mathbf{A}_L \to \mathbf{A}_{\check{L}}$.

We will equip $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ with a non-canonical O_L-algebra structure by first defining an injection $O_{L^\square} \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$, via the map $X_i \mapsto [X_i^\flat]$, and then extending it uniquely along the finite étale map $O_{L^\square} \to O_L$, to an injection $O_L \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ (see [22, Proposition 2.1]). Note that the preceding maps are Frobenius-equivariant but not Γ_L-equivariant. Moreover, this O_L-algebra structure naturally extends to a Frobenius-equivariant $O_{\check{L}}$-algebra structure on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ by sending $X_i^{1/p^n} \mapsto [(X_i^{1/p^n})^\flat]$, for all $1 \leqslant i \leqslant d$ and $n \in \mathbb{N}$. We may further extend this to a Frobenius and $\Gamma_{\check{L}}$-equivariant embedding $\mathbf{A}^+_{\check{L}} = O_{\check{L}}[\![\mu]\!] \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$.

Using the embeddings described above and following the definitions of various period rings discussed so far, we obtain a commutative diagram with injective arrows where the top horizontal arrows are Frobenius and Γ_L-equivariant and the rest are Frobenius and $\Gamma_{\check{L}}$-equivariant:

$$\begin{tikzcd}
\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} & \mathbf{B}^+_{\mathrm{rig},L} \arrow[l] \arrow[r] \arrow[d] & \mathbf{B}^\dagger_{\mathrm{rig},L} \arrow[r] \arrow[d] & \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L} \\
& \mathbf{B}^+_{\mathrm{rig},\check{L}} \arrow[r] \arrow[ul] & \mathbf{B}^\dagger_{\mathrm{rig},\check{L}}. \arrow[ur] &
\end{tikzcd}$$

Remark 2.9. Similar to Lemma 2.5 we have that,

$$\mathbf{B}^+_{\mathrm{rig},\check{L}} \xrightarrow{\sim} \Big\{ \sum_{k\in\mathbb{N}} a_k \mu^k \text{ such that } a_k \in \check{L} \text{ and for any } 0 \leqslant \rho < 1,\ \lim_{k\to+\infty} |a_k|\rho^k = 0 \Big\}.$$

The ring $\mathbf{B}^+_{\mathrm{rig},\check{L}}$ is equipped with a Fréchet topology similar to Remark 2.6. Moreover, since

$$\varphi : \check{L} \xrightarrow{\sim} \check{L} \quad \text{and} \quad \varphi(\mu) = (1+\mu)^p - 1,$$

the Frobenius endomorphism on $\mathbf{B}^+_{\mathrm{rig},\check{L}}$ is faithfully flat and finite of degree p.

Lemma 2.10. *The rings $\mathbf{B}^+_{\mathrm{rig},L}$ and $\mathbf{B}^+_{\mathrm{rig},\check{L}}$ are Bézout domains and $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\check{L}}$ is flat.*

Proof. The first claim follows from [7, Proposition 4.12]. Note that loc. cit. assumes the residue field of the discrete valuation base field (L and $\check{L}$ in our case) to be perfect, however, the proof of loc. cit. only depends on [31, 38] which are independent of this assumption. For the second claim, note that we can write $\mathbf{B}^+_{\mathrm{rig},\check{L}} = \mathrm{colim}_{i \in I} M_i$, where I is the directed index set of finitely generated $\mathbf{B}^+_{\mathrm{rig},L}$-submodules of $\mathbf{B}^+_{\mathrm{rig},\check{L}}$. Since $\mathbf{B}^+_{\mathrm{rig},\check{L}}$ is a domain, M_i is torsion-free for each $i \in I$. Now, recall that finitely generated torsion-free modules over a Bézout domain are finite projective (see [17, Chapter VII, Proposition 4.1] noting that Bézout domains are a special case of Prüfer domains), and therefore finite free by [33, Proposition 2.5]. Moreover, by a theorem of Lazard (see [43, Tag 058G]), we know that a directed colimit of finite free modules over a ring is flat. Hence, it follows that $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\check{L}}$ is flat. ∎

Lemma 2.11. *The following element converges in $\mathbf{B}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{rig},\check{L}}$:*

$$\frac{t}{\mu} = \frac{\log(1+\mu)}{\mu} = \prod_{n \in \mathbb{N}} \Big(\frac{\varphi^n([p]_q)}{p}\Big).$$

Moreover, inside $\mathbf{B}^+_{\mathrm{rig},\check{L}}$, we have that $(t/\mu)\mathbf{B}^+_{\mathrm{rig},\check{L}} \cap \mathbf{B}^+_{\mathrm{rig},L} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},L}$.

Proof. The first claim follows from [8, Exemple I.3.3] and [38, Remarque 4.12]. For the second claim let $x = \sum_{k \in \mathbb{N}} x_k \mu^k$ in $\mathbf{B}^+_{\mathrm{rig},L}$, with $x_k \in L$, and let $y = \sum_{k \in \mathbb{N}} y_k \mu^k$ in $\mathbf{B}^+_{\mathrm{rig},\check{L}}$, with $y_k \in \check{L}$, such that $ty/\mu = x$. Write $t/\mu = \sum_{k \in \mathbb{N}} a_k \mu^k$, with $a_k \in \mathbb{Q}_p$. Then, we have that

$$\Big(\sum_{k \in \mathbb{N}} a_k \mu^k\Big)\Big(\sum_{k \in \mathbb{N}} y_k \mu^k\Big) = \sum_{k \in \mathbb{N}} x_k \mu^k.$$

We will show that y_k is in L, for all $k \in \mathbb{N}$, using induction. Indeed, note that $a_0 y_0 = x_0$ in L, so $y_0 = x_0/a_0$ is in L. Let $n \in \mathbb{N}$ and assume that y_k is in L, for every $k \leqslant n$. Then, we have that $\sum_{k=0}^{n+1} a_k y_{n+1-k} = x_{n+1}$ in L and by the induction assumption we get that $y_{n+1} = (x_{n+1} - \sum_{k=0}^{n} a_k y_{n+1-k})/a_0$ is in L. Hence, we conclude that y is in $\mathbf{B}^+_{\mathrm{rig},L}$, implying that $(t/\mu)\mathbf{B}^+_{\mathrm{rig},\check{L}} \cap \mathbf{B}^+_{\mathrm{rig},L} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},L}$. ∎

Lemma 2.12. *Inside $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, we have that $(t/\mu)\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},\check{L}} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},\check{L}}$, therefore, from Lemma 2.11 we get that $(t/\mu)\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},L} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},L}$.*

Proof. Let us first note that for each $n \in \mathbb{N}_{\geqslant 1}$ we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{B}^+_{\mathrm{rig},\check{L}} \arrow[r, "\varphi^{n-1}([p]_q)"] \arrow[d] & \mathbf{B}^+_{\mathrm{rig},\check{L}} \arrow[r, "\mu \mapsto \zeta_{p^n}-1"] \arrow[d] & \check{L}(\zeta_{p^n}) \arrow[r] \arrow[d, "\phi_{\check{L}}"] & 0 \\
0 \arrow[r] & \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \arrow[r, "\varphi^{n-1}([p]_q)"] & \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \arrow[r, "\theta \circ \varphi^{-n}"] & \mathbb{C}_L \arrow[r] & 0,
\end{tikzcd}$$

where the left and middle vertical arrows are natural inclusions, the right vertical arrow is $\phi_{\breve{L}}\colon \breve{L}(\zeta_{p^n}) \xrightarrow{\sim} \breve{L}(\zeta_{p^n}) \subset \mathbb{C}_L$, given as $\sum_{k=0}^{e-1} a_k \zeta_{p^n}^k \mapsto \sum_{k=0}^{e-1} \varphi_{\breve{L}}^{-n}(a_k)\zeta_{p^n}^k$, with $e = [\breve{L}(\zeta_{p^n})\colon \breve{L}]$ and $\varphi_{\breve{L}}\colon \breve{L} \xrightarrow{\sim} \breve{L}$ and $\theta\colon \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \subset \mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}}) \to \mathbb{C}_L$ from Section 2.1.1. The top row is obviously exact and the bottom row is exact by [7, Proposition 2.11, Proposition 2.12, Remarque 2.14]. All vertical maps are injective and hence we obtain that

$$\varphi^n\big([p]_q\big)\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \cap \mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \varphi^n\big([p]_q\big)\mathbf{B}_{\mathrm{rig},\breve{L}}^+, \quad \text{for all } n \in \mathbb{N},$$

in particular, $\varphi^n([p]_q)\widetilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap \mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \varphi^n([p]_q)\mathbf{B}_{\mathrm{rig},\breve{L}}^+$. Now, let x be in $(t/\mu)\widetilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap \mathbf{B}_{\mathrm{rig},\breve{L}}^+$ and write $x = ty/\mu$, for some y in $\widetilde{\mathbf{B}}_{\mathrm{rig},L}^+$. We will show that y is in $\mathbf{B}_{\mathrm{rig},\breve{L}}^+$ by showing that it converges over each closed disk $D(\breve{L},\rho)$, for $0 < \rho < 1$. Fix some $0 < \rho < 1$ and from Lemma 2.11, we write $t/\mu = \prod_{n\in\mathbb{N}}(\varphi^n([p]_q)/p) = \upsilon \prod_{n=0}^{m}(\varphi^n([p]_q)/p)$, for a unit $\upsilon \in \mathcal{O}(D(\breve{L},\rho))^\times$ and $m \in \mathbb{N}$ depending on ρ. Then, we have that $x = ([p]_q/p)y_1$, where

$$y_1 := \upsilon \prod_{n=1}^{m} \big(\varphi^n\big([p]_q\big)/p\big)y$$

is in $\widetilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap (p/[p]_q)\mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \mathbf{B}_{\mathrm{rig},\breve{L}}^+$. Repeating the preceding argument for $1 \leqslant k \leqslant m$, we obtain elements

$$y_{k+1} := \upsilon \prod_{n=k+1}^{m} \big(\varphi^n\big([p]_q\big)/p\big)y \quad \text{in } \widetilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap \varphi^n\big(p/[p]_q\big)\mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \mathbf{B}_{\mathrm{rig},\breve{L}}^+.$$

In particular, we see that $y = \upsilon^{-1}y_{m+1}$ is in $\mathcal{O}(D(\breve{L},\rho))$. Since, $\mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \lim_\rho \mathcal{O}(D(\breve{L},\rho))$, therefore, we conclude that y must be in $\mathbf{B}_{\mathrm{rig},\breve{L}}^+$. This completes our proof. ∎

2.1.6. φ-modules over certain period rings. Let $\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^\dagger}$ denote the category of finite free modules over $\mathbf{B}_{\mathrm{rig},L}^\dagger$ equipped with an isomorphism $1 \otimes \varphi\colon \varphi^*M \xrightarrow{\sim} M$ and morphisms between objects are $\mathbf{B}_{\mathrm{rig},L}^\dagger$-linear maps compatible with $1 \otimes \varphi$ on both sides; denote by $\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^\dagger}^0$ the full subcategory of objects that are pure of slope zero in the sense of [33, Section 6.3]. Similarly, one can define the category $\varphi\text{-Mod}_{\mathbf{B}_L^\dagger}$ and denote by $\varphi\text{-Mod}_{\mathbf{B}_L^\dagger}^0$ the full subcategory of objects that are pure of slope zero (as φ-modules over a discretely valued field).

Let $\text{Eff-}\varphi\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}$ denote the category of effective and finite $[p]_q$-height $\mathbf{A}_L^+$-modules, i.e., an object in this category is a finite free $\mathbf{A}_L^+$-module N equipped with a Frobenius-semilinear endomorphism

$$\varphi\colon N \longrightarrow N$$

such that the map $1 \otimes \varphi\colon \varphi^*(N) \to N$ is injective and its cokernel is killed by a finite power of $[p]_q$; denote by $\text{Eff-}\varphi\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} \otimes \mathbb{Q}_p$ the associated isogeny category. Similarly, define $\text{Eff-}\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^+}^{[p]_q}$ to be the category of effective and finite $[p]_q$-height $\mathbf{B}_{\mathrm{rig},L}^+$-modules and $\text{Eff-}\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^+}^{[p]_q,0}$ to be the full subcategory of objects that are pure of slope zero, i.e., M such that $\mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^+} M$ is pure of slope zero.

Lemma 2.13. *The objects described above are related as follows:*

(1) *The following functor induces a natural equivalence of categories:*

$$\begin{aligned} \varphi\text{-Mod}^{0}_{\mathbf{B}^{\dagger}_{L}} &\xrightarrow{\sim} \varphi\text{-Mod}^{0}_{\mathbf{B}^{\dagger}_{\mathrm{rig},L}} \\ M &\longmapsto M \otimes_{\mathbf{B}^{\dagger}_{L}} \mathbf{B}^{\dagger}_{\mathrm{rig},L}. \end{aligned}$$

(2) *The following functor induces an exact equivalence of $\otimes$-categories:*

$$\begin{aligned} \text{Eff-}\varphi\text{-Mod}^{[p]_q}_{\mathbf{A}^{+}_{L}} \otimes \mathbb{Q}_p &\xrightarrow{\sim} \text{Eff-}\varphi\text{-Mod}^{[p]_q,0}_{\mathbf{B}^{+}_{\mathrm{rig},L}} \\ N &\longmapsto N \otimes_{\mathbf{A}^{+}_{L}} \mathbf{B}^{+}_{\mathrm{rig},L}. \end{aligned}$$

Proof. The claim in (1) follows from [34, Theorem 6.3.3]. The equivalence of $\otimes$-categories in (2) follows from (1), [35, Lemma 1.3.13] and [33, Proposition 6.5], and the exactness follows since $\mathbf{B}^{+}_{L} \to \mathbf{B}^{+}_{\mathrm{rig},L}$ is faithfully flat by Lemma 2.7. Note that in [35], Kisin assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of [35, Lemma 1.3.13] depends only on [33, Proposition 6.5] and [34, Theorem 6.3.3] which are independent of the structure of the residue field. In particular, the proof of [35, Lemma 1.3.13] applies almost verbatim to our case. We recall the quasi-inverse functor from loc. cit. that will be useful later (see Section 4.5).

Let M^{+}_{rig} be a finite height effective $\mathbf{B}^{+}_{\mathrm{rig},L}$-module which is pure of slope zero. Then,

$$M^{\dagger}_{\mathrm{rig}} := \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},L}} M^{+}_{\mathrm{rig}}$$

is pure of slope zero and (1) implies that there exists a finite free $\mathbf{B}^{\dagger}_{L}$-module $M^{\dagger}$ pure of slope zero such that

$$\mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_{L}} M^{\dagger} \xrightarrow{\sim} M^{\dagger}_{\mathrm{rig}} \xleftarrow{\sim} \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},L}} M^{+}_{\mathrm{rig}}.$$

Choose a $\mathbf{B}^{\dagger}_{L}$-basis of $M^{\dagger}$ and a $\mathbf{B}^{+}_{\mathrm{rig},L}$-basis of M^{+}_{rig}. The composite of the isomorphisms above is given by a matrix with values in $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$. By [33, Proposition 6.5], after modifying the chosen bases, we may assume the matrix to be identity, in particular, $M^{\dagger}$ and M^{+}_{rig} are spanned by a common basis. Let M denote the $\mathbf{B}^{+}_{L}$-span of this basis. Since $\mathbf{B}^{+}_{L} = \mathbf{B}^{+}_{\mathrm{rig},L} \cap \mathbf{B}^{\dagger}_{L} \subset \mathbf{B}^{\dagger}_{\mathrm{rig},L}$, we obtain that

$$M = M^{+}_{\mathrm{rig}} \cap M^{\dagger} \subset M^{\dagger}_{\mathrm{rig}},$$

and $\mathbf{B}^{+}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{+}_{L}} M \xrightarrow{\sim} M^{+}_{\mathrm{rig}}$ and $\mathbf{B}^{\dagger}_{L} \otimes_{\mathbf{B}^{+}_{L}} M \xrightarrow{\sim} M^{\dagger}$. Moreover, $M^{\dagger}$ is pure of slope zero, so there exists an $\mathbf{A}^{\dagger}_{L}$-lattice $M^{\dagger}_{0} \subset M^{\dagger}$. Let $M'_0 := M \cap M^{\dagger}_{0} \subset M^{\dagger}$ and set

$$M_0 := (\mathbf{A}^{\dagger}_{L} \otimes_{\mathbf{A}^{+}_{L}} M'_0) \cap M'_0[1/p] \subset M^{\dagger}.$$

Using [35, Lemma 1.3.13] and the discussion above, $M_0 \subset M$ is a finite free φ-stable $\mathbf{A}^{+}_{L}$-submodule such that the cokernel of the injective map $1 \otimes \varphi \colon \varphi^{*}(M_0) \to M_0$ is killed by some finite power of $[p]_q$. ∎

Remark 2.14. Let M be a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module and $N \subset M$ a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule. Then, N is finite free if and only if it is finitely generated if and only if it is a closed submodule of M. Equivalences in the preceding statement essentially follow from [35, Lemma 1.1.5]. Note that Kisin assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of loc. cit. depends on the results of [38, Sections 7–8], [33, Lemma 2.4] and [7, Proposition 4.12 and Lemme 4.13], where the proof of the latter depends on [31, 38]. Relevant results of [31, 33, 38] are independent of the structure of the residue field of L. Hence, we get the claim by using the proof of [35, Lemma 1.1.5] almost verbatim.

Next, we note some useful facts about φ-modules over $\mathbf{A}^+_L$.

Lemma 2.15. *Let $O_K := O_F$, O_L or $O_{\breve{L}}$ and let $A := O_K[\![\mu]\!]$ equipped with a Frobenius endomorphism extending the natural Frobenius on O_K by setting $\varphi(\mu) = (1+\mu)^p - 1$. Let N be a finitely generated A-module equipped with a Frobenius-semilinear endomorphism such that*

$$1 \otimes \varphi \colon \varphi^*(N)\big[1/[p]_q\big] \xrightarrow{\sim} N\big[1/[p]_q\big].$$

Then, $N[1/p]$ is finite free over $A[1/p]$.

Proof. The proof is essentially the same as [9, Proposition 4.3]. Let J denote the smallest non-zero Fitting ideal of N over A. Set $K := O_K[1/p]$ and $\bar{A} = A/J$. From loc. cit. the claim can be reduced to checking that $\bar{A}[1/p] = 0$. Note that the Frobenius endomorphism on A and finite height condition on N are different from loc. cit. Therefore, we need some modifications in the arguments of loc. cit.; we point out the differences in terms of their notations. Fix an algebraic closure $\bar{K}$ of K and consider the finite set

$$Z := \mathrm{Spec}\,\big(\bar{A}[1/p]\big)(\bar{K})$$

of $\bar{K}$-valued points of $\bar{A}[1/p]$. Let $Z' := \{x \in \mathfrak{m}$ such that $(1+x)^p - 1 \in Z\}$, where $\mathfrak{m} \subset O_{\bar{K}}$ is the maximal ideal. Then, from the equality $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$, we get that $Z \cap U = Z' \cap U$, where $U := \mathfrak{m} - \{\zeta_p - 1, \ldots, \zeta_p^{p-1} - 1\}$. Now, all the arguments from loc. cit. can be easily adapted to show that there exists some $r \in \mathbb{N}$ such that we have an isomorphism $K[\mu]/(\mu^r) \xrightarrow{\sim} K[\mu]/(\varphi(\mu)^r)$. But, then we obtain that $(\varphi(\mu)/\mu)^r$ is a unit in $K[\mu]$, whereas $\varphi(\mu)/\mu \in K[\mu]$ is an irreducible polynomial. Hence, we must have that $r = 0$ and thus $(A/J)[1/p] = 0$, allowing us to conclude. ∎

Remark 2.16. Let N be a finitely generated torsion-free $\mathbf{A}^+_L$-module. Then,

$$D := \mathbf{A}_L \otimes_{\mathbf{A}^+_L} N$$

is a finite free $\mathbf{A}_L$-module and $N \subset D$ an $\mathbf{A}^+_L$-submodule. Moreover, the $\mathbf{A}^+_L$-module $N' := N[1/p] \cap D$ is finite free. The claim essentially follows from [27, Proposition B.1.2.4]. Note that Fontaine assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of [27, Proposition B.1.2.4] only depends on [37, Chapter 5, Theorem 3.1] which is independent of the structure of the residue field of L. Therefore, one can adapt Fontaine's proof verbatim to show that N' is finite free.

Let N be a finite free $\mathbf{A}_{\breve{L}}^{+}$-module. Say that N is *effective* and of *finite* $[p]_q$*-height* if N is equipped with a Frobenius-semilinear endomorphism φ such that the natural map $1 \otimes \varphi \colon \varphi^*(N) \to N$ is injective and its cokernel is killed by some finite power of $[p]_q$.

Let $D_{\breve{L}}$ be a finite free étale φ-module over $\mathbf{A}_{\breve{L}}$. Let $\mathcal{S}(D_{\breve{L}})$ denote the set of all finitely generated $\mathbf{A}_{\breve{L}}^{+}$-submodules $M \subset D_{\breve{L}}$ such that M is stable under the induced φ from $D_{\breve{L}}$, and the cokernel of the injective map $1 \otimes \varphi \colon \varphi^*(M) \to M$ is killed by some finite power of $[p]_q$. In [27, Section B.1.5.5], Fontaine functorially attached to $D_{\breve{L}}$ an $\mathbf{A}_{\breve{L}}^{+}$-submodule $j_*^{+}(D_{\breve{L}}) := \bigcup_{M \in \mathcal{S}(D_{\breve{L}})} M \subset D_{\breve{L}}$ (Fontaine uses the notation j_*^{q} to denote the functor j_*^{+}; we change notations to avoid the obvious confusion).

Lemma 2.17. *The $\mathbf{A}_{\breve{L}}^{+}$-module $j_*^{+}(D_{\breve{L}})$ is free of rank $\leqslant \mathrm{rk}_{\mathbf{A}_{\breve{L}}} D_{\breve{L}}$. Moreover, if N is an effective $\mathbf{A}_{\breve{L}}^{+}$-module of finite $[p]_q$-height, then the cokernel of the injective map $N \to j_*^{+}(\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^{+}} N)$ is killed by some finite power of μ.*

Proof. The first claim is shown in [27, Section B.1.5.5]. For the second claim note that N is finite free over $\mathbf{A}_{\breve{L}}^{+}$ and of finite $[p]_q$-height, therefore, by the equivalence shown in [27, Proposition B.1.3.3] we get that N is p-étale in the sense of [27, Section B.1.3.1]. In particular, we get that $D_{\breve{L}} = \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^{+}} N$ is an étale φ-module and $N \in \mathcal{S}(D_{\breve{L}})$. Now, from [27, Proposition B.1.5.6], it follows that the cokernel of the injective map $N \to j_*^{+}(D_{\breve{L}})$ is killed by some finite power of μ. ∎

2.2. p-adic representations and (φ, Γ)-modules

Let T be a finite free $\mathbb{Z}_p$-representation of G_L. From the theory of (φ, Γ_L)-modules (see [5, 27]), one can functorially associate to T a finite free étale (φ, Γ_L)-module

$$\mathbf{D}_L(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_L},$$

over $\mathbf{A}_L$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$, i.e., $\mathbf{D}_L(T)$ is equipped with a continuous and semilinear action of Γ_L and a Forbenius-semilinear endomorphism φ commuting with Γ_L and such that the natural map $1 \otimes \varphi \colon \varphi^*(\mathbf{D}_L(T)) \to \mathbf{D}_L(T)$ is an isomorphism. Moreover, we have that

$$\widetilde{\mathbf{D}}_L(T) := (\widetilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_L} \xrightarrow{\sim} \widetilde{\mathbf{A}}^{H_L} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T).$$

Furthermore, by the theory of overconvergence of p-adic and $\mathbb{Z}_p$-representations (see [6, 19]), one can functorially associate to T a finite free étale (φ, Γ_L)-module,

$$\mathbf{D}_L^{\dagger}(T) := (\mathbf{A}^{\dagger} \otimes_{\mathbb{Z}_p} T)^{H_L},$$

over $\mathbf{A}_L^{\dagger}$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ and such that $\mathbf{A}_L \otimes_{\mathbf{A}_L^{\dagger}} \mathbf{D}_L^{\dagger}(T) \xrightarrow{\sim} \mathbf{D}_L(T)$. Then, we have natural isomorphisms

$$\mathbf{A} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T, \quad \mathbf{A}^{\dagger} \otimes_{\mathbf{A}_L^{\dagger}} \mathbf{D}_L^{\dagger}(T) \xrightarrow{\sim} \mathbf{A}^{\dagger} \otimes_{\mathbb{Z}_p} T, \tag{2.1}$$

compatible with (φ, Γ_L)-actions. More generally, the constructions described above are functorial and induce exact equivalence of $\otimes$-categories:

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_L) \xrightarrow{\sim} (\varphi, \Gamma_L)\text{-}\mathrm{Mod}_{\mathbf{A}_L}^{\text{ét}} \xleftarrow{\sim} (\varphi, \Gamma_L)\text{-}\mathrm{Mod}_{\mathbf{A}_L^{\dagger}}^{\text{ét}}. \tag{2.2}$$

Similar statements are also true for p-adic representations of G_L. For a p-adic representation V of G_L, set

$$\mathbf{D}^{\dagger}_{\mathrm{rig},L}(V) := \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_L} \mathbf{D}^{\dagger}_L(V),$$

which is the unique finite free (φ, Γ_L)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ of rank $= \dim_{\mathbb{Q}_p} V$ and pure of slope zero functorially attached to V (see [7, 34, 41]). Moreover, the preceding functor induces an equivalence of categories between p-adic representations of G_L and finite free (φ, Γ_L)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ which are pure of slope zero (see [41, Lemma 4.5.7]), and for any p-adic representation V of G_L we have a natural (φ, G_L)-equivariant isomorphism,

$$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},L}} \mathbf{D}^{\dagger}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V. \tag{2.3}$$

Remark 2.18. Analogous to the results mentioned above, the natural statements for p-adic (resp. $\mathbb{Z}_p$-representations) of $G_{\breve{L}}$ also hold true (see [7, 19, 27] for details).

Finally, let V be a p-adic representation of G_L and $T \subset V$ a G_L-stable $\mathbb{Z}_p$-lattice. Since $G_{\breve{L}}$ is a subgroup of G_L, therefore, by restriction V is a p-adic representation of $G_{\breve{L}}$ and $T \subset V$ a $G_{\breve{L}}$-stable $\mathbb{Z}_p$-lattice. Furthermore, we have a $\Gamma_{\breve{L}}$-equivariant embedding $\mathbf{A}_L \subset \mathbf{A}_{\breve{L}}$ (via the map $[X_i^{\flat}] \mapsto X_i$) and thus we have isomorphisms of étale $(\varphi, \Gamma_{\breve{L}})$-modules:

$$\mathbf{D}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T), \quad \widetilde{\mathbf{D}}_{\breve{L}}(T) := (\widetilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_{\breve{L}}} \xrightarrow{\sim} \widetilde{\mathbf{A}}^{H_{\breve{L}}} \otimes_{\mathbf{A}_{\breve{L}}} \mathbf{D}_{\breve{L}}(T).$$

Similar statements are also true for V.

2.3. Crystalline representations

Let us denote the category of p-adic crystalline representations of G_L (see [14, Section 3.3]) as $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_L)$ and let $\mathrm{MF}^{\mathrm{wa}}_L(\varphi, \partial)$ denote the category of weakly admissible filtered (φ, ∂)-modules over L (see [14, Définition 4.21]). Then, the following functor induces an exact equivalence of $\otimes$-categories:

$$\begin{aligned} \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_L) &\xrightarrow{\sim} \mathrm{MF}^{\mathrm{wa}}_L(\varphi, \partial) \\ V &\longmapsto \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) := \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V\big)^{G_L}, \end{aligned} \tag{2.4}$$

with an exact quasi-inverse $\otimes$-functor given as (see [14, Corollaire 4.37]),

$$D \longmapsto \mathcal{O}\mathbf{V}_{\mathrm{cris},L}(D) := \big(\mathrm{Fil}^0\big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L D\big)\big)^{\partial=0,\varphi=1}.$$

In particular, if V is a p-adic crystalline representation of G_L, then $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is a rank $= \dim_{\mathbb{Q}_p} V$, weakly admissible filtered (φ, ∂)-module over L. Moreover, as a representation of $G_{\breve{L}}$ one can functorially attach to V a rank $= \dim_{\mathbb{Q}_p} V$, weakly admissible filtered φ-module over $\breve{L}$, denoted as $\mathbf{D}_{\mathrm{cris},\breve{L}}(V)$. Now, note that since V is crystalline for G_L, therefore, we have a (φ, G_L)-equivariant isomorphism

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V.$$

By base changing the preceding isomorphism along the $(\varphi, G_{\breve{L}})$-equivariant surjection $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{L}}) = \mathbf{B}_{\text{cris}}(O_{\overline{\breve{L}}})$, sending $X_i \mapsto [X_i^\flat]$ for $1 \leqslant i \leqslant d$, we obtain the following $(\varphi, G_{\breve{L}})$-equivariant isomorphism (also see the proof of [16, Proposition 4.14]):

$$\mathbf{B}_{\text{cris}}(O_{\overline{\breve{L}}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \xrightarrow{\sim} \mathbf{B}_{\text{cris}}(O_{\overline{\breve{L}}}) \otimes_{\mathbb{Q}_p} V,$$

where on the left, the L-algebra structure on $\mathbf{B}_{\text{cris}}(O_{\overline{\breve{L}}})$ is given via the $(\varphi, G_{\breve{L}})$-equivariant composition $L \to \mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{\breve{L}}})$ with the first map being the non-canonical L-algebra structure on $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}})$ (see Section 2.1.1). By taking $G_{\breve{L}}$-invariants in the preceding isomorphism, we obtain an isomorphism of filtered φ-modules over $\breve{L}$:

$$\breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \xrightarrow{\sim} \mathbf{D}_{\text{cris},\breve{L}}(V). \tag{2.5}$$

The representation V is said to be positive if all its Hodge–Tate weights are $\leqslant 0$, and in this case we have that $\mathcal{O}\mathbf{D}_{\text{cris},L}(V) = (\mathcal{O}\mathbf{B}^+_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{G_L}$.

Lemma 2.19. *There exist natural $\mathbf{B}_{\text{cris}}(O_{\overline{L}})$-linear and Frobenius-equivariant isomorphisms,*

$$\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \big(\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V)\big)^{\partial=0} \xrightarrow{\sim} \mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V),$$

where the second isomorphism is induced by the surjective map $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{L}})$, sending $X_i \mapsto [X_i^\flat]$ for $1 \leqslant i \leqslant d$.

Proof. Let us consider the following projection map:

$$\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris}}(V) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris}}(V), \tag{2.6}$$

induced by the surjective map $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{L}})$ sending $X_i \mapsto [X_i^\flat]$, and the kernel of (2.6) is given as $J\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris}}(V)$, where

$$J\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) := p\text{-adic closure of the ideal } \big([X_1^\flat] - X_1, \ldots, [X_d^\flat] - X_d\big) \subset \mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}).$$

Moreover, recall that we have a connection $\partial \colon \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \to \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \otimes_{O_L} \Omega^1_{O_L/O_F}$, given as $\partial(x) = \sum_{i=1}^d \partial_i(x) dX_i$, for differential operators ∂_i on $\mathcal{O}\mathbf{D}_{\text{cris},L}(V)$. Then, using the non-canonical L-algebra structure on $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}})$ (see Section 2.1.1), we can give an L-linear map,

$$\begin{aligned} \mathcal{O}\mathbf{D}_{\text{cris},L}(V) &\longrightarrow \mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \\ x &\longmapsto \sum_{\mathbf{k} \in \mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d \big([X_i^\flat] - X_i\big)^{[k_i]}, \end{aligned} \tag{2.7}$$

where we write $\prod_{i=1}^d \partial_i^{k_i}(x) = \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x)$ for notational convenience. As the connection ∂ on $\mathcal{O}\mathbf{D}_{\text{cris},L}(V)$ is p-adically quasi-nilpotent, therefore, $\prod_{i=1}^d \partial_i^{k_i}(x)$ goes p-adically to 0 as $\sum_{i=1}^d k_i \to +\infty$, and thus, in (2.7), the formula on the right converges in the target, for its natural topology (see Remark 2.20). Moreover, note that the map in (2.7) extends $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linearly to a map,

$$\begin{aligned}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\\ a \otimes x &\longmapsto a \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} \partial_i^{k_i}(x) \prod_{i=1}^{d} \big([X_i^\flat] - X_i\big)^{[k_i]},\end{aligned} \tag{2.8}$$

and it provides a section to the projection in (2.6). In particular, we obtain the following $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear and direct sum decomposition compatible with the respective Frobenii and connections:

$$\begin{aligned}&\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\\ &\quad = \big(J\,\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big) \oplus \big(\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big).\end{aligned}$$

Note that the image of the map in (2.8) lies in $(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0}$. Moreover, since V is crystalline, therefore, we have the following $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear isomorphism compatible with the respective Frobenii and connections:

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L V.$$

Using the preceding isomorphism, we easily get that

$$\big(J\,\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big)^{\partial=0} = 0.$$

So, from the direct sum decomposition above, it follows that we have

$$\big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big)^{\partial=0} \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

Note that the maps in (2.6) and (2.8) are evidently compatible with the respective Frobenii, therefore, the isomorphism in the claim is also compatible with the Frobenius. This allows us to conclude. ∎

Remark 2.20. Note that the φ-module $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is a finite-dimensional L-vector space. Therefore, $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is a finite free module over $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ equipped with the natural product topology induced from the p-adic topology on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. Similarly, in Lemma 2.23, we will see that $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is a finite free module over $\mathbf{B}^+_{\mathrm{rig},L}$ and equipped with the natural product topology induced from the Fréchet topology on $\mathbf{B}^+_{\mathrm{rig},L}$ (see Remark 2.6).

Remark 2.21. Using the $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear map in (2.8) and by transport of structure, we equip $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ with a continuous action of G_L. In particular, for any g in G_L, its action on $a \otimes x$ in $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is given by the following formula:

$$g(a \otimes x) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} \partial_i^{k_i}(x) \prod_{i=1}^{d} \big(g([X_i^\flat]) - [X_i^\flat]\big)^{[k_i]}.$$

Remark 2.22. Using the description of the G_L-action on $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ in Remark 2.21, note that $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the induced action of G_L. Moreover, it is clear that the action of H_L, induced from the action of G_L described in Remark 2.21, is trivial on

$$\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

and we have that $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})^{H_L} = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ by [39, Lemma 4.32]. Therefore, we obtain that,

$$\big(\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big)^{H_L} = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \tag{2.9}$$

We equip $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ with the residual Γ_L-action.

Lemma 2.23. *For any x in $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and g in Γ_L, the following summation converges in $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$:*

$$g(x) = \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} \partial_i^{k_i}(x) \prod_{i=1}^{d} \big(g([X_i^\flat]) - [X_i^\flat]\big)^{[k_i]}.$$

In particular, $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the induced action of Γ_L.

Proof. Let $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ be topological generators of Γ_L as in Section 2.1.1, in particular, $\gamma_j([X_i]^\flat) = (1+\mu)[X_i^\flat]$, if $i = j$, and $[X_i^\flat]$, otherwise. As $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is closed for the p-adic topology and the action of Γ_L on the latter is continuous (see Remark 2.21), therefore, it is enough to show the claim for the chosen topological generators of Γ_L. For any γ_j, we can simplify the sum in the claim and rewrite it as $\sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_j^\flat] \prod_{i=1}^d \partial_i^{k_i}(x)$. Now, recall that the connection ∂ on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is p-adically quasi-nilpotent, i.e., there exists an O_L-lattice $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ stable under ∂, i.e., $\partial \colon M \to M \otimes \Omega^1_{O_L}$ such that ∂ is nilpotent modulo p. Let $\{e_1, \ldots, e_h\}$ denote an O_L-basis of M. Then, we may check that on the chosen basis we have $\varphi(M) \subset p^{-r}M$, for some fixed $r \in \mathbb{N}$. Moreover, recall that we have $L \otimes_{\varphi,L} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, so we may write $x = \sum_{j=1}^h a_j\varphi(e_j)$, for some $a_j \in L$. Since $\partial_i(\varphi(e_j)) = p\varphi(\partial_i(e_j))$, for all $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant h$, therefore, we get that,

$$\sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_j^\flat] \prod_{i=1}^{d} \partial_i^{k_i}\big(\varphi(e_i)\big) = p^{-dr} \sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_i^\flat] \prod_{i=1}^{d} p^{k_i} p^r \varphi\big(\partial_i^{k_i}(e_i)\big),$$

converges p-adically, and thus converges for the natural topology on $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ (see Remark 2.20). Therefore, by using the Leibniz rule, we are reduced to showing that the summation $\sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_j^\flat] \prod_{i=1}^d \partial_i^{k_i}(a)$ converges in $\mathbf{B}^+_{\mathrm{rig},L}$, for any a in L. This follows easily since we have $\partial_i^k(X_i^n)/k! = 0$, for $n < k$, $\partial_i^k(X_i^n)/k! = \binom{n}{k} X_i^{n-k}$, for $n \geqslant k$, and $\partial_i^k(X_i^{-n})/k! = (-1)^k \binom{n+k-1}{k} X_i^{-(n+k)}$, for $n \in \mathbb{N}$. Hence, the lemma is proved. ∎

Lemma 2.24. *The action of Γ_L on $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is trivial modulo μ.*

Proof. Note that $g(\mu) = (1+\mu)^{\chi(g)} - 1$, for any g in Γ_L and χ the p-adic cyclotomic character. Using Lemma 2.23 and Remark 2.21, for $a \otimes x$ in $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and $g \in \Gamma_L$, we have that

$$g(a \otimes x) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{x} \partial_i^{k_i}(x) \prod_{i=1}^{d} \big(g([X_i^\flat]) - [X_i^\flat]\big)^{[k_i]}.$$

Note that

$$(g-1)(a \otimes x) = \big((g-1)a\big) \otimes x + g(a) \otimes \big((g-1)x\big), \tag{2.10}$$

where $g(x)$ is given by the series in the statement of Lemma 2.23. So, we have that

$$(g-1)x = \sum_{\mathbf{k}\in\mathbb{N}^d_+} \prod_{i=1}^{x} \partial_i^{k_i}(x) \prod_{i=1}^{d} \big((g-1)[X_i^\flat]\big)^{[k_i]},$$

where $\mathbb{N}^d_+ = \mathbb{N}^d \setminus \{(0,0,\ldots,0)\}$. Using the explicit description of $\mathbf{B}^+_{\mathrm{rig},L}$ in Lemma 2.5, note that $(g-1)\mathbf{B}^+_{\mathrm{rig},L} \subset \mu\mathbf{B}^+_{\mathrm{rig},L}$, and from the proof of Lemma 2.23 note that $(g-1)[X_i^\flat]$ is in $\mu\mathbf{B}^+_L$. Therefore, an argument similar to the proof of Lemma 2.23 shows that $(g-1)x$ converges in $\mu\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. So, from (2.10) it follows that $(g-1)(a \otimes x)$ is in $\mu\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. This allows us to conclude. ∎

3. Wach modules

In this section, we will define and study Wach modules in the imperfect residue field case and finite $[p]_q$-height representations of G_L and relate them to crystalline representations. Our definition is a direct and natural generalisation of Wach modules in the perfect residue field case (see [8, Définition III.4.1]).

3.1. Wach modules over $\mathbf{A}^+_L$

In the period ring $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$, let us fix $q := [\varepsilon]$, $\mu := q - 1 = [\varepsilon] - 1$ and $[p]_q := \tilde{\xi} := \varphi(\mu)/\mu$.

Definition 3.1. Let $a, b \in \mathbb{Z}$ with $b \geqslant a$. A *Wach module* over $\mathbf{A}^+_L$ with weights in the interval $[a,b]$ is a finite free $\mathbf{A}^+_L$-module N equipped with a continuous and semilinear action of Γ_L and satisfying the following assumptions:

(1) The action of Γ_L on $N/\mu N$ is trivial.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$, commuting with the action of Γ_L, and such that $\varphi(\mu^b N) \subset \mu^b N$ and the cokernel of the injective map $(1 \otimes \varphi)\colon \varphi^*(\mu^b N) \to \mu^b N$ is killed by $[p]_q^{b-a}$.

Define the $[p]_q$-*height* of N to be the largest value of $-a$, for $a \in \mathbb{Z}$ as above. Say that N is *effective* if one can take $b = 0$ and $a \leqslant 0$. A Wach module over $\mathbf{B}_L^+$ is a finitely generated module M equipped with a Frobenius-semilinear operator $\varphi : M[1/\mu] \to M[1/\varphi(\mu)]$, commuting with the action of Γ_L, and such that there exists a φ-stable (after inverting μ) and Γ_L-stable $\mathbf{A}_L^+$-submodule $N \subset M$, with N a Wach module over $\mathbf{A}_L^+$ (equipped with the induced (φ, Γ_L)-action) and $N[1/p] = M$.

Denote the category of Wach modules over $\mathbf{A}_L^+$ as $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_L^+$-linear, Γ_L-equivariant and φ-equivariant (after inverting μ).

Definition 3.2. Let N be a Wach module over $\mathbf{A}_L^+$. Define a decreasing filtration on N called the *Nygaard filtration*, for $k \in \mathbb{Z}$, as

$$\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}.$$

From the definition, it is clear that N is effective if and only if $\mathrm{Fil}^0 N = N$. Similarly, we can define a Nygaard filtration on $M := N[1/p]$ and it satisfies $\mathrm{Fil}^k M = (\mathrm{Fil}^k N)[1/p]$.

Extending scalars along $\mathbf{A}_L^+ \to \mathbf{A}_L$ induces a functor $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} \to (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L}^{\text{ét}}$, and we make the following claim.

Proposition 3.3. *The following natural functor is fully faithful:*

$$\begin{aligned}(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} &\longrightarrow (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L}^{\text{ét}}\\ N &\longmapsto \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N.\end{aligned}$$

Proof. We need to show that for Wach modules N and N', we have a natural bijection,

$$\mathrm{Hom}_{(\varphi,\Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}}(N, N') \xrightarrow{\sim} \mathrm{Hom}_{(\varphi,\Gamma)\text{-Mod}_{\mathbf{A}_L}^{\text{ét}}}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N'). \tag{3.1}$$

Note that $\mathbf{A}_L^+ \to \mathbf{A}_L = \mathbf{A}_L^+[1/\mu]^\wedge$ is injective, in particular, the map in (3.1) is injective. To check that (3.1) is surjective, let $D_L := \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N$, $D_L' := \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N'$, and take an $\mathbf{A}_L$-linear and (φ, Γ_L)-equivariant map $f : D_L \to D_L'$. Then, by base changing f along the embedding $\mathbf{A}_L \to \mathbf{A}_{\breve{L}}$ (see Section 2.1.5), we obtain an $\mathbf{A}_{\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant map $f_{\breve{L}} : D_{\breve{L}} \to D_{\breve{L}}'$. Using the definition and notation preceding Lemma 2.17, we further obtain an $\mathbf{A}_{\breve{L}}^+$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant map $f_{\breve{L}} : j_*^+(D_{\breve{L}}) \to j_*^+(D_{\breve{L}}')$, where we abuse notations by writing $f_{\breve{L}}$ instead of $j_*^+(f_{\breve{L}})$. From Lemma 2.17, note that for some $s \in \mathbb{N}$ and $N_{\breve{L}} := \mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} N$, we have an inclusion $\mu^s N_{\breve{L}} \subset j_*^+(D_{\breve{L}})$ and the cokernel is killed by some finite power of μ. Hence, $N_{\breve{L}}[1/\mu] \xrightarrow{\sim} j_*^+(D_{\breve{L}})[1/\mu]$. Similarly, one can also show that $N_{\breve{L}}'[1/\mu] \xrightarrow{\sim} j_*^+(D_{\breve{L}}')[1/\mu]$. Now, from the map $f_{\breve{L}} : j_*^+(D_{\breve{L}}) \to j_*^+(D_{\breve{L}}')$, we obtain an induced $\Gamma_{\breve{L}}$-equivariant map

$$f_{\breve{L}} : N_{\breve{L}}[1/\mu] = j_*^+(D_{\breve{L}}')[1/\mu] \to j_*^+(D_{\breve{L}}')[1/\mu] = N_{\breve{L}}'[1/\mu],$$

and from Lemma 3.4 below, we get that $f_{\check{L}}(N_{\check{L}}) \subset N'_{\check{L}}$. It is easy to see that $N = N_{\check{L}} \cap D_L \subset D_{\check{L}}$ and $N' = N'_{\check{L}} \cap D'_L \subset D'_{\check{L}}$. So, we conclude that $f(N) \subset f_{\check{L}}(N_{\check{L}}) \cap f(D_L) \subset N'_{\check{L}} \cap D'_L = N'$. This proves the surjectivity of (3.1). ∎

Lemma 3.4. *Let N and N' be Wach modules over $\mathbf{A}^+_{\check{L}}$ and let $f\colon N[1/\mu] \to N'[1/\mu]$ be an $\mathbf{A}^+_{\check{L}}$-linear and $\Gamma_{\check{L}}$-equivariant map. Then $f(N) \subset N'$.*

Proof. The proof is similar to the proof of [3, Lemma 5.32]. Assume that $f(N) \subset \mu^{-k}N'$, for some $k \in \mathbb{N}$, and consider the reduction of f modulo μ, which is again $\Gamma_{\check{L}}$-equivariant. By definition, we have that $\Gamma_{\check{L}}$ acts trivially over $N/\mu N$, whereas $\mu^{-k}N'/\mu^{-k+1}N' \xrightarrow{\sim} N'/\mu N'(-k)$, i.e., the action of $\Gamma_{\check{L}}$ on $\mu^{-k}N'/\mu^{-k+1}N'$ is given by χ^{-k}, where χ is the p-adic cyclotomic character, in particular, $(\mu^{-k}N'/\mu^{-k+1}N')^{\Gamma_{\check{L}}} = 0$. Since we know that f is $\Gamma_{\check{L}}$-equivariant, therefore, we must have that $k = 0$, i.e., $f(N) \subset N'$. ∎

Analogous to above, one can define categories $(\varphi,\Gamma)\text{-Mod}^{[p]_q}_{\mathbf{B}^+_L}$ and $(\varphi,\Gamma)\text{-Mod}^{\text{ét}}_{\mathbf{B}_L}$ and a functor from the former to latter by extending scalars along $\mathbf{B}^+_L \to \mathbf{B}_L$. Then, passing to associated isogeny categories in Proposition 3.3, we obtain the following.

Corollary 3.5. *The natural functor $(\varphi,\Gamma)\text{-Mod}^{[p]_q}_{\mathbf{B}^+_L} \to (\varphi,\Gamma)\text{-Mod}^{\text{ét}}_{\mathbf{B}_L}$ is fully faithful.*

Composing the functor in Proposition 3.3 with the equivalence in (2.2), we obtain a fully faithful functor,

$$\begin{aligned} \mathbf{T}_L\colon (\varphi,\Gamma)\text{-Mod}^{[p]_q}_{\mathbf{A}^+_L} &\longrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(G_L) \\ N &\longmapsto (\mathbf{A} \otimes_{\mathbf{A}^+_L} N)^{\varphi=1} \xrightarrow{\sim} \big(W(\mathbb{C}^\flat_L) \otimes_{\mathbf{A}^+_L} N\big)^{\varphi=1}. \end{aligned} \tag{3.2}$$

Lemma 3.6. *Let N be Wach module of $[p]_q$-height s and let $T := \mathbf{T}_L(N)$. Then, we have a G_L-equivariant isomorphism,*

$$\mathbf{A}^+[1/\mu] \otimes_{\mathbf{A}^+_L} N \xrightarrow{\sim} \mathbf{A}^+[1/\mu] \otimes_{\mathbb{Z}_p} T. \tag{3.3}$$

Moreover, if N is effective, then we have natural G_L-equivariant inclusions

$$\mu^s(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}^+ \otimes_{\mathbf{A}^+_L} N \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T.$$

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have that

$$\mathbf{T}_L\big(\mu^r N(-r)\big) = \mathbf{T}_L(N)(-r)$$

(the twist $(-r)$ denotes a Tate twist on which Γ_L acts via χ^{-r}, where χ is the p-adic cyclotomic character). Therefore, it is enough to show both the claims for effective Wach modules. So assume that N is effective. Now, as N is finite free over $\mathbf{A}^+_L$, therefore, by using Definition 3.1 (2) and the tensor product Frobenius, we obtain a Frobenius-semilinear isomorphism

$$\varphi\colon \mathbf{A}_{\inf}(O_{\overline{L}})[1/\xi] \otimes_{\mathbf{A}^+_L} N \xrightarrow{\sim} \mathbf{A}_{\inf}(O_{\overline{L}})[1/\tilde{\xi}] \otimes_{\mathbf{A}^+_L} N.$$

Then, from [39, Proposition 6.15] we get the following G_L-equivariant inclusions:

$$\mu^s\big(\mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T\big) \subset \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \tilde{\mathbf{A}} \otimes_{\mathbf{A}_L^+} N.$$

Moreover, from (2.1), we have that $\mathbf{A} \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T$. Therefore, by taking the intersection of $\mathbf{A} \otimes_{\mathbb{Z}_p} T$ with the chain of inclusions above, inside $\tilde{\mathbf{A}} \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T$, we obtain the following G_L-equivariant inclusions:

$$\mu^s\big(\mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}\big) \otimes_{\mathbb{Z}_p} T \subset \big(\mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}\big) \otimes_{\mathbf{A}_L^+} N \subset \big(\mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}\big) \otimes_{\mathbb{Z}_p} T.$$

Since $\mathbf{A}^+ = \mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}$, therefore, we get that the natural map in (3.3) is bijective and $\mu^s(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T$ (for N effective), as desired. ∎

3.2. Finite $[p]_q$-height representations

In this section, we will generalise the definition of finite $[p]_q$-height representations from [3, Definition 4.9] in the imperfect residue field case. Let T be a finite free $\mathbb{Z}_p$-representation of G_L, $V := T[1/p]$ and set $\mathbf{D}_L^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_L}$ to be the (φ, Γ_L)-module over $\mathbf{A}_L^+$ associated to T and let $\mathbf{D}_L^+(V) := \mathbf{D}_L^+(T)[1/p]$ be the (φ, Γ_L)-module over $\mathbf{B}_L^+$ associated to V.

Definition 3.7. A *finite $[p]_q$-height $\mathbb{Z}_p$-representation* of G_L is a finite free $\mathbb{Z}_p$-module T admitting a linear and continuous action of G_L, and such that there exists a finite free $\mathbf{A}_L^+$-submodule $\mathbf{N}_L(T) \subset \mathbf{D}_L(T)$ satisfying the following:

(1) $\mathbf{N}_L(T)$ is a Wach module in the sense of Definition 3.1.

(2) We have a natural (φ, Γ_L)-equivariant isomorphism $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$.

Set the $[p]_q$-height of T to be the $[p]_q$-height of $\mathbf{N}_L(T)$. Say T is *positive* if $\mathbf{N}_L(T)$ is effective.

A finite $[p]_q$-height p-adic representation of G_L is a finite-dimensional $\mathbb{Q}_p$-vector space admitting a linear and continuous action of G_L, and such that there exists a G_L-stable $\mathbb{Z}_p$-lattice $T \subset V$ with T of finite $[p]_q$-height. Let us define $\mathbf{N}_L(V) := \mathbf{N}_L(T)[1/p] \subset \mathbf{D}_L(T)[1/p] = \mathbf{D}_L(V)$ to be a $\mathbf{B}_L^+$-submodule of $\mathbf{D}_L(V)$, and satisfying properties analogous to (1) and (2) above. Set the $[p]_q$-height of V to be the $[p]_q$-height of T. Say V is positive if $\mathbf{N}_L(V)$ is effective.

Remark 3.8. For T a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L and $r \in \mathbb{N}$, we have that $\mathbf{N}_L(T(r)) = \mu^{-r}\mathbf{N}_L(T)(r)$, in particular, property of being finite $[p]_q$-height is invariant under Tate twists.

Lemma 3.9. *Let T be a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L.*

(1) *If T is positive, then $\mu^s\mathbf{D}_L^+(T) \subset \mathbf{N}_L(T) \subset \mathbf{D}_L^+(T)$.*

(2) *The $\mathbf{A}_L^+$-module $\mathbf{N}_L(T)$ is unique, i.e., if there exists an $\mathbf{A}_L^+$-submodule $N \subset \mathbf{D}_L(T)$ satisfying the conditions* (1) *and* (2) *in Definition 3.7, then we must have that $N = \mathbf{N}_L(T) \subset \mathbf{D}_L(T)$.*

Proof. Note that $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$ and this scalar extension defines a fully faithful functor as in Proposition 3.3, in particular, we obtain that $\mathbf{T}_L(\mathbf{N}_L(T)) \xrightarrow{\sim} T$ as representations of G_L (here $\mathbf{T}_L$ is the functor defined in (3.2)). This also implies that Lemma 3.6 holds for $\mathbf{N}_L(T)$, so by taking H_L-invariants of the chain of inclusions in the final statement of Lemma 3.6, we obtain that $\mu^s \mathbf{D}_L^+(T) \subset \mathbf{N}_L(T) \subset \mathbf{D}_L^+(T)$ which proves (1). The claim in (2) follows from Proposition 3.3, or using an argument similar to [3, Proposition 4.13]. ∎

Remark 3.10. Let V be a finite $[p]_q$-height p-adic representation of G_L and $T \subset V$ a finite $[p]_q$-height G_L-stable $\mathbb{Z}_p$-lattice. Then, we have that $\mathbf{N}_L(V) = \mathbf{N}_L(T)[1/p]$ and from Lemma 3.9 we get that if V is positive then $\mu^s \mathbf{D}_L^+(V) \subset \mathbf{N}_L(V) \subset \mathbf{D}_L^+(V)$. Moreover, from Corollary 3.5 (or [3, Proposition 4.13]) it follows that $\mathbf{N}_L(V)$ is unique, i.e., if there exists a $\mathbf{B}_L^+$-module $M \subset \mathbf{D}_L(V)$ satisfying the conditions analogous to (1) and (2) in Definition 3.7, then we must have that $M = \mathbf{N}_L(V) \subset \mathbf{D}_L(V)$. In particular, it follows that $\mathbf{N}_L(V)$ is independent of the choice of the lattice $T \subset V$. Alternatively, note that since we have $\mathbf{N}_L(V(r)) = \mu^{-r}\mathbf{N}_L(V)(r)$, without loss of generality we may assume that V is positive and $T' \subset V$ another finite $[p]_q$-height G_L-stable $\mathbb{Z}_p$-lattice. Then, we have that $\mu^s \mathbf{D}_L^+(V) \subset \mathbf{N}_L(T')[1/p] \subset \mathbf{D}_L^+(V)$, and using the argument in the proof of [3, Proposition 4.13] almost verbatim gives $\mathbf{N}_L(V) = \mathbf{N}_L(T)[1/p] \xrightarrow{\sim} \mathbf{N}_L(T')[1/p]$ compatible with the respective (φ, Γ_L)-actions.

Remark 3.11. From the definition of finite $[p]_q$-height representations, Lemma 3.9 and the fully faithful functor in (3.2), it follows that the data of a finite $[p]_q$-height representation is equivalent to the data of a Wach module.

3.3. Wach modules are crystalline

The goal of this subsection is to prove Theorem 3.12 and Corollary 3.16. To prove our results, we need certain period rings similar to [3, Section 4.3.1], which we denote as $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ below. We define these as follows: let $\varpi = \zeta_{p^m} - 1$, where $m \geqslant 1$ for $p \geqslant 3$ and $m \geqslant 2$ for $p = 2$, and set

$$\mathbf{A}_{L,\varpi}^+ := \mathbf{A}_L^+\big[\varphi^{-m}(\mu)\big] \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}).$$

Restricting the map θ on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ (see Section 2.1.1) to $\mathbf{A}_{L,\varpi}^+$, we get a surjection $\theta : \mathbf{A}_{L,\varpi}^+ \twoheadrightarrow O_L[\varpi]$. Define $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of the map θ with respect to $\operatorname{Ker}\theta$. Moreover, consider the surjective map

$$\theta_L \colon O_L \otimes_{\mathbb{Z}} \mathbf{A}_{L,\varpi}^+ \twoheadrightarrow O_L[\varpi],$$

given as $x \otimes y \mapsto x\theta(y)$. Define $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of the map θ_L with respect to $\operatorname{Ker}\theta_L$. Similar to [3, Section 4.3.1], one can show that $\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, stable under the Frobenius and Γ_L-action on latter. We equip $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ with induced structures, in particular, a filtration (same as the filtration by divided powers of $\operatorname{Ker}\theta$ and $\operatorname{Ker}\theta_L$ respectively, see [3, Remark 4.23]) and a connection ∂_A on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the filtration and such that $(\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}})^{\partial_A=0} = \mathbf{A}_{L,\varpi}^{\mathrm{PD}}$. Furthermore, note that we have natural inclusions $\mathbf{B}_L^+ \subset \mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \subset \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \subset \mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}}) \subset \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(O_{\overline{L}})$, where the last term is the big de Rham period ring which is an integral domain (see [14, Proposition 2.9 and Remarque 2.10]). So, it follows that $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$ are torsion-free modules over the principal ideal domain $\mathbf{B}_L^+$, hence flat. A similar reasoning shows that $\mathrm{Fil}^k \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$ is a flat $\mathbf{B}_L^+$-module, for any $k \in \mathbb{N}$.

Theorem 3.12. *Let N be a Wach module over $\mathbf{A}_L^+$. Then, $V := \mathbf{T}_L(N)[1/p]$ is a p-adic crystalline representation of G_L.*

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have that $\mathbf{T}_L(\mu^r N(-r)) = \mathbf{T}_L(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_L acts via χ^{-r}, where χ is the p-adic cyclotomic character). Therefore, it is enough to show the claim for effective Wach modules. So assume that N is effective. Note that N is free over $\mathbf{A}_L^+$ and $\mathbf{T}_L(N)$ is a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L in the sense of Definition 3.7 (see Remark 3.11). So, the results of [3, Sections 4.3–4.5] can be adapted to the case of O_L almost verbatim as all objects appearing in loc. cit. admit a natural variation for O_L. In particular, as we explain below, the proofs of [3, Theorem 4.25 and Proposition 4.28] can be adapted to get that $V = \mathbf{T}_L(N)[1/p]$ is a crystalline representation of G_L.

Set $D_L := (\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Then, from Proposition 3.14 it follows that D_L is a finite L-vector space of dimension $= \mathrm{rk}_{\mathbf{A}_L^+} N$ equipped with a tensor product Frobenius and a connection induced from the connection on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the tensor-product filtration defined as $\mathrm{Fil}^k D_L := (\sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} \mathrm{Fil}^j N[1/p])^{\Gamma_L}$, where $N[1/p]$ is equipped with Nygaard filtration of Definition 3.2 (see after Lemma 3.15 for well-definedness of the tensor-product filtration). Moreover, from Proposition 3.14 below, note that we have a natural isomorphism $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]$. Now, consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L D_L \arrow[r, "(3.6)", "\sim"'] \arrow[d, "(3.7)"'] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N[1/p] \arrow[d, "(3.3)"', "\wr"] \\
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd} \tag{3.4}$$

where the left vertical arrow is the extension of the inclusion $D_L \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, from (3.7), along the natural map $L \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, the top horizontal arrow is the extension of the isomorphism, in Proposition 3.14, along the natural map $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, the right vertical arrow is the extension of the isomorphism (3.3), in Lemma 3.6, along the natural map $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ and the bottom horizontal arrow is the natural injective map (see [14, Proposition 3.22]). Commutativity and compatibility of the diagram with the respective (φ, G_L)-actions and connections follow from (3.7) below. Bijectivity of the top horizontal arrow and the right vertical arrow imply that the left vertical arrow and the bottom horizontal arrow are bijective as well. Hence, we obtain that V is a crystalline representation of G_L. ∎

Remark 3.13. In diagram (3.4), by taking the G_L-fixed part of the left vertical arrow, we get that,

$$D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \tag{3.5}$$

compatible with the respective Frobenii and connections. Moreover, since the bottom horizontal arrow of the diagram (3.4) is compatible with filtrations (see [14, Proposition 3.35]), an argument similar to the proof of [3, Proposition 4.49] shows that the isomorphism in (3.5) is compatible with filtrations, where we consider the Hodge filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$.

The following result was used in the proof of Theorem 3.12.

Proposition 3.14. *Let N be an effective Wach module over $\mathbf{A}_L^+$. Then, the L-vector space $D_L = (\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L}$ is of finite dimension $= \mathrm{rk}_{\mathbf{A}_L^+} N$, and naturally equipped with a Frobenius, a filtration and a connection satisfying Griffiths transversality with respect to the filtration. Moreover, we have a natural comparison isomorphism*

$$\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p], \tag{3.6}$$

compatible with the respective Frobenii, filtrations, connections and Γ_L-actions.

Proof. We will adapt the proof of [3, Proposition 4.28]. The main idea, as explained below, is to work over a new period ring $\mathcal{O}S_n^{\mathrm{PD}}$ (defined below), prove an isomorphism analogous to (3.6) (see Lemma 3.15), and then, extend the latter isomorphism over to $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ using the Frobenius. So, following [3, Section 4.4.1], for $n \in \mathbb{N}$, let us define a p-adically complete ring $S_n^{\mathrm{PD}} := \mathbf{A}_L^+\langle \frac{\mu}{p^n}, \frac{\mu^2}{2!p^{2n}}, \ldots, \frac{\mu^k}{k!p^{kn}}, \ldots\rangle$. The p-adically completed divided power ring S_n^{PD} is equipped with a continuous action of Γ_L and we have a Frobenius homomorphism $\varphi\colon S_n^{\mathrm{PD}} \to S_{n-1}^{\mathrm{PD}}$, in particular, $\varphi^n(S_n^{\mathrm{PD}}) \subset S_0^{\mathrm{PD}} \subset \mathbf{A}_{L,\varpi}^{\mathrm{PD}}$, where the latter inclusion is obvious. The reader should note that in [3, Section 4.4.1] we consider a further completion of S_n^{PD} with respect to certain filtration by PD-ideals, denoted $\widehat{S}_n^{\mathrm{PD}}$ in loc. cit. However, such a completion is not strictly necessary and all proofs of loc. cit. can be carried out without it. In particular, many good properties of $\widehat{S}_n^{\mathrm{PD}}$ restrict to good properties on S_n^{PD} as well (for example, the (φ, Γ_L)-action described above).

Now, consider the O_F-linear homomorphism of rings

$$\iota\colon O_L \longrightarrow S_n^{\mathrm{PD}},$$

sending $X_j \mapsto [X_j^\flat]$, for $1 \leqslant j \leqslant d$. Using ι define an O_F-linear morphism of rings $f\colon O_L \otimes_{O_F} S_n^{\mathrm{PD}} \to S_n^{\mathrm{PD}}$, via $a \otimes b \mapsto \iota(a)b$. Let $\mathcal{O}S_n^{\mathrm{PD}}$ denote the p-adic completion of the divided power envelope of $O_L \otimes_{O_F} S_n^{\mathrm{PD}}$ with respect to $\mathrm{Ker}\, f$. The divided power ring $\mathcal{O}S_n^{\mathrm{PD}}$ is equipped with a continuous action of Γ_L, an integrable connection and we have a Frobenius

$$\varphi\colon \mathcal{O}S_n^{\mathrm{PD}} \longrightarrow \mathcal{O}S_{n-1}^{\mathrm{PD}},$$

in particular,

$$\varphi^n(\mathcal{O}S_n^{\mathrm{PD}}) \subset \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}.$$

Moreover, we have that $O_L = (\mathcal{O}S_n^{\mathrm{PD}})^{\Gamma_L}$, and with $V_j := \frac{X_j \otimes 1}{1 \otimes [X_j^\flat]}$, for $1 \leqslant j \leqslant d$, we have p-adically closed divided power ideals

$$J^{[i]}\mathcal{O}S_n^{\mathrm{PD}} := \left\langle \frac{\mu^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d} (1 - V_j)^{[k_j]},\ \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ such that } \sum_{j=0}^{d} k_j \geqslant i \right\rangle.$$

Next, let us equip $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$ with the tensor product Frobenius and an integrable connection induced from the connection on $\mathcal{O}S_n^{\mathrm{PD}}$. Then, $D_n := (\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L}$ is an L-vector space equipped with an integrable connection and we have an induced semilinear Frobenius morphism $\varphi \colon D_n \to D_{n-1}$. In particular,

$$\varphi^n(D_n) \subset D_L = \big(\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]\big)^{\Gamma_L} \subset \big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N[1/p]\big)^{G_L},$$

where the last inclusion follows since $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) = \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})^{H_L}$ (see [39, Corollary 4.34]). Let $T = \mathbf{T}_L(N)$ be the associated finite free $\mathbb{Z}_p$-representation of G_L and $V = T[1/p]$. Then, we have that,

$$\begin{aligned} D_L &\subset \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p]\big)^{G_L} \\ &\subset \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p]\big)^{G_L} \xrightarrow{\sim} \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V\big)^{G_L} = \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \end{aligned} \tag{3.7}$$

where the isomorphism follows by taking G_L-fixed elements of the extension along $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ of the isomorphism in Lemma 3.6. Recall that $\varphi^n(D_n) \subset D_L$, or equivalently, the L-linear map $1 \otimes \varphi^n \colon L \otimes_{\varphi^n, L} D_n \to D_L$ is injective, therefore, we get that $L \otimes_{\varphi^n, L} D_n$ is a finite-dimensional L-vector space. Moreover, φ is faithfully flat and finite of degree p^d over L, so it follows that D_n is a finite-dimensional L-vector space equipped with an integrable connection. Furthermore, for $n \geqslant 1$ similar to the proof of [3, Lemmas 4.32 and 4.36], one can show that

$$\log \gamma_i = \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1},$$

converge as a series of operators on $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$, where $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ_L (see Section 2.1).

Lemma 3.15. *Let $m \geqslant 1$ (let $m \geqslant 2$ if $p = 2$), then we have a Γ_L-equivariant isomorphism via the natural map $a \otimes b \otimes x \mapsto ab \otimes x$:*

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m \xrightarrow{\sim} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]. \tag{3.8}$$

Proof. Compatibility of (3.8) with the Γ_L-action is obvious from the definitions, so we only need to check that it is bijective. We will first show that (3.8) is injective. Note that we have an injective ring homomorphism

$$\mathcal{O}S_m^{\mathrm{PD}}[1/p] \xrightarrow{\varphi^m} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}).$$

Since D_m is a finite-dimensional L-vector space, therefore, we get that the following map is injective:

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_L D_m \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, L} D_m. \tag{3.9}$$

Now, recall that $V = T[1/p]$ and consider the following composition:

$$\begin{aligned} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, L} D_m &\xrightarrow{1\otimes\varphi^m} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L D_L \\ &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \end{aligned} \tag{3.10}$$

where the first map is injective because $1 \otimes \varphi^m \colon L \otimes_{\varphi^m, L} D_m \to D_L$ is injective, and the injectivity of the second map in (3.10) follows from (3.7), in particular, (3.10) is injective. Furthermore, similar to (3.9), note that $N[1/p]$ is a finite free $\mathbf{B}_L^+$-module, so it follows that the map

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p] = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_L^+} N[1/p] \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, \mathbf{B}_L^+} N[1/p],$$

is injective as well. Also, recall that we have a $\mathbf{B}_L^+$-linear isomorphism,

$$1 \otimes \varphi \colon \mathbf{B}_L^+ \otimes_{\varphi, \mathbf{B}_L^+} N\big[1/p, 1/[p]_q\big] \xrightarrow{\sim} N\big[1/p, 1/[p]_q\big].$$

So, we get that $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, \mathbf{B}_L^+} N[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p]$, since $[p]_q$ is invertible in $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. From the preceding two observations, we get that the following composition is injective:

$$\begin{aligned} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p] &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, \mathbf{B}_L^+} N[1/p] \\ &\underset{\sim}{\xrightarrow{1\otimes\varphi^m}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p]. \end{aligned} \tag{3.11}$$

Now, consider the following diagram

$$\begin{tikzcd} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m \arrow[r, "(3.9)"] \arrow[d, "(3.8)"'] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, L} D_m \arrow[r, "(3.10)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[d] \\ \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p] \arrow[r, "(3.11)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p] \arrow[r] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V, \end{tikzcd}$$

where the right vertical arrow is the natural injective map (see [14, Proposition 3.22]) and the bottom right horizontal map is the extension of the isomorphism in Lemma 3.6 along the natural map $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. The diagram commutes by definition and from the discussion above it follows that the left vertical arrow, i.e., (3.8), is injective.

Next, let us check the surjectivity of the map (3.8). Define the following operators on $\mathcal{O}N_m^{\mathrm{PD}} := \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]$,

$$\partial_i := \begin{cases} -(\log \gamma_0)/t, & \text{for } i = 0, \\ (\log \gamma_i)/(tV_i), & \text{for } 1 \leqslant i \leqslant d, \end{cases}$$

where (see [3, Section 4.4.2]),

$$V_i = \frac{X_i \otimes 1}{1 \otimes [X_i^\flat]}, \quad \text{for } 1 \leqslant i \leqslant d.$$

Using the fact that for any g in Γ_L and x in $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$, we have

$$(g-1)(ax) = (g-1)a \cdot x + g(a)(g-1)x,$$

and from the equality $\log(\gamma_i) = \lim_{n \to +\infty}(\gamma_i^{p^n} - 1)/p^n$, it is easy to see that ∂_i satisfies the Leibniz rule, for all $0 \leqslant i \leqslant d$. In particular, the operator

$$\begin{aligned} \partial\colon \mathcal{O}N_m^{\mathrm{PD}} &\longrightarrow \mathcal{O}N_m^{\mathrm{PD}} \otimes_{\mathcal{O}S_m^{\mathrm{PD}}} \Omega^1_{\mathcal{O}S_m^{\mathrm{PD}}/O_L} \\ x &\longmapsto \partial_0(x)dt + \sum_{i=1}^{d} \partial_i(x) d[X_i^\flat], \end{aligned}$$

defines a connection on $\mathcal{O}N_m^{\mathrm{PD}}$. The connection ∂ is integrable since the operators ∂_i commute with each other (see [3, Lemma 4.38]) and using the finite $[p]_q$-height property of N it is easy to show that ∂ is p-adically quasi-nilpotent as well (see [3, Lemma 4.39]).

For any x in $N[1/p]$, similar to the proof of [3, Lemmas 4.41 and 4.43], it follows that the following sum converges in $D_m = (\mathcal{O}N_m^{\mathrm{PD}})^{\Gamma_L} = (\mathcal{O}N_m^{\mathrm{PD}})^{\partial=0}$:

$$y = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \frac{t^{[k_0]}}{p^{mk_0}} (1 - V_1)^{[k_1]} \cdots (1 - V_d)^{[k_d]}. \tag{3.12}$$

By choosing a basis of N and using the formula in (3.12) on the basis elements, we can define a linear transformation α on the finite free $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$-module $\mathcal{O}N_m^{\mathrm{PD}}$. Now, similar to the proof of [3, Lemma 4.43] it can easily be deduced that for some large enough $N \in \mathbb{N}$, we can write $p^N \det \alpha \in 1 + J^{[1]}\mathcal{O}S_m^{\mathrm{PD}}$, i.e., $\det \alpha$ is a unit in $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$ and α defines an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$. Finally, as the formula in (3.12) converges in D_m, it follows that the map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m \to \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]$ is surjective. Hence, (3.8) is bijective. ∎

We continue with the proof of Proposition 3.14. Note that D_L is an L-vector space equipped with the tensor product Frobenius and a filtration given as

$$\mathrm{Fil}^k D_L = \Big(\sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} \mathrm{Fil}^j N[1/p] \Big)^{\Gamma_L},$$

where $N[1/p]$ is equipped with the Nygaard filtration of Definition 3.2. The preceding filtration is well defined, i.e., $\mathrm{Fil}^k D_L$ is a sub vector space of D_L, for each $k \in \mathbb{N}$. Indeed, it is enough to check that $\mathrm{Fil}^i \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \otimes_{\mathbf{A}_L^+} \mathrm{Fil}^j N[1/p]$ is contained in $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_L^+} N[1/p]$ as an $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$-submodule, for each $i, j \in \mathbb{N}$. This easily follows from the fact that the following $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$-linear composition is injective:

$$\begin{aligned} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_L^+} \mathrm{Fil}^j N[1/p] &\longrightarrow \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_L^+} \mathrm{Fil}^j N[1/p] \\ &\longrightarrow \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_L^+} N[1/p], \end{aligned}$$

where the first arrow is obtained by tensoring the $\mathbf{B}_L^+$-linear inclusion

$$\mathrm{Fil}^i\, \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \subset \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$$

with the $\mathbf{B}_L^+$-module $\mathrm{Fil}^j N[1/p]$ which is flat (because it is a finite torsion-free module over the principal ideal domain $\mathbf{B}_L^+$), and the second arrow is obtained by tensoring the $\mathbf{B}_L^+$-linear inclusion $\mathrm{Fil}^j N[1/p] \to N[1/p]$ with the flat $\mathbf{B}_L^+$-algebra $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p]$ (see the discussion at the start of Section 3.3). Next, note that D_L is equipped with an integrable connection induced from the connection on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the filtration since the same is true for the connection on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$. Now, consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L,\varphi^m} D_m \arrow[r, "1\otimes\varphi^m"] \arrow[d, "(3.8)"', "\wr"] & \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \arrow[d, "(3.6)"] \\
\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+,\varphi^m} N[1/p] \arrow[r, "\sim"] & \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p],
\end{tikzcd} \tag{3.13}$$

where the left vertical arrow is the extension of the isomorphism (3.8) in Lemma 3.15 along $\varphi^m\colon \mathcal{O}S_m^{\mathrm{PD}} \to \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ and the bottom horizontal isomorphism follows from an argument similar to [3, Lemma 4.46]. By the description of the arrows it follows that the diagram in (3.13) is commutative and (φ, Γ_L)-equivariant. Taking Γ_L-invariants for the composition of the left vertical and the bottom horizontal isomorphisms gives an L-linear isomorphism $O_L \otimes_{O_L,\varphi^m} D_m \xrightarrow{\sim} D_L$. So it follows that the top horizontal arrow in the diagram (3.13) is bijective as well. The preceding observation together with the bijectivity of the left vertical and the bottom horizontal arrows imply that the right vertical arrow is bijective as well, in particular, the comparison in (3.6) is an isomorphism compatible with the respective Frobenii, connections and Γ_L-actions. Compatibility of (3.6) with filtrations follows from an argument similar to [3, Corollary 4.54] (using the filtration compatible isomorphism (3.5) in Remark 3.13). This concludes our proof. ∎

There exists another relation between the Wach module N and $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Let us equip N with a Nygaard filtration as in Definition 3.2. Then, we note that $(N/\mu N)[1/p]$ is a φ-module over L, since $[p]_q = p \bmod \mu N$ and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with the induced filtration, in particular, it is a filtered φ-module over L.

Corollary 3.16. *Let N be a Wach module over $\mathbf{A}_L^+$ and $V = \mathbf{T}_L(N)[1/p]$ the associated crystalline representation from Theorem 3.12. Then, we have that $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ as filtered φ-modules over L.*

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have that $\mathbf{T}_L(\mu^r N(-r)) = \mathbf{T}_L(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_L acts via χ^{-r}, where χ is the p-adic cyclotomic character). Therefore, it is enough to show the claim for effective Wach modules. So assume that N is effective and set $M := N[1/p]$

equipped with the induced Frobenius, Γ_L-action and Nygaard filtration. Note that the L-vector space $M/\mu M$ is equipped with a Frobenius-semilinear operator φ induced from M such that we have

$$1 \otimes \varphi : \varphi^*(M/\mu M) \xrightarrow{\sim} M/\mu M,$$

because $[p]_q = p \bmod \mu$. The filtration $\mathrm{Fil}^k(M/\mu M)$ on $M/\mu M$ is the image of $\mathrm{Fil}^k M$ under the surjective map $M \twoheadrightarrow M/\mu M$. From the discussion before Theorem 3.12, recall that we have a period ring $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ equipped with a natural Frobenius, filtration, connection and Γ_L-action. Moreover, from Theorem 3.12, we have that $D_L = (\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(V))^{\Gamma_L}$ is equipped with a natural Frobenius, filtration and connection, such that $D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ compatible with the respective Frobenii, filtrations and connections (see (3.5) in Remark 3.13). Now, consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mu M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/\mu M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathrm{Fil}^1 \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi}) \otimes_{\mathbf{A}_L^+} M \arrow[r] & \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \otimes_{\mathbf{A}_L^+} M \arrow[r] & L(\zeta_p) \otimes_L M/\mu M \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{Fil}^1 \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi}) \otimes_{O_L} D_L \arrow[r] \arrow[u, "\wr"] & \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \otimes_{O_L} D_L \arrow[r] \arrow[u, "\wr", "(3.6)"'] & L(\zeta_p) \otimes_L D_L \arrow[r] \arrow[u, "\wr"] & 0.
\end{tikzcd}$$

Note that

$$(\mathrm{Fil}^1 \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \otimes_{\mathbf{A}_L^+} M) \cap M = (\mathrm{Fil}^1 \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \cap \mathbf{A}_L^+) \otimes_{\mathbf{A}_L^+} M = \mu M,$$

so the vertical maps from the first to the second row are natural inclusions. Moreover, from the third to the second row, note that the middle vertical arrow is the isomorphism (3.6) in Proposition 3.14, from which it easily follows that the left vertical arrow is also an isomorphism, and hence, the right vertical arrow is an isomorphism as well. Taking the $\mathrm{Gal}(L(\zeta_p)/L)$-invariants of the right vertical arrow gives $M/\mu M \xleftarrow{\sim} D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, where the last isomorphism is compatible with the respective Frobenii, filtrations and connections (see (3.5) in Remark 3.13).

Note that the isomorphism $D_L \xrightarrow{\sim} M/\mu M$ is compatible with the respective Frobenii and we need to check the compatibility between the respective filtrations. In the diagram above, the middle term of the second row is equipped with the tensor product filtration, so the image of $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \otimes_{\mathbf{A}_L^+} M)$ under the surjective map from the second to the third term is given as

$$L(\zeta_p) \otimes_L \mathrm{Fil}^k(M/\mu M).$$

Similarly, the middle term of the third row is equipped with the tensor product filtration, so the image of $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{L,\varpi} \otimes_{O_L} D_L)$ under the surjective map from the second to the third term is given as

$$L(\zeta_p) \otimes_L \mathrm{Fil}^k D_L.$$

Since the isomorphism (3.6) in Proposition 3.14 is compatible with filtrations, we get that $L(\zeta_p) \otimes_L \mathrm{Fil}^k D_L \xrightarrow{\sim} L(\zeta_p) \otimes_L \mathrm{Fil}^k(M/\mu M)$. Taking $\mathrm{Gal}(L(\zeta_p)/L)$-invariants in the preceding isomorphism gives $\mathrm{Fil}^k D_L \xrightarrow{\sim} \mathrm{Fil}^k(M/\mu M)$. This allows us to conclude. ∎

4. Crystalline implies finite height

The goal of this section is to prove the following claim:

Theorem 4.1. *Let T be a finite free $\mathbb{Z}_p$-representation of G_L such that $V := T[1/p]$ is a p-adic crystalline representation of G_L. Then, there exists a unique Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$ satisfying Definition 3.7. In other words, T is of finite $[p]_q$-height.*

Before carrying out the proof of Theorem 4.1, we note the following corollaries: let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_L. Then, by combining Theorems 3.12 and 4.1 and [3, Proposition 4.14] (for compatibility with tensor products below), we obtain the following.

Corollary 4.2. *The Wach module functor induces an equivalence of $\otimes$-categories,*

$$\begin{aligned}\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L) &\xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q}\\ T &\longmapsto \mathbf{N}_L(T),\end{aligned}$$

with a quasi-inverse $\otimes$-functor given as $N \mapsto \mathbf{T}_L(N) := (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} N)^{\varphi=1}$.

Passing to associated isogeny categories, we obtain the following.

Corollary 4.3. *The Wach module functor induces an exact equivalence of $\otimes$-categories*

$$\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{B}_L^+}^{[p]_q},$$

via $V \mapsto \mathbf{N}_L(V)$, with an exact quasi-inverse $\otimes$-functor given as $M \mapsto \mathbf{V}_L(M) := (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} M)^{\varphi=1}$.

In the rest of this section, we will carry out the proof of Theorem 4.1 and Corollary 4.2 by constructing $\mathbf{N}_L(T)$ and show Corollary 4.3 as a consequence. In Section 4.1, we collect important properties of classical Wach modules, i.e., the perfect residue field case. In Section 4.2, we use ideas from [35, 36] to show that classical Wach modules are compatible with Kisin–Ren modules, and we further show that in our setting, a finite $[p]_q$-height module on the open unit disk over $\breve{L}$ descends to a finite $[p]_q$-height module on the open unit disk over L, similar to [16]. On the module thus obtained, we use results of Section 2.3 to construct an action of Γ_L and study its properties in Section 4.3. Then, in Section 4.4, we check that our construction is compatible with the theory of étale (φ, Γ_L)-modules. Finally, in Section 4.5, we construct the promised Wach module $\mathbf{N}_L(T)$ and prove Theorem 4.1 and Corollary 4.3.

For a p-adic representation of G_L, note that the property of being crystalline and of finite $[p]_q$-height is invariant under twisting the representation by χ^r, for $r \in \mathbb{N}$. So, from now onwards we will assume that V is a p-adic positive crystalline representation of G_L, i.e., all its Hodge–Tate weights are $\leqslant 0$ and we have $T \subset V$ a G_L-stable $\mathbb{Z}_p$-lattice.

4.1. Classical Wach modules

Recall that $G_{\breve{L}}$ is a subgroup of G_L, so from [16, Proposition 4.14], it follows that V is a p-adic positive crystalline representation of $G_{\breve{L}}$ and $T \subset V$ a $G_{\breve{L}}$-stable $\mathbb{Z}_p$-lattice.

Note that $\breve{L}$ is an unramified extension of $\mathbb{Q}_p$ with perfect residue field, therefore, the $G_{\breve{L}}$-representation V is of finite $[p]_q$-height (see [8, 20]). Let the $[p]_q$-height of V be $s \in \mathbb{N}$. One associates to V a finite free $(\varphi, \Gamma_{\breve{L}})$-module over $\mathbf{B}_{\breve{L}}^+$ of rank $= \dim_{\mathbb{Q}_p} V$, called the Wach module $\mathbf{N}_{\breve{L}}(V)$, and to T a finite free $(\varphi, \Gamma_{\breve{L}})$-module over $\mathbf{A}_{\breve{L}}^+$ of rank $= \dim_{\mathbb{Q}_p} V$, called the Wach module $\mathbf{N}_{\breve{L}}(T)$ (see [8,46,47] and [3, Section 4.1] for a recollection). Let $\widetilde{\mathbf{D}}_L^+(T) := (\mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T)^{H_L}$ be the (φ, Γ_L)-module over $\mathbf{A}_{\inf}(O_{L_\infty}) = \mathbf{A}_{\inf}(O_{\overline{L}})^{H_L}$ (see [5, Proposition 7.2]) associated to T and let $\widetilde{\mathbf{D}}_L^+(V) := \widetilde{\mathbf{D}}_L^+(T)[1/p]$ over $\mathbf{B}_{\inf}(O_{L_\infty}) = \mathbf{B}_{\inf}(O_{\overline{L}})^{H_L}$ associated to V.

Lemma 4.4 ([8, Lemme II.1.3, Théorème III.3.1]). *With notations as above, we have the following:*

(1) $\mathbf{N}_{\breve{L}}(T) = \mathbf{N}_{\breve{L}}(V) \cap \mathbf{D}_{\breve{L}}(T) \subset \mathbf{D}_{\breve{L}}(V)$.

(2) *We have that* $\mu^s \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \subset \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T$, *and taking H_L-invariants gives* $\mu^s \widetilde{\mathbf{D}}_L^+(T) \subset \mathbf{A}_{\inf}(O_{L_\infty}) \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \subset \widetilde{\mathbf{D}}_L^+(T)$. *Similar claims are also true for V.*

By properties of Wach modules, we have the following functorial isomorphisms of étale (φ, Γ_L)-modules:

$$\begin{gathered}
\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T) \quad \text{and} \quad \mathbf{A}_{\breve{L}}^\dagger \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}^\dagger(T),\\
\mathbf{B}_{\breve{L}} \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(V) \quad \text{and} \quad \mathbf{B}_{\breve{L}}^\dagger \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}^\dagger(V),\\
\mathbf{B}_{\mathrm{rig},\breve{L}}^\dagger \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{rig},\breve{L}}^\dagger(V),
\end{gathered} \tag{4.1}$$

where the second isomorphism in the first row follows from [8, Théorème III.3.1].

Let us set $\mathbf{N}_{\mathrm{rig},\breve{L}}(V) := \mathbf{B}_{\mathrm{rig},\breve{L}}^+ \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V)$ equipped with the induced tensor-product Frobenius-semilinear operator φ and $\Gamma_{\breve{L}}$-action. From [8, Proposition II.2.1], recall that we have a natural inclusion $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$, which extends $\mathbf{B}_{\mathrm{rig},\breve{L}}^+$-linearly to a Frobenius and $\Gamma_{\breve{L}}$-equivariant inclusion,

$$\mathbf{B}_{\mathrm{rig},\breve{L}}^+ \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V),$$

such that its cokernel is killed by $(t/\mu)^s \in \mathbf{B}_{\mathrm{rig},\breve{L}}^+$ (see [8, Propositions II.3.1 and III.2.1]). In particular, we obtain a $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism,

$$\mathbf{B}_{\mathrm{rig},\breve{L}}^+[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{rig},\breve{L}}^+[\mu/t] \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V). \tag{4.2}$$

Moreover, note that from loc. cit., we have a natural $\breve{L}$-linear isomorphism of filtered φ-modules $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu\mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \mathbf{N}_{\breve{L}}(V)/\mu\mathbf{N}_{\breve{L}}(V)$ such that the largest Hodge–Tate weight of V equals s, i.e., the $[p]_q$-height of V. Since t/μ is a unit in $\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty})$ and $\mathbf{B}_{\mathrm{rig},\breve{L}}^+ \subset \widetilde{\mathbf{B}}_{\mathrm{rig},L}^+ \subset \mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty})$, therefore, extension of scalars of (4.2) yields the following $\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty})$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism:

$$\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V). \tag{4.3}$$

Lemma 4.5. *There exists a natural $(\varphi, G_{\breve{L}})$-equivariant commutative diagram as follows:*

$$\begin{tikzcd} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[d, "\wr"] \\ \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \arrow[r, equal] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V, \end{tikzcd}$$

where the top horizontal arrow is the extension of scalars of (4.3) along $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, and the left vertical arrow is the natural isomorphism as V is a crystalline representation of $G_{\breve{L}}$.

Proof. It remains to explain the right vertical arrow and the commutativity of the diagram. From Lemma 4.4 (2), note that we have a $(\varphi, G_{\breve{L}})$-equivariant isomorphism

$$\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbb{Z}_p} T,$$

and extending this isomorphism along $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ gives the right vertical isomorphism. Then, the commutativity of the diagram follows because the top horizontal arrow is also the $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear extension of the natural inclusion $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ (see [8, Section II.2]). ∎

4.2. Kisin's construction

Our goal is to construct a Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}^+_L$. To this end, we will adapt some ideas from [16, 36], generalising the results of Kisin in [35], to first construct a finite $[p]_q$-height module over $\mathbf{B}^+_{\mathrm{rig},L}$.

Let $E(X) := \frac{(1+X)^p-1}{X}$ in $\mathbb{Z}_p[\![X]\!]$ denote the cyclotomic polynomial. We equip $\mathbb{Z}_p[\![X]\!]$ with the cyclotomic Frobenius operator φ given by identity on $\mathbb{Z}_p$ and setting $\varphi(X) = (1+X)^p - 1$, and for $n \in \mathbb{N}$ we set $E_n(X) := \varphi^n(E(X))$. In particular, $\zeta_{p^{n+1}} - 1$ is a simple zero of $E_n(X)$, where $\zeta_{p^{n+1}}$ is a primitive p^{n+1}-th root of unity. For $X = \mu$, we will write $E_n(\mu) = \tilde{\xi}_n$, for $n \in \mathbb{N}$, and $E(\mu) = \varphi(\mu)/\mu = \tilde{\xi} = \tilde{\xi}_0 = [p]_q$.

Remark 4.6. Define $\phi_L : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ to be the homomorphism given by the Frobenius φ_L on L and set $\phi_L(\mu) = \mu$, i.e.,

$$\sum_{k \in \mathbb{N}} \iota(a_k)\mu^k \longmapsto \sum_{k \in \mathbb{N}} \iota\big(\varphi_L(a_k)\big)\mu^k,$$

where we used Lemma 2.5 to represent an element of $\mathbf{B}^+_{\mathrm{rig},L}$. Then, $\mathbf{B}^+_{\mathrm{rig},L}$ is flat and finite free of rank p^d over $\mathbf{B}^+_{\mathrm{rig},L}$, via the map ϕ_L. Similarly, let $\phi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ denote the homomorphism given by the Frobenius $\varphi_{\breve{L}}$ on $\breve{L}$ and set $\phi_{\breve{L}}(\mu) = \mu$. Moreover, note that we have $\varphi_{\breve{L}} : \breve{L} \xrightarrow{\sim} \breve{L}$, since the residue field of $\breve{L}$ is perfect, and therefore, we see that $\phi_{\breve{L}}$ is bijective on $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$, with its inverse given as $\phi^{-1}_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$, sending

$$\sum_{k \in \mathbb{N}} \iota(a_k)\mu^k \longmapsto \sum_{k \in \mathbb{N}} \iota\big(\varphi^{-1}_{\breve{L}}(a_k)\big)\mu^k.$$

Furthermore, from Section 2.1.4, recall that we have an injective homomorphism $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$, which is evidently compatible with ϕ_L on the left and $\phi_{\breve{L}}$ on the right.

Remark 4.7. We have that t/μ is in $\mathbf{B}^+_{\mathrm{rig},L} \hookrightarrow \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ and we can write $t/\mu = \prod_{n\in\mathbb{N}}(\tilde{\xi}_n/p)$ (see [8, Exemple I.3.3] and [38, Remarque 4.12]). The zeros of t/μ are given as $\zeta_{p^{n+1}} - 1$, for all $n \in \mathbb{N}$. Moreover, we have that $\phi_{\breve{L}}^{-n}(t/\mu) = t/\mu$, therefore, the zeros of $\phi_{\breve{L}}^{-n}(t/\mu)$ are given by $\zeta_{p^{n+1}} - 1$ as well.

Now, let $\widehat{\mathbf{B}}_{\breve{L},n}$ denote the completion of $\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\breve{L}}$ with respect to the maximal ideal generated by $\mu - (\zeta_{p^{n+1}} - 1)$. Since $\zeta_{p^{n+1}} - 1$ is a simple root of $\tilde{\xi}_n$, therefore, we see that $(\mu - (\zeta_{p^{n+1}} - 1)) = (\tilde{\xi}_n) \subset \widehat{\mathbf{B}}_{\breve{L},n}$. The local ring $\widehat{\mathbf{B}}_{\breve{L},n}$ naturally admits an action of $\Gamma_{\breve{L}}$ induced by the tensor product action of $\Gamma_{\breve{L}}$ on $\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\breve{L}}$. We put a filtration on $\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n]$ by setting

$$\mathrm{Fil}^r \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] := \tilde{\xi}_n^r \widehat{\mathbf{B}}_{\breve{L},n}, \quad \text{for } r \in \mathbb{Z}.$$

We have inclusions $\mathbf{B}^+_{\breve{L}} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \subset \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n]$.

Let $D_L := \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and $D_{\breve{L}} := \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$, and recall that using the φ-equivariant injection $L \to \breve{L}$, we have an isomorphism of filtered φ-modules $\breve{L} \otimes_L D_L \xrightarrow{\sim} D_{\breve{L}}$ from (2.5). Note that D_L (resp. $D_{\breve{L}}$) is an effective filtered φ-module over L (resp. over $\breve{L}$), i.e., $\mathrm{Fil}^0 D_L = D_L$ (resp. $\mathrm{Fil}^0 D_{\breve{L}} = D_{\breve{L}}$), and we have a φ-equivariant inclusion $D_L \subset D_{\breve{L}}$. Now, consider a map,

$$i_n : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \xrightarrow[\sim]{\phi_{\breve{L}}^{-n} \otimes \varphi_{D_{\breve{L}}}^{-n}} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}}, \tag{4.4}$$

where $\phi_{\breve{L}}^{-1} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is well defined by Remark 4.6, and $\varphi_{D_{\breve{L}}}$ is the (bijective) Frobenius-semilinear operator on $D_{\breve{L}}$. The map i_n is evidently well defined, and it extends to a map,

$$i_n : \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}.$$

Define a $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-module as follows:

$$\mathcal{M}_{\breve{L}}(D_{\breve{L}}) := \{x \in \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} \mid \forall n \in \mathbb{N},\ i_n(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}})\},$$

where we note that $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}$ is equipped with the tensor product Frobenius and $\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}}$ is equipped with the tensor product filtration. By [35, Lemma 1.2.2] and [36, Lemma 2.2.1], the $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-module $\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is finite free of rank $= \dim_{\breve{L}} D_{\breve{L}}$, stable under φ and $\Gamma_{\breve{L}}$, and such that the cokernel of the injective map

$$1 \otimes \varphi : \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \longrightarrow \mathcal{M}_{\breve{L}}(D_{\breve{L}})$$

is killed by $\tilde{\xi}^s$ (where s = height of T = height of V), and the action of $\Gamma_{\breve{L}}$ is trivial modulo μ. Moreover, from [36, Lemma 2.2.2], there exists a unique $\breve{L}$-linear section

$$\alpha : \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \longrightarrow \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t],$$

such that the image $\alpha(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}))$ is $\Gamma_{\breve{L}}$-invariant. Furthermore, the section α is φ-equivariant and it induces an isomorphism,

$$1 \otimes \alpha \colon \mathbf{B}^{+}_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \big(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})\big) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t]. \tag{4.5}$$

Finally, from [36, Proposition 2.2.6] note that we have a natural isomorphism $D_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ compatible with the respective Frobenii and filtrations, and under the isomorphism (4.5), the image of $D_{\breve{L}}$ coincides with $\alpha(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}))$.

Next, we note that the $\mathbf{B}^{\dagger}_{\mathrm{rig},\breve{L}}$-module $\mathbf{B}^{\dagger}_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is pure of slope zero using [35, Theorem 1.3.8] and [36, Proposition 2.3.3]. Then, from [36, Corollay 2.4.2] one obtains an $\mathbf{A}^{+}_{\breve{L}}$-module $N_{\breve{L}}$ finite free of rank $= \dim_{\breve{L}} D_{\breve{L}}$, equipped with a Frobenius-semilinear endomorphism φ and semilinear and continuous action of $\Gamma_{\breve{L}}$, and such that cokernel of the injective map $1 \otimes \varphi \colon \varphi^*(N_{\breve{L}}) \rightarrow N_{\breve{L}}$ is killed by $\tilde{\xi}^s$, the action of $\Gamma_{\breve{L}}$ is trivial modulo μ and $\mathbf{B}^{+}_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{A}^{+}_{\breve{L}}} N_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ compatible with the respective $(\varphi, \Gamma_{\breve{L}})$-actions.

Lemma 4.8. *There is a natural $\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism*

$$\beta : \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V).$$

Moreover, it restricts to a $\mathbf{B}^{+}_{\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism

$$\beta : N_{\breve{L}}[1/p] \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V).$$

Proof. Recall that by definition $\mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \mathbf{B}^{+}_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^{+}_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$, and consider the following diagram:

$$\begin{tikzcd}
\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbf{N}_{\mathrm{rig},\breve{L}}(V)[\mu/t] \\
\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \big(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})\big) \arrow[r, "\sim", "1\otimes\alpha"'] & \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t], \arrow[u, "\wr", "\beta"']
\end{tikzcd} \tag{4.6}$$

where the top horizontal arrow is (4.2), the bottom horizontal arrow is (4.5) and the left vertical arrow is the extension of scalars of the isomorphism $D_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ along the natural $(\varphi, \Gamma_{\breve{L}})$-equivariant map $\breve{L} \rightarrow \mathbf{B}^{+}_{\mathrm{rig},\breve{L}}$. For the right vertical arrow β, we consider $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ and $\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ as submodules of $\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$ and construct the map as follows: note that from the discussion after (4.2), we have a natural isomorphism $D_{\breve{L}} \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ of filtered φ-modules over $\breve{L}$. Moreover, from [36, Lemma 2.1.2], note that the action of $\Gamma_{\breve{L}}$ on $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is "$\mathcal{O}$-analytic" in the sense of [36, Section 2.1.3], where $\mathcal{O} = \mathbb{Z}_p$ in our case (this is true because in the language of op. cit., we see that the Lubin–Tate group law over $O_{\breve{L}}$ that we consider here is given by the Frobenius power series $(1 + X)^p - 1$ for the uniformiser p). Therefore, from the equivalence of categories in [36, Proposition 2.2.6] and its proof, it follows that we have a natural isomorphism

$$\beta : \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}\big(\mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu\mathbf{N}_{\mathrm{rig},\breve{L}}(V)\big) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V),$$

as $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-submodules of $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}$, compatible with the $(\varphi, \Gamma_{\breve{L}})$-action, and such that the reduction modulo μ of β induces natural isomorphisms,

$$\beta \bmod \mu \colon \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} D_{\breve{L}} \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu\mathbf{N}_{\mathrm{rig},\breve{L}}(V),$$

of filtered φ-modules over $\breve{L}$, where the latter isomorphism coincides with the one mentioned above (coming from the discussion after (4.2)). Now, by composing the natural $\breve{L}$-linear inclusion

$$\big(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})\big) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \big(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})\big),$$

with the inverse of the left vertical arrow, the top horizontal arrow and the inverse of the right vertical arrow of the diagram (4.6), provides an $\breve{L}$-linear section

$$\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \longrightarrow \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t],$$

satisfying the same properties as α (see the discussion before (4.5)). Therefore, from the uniqueness of α, it follows that the diagram commutes, thus proving our first claim. For the second claim, note that as a $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}$-module,

$$\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$$

is pure of slope zero, so from [36, Corollary 2.4.2] we conclude that the isomorphism β induces an isomorphism $\beta : N_{\breve{L}}[1/p] \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$ compatible with the $(\varphi, \Gamma_{\breve{L}})$-action. This allows us to conclude. ∎

Recall that from (2.5), we have an isomorphism of filtered φ-modules $\breve{L} \otimes_L D_L \xrightarrow{\sim} D_{\breve{L}}$.

Definition 4.9. Define the following $\mathbf{B}^+_{\mathrm{rig},L}$-module:

$$\begin{aligned}\mathcal{M}_L(D_L) &:= \big\{x \in \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L \mid \forall n \in \mathbb{N},\ i_n(x) \in \mathrm{Fil}^0\big(\widehat{\mathbf{B}}_{\breve{L},n}[1/\widetilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}}\big)\big\}\\ &= \big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L\big) \cap \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}.\end{aligned}$$

From Section 2.1.5, recall that we have a φ-equivariant injective homomorphism of L-algebras $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$, therefore, by definition $\mathcal{M}_L(D_L)$ is stable under the induced tensor product Frobenius semilinear-operator φ on $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}$. Then, by using Lemma 4.8 and the discussion preceding (4.2), we have φ-equivariant inclusions,

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}.$$

Moreover, from Lemma 2.10, recall that $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat, and from Lemma 2.11 we have that

$$\mathbf{B}^+_{\mathrm{rig},L} \cap (t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}},$$

or equivalently, $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] = \mathbf{B}^+_{\mathrm{rig},L}$. So, it follows that we have φ-equivariant inclusions,

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L \subset \mathcal{M}_L(D_L) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L. \tag{4.7}$$

Therefore, similar to (4.2), we obtain a φ-equivariant isomorphism,

$$\mathcal{M}_L(D_L)[\mu/t] \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L. \tag{4.8}$$

Lemma 4.10. *For each $n \in \mathbb{N}$, the natural morphism $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \widehat{\mathbf{B}}_{\breve{L},n}$ is flat, and therefore, the composition $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \widehat{\mathbf{B}}_{\breve{L},n}$ is flat.*

Proof. Note that we have a natural isomorphism

$$\breve{L}(\zeta_{p^n+1}) \xrightarrow{\sim} \big(\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},\breve{L}}\big)/I,$$

where $I \subset \breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ denotes the maximal ideal generated by $\mu - (\zeta_{p^{n+1}} - 1)$, and let $(\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},\breve{L}})_I$ denote its localisation at I. Then, the natural map

$$\big(\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},\breve{L}}\big)_I \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n},$$

realises the target as the I-adic completion of the source which is a discrete valuation ring, in particular, the preceding map is flat. It is easy to see that the first map in the claim factors as the composition

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \longrightarrow \breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \longrightarrow \big(\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},\breve{L}}\big)_I \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n},$$

where each map is flat, hence, the composition is flat. Furthermore, recall that the natural map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat (see Lemma 2.10), therefore, the composition

$$\mathbf{B}^+_{\mathrm{rig},L} \longrightarrow \mathbf{B}^+_{\mathrm{rig},\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n}$$

is flat as well. ∎

Lemma 4.11. *Let us consider $\widehat{\mathbf{B}}_{\breve{L},n}$ as a $\mathbf{B}^+_{\mathrm{rig},L}$-algebra via the composition*

$$i_{L,n} \colon \mathbf{B}^+_{\mathrm{rig},L} \longrightarrow \mathbf{B}^+_{\mathrm{rig},\breve{L}} \xrightarrow[\sim]{\phi_{\breve{L}}^{-n}} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n}.$$

Then, we have the following:

(1) *The homomorphism*

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} (\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}} \xleftarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L D_L,$$

induced by i_n in (4.4), is an isomorphism.

(2) *The isomorphism in* (1) *induces an isomorphism,*

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \sum_{i\in\mathbb{N}} \widetilde{\xi}_n^{-i} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^i D_L.$$

(3) *Extending the* $\mathbf{B}^+_{\mathrm{rig},L}$*-linear and* φ*-equivariant inclusion* $\mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ *along the map*

$$\mathbf{B}^+_{\mathrm{rig},L} \longrightarrow \mathbf{B}^+_{\mathrm{rig},\breve{L}},$$

yields the following φ*-equivariant isomorphism of* $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$*-modules:*

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \longrightarrow \mathcal{M}_{\breve{L}}(D_{\breve{L}}).$$

Moreover, $\mathcal{M}_L(D_L)$ *is a finite free* $\mathbf{B}^+_{\mathrm{rig},L}$*-module of* $\mathrm{rank} = \dim_L D_L$.

Proof. The proof follows in a manner similar to [35, Lemma 1.2.1]. Let us first note that the linearisation of i_n along the morphism

$$i_{\breve{L},n} \colon \mathbf{B}^+_{\mathrm{rig},\breve{L}} \xrightarrow[\sim]{\phi_{\breve{L}}^{-n}} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n},$$

yields an isomorphism,

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{\breve{L},n},\mathbf{B}^+_{\mathrm{rig},\breve{L}}} (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}) \xrightarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}}.$$

Moreover, from (2.5), we have that $D_{\breve{L}} \xrightarrow{\sim} \breve{L} \otimes_L D_L$, so we can write,

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_L D_L \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}.$$

Then, by extending the composition above along $i_{\breve{L},n} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \widehat{\mathbf{B}}_{\breve{L},n}$, we obtain the isomorphism claimed in (1):

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} (\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \xrightarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}}.$$

To show (2), let us write for $k \in \mathbb{N}$,

$$\mathcal{M}_{L,k}(D_L) := \big\{x \in \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L \text{ such that } i_k(x) \in \mathrm{Fil}^0\big(\widehat{\mathbf{B}}_{\breve{L},k}[1/\widetilde{\xi}_k] \otimes_{\breve{L}} D_{\breve{L}}\big)\big\}.$$

Then, we have that

$$\mathcal{M}_L(D_L) = \bigcap_{k\in\mathbb{N}} \mathcal{M}_{L,k}(D_L) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L.$$

Moreover, using Lemmas 2.10 and 4.10, we see that the morphism $i_{L,n} \colon \mathbf{B}^+_{\mathrm{rig},L} \to \widehat{\mathbf{B}}_{\breve{L},n}$ is flat. So, we get that

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) = \bigcap_{k\in\mathbb{N}} \big(\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L)\big) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L.$$

To prove the claim, it suffices to show that the isomorphism in (1) induces the following two bijections:

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L) \xrightarrow{\sim} \sum_{r\in\mathbb{N}} \widetilde{\xi}_n^{-r}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L,$$
$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L) \xrightarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n}[1/\widetilde{\xi}_n] \otimes_L D_L, \quad \text{for } k \neq n.$$

For the first claim, note that by definition, we have a natural inclusion

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L) \hookrightarrow \sum_{r\in\mathbb{N}} \widetilde{\xi}_n^{-r}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L.$$

To show the converse, note that we have

$$\widetilde{\xi}_n^{-1} = \tfrac{1}{p}\varphi^n(\mu/t)\varphi^{n+1}(t/\mu) \text{ in } \mathbf{B}_{\mathrm{rig},L}[\mu/t] \quad \text{and} \quad \phi_{\breve{L}}^{-n}(\widetilde{\xi}_n^{-1}) = \widetilde{\xi}_n^{-1}.$$

So, for any $r \in \mathbb{N}$ and $\widetilde{\xi}_n^{-r} a \otimes d$ in $\widetilde{\xi}_n^{-r}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L$, we have that $\widetilde{\xi}_n^{-r} \otimes \varphi^n(d)$ is in $\mathcal{M}_{L,n}(D_L)$, since $i_n(\widetilde{\xi}_n^{-r} \otimes \varphi^n(d)) = \widetilde{\xi}_n^{-r} \otimes d$ is in $\mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[1/\widetilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}})$. Therefore, $\widetilde{\xi}_n^{-r} a \otimes d = a \otimes i_n(\widetilde{\xi}_n^{-r} \otimes \varphi^n(d))$ is in the image of $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L)$. For the second claim, again note that by definition, we have a natural inclusion

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L) \hookrightarrow \widehat{\mathbf{B}}_{\breve{L},n}[1/\widetilde{\xi}_n] \otimes_L D_L.$$

For the converse, take $\widetilde{\xi}_n^{-r} a \otimes d$ in $\widehat{\mathbf{B}}_{\breve{L},n}[1/\widetilde{\xi}_n] \otimes_L D_L$, for some $r \in \mathbb{N}$. Then, note that $\widetilde{\xi}_n$ is a unit in $\widehat{\mathbf{B}}_{\breve{L},k}$, since $\zeta_{p^{n+1}} - 1$ is not a root of $\widetilde{\xi}_k$ as $k \neq n$. So, we get that

$$i_k\big(\widetilde{\xi}_n^{-r} \otimes \varphi^n(d)\big) = \widetilde{\xi}_n^{-r} \otimes \varphi^{n-k}(d)$$

is in $\widehat{\mathbf{B}}_{\breve{L},k} \otimes_{\breve{L}} D_{\breve{L}} \subset \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},k}[1/\widetilde{\xi}_k] \otimes_{\breve{L}} D_{\breve{L}})$, since $\mathrm{Fil}^0 D_L = D_L$. In particular, we have that $\widetilde{\xi}_n^{-r} \otimes \varphi^n(d)$ is in $\mathcal{M}_{L,k}(D_L)$. Therefore, $\widetilde{\xi}_n^{-r} a \otimes d = a \otimes i_n(\widetilde{\xi}_n^{-r} \otimes \varphi^n(d))$ is in the image of $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_{L,n},\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L)$.

For (3), note that we have natural inclusions

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_L D_L \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s\mathbf{B}_{\mathrm{rig},\breve{L}} \otimes_L D_L,$$

where the first two inclusions follow since the map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat (see Lemma 2.10) and $\mathcal{M}_L(D_L) \subset (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ from (4.7). So, we get that $(t/\mu)^s$ kills the cokernel of the map

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \longrightarrow \mathcal{M}_{\breve{L}}(D_{\breve{L}}).$$

Moreover, note that

$$\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}$$

is a closed submodule by [35, Lemmas 1.1.5 and 1.2.2]. Now, since $\mathbf{B}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is a closed subring, therefore, we get that $\mathcal{M}_L(D_L) \subset (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ is closed and hence finite free over $\mathbf{B}^+_{\mathrm{rig},L}$ by Remark 2.14, and of rank $= \dim_L D_L$ by the isomorphism shown below.

Let us write $\mathbf{B}^+_{\mathrm{rig},L} = \lim_\rho \mathcal{O}(D(L,\rho))$ as the limit of rings of analytic functions on closed disks $D(L,\rho)$ of radius $0 < \rho < 1$ (see Remark 2.6); similarly let us write

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} = \lim_\rho \mathcal{O}\big(D(\breve{L},\rho)\big).$$

Since $\mathcal{M}_L(D_L)$ and $\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ are free modules over their respective base rings, therefore, we have that

$$\mathcal{M}_L(D_L) \xrightarrow{\sim} \lim_\rho \big(\mathcal{O}\big(D(L,\rho)\big) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)\big),$$
$$\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \lim_\rho \big(\mathcal{O}\big(D(\breve{L},\rho)\big) \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})\big).$$

Then, to show our claim, it is enough to show that the natural map,

$$\mathcal{O}\big(D(\breve{L},\rho)\big) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \longrightarrow \mathcal{O}\big(D(\breve{L},\rho)\big) \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}}), \tag{4.9}$$

is a bijection. Note that $\mathcal{O}(D(\breve{L},\rho))$ is a domain, so injectivity of (4.9) can be checked after passing to the fraction field of $\mathcal{O}(D(\breve{L},\rho))$. To check that (4.9) is surjective, let Q denote the cokernel of (4.9) and we will show that $Q = 0$. Note that Q is a finitely generated $S := \mathcal{O}(D(\breve{L},\rho))$-module killed by $(t/\mu)^s$ and S is a principal ideal domain (see [13, Chapter 2, Corollary 10]). So, by the structure theorem of finitely generated modules over S, we can write $Q = \oplus S/\mathfrak{a}_i$, where $\mathfrak{a}_i = (a_i)$ for some non-zero primary elements $a_i \in S$ and such that a_i divides $(t/\mu)^s$, for each i. Note that $\sqrt{\mathfrak{a}_i}$ is a maximal ideal of S and $Q_{\sqrt{\mathfrak{a}_i}} = S/\mathfrak{a}_i$, so to obtain that $Q = 0$, it is enough to show that $Q_{\sqrt{\mathfrak{a}_i}} = 0$. From [13, Chapter 2, Corollary 13] note that each maximal ideal $\sqrt{\mathfrak{a}_i}$ corresponds to a zero of $(t/\mu)^s$, in particular, we are reduced to showing that Q vanishes at zeros of t/μ. This follows from (2). Hence, we get that (4.9) is an isomorphism and passing to the limit over ρ we obtain that $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$, as desired. ∎

Lemma 4.12. *We have following properties for the $\mathbf{B}^+_{\mathrm{rig},L}$-module $\mathcal{M}_L(D_L)$:*

(1) *The cokernel of the injective map $1 \otimes \varphi\colon \varphi^*(\mathcal{M}_L(D_L)) \to \mathcal{M}_L(D_L)$ is killed by $[p]_q^s$.*

(2) *$\mathcal{M}_L(D_L)$ is pure of slope zero, i.e., the $\mathbf{B}^\dagger_{\mathrm{rig},L}$-module $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)$ is pure of slope zero in the sense of* [33, Section 6.3].

Proof. For (1), let us first note the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_L(D_L) & \longrightarrow & \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_{\breve{L}}(D_{\breve{L}}) & \longrightarrow & \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} & \longrightarrow & \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} Q & \longrightarrow & 0,
\end{array}$$

where Q is the cokernel of the left horizontal arrow in the first row. All the maps above are

φ-equivariant and the vertical maps are injective (see (4.7), Definition 4.9 and Lemmas 4.8 and 4.11 (3)). From Remarks 2.8, 2.9 and Lemmas 2.10, 2.11, recall that the maps

$$\varphi_L : \mathbf{B}^+_{\mathrm{rig},L} \longrightarrow \mathbf{B}^+_{\mathrm{rig},L} \quad \text{and} \quad \varphi_{\check{L}} : \mathbf{B}^+_{\mathrm{rig},\check{L}} \longrightarrow \mathbf{B}^+_{\mathrm{rig},\check{L}}$$

are faithfully flat (we write φ with subscripts to avoid confusion), the natural map

$$\mathbf{B}^+_{\mathrm{rig},L} \longrightarrow \mathbf{B}^+_{\mathrm{rig},\check{L}}$$

is flat and

$$\mathbf{B}^+_{\mathrm{rig},\check{L}} \cap \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] = \mathbf{B}^+_{\mathrm{rig},L}.$$

Then, using Lemma 4.11 (3) and that $D_{\check{L}} \xrightarrow{\sim} \check{L} \otimes_L D_L$ from (2.5), we get that

$$\begin{aligned}
\varphi_{\check{L}}^*(\mathcal{M}_{\check{L}}(D_{\check{L}})) &\xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\check{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi_L^*(\mathcal{M}_L(D_L)),\\
\varphi_L^*\big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L\big) &\xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi_L^*(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L)\\
&\subset \mathbf{B}^+_{\mathrm{rig},\check{L}}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]} \varphi_L^*\big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L\big)\\
&\xrightarrow{\sim} \varphi_{\check{L}}^*\big(\mathbf{B}^+_{\mathrm{rig},\check{L}}[\mu/t] \otimes_{\check{L}} D_{\check{L}}\big).
\end{aligned}$$

So, from the preceding discussion and the exactness of both rows in the diagram above, it follows that,

$$\begin{aligned}
\varphi_L^*\big(\mathcal{M}_L(D_L)\big) &= \big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi_L^*(\mathcal{M}_L(D_L))\big) \cap \big(\mathbf{B}^+_{\mathrm{rig},\check{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi_L^*(\mathcal{M}_L(D_L))\big)\\
&\xrightarrow{\sim} \varphi_L^*\big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L\big) \cap \varphi_{\check{L}}^*\big(\mathcal{M}_{\check{L}}(D_{\check{L}})\big)\\
&\hookrightarrow \varphi_{\check{L}}^*\big(\mathbf{B}^+_{\mathrm{rig},\check{L}}[\mu/t] \otimes_{\check{L}} D_{\check{L}}\big).
\end{aligned}$$

Now, let x in $\mathcal{M}_L(D_L) \subset \mathcal{M}_{\check{L}}(D_{\check{L}})$, then there exists some y in $\varphi_{\check{L}}^*(\mathcal{M}_{\check{L}}(D_{\check{L}}))$ such that $(1 \otimes \varphi)y = \tilde{\xi}^s x$. Recall that $1 \otimes \varphi : \varphi_L^*(D_L) \xrightarrow{\sim} D_L$ and $\varphi(\mu/t) = (\tilde{\xi}\mu)/(pt)$, therefore, the cokernel of the induced map $1 \otimes \varphi : \varphi_L^*((\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \rightarrow (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ is killed by $\tilde{\xi}^s$, in particular, $\tilde{\xi}^s x$ is in $(1 \otimes \varphi)\varphi_L^*((\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L)$. Since $1 \otimes \varphi$ is injective on $\varphi_{\check{L}}^*((\mu/t)^s \mathbf{B}^+_{\mathrm{rig},\check{L}} \otimes_{\check{L}} D_{\check{L}})$, therefore, we get that y is in

$$\varphi_L^*\big((\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L\big) \cap \varphi_{\check{L}}^*\big(\mathcal{M}_{\check{L}}(D_{\check{L}})\big) = \varphi_L^*\big(\mathcal{M}_L(D_L)\big).$$

In particular, the cokernel of the natural map $1 \otimes \varphi : \varphi_L^*(\mathcal{M}_L(D_L)) \rightarrow \mathcal{M}_L(D_L)$ is killed by $\tilde{\xi}^s$.

For (2), note that from Lemma 4.11 (3), we have that

$$\mathbf{B}^+_{\mathrm{rig},\check{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \mathcal{M}_{\check{L}}(D_{\check{L}}).$$

Moreover, from [33, Theorem 6.10] we obtain a slope filtration on $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)$ such that base changing this slope filtration along

$$\mathbf{B}^\dagger_{\mathrm{rig},L} \longrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\check{L}}$$

gives a slope filtration on $\mathbf{B}^{\dagger}_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. However, from [35, Theorem 1.3.8] and [36, Proposition 2.3.3], we know that $\mathbf{B}^{\dagger}_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is pure of slope zero. Therefore, we must have that $\mathcal{M}_L(D_L)$ is pure of slope zero. ∎

4.3. Stability under Galois action

In this section, we will define and study a finite free (φ, Γ_L)-module $\mathbf{N}_{\mathrm{rig},L}(V)$ over $\mathbf{B}^{+}_{\mathrm{rig},L}$, of slope zero, and obtained from the $\mathbf{B}^{+}_{\mathrm{rig},L}$-module in Definition 4.9. From Section 2.1.4, recall that we have identifications

$$\widetilde{\mathbf{B}}^{+}_{\mathrm{rig},L} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{H_L} = \bigcap_{n\in\mathbb{N}} \varphi^n\big(\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty})\big),$$

where the last equality follows because $\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty}) = \mathbf{B}^{+}_{\mathrm{cris}}(O_{\overline{L}})^{H_L}$ (see Section 2.1.1). Moreover, using the isomorphism in Lemma 2.19 and Remark 2.22, we see that

$$\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$$

is equipped with a continuous action of Γ_L. Note that we have $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and we claim the following.

Lemma 4.13. *The $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig},L}$-module $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the induced action of Γ_L. For any $a \otimes x$ in $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and $g \in \Gamma_L$, this action can be explicitly described by the following formula:*

$$g(a \otimes x) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} \partial_i^{k_i}(x) \prod_{i=1}^{d} \big(g([X_i^\flat]) - [X_i^\flat]\big)^{[k_i]}.$$

Proof. The non-canonical $(\varphi, G_{\breve{L}})$-equivariant L-algebra structure on $\mathcal{O}\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty})$ from Section 2.1.1, extends to a $(\varphi, G_{\breve{L}})$-equivariant $\breve{L}$-algebra structure, and thus it provides $(\varphi, G_{\breve{L}})$-equivariant L-algebra and $\breve{L}$-algebra structures on $\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty})$, via the composition $L \to \breve{L} \to \mathcal{O}\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty}) \twoheadrightarrow \mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty})$, where the last map is the projection map described before Lemma 2.19. Moreover, recall that we have

$$L \otimes_{\varphi^n, L} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \quad \text{for all } n \in \mathbb{N}.$$

So, we can write,

$$\begin{aligned}\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\xrightarrow{\sim} \mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)\\ &\xrightarrow{\sim} \mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\varphi_{\breve{L}}^{-n}, \breve{L}} \big(\breve{L} \otimes_{\varphi_{\breve{L}}^{n}, \breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)\big).\end{aligned}$$

Applying φ^n to the isomorphism above gives that

$$\varphi^n\big(\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big) \xrightarrow{\sim} \varphi^n\big(\mathbf{B}^{+}_{\mathrm{cris}}(O_{L_\infty})\big) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

Note that the Frobenius-semilinear endomorphism φ of $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ commutes with the action of Γ_L described in Remark 2.22. Therefore, the following is stable under the Γ_L-action:

$$\bigcap_{n\in\mathbb{N}} \varphi^n\big(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big) \xrightarrow{\sim} \Big(\bigcap_{n\in\mathbb{N}} \varphi^n\big(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})\big)\Big) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$$
$$= \widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

The second claim follows from Lemma 2.23. ∎

Extending the isomorphism in (4.2) along the natural map $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \to \widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t]$ (see Section 2.1.5), yields a φ-equivariant isomorphism

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V).$$

Now, recall that for any $g \in \Gamma_L$, we have that $g(t) = \chi(g)t$ and $g(\mu) = (1+\mu)^{\chi(g)} - 1$, where χ is the p-adic cyclotomic character. Now, using that

$$\breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},\breve{L}}(V),$$

we get φ-equivariant isomorphisms

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V),$$

and we equip the last term with a Γ_L-action by transport of structure via this isomorphism. In particular, the preceding discussion induces an action of Γ_L over

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V).$$

Our next goal is to show that $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is stable under the action of Γ_L induced from the Γ_L-action on

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V).$$

We will do this by embedding everything into $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$.

Let us fix some elements in $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$. For $n \in \mathbb{N}$, let $n = (p-1)f(n) + r(n)$ with $r(n), f(n) \in \mathbb{N}$ and $0 \leqslant r(n) < p-1$. Set $t^{\{n\}} := \frac{t^n}{f(n)!p^{f(n)}}$ and define

$$\Lambda := \Big\{\sum_{n\in\mathbb{N}} a_n t^{\{n\}}, \text{ with } a_n \in O_F \text{ such that } a_n \to 0 \text{ as } n \to +\infty\Big\}$$
$$= O_F\big[t, (t^{p-1}/p)^{[k]}, k \in \mathbb{N}\big]^\wedge \xrightarrow{\sim} O_F\big[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}\big]^\wedge,$$

where $^\wedge$ denotes the p-adic completion and the isomorphism is induced by the map $t \mapsto \log(1+\mu)$ with the inverse map given as $\mu \mapsto \exp(t) - 1$ (see [15, Lemme 6.2.13]). Furthermore, for $r \in \mathbb{N}$ and $A := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}), \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}), \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ or $\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$, set

$$I^{(r)}A := \big\{a \in A \text{ such that } \varphi^n(a) \in \mathrm{Fil}^r A \text{ for all } n \in \mathbb{N}\big\}. \tag{4.10}$$

Lemma 4.14. *We note the following facts:*

(1) $t^{p-1} \in p\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, $t^{\{n\}} \in \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ *and* t/μ *is a unit in* $\Lambda \subset \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$.

(2) *For any* $r \in \mathbb{N}$, *we have that*

$$I^{(r)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu^r \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \quad \text{and} \quad I^{(p-1)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu^{p-1}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}).$$

(3) *Let* $S = O_F[\![\mu]\!]$, *then we have the following natural and continuous (for the* p*-adic topology) isomorphism of* $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$*-algebras:*

$$\begin{aligned} \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \,\widehat{\otimes}_S\, \Lambda &\xrightarrow{\sim} \mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) \\ \sum_{k\in\mathbb{N}} a_k \otimes (\mu^{p-1}/p)^{[k]} &\longmapsto \sum_{k\in\mathbb{N}} a_k (\mu^{p-1}/p)^{[k]}, \end{aligned}$$

(4) *The ideal* $I^{(r)}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ *is topologically generated by* $t^{\{s\}}$, *for* $s \geqslant r$.

(5) *The following natural map is injective:*

$$\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \longrightarrow \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}),$$

and its cokernel is killed by $m!p^m$, *where* $m = \lfloor \frac{r}{p-1} \rfloor$.

Similar statements are true for $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$ *and* $\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$.

Proof. All claims except (3) follow from [28, Section 5.2] and [45, Section A3]. The proof of the claim in (3) follows in a manner similar to the proof of [15, Proposition 6.2.14]. ∎

Remark 4.15. Note that the $\mathbb{Q}_p$-algebras $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})[1/p]$, $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})[1/p]$ and $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}$ naturally embed into the $\mathbb{Q}_p$-algebra $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$, and we equip the former rings with a filtration induced from the natural filtration on $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$ (see Section 2.1.1). Then, one can define ideals similar to (4.10) for these rings and from Lemma 4.14 (5), we have the following natural isomorphisms:

$$\begin{aligned} \mathbf{B}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)}\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) &\xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}), \\ \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/I^{(r)}\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) &\xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})/I^{(r)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}). \end{aligned}$$

Proposition 4.16. *The* $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})$*-module*

$$\mathbf{N}_{\breve{L},\infty}(V) := \mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset \big(\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V\big)^{H_L} = \widetilde{\mathbf{D}}^+_L(V),$$

is stable under the residual action of Γ_L *on* $\widetilde{\mathbf{D}}^+_L(V)$ *and we equip* $\mathbf{N}_{\breve{L},\infty}(V)$ *with this action. Then, we have a natural* Γ_L*-equivariant embedding*

$$\mathbf{N}_{\breve{L},\infty}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

where we use Remark 2.21 and (2.9) to equip the right hand term with an action of Γ_L.

Proof. From Lemma 4.4 (2), consider the following exact sequence:

$$0 \longrightarrow \mu^s \widetilde{\mathbf{D}}_L^+(V) \longrightarrow \mathbf{N}_{\check{L},\infty}(V) \longrightarrow \mathbf{N}_{\check{L},\infty}(V)/\mu^s \widetilde{\mathbf{D}}_L^+(V) \longrightarrow 0, \tag{4.11}$$

where we know that $\mu^s \widetilde{\mathbf{D}}_L^+(V) \subset \widetilde{\mathbf{D}}_L^+(V)$ is stable under the action of Γ_L. Therefore, to show that the middle term above is stable under the action of Γ_L, it is enough to show that for the inclusion,

$$\begin{aligned}\mathbf{N}_{\check{L},\infty}(V)/\mu^s \widetilde{\mathbf{D}}_L^+(V) \subset \widetilde{\mathbf{D}}_L^+(V)/\mu^s \widetilde{\mathbf{D}}_L^+(V) &\subset \big(\mathbf{B}_{\mathrm{inf}}(O_{\bar{L}})/\mu^s \mathbf{B}_{\mathrm{inf}}(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V\big)^{H_L} \\ &\subset \mathbf{B}_{\mathrm{inf}}(O_{\bar{L}})/\mu^s \mathbf{B}_{\mathrm{inf}}(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V,\end{aligned}$$

the image of the first term in the last term is stable under the action of G_L.

Note that from Lemma 4.5, we have a natural $\mathbf{B}_{\mathrm{cris}}(O_{\bar{L}})$-linear and $(\varphi, G_{\check{L}})$-equivariant isomorphism $\mathbf{B}_{\mathrm{cris}}(O_{\bar{L}}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V$. In view of Remark 4.15, let us set,

$$M := \big(I^{(s)}\mathbf{B}_{\mathrm{cris}}^+(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V\big) \cap \big(\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V)\big) \subset \mathbf{B}_{\mathrm{cris}}(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V.$$

Then we obtain the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mu^s \widetilde{\mathbf{D}}_L^+(V) \arrow[r] \arrow[d] & \mathbf{N}_{\check{L},\infty}(V) \arrow[r] \arrow[d] & \mathbf{N}_{\check{L},\infty}(V)/\mu^s \widetilde{\mathbf{D}}_L^+(V) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & \mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V) \arrow[r] & \big(\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V)\big)/M \arrow[r] & 0, \end{tikzcd}$$

where the left vertical arrow is injective by Lemma 4.4 (2) and the middle vertical arrow is obviously injective. Moreover, we have the following.

Lemma 4.17. *The natural embedding* $\mathbf{N}_{\check{L},\infty}(V) \subset \mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V)$ *induces the following* $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})$*-linear and* $\Gamma_{\check{L}}$*-equivariant isomorphism:*

$$\mathbf{N}_{\check{L},\infty}(V)/\mu^s \widetilde{\mathbf{D}}_L^+(V) \xrightarrow{\sim} \big(\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V)\big)/M.$$

Proof. First, we observe that by Lemma 4.4 (2) we have that,

$$\begin{aligned} M \cap \mathbf{N}_{\check{L},\infty}(V) &= \big(I^{(s)}\mathbf{B}_{\mathrm{cris}}^+(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V\big) \cap \mathbf{N}_{\check{L},\infty}(V) \\ &\subset \big(I^{(s)}\mathbf{B}_{\mathrm{cris}}^+(O_{\bar{L}}) \otimes_{\mathbb{Q}_p} V\big) \cap \widetilde{\mathbf{D}}_L^+(V) \subset \mu^s \widetilde{\mathbf{D}}_L^+(V). \end{aligned}$$

Therefore, we get that the rightmost vertical map in the diagram above is injective. Next, we need to show that

$$\mathbf{N}_{\check{L},\infty}(V) + M = \mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V).$$

It is clear that the left expression is contained in the right. To show the converse, let x be an element of $\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) \otimes_{\mathbf{B}_{\check{L}}^+} \mathbf{N}_{\check{L}}(V)$. Then, for $m \in \mathbb{N}$ large enough, we have that $p^m x$

is in $\mathbf{A}_{\text{cris}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T)$. By the isomorphism in Lemma 4.14 (3), for $r = \lceil \frac{s}{p-1} \rceil$, $k \in \mathbb{N}$ and x_k in $\mathbf{N}_{\breve{L}}(T)$ such that $x_k \to 0$ as $k \to +\infty$, we can write

$$p^m x = \sum_{k \in \mathbb{N}} x_k (\mu^{p-1}/p)^{[k]} = \sum_{0 \leqslant k \leqslant r-1} x_k (\mu^{p-1}/p)^{[k]} + \sum_{k \geqslant r} x_k (\mu/p)^{[k]}.$$

Clearly, the first summation in the rightmost expression is in $\mathbf{N}_{\breve{L},\infty}(V)$. Moreover, from Lemma 4.14 (1) there exists some $v \in \Lambda^\times$, such that $\mu^{p-1}/p = vt^{p-1}/p$. Therefore, we obtain that the second summation is in

$$\big(I^{(s)}\mathbf{A}_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T\big) \cap \big(\mathbf{A}_{\text{cris}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T)\big) \subset M.$$

Hence, x is in $\mathbf{N}_{\breve{L},\infty}(V) + M$. ∎

Next, let consider the following diagram:

$$\begin{tikzcd}
\mathbf{B}^+_{\text{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\text{cris},\breve{L}}(V) \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{B}^+_{\text{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[d, hook] \\
\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_{\breve{L}} \mathbf{D}_{\text{cris},\breve{L}}(V) \arrow[r, "\sim"] & \mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \arrow[d, equal] \\
\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \\
\big(\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V)\big)^{\partial=0} \arrow[r, "\sim"] \arrow[u, "\wr"] & \big(\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V\big)^{\partial=0}. \arrow[u, "\wr"]
\end{tikzcd} \tag{4.12}$$

In (4.12) the bottom horizontal arrow is a (φ, G_L)-equivariant isomorphism since V is a crystalline representation of G_L. The left vertical arrow from the fourth to the third row is induced by the projection $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{L}})$, via $X_i \mapsto [X_i^\flat]$, it admits a section as in (2.8), it is evidently φ-equivariant and it is G_L-equivariant since the codomain is equipped with a G_L-action by transport of structure from the domain (see Remark 2.21). The right vertical arrow from the fourth to the third row is also induced by the projection $\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\text{cris}}(O_{\overline{L}})$, it admits a natural section

$$\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \longrightarrow \big(\mathcal{O}\mathbf{B}_{\text{cris}}(O_{\overline{L}}) \otimes V\big)^{\partial=0},$$

and it is naturally (φ, G_L)-equivariant. The horizontal arrow in the third row is the inverse of the isomorphism in Lemma 2.19, which is given as the composition of the inverse of the bottom left vertical arrow, the bottom horizontal arrow and the bottom right vertical arrow, and it is (φ, G_L)-equivariant by the preceding discussion and Remark 2.21. In particular, we get that the lower square is commutative and (φ, G_L)-equivariant. Next, the left vertical arrow from the third to the second row is an isomorphism since $\breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\text{cris},L}(V) \xrightarrow{\sim} \mathbf{D}_{\text{cris},\breve{L}}(V)$ by (2.5) and its $(\varphi, G_{\breve{L}})$-equivariance can either be checked by the explicit formula in Remark 2.21 or by observing that the non-canonical map $L \to \check{L} \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ is $(\varphi, G_{\check{L}})$-equivariant (see the proof of Lemma 4.13). The horizontal arrow in the second row is a $(\varphi, G_{\check{L}})$-equivariant isomorphism since V is a crystalline representation of $G_{\check{L}}$. Commutativity of the middle square follows since the outer square between the second and the fourth row as well as the lower square are commutative. Commutativity and $(\varphi, G_{\check{L}})$-equivariance of the top square follows from Lemma 4.5.

Furthermore, in the diagram (4.12), the image of composition of the top two left vertical maps inside $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of G_L by Remark 2.21. So the image of composition of the top two right vertical maps inside $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L, and it follows that its image

$$\big(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V)\big)/M \subset \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})/I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$$
$$\xrightarrow{\sim} \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V,$$

is stable under the action of G_L. Therefore, from Lemma 4.17, we obtain that the image of $\mathbf{N}_{\check{L},\infty}(V)/\mu^s \widetilde{\mathbf{D}}^+_{\check{L}}(V) \subset \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L. Hence, from (4.11) we conclude that $\mathbf{N}_{\check{L},\infty}(V)$ is stable under the action of Γ_L and the following natural composition is Γ_L-equivariant:

$$\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \quad (4.13)$$

This concludes our proof. ∎

Recall that $\mathbf{N}_{\mathrm{rig},\check{L}}(V) = \mathbf{B}^+_{\mathrm{rig},\check{L}} \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V)$ and we note the following.

Corollary 4.18. *The $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}$-linear extension of the Γ_L-equivariant embedding*

$$\mathbf{N}_{\check{L},\infty}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$$

from Proposition 4.16, induces an identification of the following $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}$-submodules of $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$:

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})} \mathbf{N}_{\check{L},\infty}(V) = \widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V) = \widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},\check{L}}} \mathbf{N}_{\mathrm{rig},\check{L}}(V),$$

which are stable under the induced Γ_L-action.

Proof. The equalities in the claim follows from the definitions and their compatibility with Γ_L-actions follows from (4.13). Then, by using (4.2), we see that

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\check{L}} \mathbf{D}_{\mathrm{cris},\check{L}}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\check{L}} \mathbf{D}_{\mathrm{cris},\check{L}}(V)$$
$$= \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

In particular, we obtain that

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V)$$

is a $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}$-submodule of $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, and the stability of $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V)$ under the Γ_L-action now follows from Proposition 4.16. ∎

Recall that from Definition 4.9, we have a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule $\mathcal{M}_L(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \subset \mathcal{M}_{\breve{L}}(\mathbf{D}_{\mathrm{cris},\breve{L}}(V))$ stable under the action of $(\varphi, \Gamma_{\breve{L}})$, and from Lemma 4.8, we have a $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism $\beta : \mathcal{M}_{\breve{L}}(\mathbf{D}_{\mathrm{cris},\breve{L}}(V)) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$. Let us define a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule of $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ as,

$$\mathbf{N}_{\mathrm{rig},L}(V) := \beta\big(\mathcal{M}_L\big(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big)\big) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V). \tag{4.14}$$

Note that the map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ constructed in Section 2.1.5 is $(\varphi, \Gamma_{\breve{L}})$-equivariant, therefore, from (4.14) we obtain a natural $\mathbf{B}^+_{\mathrm{rig},L}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism $\beta : \mathcal{M}_L(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},L}(V)$. In particular, from Lemma 4.11 (3), we obtain that $\mathbf{N}_{\mathrm{rig},L}(V)$ is a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module of rank $= \dim_{\mathbb{Q}_p} V$, and the natural $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linear map $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \to \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is a $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism, since β is $(\varphi, \Gamma_{\breve{L}})$-equivariant. Moreover, from Lemma 4.12, it follows that $\mathbf{N}_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height and pure of slope zero. Now, consider the following diagram:

$$\begin{tikzcd}
\mathcal{M}_L\big(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big)[\mu/t] \arrow[r, "\sim"] \arrow[rr, bend left, "\beta", "\sim"'] \arrow[d] & \mathbf{B}^+_{\mathrm{rig},L}[\mu/t]\otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[d] & \mathbf{N}_{\mathrm{rig},L}(V)[\mu/t] \arrow[l, "\sim"'] \arrow[d] \\
\mathcal{M}_{\breve{L}}\big(\mathbf{D}_{\mathrm{cris},\breve{L}}(V)\big)[\mu/t] \arrow[r, "\sim"] \arrow[rr, bend right, "\beta"', "\sim"] & \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t]\otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) & \mathbf{N}_{\mathrm{rig},\breve{L}}(V)[\mu/t]. \arrow[l, "\sim"']
\end{tikzcd} \tag{4.15}$$

In the diagram (4.15), all vertical arrows are natural inclusions. In the bottom row, the left to right horizontal arrow is the inverse of the composition of the lower horizontal arrow and the left vertical arrow of diagram (4.6), the right to left horizontal arrow is the inverse of (4.2), the curved arrow is the map β in Lemma 4.8 and the resulting triangle commutes by diagram (4.6). In the top row, the left to right horizontal arrow is the isomorphism in (4.8), the curved arrow is from (4.14), the right to left horizontal arrow is the composition of the inverse of β with the inverse of (4.8) and the resulting triangle commutes by definition. Moreover, the two inner squares commute by definition and all maps are $(\varphi, \Gamma_{\breve{L}})$-equivariant.

Using the diagram (4.15) and Definition 4.9, we can write

$$\mathbf{N}_{\mathrm{rig},L}(V) = \big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big) \cap \mathbf{N}_{\mathrm{rig},\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V),$$

in particular, we will now consider $\mathbf{N}_{\mathrm{rig},L}(V)$ as a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule of $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Furthermore, from Lemma 2.23, recall that the $\mathbf{B}^+_{\mathrm{rig},L}$-submodule

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$$

is stable under the action of Γ_L on the latter, and we equip the former with the induced Γ_L-action. Since we have that $g(t) = \chi(g)t$ and $g(\mu) = (1+\mu)^{\chi(g)} - 1$, for any $g \in \Gamma_L$ and χ the p-adic cyclotomic character, therefore, the preceding Γ_L-action naturally extends to $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$.

Proposition 4.19. *The $\mathbf{B}^+_{\mathrm{rig},L}$-submodule $\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L. Moreover, the preceding inclusion extends to a $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]$-linear and (φ, Γ_L)-compatible isomorphism,*

$$\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \tag{4.16}$$

Proof. From Corollary 4.18 and the discussion after (4.14), we have that

$$\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V),$$

which is stable under the action of Γ_L on $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Moreover, using Lemma 2.23 and the discussion after (4.15), we have a Γ_L-equivariant embedding

$$\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

Therefore, inside $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, the following intersection is stable under the action of Γ_L:

$$\begin{aligned}\big(\tilde{\mathbf{B}}^+_{\mathrm{rig},L} &\otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big) \cap \big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\big)\\ &= \big(\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big) \cap \big(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big)\\ &= \big(\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},L}[\mu/t]\big) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) = \mathbf{N}_{\mathrm{rig},L}(V).\end{aligned}$$

The first equality follows from (4.15) and the second equality follows from Lemma 2.12 and the fact that $\mathbf{N}_{\mathrm{rig},L}(V)$ is finite free over $\mathbf{B}^+_{\mathrm{rig},L}$. This proves the first claim. For the second claim, note that by definition, the $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]$-linear extension of the (φ, Γ_L)-equivariant inclusion $\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, coincides with the top right horizontal arrow of the diagram (4.15). Hence, the isomorphism in (4.16) follows. ∎

Corollary 4.20. *The action of Γ_L on $\mathbf{N}_{\mathrm{rig},L}(V)$ is trivial modulo μ.*

Proof. Note that we have $g(\mu) = (1+\mu)^{\chi(g)} - 1$ and $g(t) = \chi(g)t$, for any $g \in \Gamma_L$ and χ the p-adic cyclotomic character, in particular, $(g-1)(\mu/t) = \mu u_g(\mu/t)$, for some $u_g \in \mathbf{B}^+_L$. Therefore, using Lemma 2.24 it follows that the action of Γ_L is trivial modulo μ on $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xleftarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)$ (see (4.16)). Next, from Proposition 4.19, note that we have a (φ, Γ_L)-equivariant inclusion

$$\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

Let x be in $\mathbf{N}_{\mathrm{rig},L}(V)$, then for any $g \in \Gamma_L$, we have that $(g-1)x$ is in $\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ and $(g-1)x$ is also in $\mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)$. Now, inside $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)[\mu/t]$, we have that,

$$\begin{aligned}\mathbf{N}_{\mathrm{rig},\breve{L}}(V) &\cap \big(\mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big)\\ &= \big(\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big) \cap \big(\mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big)\\ &= \big(\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]\big) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) = \mu\mathbf{N}_{\mathrm{rig},L}(V),\end{aligned}$$

where the first equality follows from the isomorphism (see the discussion after (4.14)),

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V),$$

the second equality follows since $\mathbf{N}_{\mathrm{rig},L}(V)$ is free over $\mathbf{B}^+_{\mathrm{rig},L}$ and the last equality follows from Lemma 2.11. Hence, we conclude that $(g-1)x$ is in $\mu\mathbf{N}_{\mathrm{rig},L}(V)$, for any x in $\mathbf{N}_{\mathrm{rig},L}(V)$ and $g \in \Gamma_L$. ∎

4.4. Compatibility with (φ, Γ_L)-modules

From Section 2.2, recall that $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$ is a pure of slope zero finite free (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$, functorially associated to V. The following result is a generalisation of [7, Proposition 3.5 and Théorème 3.6] from the perfect residue field case to L.

Proposition 4.21. *There are natural (φ, G_L)-equivariant isomorphisms,*

(1) $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V$.

(2) $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V)$.

Proof. For (1), recall that from Lemma 4.13, there is a $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$-linear and (φ, G_L)-equivariant map,

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \longrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V,$$

where the isomorphism is from Lemma 2.19. Extending the isomorphism in (4.2) along $\widetilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$ and using (2.5), we obtain a φ-equivariant isomorphism,

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V).$$

The preceding isomorphism fits into a commutative diagram compatibly with (4.12),

$$\begin{tikzcd}
\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]\otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r] \arrow[d, "\wr"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})\otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[dr, "\sim"] & \\
\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]\otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[r, "\sim"] & \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p} V \arrow[r] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})\otimes_{\mathbb{Q}_p} V,
\end{tikzcd} \tag{4.17}$$

where the left horizontal arrow in the bottom row is induced from the natural isomorphism $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbb{Z}_p} T$ (see Lemma 4.4 (2)), the slanted isomorphism is the isomorphism in the third row of (4.12) and the rest are natural injective maps. Since the slanted isomorphism is (φ, G_L)-equivariant, therefore, we obtain that the isomorphism $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V$ is (φ, G_L)-equivariant, showing (1). For (2), by extending the isomorphism in (1) along $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]$ and using (2.3), we obtain (φ, G_L)-equivariant isomorphisms,

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V).$$ ∎

From the discussion after (4.14) and Proposition 4.19, note that $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)$ is a pure of slope zero finite free (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$ of rank $= \dim_{\mathbb{Q}_p} V$. Therefore, by the equivalence of categories in [41, Lemma 4.5.7], there exists a unique finite free étale (φ, Γ_L)-module $D^\dagger_L$ over $\mathbf{B}^\dagger_L$ of rank $= \dim_{\mathbb{Q}_p} V$ such that

$$\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} D^\dagger_L,$$

compatible with the (φ, Γ_L)-action.

Corollary 4.22. *There exists a natural (φ, G_L)-equivariant isomorphism*

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_L} V,$$

inducing natural (φ, Γ_L)-equivariant isomorphisms $D^\dagger_L \xrightarrow{\sim} \mathbf{D}^\dagger_L(V)$ and

$$\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger_L(V).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) & \xrightarrow{\sim} & \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) & \xrightarrow{\sim} & \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) & \xrightarrow{\sim} & \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) & \xrightarrow{\sim} & \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V.
\end{array}$$

In the top row, the left horizontal arrow is the extension along $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ of the natural isomorphism $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ (see the discussion after (4.14)), and the right horizontal arrow is the extension along $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ of the natural isomorphism $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbb{Z}_p} T$ (see Lemma 4.4 (2)). In the bottom row, the left horizontal arrow is induced by the (φ, Γ_L)-equivariant isomorphism $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ (see (4.16) in Proposition 4.19) and the right horizontal arrow is induced from Proposition 4.21 (1). The left and the right vertical arrows are natural maps and the middle vertical arrow is induced from (4.2) and (2.5). Commutativity of the left square follows from (4.15) and commutativity of the right square follows from (4.17). This shows the first claim.

For the second claim, set $V' := (\widetilde{\mathbf{B}}^\dagger \otimes_{\mathbf{B}^\dagger_L} D^\dagger_L)^{\varphi=1}$, and note that it is a p-adic representation of G_L with $\dim_{\mathbb{Q}_p} V' = \dim_{\mathbb{Q}_p} V$ (see [6, Théorème 4.35]). Moreover, we have

$$V' \subset (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_L} D^\dagger_L)^{\varphi=1} \xrightarrow{\sim} \big(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)\big)^{\varphi=1} \xrightarrow{\sim} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{\varphi=1} = V,$$

where the first isomorphism follows from the discussion before the statement of the claim above, the second isomorphism follows from the first claim proven in the previous paragraph, and the last equality follows from Lemma 2.3. Therefore, we obtain that $V' \xrightarrow{\sim} V$ as

G_L-representations and it implies that $D_L^\dagger = \mathbf{D}_L^\dagger(V') \xrightarrow{\sim} \mathbf{D}_L^\dagger(V)$ as étale (φ, Γ_L)-modules over $\mathbf{B}_L^\dagger$. It is straightforward to verify that this isomorphism is compatible with the commutative diagram above. This concludes our proof. ∎

Remark 4.23. As indicated before Corollary 4.22, for a p-adic crystalline representation of V, combining the (φ, Γ_L)-equivariant isomorphism

$$\mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^+} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_L^\dagger} \mathbf{D}_L^\dagger(V),$$

together with the inverse of the isomorphism (4.16), gives a $\mathbf{B}_{\mathrm{rig},L}^\dagger$-linear (φ, Γ_L)-equivariant isomorphism,

$$\mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_L^\dagger} \mathbf{D}_L^\dagger(V). \tag{4.18}$$

The isomorphism (4.18) generalises [7, Proposition 3.7] from the perfect residue field case to L.

4.5. Obtaining Wach module

The finite free $\mathbf{B}_{\mathrm{rig},L}^+$-module $\mathbf{N}_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height s and pure of slope zero (see Lemma (4.12)), therefore, from Lemma 2.13 (2) there exists a unique finite free $\mathbf{B}_L^+$-module of rank $= \dim_{\mathbb{Q}_p} V$ and finite $[p]_q$-height s, whose extension of scalars along $\mathbf{B}_L^+ \to \mathbf{B}_{\mathrm{rig},L}^+$ gives $\mathbf{N}_{\mathrm{rig},L}(V)$. In particular, from the proof of Lemma 2.13, we note the following.

Definition 4.24. Define $\mathbf{N}_L(V) := \mathbf{N}_{\mathrm{rig},L}(V) \cap \mathbf{D}_L^\dagger(V) \subset \mathbf{D}_{\mathrm{rig},L}^\dagger(V)$.

The $\mathbf{B}_L^+$-module $\mathbf{N}_L(V)$ is finite free of rank $= \dim_{\mathbb{Q}_p} V$ and it is equipped with an induced Frobenius-semilinear endomorphism φ such that the cokernel of the injective map $(1 \otimes \varphi)\colon \varphi^*(\mathbf{N}_L(V)) \to \mathbf{N}_L(V)$ is killed by $[p]_q^s$, since $\mathbf{N}_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height s and $1 \otimes \varphi\colon \varphi^*(\mathbf{D}_L^\dagger(V)) \xrightarrow{\sim} \mathbf{D}_L^\dagger(V)$. Moreover, we have that $\mathbf{N}_L(V) \subset \mathbf{D}_L^+(V)$ because inside $\mathbf{D}_{\mathrm{rig},L}^\dagger(V)$ we have,

$$\begin{aligned}
\mathbf{N}_L(V) = \mathbf{N}_{\mathrm{rig},L}(V) \cap \mathbf{D}_L^\dagger(V) &\subset (\tilde{\mathbf{B}}_{\mathrm{rig}}^+ \otimes_{\mathbb{Q}_p} V)^{H_L} \cap (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L} \\
&\subset \big((\tilde{\mathbf{B}}_{\mathrm{rig}}^+ \otimes_{\mathbb{Q}_p} V) \cap (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)\big)^{H_L} \\
&\subset \big((\tilde{\mathbf{B}}_{\mathrm{rig}}^+ \cap \mathbf{B}^\dagger) \otimes_{\mathbb{Q}_p} V\big)^{H_L} \\
&= (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_L} = \mathbf{D}_L^+(V).
\end{aligned}$$

Furthermore, since $\mathbf{N}_{\mathrm{rig},L}(V)$ and $\mathbf{D}_L^\dagger(V)$ are stable under the compatible action of Γ_L (see Proposition 4.19 and Corollary 4.22), we conclude that $\mathbf{N}_L(V)$ is stable under the induced Γ_L-action. In particular, from the preceding discussion and Lemma 2.13, we obtain (φ, Γ_L)-equivariant isomorphisms,

$$\mathbf{B}_{\mathrm{rig},L}^+ \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},L}(V) \quad \text{and} \quad \mathbf{B}_L^\dagger \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{D}_L^\dagger(V). \tag{4.19}$$

Lemma 4.25. *The action of Γ_L on $\mathbf{N}_L(V)$ is trivial modulo μ.*

Proof. Let $g \in \Gamma_L$ and $x \in \mathbf{N}_L(V)$. Then, $(g-1)x$ is in $\mathbf{N}_L(V) \subset \mathbf{D}_L^\dagger(V)$. Moreover, from Corollary 4.20, we have that $(g-1)x$ is in $\mu\mathbf{N}_{\mathrm{rig},L}(V)$. Therefore, inside $\mathbf{D}_{\mathrm{rig},L}^\dagger(V)$, by using (4.19) we get that,

$$(g-1)x \in \mathbf{D}_L^\dagger(V) \cap \mu\mathbf{N}_{\mathrm{rig},L}(V) = (\mathbf{B}_L^\dagger \cap \mu\mathbf{B}_{\mathrm{rig},L}^+) \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) = \mu\mathbf{N}_L(V),$$

as claimed. ■

Definition 4.26. Define the Wach module over $\mathbf{A}_L^+ = \mathbf{B}_L^+ \cap \mathbf{A}_L \subset \mathbf{B}_L$ as,

$$\mathbf{N}_L(T) := \mathbf{N}_L(V) \cap \mathbf{D}_L(T) \subset \mathbf{D}_L(V).$$

Proof of Theorem 4.1. We will show that $\mathbf{N}_L(T)$ from Definition 4.26 satisfies all axioms of Definition 3.7. From the definition, note that $\mathbf{N}_L(T)$ is a finitely generated torsion-free $\mathbf{A}_L^+$-module and an elementary computation shows that $\mathbf{N}_L(T) \cap \mu^n\mathbf{N}_L(V) = \mu^n\mathbf{N}_L(T)$, for all $n \in \mathbb{N}$, in particular, $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ is p-torsion-free. Moreover, we have that $\mathbf{N}_L(T)[1/p] = \mathbf{N}_L(V)$, and a simple diagram chase shows that $(\mathbf{N}_L(T)/p\mathbf{N}_L(T))[\mu] = (\mathbf{N}_L(T)/\mu\mathbf{N}_L(T))[p] = 0$ and

$$\big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big)/p\big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big) = \big(\mathbf{N}_L(T)/p\mathbf{N}_L(T)\big)[1/\mu].$$

So, for all $n \in \mathbb{N}$, we have that

$$\mathbf{N}_L(T)/p^n\mathbf{N}_L(T) \subset \big(\mathbf{N}_L(T)/p^n\mathbf{N}_L(T)\big)[1/\mu] = \big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big)/p^n\big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big),$$

and therefore, $\mathbf{N}_L(T) \cap p^n(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)) = p^n\mathbf{N}_L(T)$, in particular, it follows that we have $\mathbf{N}_L(V) \cap (\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)) = \mathbf{N}_L(T)$. Now, by using Remark 2.16, it follows that $\mathbf{N}_L(T)$ is a finite free $\mathbf{A}_L^+$-module of rank $= \mathrm{rk}_{\mathbf{B}_L^+}\mathbf{N}_L(V) = \dim_{\mathbb{Q}_p} V$. Alternatively, to get the preceding statement, one can also use [8, Lemme II.1.3] (the proof of loc. cit. does not require the residue field of discrete valuation base field, L in our case, to be perfect).

From the definition, it also follows that $\mathbf{N}_L(T) \cap p^n\mathbf{D}_L(T) = p^n\mathbf{N}_L(T)$, in particular, we have that $\mathbf{N}_L(T)/p^n\mathbf{N}_L(T) \subset \mathbf{D}_L(T)/p^n\mathbf{D}_L(T)$, and therefore,

$$\big(\mathbf{N}_L(T)/p^n\mathbf{N}_L(T)\big)[1/\mu] \subset \mathbf{D}_L(T)/p^n\mathbf{D}_L(T).$$

So, we get that

$$\big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big)/p^n\big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big) \subset \mathbf{D}_L(T)/p^n\mathbf{D}_L(T),$$

or equivalently, $(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)) \cap p^n\mathbf{D}_L(T) = p^n(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))$. Note that we have $(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))[1/p] = \mathbf{B}_L \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{D}_L(V)$, where the last isomorphism follows from (4.19). Therefore, we get that

$$\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) = \mathbf{D}_L(T) \cap \big(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)\big)[1/p] \xrightarrow{\sim} \mathbf{D}_L(T) \cap \mathbf{D}_L(V) = \mathbf{D}_L(T).$$

Next, note that $\mathbf{N}_L(T)$ is equipped with an induced Frobenius-semilinear endomorphism φ. We have that $\varphi\colon \mathbf{A}_L^+ \to \mathbf{A}_L^+$ is faithfully flat and finite of degree p^{d+1} and $\varphi^*(\mathbf{A}_L) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi,\mathbf{A}_L^+} \mathbf{A}_L$ and similarly $\varphi^*(\mathbf{B}_L^+) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi,\mathbf{A}_L^+} \mathbf{B}_L^+$ (see Section 2.1.2). Therefore, we get that

$$\varphi^*\big(\mathbf{N}_L(V)\big) = \mathbf{B}_L^+ \otimes_{\varphi,\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi,\mathbf{A}_L^+} \mathbf{N}_L(V),$$
$$\varphi^*\big(\mathbf{D}_L(T)\big) = \mathbf{A}_L \otimes_{\varphi,\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi,\mathbf{A}_L^+} \mathbf{D}_L(T).$$

Then, it easily follows that $\varphi^*(\mathbf{N}_L(T)) = \varphi^*(\mathbf{N}_L(V)) \cap \varphi^*(\mathbf{D}_L(T)) \subset \varphi^*(\mathbf{D}_L(V))$. Now, since $1 \otimes \varphi$ is injective on $\varphi^*(\mathbf{D}_L(V))$, $1 \otimes \varphi\colon \varphi^*(\mathbf{D}_L(T)) \xrightarrow{\sim} \mathbf{D}_L(T)$ and the cokernel of $1 \otimes \varphi\colon \varphi^*(\mathbf{N}_L(V)) \to \mathbf{N}_L(V)$ is killed by $[p]_q^s$, therefore, we get that the cokernel of the injective map $1 \otimes \varphi\colon \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$. Finally, note that $\mathbf{N}_L(T)$ is equipped with an induced Γ_L-action such that Γ_L acts trivially on $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ (follows easily from Lemma 4.25), and we have that $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$. Hence, we conclude that T is of finite $[p]_q$-height. ∎

Corollary 4.27. *There exists a natural isomorphism of étale $(\varphi, \Gamma_{\breve{L}})$-modules*

$$\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T)$$

and a natural isomorphism of Wach modules $\mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(T)$.

Proof. Note that we have an injection of étale $(\varphi, \Gamma_{\breve{L}})$-modules $\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \subset \mathbf{D}_{\breve{L}}(T)$ and isomorphisms of $G_{\breve{L}}$-representations:

$$\big(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L} \mathbf{D}_L(T)\big)^{\varphi=1} \xrightarrow{\sim} T \xleftarrow{\sim} \big(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_{\breve{L}}} \mathbf{D}_{\breve{L}}(T)\big)^{\varphi=1}.$$

So, we get that $\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T)$. Furthermore, we have a $(\varphi, \Gamma_{\breve{L}})$-equivariant injection of Wach modules $\mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \subset \mathbf{N}_{\breve{L}}(T)$. So, by the uniqueness of a Wach module attached to T (see Lemma 3.9), it follows that

$$\mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(T). \qquad ∎$$

Proof of Corollary 4.3. The equivalence of $\otimes$-categories follows from Theorem 4.1 and we are left to show the exactness of the functor $\mathbf{N}_L$ since the exactness of the quasi-inverse functor follows from Proposition 3.3 and the exact equivalence in (2.2). From Section 2.1.5, recall that $\mathbf{A}_L^+ \to \mathbf{A}_{\breve{L}}^+$ is faithfully flat, therefore, $\mathbf{B}_L^+ \to \mathbf{B}_{\breve{L}}^+$ is faithfully flat. Moreover, for a p-adic crystalline representation V of G_L, from Corollary 4.27, note that we have $\mathbf{B}_{\breve{L}}^+ \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$. So, given an exact sequence,

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0, \tag{4.20}$$

of p-adic crystalline representations of G_L, it is enough to show that the following sequence is exact:

$$0 \longrightarrow \mathbf{N}_{\breve{L}}(V_1) \longrightarrow \mathbf{N}_{\breve{L}}(V_2) \longrightarrow \mathbf{N}_{\breve{L}}(V_3) \longrightarrow 0. \tag{4.21}$$

Furthermore, note that (4.20) is exact if and only if it is exact after tensoring with $\mathbb{Q}_p(r)$, for any $r \in \mathbb{Z}$. Similarly, (4.21) is exact if and only if it is exact after tensoring with $\mu^{-r}\mathbf{B}_{\breve{L}}^{+}(r)$. So we may assume that (4.20) is an exact sequence of positive crystalline representations, i.e., the Wach modules in (4.21) are effective. Moreover, the map

$$\mathbf{B}_{\breve{L}}^{+} \longrightarrow \mathbf{B}_{\mathrm{rig},\breve{L}}^{+}$$

is faithfully flat (by an argument similar to Lemma 2.7), so it is enough to show that the following sequence is exact:

$$0 \longrightarrow \mathbf{N}_{\mathrm{rig},\breve{L}}(V_1) \longrightarrow \mathbf{N}_{\mathrm{rig},\breve{L}}(V_2) \longrightarrow \mathbf{N}_{\mathrm{rig},\breve{L}}(V_3) \longrightarrow 0.$$

Exactness of the preceding sequence follows from Lemma 4.8, [35, Theorem 1.2.15], [36, Proposition 2.2.6] and the exactness of the functor $\mathbf{D}_{\mathrm{cris},\breve{L}}$. This allows us to conclude. ■

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