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Benjamini–Schramm vs. Plancherel convergence

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Abstract. We show that Plancherel convergence is strictly stronger than Benjamini–Schramm convergence. In order to do so, we introduce two criteria for Plancherel and Benjamini–Schramm convergence in terms of counting functions of the length spectrum.

1. Introduction

Let G be a connected semisimple Lie group of non-compact type, $K \leq G$ a maximal compact subgroup and let $X = G/K$ the associated Riemannian symmetric space. A lattice $\Gamma \leq G$ gives rise to a locally symmetric space $M = \Gamma \setminus X$. A sequence $(M_j)_{j\in\mathbb{N}}$ of locally symmetric spaces $M_j = \Gamma_j \setminus X$ is said to be *Benjamini–Schramm convergent* to X if, for every $R > 0$, the ball of radius R centered at a random point in M_j is almost surely isometric to an R-ball in X as $j \to \infty$. This definition of Benjamini–Schramm convergence fits into a more general notion of convergence, the nuances of which are beyond the scope of this paper. For a survey on Benjamini–Schramm convergence and its relation with invariant random subgroups, Farber sequences and more, we refer to [14, Sections 3–4]. A family of lattices in G is *uniformly discrete* if there is a neighborhood of the identity in G that intersects all of its conjugates trivially. Benjamini–Schramm convergence has been extensively studied in relation to the asymptotic behavior of L^2-invariants [1,2] and quantum ergodicity properties [4,15,17]. Next, one can associate to any uniform lattice Γ in a locally compact group G the spectral measure μ_Γ on the unitary dual $\widehat{G}$. A sequence of lattices $\{\Gamma_j\}_{j\in\mathbb{N}}$ is said to be *Plancherel convergent* if the sequence $\{\mu_{\Gamma_j}\}_{j\in\mathbb{N}}$ converges to the Plancherel measure of $\widehat{G}$ in a certain sense, which will be specified in Section 2.1. This kind of convergence has also been studied in many instances [2,8,10,12,13,16]. It is known that Plancherel convergence implies Benjamini–Schramm convergence [9, Theorem 2.6]. Regarding the converse direction, the authors of [2] have shown that a uniformly discrete Benjamini–Schramm sequence is Plancherel convergent. Since uniform discreteness for sequences of lattices in higher rank Lie groups would follow from the Lehmer conjecture, these notions of convergence are expected to be equivalent in the higher rank case. This bears the question whether these two notions of convergence are always equivalent. We will demonstrate that this is not the case by constructing a Benjamini–Schramm

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convergent sequence, which is not Plancherel convergent. For this we start with the well-known principal congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \ (\mathrm{mod}\ N),\ b, c \equiv 0 \ (\mathrm{mod}\ N) \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$ and replace the cusps of the associated congruence surfaces $X(N) = \Gamma(N) \setminus \mathbb{H}$ with short geodesics. By identifying these geodesics in pairs we end up with compact surfaces. An explicit study of the length spectrum of these surfaces allows one to directly check whether they fulfill the above notions of convergence. This verification is based on the criteria for Benjamini–Schramm convergence (Proposition 3.2) and Plancherel convergence (Proposition 3.3) that we introduce in Section 3.

2. Preliminaries

In this Section we collect some terminology and known results about locally compact groups and hyperbolic surfaces that we will need in the following.

2.1. Plancherel and Benjamini–Schramm convergence

Let G be a connected semisimple Lie group of non-compact type and fix a Haar measure μ on G. Let $\Gamma \leq G$ be a uniform lattice and $\widehat{G}$ be the unitary dual of G. Let μ_{Pl} be the Plancherel measure on $\widehat{G}$. The right regular representation R on $L^2(\Gamma \setminus G)$ defined by $R_y\phi(x) := \phi(xy)$ decomposes into a direct sum of unitary irreducible representations,

$$L^2(\Gamma \setminus G) \cong \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi)\pi,$$

with finite multiplicities $N_\Gamma(\pi)$, see, e.g., [11, Theorem 9.2.2].

Definition 2.1 (Plancherel convergence). Let δ_π denote the Dirac measure for $\pi \in \widehat{G}$. The measure on $\widehat{G}$ defined by

$$\mu_\Gamma = \sum_{\pi \in \widehat{G}} N_\Gamma(\pi)\delta_\pi$$

is called the *spectral measure* associated with Γ. A sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of uniform lattices in G is called *Plancherel convergent* (or a *Plancherel sequence*) if for every $f \in C_c^\infty(G)$

$$\frac{1}{\mathrm{vol}(\Gamma_j \setminus G)} \mu_{\Gamma_j}(\hat{f}) \xrightarrow{j\to\infty} \mu_{\mathrm{Pl}}(\hat{f}). \tag{2.1}$$

If the sequence $\{\Gamma_j\}_{j\in\mathbb{N}}$ is Plancherel convergent, we also call the associated sequence of locally symmetric spaces $\Gamma_j \setminus G/K$ Plancherel convergent.

Let us turn to Benjamini–Schramm convergence. In the following, $\{\Gamma_j\}_{j\in\mathbb{N}}$ is always a sequence of torsion-free lattices in G.

Definition 2.2. Let $X = \Gamma \setminus G/K$ be a locally symmetric space. For $p \in X$, we define the *injectivity radius* at p as

$$\mathrm{InjRad}(p) = \sup\left\{r \in \mathbb{R} : \exp_p \colon B_r(0) \to X \text{ is injective}\right\},$$

where $\exp_p \colon T_p X \to X$ is the exponential map. The *injectivity radius* of X is defined as

$$\mathrm{InjRad}(X) = \inf_{p \in X} \mathrm{InjRad}(p).$$

For $R > 0$ we define the *R-thin part* of X as

$$X_{<R} = \left\{p \in X : \mathrm{InjRad}(p) < R\right\}.$$

Definition 2.3 (Benjamini–Schramm convergence). We say that a sequence $(X_j)_{j\in\mathbb{N}}$ of locally symmetric spaces given by $X_j = \Gamma_j \setminus G/K$ is *Benjamini–Schramm convergent* (or *BS-convergent*) to the universal cover G/K if for every $R > 0$

$$\frac{\mathrm{vol}\left((X_j)_{<R}\right)}{\mathrm{vol}(X_j)} \xrightarrow{j\to\infty} 0. \tag{2.2}$$

In this case, we will also say that the associated sequence of lattices $(\Gamma_n)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent.

Finally, we notice that for $l_{s,j}$ the length of a shortest closed geodesic on X_j, also called the *systole* of X_j, the sequence $(X_j)_{j\in\mathbb{N}}$ is uniformly discrete if and only if there is a uniform lower bound for the systoles $l_{s,j}$ for all $j \in \mathbb{N}$.

2.2. Decompositions of hyperbolic surfaces

In this section we collect known results concerning hyperbolic surfaces and their decomposition into (possibly degenerate) pairs of pants (or Y-pieces). For a complete treatment of the topic, we refer to [5]. Let X be a smooth closed hyperbolic surface. Then X is a quotient $X = \Gamma \setminus \mathbb{H}$, where Γ is a purely hyperbolic Fuchsian group and $\mathbb{H}$ is the hyperbolic plane

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

equipped with the Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. We write $g = g(X)$ for the *genus* of X and note that Gauss–Bonnet Theorem implies $\mathrm{vol}(X) = 4\pi(g-1)$. A compact topological surface has *signature* (g, n) if it is obtained from a closed topological surface of genus g by removing the interior of n disjoint topological disks. A closed hyperbolic surface of signature $(0, 3)$ is called a *Y-piece* or a *pair of pants*. For any triple of positive real numbers l_1, l_2, l_3 there exists a pair of pants Y_{l_1,l_2,l_3} with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of respective lengths l_1, l_2, l_3. Such a Y-piece is obtained by pasting together two copies of a geodesic hexagon.

Theorem 2.4 (Collar theorem). *Let X be a smooth closed hyperbolic surface of genus g. Let $\gamma_1, \dots, \gamma_m$ be pairwise disjoint simple closed geodesics on X. Then:*

(1) $m \le 3g - 3$*;*

(2) *There exist simple closed geodesics* $\gamma_{m+1}, \ldots, \gamma_{3g-3}$, *which together with* $\gamma_1, \ldots, \gamma_m$ *decompose* X *into* Y*-pieces;*

(3) *The collars*

$$\mathcal{C}(\gamma_i) = \{p \in X; \operatorname{dist}(p, \gamma_i) \le w_i\}$$

of widths

$$w_i = \sinh^{-1}\left(\frac{1}{\sinh(l_i/2)}\right)$$

are pairwise disjoint for $i = 1, \ldots, 3g-3$.

In order to construct Benjamini–Schramm convergent sequences which are not uniformly discrete, we need to extend our discussion of Y-pieces to possibly non-compact surfaces. A *degenerate* hexagon is a hexagon with either one, two or three points at infinity. We will also refer to points at infinity as *punctures*. One can paste together two degenerate hexagons to get a *degenerate* Y*-piece* (or *degenerate pair of pants*). We extend the notation Y_{l_1,l_2,l_3} to degenerate Y-pieces by writing $l_i = 0$ for any boundary component arising from a puncture. A degenerate Y-piece contains a neighborhood $\mathscr{C}$ around each puncture, which is isometric to $(-\infty, \log 2] \times \mathbb{S}^1$ equipped with the Riemannian metric

$$ds^2 = dr^2 + e^{2r} dt^2.$$

This neighborhood is called a *cusp*. A Y-piece has signature $(0, p; q)$ if it has p boundary geodesics and q punctures. A smooth hyperbolic surface of genus g is said to have *signature* $(g, p; q)$ if it has p boundary geodesics and q cusps.

Theorem 2.5. *Let* X *be a (possibly non-compact) smooth hyperbolic surface of signature* $(g, 0; q)$. *Let* $\gamma_1, \ldots, \gamma_m$ *be pairwise disjoint simple closed geodesics on* X. *Then:*

(1) $m \le 3g - 3 + q$;

(2) *There exist simple closed geodesics* $\gamma_{m+1}, \ldots, \gamma_{3g-3+q}$, *which together with* $\gamma_1, \ldots, \gamma_m$ *decompose* X *into (possibly degenerate)* Y*-pieces;*

(3) *The collars* $\mathcal{C}(\gamma_i)$, $i = 1, \ldots, 3g-3+q$ *and the cusps* $\mathscr{C}^1, \ldots, \mathscr{C}^q$ *are all pairwise disjoint.*

If we pinch a simple closed geodesic γ on a smooth hyperbolic surface X, i.e., we let $l_\gamma \to 0$, then we expect that the collar $\mathcal{C}(\gamma)$ around γ converges in a suitable sense to two copies of a cusp $\mathscr{C}$. To make this more precise we need some additional terminology (cf. [7]). Let Y_{l_1,l_2,l_3} be a non-degenerate Y-piece with boundary geodesics γ_i, $i = 1, 2, 3$. For $0 \le r_i \le w_i$, The distance sets

$$\gamma_i^{r_i} = \{p \in Y : \operatorname{dist}(p, \gamma_i) = r_i\}$$

are called *equidistant curves*. For degenerate Y-pieces the equidistant curves are given by horocycles

$$h_r = \{p \in \mathscr{C} : \operatorname{dist}(p, \partial\mathscr{C}) = r\}.$$

Now, select in each half-collar or cusp an equidistant curve β_i of length λ_i. Then the closure of the connected component of $Y_{l_1,l_2,l_3} \setminus (\beta_1 \cup \beta_2 \cup \beta_3)$ not containing any of the boundary geodesics or punctures of Y_{l_1,l_2,l_3} is called a *restricted Y-piece* and denoted $Y_{l_1,l_2,l_3}^{\lambda_1,\lambda_2,\lambda_3}$. We will write Y_{l_1,l_2}^{c} instead of $Y_{l_1,l_2,0}^{l_1,l_2,c}$. A homeomorphism $\phi\colon Y \to Y'$ of (possibly restricted) Y-pieces is called *boundary-coherent* if for corresponding boundary curves α_i of Y and α_i' of Y' in standard parametrization.[1] one has

$$\phi\big(\alpha_i(t)\big) = \alpha_i'(t), \quad \forall t \in [0,1].$$

For each boundary length l_i we let

$$P_i = \big\{p \in Y_{l_1,l_2,l_3} : \operatorname{dist}(p,\gamma_i) < \log(2/l_i)\big\}$$

if $0 < l_i < 2$ and $P_i = \emptyset$ for $l_i \geq 2$. In the degenerate case, we let P_i be the connected component of $Y_{l_1,l_2,l_3} \setminus h_1$ which contains the puncture corresponding to $l_i = 0$. Then

$$\hat{Y}_{l_1,l_2,l_3} = Y_{l_1,l_2,l_3} \setminus (P_1 \cup P_2 \cup P_3)$$

is called a *reduced Y-piece*. Let us recall that a piecewise smooth map[2] $\Psi\colon M \to N$ of Riemannian manifolds M and N is called a *quasi-isometry* if there exists $d > 0$ such that for any $p \in M$ and any $v \in T_pM$

$$\frac{1}{d}\|v\|_M \leq \big\|D\Psi(v)\big\|_N \leq d\,\|v\|_M, \tag{2.3}$$

where $\|\cdot\|_M$ and $\|\cdot\|_N$ are the norms associated to the Riemannian metrics on M and N respectively. The infimum over all possible $d > 0$ such that (2.3) holds is called the *length distortion* and is denoted by d_Ψ.

Proposition 2.6. *Let $0 < l_1, l_2$ and let $0 < \varepsilon < \frac{1}{2}$. Set $\varepsilon^* = \frac{2}{\pi}\varepsilon$. Then there exists a boundary-coherent homeomorphism*

$$\phi\colon Y_{l_1,l_2,\varepsilon} \to Y_{l_1,l_2}^{\varepsilon^*}$$

such that

(1) $\phi(\hat{Y}_{l_1,l_2,\varepsilon}) = \hat{Y}_{l_1,l_2,0}$*;*

(2) *The restriction of ϕ to $\hat{Y}_{l_1,l_2,\varepsilon}$ is boundary-coherent and has length distortion $d_\phi \leq 1 + \frac{5}{4}\varepsilon^2$.*

Proof. [7, Theorem 5.1]. ■

Remark 2.7. The above proposition can be extended in an obvious manner to Y-pieces with more than one degenerating boundary geodesic.

[1] The precise description of this terminology can be found in [7, p. 2].

[2] A piecewise smooth map is a continuous map which is smooth on the complement of a finite number of curves.

3. Criteria for Benjamini–Schramm and Plancherel convergence

Given a smooth closed hyperbolic surface X and $R > 0$, we denote by $N(X, R)$ the number of closed geodesics on X of length $\leq R$ and with $N_s(X, R)$ the number of simple closed geodesics on X with length $\leq R$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of smooth closed hyperbolic surfaces. By [18, Proposition 2.2], if

$$\frac{N(X_n, R)}{\mathrm{vol}(X_n)} \xrightarrow{n\to\infty} 0, \tag{3.1}$$

then $(X_n)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent. We improve this result by showing that it is enough to consider simple closed geodesics. This will be key in the proof of Theorem 4.1. Moreover, we show that (3.1) implies Plancherel convergence.

Proposition 3.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed hyperbolic surfaces. If for every $R > 0$*

$$\frac{N_s(X_n, R)}{\mathrm{vol}(X_n)} \xrightarrow{n\to\infty} 0, \tag{3.2}$$

then the sequence $(X_n)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent.

Proof. The proof of this result is given in Section 5.1. ■

We get the following characterization of Plancherel convergence.

Proposition 3.2. *A sequence of closed hyperbolic surfaces $(X_n)_{n\in\mathbb{N}}$ is Plancherel convergent if and only if for every $R > 0$*

$$\frac{1}{\mathrm{vol}(X_n)} \sum_{\substack{[\gamma]\neq 1 \\ l_\gamma \leq R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \xrightarrow{n\to\infty} 0, \tag{3.3}$$

where the sum runs over the length spectrum of X_n and, for each closed geodesic γ, γ_0 is the correspondent primitive geodesic.

Proof. The proof of this result is given in Section 5.2. ■

Corollary 3.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed hyperbolic surfaces. If for every $R > 0$*

$$\frac{N(X_n, R)}{\mathrm{vol}(X_n)} \xrightarrow{n\to\infty} 0,$$

then the sequence $(X_n)_{n\in\mathbb{N}}$ is Plancherel convergent.

Proof. Notice that for any closed geodesic γ on X_n,

$$\frac{l_{\gamma_0}}{\sinh\left(\frac{l_\gamma}{2}\right)} \leq 2.$$

Hence,

$$\frac{1}{\mathrm{vol}(X_n)}\sum_{\substack{[\gamma]\neq 1\\ l_\gamma\leq R}}\frac{l_{\gamma_0}}{\sinh(l_\gamma/2)}\leq 2\frac{N(X_n,R)}{\mathrm{vol}(X_n)}\xrightarrow{n\to\infty}0.$$

Therefore, $(X_n)_{n\in\mathbb{N}}$ is Plancherel convergent. ∎

4. A degenerate Benjamini–Schramm sequence

This section is to devoted to the proof of the following theorem.

Theorem 4.1. *There exists a Benjamini–Schramm convergent sequence $(X_n)_{n\in\mathbb{N}}$ of smooth closed hyperbolic surfaces, which is not Plancherel convergent.*

For this, we adapt an example from [6]. Let us recall a few facts about the principal congruence subgroups

$$\Gamma(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}) : a,d\equiv 1 \ (\mathrm{mod}\ N),\ b,c\equiv 0\ (\mathrm{mod}\ N)\right\}.$$

We denote by $X(N)=\Gamma(N)\setminus\mathbb{H}$ the congruence surface of level N. We write g_N and b_N respectively for the genus and the number of boundary components of $X(N)$. The principal congruence subgroup $\Gamma(N)$ is torsion-free for $N\geq 3$. All boundary components of $X(N)$ are punctures and we have

$$g_N=1+\frac{d_N(N-6)}{24N},\quad b_N=\frac{d_N}{2N},\tag{4.1}$$

where d_N is given by $d_2=12$ and $d_N=N^3\prod_{p|N}(1-\frac{1}{p^2})$ for $N\geq 3$ (cf. [20]). Note that the number of cusps of $X(N)$ is always even for $N\geq 3$. By [19, Lemma 2] the systole $l_{s,N}$ of $X(N)$ is given by

$$2\cosh\left(\frac{l_{s,N}}{2}\right)=N^2-2.\tag{4.2}$$

Now, decompose $X(N)$ into (possibly degenerate) pairs of pants. The boundary components of the pants are either geodesics or punctures. We keep the boundary geodesics and replace each puncture by a geodesic of length $t>0$. Let us reassemble these pieces using the old identifications. Since the number of cusps of $X(N)$ is even, we can identify the remaining geodesics in pairs. This yields a closed hyperbolic surface $X_t(N)$. Notice that the surface $X_t(N)$ depends on the pants decomposition and on the choice of which pairs of boundary geodesics of length t to identify. However, this does not affect the following argument. By counting the number of Y-pieces involved one can show that

$$g\big(X_t(N)\big)\geq g_N.\tag{4.3}$$

We also note that the surface $X_t(N)$ contains $\frac{b_N}{2}$ disjoint simple closed geodesics γ_i, $i=1,\dots,\frac{b_N}{2}$ of length t. Now, let $(N_j)_{j\in\mathbb{N}}$ be a sequence of natural numbers such that $N_j\geq 3$ and $N_j\xrightarrow{j\to\infty}\infty$. Let $(t_j)_{j\in\mathbb{N}}$ be a sequence of positive real numbers converging towards 0. Let us write $b_j = \frac{b_{N_j}}{2}$ and $g_j = g_{N_j}$. Let $(X_j)_{j\in\mathbb{N}}$ be the sequence of closed hyperbolic surfaces defined by $X_j = X_{t_j}(N_j)$. Also, let Γ_j denote the lattice in G such that $X_j = \Gamma_j \setminus \mathbb{H}$.

Proposition 4.2. *The sequence $(X_j)_{j\in\mathbb{N}}$ is Plancherel convergent if and only if t_j^{-1} grows sub-exponentially in N_j.*

Proof. Let us fix $R > 0$ and let $(t_j)_{j\in\mathbb{N}}$ be a sequence converging towards 0. We claim that for j large enough, any geodesic γ in X_j of length $l_\gamma \leq R$ is the power of one of the geodesics $\gamma_1, \dots, \gamma_{b_j}$. For a fixed $j \in \mathbb{N}$, let γ be a simple closed geodesic in X_j, which is not freely homotopic to some power of one of the geodesics $\gamma_1, \dots, \gamma_{b_j}$. If γ intersects any of the geodesics γ_i, $i = 1, \dots, b_j$ we get from [5, Corollary 4.1.2] that

$$\sinh(l_\gamma/2) \geq \frac{1}{\sinh(t_j/2)}. \tag{4.4}$$

Hence, γ can be dismissed for j large enough. If γ does not intersect any of the γ_i, $i = 1, \dots, b_j$, then we have by [5, Theorem 4.1.1] that γ lies outside of the collars $\mathcal{C}(\gamma_i)$, $i = 1, \dots, b_j$. In this case, there exists a boundary-coherent quasi-isometry

$$\phi\colon X_j \setminus \bigcup_{i=1}^{b_j} P_i \to X(N_j) \setminus \bigcup_{i=1}^{b_j} P_i'$$

given by the identity on any Y-piece where no boundary geodesic has been replaced in the above process, and the map from Theorem 2.6 in the remaining cases. Its length distortion is bounded by $d_\phi \leq 1 + \frac{5}{4}t_j^2$. Therefore, $\phi(\gamma)$ defines an element $[\phi(\gamma)] \in \Gamma(N_j)$ such that

$$l_{\phi(\gamma)} \leq \left(1 + \frac{5}{4}t_j^2\right) l_\gamma.$$

We claim that $[\phi(\gamma)]$ is covered by a hyperbolic transformation, i.e., there exists a closed geodesic in the free homotopy class of $\phi(\gamma)$. Assume otherwise that $[\phi(\gamma)]$ can be covered by a parabolic transformation. Then by [3, p. 72] the curve $\phi(\gamma)$ can be homotoped into the power of a simple loop around a puncture of $X(N_j)$. Now, applying ϕ^{-1} gives a homotopy of γ into the collar around some geodesic γ_{i_0} for $i_0 \in \{1, \dots, b_j\}$. Hence, γ is homotopic to some power of γ_{i_0}, which is a contradiction to our assumption on γ. Consequently, there exists a hyperbolic transformation $\eta_\gamma \in \Gamma(N_j)$ which covers $[\phi(\gamma)]$ and we get from (4.2) that

$$\operatorname{arccosh}\left((N_j^2 - 2)/2\right) \leq l_{\eta_\gamma} \leq l_{\phi(\gamma)} \leq \left(1 + \frac{5}{4}t_j^2\right) l_\gamma. \tag{4.5}$$

Since t_j is bounded from above, inequality (4.5) shows that for N_j large enough there are no simple closed geodesics γ in X_j of length $l_\gamma \leq R$ apart from $\gamma_1, \dots, \gamma_{b_j}$.

Next, let γ be a non-simple closed geodesic different from a power of any of the geodesics $\gamma_1, \dots, \gamma_{b_j}$. According to [5, Theorem 4.2.4], any non-simple primitive geodesic of smallest length is a figure eight geodesic δ (i.e., a closed geodesic with exactly one self-intersection) embedded into a Y-piece. Any Y-piece contains at least one boundary geodesic not belonging to $\{\gamma_1, \ldots, \gamma_{b_j}\}$, as $X(N_j)$ would otherwise not be connected. Then the length formula for δ (see [5, Equation (4.2.3)]) yields

$$l_\gamma \geq l_\delta \geq 2\left(1 + \frac{5}{4}t_j^2\right)^{-1} \operatorname{arccosh}\left((N_j^2 - 2)/2\right).$$

This proves that for j large enough any geodesic in X_j of length $\leq R$ is a power of one of the geodesics $\gamma_1, \ldots, \gamma_{b_j}$. Finally, we want to use the characterization of Plancherel convergence given in Proposition 3.2. For j large enough,

$$\sum_{\substack{[\gamma]\neq 1\\ l_\gamma \leq R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} = b_j \sum_{\substack{k\in\mathbb{N}\\ kt_j \leq R}} \frac{t_j}{\sinh(kt_j/2)} = b_j \sum_{\substack{k\in\mathbb{N}\\ kt_j \leq R}} \frac{1}{k}\frac{kt_j}{\sinh(kt_j/2)} \leq 2b_j \sum_{\substack{k\in\mathbb{N}\\ kt_j \leq R}} \frac{1}{k}, \tag{4.6}$$

where we used $x \leq \sinh(x)$ for $x \in \mathbb{R}$. Plugging into (4.6) the asymptotic expansion of the harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma_E + \mathcal{O}\left(\frac{1}{n}\right),$$

where γ_E is the Euler–Mascheroni constant, we obtain the estimate

$$\sum_{\substack{[\gamma]\neq 1\\ l_\gamma \leq R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq C_1 b_j |\log t_j| \tag{4.7}$$

for some sufficiently large constant C_1. Plugging the values for b_j and g_j from (4.1) into (4.7) yields

$$\frac{1}{\operatorname{vol}(X_j)} \sum_{\substack{[\gamma]\neq 1\\ l_\gamma \leq R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq \frac{3C_1|\log t_j|}{\pi(N_j - 6)}. \tag{4.8}$$

In particular, the right-hand side of (4.8) converges to 0 if t_j^{-1} grows sub-exponentially in N_j, in which case $(X_j)_{j\in\mathbb{N}}$ is Plancherel convergent. Similarly, we get the lower bound

$$\sum_{\substack{[\gamma]\neq 1\\ l_\gamma \leq R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} = b_j \sum_{\substack{k\in\mathbb{N}\\ kt_j \leq R}} \frac{1}{k}\frac{kt_j}{\sinh(kt_j/)} \geq C_R b_j \sum_{\substack{k\in\mathbb{N}\\ kt_j \leq R}} \frac{1}{k} \geq C_R b_j |\log t_j|, \tag{4.9}$$

for C_R some positive constant depending on R. Hence, we get

$$\frac{1}{\operatorname{vol}(X_j)} \sum_{\substack{[\gamma]\neq 1\\ l_\gamma \leq R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \geq C'_R \frac{|\log t_j|}{N_j - 6}, \tag{4.10}$$

for C'_R only depending on R. In particular, the right-hand side of (4.10) diverges if t_j^{-1} grows at least exponentially in N_j. This concludes the argument. ∎

Let us turn to Benjamini–Schramm convergence of the sequence $(X_j)_{j\in\mathbb{N}}$.

Proposition 4.3. *The sequence $(X_j)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent if t_j converges to 0 as $j \to \infty$.*

Proof. Let $R > 0$ be given. As in the proof of Proposition 4.2 we get that, for N_j large enough, there is no simple closed geodesic in X_j of length $\leq R$ except for $\gamma_1, \ldots, \gamma_{b_j}$ and they all have length t_j. Therefore, we have $N_S(X_j, R) = b_j$ in the terminology of Corollary 3.1. Thus, we have for j large enough that

$$\frac{N_S(X_j, R)}{\mathrm{vol}(X_j)} = \frac{b_j}{4\pi(g(X_j) - 1)} \leq \frac{C_0}{N_j},$$

for C_0 some positive constant, where we made use equation (4.1) and $g(X_j) \geq g_j$. Therefore, $(X_j)_{j\in\mathbb{N}}$ is Benjamini–Schramm convergent. ∎

The existence of the example from Theorem 4.1 follows from Proposition 4.2 and Proposition 4.3 by choosing a sequence $(t_j)_{j\in\mathbb{N}}$ such that t_j^{-1} grows at least exponentially in N_j.

5. Appendix

5.1. Proof of Proposition 3.1

For any simple closed geodesic γ in X we define

$$\mathcal{C}_R(\gamma) = \big\{p \in X : \mathrm{dist}(p, \gamma) \leq w_R(\gamma)\big\}, \quad w_R(\gamma) = \sinh^{-1}\left(\frac{\sinh(R)}{\sinh(l_\gamma/2)}\right).$$

Notice that thanks to the Collar Theorem [5, Theorem 4.1.1], for $\widetilde{R} = \sinh^{-1}(1)$ and $w(\gamma) = w_{\widetilde{R}}(\gamma)$, $\mathcal{C}_{\widetilde{R}}(\gamma)$ is isometric to the cylinder $C(\gamma) = [-w(\gamma), w(\gamma)] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + l^2(\gamma)\cosh^2(\rho)dt^2$. In an analogous way we define, for any simple closed geodesic γ and $R > 0$, the cylinder $C_R(\gamma) = [-w_R(\gamma), w_R(\gamma)] \times \mathbb{S}^1$, endowed with the same Riemannian metric as above. Hence,

$$\mathrm{vol}\left(C_R(\gamma)\right) = 4\pi l(\gamma)\sinh\left(w_R(\gamma)\right) = 4\pi\sinh(R)\frac{l_\gamma}{\sinh(l_\gamma/2)}. \tag{5.1}$$

Proposition 5.1. *Let X be a closed hyperbolic surface and $R > 0$. Then*

$$X_{<R} \subseteq \bigcup_{\gamma\in\mathcal{S}_{2R}(X)} \mathcal{C}_R(\gamma),$$

where $\mathcal{S}_R(X)$ is the set of simple closed geodesics on X with length $\leq R$. Moreover,

$$\mathrm{vol}(X_{<R}) \leq C_R \cdot N_s(X, 2R), \tag{5.2}$$

where $N_s(X, R)$ is the number of simple closed geodesics on X with length $\leq R$ and $C_R = 8\pi\sinh(R)$ is a constant only depending on R.

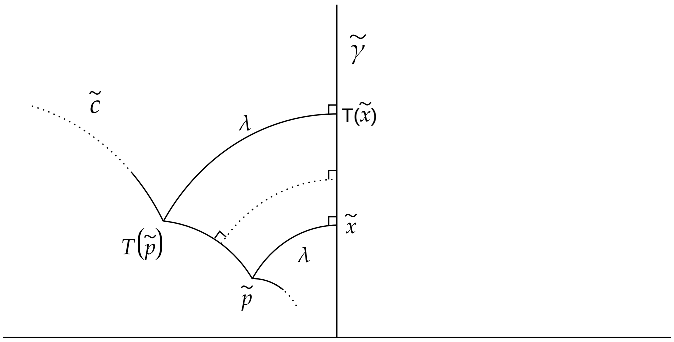

Figure 1. lift to universal cover. Up to isometry, γ can always be lifted to the imaginary axis.

Estimate (5.2) on the volume of the R-thin part of a hyperbolic surface immediately implies Proposition 3.1.

Proof of Proposition 3.1. By Proposition 5.1 we immediately get, for every $R > 0$,

$$\frac{\mathrm{vol}\left((X_n)_{<R}\right)}{\mathrm{vol}(X_n)} \underbrace{\leq}_{(5.2)} C_R \frac{N_s(X_n, 2R)}{\mathrm{vol}(X_n)} \xrightarrow{n\to\infty} 0.$$

Therefore the sequence $(X_n)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent. ∎

We are only left with the proof of Proposition 5.1. For this we will make use of the following result.

Lemma 5.2. *Let X be a closed hyperbolic surface, $p \in X$ and $R > 0$. If* $\mathrm{InjRad}(p) \leq R$, *then there exists a simple closed geodesic γ such that $l_\gamma \leq 2R$ and $\lambda = \mathrm{dist}(p, \gamma) \leq w_R(\gamma)$.*

Proof. We adapt the proof of [5, Theorem 4.1.6]. Recall that a *geodesic loop* is a geodesic arc $\gamma\colon [a, b] \to X$ such that $\gamma(a) = \gamma(b)$. Let $p \in X$ such that $\mathrm{InjRad}(p) \leq R$. There exists a simple geodesic loop c at p of length $2\,\mathrm{InjRad}(p)$, and its free homotopy class contains a unique simple closed geodesic γ [5, Theorem 1.6.6]. Since γ has minimal length in the free homotopy class of c, we immediately get $l_\gamma \leq 2R$. Let $x \in \gamma$ be the point at minimal distance to p, i.e., $\lambda = \mathrm{dist}(p, \gamma) = \mathrm{dist}(p, x)$. Let $\tilde{\gamma}$ be a lift of γ, and let T be the covering transformation with axis $\tilde{\gamma}$ closing γ. Let $\tilde{x} \in \tilde{\gamma}$ be a lift of x and $\tilde{p}$ be the lift of p at distance λ from $\tilde{x}$. Consider the geodesic quadrangle with vertices $\tilde{p}, \tilde{x}, T(\tilde{x}), T(\tilde{p})$ whose sides have length $\lambda, l_\gamma, \lambda, 2\,\mathrm{InjRad}(p)$ (see Figure 1). Dropping the common perpendicular between the geodesic segments $[\tilde{p}, T(\tilde{p})]$ and $[\tilde{x}, T(\tilde{x})]$ we obtain two isometric trirectangles. Standard results from hyperbolic trigonometry [5, Theorem 2.3.1] yield

$$\sinh\left(\mathrm{InjRad}(p)\right) = \sinh(l_\gamma/2)\cosh(\lambda).$$

Hence, $\sinh(l_\gamma/2)\sinh(\lambda) \leq \sinh(l_\gamma/2)\cosh(\lambda) \leq \sinh(R)$. Therefore,

$$\mathrm{dist}(p, \gamma) = \lambda \leq \sinh^{-1}\left(\frac{\sinh(R)}{\sinh(l_\gamma/2)}\right) = w_R(\gamma).$$ ∎

Proof of Proposition 5.1. Let $p \in X$ such that $\mathrm{InjRad}(p) \le R$. Then, by Lemma 5.2, there exists a simple closed geodesic such that $\gamma_0 \in \mathcal{S}_{2R}(X)$ and $p \in \mathcal{C}_R(\gamma_0)$. Therefore, we obtain

$$X_{<R} \subseteq \bigcup_{\gamma_0 \in \mathcal{S}_{2R}(X)} \mathcal{C}_R(\gamma_0).$$

We claim that for any simple closed geodesic γ_0 on X, and for any $R > 0$,

$$\mathrm{vol}\left(\mathcal{C}_R(\gamma_0)\right) \le \mathrm{vol}\left(C_R(\gamma_0)\right). \tag{5.3}$$

Assuming the Claim, it is immediate to see that

$$\begin{aligned}\mathrm{vol}(X_{<R}) \le \sum_{\gamma_0 \in \mathcal{S}_{2R}(X)} \mathrm{vol}\left(\mathcal{C}_R(\gamma_0)\right) &\underbrace{\le}_{(5.3)} \sum_{\gamma_0 \in \mathcal{S}_{2R}(X)} \mathrm{vol}\left(C_R(\gamma_0)\right) \\ &\underbrace{\le}_{(5.1)} \sum_{\gamma_0 \in \mathcal{S}_{2R}(X)} 4\pi \sinh(R) \underbrace{\frac{l_{\gamma_0}}{\sinh(l_{\gamma_0}/2)}}_{\le 2} \\ &\le \sum_{\gamma_0 \in \mathcal{S}_{2R}(X)} 8\pi \sinh(R) = 8\pi \sinh(R) N_s(X, 2R).\end{aligned}$$

In order to conclude the proof, we are only left with the proof of Claim. Let $\Gamma \le G$ be the uniform lattice in $\mathrm{SL}_2(\mathbb{R})$ such that $X = \Gamma \setminus \mathbb{H}$. We denote by $T_0 \in \Gamma$ the primitive element of Γ corresponding to the geodesic γ_0 in X. Let $\Gamma_{\gamma_0} = \langle T_0 \rangle$ be the cyclic subgroup of Γ generated by T_0, and consider the cylinder $C(\gamma_0)$ given by the quotient $\Gamma_{\gamma_0} \setminus \mathbb{H}$. Since $\Gamma_{\gamma_0} \subset \Gamma$ is a subgroup, then $C(\gamma_0) = \Gamma_{\gamma_0} \setminus \mathbb{H}$ is a covering of $X = \Gamma \setminus \mathbb{H}$. Let $\pi : C(\gamma_0) \to X$ be the covering projection. The image of $C_R(\gamma_0)$ under π is exactly $\mathcal{C}_R(\gamma_0)$, so that

$$\mathrm{vol}\left(\mathcal{C}_R(\gamma_0)\right) \le \mathrm{vol}\left(C_R(\gamma_0)\right),$$

which concludes the proof. ∎

5.2. Proof of Proposition 3.2

In order to prove Proposition 3.2 we need to introduce some more terminology and recall known results concerning the representation theory of $\mathrm{SL}_2(\mathbb{R})$. For the following, we refer the reader to [11, Chapter 11]. The *Hecke algebra* $\mathcal{H}$ of $G = SL_2(\mathbb{R})$ is the set of K-bi-invariant functions on G which are in $L^1(G)$. We call $\mathcal{H}\mathcal{S}$ the subalgebra of the Hecke algebra consisting of smooth functions $f \in \mathcal{H}$ such that $\varphi_f \in \mathcal{S}([0, \infty))$ is a Schwartz function. For every $f \in \mathcal{H}$, there exists a unique function φ_f on $[0, \infty)$ such that $f(x) = \varphi_f(\mathrm{tr}(x^T x) - 2)$ for every $x \in G$. To any $f \in \mathcal{H}\mathcal{S}$ we associate a function $h_f(r) = \mathrm{tr}\, \pi_{ir}(f)$, where the π_{ir}'s are, for $r \ge 0$, the unitary principal series representations of G [11, Lemma 11.2.6 and Proposition 11.2.9]. For $f \in \mathcal{H}\mathcal{S}$, one gets the following explicit expression, also known as an explicit Plancherel theorem (see [11, Theorem 11.3.1]):

$$\mu_{Pl}(\hat{f}) = f(1) = \frac{1}{4\pi} \int_{\mathbb{R}} h_f(r) \tanh(\pi r) r dr. \tag{5.4}$$

Now let $\Gamma \leq G$ be a uniform lattice and let $(r_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $ir_j \in i\mathbb{R} \cup (0, \frac{1}{2})$, with the property that π_{ir_j} is isomorphic to a subrepresentation of $(R, L^2(\Gamma \setminus G))$ and the value $r = r_j$ is repeated in the sequence $N_\Gamma(\pi_{ir})$ times. For a function $f \in \mathcal{H}$ for which the operator $R(f)$ is of trace class, one gets

$$\operatorname{tr} R(f) = \sum_{\pi\in\widehat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(f) = \sum_{j=0}^{\infty} h_f(r_j).$$

In particular, if $f \in \mathcal{HS} \cap C_c^\infty(G)$, one can apply Selberg trace formula to obtain

$$\sum_{j=0}^{\infty} h_f(r_j) = \frac{\operatorname{vol}(\Gamma \setminus \mathbb{H})}{4\pi} \int_{\mathbb{R}} h_f(r) \tanh(\pi r) r dr + \sum_{[\gamma]\neq 1} \frac{l_{\gamma_0}}{2\sinh(l_\gamma/2)} g_f(l_\gamma), \quad (5.5)$$

where

$$g_f(r) = \int_{\mathbb{R}} \varphi_f\big(2\cosh(r) - 2 + s^2\big) ds.$$

(5.4) and (5.5) yield the following result.

Lemma 5.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a Plancherel sequence of hyperbolic surfaces. Then, for any $f \in C_c^\infty(G) \cap \mathcal{HS}$,*

$$\frac{1}{\operatorname{vol}(X_n)} \sum_{[\gamma]\neq 1} \frac{l_{\gamma_0}}{2\sinh(l_\gamma/2)} g_f(l_\gamma) \xrightarrow{n\to\infty} 0, \quad (5.6)$$

We are finally able to prove Proposition 3.2.

Proof of Proposition 3.2. We start by assuming that (3.3) holds and prove Plancherel convergence. Let $f \in C_c^\infty(G)$ and let us write $K_f := \operatorname{supp}(f)$. Also, denote

$$B_n := \left| \frac{\mu_{\Gamma_n}(\hat{f})}{\operatorname{vol}(X_n)} - \mu_{Pl}(\hat{f}) \right|.$$

Notice that Plancherel convergence of (X_n) is equivalent to $B_n \xrightarrow{n\to\infty} 0$. Applying Selberg's trace formula yields

$$B_n = \frac{1}{\operatorname{vol}(X_n)} \left| \sum_{[\gamma]\neq 1} l_{\gamma_0} \int_{G_\gamma \setminus G} f(x^{-1}\gamma x) dx \right|.$$

Since the orbital integral is invariant under conjugation, we may assume

$$\gamma = \begin{pmatrix} e^{l_\gamma/2} & 0 \\ 0 & e^{-l_\gamma/2} \end{pmatrix}.$$

Let $G = ANK$ be the Iwasawa decomposition of $G = SL_2(\mathbb{R})$. By compactness of K_f, there exists $R = R(f) > 0$ such that

$$B_n \leq \frac{1}{\operatorname{vol}(X_n)} \sum_{\substack{[\gamma]\neq 1 \\ l_\gamma \leq R}} l_{\gamma_0} \int_{G_\gamma \setminus G} \left| f(x^{-1}\gamma x) \right| dx.$$

A quick computation shows that the centralizer G_γ of γ in G is $G_\gamma = A$, so that

$$\int_{G_\gamma \backslash G} |f(x^{-1}\gamma x)| dx = \frac{1}{2\sinh(l_\gamma/2)} \int_{\mathbb{R}} \int_0^{2\pi} \left| f\left(k_\theta^{-1} \begin{pmatrix} e^{l_\gamma/2} & y \\ 0 & e^{-l_\gamma/2} \end{pmatrix} k_\theta\right)\right| d\theta dy.$$

The integrand on the right-hand side can only be non-zero if

$$\begin{pmatrix} e^{l_\gamma/2} & y \\ 0 & e^{-l_\gamma/2} \end{pmatrix} \in K K_f K =: K'_f.$$

K'_f is compact. Hence, there exists $M_f > 0$ solely depending on f such that

$$\int_{G_\gamma \backslash G} |f(x^{-1}\gamma x)| dx \le \frac{M_f}{\sinh(l_\gamma/2)}.$$

Thus,

$$B_n \le \frac{M_f}{\mathrm{vol}(X_n)} \sum_{\substack{[\gamma]\neq 1 \\ l_\gamma \le R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)}.$$

(3.3) implies that $B_n \xrightarrow{n\to\infty} 0$, so that $(X_n)_{n\in\mathbb{N}}$ is Plancherel convergent. For the converse, we start by observing that for every $\varphi \in C_c^\infty([0,\infty))$ the function $f(x) = \varphi(\mathrm{tr}(x^T x) - 2)$ is such that $f \in C_c^\infty(G) \cap \mathcal{H}\mathcal{S}$ and $\varphi = \varphi_f$.

Next, for every $R < 0$, choose $\varphi \in C_c^\infty([0,\infty))$ such that $\varphi \ge 0$ and $g_f(r) > 0$ for every $r \in [0, R]$. Let us denote $m_R := \min_{[0,R]} g_f(r)$. We get,

$$\sum_{\substack{[\gamma]\neq 1 \\ l_\gamma \le R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \le \sum_{\substack{[\gamma]\neq 1 \\ l_\gamma \le R}} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \frac{g_f(l_\gamma)}{m_R} \le \frac{1}{m} \sum_{[\gamma]\neq 1} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} g_f(l_\gamma).$$

By Lemma 5.3, the last term vanishes as $n \to \infty$, and this concludes the proof. ■

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