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Graph Theory

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Abstract. The workshop provided a venue for discussing several recent major developments in graph theory, with a primary focus on results concerning the notion of twin-width and hereditary properties of graphs. Both areas have seen a very rapid development recently, as reflected in the many results presented during the workshop. In addition to many interesting talks spanning the whole breadth of graph theory, the workshop also offered valuable collaboration opportunities, which also engaged early career researchers.

Mathematics Subject Classification (2020): 05Cxx.

Introduction by the Organizers

Graph theory has seen many significant results over the past three years. The workshop offered an opportunity for the community to meet and discuss the major recent developments. Talks covering two main themes of the workshop, hereditary properties of graphs and the notion of twin-width, were complemented by more focused talks highlighting recent advancements in other areas of graph theory.

The workshop was attended by 44 participants on-site and 4 participants online. One on-line participant was unable to attend due to parenting responsibilities, one due to visa related issues and two due to health reasons. We are indebted to Oberwolfach Research Institute for making arrangements for the on-line attendance of these four participants, particularly for providing excellent technical facilities supporting the hybrid format of the workshop. We would also like to thank Marcin Briański and Tung Nguyen for taking great care of the videoconferencing facilities throughout the entire workshop.

The workshop program started on Monday morning and concluded in the early afternoon on Friday. Each of the five days began with a 50-minute overview talk, followed by a 10-minute discussion. These talks were given by Maria Chudnovsky, Eun Jung Kim, Jacob Fox, Piotr Micek and Luke Postle. In particular, Maria Chudnovsky opened the theme of hereditary properties of graphs in her Monday talk, while Eun Jung Kim surveyed the notion of twin-width in her Tuesday talk. Most of the remaining program on Monday was devoted to 3-minute minipresentations that were given by each participant; the minipresentations were intended to introduce the participants and their research areas to one another. The full Monday program concluded with an open problem session designed to spark productive discussions for the rest of the week. The scientific program on the following days included 21 short talks, in addition to the 5 long talks given at the beginning of each day. A substantial part of the afternoon each day was left free for collaborative work while Wednesday afternoon was completely free for the traditional workshop trip. We were pleased that the workshop attendees included many early career researchers and females, and this diversity was also well reflected among those giving the talks.

The two main themes were hereditary properties of graphs, i.e., those closed under taking induced subgraphs, and the notion of twin-width. Both themes have seen very major developments recently. On Monday, Maria Chudnovsky reported on her work on induced subgraph obstructions to bounded pathwidth (path and tree decompositions are powerful tools in structural and algorithmic graph theory with many applications). Later in the week, Tung Nguyen updated the participants on his results on the Erdős–Hajnal conjecture, a major open problem concerning homogeneous subgraphs in hereditary classes of graphs, and Peter Nelson discussed the conjecture in the matroid setting.

The survey talk by Eun Jung Kim in the morning on Tuesday kicked off the theme of twin-width, a recent graph parameter interlinking several seemingly different width notions. Indeed, classes of graphs with bounded twin-width generalize many well-established classes of (both sparse and dense) graphs, such as minor closed classes of graphs and classes of graphs with bounded rank-width, while preserving good algorithmic and structural properties. The twin-width parameter has already found many algorithmic applications and has been attracting substantial attention within the computer science community (as witnessed by presentations at top conferences such as FOCS, ICALP, SODA and STOC). The workshop participants were also updated on other recent developments in graph theory, including the Erdős–Pósa properties, which concern covering and packing subgraphs, the dimension of posets whose diagram is a planar graph, and the refined absorption method yielding a more compact proof of the Existence Conjecture.

The workshop participants, representing all areas of graph theory, were updated on recent major developments across graph theory. We were pleased to experience an atmosphere that fostered collaboration and we would like to thank all workshop participants for creating such an extremely stimulating scientific atmosphere. Our thanks also go to the staff of the Oberwolfach Research Institute for providing an

excellent environment and outstanding support for this workshop. We would also like to thank Emma Hogan for her assistance in preparing this report.

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Workshop: Graph Theory

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Abstracts

Equiangular lines via improved eigenvalue multiplicity

Matija Bucić
(joint work with Igor Balla)

1. Equiangular lines

A family of lines passing through the origin in an inner product space is said to be *equiangular* if every pair defines the same angle. The study of large families of equiangular lines dates back to 1947 and it has since found numerous interesting connections and applications in a wide variety of different areas. These include elliptic geometry, polytope theory, frame theory, Banach spaces, and quantum information theory with connections to algebraic number theory and Hilbert's twelfth problem.

Determining the maximum number of equiangular lines in $\mathbb{R}^r$ is widely considered as one of the founding problems of algebraic graph theory and features in most introductory courses on the subject. The well-known absolute bound (see e.g. [6]) states that the answer is at most $\binom{r+1}{2}$, which is tight up to a multiplicative constant and it is a difficult open problem to determine whether there are infinitely many r for which it holds with equality.

The more refined version of this question, where we additionally specify the common angle between the lines, was first posed by Lemmens and Seidel [6] in 1973. Given $\alpha \in [0, 1)$, we will refer to a family of lines passing through the origin as *α-equiangular* if the acute angle between any two lines equals $\arccos \alpha$. Lemmens and Seidel also gave a good partial answer to their question when $r \leq 1/\alpha^2 - 2$, but until recently, our understanding of the complementary regime $r \geq 1/\alpha^2 - 2$ has been much more limited. After a series of recent breakthroughs [3, 2, 5, 1] we now know the answer precisely for any α and r at least doubly exponential in $O(k \log \alpha^{-1})$), where k is the smallest order of a graph with largest eigenvalue equal to $\frac{1-\alpha}{2\alpha}$.

In the subexponential regime, besides the aforementioned bound of Lemmens and Seidel from 1973, [8, 4, 1] recently significantly extended the regime for which we essentially know the answer to $r \leq r_\alpha \approx \frac{1}{4\alpha^4}$. Beyond this range the best known bound is linear in r.

Our first result determines the answer up to lower order terms provided r is superpolynomial in $1/\alpha \to \infty$.

Theorem 1. *If $\alpha \to 0$ and $r \geq 1/\alpha^{\omega(1)}$, then the maximum number of α-equiangular lines in $\mathbb{R}^r$ is $r + o(r)$.*

We note that the assumption that $1/\alpha$ is growing is necessary due to the standard construction in the classical case when $1/\alpha$ is an odd integer. Furthermore, we note that the results of [5] establish this theorem when r is at least doubly exponential in $1/\alpha^2$, so the main new contribution of the above theorem is an improved requirement on r from double exponential to only super polynomial in $1/\alpha$. When r is at most exponential in $1/\alpha$, our bound is of the form $r + r/(\log_{1/\alpha} r)^{1-o(1)}$ and in fact, it already beats the previously best known (linear) bound when r is a large polynomial in $1/\alpha$. When r is at least exponential in $1/\alpha$, our bound is of the form $r + r\alpha^{1-o(1)}$.

In the exponential regime, i.e. when r is at least exponential in $1/\alpha^{O(1)}$, we can prove an even more precise result which turns out to be tight.

Theorem 2. *Given $0 < \alpha < 1$, if $r \geq 2^{1/\alpha^{O(1)}}$, then the number of α-equiangular lines in $\mathbb{R}^r$ is at most*

$$r - 1 + \left\lfloor (r-1) \cdot \frac{2\alpha}{1-\alpha} \right\rfloor .$$

The standard construction in the classical case, when the angle is one over an odd integer, shows that our theorem is tight. In particular, we get the following corollary.

Corollary 3. *Given an integer $k \geq 2$, if $r \geq 2^{k^{O(1)}}$, then the maximum number of $\frac{1}{2k-1}$-equiangular lines in $\mathbb{R}^r$ equals*

$$r - 1 + \left\lfloor \frac{r-1}{k-1} \right\rfloor .$$

This improves upon the previously best-known doubly exponential constraint on r from [5, 1].

2. Second eigenvalue multiplicity.

A crucial ingredient in the quest to solve the equiangular lines problem, which was brought to the table in [5], is a relation between it and the multiplicity of the second largest eigenvalue of a certain auxiliary graph.

In addition to the relation to the equiangular lines problem, the question of bounding the multiplicity of the second eigenvalue of a graph has a number of other connections and is a very interesting problem in spectral graph theory in its own right. For example, the case of Cayley graphs is already quite interesting due to it having connections to deep results in Riemannian geometry. For instance, following the approach of Colding and Minicozzi on harmonic functions on manifolds and Kleiner's proof of Gromov's theorem on groups of polynomial growth, Lee and Makarychev showed that in groups with bounded doubling constant, the second eigenvalue multiplicity is bounded. Owing to the fact that many algorithmic problems become much easier on graphs with few large eigenvalues, there are also interesting connections to computer science. For instance, McKenzie, Rasmussen, and Srivastava [7] do an excellent job of motivating the problem from this perspective and give a detailed history. In addition, they mention further connections to higher-order Cheeger inequalities, typical support size of random walks, and the properties of the Perron eigenvector. Finally, the second eigenvalue multiplicity question is also relevant in the study of Schrödinger operators on two-dimensional Riemannian manifolds. Indeed, the connection between the spectral theory of graphs and Schrödinger operators on surfaces is a point of view emphasized by Colin de Verdière and is the underlying motivation for the definition of his graph parameter. In particular, Letrouit and Machado were recently able to make progress on a conjecture of de Verdière by extending the ideas of [5] to Laplacians of negatively curved surfaces.

Our main result here is the following improved bound on the second eigenvalue multiplicity of an arbitrary graph.

Theorem 4. *Let G be an n-vertex graph with second eigenvalue λ_2 and maximum degree $\Delta \geq 2$. Then,*

$$m_G(\lambda_2) \leq \max\left\{n/\lambda_2^{1-o(1)}, n/(\log_\Delta n)^{1-o(1)}\right\}.$$

In the subexponential regime, our bound improves upon and significantly extends a result of McKenzie, Rasmussen, and Srivastava [7], who proved an upper bound of the form $\frac{n(\log d)^{1/4}}{(\log_d n)^{1/4-o(1)}}$ for a d-regular graph with $d = \Omega(\log^{1/4} n)$. In particular, removing the regularity assumption is key for our applications to the equiangular lines problem. Furthermore, their bound is trivial already when $d \geq 2^{\Omega(\sqrt{\log n})}$, whereas our result remains non-trivial all the way up to $d = n^{\Omega(1)}$. In fact, our bound here is the first non-trivial upper bound on the second eigenvalue multiplicity for arbitrary n-vertex graphs with maximum degree larger than $\log n$.

While the bound of Theorem 4 in the subexponential regime is behind our proof of Theorem 1, the bound it gives in the superexponential regime is just slightly insufficient to conclude Theorem 2. The additional ingredient that comes to our rescue in the equiangular lines setting, is that we can reduce to working with connected graphs. Note that Jiang, Yao, Tidor, Zhang, and Zhao [5] also made use of this assumption, showing that in an n-vertex connected graph of maximum degree Δ, the second eigenvalue multiplicity is at most $O(n/\log_\Delta \log n)$. While Theorem 4 improves upon this so long as the graph is not very sparse, we can slightly improve this and put the two bounds together in the connected case.

Theorem 5. *Let G be a connected n-vertex graph with second eigenvalue λ_2, $\Delta = \Delta(G) \geq 2$, and $\delta = \delta(G)$. Then, provided $\log n \geq \Delta^{O(1)}$, we have*

$$m_G(\lambda_2) \leq \frac{n}{\lambda_2 + \Omega(\delta \log_\Delta \log n)}.$$

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Beyond Tutte's 2-separator theorem

Johannes Carmesin
(joint work with Jan Kurkofka)

A tried and tested approach to a fair share of problems in structural and topological graph theory – such as the two-paths problem [5, 6, 7] or Kuratowski's theorem [8] – is to first solve the problem for 4-connected graphs. Then, in an intermediate step, the solutions for the 4-connected graphs are extended to the 3-connected graphs, by drawing from a theory of 3-connected graphs that has been established to this end. Finally, the solutions for the 3-connected graphs are extended to all graphs, in a systematic way by employing decompositions of general graphs along their cutvertices and 2-separators.

The intermediate step of this strategy seems curious: why should the step from 4-connected to 3-connected require an entirely different treatment than the systematic step from 3-connected to the general case? Indeed, the intermediate step carries the implicit believe that it is not possible to decompose 3-connected graphs along 3-separators in a way that is on a par with the renowned decompositions along separators of size at most two. Our main result offers a solution to this long-standing hindrance. To explain this, we start by giving a brief overview of the renowned decompositions along low-order separators.

Graphs trivially decompose into their components, which either are 1-connected or consist of isolated vertices. The 1-connected graphs are easily decomposed further, along their cutvertices, into subgraphs that either are 2-connected or K_2's which stem from bridges.

When decomposing 2-connected graphs further, however, things begin to get more complicated. Indeed, a 2-separator – a set of two vertices such that deleting the two vertices disconnects the graph – may separate the vertices of another 2-separator. Then if we choose one of them to decompose the graph by cutting at the 2-separator, we loose the other. In particular, it is not possible to decompose a 2-connected graph simply by cutting at all its 2-separators. An illustrative example for this are the 2-separators of a cycle.

There is an elegant way to resolve this problem. If two 2-separators are compatible with each other, in the sense that they do not cut through each other, then we say that these 2-separators are *nested* with each other. Let us call a 2-separator *totally-nested* if it is nested with every 2-separator of the graph. The solution is that every 2-connected graph decomposes into 3-connected graphs, cycles and K_2's, by cutting precisely at its totally-nested 2-separators. Tutte [10] found this decomposition first, but with a different description. The description via total nestedness was discovered later by Cunningham and Edmonds [3].

A *mixed-separation* of a graph G is a pair (A, B) such that $A \cup B = V(G)$ and both $A \setminus B$ and $B \setminus A$ are nonempty. We refer to A and B as the *sides* of the mixed-separation. The *separator* of (A, B) is the disjoint union of the vertex set $A \cap B$ and the edge set $E(A \setminus B, B \setminus A)$. If the separator of (A, B) has size three, we call (A, B) a *mixed 3-separation.* A *tri-separation* of a graph G is a mixed 3-separation (A, B) of G such that every vertex in $A \cap B$ has at least two neighbours in both $G[A]$ and $G[B]$.

Given a 3-connected graph G and a set N of pairwise nested tri-separations, we can say which parts we obtain by decomposing G along N. Roughly speaking, these are maximal subgraphs of G that lie on the same side of every tri-separation in N, with some edges added to represent external connectivity in G. We call the resulting minors of G the *torsos* of N, as they generalise the well known torsos of tree-decompositions from the theory of graph minors.

According to the 2-separator theorem, some of the building blocks for 2-connected graphs are K_2's. The analogue of these building blocks for 3-connected graphs turn out to be *thickened* $K_{3,m}$*'s* with $m \geq 0$: these are obtained from $K_{3,m}$ by adding edges to its left class of size three to turn it into a triangle.

Theorem 1. *Let G be a 3-connected graph and let N denote its set of totally-nested nontrivial tri-separations. Each torso τ of N is a minor of G and satisfies one of the following:*

(1) τ is quasi 4-connected;
(2) τ is a wheel;
(3) τ is a thickened $K_{3,m}$ or $G = K_{3,m}$ with $m \geq 0$.

We emphasise that the sets $N = N(G)$ obviously are canonical, meaning that they commute with graph-isomorphisms: $N(\varphi(G)) = \varphi(N(G))$ for all $\varphi\colon G \to G'$.

Applications. We provide the following applications of our work. It is well known that all Cayley graphs of finite groups are either 3-connected, cycles, or complete graphs on at most two vertices [4]. By heavily exploiting the fact that our decomposition of 3-connected graphs is canonical, we can refine this fact: every vertex-transitive finite connected graph G either is essentially 4-connected, a cycle, or a complete graph on at most four vertices. Another application comes in the form of an automatic proof of Tutte's wheel theorem [9]. In the upcoming work [2], Theorem 1 will be used to construct an FPT algorithm for connectivity augmentation from 0 to 4, and the property of total nestedness is crucial for that.

When canonicity and an explicit description matter. Recall that the tri-separation decomposition of Theorem 1 is canonical and is explicitly described so that it is uniquely determined for every 3-connected graph. These two of its aspects are *absolutely crucial* for a number of its applications:

(1) For vertex-transitive graphs, such as Cayley graphs, exploiting the combination of canonicity and total-nestedness makes up the entire proof. Just recently, this combination has also been exploited when using the Tutte-decomposition in the proof of a low-order Stallings-type theorem for finite

nilpotent groups [1]. An obvious next step in this direction would be to exploit this combination with the tri-separation decomposition.

(2) For Connectivity Augmentation, canonicity and access to an explicit description are key [2].
(3) Total-nestedness is incredibly desirable in Parallel Computing, the foundation of every supercomputer. Splitting the workload of finding the decomposition is a lot easier when all the partial solutions, which would come in the form of sets of already found totally-nested tri-separations, can always be combined without conflict.
(4) Finally, as the Tutte-decomposition is canonical and explicit, we believe that any decomposition result that claims to generalise Tutte should be both canonical and explicit.

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Induced subgraphs and pathwidth

Maria Chudnovsky
(joint work with Sepehr Hajebi, Sophie Spirkl)

Tree decompositions, and in particular path decompositions, are a powerful tool in both structural graph theory and graph algorithms. Many hard problems become tractable if the input graph is known to have a tree decomposition of bounded "width". Exhibiting a particular kind of a tree decomposition is also a useful way to describe the structure of a graph.

For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.

(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

The *width* of a tree decomposition (T, χ), denoted by width(T, χ), is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by tw(G), is the minimum width of a tree decomposition of G. A *path decomposition* is a tree decomposition (T, χ) where T is a path. The *pathwidth* of G, denoted by pw(G), is the minimum width of a path decomposition of G.

Given a graph G, a *subgraph of* G is a graph obtained from G by a sequence of vertex and edge deletions. A *minor* of G is a graph obtained from a subgraph of G by repeatedly contracting edges. In the context of simple graphs the minor operation is modified slightly: parallel edges obtained in the contraction process are deleted.

Families of bounded treewith and pathwidth have been completely characterized in terms of forbidden subgraphs (and minors) in the 1980s [7] and [8].

Theorem 1 (Robertson and Seymour [7])**.** *For every forest F, there is a positive integer p_F such that every graph G with* $\mathrm{pw}(G) \geq p_F$ *has a minor isomorphic to H. Moreover, if F is not a forest, then no such p_F exists.*

Theorem 2 (Robertson and Seymour [8])**.** *For every planar graph H, there is positive integer c_H such that every graph G with $tw(G) \geq c_H$ has a minor isomorphic to H. Moreover, if H is not planar, then no such c_H exists.*

An analogue of Theorem 1 can be stated in the language of subgraphs, rather than minors. For a positive integer $r > 1$, the *binary tree of height* r is the unique (up to isomorphism) rooted tree of radius r such that the root has degree 3 and every vertex that is neither the root nor a leaf has degree 2.

Theorem 3 (Robertson and Seymour [7])**.** *For every binary tree T there is a positive integer p_T such that every graph G with* $\mathrm{pw}(G) \geq p_T$ *has a subgraph isomorphic to a subdivision of T.*

There is also a subgraph analogue of Theorem 2, but we will not state it here.

Given a graph G, a graph H is an *induced subgraph* if G if $V(H) \subseteq V(G)$, and two vertices of H are adjacent in H if and only if they are adjacent in G. An *induced minor* of G is a graph obtained from an induced subgraph of G by contracting edges (and removing parallel edges that may have resulted from the contractions).

Studying obstructions to bounded therewith and pathwidth in connection with induced subgraph or induced minor containment relations is a relatively new research direction. Here we present a complete analogue of Theorems 1 and 3 for induced minors and induced subgraphs, respectively. This is based on [4], which in turn relies on many results proved earlier papers in the same series. We start with an analogue of Theorem 1, which is easier to state.

Theorem 4. *For every positive integer t and every forest F, there is an integer $p_{F,t}$ such that every graph G with* $\mathrm{pw}(G) \geq p_{F,t}$ *has a subgraph isomorphic to K_{t+1}, an induced minor isomorphic to $K_{t,t}$ or an induced minor isomorphic to F.*

To state the induced subgraph version, we need a few definitions. A *constellation* is a graph $\mathfrak{c}$ in which there is a stable set $S_\mathfrak{c}$ such that every component of $\mathfrak{c}\setminus S_\mathfrak{c}$ is a path, and each vertex $x \in S_\mathfrak{c}$ has at least one neighbor in each component of $\mathfrak{c} \setminus S_\mathfrak{c}$. We denote by $\mathcal{L}_\mathfrak{c}$ the set of all components $\mathfrak{c} \setminus S_\mathfrak{c}$ (each of which is a path), and denote the constellation $\mathfrak{c}$ by the pair $(S_\mathfrak{c}, \mathcal{L}_\mathfrak{c})$. For positive integers l, s, by an (s,l)-*constellation* we mean a constellation $\mathfrak{c}$ with $|S_\mathfrak{c}| = s$ and $|\mathcal{L}_\mathfrak{c}| = l$. Given a graph G, by an (s,l)-*constellation in* G we mean an induced subgraph of G which is an (s,l)-constellation.

We will need a few notions associated with a constellation $\mathfrak{c} = (S_\mathfrak{c}, \mathcal{L}_\mathfrak{c})$, which we define below:

- By a $\mathfrak{c}$-*route* we mean a path R in $\mathfrak{c}$ with ends in $S_\mathfrak{c}$ and with $R^* \subseteq V(\mathcal{L}_\mathfrak{c})$, or equivalently, with $R^* \subseteq L$ for some $L \in \mathcal{L}_\mathfrak{c}$.
- For a positive integer d we say that $\mathfrak{c}$ is d-*ample* if there is no $\mathfrak{c}$-route of length at most $d+1$. We also say that $\mathfrak{c}$ is *ample* if $\mathfrak{c}$ is 1-ample. It follows that $\mathfrak{c}$ is ample if and only if no two vertices in $S_\mathfrak{c}$ have a common neighbor in $V(\mathcal{L}_\mathfrak{c})$.
- We say that $\mathfrak{c}$ is *interrupted* if there is an enumeration $x_1, \ldots, x_s$ of all vertices in $S_\mathfrak{c}$ such that for all $i, j, k \in \mathbb{N}_s$ with $i < j < k$ and every $\mathfrak{c}$-route R from x_i to x_j, the vertex x_k has a neighbor in R.
- For a positive integer q, we say that $\mathfrak{c}$ is q-*zigzagged* if there is an enumeration $x_1, \ldots, x_s$ of all vertices in $S_\mathfrak{c}$ such that for all integers $i, k \geq s$ with $i < k$ and every $\mathfrak{c}$-route R from x_i to x_k, fewer than q vertices in $\{x_j : i < j < k\}$ are anticomplete to R in $\mathfrak{c}$.

Interrupted constellations form a slight extension of another construction from [1, 2], and zigzagged constellations are a fairly substantial generalization of a construction from [5, 6] (see [3] for further discussion). We can now state the induced subgraph analogue of Theorem 3.

Theorem 5. *For all positive integers d, r, l, l', s, s', there is an integer $p_{d,r,l,l',s,s'}$ such that if G is a graph with $\mathrm{pw}(G) \geq p_{d,r,l,l',s,s'}$, then one of the following holds.*

(a) *There is an induced subgraph of G isomorphic to K_{r+1}, $K_{r,r}$, a subdivision of a binary tree of height $2r$, or the line graph of a subdivision of a binary tree of height $2r$.*
(b) *There is a d-ample interrupted (s,l)-constellation in G.*
(c) *There is a d-ample 2^{4r+1}-zigzagged (s', l')-constellation in G.*

Theorem 5 is "qualitatively" best possible, in the sense that:

- the outcomes of 5 themselves can have arbitrarily large pathwidth; and
- the statement of 5 will be false if any of the outcomes is omitted.

The first point is straightforward to check, and the second point is easily seen to be true for 5(a). For 5(b) and 5(c), the second point follows from results of [3] and the fact that all constellations are K_4-free, and all ample constellations are $K_{3,3}$-free. Additionally, one might hope that 5(c) could be refined further to coincide with the Pohoata-Davies graphs [5, 6], but [3] shows that this is not possible.

To deduce Theorem 5 from Theorem 4 we rely on the main result of [3] that describes unavoidable induced subgraphs of large complete bipartite induced minor models.

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A width parameter that exhibits local-global structure

Reinhard Diestel

(joint work with Tara Abrishami and Paul Knappe)

Tree-decompositions of a graph exhibit its tree-like global structure. Graphs that are not globally tree-like can more fittingly be described by decompositions modelled on graphs other than trees (Figure 1). We indicate how these may be found.

Given graphs G and H, we call a family $(G_h)_{h\in H}$ of induced subgraphs G_h of G indexed by the nodes h of H an *H-decomposition* of G with *model* H and *width* $\max\{\,|G_h| : h \in H\}$ if $\bigcup_{h\in H} G_h = G$ and, for all vertices v of G, the H-subgraphs

$$H_v := H\,[\{h \in H : v \in G_h\}] \text{ are connected.} \tag{1}$$

Note the duality $v \in G_h \Leftrightarrow h \in H_v$ between the *parts* G_h and the *co-parts* H_v.

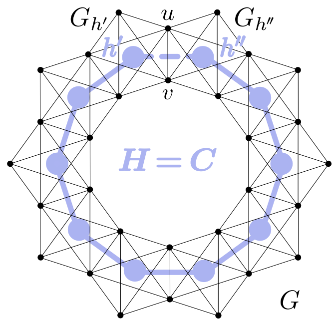

FIGURE 1. An H-decomposition of a graph G with H a cycle

Like the analogous axiom for tree-decompositions, (1) relates the structure of H to that of G: without (1), these would not be related. In Figure 1, the cyclic global structure of G is reflected in the model H of its decomposition, which is a cycle C. Deleting the edge $h'h''$ from C turns it into a path P, which no longer reflects the global structure of G as well. And indeed, $(G_h)_{h \in P}$ violates (1).

In the example we can repair (1), and thus turn $(G_h)_{h \in P}$ into a P-decomposition of G, by adding the vertices u and v to all the parts G_h. So the drop in quality as we turn our fitting cyclic decomposition into a less fitting path-decomposition is detected by a simple parameter: the decomposition's width increases from 5 to 7.

Motivated by this example, here is an intriguing idea. Might we be able to find a fitting H-decomposition of a graph G, one whose model H describes the global structure of G better than other decompositions, simply by minimising the width of those decompositions *while allowing H to vary*?

A moment's thought shows that this will not work without any requirements on H: if we allow H to be complete, for example, then (1) will always be satisfied. To rule out such H, we might try to emulate tree-decompositions a little more closely: while allowing H to contain cycles, we might still require it to be sparse in some sense. Graph minor theory suggests that, perhaps, we might require H to be embeddable in some surface of low genus depending on G.

But even requiring H to be planar allows H-decompositions we clearly do not want to allow. For example [4], consider the H-decomposition of a complete graph K of order n, where H is the $(n \times n)$-grid with rows and columns both indexed by the vertices of K. For all $h = (u, v)$ let $G_h := \{u, v\}$. This makes $(G_h)_{h \in H}$ into an H-decomposition of K of width 2, which is clearly minimum over all choices of H. But we can hardly claim that the grid H captures anything like 'global structure' of the complete graph K.

Yet while grids are sparse in the sense of containing few edges, they are not sparse in our local-global sense: unlike the graphs which Erdős [5] found to have high chromatic number for global reasons alone, they have low girth. Since our aim is that H should be a model for the global structure of G, should we require H to have large girth? This would also rule out both grids and complete graphs for H.

With a small adjustment, this does indeed work. An adjustment is needed, because large girth can be achieved for H simply by subdividing its edges, assigning to nodes subdividing an edge hh' either G_h or $G_{h'}$, which does little to improve the value of H as a model of the global structure of G. Yet what we are aiming for is not that our models H themselves be locally tree-like but that what they model in G should be. And this is indeed what our condition on H will imply: the H-decompositions of G we shall admit will induce low-width tree-decompositions everywhere locally in G. Hence those models H neither represent, nor add to, the existing local connectivity in G (which resides entirely in the parts G_h) while reflecting, by virtue of (1), all the global connectivity of G that determines how those local tree-decompositions fit together to form a uniform overall structure.

Given an integer $r \geq 0$, we say that an H-decomposition of G is *r-acyclic* if there is no cycle in any of the subgraphs $\bigcup_{v \in X} H_v$ of H generated by a set X of at most r vertices of G. Note that deleting nodes h with $G_h = \emptyset$ or edges hh' with $G_h \cap G_{h'} = \emptyset$ in H will not invalidate an r-acyclic H-decomposition, so we shall normally assume that these are non-empty.

The models H of r-acyclic decompositions are easily seen to have girth $g(H) > r$. But the condition is stronger, since vertices v of G can lie in many parts G_h, and so even a single co-part H_v might contain a cycle.

These decompositions achieve our benchmark for modelling global structure, in a way that lets us choose our desired local-global threshold by prescribing r:

Theorem 1. *Every r-acyclic H-decomposition of G induces tree-decompositions on the $(r/2)$-local neighbourhoods $B(v, r/2)$ in G of the vertices v of G.*

For odd r, the T-decomposition (say) induced on B by the H-decomposition of G is obtained from the H-decomposition $(B \cap G_h)_{h \in H}$ of $B = B(v, r/2)$ by deleting nodes h of H with $B \cap G_h = \emptyset$ and edges hh' with $B \cap G_h \cap G_{h'} = \emptyset$ to obtain T. For even r we need a small technical adjustment, but the idea is the same.

The least width of an r-acyclic decomposition of G is its *r-acyclic width*, r-aw(G). In order to find our desired local-global decomposition of a given graph G, we now *fix our desired threshold r first, and then take any r-acyclic decomposition of G of minimum width.* Its model H reflects only the global structure of G, by Theorem 1, while the G_h reflect only its local structure since they are small.

Our ability to choose r is key to adjusting the balance between two possible aims. If we choose r small we allow more variety for H, and minimising the width will produce models H that hug G so closely that they add no new perspective. Indeed, for $r = 2$ we can choose H as a subdivision of G, assigning a node h subdividing the edge uv the part $G_h \subseteq G$ induced by $\{u, v\}$. If we let $r \geq |G|$, on the other hand, then no cycles are allowed in H. Then the r-acyclic width of G equals its tree-width (plus 1), and even a minimum-width decomposition will have parts so large that no interesting connectivity structure lies between them, modelled by H.

In the remainder of these notes, we list some known properties of these decompositions. For a start, note that the r-acyclic width of a graph G increases with r: from at most 2 for $r = 2$ up to the tree-width of G (plus 1) for $r \geq |G|$, as noted.

For $r \geq 3$ we have r-aw$(G) \leq 2$ if and only if $g(G) > r$, and r-aw$(G) \geq \omega(G)$ for all G, where g and ω denote girth and clique number, respectively. Indeed, unlike in our earlier unnatural grid-decomposition of a complete graph we have the following more natural property:

Theorem 2. *Every clique in G lies in a part of any r-acyclic decomposition of G if $r \geq 3$. In particular, the cycle-decomposition in Figure 1 has minimum width amongst all r-acyclic graph-decompositions of G for $3 \leq r \leq 9$.*

The tree-width of a graph G is well known to equal the least clique number (minus 1) of a chordal supergraph of G, which can be obtained by adding edges to the parts of any minimum-width tree-decomposition to make them complete. This fact extends to r-acyclic decompositions: making the parts of any r-acyclic decomposition of G complete yields a supergraph of G that is *r-locally chordal* in the sense that all the $(r/2)$-local neighbourhoods $B(v, r/2)$ of its vertices v are chordal. In fact, we have the following more general equivalence:

Theorem 3. [1] *Given $r \geq 3$, a finite graph has an r-acyclic decomposition into cliques if and only if it is r-locally chordal.*

By Theorem 3, the r-acyclic width of G is thus equal to the least clique number of an r-locally chordal supergraph of G.

Recall that the r-local width of a grid, however large, becomes 2 if we subdivide it sufficiently to give it girth $> r$. Grids whose faces are bounded by r-cycles, on the other hand, have unbounded r-width:

Theorem 4. *Given $r \geq 3$, the r-acyclic width of any grid whose faces are bounded by cycles of length at most r is at least its tree-width.*

Proof sketch. We start from any optimal r-acyclic decomposition of the given graph G. Making its parts complete we obtain an r-locally chordal supergraph G' of G by Theorem 3. The covering G'_r of G' for the subgroup of $\pi_1(G')$ generated by the cycles of length at most r as characteristic subgroup, see [3], is a chordal graph [1]. Since all the cycles of $G \subseteq G'$ lie in this subgroup, G'_r contains a copy of G and thus has tree-width at least that of G. As G'_r is chordal, its tree-width (plus 1) equals its clique number $\omega(G'_r)$, which in turn equals $\omega(G')$ and thereby the width of our original decomposition of G. □

The proofs [2] of Theorems 1, 3 and 4 rely on our theory of local-global structure theory for finite graphs based on tangle analysis of their *r-local* covering spaces [3].

Our new width parameter gives rise to some immediate questions, both technical and more fundamental. Technically, we would like to know which operations on G that make it smaller will also reduce its r-acyclic width, or at least cannot increase it. Deleting vertices or edges of G leave any r-acyclic decomposition intact, and hence cannot increase the width. We thus have, in principle, Kuratowski-type characterisations of all the classes $\mathcal{G}^r_k$ of graphs of r-acyclic width at most k, given by the $\subset$-minimal graphs of r-acyclic width $> k$ as 'forbidden subgraphs'.

Contracting edges, however, can increase the r-acyclic width: recall that for C^{r+1} this is 2, while for C^r it is 3, the tree-width of C^r (plus 1).

Problem 5. *Which edge contractions cannot increase the r-acyclic width of G?*

Note that H-decompositions of G induce H-decompositions on minors of G, as with tree-decompositions. The induced decompositions just need not be r-acyclic if the original was. They will, however, be (r/s)-acyclic if the minor's branch sets have diameter at most about $s \leq r$ [2].

A more fundamental question is what certificates of large r-acyclic width, if any, will necessarily occur in graphs of large enough r-acyclic width. For example:

Problem 6. *Given $r \geq 3$, is there an $\mathbb{N} \to \mathbb{N}$ function f such that every graph of r-acyclic width at least $f(k)$ contains a subdivided grid of r-acyclic width at least k?*

Although we noted that requiring H to be sparse in terms of edge density is not enough to ensure that H-decompositions separate out the global-local structure of a graph as intended, one can still ask how sparse our models H can be in that sense:

Problem 7. *Is there an $\mathbb{N} \to \mathbb{R}$ function $h \to 0$ such that $\|H\| \in O(|G|^{1+h(r)})$ for some minimum-width r-acyclic H-decompositions of graphs G and $r \geq 3$?*

Finally, we can ask how our global-local distinction works with graph invariants which, like the chromatic number, can be large for local or global reasons alone. For example, if G has large chromatic number and $(G_h)_{h\in H}$ is a minimum-width r-acyclic decomposition of G, must either some 'local' G_h or some 'global' large-girth subgraph of H have large chromatic number?

Problem 8. *Given $r \geq 3$, let H be the model of any r-acyclic decomposition of width $w = r$-$aw(G)$ of a graph G. Let k be the largest chromatic number of any topological minor H' of H of girth $> r$. Can G be coloured with wk colours?*

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Long induced paths

Louis Esperet

(joint work with Julien Duron, Jean-Florent Raymond)

For a graph class $\mathcal{G}$, and an integer n, let $f_{\mathcal{G}}(n)$ be the largest integer k such that any graph $G \in \mathcal{G}$ with a path on n vertices contains an induced path on k vertices. What natural conditions on $\mathcal{G}$ ensure that $f_{\mathcal{G}}(n) \to \infty$? Complete bipartite graphs must be forbidden as subgraphs, since these graphs have long paths but no induced paths of order 3. A classical result of Galvin, Rival, and Sands [7] states that if G contains an n-vertex path and is $K_{t,t}$-subgraph-free, then it contains an induced path of order $\Omega((\log\log\log n)^{1/3-o(1)})$. In [3], we recently improved this bound to $\Omega\left((\frac{\log\log n}{\log\log\log n})^{1/5}\right)$. The main idea of our proof is to use Ramsey's theorem for triples, instead of Ramsey's theorem for quadruples in the proof of [7]. Our result was recently improved to $\Omega(\frac{\log\log n}{\log\log\log n})$ by Hunter, Milojević, Sudakov, and Tomon in [8]. This is close to best possible, as Defrain and Raymond [4] recently constructed an infinite family of 2-degenerate (and thus $K_{3,3}$-subgraph-free) Hamiltonian n-vertex graphs without induced path of order $\Omega((\log\log n)^2)$.

The study of $f_{\mathcal{G}}(n)$ has also been investigated extensively when $\mathcal{G}$ is the class of outerplanar graphs, planar graphs or graphs of bounded genus [1, 5, 6], graphs of bounded pathwidth or treewidth [5, 9], degenerate graphs [4, 10], and graphs excluding a minor or topological minor [9].

In [3], we recently observed that most of the known results on this question can be recovered in a simple and unified way by considering the following variant of the problem. Consider that the vertices of the n-vertex path P in G are ordered following their occurrence in P. What forbidden ordered subgraphs in $G - E(P)$ force the existence of a long induced path in G? In [3], we showed that forbidding ordered matchings yields an induced path of order poly(n) or polylog(n), depending on the structure of the matching. This is enough to imply most existing lower bounds in the area. For instance, by considering a specific matching on 4 edges, we can deduce that every planar graph with a path on n vertices has an induced path on $\Omega(\sqrt{\log n})$ vertices, recovering a result from [5] (whose proof was quite intricate). The best known upper bound in the case of planar graphs is $O(\log n/\log\log n)$ and it is an interesting problem to close the gap between the lower and upper bounds.

In [2] we completely characterized the forbidden ordered subgraphs yielding induced paths of order polylog(n). These graphs are star forests with a specific vertex ordering, which we call *constellations*. Our proof has two parts: we first show that forbidding a constellation yields induced paths of order polylog(n), and we then construct a graph without any of these constellations, which has no induced path of order $\Omega((\log\log n)^2)$. The construction is inspired by the recent construction of [4] of a 2-degenerate Hamiltonian n-vertex graph without induced path of order $\Omega((\log\log n)^2)$.

As a direct consequence of our result, we obtain that graphs which do not contain K_t as a topological minor, and which contain a path on n vertices also contain an induced path of order $(\log n)^{\Omega(1/t\log^2 t)}$. This simplifies and improves upon an earlier result of [9], in which the exponent was an unspecified function of t (relying on structure theorems of Robertson and Seymour, and Grohe and Marx). In the particular case of forbidden K_t minors, this also improves upon the bound of order $(\log n)^{\Omega(1/t^2)}$ obtained in [3] using forbidden ordered matchings (we note that the proof of the weaker bound in [3] is significantly simpler than the proof of the stronger bound obtained in the present paper).

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Extremal graph theory methods for forbidden induced subgraphs

Jacob Fox

(joint work with Matija Bucić, Huy Tuan Pham)

What can we say about the structure of induced H-free graphs G? More generally, what about graphs in a hereditary family F? In this talk, we discuss some of the major results and open problems in this area, focusing on methods that come from extremal graph theory.

A very basic example which shows the difficulty is to study the structure of triangle-free graphs. Some basic questions remain open, including the following two conjectures.

Conjecture 1. *There is $c > 0$ such that every graph G on $n \geq 3$ vertices with $\alpha(G) = 2$ contains the complete bipartite graph $K_{n^c,cn}$.*

Conjecture 2. *There is $c > 0$ such that every graph G on $n \geq 3$ vertices with $\alpha(G) = 2$ contains a connected matching of size cn.*

Some powerful extremal graph theory methods include Szemerédi's regularity method, dependent random choice, the containers method, and the greedy embedding method. Using Szemerédi's regularity lemma one can prove the following result, which is a strengthening of the graph removal lemma.

Theorem 3. *If G is an induced H-free graph, then G has edit distance at most $\varepsilon|V(G)|^2$ from a graph G' such that G' has an induced homomorphism to an induced H-free graph F with at most $M_H(\varepsilon)$ vertices.*

Understanding how small $M_H(\varepsilon)$ is as a function of ε is an important problem with various applications in computer science, combinatorics, discrete geometry, and number theory. See the paper [3] by Zhao and the speaker for more about this problem.

Another major open problem in this area is the Erdős-Hajnal conjecture:

Conjecture 4. *For each graph H there is $c(H) > 0$ such that every induced H-free graph on n vertices contains a clique or independent set of size at least n^c.*

It is well-known that polynomial versions of theorems of Rödl and Nikiforov, as conjectured by Fox and Sudakov and Nguyen, Scott and Seymour imply the classical Erdős-Hajnal conjecture. In the recent paper [1], Bucić, Pham, and the speaker proved that these three conjectures are in fact equivalent, extending several previous particular results in this direction by Fox, Nguyen, Scott and Seymour; Nguyen, Scott and Seymour and Gishboliner and Shapira. We deduce that the family of string graphs satisfies the polynomial Rödl conjecture. The proof utilizes the graph container method, a powerful method in extremal graph theory. In this talk, we also presented a simple inductive proof of the graph container lemma from [1]. We also derived analogous results for hypergraphs, tournaments, ordered graphs, and colored graphs.

In the direction of the Erdős-Hajnal conjecture, Erdős and Hajnal used the greedy embedding method to prove the following result. For every induced H-free graph G on $n^{C(H)}$ vertices with $n \geq 2$, G or its complement $\bar{G}$ contains $K_{n,n}$ as a subgraph. The greedy embedding method is a general technique that is used in many extremal graph theory problems. The basic idea is to try to embed a copy of a graph one vertex at a time, keeping the sets of future potential vertices as large as possible. Using the powerful probabilistic method dependent random choice, Sudakov and the speaker [2] improved this to the following result in the direction of the Erdős-Hajnal conjecture. For every induced H-free graph G on $n^{C(H)}$ vertices, G contains an independent set of order n or $K_{n,n}$ as a subgraph.

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Discrepancy of spanning substructures in hypergraphs

Stefan Glock
(joint work with Lior Gishboliner, Peleg Michaeli, Amedeo Sgueglia)

The basic question in combinatorial discrepancy theory can be phrased as follows: Given an integer $r \geq 2$, a finite ground set Ω and a family $\mathcal{P}$ of subsets of Ω, can we partition Ω into r parts such that every set $P \in \mathcal{P}$ contains (roughly) the same number of elements from each part? Or is there always some "discrepancy" no matter how the partition is made? In the context of graph theory, a well-studied question is whether for a given k-uniform graph (k-graph for short) and an integer $r \geq 2$, any r-colouring of its edges must contain a certain substructure with *high discrepancy*, meaning that within this substructure one of the colour classes is significantly larger than the others. Classical results include the works of Erdős and Spencer [6] on cliques and of Erdős, Füredi, Loebl and Sós [5] on spanning trees. By now, there is a large collection of results for 2-graphs, but very little is known for hypergraphs. Here, we present some recent results for spanning substructures of hypergraphs such as perfect matchings, tight Hamilton cycles and Steiner triple systems.

Tight Hamilton cycles. *(Joint work with Lior Gishboliner and Amedeo Sgueglia)* A *tight Hamilton cycle* of a k-graph G is a cyclic ordering $v_1, \dots, v_n$ of the vertices of G such that $v_i v_{i+1} \dots v_{i+k-1}$ is an edge for every $1 \leq i \leq n$, with indices taken modulo n. For a k-graph G and a set $S \subseteq V(G)$, we say that the *degree* of S in G, denoted by $d_G(S)$, is the number of edges containing S. We use $\delta(G)$ to denote the *minimum* $(k-1)$*-degree*, which is the minimum of $d_G(S)$ over all $(k-1)$-sets $S \subseteq V(G)$.

We determine the optimal minimum $(k-1)$-degree condition for a tight Hamilton cycle of high discrepancy for any uniformity $k \geq 3$ and any number of colours $r \geq 2$.

Theorem 1. *For all $k, r \in \mathbb{N}$ with $k \geq 3$ and $r \geq 2$, and all $\varepsilon > 0$, there exists $\mu > 0$ such that the following holds for all sufficiently large n. Let G be an n-vertex k-graph with $\delta(G) \geq (1/2 + \varepsilon)n$ whose edges are r-coloured. Then there exists a tight Hamilton cycle in G which contains at least $(1+\mu)\frac{n}{r}$ edges of the same colour.*

This establishes a discrepancy version of the celebrated theorem of Rödl, Ruciński and Szemerédi [10], who proved that $1/2$ is the (asymptotic) threshold for the existence of a tight Hamilton cycle. The graph case $k = 2$ was already settled in [2, 7, 9], in which case the discrepancy threshold is strictly larger than the existence threshold and decreases as r increases. Our proof combines various structural techniques such as Turán-type problems and hypergraph shadows with probabilistic techniques such as random walks and the nibble method.

Note that if n is divisible by k, then a tight Hamilton cycle decomposes into k perfect matchings. Therefore, under the same hypotheses we also find a perfect matching with high discrepancy and, again, the constant $1/2$ is asymptotically best possible. We remark that this corollary was proved independently by Balogh, Treglown and Zárate-Guerén [3].

Steiner triple systems. *(Joint work with Lior Gishboliner and Amedeo Sgueglia)* A *Steiner triple system* (STS for short) of order n is a set $\mathcal{S}$ of 3-subsets of $[n]$ such that every 2-subset of $[n]$ is contained in exactly one of the triples of $\mathcal{S}$. STSs are one of the central objects of study in the theory of combinatorial designs and graph decompositions, and also have applications in other areas. By a famous result of Kirkman, an STS of order n exists if and only if $n \equiv 1, 3 \pmod 6$.

The following construction shows that there is a 2-edge-colouring of the 3-subsets of $[n]$ (equivalently of the edges of $K_n^{(3)}$, i.e. the complete 3-graph on n vertices) where every STS has roughly the same number of triples of each colour (recall that an STS has $\binom{n}{2}/3$ triples).

Example 2. *Partition $[n]$ into two sets X and Y, and assign colour blue to all triples touching both X and Y, and colour red to all the other triples. In an STS, every edge between X and Y must be covered by a blue triple, and every blue triple covers exactly two of these edges. Hence, in any STS, the number of blue triples is $\frac{|X||Y|}{2}$. By choosing $|X| = \lfloor \frac{3+\sqrt{3}}{6} n \rfloor$, one can show that $\frac{|X||Y|}{2}$ is roughly $\binom{n}{2}/6$. Thus, in this colouring, any STS is roughly balanced.*

However, we can show that this is essentially the only 2-edge-colouring of $K_n^{(3)}$ which does not contain an STS with high discrepancy.

Theorem 3. *For every $\eta > 0$, there exists $\mu > 0$ such that the following holds for all sufficiently large n with $n \equiv 1, 3 \pmod 6$. Every 2-edge-colouring of $K_n^{(3)}$ either contains a Steiner triple system with at least $(1/2 + \mu)\binom{n}{2}/3$ triples of the same colour, or it can be obtained from the colouring in Example 2 by switching the colour of at most ηn^3 triples.*

In contrast with the 2-colour case, when the number of colours is three or more, we can show that an STS with high discrepancy always exists.

Theorem 4. *For all $r \in \mathbb{N}$ with $r \geq 3$, there exists $\mu > 0$ such that the following holds for all sufficiently large n with $n \equiv 1, 3 \pmod 6$. Every r-edge-colouring of $K_n^{(3)}$ contains a Steiner triple system with at least $(1/r + \mu)\binom{n}{2}/3$ triples of the same colour.*

Defect and transference versions of the Alon–Frankl–Lovász theorem. *(Joint work with Lior Gishboliner, Peleg Michaeli and Amedeo Sgueglia)* Confirming a conjecture of Erdős [4] on the chromatic number of Kneser hypergraphs, Alon, Frankl and Lovász [1] proved that in any r-colouring of the edges of the complete k-graph, there exists a monochromatic matching of size $\lfloor \frac{n+r-1}{k+r-1} \rfloor$. This bound is best possible. Moreover, by arbitrarily adding edges to this monochromatic matching, we can complete it to a perfect matching with high discrepancy (assuming $k \mid n$ of course). In this sense, the AFL Theorem is a discrepancy result for perfect matchings.

Many classical theorems concerning dense graphs (or hypergraphs) have corresponding analogues in the setting of (sparse) random graphs. Such results are usually known as *transference* theorems. We prove a transference version of the

AFL Theorem. The random hypergraph model we consider is the binomial random k-graph $\mathbb{G}^{(k)}(n,p)$, which has n vertices, and where each k-set of vertices forms an edge independently with probability p.

Theorem 5. *For all $k, r \in \mathbb{N}$ with $k, r \geq 2$, and all $\mu > 0$, there exists $C > 0$ such that, provided $p \geq Cn^{-k+1}$, w.h.p. the following holds for $G \sim \mathbb{G}^{(k)}(n,p)$: For any r-colouring of the edges of G, there exists a monochromatic matching of size at least $(1-\mu)\frac{n}{k+r-1}$.*

The size of the matching is asymptotically best possible since one cannot do better even in the complete k-graph. The graph case $k = 2$ was already proved in [8].

We prove Theorem 5 by combining the sparse hypergraph regularity method with the following "defect" version of the AFL Theorem. It shows that, for large n, the conclusion of the AFL theorem approximately holds even for edge-colourings of *almost* complete k-graphs.

Theorem 6. *For all $k, r \in \mathbb{N}$ with $k, r \geq 2$, and all $\mu > 0$, there exists $\varepsilon > 0$ such that the following holds for all sufficiently large n. Let G be an n-vertex k-graph whose edges are r-coloured and assume $e(G) \geq (1-\varepsilon)\binom{n}{k}$. Then G contains a monochromatic matching of size at least $(1-\mu)\frac{n}{k+r-1}$.*

The main challenge for proving this result is that the proof of Alon, Frankl and Lovász uses the topological method, which does not seem to work in the almost-complete setting. Instead, we pursue a novel approach using tools from extremal set theory developed in the study of the Erdős matching conjecture.

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Twin-width

Eun Jung Kim
(joint work with Édouard Bonnet, Stéphan Thomassé, Rémi Watrigant)

Twin-width is a notion introduced in 2020 by Bonnet, Kim, Thomassé and Watrigant [1] which provides a unified perspective on a range of important graph classes, encompassing both sparse and dense classes. A *contraction* sequence of an n-vertex graph G is a sequence $G_n, G_{n-1}, \cdots, G_1$ of so called *trigraphs*, in which a graph has three binary relations, namely (black) edges, non-edges and *red edges*, such that $G_n = G$, G_1 is a single-vertex graph and each G_{i-1} is obtained from G_i by identifying two vertices u, v to a new vertex z and defining the relation between z and $w \in V(G_i) \setminus \{u, v\}$ as follows: zw is a black edge (respectively, non-edge) if and only if both uw and vw are black edges (respectively, non-edges); in all other cases, zw becomes a red edge. Twin-width of a graph G is the maximum *red degree* of a trigraph, i.e. the maximum number of red edges incident with a vertex in a trigraph, in a contraction sequence taken over all contraction sequences of G.

The notion of twin-width was strongly inspired by the work of Guillemot and Marx [3], which proposed a linear-time algorithm to decide if a given permutation π contains a permutation pattern σ as a subpattern. A key insight of their work was that one of the following possibilities occur. If the permutation matrix of π contains a $|\sigma|$-grid, namely a partition of the columns and rows into $|\sigma|$ parts consisting of consecutive rows and columns (*division*) so that each *cell* contains an entry 1, then π contains any permutation of length $|\sigma|$ as a subpattern. If the permutation matrix of π does not contain $|\sigma|$-grid, then there is a sequence of merging adjacent row parts and column parts, starting from the finest division of the permutation matrix into the coarsest division, in such a way that the division sequence can be used to design an algorithm for detecting σ as a subpattern. The classic result of Markus and Tardos [4] which bounds the density of 1-entries in a (0,1)-matrix excluding a grid of fixed size crucially underlies their result.

An underlying idea of twin-width is that the matrix viewpoint and the use of *merge sequence* for either detecting an obstruction (such as σ, if you are interested in a permutation class avoiding σ as a permutation) or for finding a well-behaved sequence can be exploited for graphs and binary structures in general. Many graph classes ranging from planar graphs, H-minor-free graphs to proper interval graphs and graphs of bounded cliquewidth have bounded twin-width. This new perspective also allows us to establish powerful properties such as (polynomial) χ-boundedness [5, 2] and tractability of First-Order model checking on many graph classes [1] in a unified way. Twin-width is now an important part of the toolbox for structural graph theory, algorithms design, logic on finite graphs and data structure.

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A coarse Erdős-Pósa theorem

O-joung Kwon

(joint work with Jungho Ahn, Pascal Gollin, Tony Huynh)

An *induced packing* of cycles in a graph is a set of vertex-disjoint cycles with no edges between them. We generalise the classic Erdős-Pósa theorem to induced packings of cycles. More specifically, we show that there exist functions $f(k,\ell) = \mathcal{O}(\ell k \log k)$ and $g(k) = \mathcal{O}(k \log k)$ such that for all integers $k \geq 1$ and $\ell \geq 3$, every graph G contains either an induced packing of k cycles of length at least ℓ, not necessarily induced cycles, or sets X_1 and X_2 of vertices with $|X_1| \leq f(k,\ell)$ and $|X_2| \leq g(k)$ such that, after removing the closed neighbourhood of X_1 or the ball of radius ℓ around X_2, the resulting graph has no cycle of length at least ℓ in G. Our proof is constructive and yields a polynomial-time algorithm finding either the induced packing or the sets X_1 and X_2 when ℓ is a constant. Furthermore, we show that for every positive integer d, if a graph G does not contain two cycles at distance more than d, then G contains sets X_1 and X_2 of vertices with $|X_1| \leq 12(d+1)$ and $|X_2| \leq 12$ such that, after removing the ball of radius $2d$ around X_1 or the ball of radius $3d$ around X_2, the resulting graphs are forests.

As a corollary, we prove that every graph with no $K_{1,t}$ induced subgraph and no induced packing of k cycles of length at least ℓ has tree-independence number at most $\mathcal{O}(t\ell k \log k)$, and one can construct a corresponding tree-decomposition in polynomial time when ℓ is a constant. This resolves a special case of a conjecture of Dallard et al. (arXiv:2402.11222), and implies that on such graphs, many NP-hard problems, such as Maximum Weight Independent Set, Maximum Weight Induced Matching, Graph Homomorphism, and Minimum Weight Feedback Vertex Set, are solvable in polynomial time. On the other hand, we show that the class of all graphs with no $K_{1,3}$ induced subgraph and no two cycles at distance more than 2 has unbounded tree-independence number.

We conjecture that the following generalisation of our main result holds.

Conjecture 1. *There exist functions $f(k) = \mathcal{O}(k \log k)$ and $g(d) = \mathcal{O}(d)$ such that for all positive integers k and d, every graph G contains either a family of k cycles* *whose pairwise distance is more than d, or a set X of at most $f(k)$ vertices such that $G - B_G(X, g(d))$ is a forest.*

Palettes determine uniform Turán density

Ander Lamaison

The *Turán density* of an r-uniform hypergraph F (or r-graph for short) is the largest value of d such that there exist arbitrarily large r-graphs F, with edge-density arbitrarily close to d, not containing F as a subgraph.

Most of the extremal constructions for Turán problems in the hypergraph setting have large independent sets, i.e., linear-sized sets of vertices with no edges. This lead Erdős and Sós [1, 3] to propose studying the *uniform Turán density* of hypergraphs, which is the density threshold for the existence of a hypergraph with the additional requirement that the edges of the host hypergraph are distributed uniformly. We define this notion as follows:

Definition 1. *A hypergraph H is said to be* locally (d, ε)-dense *if any subset $S \subseteq V(H)$ of size at least $\varepsilon|V(H)|$ has edge density at least $d - \varepsilon$. A sequence of hypergraphs $\{H_i\}_{i=1}^{\infty}$ is said to be* locally d-dense *if each H_i is locally (d, ε_i)-dense with $\varepsilon_i \to 0$, and $|V(H_i)| \to \infty$.*

The uniform Turán density $\pi_u(F)$ *of an r-graph F is defined as the supremum of the values of d, for which there exists a locally d-dense sequence of r-graphs which do not contain F as a subgraph.*

In all exact results on the uniform Turán density of 3-graphs, palette constructions, which were introduced by Reiher [5], extending constructions by Erdős and Hajnal [2] and Rödl [7], act as the lower bound construction. Palettes can be seen a way to generate locally dense sequences of 3-graphs.

Definition 2. *A* palette *$\mathcal{P}$ is a pair $(\mathcal{C}, \mathcal{A})$, where $\mathcal{C}$ is a finite set (whose elements we call* colors*) and a set of (ordered) triples of colors $\mathcal{A} \subseteq \mathcal{C}^3$, which we call the* admissible triples*. The density of $\mathcal{P}$ is $d(\mathcal{P}) := |\mathcal{A}|/|\mathcal{C}|^3$.*

If $\mathcal{P}$ is a palette with density d, one can construct a locally d-dense sequence of 3-graphs $\{H_i\}_{i=1}^{\infty}$. To generate H_n, take $[n]$ as the vertex set. Color each pair of vertices uv with a color $\varphi(uv) \in \mathcal{C}$ chosen uniformly at random. The edges of H_n are the triples $u < v < w$ such that $(\varphi(uv), \varphi(uw), \varphi(vw)) \in \mathcal{A}$. With high probability, the resulting sequence is locally d-dense.

Definition 3. *We say that a 3-graph F* admits *a palette $\mathcal{P}$ if there exists an order $\preceq$ on $V(F)$ and a function $\varphi : \binom{V(F)}{2} \to \mathcal{C}$ such that for every edge $uvw \in E(F)$ with $u \prec v \prec w$ we have $(\varphi(uv), \varphi(uw), \varphi(vw)) \in \mathcal{A}$.*

It is easy to see that any subgraph of H_n admits the palette $\mathcal{P}$. Therefore, if a 3-graph F does not admit $\mathcal{P}$, then it is not a subgraph of the locally d-dense sequence $\{H_i\}_{i=1}^{\infty}$, and so $\pi_u(F) \geq d = d(\mathcal{P})$.

All the lower bound constructions for the known tight results on uniform Turán density mentioned above are derived from palettes via this procedure. This motivated the question, which had circulated in the community and is explicitly discussed in [5, Section 3], of whether all lower bounds on uniform Turán density can be obtained or approximated arbitrarily with palette constructions. Our main theorem answers this question in the affirmative.

Theorem 4. *For every* 3*-graph* F*,*

$$\pi_u(F) = \sup\{d(\mathcal{P}) : \mathcal{P} \text{ palette, } F \text{ does not admit } \mathcal{P}\}. \tag{1}$$

The proof of the main theorem is based on a method developed by Reiher [5], using some auxiliary structures called reduced hypergraphs. This method is itself based on the hypergraph regularity method, where the properties of locally dense sequences of hypergraphs are used to simplify the structure of their regularity partitions. To prove the main theorem, we show that every large enough reduced hypergraph, with edge density d, contains large repetitive substructures (which can be associated with the regularity partitions of palette constructions) with density at least $d - \varepsilon$. The proof is based on an iterative algorithmic approach, in which randomized vertex selections are alternated with applications of the hypergraph Ramsey theorem.

An important reason why Theorem 4 is significant is that, judging from all the proofs available so far, the right hand side of (1) is considerably easier to compute than the left hand side. The simplicity of the palette method compared to the regularity method has allowed for more exact values of π_u to be found. Previous to Theorem 4, the only values of d for which hypergraphs F with $\pi_u(F) = d$ were known to exist were $0, 1/27, 4/27, 1/4$ and $8/27$. Using Theorem 4, for each $k \geq 2$ we constructed a hypergraph F_k with $\pi_u(F_k) = \frac{1}{2} - \frac{1}{2k}$.

The k-star S_k is a 3-graph on $k+1$ vertices $u, v_1, v_2, \ldots, v_k$, where the edges are all the triples of the form uv_iv_j with $1 \leq i < j \leq k$. Reiher, Rödl and Schacht [6] studied the uniform Turán density of stars, and proved that

$$\frac{k^2 - 5k + 7}{(k-1)^2} \leq \pi_u(S_k) \leq \left(\frac{k-2}{k-1}\right)^2.$$

In [4], Wu and the author used Theorem 4 to prove that $\pi_u(S_k) = \frac{k^2-5k+7}{(k-1)^2}$ for all $k \geq 48$, providing another infinite family of values of the uniform Turán density.

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Disjoint paths problem with group-expressable constraints

Chun-Hung Liu
(joint work with Youngho Yoo)

The *Disjoint Paths Problem* is an algorithmic problem that gives a graph G and k pairs of vertices $\{s_1, t_1\}, \dots, \{s_k, t_k\}$ in G and asks us to determine whether there exists a set of k disjoint paths $P_1, \dots, P_k$ such that P_i joins s_i and t_i for all $i \in [k]$. This problem is known to be NP-complete [3]. On the other hand, Robertson and Seymour [7] gave an $O(n^3)$ time algorithm for the k*-Disjoint Paths Problem* where k is fixed. This running time was later improved to $O(n^2)$ in [4] and, more recently, to almost-linear time in [6].

In this talk, we consider a variant of the k-Disjoint Paths Problem that not only asks for paths linking the given pairs of terminals but also asks those paths to satisfy additional constraints.

For an integer f, an f*-group-expressable property* is a property $\mathcal{Q}$ for paths and cycles in G such that there exists an abelian group Γ, a subset F of Γ with $|F| \leq f$, and a function $\phi : E(G) \to \Gamma$ such that a path or cycle P in G satisfies the property $\mathcal{Q}$ if and only if $\sum_{e \in E(P)} \phi(e) \in \Gamma - F$. For integers d and f, we say that a property $\mathcal{Q}$ is (d, f)*-group-expressable* if there exist an integer d' with $0 \leq d' \leq d$ and f-group-expressable properties $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_{d'}$ such that $\mathcal{Q} = \bigcap_{i=1}^{d'} \mathcal{Q}_i$.

Examples of (d, f)-group-expressable properties include:

- The property "having length ℓ modulo m", where ℓ and m are nonnegative integers with $m \geq 2$.
- Let S be a subset of $E(G)$ or $V(G)$, and let ℓ and m be integers with $m \geq 2$.
  - The property "passing through S at least ℓ times".
  - The property "passing through S x times, where $x \equiv \ell \pmod{m}$".
- The property "having length at least $\text{dist}(s, t) + \ell$" for paths between two fixed vertices s and t, where ℓ is a nonnegative integer and $\text{dist}(s, t)$ is the distance between s and t.
- The property "having length at least $g + \ell$" for cycles, where g is the girth and ℓ is a nonnegative integer.

The following is the main result of this talk.

Theorem 1 (Liu, Yoo). *For any positive integers k, d, f, and for any (d, f)-group-expressable properties $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_k$, there exists an algorithm which, given* *as input a simple graph G and k pairs $\{s_1, t_1\}, ..., \{s_k, t_k\}$ of vertices of G, either finds k disjoint paths $P_1, P_2, ..., P_k$ in G such that P_i is a path between s_i and t_i satisfying $\mathcal{Q}_i$ for every $i \in [k]$, or concludes that no such k disjoint paths exist, with running time $O(|V(G)|^8)$.*

Theorem 1 generalizes results in [2, 5] on polynomial time algorithms for the k-Disjoint Paths Problem with parity constraints, though with worse running time.

The $k = 1$ case of Theorem 1 immediately leads to the following corollary that solves an open problem stated in [1].

Corollary 2 (Liu, Yoo). *If ℓ and m are nonnegative integers with $m \geq 2$, then there exists an $O(n^8)$ time algorithm that determines whether an input n-vertex simple graph with specified vertices s and t contains a path between s and t of length ℓ modulo m.*

Our algorithm for the k-Disjoint Paths Problem with (d, f)-group-expressable properties also leads to the following algorithm for testing the existence of a subdivision of a fixed graph with requirements on the paths between branch vertices.

Corollary 3 (Liu, Yoo). *For any graph H, integers d, f and a collection $\{\mathcal{Q}_e : e \in E(H)\}$ of (d, f)-group expressable properties, there exists an algorithm which, given as input a simple graph G, either finds a subgraph of G isomorphic to a subdivision of H such that for every $e \in E(H)$, the path in the subdivision corresponding to e satisfies $\mathcal{Q}_e$, or concludes that no such a subgraph exists, with running time $O(|V(G)|^{|V(H)|+8})$.*

Taking H to be a loop in Corollary 3 immediately leads to the following corollary, which solves another open problem in [1].

Corollary 4 (Liu, Yoo). *If ℓ and m are nonnegative integers with $m \geq 2$, then there exists an $O(n^9)$ time algorithm that determines whether an input n-vertex simple graph contains a cycle of length ℓ modulo m.*

The case $\ell = 0$ in Corollary 4 were proved independently in [1, 8].

Problem 5. *Let ℓ and m be nonnegative integers with $m \geq 2$.*

(1) Does there exist a polynomial time algorithm which either finds a shortest path in an input graph between two specified vertices s and t of length ℓ modulo m, or conclude that no such a path exists?

(2) Does there exist a polynomial time algorithm which either finds a shortest cycle in an input graph of length ℓ modulo m, or concludes that no such a cycle exists?

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Monadically stable graph classes

Rose McCarty

(joint work with Jan Dreier, Eleftheriadis, Jacob Gajarský, Nikolas Mählmann, Pierre Ohlmann, Michal Pilipczuk, Wojciech Przybyszewski, Sebastian Siebertz, Marck Sokołowski, Szymon Toruńczyk)

Introduction. Stability is a notion which originates in model theory; it was first studied by Baldwin and Shelah [3], and it played a significant role in Shelah's classification theory [14]. Later on, Adler and Adler [2] noticed a surprising connection between stability and the theory of "sparsity" of Nešetřil and De Mendez [13]. This theorem gives a wonderful connection between model theory and graph minors, a fundamental subject in graph theory.

Moreover, stability provides a dividing line in computational complexity. In 2017, Grohe, Kreutzer, and Siebertz [12] proved that a subgraph-closed class of graphs $\mathcal{F}$ is stable if and only if there exists an "efficient" algorithm to determine whether an n-vertex graph in $\mathcal{F}$ satisfies or "models" a given first-order sentence ϕ. (To be more precise, this theorem holds under the widely-believed assumption that $FPT \neq AW[*]$. We omit further discussion of the technical aspects for now.) To prove this theorem, Grohe, Kreutzer, and Siebertz also introduced a combinatorial game called the "Splitter Game". They showed that the Splitter Game can be used to characterize which subgraph-closed classes of graphs are stable.

Unfortunately, all of the theorems discussed above only work for classes of graphs that are closed under taking subgraphs. These theorems do not hold under the weaker assumption that the class of graphs is closed under taking *induced* subgraphs. This has been a major focus of study in recent years. That is, how can we generalize this new combinatorial theory of stability to classes which are closed under taking induced subgraphs (but are not necessarily closed under taking subgraphs)? The talk focused on recent developments in this area, drawing parallels to the theorems discussed above.

Characterizations of stability. In this section we provide a brief overview of several new combinatorial characterizations of stable graph classes.

We note that, in the area, it is typical to discuss a slightly more technical notion of *monadic stability* rather than stability. However, Braunfeld and Laskowski [6] showed that the two notions coincide for classes of graphs that are closed under taking induced subgraphs. So we just discuss stability. Roughly, a class of graphs $\mathcal{F}$ is *stable* if there is no first-order formula that defines arbitrarily large linear orders when applied to graphs in $\mathcal{F}$. See, for instance, [15, Chapter 8] for a full introduction to stability.

For a class of graphs $\mathcal{F}$ that is subgraph-closed, Adler and Adler [2] showed that $\mathcal{F}$ is stable if and only if for each $r \in \mathbb{N}$, there exists $c = c(r) \in \mathbb{N}$ such that no graph in $\mathcal{F}$ contains a clique on c vertices as an r-shallow graph minor. Informally, a graph minor is *r-shallow* if it has a minor model where each bag induces a subgraph of radius at most r.

Buffière, Kim, and Ossona de Mendez [7] recently proved an analogous theorem without the assumption that $\mathcal{F}$ is subgraph-closed. They proved that a class of graphs $\mathcal{F}$ is stable if and only if for each $r \in \mathbb{N}$, there exists $c = c(r) \in \mathbb{N}$ such that no graph in $\mathcal{F}$ contains a half-graph of order c as an r-shallow vertex-minor. The *half-graph of order c* is the graph with $2c$-many vertices $a_1, a_2, \ldots, a_c, b_1, b_2, \ldots, b_c$, where a_i is adjacent to b_j if $j \geq i$. (Intuitively, these graphs encode a linear order.) An *r-shallow vertex-minor* of a graph G is any graph which can be obtained from G by performing the following operation at most r times. *Locally complementing* on an independent set $S \subseteq V(G)$ switches the adjacency between two vertices $u, v \in V(G) \setminus S$ if u and v have an odd number of common neighbors in S.

For a class of graphs $\mathcal{F}$ that is subgraph-closed, Grohe, Kreutzer, and Siebertz [12] proved that $\mathcal{F}$ is stable if and only if for each $r \in \mathbb{N}$, Splitter wins the radius-r Splitter Game on $\mathcal{F}$. This characterization is more constructive, because it means that we can verify a class is stable by exhibiting a winning strategy for a certain player in a sequence of two-player combinatorial games. Moreover, this game is a key step in Grohe, Kreutzer, and Siebertz's algorithm for first-order model-checking.

Recently, Gajarský, Mählmann, McCarty, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, and Toruńczyk [11] found an analogous game in the more general setting. They proved that for a class of graphs $\mathcal{F}$ which is closed under induced subgraphs, $\mathcal{F}$ is stable if and only if for each $r \in \mathbb{N}$, Flipper wins the radius-r Flipper Game on $\mathcal{F}$.

Model-checking Algorithms. On the algorithmic side, we say that first-order model-checking is *fixed-parameter tractable (FPT)* on a graph class $\mathcal{F}$ if for each first-order sentence ϕ, there is an algorithm which can determine whether an n-vertex graph $G \in \mathcal{F}$ satisfies ϕ in time $\mathcal{O}_\phi(n^c)$, where $c > 0$ is some constant that is allowed to depend on $\mathcal{F}$, but not on the sentence ϕ or the graph G.

Recall that Grohe, Kreutzer, and Siebertz [12] proved that first-order model-checking is FPT on any graph class which is stable and is closed under taking subgraphs. Very recently, Dreier, Mählmann, and Siebertz [9] used the Flipper Game to show fixed-parameter tractability on monadically stable classes under the assumption that a second key ingredient — *sparse neighborhood covers* — is also present. Finally, Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, and

Toruńczyk [8] showed that every monadically stable class admits sparse neighborhood covers. Thus, putting the results from [6, 8, 9, 11] together, we have that first-order model-checking is FPT on any class which is stable and is closed under taking induced subgraphs.

We note that the hardness direction does not hold in general. This, for instance, follows from results of Bonnet, Kim, Thomassé, and Watrigant [5] about twin-width. The folklore conjecture is that a different notion from model theory, called *monadic dependence*, is the right dividing line for the computational complexity of first-order model-checking.

Conjecture 1 (see e.g. [1, 4, 9, 10])**.** *Let $\mathcal{F}$ be a class of graphs which is closed under taking induced subgraphs. If $\mathcal{F}$ is monadically dependent, then first-order model checking is FPT on $\mathcal{F}$. Otherwise, first-order model checking is* AW[$*$]*-hard on $\mathcal{F}$.*

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Planarity and Dimension

Piotr Micek
(joint work with Heather S. Blake, Jędrzej Hodor, Michał Seweryn, and William T. Trotter)

We study finite partially ordered sets, called *posets* for short. *Dimension* is a key measure of complexity for posets, and the dimension of a poset P, denoted by $\dim(P)$ is the least positive integer d such that P is isomorphic to a subposet of $\mathbb{R}^d$ equipped with the product order. The problem of testing whether dimension is at most d (for $d \geq 3$) appeared on the famous Garey-Johnson list of problems [6]. Nowadays, we know that dimension is NP-hard to compute [13, 3] and hard to approximate [1]. The dimension captures important graph concepts like planarity [9] or nowhere denseness [7].

Dimension was introduced in a foundational paper [2] by Dushnik and Miller in 1941. This paper also includes the canonical structure in posets forcing dimension to be large, namely, the family of standard examples. For each integer n with $n \geq 2$, the *standard example of order* n, denoted by S_n, is a poset on $2n$ elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $a_1, \ldots, a_n$ are pairwise incomparable, $b_1, \ldots, b_n$ are pairwise incomparable, and for all $i, j \in \{1, \ldots, n\}$, we have $a_i < b_j$ in S_n if $i \neq j$ and $a_i \parallel b_j$ if $i = j$. It is one of the first exercises in dimension theory to show that $\dim(S_n) = n$ for all n with $n \geq 2$. Since dimension is a monotone parameter, $\dim(P) \geq n$ whenever P contains a subposet isomorphic to S_n.

However, large standard examples are not the only way to drive dimension up. There are families of posets with arbitrarily large dimension such that for some integer d with $d \geq 2$, no poset in the family contains S_d, e.g., incidence posets of complete graphs (as proved by Dushnik and Miller [2]), interval orders (see a tight asymptotic bound on their dimension by Füredi, Hajnal, Rödl, and Trotter [5]), adjacency posets of triangle-free graphs with large chromatic number (as shown by Felsner and Trotter [4]). These results motivate the following definitions. The *standard example number* of a poset P, denoted $\mathrm{se}(P)$, is set to be 1 if P does not contain a standard example; otherwise, $\mathrm{se}(P)$ is the maximum order of a standard example contained in P. Clearly, for every poset P, we have $\mathrm{se}(P) \leq \dim(P)$. A class of posets $\mathcal{C}$ is dim-*bounded* if there is a function f such that $\dim(P) \leq f(\mathrm{se}(P))$ for every P in $\mathcal{C}$. As we discussed, the class of all posets is not dim-bounded.

The dimension of a poset P can be defined equivalently as the chromatic number of the hypergraph on the set of all incomparable pairs of P with the edge set given by the set of all alternating cycles in P. This establishes an analogy between dimension of posets and chromatic number of graphs. The inequality $\mathrm{se}(P) \leq$

$\dim(P)$ for all posets P parallels the inequality $\omega(G) \leq \chi(G)$ for all graphs G. Both inequalities are far from tight. A class of graphs $\mathcal{C}$ is *χ-bounded* if there is a function f such that for every G in $\mathcal{C}$, we have $\chi(G) \leq f(\omega(G))$. We refer readers to the recent survey by Scott and Seymour [10] on the extensive body of research done on this topic. The analogy breaks with the celebrated Four Color Theorem, which states that planar graphs have chromatic number at most four.

An element y in a poset P *covers* an element x in P if $x < y$ in P and there is no z in P with $x < z < y$ in P. The *cover graph* of a poset P is the graph whose vertices are the elements of P and two elements are adjacent if one covers the other. Somewhat unexpectedly, posets with planar cover graphs can have arbitrarily large dimension, as we learned[1] in 1978, see [11]. A *diagram* of P is a drawing of the cover graph of P in the plane such that whenever xy is an edge in the cover graph and $x < y$ in P, the relation is represented by a curve from x to y going upwards. In 1981, Kelly [8] published a viral construction of posets with planar diagrams and arbitrarily large dimension. However, all known constructions of planar posets with large dimension contain large standard examples. Thus, since the early 1980's, it remained a challenge and perhaps the most important problem in poset theory to settle the following.

Conjecture 1.

(1) The class of posets with planar diagrams is dim-*bounded.*
(2) The class of posets with planar cover graphs is dim-*bounded.*

We believe that the first published reference to Conjecture 1(1) is an informal comment on page 119 in [12] published in 1992. However, the conjecture was circulating among researchers soon after the constructions appeared. Accordingly, Conjecture 1(1) is more than 40 years old and obviously Conjecture 1(2) is a stronger statement. In this talk, we announce that both statements are true and that we proved the following theorem.

Theorem 2. *For every poset P with a planar cover graph,* $\dim(P) \leq 256(\mathrm{se}(P)+1)^8 + 264$.

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Three-edge-coloring cubic graphs on surfaces of small Euler genus

Bojan Mohar

(joint work with Yuta Inoue, Ken-ichi Kawarabayashi, Atsuyuki Miyashita, Tomohiro Sonobe)

I. The Four-Color Theorem, nowhere-zero flows, snarks. The Four-Color Theorem (4CT) states that every loopless planar graph is 4-colorable. It was first shown by Appel and Haken [1, 2] in 1977, and a simplified version of the proof was given in 1997 by Robertson et al. [11].

Tutte proved that a plane graph is k-colorable (for any $k \geq 2$) if and only if its dual graph admits a nowhere-zero k-flow.Since having a nowhere-zero 4-flow in a cubic graph is the same as having a 3-edge-coloring, this gives the following equivalent formulation of the 4CT (this equivalence was first observed by Tait in 1880 [13]).

Theorem 1. *Every* 2*-connected cubic planar graph is* 3*-edge-colorable.*

While the 4CT has no extension outside the realm of planar graphs, the version of Theorem 1 has some chance to hold under some additional assumptions. In particular, in 1966 Tutte [14] proposed his famous 4-*Flow Conjecture.*

Conjecture 2 (Tutte, [14])**.** *Every* 2*-connected graph without a Petersen graph as a minor admits a nowhere-zero* 4*-flow.*

Although this conjecture is still open, its special case restricted to cubic graphs was proved in 1997 by Robertson, Seymour, and Thomas [12]. The proof of this major achievement is based on two nontrivial results, one about doublecross graphs [4], and the other one about apex cubic graphs. (The full proof of the latter case was not published.)

When considering 3-edge-colorings of cubic graphs, it is easy to make a reduction to cyclically 4-edge-connected cubic graphs. These are 3-connected cubic graphs in which every 3-edge-cut is formed by three edges incident with a single vertex. Moreover, one can eliminate presence of cycles of length 4. Cyclically 4-edge-connected cubic graphs with girth of at least 5 that are not 3-edge-colorable are called *snarks*.

II. Snarks on the projective plane. The Petersen graph is the only known snark that can be embedded in the projective plane. Mohar conjectured in 2004 (see [9]) that there are no other projective planar snarks. In our paper presented at FOCS 2024 [6], we proved this conjecture.

Theorem 3 ([6])**.** *The only snark embeddable in the projective plane is the Petersen graph.*

Theorem 3 has two notable consequences. The first one is a strengthening of the Tutte 4-Flow Conjecture 2 for graphs on the projective plane. We say that a graph G is *Petersen-like* if it can be obtained from the Petersen graph by replacing some of its vertices by connected planar graphs.

Theorem 4 ([6])**.** *A 2-connected graph embedded in the projective plane admits a nowhere-zero 4-flow if and only if it is not Petersen-like.*

Theorem 4 in particular resolves the following conjecture of Neil Robertson from mid-1990s (see Problem 5.5.19 in [10]).

Conjecture 5 (Robertson, 1994)**.** *Every 2-edge-connected graph on the projective plane whose face-width is at least 4, has a nowhere-zero 4-flow.*

Robertson made a more general conjecture:

Conjecture 6 (Robertson, 1994)**.** *For every surface S, there is a constant r such that every 2-edge-connected graph embedded in S with face-width at least r has a nowhere-zero 4-flow.*

It is possible that there exists r such that Conjecture 6 holds for every surface with the same value of r.

The coloring-flow duality that holds for planar graphs no longer holds for graphs on any other surface. However, an unexpected corollary of our Theorem 3 is that there is a similar duality on the projective plane. Namely, the following result is equivalent to Theorem 3.

Theorem 7. *A cubic graph embedded in the projective plane is 3-edge-colorable if and only if its dual is vertex 5-colorable.*

Theorem 7 is indeed a surprising outcome since it was believed that there is no coloring-flow duality for graphs on nonorientable surfaces (see [10] or [3]).

III. Snarks on the torus. A well-known conjecture by Grünbaum from 1968 [5] asserts that every cubic graph with a polyhedral embedding (i.e., graphs whose dual triangulation is a simple graph) in an orientable surface is 3-edge-colorable. This conjecture was disproved by Kochol [8], who found a counterexample of genus 5. However, it was left unresolved whether the conjecture holds for surfaces of smaller genus. The main version for the torus can be restated as follows.

Conjecture 8 (Grünbaum, 1968)**.** *If G is a* 2*-connected cubic graph embedded in the torus and G is not* 3*-edge-colorable, then G contains two edges whose removal gives a planar graph.*

Vodopivec [15] proved that there are infinitely many snarks that can be embedded in the torus. They are certain dot products of copies of the Petersen graph; each of them contains two edges whose removal yields a planar graph. In our recent work [7], we indeed show that the provided examples are the only toroidal snarks, which implies the conjecture of Grünbaum [5].

Theorem 9. *The only snarks embeddable in the torus are the Petersen graph and the Blanuša-Vodopivec dot products of copies of the Petersen graph.*

Theorem 9 yields a strong version of the Tutte 4-Flow Conjecture for graphs on the torus.

Theorem 10. *Every cyclically-*4*-edge-connected toroidal graph different from the Petersen graph and the Blanuša-Vodopivec snark family admits a nowhere-zero* 4*-flow.*

Theorem 10 implies Robertson's Conjecture 6 for the torus with $r = 3$.

Theorem 11. *Every* 2*-edge-connected graph in the torus whose face-width is at least* 3*, has a nowhere-zero* 4*-flow.*

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Erdős–Pósa property of cycles that are far apart

Pat Morin
(joint work with Vida Dujmović, Gwenaël Joret, Piotr Micek)

We prove that there exist functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for all nonnegative integers k and d, for every graph G, either G contains k cycles such that vertices of different cycles have distance greater than d in G, or there exists a subset X of vertices of G with $|X| \leq f(k)$ such that $G - B_G(X, g(d))$ is a forest, where $B_G(X, r)$ denotes the set of vertices of G having distance at most r from a vertex of X.

Turán densities for daisies and hypercubes

Imre Leader
(joint work with David Ellis, Maria Ivan)

An r-daisy is an r-uniform hypergraph consisting of the six r-sets formed by taking the union of an $(r-2)$-set with each of the 2-sets of a disjoint 4-set. Bollobás, Leader and Malvenuto, and also Bukh, conjectured that the Turán density of the r-daisy tends to zero as $r \to \infty$. In this paper we disprove this conjecture. Adapting our construction, we are also able to disprove a folklore conjecture about Turán densities of hypercubes. For fixed d and large n, we show that the smallest set of vertices of the n-dimensional hypercube Q_n that intersects every copy of Q_d has asymptotic density strictly below $1/(d+1)$, for all $d \geq 8$. In fact, we show that this asymptotic density is at most cd, for some constant $c < 1$. As a consequence, we obtain similar bounds for the edge-Turán densities of hypercubes. We also answer some related questions of Johnson and Talbot, and disprove a conjecture made by Bukh and by Griggs and Lu on poset densities.

Recent work on the Erdős–Hajnal conjecture

Tung Nguyen
(joint work with Alex Scott, Paul Seymour)

A cornerstone of Ramsey theory, due to Erdős and Szekeres from 1935 [7], states that every n-vertex graph has a clique or stable set of size at least $\frac{1}{2}\log n$. This cannot be asymptotically improved as Erdős [8] proved by a foundational probabilistic argument that a typical n-vertex graph has no clique or stable set with at least $2\log n$ vertices. A 1977 conjecture of Erdős and Hajnal [5] predicts a very different behaviour in graphs with a forbidden induced subgraph; formally, their conjecture says that:

Conjecture 1 (Erdős–Hajnal). *For every graph H, there exists $c > 0$ depending on H only such that every n-vertex graph with no induced copy of H has a clique or stable set of size at least n^c.*

This conjecture remains open despite a substantial body of work. It lies at the intersection of graph Ramsey theory and structural graph theory, and is connected to other mathematical areas such as geometry, model theory, and probability. In joint work with Alex Scott and Paul Seymour, we have obtained several partial results towards the conjecture:

(1) A proof of the bound $2^{c\sqrt{\log n \log\log n}}$ for every forbidden graph H (also joint with Matija Bucić) [2]. This asymptotically improves the general bound $2^{c\sqrt{\log n}}$ proved by Erdős and Hajnal from 1977 [5, 6].
(2) A proof of the conjecture when H is the five-vertex path P_5 [12]. This was the last open case of the problem of deciding the conjecture for every excluded graph H on five vertices, which was first posed by Gyárfás in 1997 [10].
(3) A proof of the conjecture in the setting of graphs of bounded VC-dimension [13]. This extends and unifies a number of previous results in geometric graph theory and model theory, and resolves a problem posed independently by Fox–Pach–Suk [9] and Chernikov–Starchenko–Thomas [3].
(4) A construction of infinitely many prime graphs $\{H_k\}_{k\geq 1}$ such that the conjecture holds for $H = H_k$ for every $k \geq 1$ [11], which settles a problem of Chudnovsky in 2014 [4]. Here, a graph is *prime* if it cannot be obtained by blowing up smaller graphs, and a theorem of Alon–Pach–Solymosi [1] reduces the conjecture to the case when H is prime.

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Rainbow triangles and the Erdős-Hajnal problem in projective geometries

Peter Nelson

(joint work with Carolyn Chun, James Dylan Douthitt, Wayne Ge, Tony Huynh, Matthew E. Kroeker)

We formulate a geometric version of the Erdős-Hajnal conjecture that applies to finite projective geometries rather than graphs, in both its usual 'induced' form and the multicoloured form.

An *s-colouring* of a finite projective geometry $\mathrm{PG}(n-1,q)$ is a function from the points of G to $\{1,\dots,s\}$. If $n \geq n_0$, we say that an s-colouring c of $G = \mathrm{PG}(n-1,q)$ contains an s-colouring c_0 of $G_0 = \mathrm{PG}(n_0-1,q)$ if there is a colour-preserving embedding of G_0 in G. The following is a slightly simplified version of our main conjecture.

Conjecture 1 (Multicoloured geometric Erdős-Hajnal conjecture)**.** *For every prime power q, all $k, s \in \mathbb{N}$ and every s-colouring c_0 of $\mathrm{PG}(k-1,q)$, there exists $\delta, C > 0$ such that, for all $n \in \mathbb{N}$ and every c_0-free colouring c of $G \cong \mathrm{PG}(n-1,q)$, there is a subspace of G with dimension at least Cn^δ that uses at most $s-1$ colours.*

In other words, an s-colouring omitting all copies of any fixed small s-colouring must be very 'far from random' in the sense of containing a large subspace in which some colour fails to appear.

Our main results resolve the conjecture in all the cases where $(k,q) = (2,2)$, and certain cases with $(k,q,s) = (2,3)$, and $(k,q,s) = (3,2,2)$.

If $(k,q) = (2,2)$, then c_0 is a colouring of a three-element 'triangle', and there are three essentially different cases, all of which we resolve. We derive both the cases where c_0 assigns the same colour to two different elements from a recent breakthrough result in additive combinatorics due to Kelley and Meka; this also implies a particular case with $(k, q, s) = (3, 2, 2)$, via an existing structure theorem.

The main contribution of the work is the case where $(k, q, s) = (2, 2, 3)$. In this case, c_0 is a 'rainbow' colouring of the three-point 'triangle' in binary projective space. We resolve this via a structure theorem, which prove that rainbow-triangle-free colourings of projective geometries are exactly those that admit a certain decomposition into two-coloured pieces. This theorem is closely analogous to a theorem of Gallai on rainbow-triangle-free coloured complete graphs.

The structure theorem, which is in fact proved at the generality of arbitrary modular matroids rather than binary projective geometries, is a little too technical to state here, but has the following consequence, which is enough to resolve Conjecture 1 in the $(k, q, s) = (2, 2, 3)$ case.

Theorem 2. *Let $n \in \mathbb{N}$, let q be a prime power, and c be a colouring of the points of $G \cong \mathrm{PG}(n-1, q)$ so that $|c(L)| \leq 2$ for each two-dimensional subspace L of G. Then there are subspaces $\emptyset = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_k = G$ of G such that $|c(W_{i+1} - W_i)| \leq 2$ for all $0 \leq i < k$.*

Colouring t-perfect graphs

Sang-il Oum

(joint work with Maria Chudnovsky, Linda Cook, James Davies, Jane Tan)

A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise nonadjacent vertices. The *stable set polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of stable sets of G, where the incidence vector of a set $S \subseteq V$ is the vector $x \in \mathbb{R}^V$ such that $x_v = 1$ if $v \in S$ and $x_v = 0$ otherwise. The *fractional stable set polytope* of a graph is the set of all vectors x in $\mathbb{R}^V$ that satisfy the following linear inequalities:

(1) (Nonnegativity) $x_v \geq 0$ for all $v \in V$,
(2) (Clique inequalities) $\sum_{v \in K} x_v \leq 1$ for all cliques K of G.

It is easy to see that the stable set polytope of a graph is contained in its fractional stable set polytope. Remarkably, it has been known since 1970s that the stable set polytope of a graph is equal to its fractional stable set polytope if and only if the graph is perfect. A graph is *perfect* if for every induced subgraph H of G, the chromatic number of H, denoted by $\chi(H)$, is equal to the size of the largest clique of H, denoted by $\omega(H)$.

Motivated by perfect graphs, in 1970s, Chvátal introduced t-perfect graphs. A graph is *t-perfect* if its stable set polytope is equal to the set of all vectors x in $\mathbb{R}^V$ that satisfy the following linear inequalities:

(1) (Nonnegativity) $x_v \geq 0$ for all $v \in V$,
(2) (Edge inequalities) $x_u + x_v \leq 1$ for all edges uv of G,
(3) (Odd cycle inequalities) $\sum_{v \in V(C)} x_v \leq \frac{|V(C)|-1}{2}$ for every odd cycle C of G.

In 1984, Sbihi and Uhry introduced h-perfect graphs. A graph is *h-perfect* if its stable set polytope is equal to the set of all vectors x in $\mathbb{R}^V$ that satisfy the following linear inequalities:

(1) (Nonnegativity) $x_v \geq 0$ for all $v \in V$,
(2) (Clique inequalities) $\sum_{v \in K} x_v \leq 1$ for all cliques K of G,
(3) (Odd cycle inequalities) $\sum_{v \in V(C)} x_v \leq \frac{|V(C)|-1}{2}$ for every odd cycle C of G.

One can easily deduce that every perfect graph is h-perfect and t-perfect graphs are precisely h-perfect graphs without K_4 subgraphs.

If a graph is t-perfect, then its stable set polytope contains $\frac{1}{3}\mathbf{1}$, where $\mathbf{1}$ is the all-one vector. This implies that the fractional chromatic number of a t-perfect graph is at most 3. This motivates a problem of determining an upper bound for the chromatic number of t-perfect graphs, proposed by Shepherd in 1990s. So far, we have only two 4-critical t-perfect graphs; the complement of the line graph of the complement of C_6 found by Laurent and Seymour [4, p. 1207] and the complement of the line graph of the 5-wheel found by Benchetrit [1, 2]. A computer search shows that there are no other 4-critical t-perfect graphs with at most 11 vertices.

In 1995, Shepherd wrote that

> for every $k \geq 4$, it is not known whether each t-perfect graph is k-colourable.

We prove that every t-perfect graph is 199053-colourable. This is the first finite bound on the chromatic number of t-perfect graphs. Our proof also shows that every h-perfect graph G is $(\omega(G) + 199050)$-colourable.

Still, we do not know whether every t-perfect graph is 4-colourable.

This is joint work with Maria Chudnovsky, Linda Cook, James Davies, and Jane Tan [3].

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Refined Absorption: A New Proof of the Existence Conjecture

Luke Postle
(joint work with Michelle Delcourt)

The study of combinatorial designs has a rich history spanning nearly two centuries. In a recent breakthrough, the notorious Existence Conjecture for Combinatorial Designs dating back to the 1800s was proved in full by Keevash via the method of randomized algebraic constructions. Subsequently, Glock, Kühn, Lo, and Osthus provided an alternate purely combinatorial proof of the Existence Conjecture via the method of iterative absorption. We introduce a novel method of *refined absorption* for designs; here as our first application of the method we provide a new alternate proof of the Existence Conjecture.

A *Steiner system* with parameters (n, q, r) is a set S of q-subsets of an n-set X such that every r-subset of X belongs to exactly one element of S. More generally, a *design* with parameters (n, q, r, λ) is a set S of q-subsets of an n-set X such that every r-subset of X belongs to exactly λ elements of S.

The notorious Existence Conjecture originating from the mid-1800's asserts that designs exist for large enough n provided the obvious necessary divisibility conditions are satisfied as follows: Let $q > r \geq 2$ and $\lambda \geq 1$ be integers. If n is sufficiently large and $\binom{q-i}{r-i} \mid \lambda\binom{n-i}{r-i}$ for all $0 \leq i \leq r-1$, then there exists a design with parameters (n, q, r, λ).

In 1847, Kirkman [9] proved this when $q = 3$, $r = 2$ and $\lambda = 1$. In the 1970s, Wilson [11, 12, 13] proved the Existence Conjecture for graphs, i.e. when $r = 2$ (for all q and λ). In 1985, Rödl [10] introduced his celebrated "nibble method" to prove that there exists a set S of q-subsets of an n-set X with $|S| = (1 - o(1))\binom{n-r}{q-r}$ such that every r-subset is in at most one element of S, thereby settling the approximate version of the Existence Conjecture (known as the Erdős-Hanani Conjecture [6]). Only in the last decade was the Existence Conjecture fully resolved as follows.

Namely in 2014, Keevash [8] proved the Existence Conjecture using *randomized algebraic constructions*. Thereafter in 2016, Glock, Kühn, Lo, and Osthus [7] gave a purely combinatorial proof of the Existence Conjecture via *iterative absorption*. Both approaches have different benefits and each has led to subsequent work using these approaches.

Here [5] we introduce our method of *refined absorption* and then to provide a new alternate proof of the Existence Conjecture via said method. We note that our proof assumes the existence of K_q^r-absorbers (as established by Glock, Kühn, Lo, and Osthus [7], a short proof inspired by Keevash was recently given by Delcourt, Kelly and Postle [2]) but sidesteps the use of iterative absorption. Our approach has some of the benefits of both of the previous proofs, namely we provide a purely combinatorial approach but one that utilizes single-step absorption and has the potential to be useful for a number of applications. For example, in a follow-up paper [4], we used this new approach to prove the existence of high-girth Steiner systems.

Refined Absorption. A transformative concept that upended design theory and many similar exact structural decomposition problems is the *absorbing method*, a method for transforming almost perfect decompositions into perfect ones, wherein a set of absorbers are constructed with the ability to 'absorb' any particular uncovered 'leftover' of a random greedy or nibble process into a larger decomposition.

The Existence Conjecture may be rephrased in graph theoretic terms, namely an (n, q, r)-Steiner system is equivalent to a K_q^r-decomposition of K_n^r. For K_q^r-decompositions, the key definition of an absorber is as follows: Let L be a K_q^r-divisible hypergraph. A hypergraph A is a K_q^r*-absorber* for L if $V(L) \subseteq V(A)$ is independent in A and both A and $L \cup A$ admit K_q^r-decompositions.

Glock, Kühn, Lo, and Osthus [7] proved that a K_q^r-absorber A_L exists for any K_q^r-divisible graph L. The main use of absorbers is to absorb the potential 'leftover' K_q^r-divisible graph of a specific set/graph X after a nibble/random-greedy process constructs a K_q^r-decomposition covering all edges outside of X. To that end, we make the following definition: Let $q > r \geq 1$ be integers. Let X be a hypergraph. A hypergraph A is a K_q^r*-omni-absorber* for X if $V(X) = V(A)$, X and A are edge-disjoint, and for every K_q^r-divisible subgraph L of X, there exists a K_q^r-decomposition of $L \cup A$.

In particular note that since the empty subgraph is K_q^r-divisible, the definition above implies that A admits a K_q^r-decomposition. Moreover, since an absorber A_L exists for every L, we have that a K_q^r-omni-absorber for X exists provided that X has enough isolated vertices compared to its number of edges; the construction is simply taking A to be the disjoint union of the A_L for every K_q^r-divisible subgraph L of X.

A natural question then is how efficient an omni-absorber can we build? Taking disjoint absorbers for every L yields an omni-absorber A with $e(A) = 2^{\Omega(e(X))}$. One main purpose of our work was to show that an extremely efficient K_q^r-omni-absorber A for X exists, in particular such that $\Delta(A) = O(\Delta(X))$ provided that $\Delta(X)$ is large enough (where $\Delta(A)$ denotes the $(r-1)$-degree of A, also written as $\Delta_{r-1}(A)$, that is defined as the maximum over all $(r-1)$-sets S of the number of edges of A containing S).

Here is the key definition for proving their existence: Let $C \geq 1$ be real. We say a K_q^r-omni-absorber A for a hypergraph X is C*-refined* if every edge is in at most C cliques among the K_q^r-decompositions of $L \cup A$ taken over all K_q^r-divisible subgraphs L of X.

Our main result for omni-absorbers is the following: For all integers $q > r \geq 1$, there exist an integer $C \geq 1$ and real $\varepsilon \in (0, 1)$ such that the following holds: Let G be an r-uniform hypergraph on n vertices with $\delta(G) \geq (1-\varepsilon)n$. If X is a spanning subhypergraph of G with $\Delta(X) \leq \frac{n}{C}$ and we let $\Delta := \max\left\{\Delta(X),\ n^{1-\frac{1}{r}} \cdot \log n\right\}$, then there exists a C-refined K_q^r-omni-absorber $A \subseteq G$ for X such that $\Delta(A) \leq C \cdot \Delta$.

We note that the above theorem with its extremely efficient omni-absorbers is the key to our new proof of the Existence Conjecture. Namely, we will take X

to be a random subset of the edges of K_n^r (taken independently with some well-chosen small probability p), apply the above theorem to find a K_q^r-omni-absorber A for X, and then use the Boosting Lemma of Glock, Kühn, Lo, and Osthus [7] combined with the nibble method to find a K_q^r-packing of $K_n^r \setminus A$ that covers all edges in $K_n^r \setminus (X \cup A)$. By the definition of omni-absorber then, any leftover L of X can be absorbed into $L \cup A$, thereby completing the K_q^r-decomposition.

The inquiring reader may wonder then whether it is necessary to prove that the K_q^r-omni-absorber is C-refined for our proof of the Existence Conjecture. Technically, the refinedness property is not needed and only the high efficiency ($\Delta(A) \leq C \cdot \Delta$) is required. However, the property of being C-refined is key to the other applications in our work; furthermore, it is necessary for our proof of the above theorem which proceeds by induction on the uniformity and uses the C-refinedness inductively.

The true benefit of our new proof of the Existence Conjecture is in the robustness of the proof structure. One may use our omni-absorber theorem as a structural black box - a template that one can modify to prove various generalizations or variations of the Existence Conjecture. By embedding various gadgets on to the cliques of the decompositions of the omni-absorber (what we generally call *boosters*), we can build omni-absorbers suited to other settings. Note this is only possible since our omni-absorbers are C-refined and hence the number of cliques in the decomposition family is also small. Indeed, we used this proof structure and black-box theorem in our other work, namely in the proof of the High-Girth Existence Conjecture with Delcourt [4], in finding clique decompositions of random graphs with Delcourt and Kelly [1] and in improving the upper bound for the threshold of $(n, q, 2)$-Steiner systems with Delcourt and Kelly [3].

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Colouring t-perfect graphs

Clément Rambaud

For every $k \in \mathbb{N}_{>0} \cup \{+\infty\}$, we define the *k-treedepth* as the largest graph parameter td_k satisfying

(1) $\mathrm{td}_k(\emptyset) = 0$,
(2) $\mathrm{td}_k(G) \leq 1 + \mathrm{td}_k(G - u)$ for every graph G and every vertex $u \in V(G)$, and
(3) $\mathrm{td}_k(G) \leq \max\{\mathrm{td}_k(G_1), \mathrm{td}_k(G_2)\}$ if G is a $(< k)$-clique-sum of G_1 and G_2, for all graphs G_1, G_2.

A $(< k)$-clique-sum of G_1 and G_2 is a graph obtained from the disjoint union of G_1 and G_2 by identifying two cliques respectively in G_1 and G_2 of the same size, which is less than k, and then possibly removing some edges in the resulting clique. In particular, a (< 1)-clique-sum of G_1 and G_2 is always the disjoint union of G_1 and G_2. This gives a well-defined parameter because if $\mathcal{P}$ is the family of all the graph parameters satisfying 1-3, then $\mathrm{td}_k \colon G \mapsto \max_{\mathrm{p} \in \mathcal{P}} \mathrm{p}(G)$ also satisfies 1-3.

Note that the 1-treedepth is the usual treedepth, the 2-treedepth coincides with the homonymous parameter td_2 introduced by Huynh, Joret, Micek, Seweryn, and Wollan in [HJM+21], and that the $+\infty$-treedepth coincides with the treewidth plus 1. Hence, for every graph G,

$$\mathrm{td}(G) = \mathrm{td}_1(G) \geq \mathrm{td}_2(G) \geq \cdots \geq \mathrm{td}_{+\infty}(G) = \mathrm{tw}(G) + 1.$$

We characterize classes of graphs having bounded k-treedepth in terms of excluded minors.

Theorem 1. *Let k be a positive integer. A class $\mathcal{C}$ of graphs has bounded k-treedepth if and only if there exists an integer ℓ such that for every tree T on k vertices, no graph in $\mathcal{C}$ contains $T \square P_\ell$ as a minor.*

Here, $\square$ denotes the Cartesian product: for all graphs G_1, G_2, $G_1 \square G_2 = \big(V(G_1) \times V(G_2), \{(u,v)(u',v') \mid (u = v \text{ and } u'v' \in E(G_2)) \text{ or } (uv \in E(G_1) \text{ and } u' = v')\}\big)$; and P_ℓ denotes the path on ℓ vertices. For $k = 1$, this theorem implies that a class of graphs has bounded treedepth if and only if it excludes a path as minor, and for $k = 2$ that a class of graphs has bounded 2-treedepth if and only if it excludes a ladder as a minor, as proven by Huynh, Joret, Micek, Seweryn, and Wollan [HJM+21].

As a corollary of Theorem 1, we obtain the following structural property for the graphs excluding the $k \times \ell$ grid as a minor, when k is small compared to ℓ. This is a qualitative strengthening of the celebrated Grid-Minor Theorem by Robertson and Seymour [RS86].

Corollary 2. *There is a function $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all positive integers k, ℓ, for every graph G, if the $k \times \ell$ grid is not a minor of G, then*

$$\mathrm{td}_{2k-1}(G) \leq f(k, \ell).$$

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Essentially tight bounds for rainbow cycles in proper edge-colourings

Lisa Sauermann

(joint work with Noga Alon, Matija Bucić, Dmitrii Zakharov, Or Zamir)

It is very well-known that every graph with average degree at least 2 must contain a cycle. This talk concerned a rainbow version of this classical result: What average degree is needed in a properly edge-coloured n-vertex graph in order to guarantee a rainbow cycle, i.e. a cycle in which every colour appears at most once? This problem was raised by Keevash, Mubayi, Sudakov and Verstraëte [7] in 2007 and has resisted numerous attempts over the years. It is known, already due to [7], that for an n-vertex graph an average degree of at least $\Omega(\log n)$ is needed to guarantee a rainbow cycle. This remains the best-known lower bound (up to the implicit constant factor). The upper bound has been the subject of a series of improvements, starting with an initial upper bound of Keevash, Mubayi, Sudakov and Verstraëte [7], who showed that any properly edge-coloured graph on n vertices without a rainbow cycle has average degree at most $O(n^{1/3})$ (i.e. it has at most $O(n^{4/3})$ edges). The first improvement of this bound was due to Das, Lee and Sudakov [2], showing an upper bound of the form $e^{(\log n)^{1/2+o(1)}}$ for the average degree. This was in turn improved by Janzer [4] to $O(\log^4 n)$ and subsequently to a bound of the form $(\log n)^{2+o(1)}$ by Tomon [8]. Very recently, the $o(1)$ term in the exponent was removed independently by Janzer and Sudakov [5] and by Kim, Lee, Liu and Tran [6], showing the current state-of-the-art bound of $O(\log^2 n)$. Our main result is an essentially tight answer to the question of Keevash, Mubayi, Sudakov and Verstraëte, determining the average degree needed in a properly edge-coloured graph on n vertices in order to guarantee a rainbow cycle up to lower order terms.

Theorem 1. *There exists a constant $C > 0$ such that every properly edge-coloured graph on $n \geq 3$ vertices with average degree at least $C \cdot \log n \cdot \log\log n$ contains a rainbow cycle.*

Our result is tight up to the $\log\log n$ factor. In particular, our result shows that every properly edge-coloured graph on n vertices without a rainbow cycle must have average degree at most $(\log n)^{1+o(1)}$, which is tight up to the $o(1)$-term. Our result can also be rephrased as saying that for a certain absolute constant $C > 0$, every properly edge-coloured graph on n vertices with at least $C \cdot n \cdot \log n \cdot \log \log n$ edges contains a rainbow cycle. The maximal possible number of edges in a properly edge-coloured graph on n vertices without a rainbow cycle may be viewed as the *rainbow Turán number* of the family of all cycles. Our result shows an upper bound of $O(n \cdot \log n \cdot \log \log n)$ for this rainbow Turán number, which is optimal up to the $\log \log n$ factor.

The previous work on this question can be split into two fairly different proof approaches. The previous state-of-the-art results [5, 6] both follow the "homomorphism counting" approach pioneered by Janzer in [4]. The other approach of "passing to an expander" was first applied to this problem by Das, Lee and Sudakov [2] and later significantly refined by Tomon in [8]. Our argument falls in the latter camp, and in the first step of our proof we pass to a subgraph which is a robust sublinear expander of essentially the same average degree as our initial graph. We note that our expander subgraph does not have quantitatively stronger expansion properties compared to that of [8], and our method of finding it is actually very similar to that of [8]. The crucial difference in our argument compared to [8] is that in our setup the expansion properties are much more robust, in a sense that was first introduced in [3] and subsequently used and further developed in [1]. We introduce various new tools for working with such expanders, which may also be useful for other applications. The main part of our argument for finding a rainbow cycle in such an expander subgraph is a carefully designed random process that we analyse relying on the (robust) expansion properties and delicate multiple exposure arguments. In particular, a key point of our argument is a setup where we condition on the current state of the random process while keeping both the past and the future sufficiently random. At every step, we then show that (with high likelihood) we have significant expansion of a certain "rainbow-reachable" vertex set when going to the next step, or we had significant expansion when compared to the previous step. Interestingly, we cannot guarantee significant expansion of this "rainbow-reachable" vertex set for every step, but only going forwards *or* going backwards from any given step. Then, when comparing every other step, we have significant expansion and can complete our argument.

The bound obtained in Theorem 1 represents a hard limit for our approach. However, it is unclear whether the $\log \log n$ is actually necessary:

Question 2. *Is the* $\log \log n$ *factor in the bound for the average degree in Theorem 1 necessary in order to guarantee a rainbow cycle?*

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Resolution of the Kohayakawa-Kreuter conjecture

Raphael Steiner
(joint work with Micha Christoph, Anders Martinsson, Yuval Wigderson)

The talk concerned thresholds for Ramsey properties of graphs. Given an r-tuple $(H_1, \ldots, H_r)$ of graphs, a graph G is said to be *Ramsey* for $(H_1, \ldots, H_r)$ if every r-coloring of the edges of G contains a monochromatic copy of H_i in color i, for some $1 \le i \le r$. Equivalently, in the language above, G is Ramsey for $(H_1, \ldots, H_r)$ if there is no decomposition of G into the edge-union of r graphs, the ith of which is H_i-free. In general, the fundamental question of graph Ramsey theory is to understand which graphs G are Ramsey for a given r-tuple; for more information, see e.g. the survey [3]. An important special case of this question, which came to prominence in the late 1980s thanks to pioneering work of Frankl–Rödl [5] and Łuczak–Ruciński–Voigt [14], is the question of when a *random* graph is Ramsey for a given r-tuple with high probability. More precisely, if $G_{n,p}$ denotes the binomial random graph with edge density p, then the question is for which values of $p = p(n)$ one has that $G_{n,p}$ is Ramsey for $(H_1, \ldots, H_r)$ a.a.s. In the symmetric case $H_1 = \cdots = H_r$, this question was completely resolved in seminal work of Rödl and Ruciński [20, 18, 19]. For a graph J, let us denote by $\mathrm{v}(J), \mathrm{e}(J)$ its number of vertices and edges, respectively, and let us define the *maximal 2-density* of a graph H with $\mathrm{e}(H) \ge 2$ to be

$$m_2(H) := \max\left\{\frac{\mathrm{e}(J)-1}{\mathrm{v}(J)-2} : J \subseteq H, \mathrm{v}(J) \ge 3\right\}.$$

With this notation, the random Ramsey theorem of Rödl and Ruciński [20, 18] is as follows.

Theorem 1 (Rödl–Ruciński)**.** *Let H be a graph which is not a forest and let $r \ge 2$ be an integer. There exist constants $C > c > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \text{ is Ramsey for } (H, \ldots, H)) = \begin{cases} 1 & \text{if } p \ge Cn^{-1/m_2(H)}, \\ 0 & \text{if } p \le cn^{-1/m_2(H)}. \end{cases}$$

There is a simple heuristic explanation for why the threshold for the Ramsey property is controlled by the quantity $m_2(H)$. To explain it, let us suppose for simplicity that H is *strictly 2-balanced*, meaning that $m_2(J) < m_2(H)$ for any proper subgraph $J \subset H$. Then one can easily verify that at the regime $p \asymp n^{-1/m_2(H)}$, an average edge of $G_{n,p}$ lies in a constant number of copies of H. Thus, if $p \le cn^{-1/m_2(H)}$ where $c \ll 1$, then a typical edge lies in no copy of H, and one expects the copies of H in $G_{n,p}$ to be "well spread-out". Thus, it is reasonable to expect that one can r-color the edges without creating monochromatic copies of H. On the other hand, if $p \ge Cn^{-1/m_2(H)}$ where $C \gg 1$, then a typical edge lies in a large (constant) number of copies of H. In this regime, we expect a lot of interaction between different H-copies, and it should be difficult to avoid creating monochromatic copies.

Theorem 1 provides a very satisfactory answer to the question "when is $G_{n,p}$ Ramsey for $(H_1, \ldots, H_r)$?" in the case that $H_1 = \cdots = H_r$, but says nothing about the general case. However, nearly thirty years ago, Kohayakawa and Kreuter [9] formulated a conjecture for the threshold for an arbitrary r-tuple of graphs. Given two graphs H_1, H_2 with $m_2(H_1) \ge m_2(H_2)$, they defined the *mixed 2-density* to be

$$m_2(H_1, H_2) := \max\left\{\frac{\mathrm{e}(J)}{\mathrm{v}(J) - 2 + 1/m_2(H_2)} : J \subseteq H_1, \mathrm{v}(J) \ge 2\right\}.$$

It is well-known and easy to verify (see e.g. [12, Lemma 3.4] or [2, Proposition 3.1]) that $m_2(H_1) \ge m_2(H_1, H_2) \ge m_2(H_2)$, and that both inequalities are strict if one is.

Conjecture 2 (Kohayakawa–Kreuter)**.** *Let $H_1, \ldots, H_r$ be graphs, and suppose that $m_2(H_1) \ge \cdots \ge m_2(H_r)$ and $m_2(H_2) > 1$. There exist constants $C > c > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \text{ is Ramsey for } (H_1, \ldots, H_r)) = \begin{cases} 1 & \text{if } p \ge Cn^{-1/m_2(H_1,H_2)}, \\ 0 & \text{if } p \le cn^{-1/m_2(H_1,H_2)}. \end{cases}$$

Just as in the case of Theorem 1, there is a simple (and mostly analogous) heuristic explanation for why the function $m_2(H_1, H_2)$ controls the threshold for the asymmetric Ramsey property of $G_{n,p}$. Conjecture 2 has received a great deal of attention over the past three decades [9, 15, 10, 12, 2, 11]. For many years, most papers on the topic aimed to prove the Kohayakawa–Kreuter conjecture for certain special families of $H_1, \ldots, H_r$; such as when the graphs are cycles or cliques. More recent works have proved results in greater generality. Notably, Mousset, Nenadov, and Samotij [16] established the 1-statement of Conjecture 2 for all $(H_1, \ldots, H_r)$. Subsequently, Bowtell–Hancock–Hyde [2] and Kuperwasser–Samotij–Wigderson [12] took a major leap forward by showing that Conjecture 2 reduces to a necessary deterministic graph decomposition statement. To understand this, note that apart from the previously mentioned global reason, a random graph could also be Ramsey for $(H_1, \ldots, H_r)$ due to a potential "local" reason: for any fixed graph G that is Ramsey for $(H_1, \ldots, H_r)$, if G is a subgraph of $G_{n,p}$, then certainly $G_{n,p}$ is Ramsey for $(H_1, \ldots, H_r)$. Therefore, in order to prove the 0-statement in Theorem 1, one necessarily has to prove that when $p \le cn^{-1/m_2(H)}$, then $G_{n,p}$ a.a.s. does not contain any fixed G which is Ramsey for $(H, \ldots, H)$. It is well-known that the threshold for appearance of G in $G_{n,p}$ is determined by the *maximal density* of G, defined as

$$m(G) := \max\left\{\frac{\mathrm{e}(J)}{\mathrm{v}(J)} : J \subseteq G, \mathrm{v}(J) \geq 1\right\}.$$

Thus, a necessary condition for the 0-statement in Theorem 1 is that every Ramsey graph G for $(H_1, \ldots, H_r)$ satisfies $m(G) > m_2(H_1, H_2)$. As alluded to before, Bowtell–Hancock-Hyde and Kuperwasser-Samotij-Wigderson established that this is the only remaining obstacle towards Conjecture 2.

Theorem 3 (Bowtell–Hancock–Hyde, Kuperwasser–Samotij–Wigderson)**.** *Suppose that for every pair of graphs (H_1, H_2) with $m_2(H_1) > m_2(H_2) > 1$, the following holds: if G is Ramsey for (H_1, H_2), then $m(G) > m_2(H_1, H_2)$. Then Conjecture 2 is true.*

Theorem 3 is a generalization of the Rödl–Ruciński probabilistic lemma discussed above. Indeed, the condition in Theorem 3 is clearly necessary for the 0-statement of Conjecture 2 to hold, as if there were some G with $m(G) \leq m_2(H_1, H_2)$ which is Ramsey for (H_1, H_2), then that G appears with positive probability in $G_{n,cn^{-1/m_2(H_1,H_2)}}$ for any constant $c > 0$, and hence the 0-statement of Conjecture 2 would be false. Theorem 3 then says that this necessary condition is also sufficient. Thus, the validity of Conjecture 2 is reduced to a deterministic graph decomposition question. In both [2, 12], this deterministic condition was verified for most pairs (H_1, H_2), but its verification for all pairs remained open.

Our main result confirms that this condition always holds, thus completing the proof of Conjecture 2.

Theorem 4. *Let H_1, H_2 be graphs with $m_2(H_1) > m_2(H_2) > 1$. If a graph G is Ramsey for (H_1, H_2), then $m(G) > m_2(H_1, H_2)$.*

Equivalently, in the graph decomposition language, Theorem 4 states that if $m(G) \leq m_2(H_1, H_2)$, then G can be edge-partitioned into an H_1-free graph and an H_2-free graph.

In fact, we prove two more general graph decomposition theorems, Theorems 6, 5 below, which we expect to be of independent interest. It is not hard to see (and we show this in the next section) that these two results, plus simple well-known arguments, suffice to prove Theorem 4.

Recall that a *pseudoforest* is a graph in which every connected component contains at most one cycle. If F is a subgraph of G, we denote by $G - F$ the subgraph of G comprising all edges not in F.

Theorem 5. *Every graph G contains a pseudoforest $F \subseteq G$ such that $m_2(G-F) \leq m(G)$.*

On its own, Theorem 5 already suffices to prove Cpnjecture 2 in almost all cases, namely for all tuples $(H_1, \ldots, H_r)$ where H_2 contains a strictly 2-balanced subgraph H_2' with $m_2(H_2) = m_2(H_2')$ such that H_2' is not a cycle; this includes almost all cases that were known before. However, Theorem 5 cannot be used to resolve the remaining cases, so to prove Theorem 4 we need another decomposition result, which needs a new density parameter called the *maximal* $\frac{4}{3}$*-density* of G. We omit the technical definition.

Theorem 6. *Let $m > \frac{3}{2}$ be a real number. Every graph G with $m(G) \leq m$ contains a forest $F \subseteq G$ such that $m_{\frac{4}{3}}(G - F) < m$.*

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Hamiltonicity of expanders: optimal bounds and applications

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(joint work with Nemanja Draganić, Richard Montgomery, David Munhá Correia, Alexey Pokrovskiy)

1. Introduction

A Hamilton cycle in a graph G is a cycle that contains all the vertices of G. The presence of such a cycle categorizes G as Hamiltonian. This fundamental concept in Graph Theory has been extensively studied. Deciding whether a graph is Hamiltonian or not is an NP-complete problem, and thus it is an important area of research to find simple conditions which imply Hamiltonicity. One classic example is Dirac's theorem [11] that any graph with $n \geq 3$ vertices and minimum degree at least $n/2$ is Hamiltonian. Almost all of the other famous conditions also imply that the graph is dense. Hence, it is of particular interest to find Hamiltonicity conditions which also apply to sparse graphs.

Over the past 50 years, a central focus in random graph theory has been determining when sparse random graphs are Hamiltonian. Pósa [24] introduced the rotation-extension technique and established the threshold for Hamiltonicity in $G(n,p)$. This result was later refined by Bollobás [5] and Komlós and Szemerédi [18], who independently showed that $G(n,p)$ is almost surely Hamiltonian when $p = \frac{\log n + \log\log n + \omega(1)}{n}$, whereas a smaller p leads to the presence of isolated vertices. Random regular graphs can be even sparser yet still Hamiltonian with high probability for all $3 \leq d \leq n-1$, see e.g. Cooper, Frieze, Reed [9] and Krivelevich et al. [21].

The well-established understanding of Hamiltonicity in random graphs presents an important step towards the search for simple properties of sparse graphs which imply Hamiltonicity. It points to considering natural 'pseudorandom' conditions which are required by a deterministic graph to resemble a random graph. However, forgoing the randomness of $G(n,p)$ and relying only on these pseudorandom properties to find a Hamilton cycle presents a significantly firmer challenge, similar to the generalisation of other problems from random to pseudorandom graphs. Pseudorandom graphs have been systematically studied since work by Thomason [26] in the 1980's and their history can be found in the survey by Krivelevich and Sudakov [20]. The most studied class of pseudorandom *regular* graphs was introduced by Alon and is defined using spectral properties. Recalling that if a graph G is d-regular then its largest eigenvalue is d, we denote the second largest eigenvalue of G in absolute value by $\lambda(G)$. Then, a graph G is an (n,d,λ)*-graph* if it is d-regular with n vertices and satisfies $|\lambda(G)| \leq \lambda$.

The first major step towards understanding the Hamiltonicity of such pseudorandom graphs was made by Krivelevich and Sudakov in 2003 in their influential paper [19]. They showed that if $d/\lambda > \log^{1-o(1)} n$, then the graph is Hamiltonian. In the same paper, Krivelevich and Sudakov made the beautiful conjecture that the bound can be replaced by $\frac{d}{\lambda} \geq C$ for some large constant C, as follows.

Conjecture 1. *There exists $C > 0$ such that if $\frac{d}{\lambda} \geq C$, then every (n, d, λ)-graph is Hamiltonian.*

That is, if the absolute value of every other eigenvalue of a regular graph is at most a small constant fraction of the largest eigenvalue, then the graph should be Hamiltonian.

Considering the result in [19] in the context of random graphs allows us to benchmark this progress in a broad sense. That is to say, the result of [19] is strong enough to prove the likely Hamiltonicity of the random regular graph $G_{n,d}$ when $d \geq \log^{2-o(1)} n$, while the ultimate goal, Conjecture 1, would be strong enough to prove the likely Hamiltonicity of the random regular graph $G_{n,d}$ when d is at least some large constant. Despite a great deal of attention, through various relaxations and generalisations of the problem, as well as many incentivising applications, the bound established in [19] remained unchallenged for 20 years. Only recently, Glock, Munhá Correia and Sudakov [14] finally improved this, significantly strengthening the result of [19] by showing that, for some large constant $C > 0$, $d/\lambda \geq C \log^{1/3} n$ suffices to imply Hamiltonicity. Moreover, they showed that Conjecture 1 holds in the special case when $d \geq n^{\alpha}$, for any fixed α.

Krivelevich and Sudakov applied their bound to several other problems on the Hamiltonicity of sparse graphs; these and other applications are discussed in Section 1.1. To allow applications to non-regular graphs, other pseudorandom conditions which imply Hamiltonicity were also studied. Motivated by this, shortly after Conjecture 1 was stated, several papers considered an even stronger conjecture, singling out the key properties of (n, d, λ)-graphs thought to give some potential for proving Hamiltonicity. To state this even stronger conjecture, a variant of which appeared for example in [6], we need the following definition.

Definition 2. *An n-vertex graph G with $n \geq 3$ is a C-expander if: (a) $|N(X)| \geq C|X|$ for all vertex sets $X \subseteq V(G)$ with $|X| < n/2C$; (b) there is an edge between any disjoint vertex sets $X, Y \subseteq V(G)$ with $|X|, |Y| \geq n/2C$.*

Conjecture 3. *For every sufficiently large $C > 0$, every C-expander is Hamiltonian.*

In 2012, Hefetz, Krivelevich and Szabó [16] made progress on this problem; the precise expansion conditions used in their result can be found in Theorem 1.1 of [16] (in particular weakening (a) in our Definition 2), but imply that every $(\log^{1-o(1)} n)$-expander is Hamiltonian.

We prove Conjecture 3, thus completing an extensive line of research on Hamiltonicity problems.

Theorem 4. *For every sufficiently large $C > 0$, every C-expander is Hamiltonian.*

This result has a large number of applications, as discussed below. In particular, it is a standard exercise to show that for every $C > 0$ there exists a constant C_0 such that, if $\frac{d}{\lambda} \geq C_0$, then every (n, d, λ)-graph is a C-expander. Thus, clearly Conjecture 1 is implied by Theorem 4, giving the following.

Theorem 5. *There is a constant $C > 0$ such that if $\frac{d}{\lambda} \geq C$ then every (n, d, λ)-graph is Hamiltonian.*

We note here that in fact a stronger result than Theorem 4 holds – such graphs are not only Hamiltonian, but Hamilton-connected. Furthermore, the Hamilton cycle in Theorem 5 can be found in polynomial time. We finish by giving some examples of the applications of Theorems 4 and 5.

1.1. **Applications.** Pseudorandom conditions for Hamiltonicity have found a large variety of applications, and Theorems 4 and 5 immediately improve the bounds required for many such applications. Notable examples are solving a conjecture of Pak and Radoičić on random Cayley graphs [23], which is a probabilistic version of Lovász's famous Hamiltonicity conjecture [22], then giving improved bounds to a problem of Alon and Bourgain on additive patterns in multiplicative subgroups [1] and other topics such as the Hamiltonicity in well-connected graphs (see [14]), Hamilton cycles with few colours in edge-coloured graphs (see [14]), positional games (see, e.g., [15]), coverings and packings of Hamilton cycles in random and pseudorandom graphs (see, e.g., [12, 17]) and Hamiltonicity thresholds in different random graph models (see, e.g., [2, 13]). Below we further describe the first example.

In 1969, Lovász [22] made the following famous conjecture about the Hamiltonicity of vertex-transitive graphs, which are graphs in which any vertex can be mapped to any other vertex by an automorphism.

Conjecture 6. *Every connected vertex-transitive graph contains a Hamilton path, and, except for five known examples, a Hamilton cycle.*

As Cayley graphs are vertex-transitive and none of the five known exceptions in Conjecture 6 are Cayley graphs, Lovász's conjecture implies the following earlier conjecture, posed in 1959, by Strasser [25].

Conjecture 7. *Every connected Cayley graph is Hamiltonian.*

Conjecture 7 is known to be true when the underlying group is abelian, but the only progress towards the conjectures in general is a result of Babai [4] that every vertex-transitive n-vertex graph contains a cycle of length $\Omega(\sqrt{n})$ (see [10] for a recent improvement by DeVos) and a result of Christofides, Hladký and Máthé [7] that every vertex-transitive graph of linear minimum degree contains a Hamilton cycle. The "random version" of Conjecture 7 is a natural relaxation of the original problem. Alon and Roichman [3] showed that there is a constant $C > 0$ for which, for any group G, the Cayley graph generated by a random set S of $C \log |G|$ elements, $\Gamma(G, S)$ say, is almost surely connected. Hence, an important instance of Conjecture 7 is to show that $\Gamma(G, S)$ is almost surely Hamiltonian. This problem was also stated as a conjecture by Pak and Radoičić [23]. Since then, several papers have made progress on this problem. Krivelevich and Sudakov [19] showed that $O(\log^5 n)$ generators suffice, Christofides and Markström [8] improved this bound to $O(\log^3 n)$, and Glock, Munhá Correia, and Sudakov [14] further refined it to $O(\log^{5/3} n)$.

Theorem 5 resolves this conjecture. Indeed, Alon and Roichman [3] showed that if $|S| \geq C \log |G|$ for some large constant C, then $\Gamma(G, S)$ is almost surely an (n, d, λ)-graph with $d/\lambda \geq K$ for some large constant K. Thus, we obtain the following.

Theorem 8. *Let C be a sufficiently large constant. Let G be a group of order n and $d \geq C \log n$. If $S \subseteq G$ is a set of size d chosen uniformly at random, then, with high probability, $\Gamma(G, S)$ is Hamiltonian.*

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Factoring pattern-free permutations into separable ones

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(joint work with Édouard Bonnet, Romain Bourneuf, Colin Geniet)

1. Introduction

Given a class $\mathcal{C}$ of discrete structures, the arguably preeminent algorithmic task is the one of recognition: does the input belong to $\mathcal{C}$? This problem is often tied with an effective *construction* of the class $\mathcal{C}$ by performing elementary operations on some basic building blocks. For instance, totally unimodular matrices [10], minor-closed classes [9], and perfect graphs [6] can be constructed from simpler objects, respectively, network matrices, graphs embeddable on low-genus surfaces, and variants of bipartite graphs. In this talk we show that *strict* classes of permutations, that is, those avoiding a fixed pattern, can be constructed from separable permutations (the basic class) via a bounded number of compositions (the elementary operation).

Given a positive integer n, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. Let $n \leq m$ be two integers, we say that a permutation $\pi \in \mathfrak{S}_n$ is a *pattern* of $\sigma \in \mathfrak{S}_m$ if there is an increasing function f from $[n]$ to $[m]$ such that $\pi(i) < \pi(j)$ if and only if $\sigma(f(i)) < \sigma(f(j))$ for all $i, j \in [n]$. Another way of characterizing patterns is to associate to a permutation $\sigma \in \mathfrak{S}_n$ its $n \times n$ matrix $A(\sigma) = (a_{ij})$ with $a_{ij} = 1$ if $j = \sigma(i)$, and $a_{ij} = 0$ otherwise. Observe that π is a pattern of σ if and only if $A(\pi)$ is a submatrix of $A(\sigma)$. For instance, 12345 is a pattern of σ if it contains an increasing subsequence of length five. A crucial achievement in permutation patterns is the Guillemot–Marx algorithm, which decides if a permutation π is a pattern of σ in time $f(\pi) \cdot |\sigma|$, where $|\sigma|$ is the size of σ.

Patterns readily offer a complexity notion for permutations: A permutation is "simple" if it does not contain a fixed small pattern. We will consider *classes* of permutations, which are assumed closed under taking patterns. The existence of a gap between the class of all permutations and any strict class is illustrated by the Marcus–Tardos theorem, answering the Stanley–Wilf conjecture: Every strict class of permutations has at most $2^{O(n)}$ permutations of size n, whereas the class of all permutations obviously has $n! = 2^{\Theta(n \log n)}$ such permutations. From an algorithmic perspective, sequences avoiding a fixed pattern can be comparison-sorted in almost linear time $O\left(n \cdot 2^{(1+o(1))\alpha(n)}\right)$ where α is the inverse Ackermann function [4, 8, 5], while linear algorithms when excluding some specific small patterns have been long known [7, 1]. Furthermore, as a generalisation of Guillemot–Marx algorithm, any property defined using first-order logic (FO) can be tested inside any strict permutation class $\mathcal{C}$ in linear time [2]. For instance, given a fixed permutation τ, one can decide in linear time if an input n-permutation σ in $\mathcal{C}$ is such that every pair of elements $i, j \in [n]$ is contained in a pattern τ of σ. Observe that even the existence of a linear-size positive certificate for this seemingly quadratic problem is far from obvious.

Strict classes of permutations are thus significantly simpler, both algorithmically and in terms of growth. The next question is to construct them from a basic class using some simple operations. In the case of permutations, possibly the most natural elementary operation is the product (or composition). Furthermore, the class of *separable permutations* is basic in several ways. It consists of those permutations whose permutation graph is a cograph; an elementary graph class (which coincides with graphs of twin-width 0). Like cographs have a natural auxiliary tree structure (the *cotrees*), separable permutations inherit their own tree structure, the so-called *separating tree* [3]. Separable permutations are originally themselves defined from the trivial permutation 1, by successively applying direct sums (setting two permutation matrices as diagonal blocks of the new permutation matrix) or skew sums (the same with antidiagonal blocks), or equivalently by closing $\{12, 21\}$ under substitution. They are well known to be the permutations avoiding the patterns 2413 and 3142 [3].

As our main result, we show:

Theorem 1. *For any pattern π, there exists $k_\pi = 2^{2^{O(|\pi|)}}$ such that every permutation avoiding π is a product of at most k_π separable permutations.*

Theorem 1 with the definition of separable permutations, every permutation of $Av(\pi)$, the set of permutations avoiding the pattern π, can be built from the trivial permutation 1 via direct and skew sums, followed by a bounded-length product. Conversely, remark that for any c, the class of products of c separable permutations avoids some pattern, since it contains only $2^{O(cn)}$ permutations on n elements.

The proof is effective, and yields a fixed-parameter tractable (FPT) algorithm to compute the factorisation. With some more work, we show how to implement it in linear time.

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A tamed family of triangle-free graphs with unbounded chromatic number

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(joint work with Édouard Bonnet, Romain Bourneuf, Julien Duron, Colin Geniet and Stéphan Thomassé)

One of the main questions on the chromatic number $\chi(G)$ of a graph G is how it compares to the clique number $\omega(G)$. Indeed, while $\omega(G) \leq \chi(G)$, early constructions by Blanche Descartes [4], Zykov [12], and Mycielski [9] show that there are triangle-free graphs with arbitrarily large χ. Such graphs have been an important source of inspiration in graph theory. For instance, a distinctive early success of the probabilistic method was the construction by Erdős [5] of graphs with large girth and large chromatic number. Another example is the proof by Lovász [8] of the Kneser conjecture [7], a cornerstone of the introduction of topological methods to combinatorial problems.

There is also an interesting interplay of these graphs with discrete geometry in the plane. For instance, triangle-free segment intersection graphs were shown to have unbounded chromatic number [10], disproving a question of Erdős and Gyárfás [6]. The proof consists of astutely representing Burling graphs (another class of triangle-free graphs of unbounded chromatic number that are intersection graphs of boxes of $\mathbb{R}^3$) [2] as intersection graphs of segments in the plane.

We present in this talk a new explicit sequence of triangle-free graphs G_k, which we call *twincut graphs*, satisfying $\chi(G_k) = k$ with the following striking property: all their induced subgraphs have non-adjacent twins (two vertices with the same neighborhood), or an edgeless vertex cutset of size at most two. This is very surprising since both situations are, when considered individually, particularly favourable to keeping the chromatic number low. On the one hand, creating twins does not change the chromatic number. On the other hand, Alon, Kleitman, Saks, Seymour and Thomassen [1] proved that the closure of any basic class under gluing along bounded subsets of vertices preserves bounded chromatic number. This was later refined by Penev, Thomassé and Trotignon [11] who showed that such closure admits extreme decompositions: a small vertex cutset isolates a basic subgraph of the final graph, hence allowing a coloring with few colors.

A natural question is to consider two different types of closure, each behaving well with respect to the chromatic number, and try to combine them. Along those lines, Chudnovsky, Penev, Scott, and Trotignon [3] asked whether the closure of a χ-bounded class under substitutions and bounded cutsets could remain χ-bounded, where a *χ-bounded class* is a hereditary class of graphs such that there exists a function f satisfying $\chi(G) \leq f(\omega(G))$ for all graphs of the class.

It may seem at first that this is just a matter of finding the right induction hypothesis, but twincut graphs show that the answer is negative in the seemingly easiest case: the closure $\mathcal{C}$ of $\{K_1, K_2\}$ (the 1-vertex graph, and the edge) under the two operations of vertex replication (i.e., creating a non-adjacent twin) and gluing two graphs on up to two non-adjacent vertices. To our surprise, the class $\mathcal{C}$ turned out to contain all twincut graphs.

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Optimally edge coloring multigraphs

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(joint work with Guantao Chen, Yanli Hao, Wenan Zang)

Let G be a loopless multigraph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let $\Delta(G)$ denote the maximum degree of G and let $\chi'(G)$ denote the chromatic index of G. Vizing's classical theorem on edge coloring states that $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ when G is a simple graph. (However, deciding whether $\chi'(G) = \Delta(G)$ is NP-complete, see [6].) This result no longer holds for multigraphs. Shannon [12] showed that $\chi'(G) \leq \frac{3}{2}\Delta(G)$; and Vizing [15] and Gupta [4] proved, independently, that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum number of edges between two vertices of G. Let

$$\Gamma^*(G) := \max\{|E(H)|/\lfloor |V(H)|/2 \rfloor : H \subseteq G \text{ and } |V(H)| \geq 2\}, \text{ and}$$
$$\Gamma(G) := \lceil \Gamma^*(G) \rceil,$$

where $H \subseteq G$ means that H is a subgraph of G. When $|V(G)| \geq 3$ there exists $H \subseteq G$ such that $|V(H)|$ is odd and $\Gamma(G) = \left\lceil \frac{|E(H)|}{(|V(H)|-1)/2} \right\rceil$. It is also easy to see that $\Gamma(G) \leq \frac{3}{2}\Delta(G)$.

Edmonds' matching polytop theorem [2] implies that $\chi'^*(G) := \max\{\Delta(G), \Gamma^*(G)\}$ is the *fractional chromatic index* of G. (See Seymour [11] for a shorter proof.) By definition, $\chi'(G) \geq \lceil \chi'^*(G) \rceil = \max\{\Delta(G), \Gamma(G)\}$. Padberg and Rao [10] proved that $\chi'^*(G)$ can be computed in polynomial time. Goldberg ([3], 1973) and Seymour ([11], 1974) independently conjectured that, for any multigraph G, $\chi'(G) \leq \max\{\Delta(G)+1, \Gamma(G)\}$.

The Goldberg-Seymour conjecture has been studied extensively; see the book [13] by Stiebitz, Scheide, Toft, and Favrholdt, which has four chapters devoted to this conjecture. It has also been studied algorithmically. Hochbaum, Nishizeki and Shmoys [5] conjectured in 1986 that a $\max\{\Delta(G)+1, \chi'(G)\}$-edge-coloring of G can be found in polynomial time. Iliopoulos and Sinclair [7] recently gave a polynomial time algorithm for Kahn's result [9] that Goldberg-Seymour conjecture holds asymptotically.

A proof of the Goldberg-Seymour conjecture was announced in 2019 [1] and a more recent proof [8] was announced; both proofs are long and have not been independently verified, and the proof in [8] is essentially the same as in [1] by eliminating one single case. We give a combinatorial algorithm that finds in polynomial time a $\max\{\Delta(G)+1, \Gamma(G)\}$-edge-coloring, establishing the Hochbaum-Nishizeki-Shmoys conjecture as well as giving a different and much shorter (and manageable) proof of the Goldberg-Seymour conjecture.

Theorem 1. *For multigraphs G, one can find a* $\max\{\Delta(G)+1, \Gamma(G)\}$*-edge-coloring of G in $O(|V(G)|^{11}|E(G)|^3)$ time.*

We use natural numbers to denote colors. For a positive integer k, let $[k] := \{1, \ldots, k\}$. For a graph G and a set $S \subseteq E(G)$, a *partial edge coloring* of G with *support* S is a function $\varphi : S \to [k]$ (for some positive integer k) such that no two adjacent edges of G receive the same color. (When $S = E(G)$, we have the usual edge coloring.) For any $v \in V(G)$, we write $\overline{\varphi}(v) := [k] \setminus \varphi(E(v) \cap S)$, where $E(v)$ is the set of edges of G incident with v. For any $U \subseteq V(G)$, let $\overline{\varphi}(U) := \bigcup_{v \in U} \overline{\varphi}(v)$. For any subgraph H of G, let $\overline{\varphi}(H) := \overline{\varphi}(V(H))$ and $\varphi(H) := \varphi(E(H))$, and we use $G[H]$ to denote the subgraph of G induced by $V(H)$ and $\partial_G H$ (or simply ∂H) to denote the set of edges of G with exactly one incident vertex in H.

We wish to understand the structure of a multigraph G that would imply $\chi'(G) = \Gamma(G)$. So suppose $\chi'(G) = \Gamma(G)$. Let $\varphi : E(G) \to [\Gamma(G)]$ be an edge coloring of G and assume that $H \subseteq G$ such that $\frac{|E(H)|}{(|V(H)|-1)/2} = \Gamma(G)$ and $|V(H)|$ is an odd integer. Then for every $\alpha \in [\Gamma(G)]$, $\varphi^{-1}(\alpha) \cap E(H)$ is a matching in H and has size $(|V(H)| - 1)/2$; thus,

- **(a)** for distinct vertices u, v of H, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$;
- **(b)** $\overline{\varphi}(H) \cap \varphi(\partial H) = \emptyset$;
- **(c)** for any $\alpha \in [\Gamma(G)] \setminus \overline{\varphi}(H)$, $|\varphi^{-1}(\alpha) \cap \partial H| = 1$.

We say that a subgraph H of G is: *φ-elementary* if (a) holds; *φ-closed* if (b) holds; *φ-maximal* if both (b) and (c) hold.

Let G be a multigraph and $k := \max\{\Delta(G) + 1, \Gamma(G)\}]$. We greedily color the edges of G using colors from $[k]$ and let φ denote the resulting partial edge coloring. If $E(G) = \varphi^{-1}([k])$ then φ gives the desired coloring. So assume $g \in E(G)\backslash\varphi^{-1}([k])$. We will find a partial edge coloring π such that $\pi^{-1}([k]) = \varphi^{-1}([k]) \cup \{g\}$, by growing a tree T starting from g in a structured way. Observe that such T cannot be both φ-elementary and φ-maximal; since, otherwise, for every $\alpha \in [k]$, $\varphi^{-1}(\alpha) \cap E(G[T])$ is a matching of size $(|V(T)| - 1)/2$ and, hence, $\Gamma(G) > \frac{|E(G[T])|-1}{(|V(T)|-1)/2} \geq k$, a contradiction.

Such a tree T is constructed in stages, which are trees $T_0, T_1, \ldots, T_{n+1}$ such that $T_{i-1} \subseteq T_i$ for $i \in [n+1]$. This step terminates when T_{n+1} is not φ-elementary but T_n is φ-elementary. We then refine the construction from T_n to T_{n+1} by constructing from $T_{n,0} := T_n$ trees $T_{n,0}, T_{n,1}, \ldots, T_{n,m}, T_{n,m+1}$, such that $T_{n,i-1} \subseteq T_{n,i}$ for $i \in [m+1]$. This step terminates when $T_{n,m+1}$ is not φ-elementary but $T_{n,m}$ is φ-elementary. We further refine the construction from $T_{n,m}$ to $T_{n,m+1}$ by constructing from $T_{n,m,0} := T_{n,m}$ trees $T_{n,m,0}, \ldots, T_{n,m,q}, T_{n,m,q+1}$ such that $T_{n,m,i-1} \subseteq T_{n,m,i}$ for $i \in [q+1]$. This step terminates when $T_{n,m,q+1}$ is not φ-elementary but $T_{n,m,q}$ is φ-elementary. Choose T to be a minimal subtree of $T_{n,m,q+1}$ that is not φ-elementary. We define $s(T) := n$, $\ell(T) := m$, and $p(T) := q$. We also define additional parameters $t(T)$ and $b(T)$ similar to what was used before by Tashkinov [14].

We perform Kempe changes based on the structure of T to obtain a tree T' and a coloring π from φ such that T' is not π-elementary and $(s(T'), \ell(T'), p(T'), t(T'), b(T'))$ precedes $(s(T), \ell(T), p(T), t(T), b(T))$ under the lexicographic ordering. This step takes $O(|V(G)|^6|E(G)|^2)$ time. By repeating this step $O(|V|^5)$ times, we arrive at a partial edge coloring that is Kempe equivalent to φ for which there is a color β not used by any edge incident with g; so we can extend π by assigning β to g. We repeat the above process $O(|E(G)|)$ time to obtain the desired edge coloring.

There are many interesting open problems about edge-coloring, and we refer the reader to the excellent book [13] by Stiebitz, Scheide, Toft, and Favrholdt. In particular, the book mentions *Twenty Pretty Edge Coloring Conjectures*, and several of those are related to Theorem 1.

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