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Set Theory

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Abstract. Set theory has experienced rapid development in the recent years, both in pure set theory and in its applications to other fields of mathematics. We have seen breakthroughs in combinatorial set theory, forcing, inner models, descriptive set theory, Borel combinatorics, and moreover, exciting new connections between some of these areas. The workshop succeeded in presenting and discussing the important developments, fostering interaction between researchers, and stimulating the field.

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Introduction by the Organizers

This workshop covered the state of the art in set theory developments. In pure set theory topics included forcing axioms, infinitary combinatorics, inner model theory and the core model induction, HOD (the hereditarily ordinal definable sets). In applied set theory there were talks on Borel combinatorics, ergodic theory, and applications to group theory, dynamics, and topology.

Lietz talked about exciting new connections between between forcing axioms like PFA and large cardinals. His work moves us closer to addressing the long standing problem on whether PFA and supercompact cardinals are equiconsistent. Lietz introduced promising methods to attack this. In another line, also motivated by this problem, and more generally, on lower bounds of consistency strengths, Müller and Schlutzenberg gave insightful talks in inner model theory. Müller reported on recent applications of genericity iterations to generic absoluteness and

LSA (the Largest Suslin Axiom). And Schlutzenberg uncovered new properties about how "correct" definable sets of reals in the model $L(\mathbb{R})$ can be, assuming the axiom of determinacy. In a related theme, Jackson reported new results on the comnbinatorics of models of determinacy.

Moving to combinatorial set theory, Adkisson presented new theorems about the tree property and successors at singular cardinals. The tree property is a key compactness principle in set theory; it "captures" the combinatorial essence of large cardinals. Related compactness-type properties are reflections principles. Hamkins reported on recent connections between reflection and large cardinals. Levine explored ways in which forcing can non-trivially change the cofinality of cardinals, from large cardinals, focusing on Namba forcing. Gitik showed a new way of increasing the powerset of a measurable cardinal, while preserving its measurability, using Matthias style forcing. And (originally) motivated by Woodin's HOD dichotomy, Poveda discussed exciting new results, many joint with Goldberg, on compactness in HOD and to what extend cardinals are computed in HOD correctly. Circling back to compactness, Viale showed the existence of higher analogues of the compactness theorem from first order logic to $\mathcal{L}_{\infty,\infty}$.

Regarding applications of PFA, Moore announced an amazing result pinpointing the consistency strength of Shelah's conjecture, which says that every Aronszajn line contains a Countyman suborder. This result falls into the category of using PFA to analyze the combinatorics of $\aleph_1$. In contrast, Rinot showed that ZFC implies that this conjecture cannot generalize to higher cardinals.

Another hot topic was the interplay between large cardinals and ultrafilters. Foreman presented new ways to analyze ineffable cardinals and various strengthenings (all of which are types of large cardinals) and games on ultrafilters and ideals. Also, on the topic of ultrafilters, Benhamou reported on striking new theorems about the Tukey order on measurable cardinals, using combinatorial principles, such as the the Galvin property and Prikry style, He also presented connections between Tukey orders and Goldberg's Ultrapower Aioms (UA). Another tour-de-force talk was the one by Dobrinen, who reported on the state of the art of infinitary Ramsey theory. She discussed big Ramsey degrees and constructions ranging from classical coloring problems to Galvin-Prikry style theorems. Her talk also went over connections with computability and Borel combinatorics.

At the applied end of set theory, we had talks about the current developments of descriptive set theory, Borel classifications, and applications of set theoretic techniques to dynamics and group theory. In Borel combinatorics, there has been a number of new developments on countable Borel equivalence relations (CBERs). Tserunyan discussed treeable CBERs (i.e. those that admit an acyclic graphing) and presented the striking theorem that quasi-treeable already implies treeable. Her methods involve graph theory, Stone duality, and ultrafilters. Also on the topic of CBERs, Iyer showed that groups which are locally of finite asymptotic dimension give rise to hyperfinite actions. Hyperfinite is a strengthening of treeable; both are prominent in the theory of Borel classification of equivalence relations and are considered "well-behaved". On the other end of the spectrum are more complicated equivalence relations, and that was the subject of the Sabok's talk. He reported on various theorems on how conjugacy relations can be very "complicated", for example, turbulent in some cases.

Many new developments in ergodic theory and descriptive set theoretic applications to group theory were also discussed. Zomback talked about ergodic theorems for Borel probability measure preserving (pmp) actions. Also discussing pmp actions, Tsankov provided new insights on the ergodic theory of infinite permutation groups.

There were also talks on set theoretic applications to topology. Vaccaro presented theorems about properties of compact manifolds, more precisely, which compact surfaces have *generic chains*. In a different vein, Solecki analyzed *filtrations of topologies*, motivated by connecting κ-Borel sets and κ-Bairness with respect to varying topologies. The broader context is studying Polishable equivalence relations. Finally, Calderoni discussed new results on idealistic equivalence relations. While every orbit equivalence relations is idealistic, he showed that the converse is false, in joint work with Motto Ros. Moreover, (assuming some determinacy) they prove that there are continuum many $\leq_B$-incomparable idealistic equivalence relations. While orbit equivalence relations have been studied a lot, fairly little is known about idealistic equivalence relations. The work of Calderoni and Motto Ros ushers in a new promising research topic.

This workshop featured a lot of early career mathematicians: Andreas Lietz, Hannes Jakob, William Adkisson, Eliott Glazer, Andreas Vaccaro, Jenna Zomback, Eyal Kaplan, Tom Benhamou, and others. It was exciting to hear new research progress – both motivated by some of classical problems in set theory, but with new techniques and ideas, and also work on brand new topics. There was a lot of fruitful interaction between students, postdocs, and more established experts, all in the informal and productive Oberwolfach environment.

Workshop: Set Theory

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Abstracts

Equiconsistencies between strengthenings of PFA and strengthenings of supercompact cardinals

Andreas Lietz

Determining the exact consistency strength of the Proper Forcing Axiom (PFA) is one of the holy grails of Set Theory. It is widely believed to be the known upper bound, a supercompact cardinal, based on the following compelling evidence.

Theorem (Magidor[2], Viale [3]). *Relative to a supercompact cardinal, it is consistent that κ is supercompact and*

$$V_\kappa \models \text{“PFA } \textit{fails in all forcing extensions}\text{”}.$$

Theorem (Viale-Weiß [4]). *Suppose κ is inaccessible and $\mathbb{P}$ is a proper standard forcing iteration of length κ which forces* PFA *and turns κ into ω_2. Then κ is supercompact.*

The best known lower bound for the consistency strength of PFA is provided by the Inner Model Theory Program and is in the region of many Woodin cardinals.

We show that under mild additional assumptions, it is possible to determine the exact strength of stronger forms of PFA which were introduced by Ben Goodman [1].

Definition (Goodman). *Suppose Γ is a class of formulas. Then Γ-CPFA holds iff for any $\varphi(x) \in \Gamma$ so that*

$$\text{ZFC} \vdash \forall x(\varphi(x) \to \forall \textit{proper } \mathbb{P} \Vdash_{\mathbb{P}} \varphi(\check{x})),$$

any regular sufficiently large θ, proper $\mathbb{P}$, any $\mathbb{P}$-name $\dot{a} \in H_\theta$, there is some $X \prec H_\theta$ of size ω_1 with $\omega_1 \cup \{\mathbb{P}, \dot{a}\} \subseteq X$ so that if $\pi\colon X \to M$ is the transitive collapse then there is a $\pi(\mathbb{P})$-generic filter g over M so that $V \models \varphi(\pi(\dot{a})^g)$.

PFA is equivalent to Σ_1-CPFA and Σ_2-CPFA holds in the standard model of PFA, see [1]. For sufficiently large Γ it is possible to recover very large cardinals from Γ-CPFA in the mantle, assuming the Bedrock Axiom (BA).

Theorem 1. *Suppose $n \geq 4$, Π_n-CPFA and* BA *hold. Then ω_2 is supercompact for $C^{(n)}$ in the mantle.*

See [1] for a definition of supercompact for $C^{(n)}$ cardinals. Combining this with the proof of consistency of Σ_n-CPFA from large cardinals of Goodman [1], we arrive at an equiconsistency.

Corollary 2 (Goodman, L.). *For $n \geq 4$, the following theories are equiconsistent:*

(1) $\text{ZFC} + \exists\kappa$ “*κ is supercompact for $C^{(n)}$*”.
(2) $\text{ZFC} + \Sigma_{n+1}\text{-CPFA} + \text{BA}$.
(3) $\text{ZFC} + \Pi_n\text{-CPFA} + \text{BA}$.

Without assuming BA, we can prove a dichotomy reminiscent of Woodin's HOD-dichotomy [5].

Theorem 3. *Suppose* Π_4-CPFA *holds. Let* N *be the inner model obtained by constructing with* $\kappa \cap \mathrm{cof}(\omega)$ *and* $\mathrm{NS}_\kappa \restriction \mathrm{cof}(\omega)$ *for all* κ *of uncountable cofinality over the mantle. Exactly one of the following holds.*

(1) Every singular cardinal λ *is singular in* N *and* $(\lambda^+)^N = \lambda^+$.
(2) Every regular cardinal $\lambda \geq \omega_2$ *is measurable in* N.

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Quasi-treeable equivalence relations are treeable

Anush Tserunyan

(joint work with Ruiyuan Chen, Antoine Poulin, Ran Tao)

Countable Borel equivalence relations (CBERs) on standard Borel spaces are exactly the orbit equivalence relations of Borel actions of countable groups. Thus, we may view CBERs as a generalization of groups in the Borel context – indeed, CBERs are Borel groupoids.

The analogues of Cayley graphs of groups in the context of CBERs are graphings, namely, a **graphing** of a CBER E on a standard Borel space X is a Borel graph G on X whose connected components are exactly the E-classes. Consequently, the analogue of free groups are the **treeable** CBERs, i.e. those which admit acyclic graphings, called **treeings**. Treeable CBERs form a special class among CBERs that is rich in terms of Borel reducibility [3], but rather fragile otherwise: it is not closed under products or countable increasing unions, and whether it is closed under finite-index extensions remains a notoriously open question [4, 6.4(B)]. It is, however, a theorem of Jackson, Kechris, and Louveau [4, 3.4] that the orbit equivalence relations of free Borel actions of virtually free groups are treeable.

A well-known result from geometric group theory [2, 7.19] states that if a Cayley graph of a finitely generated group is a **quasi-tree** (i.e. is quasi-isometric to a tree), then the group is virtually free. It was asked by R. Tucker-Drob in 2015 whether the analogue of this holds in the context of CBERs. We give a positive answer to this question in [1]:

Theorem 1. *If a CBER E admits a locally finite graphing G, each of whose connected components is a quasi-tree, then E is treeable.*

We also prove another version of this theorem, replacing the geometric notion of being a quasi-tree with the combinatorial notion of having bounded tree width. This version was also independently proven by H. Jardón-Sánchez [5] via a different method.

Theorem 2. *If a CBER E admits a locally finite graphing G, each of whose connected components has bounded tree width, then E is treeable.*

One should emphasize that in the hypotheses of these theorems, we *do not* assume that there are uniformly Borel witnesses to each component being a quasi-tree or having bounded tree width. Rather, these assumptions are abstract and apply separately to each component; part of our proofs involves translating these abstract extrinsic hypotheses into intrinsic Borel information about the graph G. Due to this translation, both of these theorems are derived from the following main result:

Theorem 3. *If a CBER E admits a locally finite Borel graphing G and a Borel family $\mathcal{H}$ of half-spaces (cuts) in G, which is finitely separating and dense towards ends in each component of G, then E is treeable.*

By a **half-space** in a graph G, we mean a connected and co-connected (within the same connected component) set of vertices with finite edge-boundary. Clearly, the complement of a half-space within the same connected component is again a half-space, so in 3 we may assume that $\mathcal{H}$ is closed under such complements, and thus forms a **pocset**: a poset under $\subseteq$ with the complement operation.

Our proof of 3 exploits the Stone duality between pocsets and median graphs (1-skeleta of CAT(0) cube complexes). More precisely, we construct a treeing not on X directly, but on the space of all clopen ultrafilters (called **orientations**) on the pocset $\mathcal{H}$. We then transfer this treeing back to X through the principle ultrafilter map.

The natural relation of nearness on these clopen ultrafilters defines a **median graph**, i.e. a graph where every triple of vertices admits a *unique* median; in other words, the intersection of all geodesics between these three vertices is a singleton. The hypothesis on $\mathcal{H}$ ensures that this median graph is standard Borel (equivalently, locally finite) and has finite hyperplanes. Using the convexity properties of median graphs and the finiteness of hyperplanes, we construct a spanning tree in each component of this median graph in a uniformly Borel manner, proving 3.

In [1], we also analyze the cases in 1, 2, 3 where G is one-ended, or more generally, there is a Borel selection of one end in each component. In this case, we show that E is, in fact, hyperfinite.

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Ergodic theory of infinite permutation groups

Todor Tsankov

Exchangeability theory in probability is concerned with classifying all possible distributions of a collection of random variables $(X_a : a \in M)$ invariant under a certain permutation group $G \leq \mathrm{Sym}(M)$. Classical theorems of de Finetti, Ryll-Nardzewski, Aldous and Hoover, and Kallenberg give a complete answer when $G = S_\infty$ or $G = \mathrm{Aut}(\mathbf{Q})$ (for all possible actions) but the general situation remains largely mysterious.

I will discuss some recent results concerning probability measure preserving and measure class preserving actions of large permutation groups. The first result is that if G is a Roelcke precompact subgroup of S_∞ (for example, an oligomorphic permutation group), $G \curvearrowright X$ is a Borel action on a standard Borel space, and μ is a Borel measure on X such that $g_*\mu$ is equivalent to μ for every $g \in G$, then there is a σ-finite measure ν equivalent to μ, which is G-invariant. This generalizes a result of Nessonov [3] for S_∞.

The second result a generalization of de Finetti's theorem. It gives more or less optimal conditions under which, for a given permutation group $G \curvearrowright M$, the only ergodic G-invariant measures on $[0,1]^M$ are the product measures. The necessary conditions are primitivity of the action and the lack of algebraicity (this means that for every finite $A \subseteq M$, the stabilizer G_A has infinite orbits outside of A). However, as an example of Jahel and Perruchaud [1] shows, these conditions are not sufficient. The result that I will discuss proves the conclusion under a slight strengthening of either of the two hypotheses, which are satisfied in the presence of compactness (for example if G is Roelcke precompact). This extends previous results of Jahel and Tsankov [2].

The results in this talk are from [4].

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Non-structure theorems for higher Aronszajn lines

Assaf Rinot
(joint work with Tanmay Inamdar)

One of the great successes of set theory is the development of consistent axioms asserting that the universe is saturated enough to the extent that objects that may be added by forcing must already exist. These axioms enable *rough classification* results for structures of size the first uncountable cardinal, $\aleph_1$. Recall that by a 'rough classification' of a class $\mathfrak{C}$ of structures equipped with a quasi-ordering $\prec$ we refer to the study of global properties of $(\mathfrak{C}, \prec)$ such as the existence of universal and minimal elements, of a small basis, well foundedness, etc.

Classes of countable structures tend to be 'tame', whereas classes of structures of size continuum tend to be 'chaotic'. It follows that the Continuum Hypothesis implies that classes of structures of size $\aleph_1$ are chaotic, but using forcing axioms, some of them may be tamed.

The first forcing axiom came to life [ST71] in the course of proving the consistency of the Souslin Hypothesis [Sou20] asserting that '$(\mathbb{R}, <)$ *is the unique complete linear order without end points and such that every pairwise disjoint collection of open intervals is countable'*. Back when Kurepa was working on this problem, he discovered a duality between uncountable linear orders and transfinite trees. His conversations with Aronszajn on the topic led to the discovery of an *Aronszajn tree* which, by duality, gives rise to an *Aronszajn line* — an uncountable linear order that is far from all previously known examples: uncountable sets of reals, uncountable well-orders, and uncountable anti-well-orders. Curiously, this new type of uncountable linear order was later rediscovered by Specker and then again by Countryman.

A linear order $(L, <)$ is *monotone* iff for every injection f from some $X \subseteq L$ of full size to L, there exists a $Y \subseteq X$ of full size on which f is either strictly increasing or strictly decreasing. In his work on topological spaces having a σ-monotone base [Cou70], Countryman proved that an uncountable linear order $(L, <)$ that is uniformly monotone in the sense that $L \times L$ may be covered by countably many sets C such that for all $(x, y), (x', y') \in C$, $x < x' \to y \leq y'$, must be far from uncountable separable orders, ω_1 and its reverse ω_1^* (i.e., must be an Aronszajn line). He conjectured that these uniformly monotone lines are too good to be true, but then Shelah [She76] ingeniously constructed one. At the end of his paper, Shelah proposed a strong form of Souslin's Hypothesis asserting that every Aronszajn line must contain a Countryman line. Shelah may have anticipated that together with Baumgartner's Theorem [Bau73] that under the *Proper Forcing Axiom* (PFA) the class of uncountable separable orders admits a one element basis, a solution of this problem would pave the way to proving that the class of uncountable linear orders may consistently admit a five element basis. Three decades later, Shelah's problem was solved by Moore [Moo06] who proved that under PFA the class of Aronszajn lines admits a two element basis consisting of a Countryman line and its reverse.

In a previous installment of this workshop in 2011, Neeman presented a new proof of the consistency of PFA using a finite support iteration. His method suggested a way to obtain higher analogs of consequences of PFA. Subsequently, the topic of 'taming' $\aleph_2$ has been the subject of numerous workshops followed by quite a few successful works.

In the proceedings [Mat16] to a workshop at the American Institute of Mathematics in 2016 dedicated to the development of forcing axioms at $\aleph_2$ and beyond, Moore asked about a higher analog of his result. Here we solve this problem in the negative. Interpreted positively, our work shows that ZFC is powerful enough to decide Shelah's strong form of Souslin's Hypothesis at *all* successors of regular cardinals except $\aleph_1$.

Theorem 1. *For every regular uncountable cardinal μ, if there is a μ^+-Aronszajn line, then there is one without a μ^+-Countryman subline.*

More generally, for the class of special Aronszajn lines we find that in the trichtomy between 'trivial structure', 'tame structure', and 'chaotic structure' that may exist at successors of uncountable regular cardinals, there is in fact a dichotomy between the two extreme possibilities.

Theorem 2. *Suppose that $\kappa = \mu^+$ for a regular uncountable cardinal μ. Then all of the following are equivalent:*

- *There is special κ-Aronszajn line;*
- *The class of special κ-Aronszajn lines is not well-founded;*
- *There exist a special κ-Aronszajn line that is monotone and a special κ-Aronszajn line with no monotone subline;*
- *Any basis for the class of special κ-Aronszajn lines has size 2^κ.*

Analogous results are also obtained for successors of singulars and inaccessibles.

In order to secure the chaotic behaviour of the class of Aronszajn lines, we formulated an anti-Ramsey relation $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^n_\theta$ and proved that it holds for various canonical trees $\mathbf{T}$. Its impact is exemplified by the following fact.

Fact 3. *For κ is a regular uncountable cardinal, θ is an infinite cardinal, and $n \geq 2$, Each of the following clauses implies the next:*[1]

1. *There exists a lexicographically ordered κ-Aronszajn tree $\mathbf{T}$ such that $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^n_\theta$ holds;*
2. *There exists a (κ, n)-entangled sequence of 2^θ-many κ-Aronszajn lines none of which contains a monotone subline;*
3. *There are 2^θ-many pairwise monotonically far κ-Aronszajn lines;*
4. *No basis for the class of κ-Aronszajn lines has size less than 2^θ.*

Our most general theorem yields anti-Ramsey relations from optimal anti-large-cardinal hypotheses, where the combinatorial hypothesis $\square(\kappa)$ (resp. $\square_<(\kappa)$) of the upcoming theorem hold true for every regular uncountable cardinal κ that is not weakly compact (resp. Mahlo) in Gödel's constructible universe.

[1]This remains true once adding 'special' before 'κ-Aronszajn' in all four clauses.

Theorem 4. *Let κ be a regular cardinal bigger than $\aleph_1$.*

- *If $\square(\kappa)$ holds, then there is a lexicographically ordered κ-Aronszajn tree $\mathbf{T}$ such that $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^n_\kappa$ holds for every positive integer n;*
- *If $\square_<(\kappa)$ holds, then there is a lexicographically ordered special κ-Aronszajn tree $\mathbf{T}$ such that $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^n_\kappa$ holds for every positive n.*

The trees that satisfy our anti-Ramsey relation are obtained from walks on ordinals. For this, we are chaining several steps together: first, finding sufficient conditions on a C-sequence $\vec{C}$ to obtain a strong colouring for ω-many colours of the corresponding κ-tree $T(\rho_0^{\vec{C}})$, then, via further conditions on $\vec{C}$ to step the number of colours up into the maximal number of colours κ, and finally to obtain suitable $\vec{C}$ meeting all of these conditions. Each of these steps is divided into multiple cases in order to address all possible cardinals and cardinal arithmetic configurations that may arise in ZFC. The third step is the most demanding one and we now have a new vocabulary backed up by a factory for producing C-sequences of prescribed characteristics. As a by-product, we were able to also answer a question of Todorčević [Tod07, Question 2.2.18] concerning walks on countable ordinals. He asked for a condition to put on $\vec{C}$ to ensure that $T(\rho_1^{\vec{C}})$ be special. We found one and also addressed the problem of when $T(\rho_2^{\vec{C}})$ is special. As a corollary, we get the following equivalence between special Aronszajn trees and canonical special Aronszajn trees.

Theorem 5. *For every regular cardinal μ, the following are equivalent:*

- *There exists a special μ^+-Aronszajn tree;*
- *There exists a μ-bounded C-sequence $\vec{C}$ over μ^+ for which $T(\rho_0^{\vec{C}})$, $T(\rho_1^{\vec{C}})$ and $T(\rho_2^{\vec{C}})$ are all special μ^+-Aronszajn trees.*

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Generic absoluteness for Chang-type models of determinacy and LSA

Sandra Müller

Universally Baire sets originate in work of Schilling and Vaught [5], and they were first systematically studied by Feng, Magidor, and Woodin [4]. Since then they play a prominent role in many areas of set theory. Recall that a set of reals is universally Baire if all of its continuous preimages in compact Hausdorff spaces have the property of Baire.

Assuming a proper class of Woodin cardinals, the study of the model $L(A,\mathbb{R})$, where A is a universally Baire set, is completely parallel to the study of $L(\mathbb{R})$, and most of the theorems proven for $L(\mathbb{R})$ can be easily generalized to $L(A,\mathbb{R})$. For example, assuming a proper class of Woodin cardinals, generalizing the proof for $L(\mathbb{R})$, Woodin showed that the theory of the model $L(A,\mathbb{R})$ cannot be changed by forcing, and in fact, for every V-generic g and $V[g]$-generic h, there is an elementary embedding

$$j : L(A_g,\mathbb{R}_g) \to L(A_{g*h},\mathbb{R}_{g*h})$$

such that $j \restriction \mathbb{R}_g = id$ and $j(A_g) = A_{g*h}$, and also $L(A,\mathbb{R}) \models \mathsf{AD}^+$.

However, the model $L(\Gamma^\infty,\mathbb{R})$ is much harder to analyze, where Γ^∞ denotes the set of all universally Baire sets of reals. For example, it is not clear that assuming large cardinals, $L(\Gamma^\infty,\mathbb{R}) \models \mathsf{AD}^+$ or even

$$\wp(\mathbb{R}) \cap L(\Gamma^\infty,\mathbb{R}) = \Gamma^\infty.$$

Woodin's $\mathsf{Sealing}$ deals with the aforementioned issues.

Definition 1 (Woodin)**.** $\mathsf{Sealing}$ *is the conjunction of the following statements.*

(1) For every set generic g over V, $L(\Gamma^\infty_g,\mathbb{R}_g) \models \mathsf{AD}^+$ and $\wp(\mathbb{R}_g)\cap L(\Gamma^\infty_g,\mathbb{R}_g) = \Gamma^\infty_g$.

(2) For every set generic g over V and set generic h over $V[g]$, there is an elementary embedding

$$j\colon L(\Gamma^\infty_g,\mathbb{R}_g) \to L(\Gamma^\infty_{g*h},\mathbb{R}_{g*h})$$

such that for every $A \in \Gamma^\infty_g$, $j(A) = A_h$.

Inspired by the work done in [3] and [2], we, in joint work with Grigor Sargsyan, introduce a new technique for establishing generic absoluteness results for models containing Γ^∞ in [1]. Our main technical tool is an iteration that realizes Γ^∞ as the sets of reals in a derived model of some iterate of V. We show, from a supercompact cardinal κ and a proper class of Woodin cardinals, that whenever $g \subseteq \mathrm{Col}(\omega, 2^{2^\kappa})$ is V-generic and h is $V[g]$-generic for some poset $\mathbb{P} \in V[g]$, there is an elementary embedding $j : V \to M$ such that $j(\kappa) = \omega_1^{V[g*h]}$ and $L(\Gamma^\infty,\mathbb{R})$ as computed in $V[g*h]$ is a derived model of M at $j(\kappa)$. Here j is obtained by iteratively taking ultrapowers of V by extenders with critical point κ and its images.

As a corollary we obtain that $\mathsf{Sealing}$ holds in $V[g]$, which was previously demonstrated by Woodin using the stationary tower forcing. Also, using a theorem of

Woodin, we conclude that the derived model of V at κ satisfies $\mathsf{AD}_{\mathbb{R}}$ + "Θ is a regular cardinal".

Inspired by core model induction, we introduce the definable powerset $\mathcal{A}^\infty$ of Γ^∞ and use our derived model representation mentioned above to show that the theory of $L(\mathcal{A}^\infty)$ cannot be changed by forcing. Working in a different direction, we also show that the theory of $L(\Gamma^\infty, \mathbb{R})[\mathcal{C}]$, where $\mathcal{C}$ is the club filter on $\wp_{\omega_1}(\Gamma^\infty)$, cannot be changed by forcing. Proving the two aforementioned results is the first step towards showing that the theory of $L(Ord^\omega, \Gamma^\infty, \mathbb{R})([\mu_\alpha : \alpha \in Ord])$, where μ_α is the club filter on $\wp_{\omega_1}(\alpha)$, cannot be changed by forcing.

In joint work in progress with Lukas Koschat and Grigor Sargsyan, we consider extensions of the model $L(\mathcal{A}^\infty)$ of the form $L((\eta^\infty)^\omega, \mathcal{A}^\infty)$ where η^∞ is the supremum of all ordinals onto which there is an OD in the universally Baire sets surjection of $\wp(\mathbb{R})$. Based on this we argue that assuming there are two supercompact cardinals $\kappa_0 < \kappa_1$, in a $\mathrm{Col}(\omega, 2^{\kappa_0})$-generic extension (and because of Sealing in any further generic extension) there are LSA pointclasses cofinally in the universally Baire sets.

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Surfaces and other Peano Continua with no Generic Chains

Andrea Vaccaro

(joint work with Gianluca Basso, Alessandro Codenotti)

Let X be a metrizable compact connected space. A *maximal chain of compact connected sets of* X – simply *chain* from now on – is a collection of compact connected subsets of X which is linearly ordered by inclusion and maximal with respect to this property. Equivalently, a chain can be described as a homeomorphic image of the interval $[0, 1]$ in the hyperspace of compact connected subsets of X such that 0 is mapped to a singleton, 1 to X and, for each $s < t$, the image of s is a subset of the image of t. Broadly speaking, the collection $\Phi(X)$ of all such chains represents all the different ways in which it is possible to start, say at time 0, from a point of X, and then continuously grow out of it until, at time 1, the whole space X has been covered.

The question I would like to address is the following: for which spaces is there *essentially* only one chain? A more rigorous way to formulate this question is asking whether, given X, there is $\mathcal{C} \in \Phi(X)$ such that its orbit under the action $\mathrm{Homeo}(X) \curvearrowright \Phi(X)$ is comeager in $\Phi(X)$, endowed with its natural compact hyperspace topology. In such case, $\mathcal{C}$ is a *generic chain* of X.

Consider compact manifolds: on the circle S^1, the chain consisting of all intervals centered around some point x_0 is generic in $\Phi(S^1)$. A similar statement is true for the closed interval. On the other hand, Gutman, Tsankov and Zucker proved in [1] that no closed manifold of dimension at least 3 has generic chains, and the same is true for the Hilbert cube. The case of surfaces was left open.

The following is a corollary of the main theorem presented in this talk.

Theorem 1. *If X is a compact surface other than the sphere or the real projective plane, then X has no generic chain.*

The methods used to prove Theorem 1 are purely combinatorial. They allow not only to recover the aforementioned result from [1], but cover moreover a vast class of *Peano continua* – i.e. metrizable compact connected spaces which are locally connected – going well beyond the setting of manifolds.

Theorem 2. *Let X be a Peano continuum with no locally separating points. If X either*

(1) has a locally non-planar open subset,
(2) has a planar open set containing a simple closed curve which is not locally separating,
(3) has a circular covering,

then X has no generic chain.

Instead of introducing all definitions appearing in the theorem, I will discuss how they translate to manifolds: a compact manifold has a locally separating point if and only if it is 1-dimensional, and it has a locally non-planar open subset if and only if it is at least 3-dimensional. A boundary of a surface is an example of a simple closed curve which is not locally separating and which lies in a planar open set. Finally, the sphere and the real projective plane are the only closed surfaces without a circular covering, which roughly means that they admit no finite covering whose elements are regular open, connected subsets and whose nerve graph is a cycle.

Notable examples covered by Theorem 2 that are not manifolds include the Sierpiński carpet, the Menger curve and all universal k-dimensional compacta μ^k, as well as any compact connected μ^k-manifold, for $k \geq 1$.

The following questions are still open

Question 3 ([1]*Question 1.3)**.** *Is there a generic chain on the sphere? What about the real projective plane?*

As previously mentioned, the proof of Theorem 2 is combinatorial: the result is in fact proved by first establishing a correspondence between Peano continua and open sets of chains on one side, and finite connected graphs and walks on these graphs on the other. After that, a combinatorial necessary condition for the existence of a generic chain on a Peano continuum is identified – based on a well-known criterion due to Rosendal (see [2, Proposition 3.2]) – and proved to fail under the assumptions of Theorem 2.

More precisely, the combinatorial condition is an *off-by-one* weak amalgamation principle, stating that each walk can be refined by another walk in such a way that any two different walks refining it can be reconciled by a small perturbation. Its failure is obtained by producing irreconcilable walks that wind enough times in opposite directions around circular subgraphs, with a combinatorial adaptation of an idea that goes back to [1].

Theorem 2 has relevant consequences regarding the dynamics of the groups of homeomorphisms of the spaces involved. If G is a topological group, a *G-flow* is a continuous action $G \curvearrowright X$ on a compact Hausdorff space X. A G-flow is *minimal* if every orbit is dense. Both classifying minimal flows of a group G and, vice versa, classifying groups based on the properties of their minimal flows – often described with a single object known as the *universal minimal flow* – are central goals in abstract topological dynamics.

This classification is based on some fundamental dividing lines: a topological group G is *extremely amenable* if it has only one minimal flow, the trivial one, or, equivalently, if all G-flows have a fixed point. Beyond the extremely amenable case, another measure of complexity for Polish groups is understanding whether all minimal flows are metrizable, and a further dividing line is given by the so called *generic point property* – implied by the aforementioned metrizability condition in the Polish case, see [2] – which asserts that all minimal G-flows have a comeager orbit. On a conceptual level, the latter condition isolates those Polish groups whose minimal dynamics is *tractable* (see [4]).

After Uspenskij's seminal work [5], it became clear that, given a flow $G \curvearrowright X$, the study of the induced action $G \curvearrowright \Phi(X)$ often gives definitive information on the dynamics of G. Building upon this insight, in [1] the authors prove that $\mathrm{Homeo}(X)$ does not have the generic point property whenever X is a closed n-manifold for $n \geq 3$. They show moreover that the subgroup $\mathrm{Homeo}_0(X)$ of homeomorphisms isotopic to the identity does not have metrizable minimal flows, for any closed surface X. It was however left open whether such groups have the generic point property. Since for a closed surface X the flow $\mathrm{Homeo}_0(X) \curvearrowright \Phi(X)$ is minimal, Theorem 1 allows to cover all but two of these cases.

Corollary 4. *If X is a closed surface which is not the sphere or the real projective plane, then $Homeo_0(X)$ does not have the generic point property.*

The following open problem is tightly related to Question 3.

Question 5 ([1]*Question 1.3)**.** *Does $Homeo_0(S^2)$, where S^2 is the sphere, have the generic point property? What about the real projective plane?*

Theorem 2 can be used to deduce the failure of the generic point property for $\mathrm{Homeo}(X)$ for spaces X (not necessarily manifolds) for which the action

Homeo(X) $\curvearrowright$ $\Phi(X)$ is minimal. Such condition is verified if X satisfies a strong form of homogeneity called *strong local homogeneity*.

Corollary 6. *If X is a strongly locally homogeneous Peano continuum (e.g. a closed manifold or the Menger curve) which is not the circle, the sphere, or the real projective plane, then Homeo(X) does not have the generic point property.*

Other examples which fit the hypotheses of Corollary 6 are the universal k-dimensional compacta μ^k and any compact connected μ^k-manifold, for $k \geq 1$.

The Sierpiński carpet S is not homogeneous, but the minimality of the action Homeo(S) $\curvearrowright$ $\Phi(S)$ can be proved by hand. Theorem 2 thus also gives the following.

Corollary 7. *Let S be the Sierpiński carpet. The group Homeo(S) does not have the generic point property.*

Finally, for spaces as in Corollary 6 it is moreover possible to prove that the action Homeo(X) $\curvearrowright$ $\Phi(X)$ is *generically turbulent*, which implies a strong non-classifiability result for the orbit equivalence relation of the action, thanks to a fundamental theorem due to Hjorth ([6]).

Corollary 8. *If X is a strongly locally homogeneous Peano continuum (e.g. a closed manifold or the Menger curve) which is not the circle, the sphere, or the real projective plane, then chains on X are not classifiable by countable structures.*

All these results are part of the joint project [3] with Gianluca Basso and Alessandro Codenotti.

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Tree Properties at Successors of Singulars of Many Cofinalities

William Adkisson

An old problem of Magidor is to obtain the tree property at every regular cardinal greater than $\aleph_1$. If there is to be a positive answer to this question, we must obtain the tree property at many successors of singular cardinals; in particular, we must obtain the tree property at successors of singular cardinals of many different cofinalities. This is easy to arrange near large cardinals, since the successor of a limit of supercompacts always has the tree property, but is much more difficult to obtain at small cardinals.

The primary difficulty is that in the standard constructions to obtain the tree property at the successor of a singular (e.g. [2]), the forcing selects a new cardinal to become the successor of the desired cofinality. That is, if we want to obtain the tree property at $\aleph_{\omega+1}$, we will have to select a new ω_1. To obtain the tree property at $\aleph_{\omega_1+1}$, we need to select a new ω_2. If we are forcing the tree property at both cardinals, we will need to select a new ω_1 and ω_2. These selections cannot be made in advance, and require complete knowledge of the forcing (other than the selection at hand); this makes making multiple such selections difficult.

In previous work [1], the author developed a construction that, for any fixed natural number n, could obtain the tree property at $\aleph_{\omega_i+1}$ for all $i < n$. We present that technique, and discuss it's fundamental limitations. In particular, it cannot obtain the tree property at successors of singulars of infinitely many different cofinalities simultaneously.

In this talk, we present a new techniques that can be used to obtain the tree property at successors of singulars with infinitely many different cofinalities simultaneously. In particular, we prove the following theorem:

Theorem 1. *Let $\langle \kappa_\alpha \mid \alpha < \kappa_0 \rangle$ be a sequence of supercompact cardinals. Then there is a forcing extension in which the tree property holds at $\aleph_{\omega+\omega+1}$ and at $\aleph_{\omega_i+1}$ for all $0 < i < \omega$ simultaneously.*

This technique inductively selects a new ω_i that will obtain the tree property at $\aleph_{\omega+\omega_i+1}$ no matter what we choose for the later elements in the sequence. This removes any possible interference between the cardinals, allowing us to choose a new ω_i for all i without disrupting any previous choices.

In the past two decades there has been a resurgence of interest in strengthenings of the tree property, and the Magidor's question can be posed for these principles as well. We focus on the strong tree property, a strengthening of the tree property that is closely linked with strongly compact cardinals. In particular, an inaccessible cardinal is strongly compact if and only if the strong tree property holds. The constructions to force the strong tree property at successors of singular cardinals are similar to the tree property, but more delicate techniques are required. We show how to use those techniques to generalize our result to the strong tree property:

Theorem 2. *Let $\langle \kappa_\alpha \mid \alpha < \kappa_0 \rangle$ be a sequence of supercompact cardinals. Then there is a forcing extension in which the strong tree property holds at $\aleph_{\omega+\omega+1}$ and at $\aleph_{\omega_i+1}$ for all $0 < i < \omega$ simultaneously.*

Finally, we discuss how to extend these results to uncountably many different cofinalities at once. These sequences can be very large, bounded above only by the least supercompact used in the construction.

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Mouse sets and correctness in $L(\mathbb{R})$

Farmer Schlutzenberg

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Work in ZF + AD + $V = L(\mathbb{R})$. Given a nicely definable countable set X of reals, it is natural to ask whether X is a mouse set; that is, whether there is a mouse M such that $X = \mathbb{R} \cap M$. If so, can the mouse M be characterized? Secondly, how correct is X (how elementary is X in $\mathbb{R}$, etc)?

For example, the set of all OD reals is a mouse set (see [9]). We consider more local versions of ordinal definability. At the projective level, $\mathbb{R} \cap M_n$ is exactly the set of reals which are Δ^1_{n+2} in a countable ordinal, where M_n is the canonical proper class mouse with n Woodin cardinals (here $M_0 = L$). Moreover, M_{2n} is Σ^1_{2n+2}- but not Σ^1_{2n+3}-correct, and M_{2n+1} is also Σ^1_{2n+2}- but not Σ^1_{2n+3}-correct. (Define $M_{-1} = L_{\omega_1^{\mathrm{ck}}}$. Then the same results hold, except that $\mathbb{R} \cap L_{\omega_1^{\mathrm{ck}}}$ is the set of Δ^1_1 reals (in no parameters).) Moreover, writing $<^{M_n}$ for the usual order of constructibility of M_n, $<^{M_n}\restriction \mathbb{R}^{M_n}$ is (a wellorder of $\mathbb{R}^{M_n}$ which is) $(\Delta^1_{n+2})^{M_n}$-definable.

Although $M = L_{\omega_1^{\mathrm{ck}}}$ is not Σ^1_1-correct, it can define Σ^1_1 truth, via the following *anti-correctness* phenomenon, due to Spector-Gandy and Ville:

– $(\Pi^1_1)^V \restriction M$ is uniformly $(\Sigma^1_1)^M$, meaning that there is a recursive $\varphi \mapsto \psi_\varphi$ sending Π^1_1 formulas φ to Σ^1_1 formulas ψ_φ, such that for all $x \in \mathbb{R} \cap M$,

$$\varphi(x) \iff M \models \psi_\varphi(x).$$

– Similarly, $(\Pi^1_1)^M$ is uniformly $(\Sigma^1_1)^V \restriction M$.

Higher up, $(\Sigma^1_3)^V \restriction L$ is not definable, and in fact, there are Δ^1_3 reals which are not in L. But the anti-correctness for $L_{\omega_1^{\mathrm{ck}}}$ has an analogue for M_1, Π^1_3:

- $(\Pi^1_3)^V \restriction M_1$ is uniformly $(\Sigma^1_3)^{M_1}$ (Woodin, via genericity iterations),
- $(\Pi^1_3)^{M_1}$ is uniformly $(\Sigma^1_3)^V \restriction M_1$ (Martin-Mitchell-Steel).

The $L(\mathbb{R})$ language is the language of set theory augmented with a constant symbol for $\mathbb{R}$. For definability over levels $\mathcal{J}_\alpha(\mathbb{R})$ of $L(\mathbb{R})$, we use this language. We also write $\Sigma^{\mathbb{R}}_1$ for Σ_1, and for integers $n > 0$, $\Pi^{\mathbb{R}}_n$ denotes $\neg\Sigma^{\mathbb{R}}_n$, and $\Sigma^{\mathbb{R}}_{n+1}$ denotes $\exists^{\mathbb{R}}\Pi^{\mathbb{R}}_n$. For ordinals $\alpha > 0$ and integers $n > 0$, $\mathrm{OD}_{\alpha n}$ denotes the set of reals y such that for some $\xi < \omega_1$ and some Σ_n formula φ of the $L(\mathbb{R})$ language,

$$y = \text{the unique real } z \text{ such that } \mathcal{J}_\alpha(\mathbb{R}) \models \varphi(w, z)$$

whenever w is a real coding wellorder of ordertype ξ. Define $\mathrm{OD}^{\mathbb{R}}_{\alpha n}$ likewise, but with $\Sigma^{\mathbb{R}}_n$ formulas φ replacing Σ_n. Woodin showed in the 1990s that for each ordinal $\lambda > 0$, $\mathrm{OD}_{\lambda 1} = \mathrm{OD}^{\mathbb{R}}_{\lambda 1}$ is a mouse set; see [8].

Recall [7] that a Σ_1 gap is an ordinal ordinal $[\alpha, \beta]$ which is maximal such that $\mathcal{J}_\alpha(\mathbb{R}) \preccurlyeq_{\Sigma_1} \mathcal{J}_\beta(\mathbb{R})$. Recall a gap $[\alpha, \beta]$ is projective-like if $\mathcal{J}_\alpha(\mathbb{R})$ is non-admissible. (In this case $\alpha = \beta$.) Say $[\alpha, \beta]$ is admissible if $\mathcal{J}_\alpha(\mathbb{R})$ is admissible. Admissible gaps are divided further into strong and weak varieties. Say a projective-like gap $[\alpha, \alpha]$ is *scale-cofinal* if α is not of form $\gamma + 1$ where γ ends a strong gap.

Rudominer and Steel (see [3] and [1]) showed that if $[\alpha, \alpha]$ is projective-like with α of uncountable cofinality and $n \geq 1$, then $\mathrm{OD}^{\mathbb{R}}_{\alpha n} = \mathrm{OD}_{\alpha n}$ is a mouse set.

Regarding projective-like gaps in general, Rudominer asked whether $\mathrm{OD}_{\alpha n} = \mathrm{OD}^{\mathbb{R}}_{\alpha n}$, and Rudominer and Steel conjectured that $\mathrm{OD}^{\mathbb{R}}_{\alpha n}$ is a mouse set.

Rudominer defined the *ladder mouse* M_{ld} in the 1990s, slightly beyond the projective in complexity. It is the least mouse M such that for each $n < \omega$ there is an M-cardinal δ such that $M^{\#}_n(M|\delta) \lhd M$ and δ is Woodin in $M^{\#}_n(M|\delta)$. Rudominer showed that $\mathbb{R} \cap M_{\mathrm{ld}} \subseteq \mathrm{OD}_{12}$. Woodin showed in 2018 [2] that $\mathbb{R} \cap M_{\mathrm{ld}} = \mathrm{OD}_{12}$ (so this is a mouse set). He also showed that M_{ld} can compute $(\Sigma^V_2) \restriction \mathbb{R}^{M_{\mathrm{ld}}}$, but did not establish the optimal anti-correctness result. His proof used the stationary tower.

Theorem 1 (S. [4], 2024)**.** *Let* $[\alpha, \alpha]$ *be a scale-cofinal projective-like gap. Let* $n \geq 1$*. Then* $\mathrm{OD}_{\alpha n} = \mathrm{OD}^{\mathbb{R}}_{\alpha n}$ *is a mouse set.*

The proof gives a new proof that $\mathrm{OD}_{12} = \mathbb{R} \cap M_{\mathrm{ld}}$, avoiding the stationary tower. It also gives anti-correctness for M_{ld}:

Theorem 2 (S., [4], 2024)**.** *Anti-correctness holds for* $\Pi^{\mathcal{J}(\mathbb{R})}_2$ *and* $M = M_{\mathrm{ld}}$*. There is a unique* Σ_1*-elementary* $\sigma : \mathcal{J}(\mathbb{R}^M) \to \mathcal{J}(\mathbb{R})$ *and moreover:*

- $\Pi^{\mathcal{J}(\mathbb{R})}_2$ *is uniformly* $\Sigma^{\mathcal{J}(\mathbb{R}^M)}_2$,
- $\Pi^{\mathcal{J}(\mathbb{R}^M)}_2$ *is uniformly* $\Sigma^{\mathcal{J}(\mathbb{R})}_2$.

There is also a version for all scale-cofinal projective-like gaps, on a cone.

Now consider admissible gaps $[\alpha, \beta]$. Then $\mathrm{OD}_{\xi n} = \mathrm{OD}_{\alpha 1}$ for all $\xi \in [\alpha, \beta)$ and $n \geq 1$. Moreover, if $[\alpha, \beta]$ is strong, Martin's argument in [6] adapts in a straightforward manner to give the same when $\xi = \beta$.

This leaves $\mathrm{OD}_{\beta n}$ and $\mathrm{OD}^{\mathbb{R}}_{\beta n}$ when $[\alpha, \beta]$ is weak and $n \geq 2$ (the $n = 1$ case is uninteresting), and $\mathrm{OD}_{\beta+1,n}$ and $\mathrm{OD}^{\mathbb{R}}_{\beta+1,n}$ when $[\alpha, \beta]$ is strong and $n \geq 2$.

Theorem 3 (S., [4], 2024). *Let $[\alpha,\gamma]$ be a weak gap, or $\gamma=\beta+1$ where $[\alpha,\beta]$ is a strong gap. Then for a cone of reals x, there is a "γ-ladder" x-mouse $M_{\mathrm{ld}}^{\gamma}(x)$ definable over x from $\mathcal{J}_\gamma(\mathbb{R})$, analogous to M_{ld} over $\mathcal{J}(\mathbb{R})$.*

Theorem 4 (S., [4], 2024). *Let $[\alpha,\gamma]$ be a weak gap, or $\gamma=\beta+1$ where $[\alpha,\beta]$ is a strong gap. Let e be least such that $\rho_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})}=\mathbb{R}$. Then $\mathrm{OD}_{\gamma n}$ is a mouse set for $n\geq e+3$, and for $n\leq e+1$.*

So this leaves $n=e+2$ open, and somewhat more naturally, $\mathrm{OD}_{\gamma,e+2}(\{\vec{p}_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})}\})$, though this is still probably not quite the right question. There is a more natural variant of $\mathrm{OD}^{\mathbb{R}}_{\gamma,e+n}$ in the current context; call it $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+n}$. Then we have:

Theorem 5 (S., [4], 2024). *Let γ,e be as above. Then:*

(1) For $n\geq 3$, $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+n}=\mathrm{OD}_{\gamma,e+n}$ is a mouse set.
(2) For a cone of reals x, $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+2}(x)=\mathbb{R}\cap M_{\mathrm{ld}}^{\gamma}(x)$ is a mouse set.

In sufficiently "canonical" cases, the cone is not needed, and $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+2}$ is the mouse set $\mathbb{R}\cap M_{\mathrm{ld}}^{\gamma}$. But in general, it remains open whether $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+2}$ is a mouse set (without the cone).

These results also relate to the correctness conjecture of Rudominer and Steel in [1], that, also assuming ZF + AD + $V=L(\mathbb{R})$, given any $(0,\omega_1+1)$-iterable mouse M, there are ordinals $\bar{\gamma},\gamma$ and $n<\omega$ such that:

(i) $\mathbb{R}^M=\mathcal{J}_{\bar{\gamma}}(\mathbb{R}^M)$,
(ii) $\pi:\mathcal{J}_{\bar{\gamma}}(\mathbb{R}^M)\to\mathcal{J}_\gamma(\mathbb{R})$ is Σ_{n+1}-elementary (for the $L(\mathbb{R})$ language),
(iii) there is wellorder of $\mathbb{R}^M$ which is Σ_{n+2}-definable over $\mathcal{J}_{\bar{\gamma}}(\mathbb{R}^M)$ from a parameter $p\in\mathbb{R}^M$.

Rudominer and Steel proved the conjecture in some cases, but not in general. The conjecture has now almost been confirmed:

Theorem 6 (Steel, S., [5], 2024). *The following variant of the Rudominer-Steel correctness conjecture holds: in property (iii) of the conjecture, allow an arbitrary parameter $p\in\mathcal{J}_{\bar{\gamma}}(\mathbb{R}^M)$, instead of demanding $p\in\mathbb{R}^M$.*

(The current versions of the preprints [4] and [5] contain only parts of the results mentioned above. The full results will be made available in due course.)

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The Tukey Order on Measurable Cardinals

Tom Benhamou

We say that a poset P is Tukey below Q if there is a function from P to Q which maps unbounded subsets of P to unbounded subsets of Q. In this talk we will be interested in particular posets– ultrafilters ordered by reverse inclusion and reverse inclusion modulo the bounded ideal. This has been studied by many including Dobrinen, Isbell, Milovich, Raghavan, Shelah, and Todorcevic. The results mostly concentrate on ultrafilters on ω.

In the first part of the talk we will present results connecting the Tukey order and recent developments in Prikry-type theory. More specifically, we connect the Galvin property and Tukey-top ultrafilters. The Galvin property of an ultrafilter U over κ says that given any 2^κ-many sets in U, there are κ-many whose intersection form a U-large set. In a joint work with Dobrinen we establish the equivalence between being a κ-complete non-Galvin ultrafilter with the property of being Maximl in the Tukey order among κ-directed posets of cardinality at most 2^κ. In Prikry theory this is used for example to characterize when are old sets of cardinality κ are dense in sets of cardinality 2^κ in a Prikry extension with the ultrafilter.

This observation initiated a joint project with Dobrinen in which we generalized many results from the countable to the measurable settings. Surprisingly, we also found some discrepancies and I will present one of them in this talk: for any two κ-complete ultrafilters U, W, the Fubini product of U and W is Tukey equivalent to the Cartesian product of U and W.

We will also discuss generalizations of Galvin's theorem which says that normal ultrafilters have the Galvin property. We will present a generalization that all the ultrafilters in the class of finite iterated sums of p-points have the Galvin property. In particular, in $L[U]$ every ultrafilter has the Galvin property. Also, Supercompact cardinals always carry a non-Galvin ultrafilter. This raises a natural question: can non-Galvin ultrafilters exist in the canonical inner models?

I will present a joint result with G. Goldberg resolving this question, we classify the Galvin ultrafilters ultrafilters under UA+"every irreducible is Dodd-sound". These assumptions hold in the known canonical inner models.

In the next part of the talk we provide an ultrapower characterization of the Tukey order: We prove that a poset P is Tukey below an ultrafilter U if in the ultrapower M_U there is a thin cover of $j''_U P$. A thin cover is a cover which does not contain $j_U(\mathcal{A})$ for any $\mathcal{A} \subseteq P$ unbounded. We use that to derive a characterization for the Galvin property: U is non-Galvin if and only if there is a cover of $j''_U 2^\kappa$ which does not contain $j_U(A)$ for any set A of size κ.

Finally, we improve a theorem of Kanamori that if $2^\kappa = \kappa^+$ then every uniform ultrafilter over κ is not (κ^+, κ^+)-cohesive and show that in general every uniform ultrafilter over κ is not $(cf(ch(\kappa)), cf(ch(\kappa)))$-cohesive. We then use the notion of a P_λ-point at a measurable cardinal to answer a question of Kanamori regarding the existence of a (κ^+, κ^+)-cohesive ultrafilter.

Particularly, we show that the existence of a (κ^+, κ^+)-cohesive ultrafilter is equivalent to the existence of a $P_{\kappa^{++}}$-point and use known techniques to get such an ultrafilter. Then we show that there is a non-trivial lower bound to the existence of a $P_{\kappa^{++}}$-point, namely a 2-strong cardinal.

The compactness theorem for $\mathrm{L}_{\infty\infty}$

Matteo Viale
(joint work with Juan Manuel Santiago Suárez)

The infinitary logic $\mathrm{L}_{\kappa\lambda}$ admits as formulae those constructed from the atomic formulae of the signature L using negation, conjunctions and disjunctions of size less than κ, and blocks of quantifiers $\exists(x_i : i < \gamma)$, $\forall(x_i : i < \gamma)$ on a string of variables indexed by some $\gamma < \lambda$. The $\mathrm{L}_{\kappa\lambda}$-formulae can only have less than λ-many free variables.[1]

The logic $\mathrm{L}_{\infty\lambda}$ is the union of the logics $\mathrm{L}_{\kappa\lambda}$ for κ a (regular) cardinal; the logic $\mathrm{L}_{\infty\infty}$ is the union of the logics $\mathrm{L}_{\kappa\lambda}$ for κ, λ (regular) cardinals. Our focus in this note is on the logics $\mathrm{L}_{\infty\omega}$ and $\mathrm{L}_{\infty\infty}$.

We present the generalization to $\mathrm{L}_{\infty\infty}$ of the usual compactness theorem for first order logic. While doing so we elaborate a bit on what are the right semantics for these logics.

The Tarski semantics for the sublogics of $\mathrm{L}_{\infty\infty}$ in an L-stucture $\mathfrak{M}$ is defined as expected. Any usual proof system for first order logic can be naturally generalized to $\mathrm{L}_{\infty\infty}$, and it is straightforward to establish the correctness theorem relative to the Tarski semantics for $\mathrm{L}_{\infty\infty}$. However the completeness theorem for these natural proof systems fails relative to Tarski semantics; specifically there is an $\mathrm{L}_{\omega_2\omega}$-sentence ψ which is provably consistent but does not hold in any Tarski model (in a suitable signature L, ψ asserts that there is a bijection of ω onto ω_1^V).

So one is left with two options: either change the semantics or change the proof system.

It is the case that the main developments in the analysis of (the infinitary sublogics of) $\mathrm{L}_{\infty\infty}$ followed the second path, i.e. most authors sticked to Tarski

[1]Hence infinitary disjunctions/conjunctions on a set of $\mathrm{L}_{\kappa\lambda}$-formulae are allowed only if the formulae have all their free variables occurring in a fixed set of size less than λ.

semantics, and either did not pay particular attention to syntactic proofs, or devised new proof systems which yield a completeness theorem for this semantics (e.g. this is the approach taken in the monographs on infinitary logics [2, 5]).

We address the discrepancy between the above proof system and Tarski semantics for $\mathrm{L}_{\infty\infty}$ following the alternative pattern of changing the semantics. This approach has already been pursued by Mansfield and Karp independently [1, 12], who proved a completeness theorem relative to boolean valued semantics.

Below are the relevant definitions and results. We refer the reader to [8, 9] for a deeper analysis of the merits of our approach in the study of $\mathrm{L}_{\infty\infty}$ and $\mathrm{L}_{\infty\omega}$ and for the definition of boolean valued model and of mixing model (the latter defines a category which is equivalent to that of sheaves on a complete boolean algebras for the sup-topology [4, Example (e), p. 115]), see [3].

Definition 1 (Santiago & V. 2024). *Let ψ_0, ψ_1 be $\mathrm{L}_{\infty\infty}$-sentences such that $\psi_1 \vdash \psi_0$.*

ψ_1 is a conservative strengthening of ψ_0 *if for all finite sets s of subsentences of ψ_0, $\psi_0 \wedge \bigwedge s$ is consistent if and only if so is $\psi_1 \wedge \bigwedge s$.*

Definition 2 (Santiago & V. 2024). *Let $\{\psi_i : i \in I\}$ be a family of consistent $\mathrm{L}_{\infty\infty}$-sentences.*

$\{\psi_i : i \in I\}$ is finitely conservative *if for all u finite subsets of I $\bigwedge_{i\in u} \psi_i$ is a conservative strengthening of all the ψ_i with i in u.*

Note that the following is a finitely consistent (but not finitely conservative) $\mathrm{L}_{\omega_1\omega}$-theory which is not boolean consistent:

$$\mathrm{T} = \{ \bigvee_{n<\omega} c_\omega = c_n\} \cup \ \{c_n \neq c_m : n < m \leq \omega\}.$$

$t = \{\bigvee_{n<\omega} c_\omega = c_n, c_0 \neq c_\omega\}$ is not finitely conservative: $c_0 = c_\omega$ is consistent with $\bigvee_{n<\omega} c_\omega = c_n$, but not with $\bigwedge t$.

Theorem 1 (Compactness for $\mathrm{L}_{\infty\infty}$, Santiago & V. 2024). *Let $\{\psi_i : i \in I\}$ be a family of $\mathrm{L}_{\infty\infty}$-sentences.*

Then $\bigwedge_{i\in I} \psi_i$ ihas a boolean valued model if and only if $\{\psi_i : i \in I\}$ is finitely conservative.

Furthermore if each ψ_i is an $\mathrm{L}_{\infty\omega}$-sentence, $\bigwedge_{i\in I} \psi_i$ has a mixing model.

The theorem generalizes first order compactness in view of the following (combined with [6, Thm. 6.3.7] and the observation that mixing models are full):

Theorem 2 (Santiago & V. 2024). *Let T be a finitely consistent first order theory. Then there exists T^* such that :*

- *T^* is provably equivalent to T, and*
- *T^* is finitely conservative.*

We can also prove that our compactness theorem (almost) implies the Barwise compactness theorem from [7].

The proofs of these results and other applications of the $\mathrm{L}_{\infty\infty}$-compactness theorem appear (or will appear) in [8, 9, 10, 11].

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The second to last word on the consistency of Shelah's conjecture

Justin Moore

(joint work with John Krueger)

In 2006, Moore proved that the Proper Forcing Axiom (PFA) implies that every Aronszajn line contains a Countryman suborder; this later statement is sometimes known as *Shelah's Conjecture.* Unlike nearly all previous applications of PFA to the combinatorics of $\aleph_1$, this use seemed to need a significant amount of the consistency strength of PFA. Also, PFA itself was needed in order to prove the existence of a proper poset that introduces a Countryman suborder to an Aronszajn line. In particular, it was not clear if the Bounded Proper Forcing Axiom (BPFA) implied Shelah's Conjecture.

In 2008, the consistency strength of Shelah's Conjecture was reduced to something less than the existence of a Mahlo cardinal by König, Larson, Moore, and Veličković. Their arguments, however, still required something more than the existence of a reflecting cardinal—the exact consistency strength of BPFA. On the other hand, the arguments showed that BPFA together with the assertion that all Aronszajn trees are *saturated* does imply Shelah's Conjecture. Here an Aronszajn tree is saturated if any family of (uncountable) subtrees with pairwise countable intersection has cardinality at most $\aleph_1$.

In this talk, we announce that if there is an inaccessible cardinal κ, then there is a κ-c.c. proper forcing which forces the conjunction of Shelah's conjecture, PFA for posets of cardinality $\aleph_1$, Moore's measuring principle, the assertion that all Aronszajn trees are saturated, and the assertion that any two $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic. In particular, if L is an Aronszajn line, there is a proper partial order in V_κ which forces the existence of a Countryman suborder of L. This also shows that the consistency strength of Aronszajn tree saturation and, e.g., $\mathrm{MA}_{\aleph_1}$, is exactly that of an inaccessible cardinal, answering a question of Moore. This is joint work with John Krueger.

Consistency results regarding the Ketonen order and the Lipschitz order

Eyal Kaplan

In recent years, Goldberg discovered and thoroughly analyzed the Ultrapower Axiom (UA), which states that any pair of ultrapowers formed via σ-complete ultrafilters can be compared by taking further internal ultrapowers. Goldberg's study of the UA has led to a series of striking results regarding the structure of the set-theoretic universe (see [2]). For instance :

Theorem 1 (Goldberg)**.** *Assume UA. Then the least strongly compact cardinal is supercompact.*

Theorem 2 (Goldberg)**.** *Assume UA. Then GCH holds above a strongly cardinal.*

Motivated by the second theorem, Goldberg asked whether the UA is consistent with the violation of GCH on a measurable cardinal. This was lately answered in [1]:

Theorem 3 (Ben-Neria, Kaplan)**.** *It is consistent (from large cardinals) that UA holds, and GCH is violated on a measurable cardinal.*

Theorem 3 demonstrates that forcing could be useful to obtain consistency results regarding UA and related concepts. One of the fundamental UA-related concepts is the *Ketonen order*.

In his study of the UA, goldberg observed that UA implies that the class of σ-complete ultrafilters is well-ordered with respect to the Ketonen order. The modern formulation of the order is due to Goldberg, building on earlier work by Ketonen, who introduced this order restricted to the class of weakly normal ultrafilters [3].

Definition 1 (The Ketonen order)**.** *Let U, W be σ-complete ultrafilters. We say that U is Ketonen below W, and denote $U <_k W$, if and only if there exists $I \in W$ and a sequence $\langle U_\xi \colon \xi \in I\rangle$ of σ-complete ultrafilters, such that each U_ξ is a uniform ultrafilter on some ordinal $\delta_\xi \leq i$, and, for every $X \subseteq \kappa$ (where κ is the underlying ordinal of U),*

$$X \in U \iff \{\xi \in I \colon X \cap \delta_\xi \in U_\xi\} \in W.$$

The Ketonen order is a strict well-founded order on the class of all σ-complete ultrafilters (see [2, Subsection 3.3.2]). When restricted to normal measures, the Ketonen order coincides with the Mitchell order (see [2, Theorem 3.4.1]). Goldberg observed that the Ketonen order is especially relevant in the study of the UA. In fact, he proved:

Theorem 4 (Goldberg)**.** *UA is equivalent to the linearity of the Mitchell order.*

Goldberg also observed that Ketonen relations between pairs of ultrafilters imply that certain games involving these ultrafilters are determined. The determinacy of these games is commonly used in descriptive set theory to order subsets of the Cantor space according to the *Lipschitz order*. We provide below the definition of the Lipschitz order only between σ-complete ultrafilters on some ordinal κ. For that, we describe the Lipschitz game $G_\kappa(W, U)$. The game is being held between two players, I and II, and consists of κ stages. On the i-th stage ($i < \kappa$), Player I chooses a bit $a(i) \in \{0, 1\}$, and Player II replies with a bit $b(i) \in \{0, 1\}$. Since Player I moves first, they are aware to the sequences $\langle a(j) \colon j < i\rangle, \langle b(j) \colon j < i\rangle$ constructed so far; Player II moves second, being aware to the values of $\langle a(j) \colon j \leq i\rangle, \langle b(j) \colon j < i\rangle$. After κ-many rounds, the players have constructed a pair of subsets of κ,

$$a = \{i < \kappa \colon a(i) = 1\}$$

$$b = \{i < \kappa \colon b(i) = 1\}.$$

Player II wins if $a \in W \leftrightarrow b \in U$. Otherwise, Player I wins.

Definition 2 (The Lipschitz order)**.** *Let U, W be σ-complete ultrafilters on some ordinal κ. We say that U is Lipschitz below W, and denote $U <_L W$, if player I has a winning strategy in $G_\kappa(W, U)$.*

The Lipschitz order is a strict partial order on the class of σ complete ultrafilters. It is not known (in ZFC) whether it must be well-founded. Goldberg observed that it is under UA:

Theorem 5 (Goldberg)**.** *Assume that U, W are σ-complete ultrafilters on κ. Then $U <_k W$ implies $U <_L W$.*

Goldberg raised the question whether the Ketonen and the Lipschitz orders could be separated (see [2, Question 9.2.10]). We will sketch the proof of the following:

Theorem 6 (Kaplan)**.** *There exists a cardinal preserving forcing extension of $L[U]$ in which there is a pair of Ketonen-incomparable σ-complete ultrafilters, $\mathcal{V}, \mathcal{W}$, such that $\mathcal{V}$ is Lipschitz below $\mathcal{W}$.*

The same result can be obtained in a model in which every σ-complete ultrafilter is a finite product of the normal measures U_0, U_1.

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Filter Games and Ineffable Cardinals

Matthew Foreman

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Filter games are a type of Challenge-Response game where Player I plays a κ complete subalgebras $\mathcal{A}$ of $\mathcal{P}(\kappa)$ and Player II plays a κ-complete ultrafilter on $\mathcal{A}$. In previous work Foreman, and Magidor, jointly with Zeman showed that if Player II wins the game of length γ a regular uncountable countable cardinal, then there is a precipitous ideal on κ with a dense tree of height γ that is closed under decreasing sequences of length less than γ.

However, that construction did not control which sets the ideal concentrates on. This talk is about an ideal, the *superineffable ideal.* A main result proves that if A_0 is positive for this ideal, and Player II wins the game where she is required to play normal, κ complete filters then the precipitous ideal concentrates on A_0. In particular this technique can produce precipitous ideals on $\{\alpha : \alpha \text{ is measurable of order } \alpha^+\}$ or $\{\alpha : \alpha \text{ is supercompact}\}$.

Other results include:

- the necessity for the superineffable ideal to be a proper ideal for the game to go even ω steps.
- The levels in the construction of the superineffable ideal intertwine with the notion due to Baumgartner of n-ineffability. (n-ineffability is equivalent to the statement that any partition $f : [\kappa]^{n+1} \to 2$ has a stationary homogeneous set.)

The last results are that the superineffable ideal on κ is proper if and only if κ is *completely ineffable* and that this is not downwards absolute.

Some new ways of blowing up the power of a measurable cardinal

Moti Gitik

Which forcing notions can be used to blow up the power of a measurable cardinal preserving its measurability? Starting with a Laver indestructible supercompact cardinal κ, basically any $\kappa-$directed closed forcing can be used. The situation changes drastically if instead of a supercompacts we work under weaker assumptions, say no inner model with a strong. H. Woodin was the first to show that it is possible. He used the Cohen forcing for this. The main difficulty here is to obtain a generic object over the ultrapower with an extender which is not closed enough. Later S. Friedman, K. Thompson and S. Friedman, L. Zdomskii showed that it is possible to use generalized versions of Sacks and Miller forcings for this purpose, as well. O. Ben-Neria and the author used a non-stationary Cohen forcing. In this constructions, in contrast to Woodin's, a generic object over the ultrapower with an extender is already generated by pointwise image of those over V. C. Merimovich used the extender based Radin forcing.

The Woodin construction starts with a GCH model V, $j : V \to M, {}^{\kappa}M \subseteq M, (\kappa^{++})^M = \kappa^{++}$. The Cohen forcing which adds ν^{++} subsets to each inaccessible $\nu \leq \kappa$ is iterated with the Easton support. Denote by G a generic set for this iteration. The final stage is to find an M-generic G^* such that $j''G \subseteq G^*$ and j extends to $j^* : V[G] \to M[G^*]$. The difficulty here is that M is not closed under κ^+-sequnces. Woodin forced with $Cohen(\kappa^+, \kappa^{++})$ in order to produce such G^*. Y. Ben Shalom, O. Keshet and the author showed that there is no need in this additional forcing and G^* can be constructed already in $V[G]$.

A question on the strength of a small number of generators of a normal ultrafilter over a measurable κ was addressed. The following was shown:

Theorem 1. *Starting with $o(\kappa) = \kappa^{+3}$, it is possible to force a model in which κ is a measurable, $2^\kappa = \kappa^{++}$, there is a normal ultrafilter over κ generated by mod Cub_κ by κ^+-many sets.*
It is possible also to have two normal ultrafilters - one generated (mod Cub_κ) by κ^+-many sets and another by $\kappa^{++}-$many sets.

Using Mathias type forcing suggested by T. Benhamou, we can prove the following:

Theorem 2. *Assume $o(\kappa) = \kappa^{++}$. There is a forcing extension in which κ is a measurable, $2^\kappa = \kappa^{++}$ and there is $\subseteq^*$ $-$decreasing of clubs of κ.*

Idealistic equivalence relations remastered

Filippo Calderoni
(joint work with Luca Motto Ros)

An equivalence relation E on a standard Borel space X is *idealistic* if there is a Borel map $C \mapsto I_C$, assigning to each equivalence class a ccc σ-ideal of subsets of C, which is Borel-on-Borel: For each Borel set $A \subseteq X^2$, the set A_I defined by

$$x \in A_I \iff \{y \in [x]_E : (x,y) \in A\} \in I_{[x]_E}$$

is a Borel subset of X.

If a Polish group G acts on the standard Borel space X in a Borel fashion, then the *orbit equivalence relation* E_G^X induced by the action is analytic. As observed in [Kec92, Section 1.II] all orbit equivalence relations are idealistic.

Although orbit equivalence relations have been widely studied in the literature, very little is known about idealistic equivalence relations and their structure up to the pre-order of Borel reducibility. (E.g., see [KecLou].) The following fundamental question is still open and motivates our work.

Question 1. *Is every idealistic equivalence relation on a standard Borel space is Borel bi-reducible to an orbit equivalence relation?*

In this talk, we discuss the following result, which is heavily based on the unpublished result of Becker [Bec01].

Theorem 2. *Assume $\mathbf{\Sigma}^1_1$-determinacy. There is a $\mathbf{\Sigma}^1_1$ idealistic equivalence relation E with Borel equivalence classes, that is not class-wise Borel embeddable into any orbit equivalence relation.*

To clarify the above statement, we explain the definition of class-wise Borel reducibility, a strengthening of the more classical notion of Borel reducibility. Given two analytic equivalence relations E, F on standard Borel spaces X, Y, respectively, we say that E is *classwise Borel isomorphic* to F, in symbols $E \simeq_{cB} F$, if there is a bijection $f \colon X/E \to Y/F$ such that both f and f^{-1} admit Borel liftings. Moreover, we say that E *classwise Borel embeds* into F, in symbols $E \sqsubseteq_{cB} F$, if there is a Borel F-invariant subset $A \subseteq Y$ such that $E \simeq_{cB} F \restriction A$.

Our method also shows that the structure of the equivalence relations as in the statement of our main theorem is positively complicated. Let $\mathcal{I}$ be the collection of all idealistic analytic equivalence relations which have only Borel equivalence classes (like all orbit equivalence relations), yet they are not classwise Borel isomorphic to an orbit equivalence relation.

Theorem 3. *Assume $\mathbf{\Sigma}^1_1$-determinacy. Then there is an embedding of $(\mathcal{O}, \leq_B)$ into $(\mathcal{I}, \leq_B)$, where $\mathcal{O}$ is the class of all Borel orbit equivalence relations with uncountably many orbits.*

In particular, there are continuum many $\leq_B$-incomparable idealistic equivalence relations which are not classwise Borel isomorphic to an orbit equivalence relation.

This is a joint work with Luca Motto Ros.

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A hypertalk

Elliot Glazer

This talk begins with a summary of William Zwicker's hypergame paradox, regarding a self-referential game in which the first player chooses a finite game, and then the second player becomes first player of that game. The question of whether the hypergame is a finite game is paradoxical in a manner similar to that of Russell's paradox. We emulate this paradox by allowing the audience to choose one of several talks to present, include the hypertalk. The audience chose nonconservativity of Global Choice over subsystems of ZFC as the topic.

We recently proved that Global Choice (GC) is not conservative over Z or over ZFC - Fnd (where Fnd is the Axiom of Foundation). After using the easier case of Z to demonstrate some basic techniques for building foundationless set theoretic models, we discussed a surprisingly concrete instance of nonconservativity of GC over ZFC - Fnd. Namely, we proved that the classical topology result that there are exactly four homeomorphism classes of connected 1-dimensional manifolds (not necessarily second countable or set-sized) is not a theorem in ZFC - Fnd, but is provable from this theory plus GC. From a choice of one point from every subset of a class curve $\mathcal{C}$ and a well-ordering of $V_{\omega\cdot 2}$, one can define a surjection from $\mathbb{R}$ to $\mathcal{C}$.

Even simple variations of the main result remain open. For example, a natural generalization is that ZFC - Fnd + GC proves that every topological manifold is a set, but the methods for proving the 1-dimensional case do not seem to suffice for the 2-dimensional case. Of course, it is enough to assume the stronger principle that there is a global bijection of the universe with Ord (this suffices to prove all structural consequences of Fnd).

A model W of ZFC - Fnd where the classification of curves fails can be built by considering the ZFCA model generated by any $V \models$ ZFC and an ordered set of atoms which externally satisfy $(A, \leq) \cong (\mathbb{R})$, but internally only the proper initial segments are sets. Replace the atoms with Quine atoms, and use ill-founded coding to define the ordering of A. Then $(A, \leq)$ is a proper class long line in this model. We obtain a sentence σ which holds in W but is inconsistent with GC by defining the underlying manifold structure of A and declaring it to be a proper class manifold.

Focusing on the nonconservativity aspect of this result, natural strengthenings can be proven with a little extra effort. Global Binary Choice is nonconservative over ZFC, with an example obtained by strengthening the hypothesis of σ to include the assertion that there is a definable map assigning to every x some x'

and a map from {tournaments on x'} to {choice functions on x}. This trick can be used to prove most global choice principles to be nonconservative over Z or ZFC - Fnd.

A more interesting improvement is to only extend the Comprehension Schema, and not the Replacement Schema, with formulas which include the Global Choice symbol. In this system, it is consistent to have the continuum map onto some proper class long line $\mathcal{L}$, but a contradiction can still be obtained by further positing a definable map from a parameterized collection of subclasses of $\mathcal{L}$ onto $2^{\mathfrak{c}^+}$. This implies a failure of Comprehension on $2^{\mathfrak{c}}$. More philosophically, this shows a genuine structural distinction between the proper class long line and the set long line of this model.

We briefly discussed the foundationless multiverse, sketching a proof in ZFC that V is the pure part of some ZFC - Fnd model in which every set is a surjective image of the power class of some definable class long line. In the language of modal logic, we conclude "Global Choice possibly necessarily fails."

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On the covering reflection principle

Joel David Hamkins

(joint work with Nai-Chung Hou, Andreas Leitz, Farmer Schlutzenberg)

The covering reflection principle asserts that every large structure is covered by elementary images of a suitable fixed small structure. The principle, we shall prove, has a remarkable large cardinal strength. This is joint work originating in a robust exchange on MathOverflow [1, 2, 3, 4], which lead to our current joint paper in progress [6].

The main principle is the following:

Definition 1. *The* covering reflection principle *(CRP_δ) holds of a cardinal δ if for every first-order structure B in a countable language $\mathcal{L}$, there is an $\mathcal{L}$-structure A of size less than δ, such that B is covered by the elementary images of A in*

B. That is, every element $b \in B$ is in the range of some elementary embedding $j : A \to B$.

The main questions are:

Question 1. *Does covering reflection occur? How large is the smallest cardinal exhibiting covering reflection? Is the covering reflection principle consistent? What is the consistency strength?*

It is easy to see that the least δ with covering reflection is uncountable and indeed strictly larger than the continuum, in light of the structure of the real field $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$, which is not covered by images of any proper substructure. But also, one can show that δ must have cofinality exceeding the continuum and much more. Ultimately, the least cardinal δ with covering reflection will be regular and truly enormous.

The covering reflection principle is equivalently formulated for finite languages, for languages of size continuum, and for much larger languages. The principle is also equivalently formulated for structures B only of size at most $2^{<\delta}$, and from this it follows that the covering reflection principle for a regular cardinal δ is Π^1_1-expressible in V_δ. Consequently, the least such δ is not weakly compact and is below the first Σ_2 correct cardinal. In particular, covering reflection begins below the first strong cardinal, if at all. The covering reflection principle is also equivalently formulated by requiring that for every structure B in a countable language there is a small structure A such that every countable subset $X \subseteq B$ is covered by some elementary image of A. And it is similarly equivalent to insist that sets X of size continuum or indeed much larger sets are all covered by elementary images of A in B. Model theorists may find it natural to consider the covering reflection principle in connection with fixed particular theories or for classes of structures of some particular kind. For example, we prove that if a countable theory T is κ-categorical in some power κ, then every uncountable model of T is covered by a fixed countable model.

We begin to establish the remarkable strength of the covering reflection principle by proving large-cardinal lower bounds very high in the large cardinal hierarchy.

Theorem 1. *If the covering reflection principle holds for δ, then there is a measurable cardinal λ below δ that is a limit of cardinals that are λ-extendible cardinals. In particular, V_λ is a model of ZFC with a proper class of fully extendible cardinals.*

The proof proceeds by considering the structure $B = \langle V_{\delta+1}, \in \rangle$. By covering reflection there is a small covering structure $\langle A, \in \rangle$, which admits many elementary embeddings $j : A \to B$. By analyzing the nature of the critical points and covering sets that arise, we show that A includes a large rank-initial segment of the universe V_λ, where λ is measurable, and there are numberous embeddings $j : A \to B$ whose critical points are extendible cardinals inside V_λ. Generalizations of the method establish as lower bounds the strength of extendible limits of extendible cardinals and limits of limits and so forth.

For an upper bound, we prove:

Theorem 2. *If κ is huge, then the covering reflection principle holds of κ. The least cardinal δ exhibiting covering reflection is therefore strictly less than κ.*

For this, we argue with a delightful trick. Namely, if $j : V \to M$ is a hugeness embedding and B is a counterexample to covering reflection, then $j(B)$ is a counterexample at $j(\kappa)$ in M, but also in V by hugeness. In particular, the small structure B does not cover $j(B)$ in V, omitting some point $x \in j(B)$. By elementarity, $j(B)$ does not cover $j(j(B))$ in M, missing $j(x)$. Except that by hugeness, $j \upharpoonright j(B) : j(B) \to j(j(B))$ is available in M and it is elementary and sends x to $j(x)$, a contradiction.

Finally, aiming at an exact equiconsistency, we introduce the following large cardinal notion.

Definition 2. *A cardinal κ is an* anchor *cardinal κ if for every $X \subseteq V_\kappa$ there is $\kappa_0 < \kappa_1 < \kappa$ and elementary embedding $j : \langle V_{\kappa_1}, \in, X \cap V_{\kappa_1} \rangle \to \langle V_\kappa, \in, X \rangle$ with $\kappa_0 = cp(j)$ and $j(\kappa_0) = \kappa_1$.*

This notion is related to *links* and *chains* in [5]. Every huge cardinal, it turns out, has a normal measure concentrating on anchor cardinals. Ultimately we settle the exact consistency strength with the following theorem:

Theorem 3. *The least cardinal δ with covering reflection is exactly the least anchor cardinal.*

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Classification of Cardinal Exponentiation and the Exponent ω.

William Chan

This talk will determine the cardinality relation between any two cardinal exponentiation below Θ, the supremum of the ordinals onto which $\mathbb{R}$ surjects, under determinacy hypothesis. Some additional cardinality and combinatorial properties concerning ω-sequences of ordinals, which is the smallest nonwellorderable cardinal exponentiation, will also be presented.

The following are the main results.

(1) Assume AD^+. If $\omega \leq \alpha \leq \beta < \Theta$ and $\omega \leq \gamma \leq \delta < \Theta$ are cardinals, then $|{}^\alpha\beta| \leq |{}^\gamma\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.
(2) Assume AD^+. If $\omega < \kappa < \Theta$ is a cardinal and $\epsilon < \kappa$ is an ordinal, then $\mathcal{P}_B(\kappa)$, the set of subsets of κ which are bounded below κ, does not inject into ${}^\epsilon\mathsf{ON}$, the class of ϵ-sequences of ordinals.
(3) Assume AD^+. If $\omega \leq \kappa < \Theta$ is a cardinal and $\delta < \kappa$, then $B(\omega, \kappa)$ (the set of bounded ω-sequences through κ) and ${}^\omega\kappa$ do not inject into $\mathcal{P}(\delta) \times \mathsf{ON}$.
(4) Assume AD^+. Let $\kappa < \Theta$ be such that $\mathrm{cof}(\kappa) = \omega$. Then ${}^\omega\kappa$ is ON-regular: For all $\Phi : {}^\omega\kappa \to \mathsf{ON}$, there is an $\alpha \in \mathrm{ON}$ so that $|\Phi^{-1}[\{\alpha\}]| = |{}^\omega\kappa|$.
(5) Assume AD^+. Let $\omega < \kappa < \Theta$ be such that $\mathrm{cof}(\kappa) = \omega$. Then $B(\omega, \kappa)$ is not ω-regular but for all $n < \omega$, $B(\omega, \kappa)$ is n-regular: There is a $\Psi : B(\omega, \kappa) \to \omega$ so that for all $m \in \omega$, $|\Psi^{-1}[\{m\}]| < |B(\omega, \kappa)|$ but for all $n \in \omega$ and $\Phi : B(\omega, \kappa) \to n$, there is some $m < n$ so that $|\Phi^{-1}[\{m\}]| = |B(\omega, \kappa)|$.

Suppose $\beta \leq \alpha < \Theta$ are cardinals. If α is finite, then ${}^\alpha\beta$ is finite exponentiation. Suppose α is infinite. If $\beta = 1$, then $|{}^\alpha\beta| = |\alpha|$. If $2 \leq \beta$, then $|{}^\alpha\beta| = |\mathcal{P}(\alpha)|$. Thus assume $\alpha \leq \beta < \Theta$. If β is finite, then ${}^\alpha\beta$ is again finite exponentiation. Suppose β is infinite but α is finite, then $|{}^\alpha\beta| = |\beta|$. By these reduction, the complete classification of cardinal exponentiation is obtained by comparing cardinals exponentiations of the form ${}^\alpha\beta$ when $\omega \leq \alpha \leq \beta < \Theta$. In this way, result (1) provides a complete classification of all cardinal exponentiation below Θ.

Steel [1] and Woodin showed that AD^+ implies boldface GCH holds below Θ, which is the statement that for all infinite cardinal κ, there is no injection of κ^+ into $\mathcal{P}(\kappa)$. The boldface GCH below Θ and result (2) imply result (1) as follows: Assume $\alpha \leq \beta < \Theta$ and $\gamma \leq \delta < \Theta$. It is clear that if $\alpha \leq \gamma$ and $\beta \leq \delta$, then ${}^\alpha\beta$ injects into ${}^\gamma\delta$. Suppose $\delta < \beta$ and there is an injection of ${}^\alpha\beta$ into ${}^\gamma\delta$. Then $|\beta| \leq |{}^\alpha\beta| \leq |{}^\gamma\delta| \leq |{}^\delta\delta| = |\mathcal{P}(\delta)|$ where the second inequality is witnessed by the injection. Since $\delta < \beta$, $\delta^+ \leq \beta$. Hence there is an injection of δ^+ into $\mathcal{P}(\delta)$ which violates boldface GCH at δ. Suppose $\gamma < \alpha$ and there is an injection of ${}^\alpha\beta$ into ${}^\gamma\delta$. $|\mathcal{P}_B(\alpha)| \leq |\mathcal{P}(\alpha)| \leq |{}^\alpha 2| \leq |{}^\alpha\beta| \leq |{}^\gamma\delta|$. Thus there is an injection of $\mathcal{P}_B(\alpha)$ into ${}^\gamma\mathrm{ON}$ where $\gamma < \alpha$ in violation of result (2).

Next, one will sketch the main ideas for the proof of result (2): For simplicity assume $V = L(\mathbb{R})$. Suppose result (2) fails. By Σ_1-reflection, there are ordinals $\omega < \kappa < \boldsymbol{\delta}^2_1$, $\epsilon < \kappa$, and an injection $\Phi : \mathcal{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$. In $L(\mathbb{R})$, all sets are ordinal definable from some real. Without loss of generality, suppose Φ is OD. Consider the forcing $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ of partial functions from ϵ^+ into ϵ^{++} as defined within HOD. One will need to solve the following two problems:

- (Generic Existence Problem) There is a $G \subseteq \mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ in the real world, $L(\mathbb{R})$, which is generic over HOD.
- (Capturing Problem) There is a $G \subseteq \mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ generic over HOD such that $\mathrm{HOD}[G] = \mathrm{HOD}_{\{G\}}$.

Assuming the solution to these two problems, one can show result (2). Let $G \subseteq \mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ generic over HOD with the above two properties. G is essentially an element of $\mathcal{P}_B(\kappa)$. $\Phi(G)$ is clearly $\mathrm{OD}_{\{G\}}$ and thus $\Phi(G) \in \mathrm{HOD}_{\{G\}} = \mathrm{HOD}[G]$. Since $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ is $< \epsilon^+$-closed within HOD, G adds no new ϵ-sequences of ordinals over HOD. Since $\Phi(G) \in {}^{\epsilon}\mathrm{ON}$, one must have $\Phi(G) \in \mathrm{HOD}$ and thus $\Phi(G)$ is ordinal definable. Then $G = \Phi^{-1}(\Phi(G))$ is ordinal definable. Thus $G \in \mathrm{HOD}$ which is impossible since G is generic over HOD.

The solution to both the Generic Existence Problem and the Capturing Problem requires the analysis of HOD by inner model theory established by Steel [1].

(Generic Existence Problem) Since $\mathrm{HOD} \models \mathsf{GCH}$, the collection $\mathcal{D}$ of all dense subsets of $\mathrm{Coll}(\epsilon^+, \epsilon^{++})$ in HOD has cardinality ϵ^{+++} in HOD. By the Steel's HOD analysis, the real world cofinality of all successor cardinals of HOD is ω. Using the fact that $\mathrm{cof}^{L(\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}}) = \omega$, one can patch together ω-many surjection of $(\epsilon^+)^{\mathrm{HOD}}$ onto various ordinals cofinal through $(\epsilon^{++})^{\mathrm{HOD}}$ to obtain a HOD-amenable surjection of $g : (\epsilon^+)^{\mathrm{HOD}} \to (\epsilon^{++})^{\mathrm{HOD}}$, which means that g restricted to any $\alpha < (\epsilon^+)^{\mathrm{HOD}}$ belong to HOD. Repeating this procedure again, one obtained a HOD-amenable surjection of $(\epsilon^+)^{\mathrm{HOD}}$ onto $(\epsilon^{+++})^{\mathrm{HOD}}$. Thus there is a HOD-amenable surjection h of $(\epsilon^+)^{\mathrm{HOD}}$ onto $\mathcal{D}$, the collection of dense subset of $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ in HOD. With this HOD-amenable surjection h, one can build a generic G for $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ in $(\epsilon^+)^{\mathrm{HOD}}$-steps by meeting the dense sets enumerated by the surjection h and using the internal $< \epsilon^+$-closedness of the forcing $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ within HOD. The HOD-amenability of h is important to getting each step of the construction internal to HOD.

(Capturing Problem) Woodin and Ikegami-Trang showed that if $f \in {}^{\omega}\kappa$ where $\kappa < \Theta$, then $\mathrm{HOD}[f] = \mathrm{HOD}_f$. Woodin showed that under AD^+, there are always uncountable $A \subseteq \omega_1$ so that $\mathrm{HOD}_{\{A\}} \neq \mathrm{HOD}[A]$. A generic filter for $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ is generally an uncountable set of ordinals. However, the fact that the real world cofinality of successor cardinals of HOD is ω implies that every generic filter for $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}}$ is countably generated in the real world. This means such a generic filter G has a countable set $\sigma \subseteq G$ in the real world so that G is the upward closure of σ by the forcing relation. σ contains all information about G and the idea is to pick an ω-enumeration f of σ, then one can try to use the Woodin and Ikegami-Trang result that $\mathrm{HOD}_{\{f\}} = \mathrm{HOD}[f]$. However, there is no uniform way to pick a countably generating set σ for G and to pick an ω-enumeration of σ. Since $\epsilon < \kappa < \boldsymbol{\delta}^2_1$ which is the largest Suslin cardinal of $L(\mathbb{R})$, Harrington-Kechris [2] showed there is a supercompact measure ν on $|\mathrm{Coll}(\epsilon^+, \epsilon^+ +)|^{\mathrm{HOD}}$ and Woodin [3] showed this supercompact measure ν is OD. The idea is now to look at the ν-large set of countable generating set σ for G. The ω-enumeration of σ will be obtained by going into a $\mathrm{Coll}(\omega, \sigma)$ forcing extension. The real world truth about ordinals and G will be expressed inside $\mathrm{HOD}[G]$ by integrating over the supercompact measure ν some statement involving forcing with $\mathrm{Coll}(\omega, \sigma)$ (for all σ in a ν-large set) and the OD ∞-Borel code forcing (a highly absolute variation of the Vopěnka forcing). The uniqueness of the supercompact measure is important since it implies ν is OD so the ultrapower by ν does not contribute any additionally complexity. These ideas will show that $\mathrm{HOD}_{\{G\}} = \mathrm{HOD}[G]$ which resolves the capturing problem.

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Filtrations between topologies

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Filtrations are certain transfinite sequences of topologies increasing in strength and interpolating between two given topologies σ and τ, with τ being stronger than σ. We prove general results on stabilization at τ of filtrations interpolating between σ and τ. These topological results involve an interplay between κ-Borel sets with respect to the topology σ and κ-Baireness of the topology τ. They are part of a broader project of analyzing Polishable equivalence relations, which is, in turn, connected with the conjectural $\mathbb{E}_1$-dichotomy.

Let $\sigma \subseteq \tau$ be topologies, and let ρ be an ordinal. A transfinite sequence $(\tau_\xi)_{\xi<\rho}$ of topologies is called a **filtration from σ to τ** if

$$\sigma = \tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_\xi \subseteq \cdots \subseteq \tau$$

and, for each $\alpha < \rho$, if F is τ_ξ-closed for some $\xi < \alpha$, then

$$\mathrm{int}_{\tau_\alpha}(F) = \mathrm{int}_\tau(F). \tag{1}$$

Condition (1) asserts that τ_α computes correctly, that is, in agreement with τ, the interiors of sets that are simple from the point of view of α, that is, sets that are τ_ξ-closed with $\xi < \alpha$. A filtration from σ to τ can be thought of as a walk from σ to τ through intermediate topologies, where each step is required to contribute a nontrivial advance towards τ, if such an advance is at all possible.

The following question about filtrations immediately presents itself. Given a filtration $(\tau_\xi)_{\xi<\rho}$ from σ to τ, does the filtration actually reach its goal τ and, if so, at what stage? That is, does $\tau_\xi = \tau$ for some $\xi < \rho$ and is there an upper estimate on such ξ?

One expects a positive answer to the question above under an appropriate assumption—τ should be at a visible distance from σ. We phrase this as follows. Let κ be an infinite cardinal such that τ is κ-Baire. (An infinite cardinal κ like that exists for each topology τ.) The assumption on the pair of topologies $\sigma \subseteq \tau$ is then for τ to have a neighborhood basis that consists of sets that are κ-Borel with respect to σ. In this situation, the distance from σ to τ is quantified by the complexity of sets in the neighborhood basis of τ as measured by the classical descriptive set theoretic hierarchy of κ-Borel sets.

Here is our main theorem. The notions involved in it are defined below. We write $(\tau_\xi)_{\xi\leq\rho}$ for $(\tau_\xi)_{\xi<\rho+1}$.

Theorem. *Let $\sigma\subseteq\tau$ be topologies with τ semiregular. Let κ be an infinite cardinal such that τ is κ-Baire, and let $1\leq\alpha\leq\kappa$ be an ordinal. Assume that τ has a neighborhood basis consisting of sets in $\bigcup_{\xi<\alpha}\mathbf{\Pi}^{\kappa;0}_{1+\xi}$ with respect to σ. If $(\tau_\xi)_{\xi\leq\alpha}$ is a filtration from σ to τ, then $\tau=\tau_\alpha$.*

We define now the notions involved in the theorem above.

A topology is called **semiregular** if it is generated by its regular open sets, that is, by sets that are interiors of their closures. To compare semiregularity with regularity of a topology, note that a topology is regular if and only if, for each open set U, there exists a family $\mathcal{F}$ of closed sets such that

$$U=\bigcup\{F\mid F\in\mathcal{F}\}=\bigcup\{\mathrm{int}(F)\mid F\in\mathcal{F}\}.$$

If we recall that a set is regular open precisely when it is equal to the interior of a closed set, it becomes clear that a topology is semiregular if and only if, for each open set U, there exists a family $\mathcal{F}$ of closed sets such that

$$U=\bigcup\{\mathrm{int}(F)\mid F\in\mathcal{F}\}.$$

In particular, it follows that each regular topological space is semiregular. The notion of semiregularity goes back to M. H. Stone.

A set is called **κ-meager** if it is the union of $<\kappa$ many τ-nowhere dense sets. Observe that ω-meager sets are τ-nowhere dense sets. The topology τ is called **κ-Baire** if the complement of each κ-meager set is dense.

We define κ-Borel sets and their stratification into $\mathbf{\Pi}$ classes. Some care in formulating these definitions is needed as we want to incorporate the case of singular cardinals κ in the right way. In the case of regular κ, the definitions we give coincide with the naive definitions—see the observations below.

First, for a cardinal ν, let

$$\mathrm{bor}(\nu)$$

be the smallest family of sets containing all closed sets and closed under taking complements and unions, and therefore also intersections, of families of cardinality $\leq\nu$. Define **κ-Borel sets** as

$$\bigcup\{\mathrm{bor}(\nu)\mid \nu \text{ a cardinal number, } \nu<\kappa\}.$$

Observe that ω-Borel sets form the algebra of sets generated by closed or open sets. The case $\kappa=\omega_1$ is the case studied in classical Descriptive Set Theory.

Observation. *Assume κ is regular. The family of κ-Borel sets is equal to the smallest family containing all closed sets and closed under taking complements and unions, and therefore also intersections, of families of cardinality $<\kappa$.*

To clarify further our definition of κ-Borel sets, note that if κ is singular, then the family of sets defined by the condition in the observation above is equal to κ^+-Borel sets and not κ-Borel sets.

We describe now a well known stratification of the family of κ-Borel sets. We define classes $\mathbf{\Pi}^{\kappa,0}_{1+\xi}$, for $\xi < \kappa$. To incorporate singular cardinals κ, the same type of care is needed here as in the definition of κ-Borel sets. When $\kappa = \omega_1$, the classes above form the well-studied hierarchy of Borel sets from Descriptive Set Theory. We conform to the tradition of enumerating these classes starting with 1 rather than 0.

For a cardinal $\nu > 0$ and an ordinal $\xi < \nu^+$, we define families

$$\mathbf{P}^{\nu}_{\xi}.$$

Let $\mathbf{P}^{\nu}_{0}$ be the family of all closed sets and, for $0 < \xi < \nu^+$, let $\mathbf{P}^{\nu}_{\xi}$ consist of intersections of subfamilies of cardinality $\leq \nu$ of complements of sets in $\bigcup_{\gamma<\xi} \mathbf{P}^{\nu}_{\gamma}$. For an ordinal $\xi < \kappa$, define

$$\mathbf{\Pi}^{\kappa,0}_{1+\xi} = \bigcup \{\mathbf{P}^{\nu}_{\xi} \mid \nu \text{ a cardinal, } \nu < \kappa,\ \xi < \nu^+\}$$

Observation. *(i)* $\mathbf{\Pi}^{\kappa,0}_{1}$ *is the class of all closed sets.*

(ii) If κ *is regular, then, for* $0 < \xi < \kappa$, $\mathbf{\Pi}^{\kappa,0}_{1+\xi}$ *consists of all intersections of* $< \kappa$ *many complements of sets in* $\bigcup_{\gamma<\xi} \mathbf{\Pi}^{\kappa,0}_{1+\gamma}$.

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Generic actions of countable groups

Sumun Iyer
(joint work with Forte Shinko)

Let G be a countable group. We will consider the space $\mathrm{Act}(G, 2^{\mathbb{N}})$ of all continuous actions of G on Cantor space. Let $\mathrm{Homeo}(2^{\mathbb{N}})$ be the Polish group of all homeomorphisms of Cantor space with the uniform convergence topology. We think of $\mathrm{Act}(G, 2^{\mathbb{N}})$ as a subspace of the space of all functions $G \to \mathrm{Homeo}(2^{\mathbb{N}})$. With the inherited topology, $\mathrm{Act}(G, 2^{\mathbb{N}})$ is a Polish space. The motivation for studying this space comes from work of Hochman and Frisch-Kechris-Shinko-Vidnyanszky [2, 3]. In particular, Hochman proved that any property which is generic in $\mathrm{Act}(G, 2^{\mathbb{N}})$ is generic in the space of *all* continuous actions of G on compact Polish spaces [3].

Given a continuous action of a countable group G on a Polish space X, the orbit-equivalence relation $\{(x, y) \ : \ \exists g \in G \ g \cdot x = y\}$ is a *countable Borel equivalence relation (CBER)* i.e., the relation is Borel in $X \times X$ and each equivalence class is countable. A CBER is *hyperfinite* if it is the increasing union of finite Borel equivalence relations. Hyperfinite relations are the simplest non-trivial ones amongst the CBERs. A CBER E on space X is *measure hyperfinite* if for every probability measure μ on X there is a μ-co-null E-invariant set A such that $E \restriction_{A \times A}$ is hyperfinite. It is currently not known if there is an example of a measure hyperfinite CBER which is not hyperfinite [5]. This question is related to an important long open question about hyperfiniteness and amenability of groups, see [5].

A group G is *exact* if its reduced C^*-algebra is exact. Exact groups are a broad class of groups which include the amenable groups and the free groups. It is a theorem of Suzuki that if G is exact then the generic element of $\mathrm{Act}(G, 2^{\mathbb{N}})$ is measure hyperfinite [6]. This raises the natural question: for G exact, is the generic element of $\mathrm{Act}(G, 2^{\mathbb{N}})$ hyperfinite?

Our main theorem is:

Theorem 1. *If G is a group which is locally of finite asymptotic dimension, then the generic element of $Act(G, 2^{\mathbb{N}})$ is hyperfinite.*

We now explain the condition in the theorem. Let G be a finitely generated group and let d_G be a proper left-invariant metric on G. The group G has *asymptotic dimension at most n* if for any $R > 0$ there exists a coloring c of the group by $n + 1$ colors such that the sizes of connected components of the graph on G given by $\{(g, h) : d_G(g, h) < R \text{ and } c(g) = c(h)\}$ are uniformly bounded. This notion is originally due to Gromov. For example, the group $\mathbb{Z}^n$ has asymptotic dimension n and finite rank free groups have asymptotic dimension 1. A group is *locally* of finite asymptotic dimension if every finitely generated subgroup of it has finite asymptotic dimension. From Theorem 1 we get the following corollary which answers a question from [2]:

Corollary 2. *The generic action of a free group on Cantor space is hyperfinite.*

The proof of Theorem 1 uses the connection between finite *Borel* asymptotic dimension and hyperfiniteness developed in [1]. Following Theorem 1, there are still many groups for which we do not know if their generic action is hyperfinite, even among the amenable groups. For example the following is open:

Question 3. *Is the generic action of $\mathbb{Z} \wr \mathbb{Z}$ on Cantor space hyperfinite?*

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All you need is HOD

Alejandro Poveda

A set X is *ordinal definable* (in symbols, $X \in \mathrm{OD}$) if there is a formula $\varphi(x, x_0, \ldots, x_n)$ of the language of set theory which, together with ordinals $\alpha_0, \ldots, \alpha_n$, defines X; namely, $X = \{x : \varphi(x, \alpha_0, \ldots, \alpha_n)\}$. Additionally, X is *Hereditarily Ordinal Definable* (in symbols, $X \in \mathrm{HOD}$) if $X \in \mathrm{OD}$ and the transitive closure of $\{X\}$. HOD is a transitive class containing all the ordinals and satisfying ZFC. In modern set-theoretic terminology, HOD is an *inner model* – in fact, it is the broadest of all inner models. Woodin's work on HOD [Woo10] concerns the question of whether HOD is close to resembling the mathematical universe. In the presence of strong enough large cardinals, his *HOD hypothesis* implies that the answer is yes: the two models closely resemble one another in terms of their large cardinal structure.

Definition ([Woo17])**.** *The* HOD *hypothesis states that there is a proper class of regular cardinals that are not ω-strongly measurable in* HOD*.*

Theorem (HOD Dichotomy, [Woo17])**.** *If δ is an extendible cardinal then exactly one of the following holds:*

(1) HOD *is a weak extender model for the supercompactness of δ.*
(2) Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD*.*

The HOD dichotomy has placed the foundations of mathematics at a critical crossroad. Namely, either there are reasonable prospects for completing Gödel's inner model program, or there is no fine-structural insight into the mathematical universe V. Therefore, which of these scenarios prevails? The HOD hypothesis implies that the prevailing scenario is the first one.

In this presentation we report on some recent results **joint with Gabe Goldberg** following this line of research. Our work focuses on two aspects; namely, the optimality of Woodin's HOD Dichotomy [GOP24] and the study of compactness phenomena in HOD [GP24].

On the optimality of the HOD dichotomy. Woodin showed that if δ is the first extendible cardinal and the HOD hypothesis holds then δ is supercompact in HOD. Also, under the same assumptions, he showed that if $\kappa > \delta$ is (say) extendible then it remains so in HOD. A natural question thus emerges – What can be said about the first extendible cardinal in HOD? More specifically, must the first extendible be extendible in HOD? The answer turns to be negative, as we show next:

Theorem 1 (Goldberg, P.)**.** *Assume that δ is an extendible cardinal. Then there is a generic extension where:*

(1) The HOD *hypothesis holds.*
(2) δ is the first extendible cardinal.
(3) HOD $\models$ "*δ is the least strongly compact cardinal*"*.*

By our comments above this result is best possible. Bearing on results from [Pov24], we also improve a theorem of Woodin by showing that the first extendible must be a "large" supercompact cardinal in HOD:

Theorem 2 (Goldberg, P.). *Assume the* HOD *hypothesis holds. Then, the first extendible cardinal is* $C^{(1)}$*-supercompact in* HOD.

Following up on this vein, we also investigate *cardinal correct extendible* cardinals (for short, cce) in HOD. This large cardinal notion was introduced in [GO24] in connection to Löwenheim–Skolem-type numbers of the equicardinality logic. Our main result on this respect is that, assuming a bit more than the HOD hypothesis, every extendible cardinal is cce:

Theorem 3 (Goldberg, Osinski, P.). *Assume* HOD *is cardinal correct. If* δ *is extendible then it is* cce *in* HOD.

Using this one can show that the first cce can be the first strongly compact. This answers a question from [GO24].

An immediate corollary of the HOD dichotomy theorem is that, if δ is extendible and the HOD hypothesis holds then HOD is a weak extender model for "δ is supercompact". As it turns, Woodin showed that the same conclusion can be gotten just from a HOD-supercompact. Could this be proven if δ is just supercompact? Yet again, the answer is no:

Theorem 4 (Goldberg, P.). *Assume* δ *is supercompact. Then, there is a generic extension where:*

(1) The HOD *Hypothesis holds.*
(2) δ *is a supercompact cardinal.*
(3) HOD $\models$ "δ *is strongly compact yet not* 2^δ*-supercompact*".

Incidentally, this answers a question by Cheng–Hamkins–Friedman [CFF15].

Starting from (much) weaker assumptions than extendibility, Goldberg proved the following variation of the HOD dichotomy theorem:

Theorem (Goldberg). *Assume that* δ *is supercompact. Then exactly one of the following hold:*

(1) If $\kappa \geq \delta$ *is singular,* κ *is singular in* HOD *and* $\kappa^{+\mathrm{HOD}} = \kappa^+$.
(2) All sufficiently large regular cardinals are ω*-strongly measurable in* HOD.

A natural question is whether one can improve even further the large cardinal assumptions needed to prove the HOD dichotomy. Building upon a previous work by Cummings–Friedman–Golshani [CFG15] we show that, yet again, the answer is no:

Theorem 5 (Goldberg-P.). *Assuming the* GCH *and the existence of a supercompact cardinal, there is a model of* ZFC *where the* HOD *hypothesis holds,* δ *is the first supercompact cardinal and there is a club* $D \subseteq \delta$ *of cardinals* κ *such that* $\kappa^{+\mathrm{HOD}} < \kappa^+$ *and* HOD $\models$ "κ *is regular*".

In the same paper, Cummings–Friedman–Golshani asked whether or not below the first supercompact cardinal δ, $\kappa^{+\mathrm{HOD}} < \kappa^+$ for all cardinals κ. We answer this is negatively, with the proviso that the HOD hypothesis holds:

Theorem 6 (Goldberg–P.). *Assume the* HOD *hypothesis. If δ is supercompact then there is a stationary set $S \subseteq \delta$ such that for each cardinal $\theta \in S$,* HOD *is a weak extender model for "θ is $\theta^{+\omega+1}$-supercompact". In particular,*

(1) $\theta^{+\omega+1} = (\theta^{+\omega+1})^{\mathrm{HOD}}$ and $\theta^{+\omega}$ is singular in HOD.
(2) Under the HOD *hypothesis, every supercompact cardinal is a limit of* HOD*-measurables.*

Compactness phenomena in HOD

Compactness is the phenomenon by which the local properties of a mathematical structure determine the nature of the structure itself. A famous compactness theorem in set theory, due to Silver, establishes that the *Singular Cardinal Hypothesis* cannot first fail at a singular cardinal of uncountable cofinality. Thus, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$ then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

With Goldberg [GP24], we prove the following:

Theorem 7 (Goldberg-P.). *Suppose that κ is a singular cardinal with $\mathrm{cf}(\kappa) \geq \omega_1$ and $\{\delta < \kappa \mid \mathrm{cf}^{\mathrm{HOD}}(\delta) < \delta\}$ is stationary. Then $\mathrm{cf}^{\mathrm{HOD}}(\kappa) < \kappa$.*

Theorem 8 (Goldberg-P.). *Suppose that κ is a singular strong limit cardinal with $\mathrm{cf}(\kappa) \geq \omega_1$ and $\{\delta < \kappa \mid \delta^{+\mathrm{HOD}} = \delta^+\}$ is stationary. Then $\kappa^{+\mathrm{HOD}} = \kappa^+$.*

Moreover, we can show that these results are provably optimal.

Theorem 8 seems relevant for a possible refutation of Woodin's HOD conjecture. For singular cardinals of countable cofinality, Poveda [Pov23] shows that starting from large cardinal hypothesis in the realm of I_2 one can force "local failures" of the HOD conjecture:

Theorem 9 (P.). *Suppose that $j\colon V \to M$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$, the critical sequence $\langle j^n(\kappa) \mid n < \omega\rangle \to \lambda$ consists of supercompact cardinals, $V_\lambda \subseteq M$ and there is an inaccessible cardinal $> \lambda$. Then, there is a model of* ZFC *where:*

(1) The HOD *hypothesis holds.*
(2) κ is $<\lambda$-extendible.
(3) λ is a strong limit cardinal with $\mathrm{cf}(\lambda) = \omega$.
(4) $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$.

The situation for uncountable cofinalities is pretty much open. In fact, in light of Theorem 8, an analogous configuration for singulars of uncountable cofinalities will be very close to refuting the HOD Conjecture. Thus, we ask:

Question 10 (Goldberg-P.). *Is the above configuration consistent (with ZFC) for a cardinal λ with $\mathrm{cf}(\lambda) = \omega_1$?*

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On Namba Forcing and Minimal Collapses

Maxwell Levine

One way to study the properties of the infinite cardinals is to examine the extent to which they can be changed by forcing. In 1969 and 1970, Bukovský and Namba independently showed that $\aleph_2$ can be forced to be an ordinal of cofinality $\aleph_0$ without collapsing $\aleph_1$. The forcings they used and their variants are now known as Namba forcing. Shelah proved that Namba forcing collapses $\aleph_3$ to an ordinal of cardinality $\aleph_1$. In a 1990 paper, Bukovský and Copláková showed that Namba forcing is $|\aleph_2^V| = \aleph_1$-minimal—meaning that it forces $\aleph_2^V$ to have cardinality $\aleph_1$, but there are no strictly intermediate models between the ground model and the generic extension collapsing $\aleph_2^V$ to $\aleph_1$ [1]. They also asked whether there can be an $|\aleph_2^V| = \aleph_1$-minimal extension preserving regularity of $\aleph_3^V$. We show up to the consistency of a measurable cardinal that there is such an extension.

In a slightly more general phrasing, we have the following:

Theorem 1. *Assume the consistency of a measurable cardinal μ. Then it is consistent that there is a model V in which there is a regular cardinal ν such that $\mu = \nu^{++}$ such that there is a also an extension $W \supset V$ that is $|\mu| = \nu^+$-minimal and which preserves regularity of μ^+.*

This theorem can be found in a recent preprint. (See [3]; one can also find updated versions of my preprints on my website.) The idea of the proof is to use a strengthening of precipitousness due to Laver that we denote $\mathsf{LIP}(\nu^{++}, \nu)$ [2]. Laver's property provides us with a dense subset D of $\nu^{++} \cap \mathrm{cof}(\nu^+)$ that has a lot of closure and a lot of flexibility for obtaining partitions. Then we use a Namba-style forcing of height ν^+ and Miller-style splitting in which the successors of splitting nodes are taken from from the dense set D provided by $\mathsf{LIP}(\nu^{++}, \nu)$. The crux is a sort of "sweeping argument" that seems to have a number of applications and does not seem to appear in the earlier literature in any overt form.

Similar arguments can be employed to obtain approximation-like properties for a variety of Namba-like forcings (see [4], [5, Section 2.1], [5, Section 4.1]), and it seems reasonable to expect more applications.

We conclude with some questions.

Question 2. *Does the conclusion of Theorem 1 require consistency of a measurable cardinal?*

It does seem that the answer is plausibly yes. If $\mu = \nu^{++}$ is not measurable in any inner model, then we would have a sequence of length μ^+ witnessing the non-saturation of $\mu \cap \mathrm{cof}(\nu^+)$. Then one may be inclined to use the ideas of the proof of Shelah that you cannot singularize μ without collapsing μ^+. However, it is not clear where the hypothesis of minimality would be used in such an argument.

There is also the question of the extent to which the main theorem can be stratified.

Question 3. *Assume* $\mathsf{LIP}(\lambda, \mu)$ *where* $\omega < \kappa < \lambda < \mu$ *are regular cardinals. Can we find a forcing like the one used in Theorem 1 that preserves cardinals* $\nu \leq \lambda$*?*

It should be noted that there are a number of open questions—even besides those highlighted by Bukovský and Copláková—regarding whether Namba-style forcings produce cardinal collapses and whether they embed certain forcing orders (i.e. at the end of [6]). Such questions are sensitive to the dimensions and splitting behavior of the Namba forcing in question.

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Ergodic theorems, weak mixing, and chaining

JENNA ZOMBACK
(joint work with Anush Tserunyan)

In general, an ergodic, probability-measure-preserving (pmp) action of a countable (discrete) group G on a standard probability space (X, μ) is said to have the *pointwise ergodic property* along a sequence (F_n) of finite subsets of G, if for every $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(g \cdot x) = \int f d\mu.$$

It is a celebrated theorem of Lindenstrauss [2] that the pointwise ergodic property is true for the pmp actions of all countable amenable groups along tempered Følner sequences.

The fact that the Følner sets F_n have small boundary relative to their size is essential for the pointwise ergodic property as it ensures that the limit of averages is an invariant function. Hence, to obtain versions of the pointwise ergodic theorem for nonamenable groups, e.g. for the nonabelian free groups $\mathbb{F}_r$, one has to imitate the almost invariance of finite sets by taking *weighted* averages instead, so that the weight of each sphere in $\mathbb{F}_r$ is equal to 1. The first instance of this was proven by Grigorchuk [1], and independently by Nevo and Stein [3]:

Theorem 1 (Grigorchuk 1987; Nevo–Stein 1994). *Let $r < \infty$ and let $\mathbb{F}_r \curvearrowright^\alpha (X, \mu)$ be a (not necessarily free) ergodic pmp action of the free group $\mathbb{F}_r$. For any $f \in L^1(X, \mu)$, for a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{w \in B_n} f(w \cdot x)\mathfrak{m}(w) = \int f(x)\, d\mu(x),$$

where B_n is the (closed) ball of radius n in the standard symmetric (left) Cayley graph of $\mathbb{F}_r$, $\mathfrak{m}_u$ is the uniform Markov measure on $\mathbb{F}_r$.

We strengthen the previous theorem in [4] by expanding the class of measures on the free group, and generalizing the sequences where we calculate the ergodic averages. Below, $\partial\mathbb{F}_r$ is the space of infinite, reduced words in the generators of $\mathbb{F}_r$, and the boundary action is by concatenation, with cancellation if necessary.

Theorem 2 (Tserunyan–Z. 2020). *Let $\mathbb{F}_r$ be the free group on $2 \leq r < \infty$ generators and $\mathfrak{m}$ be any stationary Markov measure on $\mathbb{F}_r$ such that the induced boundary action of $\mathbb{F}_r$ on $\partial\mathbb{F}_r$ is weakly mixing. Let (τ_n) be an arbitrary sequence of finite subtrees of the (left) Cayley graph of $\mathbb{F}_r$ containing the identity such that $\lim_n \mathfrak{m}(\tau_n) = \infty$. Then for every $f \in L^1(X, \mu)$, for a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\mathfrak{m}(\tau_n)} \sum_{w \in \tau_n} f(w \cdot x)\mathfrak{m}(w) = \int f(x)\, d\mu(x).$$

Of course, it is natural to ask which stationary Markov measures induce a weakly mixing boundary action, and in turn yield ergodic pmp actions that satisfy the pointwise ergodic property. Our main result is a complete characterization of such measures. It turns out that they are exactly the Markov measures arising from strictly irreducible transition matrices, a condition introduced by Bufetov in 2000 for a different purpose. The proof of this characterization goes through proving equivalences with a new combinatorial condition on the action that we call chaining:

We say that a group action $\Gamma \curvearrowright (X, \mu)$ is **k-chaining** if for any positive measure sets A and B, there are group elements γ_i $(i \leq k)$ so that $\gamma_i A \cap \gamma_{i+1} A$, $\gamma_0 A \cap A$, and $\gamma_k A \cap B$ have positive measure.

Theorem 3. *The following are equivalent for a stationary Markov measure on $\mathbb{F}_r$, $r < \infty$:*

(A) The boundary action is weakly mixing.
(B) The boundary action is $2r$-chaining.
(C) P is strictly irreducible.

Finally, we note that the condition of k-chaining is not specific to boundary actions of free groups, and can be applied to general group actions. Our proof of the previous theorem yields the following for general pmp group actions.

Theorem 4. *The following are equivalent for any Borel **pmp** action of a countable group $\Gamma \curvearrowright (X, \mu)$:*

(A) α is weakly mixing.
(B) α is chaining (i.e., k-chaining for some k).
(C) α is 1-chaining.

However, in the null-preserving setting, we do not know whether weak mixing is equivalent to chaining, nor whether 1-chaining is equivalent to chaining.

Question 5. *Is chaining equivalent to weak mixing for measure-class-preserving actions?*

Question 6. *Are there measure-class-preserving actions that are chaining but not 1-chaining?*

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On Friedman's Property

Hannes Jakob

For κ regular and $\lambda \leq \kappa$ a cardinal, let $F(\lambda, \kappa)$ state that any regressive function from κ into λ is constant on a closed set of ordertype ω_1. For $(D_i)_{i\in\omega_1}$ a partition of ω_1, let $F^+((D_i)_{i\in\omega_1}, \kappa)$ state that for any sequence $(A_i)_{i\in\omega_1}$ of stationary subsets of $\kappa \cap \mathrm{cof}(\omega)$ there is a normal function $f\colon \omega_1 \to \kappa$ such that for every $i \in \omega_1$, $f[D_i] \subseteq A_i$. $F(\lambda, \kappa)$ is a parametrized variant of the property $F(\kappa)$ which was introduced by H. Friedman in [1], while $F^+((D_i)_{i\in\omega_1}, \kappa)$ was previously studied by Foreman-Magidor-Shelah [2], Feng-Jech [3] and Fuchs [4].

The strongest possible form of $F^+((D_i)_{i\in\omega_1}, \kappa)$, where $D_i = \{i\}$, follows from the Strong Reflection Property, a consequence of Martin's Maximum. The other extreme, the failure of $F(2, \kappa)$ for every regular $\kappa \geq \omega_2$, can be forced simply by collapsing ω_1 to ω using finite conditions, as has been observed by Silver (see again [1]).

In this talk, we introduce posets which add witnesses to the failure of any such property by forcing with initial segments as conditions. These orders force F and F^+ to fail in a more gentle way and allow the lifting of ground-model embeddings and thus the preservation of large cardinals. Additionally, forcing with these posets over the standard model of Martin's Maximum yields the consistency of maximal versions of MM which are compatible with some failure of F or F^+ and will also be introduced in the talk.

The most prominent usage of these forcing axioms is the separation of instances of F and F^+: We show that for any cardinal $\lambda \leq \omega_2$, it is consistent that $F(\lambda, \omega_2)$ fails but $F(\lambda', \omega_2)$ holds for any $\lambda' < \lambda$, so it is e.g. consistent for any $n \in \omega$, $n \geq 1$ that every partition of ω_2 into n pieces must have one piece which includes a closed set of ordertype ω_1 while there is a partition of ω_2 into $n+1$ pieces where no piece includes a closed set of ordertype ω_1. For F^+, we show that implications between instances of $F^+(\overline{D}, \omega_2)$ and $F^+(\overline{E}, \omega_2)$ are almost perfectly characterized by the relation $\overline{D} \leq^* \overline{E}$ stating that $\overline{D}$ refines $\overline{E}$ on a club subset of ω_1. We show that whenever $\overline{D}$ and $\overline{E}$ are partitions of ω_1 in the standard model of MM and $\overline{D} \not\leq^* \overline{E}$, there is a forcing extension where $F^+(\overline{D}, \omega_2)$ holds but $F^+(\overline{E}, \omega_2)$ fails. This additionally resolves a question of Fuchs from [4] in a very strong manner.

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Recent Results in the Combinatorics of Determinacy Models

Stephen Jackson

We present some joint work with William Chan and Nam Trang concerning the combinatorics of determinacy models. A common theme amongst several of these results is a result/conjecture which we call the "$ABCD$ conjecture." This describes the relations between the possible cardinalities of the form α^β where α, β are ordinals (enough to consider cardinals) below Θ (the supremum of the lengths of the prewellorderings of $\mathbb{R}$). Specifically the conjecture is the following. Here we write $|X| \leq |Y|$ for sets X, Y to mean there if an injection from X into Y.

Conjecture ($ABCD$ conjecture)**.** *Assume* $\mathsf{ZF} + \mathsf{AD}$*. Suppose* $\alpha, \beta, \gamma, \delta < \Theta$ *are ordinals and* $\beta \leq \alpha$*,* $\delta \leq \gamma$*. Then* $|\alpha^\beta| \leq |\gamma^\delta|$ *iff* $|\beta| \leq |\delta|$ *and* $|\alpha| \leq |\gamma|$*.*

In full generality, that is, just assuming $\mathsf{ZF} + \mathsf{AD}$, the conjecture is still open, although special cases have been established just from this hypothesis (which we mention below). However, assuming Woodin's axiom AD^+ (a strengthening of AD, although it is not known to be strictly stronger, and in all known natural models of $\mathsf{ZF} + \mathsf{AD}$ we have that AD^+ holds; in particular in all derived models of AD from large cardinals AD^+ holds) Chan has recently shown the conjecture holds:

Theorem (Chan)*.* Assume AD^+. Then the $ABCD$ conjecture holds.

Recently another, somewhat simpler, proof of the $ABCD$ conjecture from AD^+ has been found and which makes use of a result about generalized ∞-Borel sets which may be of independent interest. To state this, first recall that part of the axioms AD^+ asserts that every set of reals $A \subseteq \omega^omega$ is ∞-Borel. This means that there is a wellfounded tree on $\lambda \in \mathrm{On}$ whose terminal nodes are attached codes for basic open sets in ω^ω, and a non-terminal node is assigned the union or intersection of the sets corresponding to nodes below it according to some convention. A theorem of Woodin says that this is equivalent to having a set of ordinals S such that $x \in A$ iff $L[S, x] \models \varphi(S, x)$ for some formula φ.

We can generalize the notion of an ∞-Borel code to sets $A \subseteq \mathcal{P}(\kappa)$ for $\kappa > \omega$. An analog of Woodin's theorem applies, and we can take either form of the definition of A being ∞-Borel. Woodin showed that even for $\kappa = \omega_1$ that not every $A \subseteq \mathcal{P}(\kappa)$ is ∞-Borel, so we cannot extend AD^+ to all $A \subseteq \mathcal{P}(\kappa)$ for $\kappa > \omega$. However, we can show that certain sets $A \subseteq \mathcal{P}(\kappa)$ are ∞-Borel. Namely, define a topology τ on $\mathcal{P}(\kappa)$ as follows. Given $f \subseteq \kappa$ (or $f\colon \kappa \to 2$) and a countable set $\sigma \subseteq \kappa$, a τ basic open set about f is a set is $N_\sigma(f) = \{g\colon \kappa \to 2\colon \forall \alpha \in \sigma\ f(\alpha) = g(\alpha)\}$. We then have the following.

Theorem. Assume AD^+ and let $\kappa < \Theta$. If $A \subseteq \mathcal{P}(\kappa)$ is Borel in the τ-topology, then A is ∞-Borel.

The current authors used this result about generalized ∞-Borel sets to give a fairly simple proof of the $ABCD$ conjecture.

We note that the $ABCD$ conjecture completely describes the relations between cardinals of the form α^β, where $\alpha, \beta < \Theta$. The statement of the conjecture requires

$\beta \leq \alpha$ and $\delta \leq \gamma$ but the other other cases easily follow from AD^+ or can be reduced to the stated case.

There are special cases of the *ABCD* conjecture that follow just from AD, and in fact follow just from partition arguments. The authors have shown various combinatorial properties that hold at partition cardinals, and these arguments can be used to establish instances of the *ABCD* conjecture (the partition properties follow from just AD).

The partition relation $\kappa \to (\kappa)^\kappa$ defines a measure on the set of functions $f\colon \kappa \to \kappa$ of the correct type (increasing, discontinuous, and of uniform cofinality ω). Likewise, the relation $\kappa \to (\kappa)^\epsilon$ gives a measure on the functions $f\colon \epsilon \to \kappa$ of the correct type. There are a number of interesting open problems about these functions space measures (for example computing $j_\nu(\omega_2)$ where ν is the function space measure on $\omega_1^{\omega_1}$; Chan has shown that $j_\nu(\omega_1) < \boldsymbol{\delta}^1_3$). We have the following monotonicity results for functions from these function spaces into the ordinals.

Theorem. Suppose $\kappa \to (\kappa)^\kappa$. Let $\Phi\colon \kappa^\kappa \to \mathrm{On}$. Then there is a measure one set on which Φ is monotonically increasing. That is, there is a c.u.b. $C \subseteq \kappa$ such that if $f, g\colon \kappa \to C$ are of the correct type and $f(\alpha) \leq g(\alpha)$ for all $\alpha < \kappa$, then $\Phi(f) \leq \Phi(g)$.

A similar theorem holds for the function space κ^ϵ assuming the corresponding partition relation.

Another result which is similar to the above theorem, and is proved using similar techniques, is the following.

Theorem. Suppose $\kappa \to \kappa^{\epsilon\cdot\epsilon}$ where $\epsilon < \kappa$ is additively indecomposable and $\mathrm{cof}(\epsilon) = \omega$. Suppose $\Phi\colon [\kappa]^\epsilon_* \to \mathrm{On}$. Then there is a club $C \subseteq \kappa$ and a $\delta < \kappa$ such that for all $f, g \in [\kappa]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

Thus, almost everywhere a function $\Phi(f)$ can only depend on a bounded part of f and $\sup(f)$.

The previous theorem is related to the *ABCD* conjecture, and gives instances of it. Specifically, we have the following corollary.

Corollary. *Suppose κ has the weak partition property, that is, $\forall \epsilon < \kappa\ \kappa \to (\kappa)^\epsilon$. Then for all $\delta < \kappa$, $\kappa^{<\kappa}$ does not inject into On^δ.*

The proof of the Corollary shows that if $\delta_1 < \delta_2 < \kappa$ with δ_2 regular, then κ^{δ_2} does not inject into On^{δ_1}. This gives an instance of the *ABCD* conjecture just from partition properties of κ.

Infinite Structural Ramsey Theory: A progress report

Natasha Dobrinen

The simplest form of the infinite Ramsey theorem states that, given any coloring of all pairs of natural numbers into two colors, there is an infinite subset of natural numbers in which all pairs have the same color. When moving from sets to structures, some suprising pheomena occur: For example, there is a coloring of pairs of rational numbers into two colors such that both colors persist in any subset of the rationals forming a dense linear order (Sierpiński 1933); likewise, for edge colorings in the Rado graph (Erdős–Hajnal–Posá 1975). These bounds were later shown to be optimal by Galvin (unpublished) and by Pouzet–Sauer (1996), respectively. The study of optimal bounds for colorings of copies (or embeddings) of a finite substructure inside an infinite structure is the subect of big Ramsey degrees. Optimal bounds are connected with optimal structural expansions which produce analogues of the infinite Ramsey theorem.

An infinite structure $\mathbf{S}$ is said to have *finite big Ramsey degrees* iff for each finite substructure $\mathbf{A}$ of $\mathbf{S}$, there is some positive number n such that for any coloring of the copies of $\mathbf{A}$ in $\mathbf{S}$ into finitely many colors, there is a subcopy $\mathbf{S}'$ of $\mathbf{S}$ in which the copies of $\mathbf{A}$ take no more than n colors. The least such n is called the *big Ramsey degree* of $\mathbf{A}$ in $\mathbf{S}$, denoted by $\mathrm{BRD}(\mathbf{A}, \mathbf{S})$ or just $\mathrm{BRD}(\mathbf{A})$, the terminology being coined in (Kechris–Pestov–Todorcevic 2005). Often $\mathbf{S}$ is taken to be the Fraïssé limit of Fraïssé class that has a Ramsey expansion (i.e., small Ramsey degrees).

The rationals were the first structure for which a full characterization of big Ramsey degrees was accomplished. Laver (unpublished) proved that the rationals as a dense linear order have finite big Ramsey degrees. D. Devlin characterized the degrees in his 1979 PhD thesis. The big Ramsey degrees of the Rado graph were characterized in a series of two papers by Sauer (2006) and Laflamme–Sauer–Vuksanovic (2006). In fact, they characterized the big Ramsey degrees for free amalagmation homogeneous structures with finitely many binary relations and no forbidden substructures of size greater than two, which includes the homogeneous digraph and random graphs with finitely many different edge relations. Their proof that the big Ramsey degrees are finite utilized Milliken's Theorem. While this method works well for so-called unrestricted structures, it does not work when there are forbidden substructures of size greater than the arity of the relations in it, for instance K_n-free graphs where $n \geq 3$.

In order to prove that all finite triangle-free graphs have finite big Ramsey degrees, Dobrinen devised the notion of coding tree of 1-types induced by a well-ordered homogeneous structure and developed forcings on them in order to prove new Ramsey theorems for colorings of particular level sets of the trees. In turn, she used those theorems to prove that the triangle-free and then all k-clique-free homogeneous graphs have finite big Ramsey degrees. These new methods led to a rapid expansion of new results on big Ramsey degrees, some of which are listed below in chronological order of their arxiv dates with journal citations where available.

- 2017. Triangle-free Henson graphs: Upper Bounds. Dobrinen, JML 2020.
- 2018. Certain homogeneous metric spaces: Upper Bounds. Mašulović, J. Combin. Theory Ser. A, 2020.
- 2019. k-clique-free Henson graphs: Upper Bounds. Dobrinen, JML 2023.
- 2019. Big Ramsey degrees for countable ordinals as linear orders. Mavsulović and Šobot, Combinatorica 2021.

- 2019. 3-uniform hypergraphs: Upper Bounds. Balko, Chodounský, Hubička, Konečný, and Vena, Combinatorica 2022.
- 2020. SDAP$^+$ implies Exact BRD characterized by diagonal antichains. Coulson, Dobrinen, and Patel, 2 papers submitted.
- 2020. Binary rel. Forb($\mathcal{F}$), finite $\mathcal{F}$: Upper Bounds. Zucker, Adv. Math., 2022.
- 2020. Circular directed graphs: Exact BRD Computed. Dasilva Barbosa, submitted.
- 2020. Homogeneous partial order: Upper Bounds. Hubička, submitted.
- 2021. Homogeneous partial order: Exact BRD. Balko, Chodoun- ský, Dobrinen, Hubička, Konečný, Vena, Zucker, Trans. AMS, to appear.
- 2021. Homogenous graphs with forbidden cycles and metric spaces: Upper Bounds. Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena, submitted.
- 2021. Binary rel. Forb($\mathcal{F}$), finite $\mathcal{F}$: Exact BRD. Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena, Zucker, JEMS, to appear.
- 2023. Big Ramsey degrees for unrestricted relations on possibly infinite languages. Braunfeld, Chodounský, de Rouncourt, Hubička, Kawach, Konečný, submitted.
- 2023. Type-respecting amalgamation and big Ramsey degrees. Aranda, Braunfeld, Chodounský, Hubička, Konečný, Nešetřil, Zucker, 2023 EuroComb extended abstract.
- 2023. Big Ramsey degrees in the metric space setting. Bice, de Rancourt, Hubička, Konečný, 2023 EuroComb extended abstract.
- 2023. A new method for generating topological Ramsey spaces with a theorem which recovers the Carlson-Simpson and Milliken Tree Theorems as corollaries. Gives upper bounds for finite big Ramsey degrees for many structures. Balko, Chodounský, Dobrinen, Hubička, Konečný, Nešetřil, Vena, Zucker, arxiv preprint.

Work on big Ramsey degrees is ongoing.

In very recent work with Chodounský, Eskew, and Weinert, we showed that the two-branching countable pseudotree has finite big Ramsey degrees for finite chains, but antichains even of size two have infinite big Ramsey degrees. This is the first example of a homogeneous structure where some finite substructures have finite big Ramsey degrees while others do not.

Answering questions of Dobrinen, computability theorists have started working on big Ramsey degree questions from the standpoint of how many Turing jumps are needed to obtain the results. This has implications for reverse mathematics as well. The first body of work in this area turned into an AMS Memoirs book by Angles d'Auriac, Cholak, Dzafarov, Monin, and Patey, 2023. Work in this area continues.

Infinite-dimensional structural Ramsey theory came into focus by a question in Kechris–Pestov–Todorcevic 2005. The general question is whether, given a homogeneous structure **H**, there is some topology on the set of its subcopies so that an analogue of the Galvin–Prikry or Ellentuck theorems on $[\omega]^\omega$ hold on the space of the subcopies **H**. The first work on this was due to Dobrinen in 2019 on the Rado graph, and will appear in the Prague Ramsey Theory DocCourse special issue. This use the coding trees and forcing to produce a Galvin-Prikry style theorem.

At Oberwolfach in 2022, I gave a talk on preliminary results on infinite-dimensional Ramsey theory of homogeneous structures satisfying the Substructure Disjoint Amalgamation Property$^+$. This work is now finished and is submitted. The SDAP^+ is a property that holds for the rationals, $\mathbb{Q}_\mathbb{Q}$, k-partite graphs, the generic tournament, and many other structures. However, it does not hold in the k-clique free Henson graps. Building on this and other prior work, Dobrinen and Zucker proved infinite-dimensional Ramsey theorems for all free amalgamation classes with finitely many relations of arity at most two, with finitely many forbidden finite irreducible substructures. An unexpected outcome of this work was the proof that so-called A.3(2) ideals suffice in place of the stronger axiom A.3(2) to obtain the conclusion of Todorcevic's Abstract Ramsey Space Theorem. It is not known if this reduced axiom is necessary or whether it can be further weakened. The following papers develop infinite-dimensional Ramsey theorems, the latter two of which also directly recover the known exact big Ramsey degree theorems.

- 2019. Infinite-dimensional Ramsey theory for Borel sets of Rado graphs. Dobrinen, Prague Ramsey Theory DocCourse. To appear.
- 2022. ∞-dimensional Ramsey theory of homogeneous structures with SDAP^+. (recovers exact BRD). Dobrinen, submitted.
- 2023 ∞-dimensional Ramsey theory for binary relational Flim(Forb($\mathcal{F}$)), finite $\mathcal{F}$. (recovers exact BRD). Dobrinen, Zucker, submitted.

Combinatorics of MAD families

Michael Hrušák

(joint work with J. Brendle, O. Guzmán, D. Raghavan)

An infinite family $\mathcal{A}$ of infinite subsets of a countable set ω is *almost disjoint* if any two distinct elements of $\mathcal{A}$ have finite intersection. $\mathcal{A}$ is *maximal almost disjoint (MAD)* if it is maximal under inclusion or, equivalently, if for every infinite subset X of ω there is an $A \in \mathcal{A}$ such that $X \cap A$ is inifinite.

Combinatorial properties of maximal almost disjoint (MAD) families are introduced and studied here. The initial motivation for this research is the following problem (still unsolved) attributed to Judy Roitman:

Problem. (Roitman) Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

and the closely related problem:

Problem. For every MAD family, is there a proper forcing that destroys it and does not add unbounded reals?

In this way, we are naturally led to the study of the indestructibility of MAD families. Once again, there are two very relevant open questions, due to Steprāns and the third author, which are the second motivation for this work:

Problem. (Steprāns) Is there a MAD family that is Cohen indestructible?

Problem Is there a MAD family that is Sacks indestructible?

Having these questions in mind, we decided to find the strongest notion of indestructibility for maximal almost disjoint families that we could imagine. This notion is what we call Shelah-Steprāns.

Definition. A MAD family $\mathcal{A}$ is *Shelah-Steprāns* if for every sequence X of non-empty finite subsets of ω there is an element of the ideal $\mathcal{I}(\mathcal{A})$ which either intersects every element of X or contains infinitely many elements of X.

This notion is strong enough to imply the indestructibility of the MAD family by Cohen, Sacks, Miller, random forcings, and much more. Nevertheless, we prove that Shelah-Steprāns MAD families can be destroyed without adding dominating or unsplit reals.

We also study properties of the MAD family generically added by countable approximations.

We would like to know if there is a combinatorial property that characterizes the generic MAD families over $\mathsf{L}(\mathbb{R})$. We note that generic MAD families are Shelah-Steprāns, in fact, they possess a substantial strengthening of this property we call raving.

We consider non-existence of certain types of MAD families in specific models, existence of such families under additional axioms, as well as examples for non-implications between various properties.

As an application we study topological properties of the weak* duals of the *Johnson–Lindenstrauss* Banach spaces naturally associated to the almost disjoint families. In particular, we show that $\mathcal{Z}$-*MAD* families produce Johnson–Lindenstrauss spaces whose duals are not *Efremov*, i.e. there are convex sets whose weak* closures cannot be reached by converegent sequences. This last part is a joint work with Luis David Reyes Saenz.

Bowen's Problem 32 and the conjugacy problem for systems with specification

Marcin Sabok

(joint work with Konrad Deka, Dominik Kwietnia, Bo Peng)

The methods of mathematical logic can be useful in establishing impossibility results. For example, a descriptive set-theoretic complexity argument was used by Wojtaszczyk and Bourgain who solved Problem 49 from the Scottish book by establishing the non-existence of certain types of Banach space. In a similar vein, the theory of complexity of Borel equivalence relations can be used to demonstrate

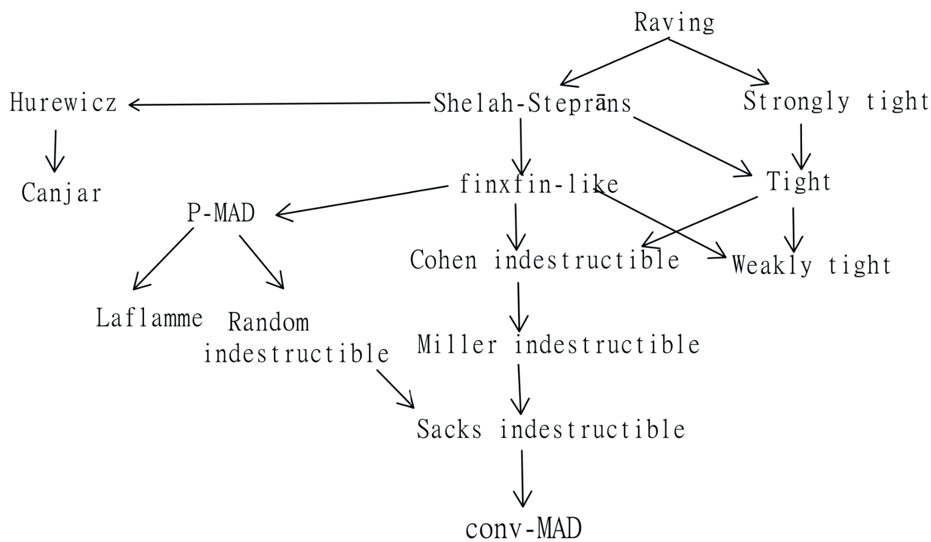

Figure 1. Implications for MAD families

the impossibility of classifying certain mathematical objects. An equivalence relation E on a standard Borel space X is **smooth** if there exists a Borel assignment $f\colon X \to Y$ of elements of another standard Borel space Y to elements of X which provides a complete classification of the equivalence relation E, i.e., two elements $x, y \in X$ are E-related if and only if $f(x) = f(y)$. The definition is broad enough to justify a Borel version of the Church–Turing thesis, namely that an isomorphism relation admits a concrete classification if and only if it is smooth.

Smooth equivalence relations are actually only at the beginning of a larger hierachy of descriptive set-theoretic complexity. Given two equivalence relations E and F on standard Borel spaces X and Y, respectively, we say that E is **Borel-reducible** to F, written $E \leq_B F$ if there exists a Borel map $f\colon X \to Y$ such that for $(x_1, x_2) \in X \times X$ we have $(x_1, x_2) \in E$ if and only if $(f(x_1), f(x_2))) \in F$. The Borel complexity of an isomorphism problem measures how complicated the problem is in comparison with other equivalence relations, when we compare equivalence relations using Borel reductions. Equivalence relations which are Borel-reducible to the equality $=$ on the real numbers, or anything that can be coded by real numbers, are exactly the smooth ones. However, the hierarchy goes much higher. The next step is formed by the **hyperfinite** equivalence relations, that is those which are induced by Borel actions of the group $\mathbb{Z}$. Another successor of $=$ is defined in terms of the Friedman–Stanley version of the Turing **jump**, and is denoted by $=^+$. The jump can be iterated and all of the countable iteration of $^+$ on $=$ are induced by Borel actions of the group S_∞ of permutations of $\mathbb{N}$. The class of equivalence relations reducible to actions of S_∞ is quite large (cf. the recent result of Paolini and Shelah [56]) but not every Borel equivalence relation is in that class. Hjorth developed the theory of turbulence and showed that turbulent group actions are not Borel reducible to any Borel S_∞-action. The theory of complexity of equivalence relations also continues in the class of equivalence relations induced by actions of more general groups than S_∞ (cf. [29, 30, 34, 35]). A lot of effort has been put in measuring the complexity of problems arising in dynamical systems, where the classification problems and their complexity are understood in the same way as in descriptive set theory. The breakthrough for measure-preserving systems came with the results of Foreman and Weiss [38] and of Foreman, Rudolph and Weiss [37]. In [38] Foreman and Weiss showed that the conjugacy relation of ergodic transformations is turbulent and in [37] Foreman, Rudolph and Weiss showed that the conjugacy relation of ergodic transformations is not Borel, and thus the classification problem in ergodic theory is intractable. In topological dynamics, Camerlo and Gao proved that the conjugacy of Cantor systems is the most complicated among equivalence relation induced by actions of S_∞ and for minimal Cantor systems it is shown in [46] that the relation is not Borel.

In contrast, the conjugacy relation of symbolic systems is quite simple from the point of view of descriptive set theory. Since every isomorphism between symbolic systems is given by a block code, the conjugacy of symbolic systems is a **countable Borel equivalence relation**, i.e. a Borel equivalence relation whose equivalence classes are countable. Clemens proved that the topological conjugacy of symbolic subshifts is a universal countable Borel equivalence relation. Gao, Jackson and Seward generalized it to G-subshifts any countable group G which is not locally finite, while for a locally finite group G they showed that the conjugacy of symbolic G-subshifts is hyperfinite. In case of minimal systems, Gao, Jackson and Seward proved that the conjugacy relation of minimal symbolic G-subshifts is not smooth for any countable infinite group G, and Thomas showed that the conjugacy relation of Toeplitz symbolic subshifts is not smooth. It remains unknown whether the conjugacy of minimal symbolic subshifts is hyperfinite, which is connected to a conjecture of Thomas on the isomorphism of complete groups. In fact, it is not known [55, Question 1.3] whether the conjugacy relation restricted to Toeplitz systems is hyperfinite or not.

A special class of symbolic systems is formed by systems with **specification**, considered by Bowen [1]. A dynamical system satisfies the specification property if for every $\varepsilon > 0$ we can find $k \in \mathbb{N}$ such that given any collection of finite fragments of orbits, there exists a point which is ε-closely following these orbit segments and takes k steps to switch between consecutive orbit segments. Nowadays, the specification property in symbolic systems for general discrete groups goes also under the name of **strong irreducibility**. Around 1970's Bowen wrote an influential list of open problems, which is now maintained on the webpage [32]. Problem 32 [31] asks to

classify symbolic systems with specification.

Several results in the direction of a classification have been obtained, as reported on the website [31]. For example Bertrand proved that every symbolic system with specification is synchronized and Thomsen found a connection with the theory of countable state Markov chains. However, a complete classification has not been obtained. Buhanan and Kwapisz proved a result suggesting that systems with specification are quite complicated. They considered the cocyclic shift spaces, which is a countable family of symbolic systems with specification and proved that the problem of equality of cocyclic shift spaces is undecidable. However, the equality relation is a much simpler relation than the conjugacy on such systems. In in talk we discuss the following result, which shows that a classification with any concrete invariants is impossible.

Theorem 1. *The conjugacy relation of symbolic systems with the specification property is not smooth.*

In fact,we prove that the conjugacy relation of symbolic systems with specification is not hyperfinite and essentially the same proof shows that it is not treeable.

Next, we look the conjugacy relation of pointed systems with the specification property and consider its complexity. It turns out that in order to compute the complexity for pointed Cantor systems with the specification property, we need to solve a problem posed in a paper of Ding and Gu [45].

In [45] Ding and Gu consider the equivalence relation E_{cs} defined on the space of metrics on $\mathbb{N}$, where two metrics are equivalent if the identity map on $\mathbb{N}$ extends to a homeomorphism of the completions of $\mathbb{N}$ with respect to those two metrics. The restriction of E_{cs} to the set of metrics whose completion is compact is denoted by E_{csc}. This is a natural equivalence relation from the point of view of descriptive set theory, and it is interesting to ask what is its complexity. Indeed, Ding and Gu ask [45, Question 4.11] whether for a given countable ordinal α and a natural number n, the restriction of E_{csc} to the metrics whose completion is homeomorphic to $\omega^{1+\alpha} \cdot n + 1$ is Borel-reducible to $=^+$. Even though this question does not seem directly connected to the topological conjugacy of systems with specification, we find a connection between the relation E_{csc} and Cantor systems, using a construction coming from the work of Williams and the work of Kaya and we answer it in the positive, by showing a slightly stronger statement. By $\mathbb{X}_{\text{0-dim}}$ we denote the set of metrics on $\mathbb{N}$ whose completion is zero-dimensional. The result below implies in particular a positive answer to [45, Question 4.11].

Theorem 2. *The relation E_{csc} restricted to $\mathbb{X}_{0\text{-}dim}$ is Borel bi-reducible with $=^+$.*

Finally, in [44] Bruin and Vejnar also studied the conjugacy relation of pointed transitive systems and asked [44, Table 1, Question 5.6] about the complexity of the conjugacy of pointed transitive homeomorphisms of the Hilbert cube. It turns out that the conjugacy of pointed transitive homeomorphisms of the Hilbert cube

	homeomorphisms	pointed transitive homeomorphisms
interval	Borel-complete	$\emptyset$
circle	Borel-complete	$=$
Cantor set	Borel-complete	$=^+$
Hilbert cube	complete orbit e.r.	?

TABLE 1. Source: [44, Table 1].

has the same complexity as the conjugacy relation of Hilbert cube systems with the specification property and in this paper we answer this question as follows.

Theorem 3. *The conjugacy relation of pointed transitive Hilbert cube systems is Borel bireducible with a turbulent group action.*

In particular, the conjugacy of pointed transitive Hilbert cube systems is not classifiable by countable structures, as opposed to most other relations considered in [44].

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