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Mini-Workshop: New Directions in Correlated Quantum Systems

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Abstract. The mathematical study of quantum many-body physics has seen significant breakthroughs in the past 15–20 years. Mathematical physicists have approached these systems from different perspectives with different methods. The mini-workshop aimed to connect two rapidly growing fields: the mathematical analysis of many-body systems in scaling regimes (which has led to precise asymptotic formulas of the ground state energy and time evolution, e.g., of Bose gases) and the study of strongly correlated quantum systems via Lieb-Robinson bounds (which provide fundamental insights into the spreading of quantum information).

Mathematics Subject Classification (2020): 82C10, 82C26, 82C22.

Introduction by the Organizers

The mini-workshop *New Directions in Correlated Quantum Systems* was organized by Emanuela L. Giacomelli (Milan), Jinyeop Lee (Basel) and Marius Lemm (Tübingen) and brought together 16 participants (including the organizers and one remote attendee). The group consisted of experts of different academic seniority and was well-balanced in various dimension, including gender, geographical representation, and scientific background. The workshop focused on both the dynamical and static properties of quantum many-body systems, with particular emphasis on the role of correlations. The aim was to foster a dialogue between researchers from two different areas of mathematical physics, both of which have recently seen significant progress:

(I) The mathematical analysis of quantum many-body systems in scaling regimes, which has led to precise asymptotic expansions of ground state energy and time evolution, in particular through higher-order Bogoliubov theory.
(II) The study of strongly correlated quantum systems using techniques from quantum information theory. A key tool in this area is the Lieb-Robinson bound, which provides fundamental insights into the propagation of quantum correlations.

The primary goal of the workshop was to learn about the key ideas of the respective other field. The secondary goal was to explore novel connections and to generate new research ideas.

These goals were reflected in the program. The workshop included four one-hour tutorials, nine one-hour talks, several open discussion sessions, an open problem session, and a wrap-up session. Throughout the week, all participants gave a talk, focusing on conveying the central conceptual and methodological points, especially to the participants from the further away area. Key themes included particle propagation and Lieb-Robinson bounds, effective dynamics of quantum system, positive temperature systems, and ground state energy expansions.

To ensure a shared basis of understanding, Monday consisted of introductory and advanced tutorials across these areas. Experts Andrew Lucas and Tomotaka Kuwahara presented on bosonic Lieb-Robinson bounds, while Arnaud Triay and Marcello Porta covered the derivation of effective evolution equations for many-body quantum systems.

Following the tutorial sessions, Tan Van Vu provided insights into the maximal speed of particle transport in closed and open quantum systems, while Carla Rubiliani presented recent progress on propagation bounds for lattice bosons under long-range interactions. For positive-temperature systems, Robert Seiringer discussed results on the Heisenberg ferromagnet which linked spin systems to dilute Bose gases. Phan Thấn Nam presented recent advances in the description of the Gibbs state of the mean-field Bose gas. Lieb-Robinson bounds were examined from different perspectives: Oliver Siebert addressed their formulation in the continuum, while Tom Wessel discussed their application to study equilibrium physics in lattice models. Bridging the two areas of research, Jingxuan Zhang presented on local enhancement of the mean-field approximation for bosons, while Simone Rademacher presented an asymptotic formula for the out-of-time ordered correlators for mean-field bosons. The ground state energy properties of quantum systems were also explored, with Christian Hainzl discussing recent advances in understanding the low-density Fermi gas.

On Wednesday morning, the open problem session provided a platform for in-depth discussions on cutting-edge methodologies and future research directions. The session was structured in two parts: first, participants formed small groups – each with a balanced mix of senior and junior researchers – to explore new ideas regarding specific topics. The topical groups were called *Bose-Hubbard type Hamiltonians*, *dynamical bounds for continuum systems*, and *positive temperature* *systems.* After about two hours of discussion, in the second part, the different groups came together again, to share their insights, which led to further comments from other participants and a productive discussion.

The mini-workshop concluded with a wrap-up session, in which participants' talks were located in a Venn diagram of the two fields. Connections that were found during the week were drawn into the diagram. Afterwards, a list of interesting open problems was compiled.

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Mini-Workshop: New Directions in Correlated Quantum Systems

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Abstracts

Introduction to the Lieb-Robinson Theorem

Andrew Lucas

In special relativity, the speed of light c sets a fundamental limit on how quickly information can be sent in any physical system. But many physical systems have emergent speed limits on information transfer; for example, sound waves propagating through air travel at $\sim 10^{-6}c$. The Lieb-Robinson Theorem, first proved in 1972 [3], demonstrates that many-body quantum lattice models can also have such emergent speed limits.

I presented a tutorial on the Lieb-Robinson Theorem, beginning by motivating how the transfer of quantum information and correlations is quantified through the operator norm of the commutator of two local operators: $\|[A_x(t), B_y]\|$, where A_x and B_y act locally on degrees of freedom at lattice site x and y respectively, and time-evolution is generated by the Heisenberg equation of motion

$$\dot{A}_x(t) = \mathrm{i}[H(t), A_x(t)].$$

A typical Hamiltonian of interest takes the form

$$H = \sum_{x\sim y} H_{xy}(t)$$

where H_{xy} acts on the degrees of freedom at two sites $x \sim y$ which are adjacent on a suitable interaction graph. Following [1], and assuming H is time-independent, I then showed how to expand $[A_x(t), B_y]$ as a Taylor series in t, using a generalization of the Duhamel identity that keeps track of the "irreducible path" between the points x and y. The resulting Lieb-Robinson Theorem took the form

$$\frac{\|[A_x(t), B_y]\|}{2\|A_x\|\|B_y\|} \le \sum_{l=0}^{\infty} \frac{(2t)^l}{l!} \sum_{\text{self-avoiding } x \to v_1 \cdots v_{l-1} \to y} \prod_{j=1}^{l} \|H_{v_{j-1}v_j}\|$$

where $x = v_0$ and $y = v_l$ in a path of length l. Emphasis in the derivation was placed on how unitarity of quantum mechanics is crucial in the bound; this is not a purely combinatorial result.

Much of the subsequent workshop was then dedicated to a better understanding of how these Lieb-Robinson bounds can and have been generalized to other problems [2]. We also discussed some more fundamental prospects, including Lieb-Robinson bounds which are tailored to specific states, or which use specific features of a state, such as its quantum metastability [4].

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Advanced Tutorial on the bosonic Lieb-Robinson bound

Tomotaka Kuwahara

(joint work with Tan Van Vu, Keiji Saito)

The Lieb-Robinson bound provides a fundamental constraint on the speed of information propagation in quantum many-body systems, playing a crucial role in understanding entanglement structure, quantum simulations, and computational complexity. While the original Lieb-Robinson bound is well-established for systems with short-range interactions and finite local energy, its extension to bosonic systems presents significant theoretical challenges due to the possibility of unbounded local energy [1, 2, 3, 4, 5, 6].

In this tutorial, we review the historical development and key results of the Lieb-Robinson bound [5], highlighting its further generalization and remaining open problems. We discuss the breakdown of conventional techniques in bosonic systems and present recent advances addressing this issue. In particular, we outline the optimal light cone for Bose-Hubbard-type models and the corresponding constraints on information propagation.

The lecture covers the following main topics:

(1) **Motivation and Background**: The role of the Lieb-Robinson bound in quantum many-body physics and its relevance in practical quantum simulations.

(2) **Challenges in Interacting Bosonic Systems**: The impact of unbounded local energy on conventional proofs and the necessity of alternative approaches.

(3) **Recent Advances**: New techniques that establish finite-speed information propagation under physically relevant conditions.

(4) **Remaining open problems**: Generalization to long-range interacting systems and improving the light cone in translation invariant systems [7].

Our discussion also explores implications for quantum computation, particularly in estimating circuit depth for bosonic simulations. The tutorial aims to provide both an accessible introduction for newcomers and an in-depth technical perspective for researchers interested in quantum information and many-body physics.

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Basic tutorial: Effective equations around Bose-Einstein condensation

Arnaud Triay

In this tutorial, I explained the main intuitions behind the validity of the Hartree (or Gross-Pitaevskii) and Bogoliubov theories in the effective description of the evolution of Bose-Einstein condensates. This is based on results and methods of [3, 2, 1].

Let us consider N weakly interacting bosons, their Hamiltonian is given by

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N-1} \sum_{1\le j<k\le N} V(x_j - x_k)$$

acting on the Bosonic space $L^2_{\mathrm{s}}(\mathbb{R}^{dN})$ $(N, d \geq 1)$ of square integrable functions which are symmetric under permutations of their N variables. The state of the system is given by a wave function $L^2_{\mathrm{s}}(\mathbb{R}^{dN})$ and its evolution is governed by the Schrödinger equation

$$\mathrm{i}\partial_t \Psi_N(t) = H_N \Psi_N(t).$$

For initial data satisfying Bose-Einstein condensation, one can show that condensation is preserved at later time

$$\Psi_N(0) \simeq u(0)^{\otimes N} \implies \Psi_N(t) \simeq u(t)^{\otimes N} \tag{1}$$

where $u(t)$ follows the Hartree equation. To see this, observe that the energy of such states is given by the Hartree energy

$$\mathcal{E}_H(u) \coloneqq \int_{\mathbb{R}^d} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^d} V * |u|^2 |u|^2 = N^{-1} \langle H_N \rangle_{u^{\otimes N}} \tag{2}$$

and the Hartree dynamics is given by the Hamiltonian evolution of $\mathcal{E}_H$, that is

$$\mathrm{i}\partial_t u(t) = \left(-\Delta + V * |u(t)|^2\right) u(t). \tag{3}$$

Note that the approximation (1) has little chance to hold in norm, since this would require that every particle is close to the condensate wave function, observe for instance that $\|u^{\otimes N} - u^{\otimes N-1} \otimes v\|_{L^2(\mathbb{R}^{dN})} = \|u - v\|_{L^2(\mathbb{R}^d)}$. However, (1) holds in a statistical way, and a precise statement can be made using the density matrices of the system

$$\gamma^{(k)}_{\Psi_N(t)} \coloneqq \mathrm{Tr}_{k+1\to N}\left(|\Psi_N(t)\rangle\langle\Psi_N(t)|\right) \underset{N\to\infty}{\longrightarrow} |u^{\otimes k}(t)\rangle\langle u^{\otimes k}(t)|$$

if true for $t = 0$.

In fact, the Hartree theory (2) also predicts correctly the dynamics of the fluctuations around $u(t)^{\otimes N}$ and allows for an approximation in norm. To focus on the fluctuations, we write

$$\Psi_N = \sum_{k=0}^{N} \varphi_k \otimes_s u^{N-k}$$

and define

$$\Phi_N \coloneqq \{\varphi_k\}_k \in \mathcal{F}_+^{\leq N}(u^\perp) = \bigotimes_{k=0}^{N} \{u^\perp\}^{\otimes_s k}.$$

Note that the map $\Psi_N \mapsto \Phi_N$ is isometric. The dynamics of the fluctuations $\mathrm{i}\partial_t \Phi(t) = \mathbb{H}(t)\Phi(t)$ is governed by the second quantization of the Hessian of $\mathcal{E}_H$

$$\mathbb{H} = \int a_x^* \left(-\Delta_x + V * |u(t)|^2 + K_1(t)\right) a_x + \frac{1}{2} \iint K_2(t,x,y) a_x^* a_y^* + h.c.$$

where K_j are the projection onto $u^\perp$ of $\widetilde{K}_1(t,x,y) = u(t,x)V(x-y)\overline{u(t,y)}$ and $\widetilde{K}_2(t,x,y) = u(t,x)V(x-y)u(t,y)$. Indeed, we have that for all $t \in \mathbb{R}$

$$\lim_{N\to\infty} \|\Phi_N(t) - \Phi(t)\|_{\mathcal{F}_+^{\leq N}} = 0.$$

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Dynamics of high density Fermi gases

Marcello Porta

(joint work with Luca Fresta, Benjamin Schlein)

We consider the dynamics of N interacting fermionic particles, initially confined in a region $\Lambda \subset \mathbb{R}^3$, with density $N/|\Lambda| =: \varrho \gg 1$. Let us introduce the parameter $\varepsilon = \varrho^{-1/3}$. The Hamiltonian of the system is

$$H_N = \sum_{j=1}^{N} K_j + \varepsilon^3 \sum_{i<j}^{N} V(x_i - x_j) \qquad \text{on } L^2_{\mathrm{a}}(\mathbb{R}^{3N}) \tag{1}$$

with $K_j = -\varepsilon^2 \Delta_j$ (non-relativistic case) or $K_j = \sqrt{1 - \varepsilon^2 \Delta_j}$ (pseudo-relativistic case). The ε-dependence of the Hamiltonian defines the high density scaling, and it is introduced in order to guarantee that kinetic and interaction energy are balanced as $\varrho \gg 1$. The mean-field regime is a special case, equivalent to $\varrho \sim N$ (and hence $|\Lambda| \sim 1$).

We are interested in the dynamics of initial data ψ_N under the evolution generated by (1), $\psi_{N,t} = e^{-iH_N t/\varepsilon}\psi_N$. In particular, we shall focus on the evolution of the one-particle density matrix, defined as

$$\gamma^{(1)}_{N,t} = N \operatorname{tr}_{2,3,\ldots,N} |\psi_{N,t}\rangle\langle\psi_{N,t}|, \tag{2}$$

an operator on $L^2(\mathbb{R}^3)$. We would like to compare this operator with the solution of the time-dependent Hartree equation, defined as

$$i\varepsilon\partial_t \omega_{N,t} = [K + \varepsilon^3(\rho_t * V), \omega_{N,t}], \qquad \omega_{N,0} = \omega_N, \tag{3}$$

with $K = -\varepsilon^2\Delta$ or $K = \sqrt{1-\varepsilon^2\Delta}$, and $\rho_t(x) = \omega_{N,t}(x;x)$. At high density, and choosing ω_N close to γ_N in a suitable topology, eq. (3) is expected to effectively describe the dynamics of the one-particle density matrix (2) of the system. A better approximation should be obtained considering the Hartree-Fock equation, differing from the Hartree equation by the presence of the exchange term, which is however subleading at high density.

In the mean-field regime, the first rigorous derivation of the Hartree equation from the many-body Schrödinger equation in the mean-field/semiclassical scaling has been obtained in [1], for short times. A different method has then been introduced in [3], that allows to prove convergence for all times, and for a larger class of interaction potentials. The work [3] allows to consider the dynamics of initial data that enjoy a suitable semiclassical structure, expected to hold for the ground state of confined systems.

Recently, in [4, 5] we extended the method of [3], in order to give a rigorous derivation of the time-dependent Hartree equation for the dynamics of *extended* fermionic systems at high density, that is for which N and $|\Lambda|$ are arbitrarily large, and where $N/|\Lambda| = \varrho \gg 1$ is fixed (and N-independent). Let us informally describe the results. In the work [4] we gave a derivation of the Hartree equation for non-relativistic and for pseudo-relativistic fermions, in a global sense. We proved that, choosing for simplicity $\gamma_N = \omega_N$ with $\omega_N = \omega_N^2$ (Slater determinant), and assuming a suitable, local semiclassical structure for ω_N

$$\operatorname{tr} |\gamma^{(1)}_{N,t} - \omega_{N,t}|^2 \le C(t) N\varepsilon, \tag{4}$$

for all times t smaller than a suitable $T_* > 0$, which is $O(1)$ and finite in the non-relativistic case, and arbitrary in the pseudo-relativistic case. The result (4) allows to prove the closeness of many-body and Hartree dynamics, at the level of expectation values of Hilbert-Schmidt operators (recall that the trivial bound for the Hilbert-Schmidt norms squared of the density matrices is N).

It is a natural question to try to resolve the Hartree dynamics on a *local scale*, that is comparing expectation values of observables which are localized in space. This has been achieved in [5], for the pseudo-relativistic dynamics of high-density Fermi gases. Let $O_z(x) \equiv O(x-z)$ be a smooth and fast decaying function, centered at z. Under the assumption that ω_N satisfies a suitable local semiclassical structure, we proved

$$\left| \operatorname{tr} O_z(\hat{x})(\gamma^{(1)}_{N,t} - \omega_{N,t}) \right| \le C(t)\varepsilon^{-2}, \tag{5}$$

for all times. Being much smaller than the density $\varrho = \varepsilon^{-3}$, this bound proves closeness of many-body and Hartree dynamics on a local scale.

The method of proof of [5] is based on the control of the local fluctuations, that quantify the deviation in time between Hartree and many-body dynamics locally in space. In order to keep track of the locality of the fluctuation dynamics, a crucial role is played by a suitable regularization procedure, that replaces the initial zero temperature state by a state at temperature of order ε, using the Araki-Wyss representation of mixed states of [2]. The introduction of a small positive temperature improves the decay properties of the density matrix ω_N, which turn out to be stable against the Hartree flow. The improved decay properties of the solution of the Hartree dynamics imply better locality properties for the fluctuation dynamics, which ultimately allow to prove that the evolution of local fluctuations stays local.

An advantage of considering pseudo-relativistic fermions is that the group velocity of the particles is bounded. It is an interesting open problem to extend the result (5) to the case of non-relativistic fermions, in which the single particle velocity is a priori unbounded.

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Maximal speed of particle transport in closed and open quantum systems

Tan Van Vu

(joint work with Tomotaka Kuwahara, Keiji Saito, Hongchao Li, Cheng Shang)

The speed at which particles can be transported places fundamental constraints on the dynamics of both isolated and open quantum systems. In this talk, I present recent advances in understanding the maximal speed of macroscopic particle transport in bosonic systems with long-range hopping and interactions. We develop a framework that unifies the quantum speed limit with optimal transport theory. Focusing on generalized Bose-Hubbard models, we derive a universal lower bound on the operational time required to transfer a macroscopic number of bosons between spatially separated regions. Our approach captures the geometric structure of transport dynamics and provides a rigorous classification of the effective light cone governing particle transport, thereby resolving a long-standing question about transport speed in generic bosonic systems. I also briefly discuss how this framework could be extended to open quantum systems, where dissipative particle loss further limits transport speed.

We consider a generic model of bosons on an arbitrary D-dimensional lattice, wherein bosons can hop between arbitrary sites and interact with each other. The time evolution of the system's density matrix is given by $\dot{\varrho}_t = -\mathrm{i}[H_t, \varrho_t]$, where the time-dependent Hamiltonian is of the form

$$H_t = -\sum_{i\neq j\in\Lambda} J_{ij}(t)\hat{b}_i^\dagger\hat{b}_j + \sum_{Z\subseteq\Lambda} h_Z(t). \tag{1}$$

Here, Λ denotes the set of lattice sites, $\hat{b}_i^\dagger$ and $\hat{b}_i$ are bosonic creation and annihilation operators at site i, and $h_Z(t)$ is an arbitrary function of $\{\hat{n}_i\}_{i\in Z}$, where $\hat{n}_i := \hat{b}_i^\dagger\hat{b}_i$ is the number operator. The hopping amplitudes $J_{ij}(t)$ decay according to a power law, $|J_{ij}(t)| \leq J/\|i-j\|^\alpha$, where $\|\cdot\|$ denotes the Euclidean norm and $\alpha > D$.

To date, the Lieb-Robinson bound [6], which characterizes the optimal light cone for information propagation, has been comprehensively studied for long-range interacting spin and fermionic systems [3, 11]. However, bosonic systems present additional challenges due to their unbounded particle occupation numbers. Although some results are available for systems with short-range hopping [9, 4], the case of long-range hopping remains underexplored. Regarding the problem of macroscopic particle transport, Faupin et al. [1] recently demonstrated the existence of a linear light cone for $\alpha > D + 2$, but the case $\alpha > D$ has remained unresolved. In what follows, we close this gap by establishing results for the entire range $\alpha > D$.

Using optimal transport theory, we define the discrete L^1-Wasserstein distance [12] between two distributions $\vec{x}$ and $\vec{y}$ as

$$\mathcal{W}(\vec{x},\vec{y}) := \min_{\pi\in\Pi(\vec{x},\vec{y})} \sum_{i,j} c_{ij}\pi_{ij}. \tag{2}$$

Here, $\Pi(\vec{x},\vec{y})$ denotes the set of couplings π (i.e. joint probability distributions of $\vec{x}$ and $\vec{y}$). The cost matrix $[c_{ij}]$ can be specified arbitrarily, as long as symmetry ($c_{ij} = c_{ji}$) and the triangle inequality ($c_{ij} + c_{jk} \geq c_{ik}$) are fulfilled. For any time-dependent nonnegative vector $\vec{x}_t$ evolving as $\dot{x}_i(t) = \sum_{j(\neq i)} f_{ij}(t)$, where $f_{ij} = -f_{ji}$, we can prove that the minimum time required to transform $\vec{x}_0$ into $\vec{x}_\tau$ is always lower bounded as [13]

$$\tau \geq \frac{\mathcal{W}(\vec{x}_0,\vec{x}_\tau)}{\left\langle \sum_{i>j} c_{ij}|f_{ij}(t)|\right\rangle_\tau}. \tag{3}$$

Here, $\langle z_t\rangle_\tau := \tau^{-1}\int_0^\tau \mathrm{d}t\, z_t$ is the time-average quantity of z_t. The unified speed limit (3) includes all the essence to extend the use of the conventional quantum speed limit [8] to a wide range of quantum many-body systems with various geometric structures.

The problem of macroscopic particle transport can be fully resolved by assigning the vector of boson concentrations to $\vec{x}_t$ (i.e. $x_i(t) = \mathcal{N}^{-1}\,\mathrm{tr}(\hat{n}_i\varrho_t)$, where $\mathcal{N}$

denotes the total number of bosons) and by considering the cost matrix $c_{ij} = \|i-j\|^{\min(1,\alpha-D-\varepsilon)}$. Consider a situation where a fraction $\mu \in (0,1]$ of all bosons is transported from region X to a distant region Y, separated by a distance d_{XY}. Then, the speed limit (3) derives that the operational time τ required for this macroscopic bosonic transport is lower bounded by the distance between the two regions as [13]

$$\tau \geq \kappa_1^{\varepsilon}\, d_{XY}^{\min(1,\alpha-D-\varepsilon)}, \tag{4}$$

where $0 < \varepsilon < \alpha - D$ is an arbitrary number and $\kappa_1^{\varepsilon} > 0$ is a constant independent of the system size. This result holds for arbitrary initial states, including both pure and mixed states, and for the entire range of the power decay $\alpha > D$. The bound (4) indicates the existence of a linear light cone for bosonic transport when $\alpha > D+1$. Furthermore, it is optimal in the sense that there exist transport protocols such that bosonic transport can be accomplished within time $\tau = O(d_{XY}^{\min(1,\alpha-D)})$ [10].

Thus far, we have exclusively considered particle transport within closed quantum systems. In the following, we extend the framework to open quantum systems, where the dynamics is governed by the Gorini-Kossakowski-Sudarshan-Lindblad equation [2, 7]

$$\dot{\varrho}_t = -\mathrm{i}[H_t, \varrho_t] + \sum_k (L_k \varrho_t L_k^\dagger - \{L_k^\dagger L_k, \varrho_t\}/2). \tag{5}$$

Here, $L_k \propto b_k^{n_k}$ (with $n_k \geq 1$) are jump operators that characterize particle loss. In contrast to the closed-system case, the total particle number is not conserved in this open setting. To address this, we introduce a modified Wasserstein distance for unbalanced distributions $\vec{x}$ and $\vec{y}$, which satisfy $\|\vec{x}\|_1 \geq \|\vec{y}\|_1$, defined as

$$\widetilde{\mathcal{W}}(\vec{x}, \vec{y}) = \min_{\vec{x} \succeq \vec{x}' \succeq \vec{0}, \vec{y}' \succeq \vec{y}, \|\vec{x}\|_1 = \|\vec{y}\|_1} \mathcal{W}(\vec{x}', \vec{y}'). \tag{6}$$

Here, $\vec{x} \succeq \vec{y}$ means that $x_i \geq y_i$ for all i and $\|\vec{x}\|_1 := \sum_i x_i$. Once again, considering the vector of boson concentrations $\vec{x}_t$, we can derive $\dot{x}_i(t) = d_i(t) + \sum_{j(\neq i)} f_{ij}(t)$, where $d_i \leq 0$ accounts for particle loss caused by the jump operators. Notably, an analogous version of the speed limit (3) can be obtained, with $\mathcal{W}$ replaced by $\widetilde{\mathcal{W}}$. Following the same reasoning, we recover the same bound (4) for the open-system case [5]. This result implies that particle loss cannot enhance the transport speed. The appearance of the same bound stems from the existence of decoherence-free subspaces. For example, when $\min_k n_k > 1$ and the system contains only a single particle, the open system effectively behaves like a closed system.

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Particle Propagation Bounds for Lattice Bosons under Long-Range Interactions

Carla Rubiliani

(joint work with Marius Lemm, Jingxuan Zhang)

The focus of this talk are particle propagation bounds for bosonic systems and their applications. The talk starts with a recap on Lieb-Robinson bounds (LRB), which are bounds on the propagation of information, first proven in 1972 [1] for spin-systems governed by a Hamiltonian H given by sum of local and bounded terms. They state that there exists a system dependent constant v_{LR}, the so-called Lieb-Robinson velocity, such that, given two bounded operators A and B supported on two disjoint regions of the lattice X and Y, $X \cap Y = \emptyset$,

$$\|[\mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH}, B]\| \leq \|A\|\|B\|\mathrm{e}^{-c(d(X,Y)-v_{\mathrm{LR}}t)}. \tag{1}$$

Intuitively, at time $t = 0$ the commutator between the two operators vanishes as their supports do not overlap, as we let one of them time evolve under the Heisenberg evolution, such commutator will not vanish anymore as the support of the time-evolved operator will spread on the entire lattice, but still, if the distance between the initial supports is big compared to time, the commutator stays small. In other word, the support of the operator grows linearly in time, up to exponentially decaying tails. Starting from the early 2000s, as Hastings, in a series of papers [2]–[5], showed the utility and versatility of LRB, many efforts were invested into extending them to more general settings. A question that still remains not fully solved, is the extension of such bounds to bosonic systems, which are characterized by unbounded interactions, due to the possible accumulation of bosons in a finite region of the lattice. Because of this issue, the tools previously developed to deal with spin and fermionic systems can not be applied in this setting. The main difference compared to those settings is that, for bosonic systems, one can only hope to control expectation values, as there may be very badly behaved initial states, with very high concentration of bosons on a few lattice sites, that could lead to accelerated dynamics. The goal is then to obtain such bounds for the most general initial state possible or to add interesting further assumptions to improve the bounds. Additionally, Eisert and Gross [6] in 2009 showed how, for special bosonic systems that are one-dimensional, translation invariant and time-independent, the velocity scales exponentially with time. This result highlights how deriving LRB for bosonic systems is not at all a trivial question. The idea on how to derive LRB for bosonic systems, due to Schuch, Harrison, Osborne, and Eisert [7], is to first control the propagation of particles, since if we start from a nice initial state with controlled density, if we know how fast bosons can propagate, we can be sure that until a certain time the state stays well-behaved, not displaying big accumulation of bosons, and one can then hope to obtain LRB until such time. This shows the possible applicability of bosons propagation bounds. In our work, Marius Lemm, Jingxuan Zhang and me, were able to obtain such particle propagation bounds for Bose-Hubbard type Hamiltonians

$$H = \sum_{x,y\in\Lambda} J_{xy} b_x^\dagger b_y + V \tag{2}$$

where $b_x^\dagger$, b_x are the bosonic creation and annihilation operator. The first term, the hopping term, describes one particle jumping from site y to site x with jumping rate given by the hopping matrix (J_{xy}). The second term is an arbitrary real potential $V = \Phi(\{n_x\}_{x\in\Lambda})$ with $n_x := b_x^\dagger b_x$ the local number operator. Assuming polynomial decay of the hopping matrix,

$$|J_{xy}| \leq C_J |x-y|^{-\alpha} \qquad \text{for } \alpha > d+1 \tag{3}$$

and controlled density for the state at initial time,

$$\exists\, \lambda_2 > \lambda_1 > 0 \quad \text{s.t.} \quad \lambda_1^q \leq \langle n_x^q \rangle_0 \leq \lambda_2^q \quad \forall x \in \Lambda \tag{4}$$

we were able to show

$$\begin{aligned} \left\langle N_{B_r}^p \right\rangle_t &\leq \left\langle N_{B_R}^p \right\rangle_0 \exp\left\{ \frac{C}{R^d} + \frac{vt}{R-r} \right\} & (vt \leq R-r), \\ \left\langle N_{B_R}^p \right\rangle_t &\geq \left\langle N_{B_r}^p \right\rangle_0 \exp\left\{ -\left(\frac{C}{R^d} + \frac{vt}{R-r} \right) \right\} & (vt \leq R-r), \end{aligned}$$

for any $R > r > 1$ satisfying appropriate assumptions. Such result is very interesting as the decay condition for the hopping term is believed to be optimal, as for the non-interacting case the same threshold appears, and we do not expect the interacting systems behaving better than the non-interacting one. Furthermore, is important to notice that the decay of the hopping does not depend on the moment of the number operator we aim to bound, this is especially interesting as to obtain LRB from particle propagation bounds one needs to be able to control high moments, and if the decay threshold would be coupled with the momentum parameter than one would require non-optimal decay of the hopping term for the LRB. The main tools we employ to show such particle propagation bounds are the ASTLOs (adiabatic space-time localization observables), which allow us to track particles moving across the lattice smoothly and for which we are able to obtain time-evolution estimates. The second important tool is a multiscale analysis, we set up a downwards multiscale induction to control the smaller scales, the ones we are interested in controlling, thanks to the higher ones, that are well-behaved. This allows us to control particles very far away from the regions of interest, particles that intuitively should not be significant as they are very far and would require a lot of time to reach the regions we are looking at. We conclude the talk by exploring a new research direction focused of substituting particle propagation bounds with energy propagation bounds in order to improve existing LRB, the intuition being that bad states with high accumulation of particles should not be energetically favoured under a repulsive potential.

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Local enhancement of the mean-field approximation for bosons

Jingxuan Zhang

(joint work with Marius Lemm, Simone Rademacher)

The nonlinear Hartree equation describes the macroscopic dynamics of initially factorized N-boson states as $N \to \infty$. Global estimates on the rate of convergence of the microscopic quantum mechanical evolution towards the limiting Hartree dynamics have been derived in the seminal works of Erdős-Schlein-Yau [1], Rodnianski-Schlein [2], etc. A benchmark result in this direction is due to [2] (see Thm. 1.1 therein). Consider the following N-body Hamiltonian acting on $\mathbb{R}^{3N}$

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \frac{1}{N}\sum_{i<j}^{N} V(x_i - x_j). \tag{1}$$

Under suitable regularity assumptions on the interaction V, the N-body evolution $\psi_{N,t} = \mathrm{e}^{-\mathrm{i}H_N t}\psi_{N,0}$ with a purely factorized initial state $\psi_{N,0} = \varphi_0^{\otimes N}$, where

$\varphi_0 \in H^1(\mathbb{R}^3)$, satisfies

$$\left|\mathrm{Tr}(\gamma_{\psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|)\right| \le \frac{Ce^{Kt}}{N^{1/2}}. \tag{2}$$

Here $\gamma_{\psi_{N,t}}$ is the one-body reduced density operator associated with $\psi_{N,t}$, and φ_t solves the following Hartree equation with initial condition $\varphi_t|_{t=0} = \varphi_0$

$$\mathrm{i}\partial_t\varphi_t = -\Delta\varphi_t + (V * |\varphi_t|^2)\varphi_t, \quad \text{on } \mathbb{R}^d. \tag{3}$$

While prior approximation results probe the quantum gas globally in space, we may expect that the mean-field approximation can be locally enhanced based on the physical principle of spatial locality. Informally, this means that quantum information (similar to typical classical physical quantities) should only propagate with some bounded speed, up to small errors due to quantum effects.

To this end, in our work [4], we seek *localization conditions* on the initial states $\psi_{N,0}$ and φ_0 that lead to an improved approximation error bound at time $t > 0$. To ensure a bounded one-body group velocity and avoid technical complications due to energy cutoffs, we consider the Bose-Hubbard type Hamiltonian acting on the integer lattice $\mathbb{Z}^{dN}$, $d \ge 3$,

$$H_N = \sum_{x\in\mathbb{Z}^d}\sum_{y\sim x} a_x^* a_y + \frac{\lambda}{2N}\sum_{x\in\mathbb{Z}^d} a_x^* a_x^* a_x a_x. \tag{4}$$

Here λ is a real, order 1 parameter. The corresponding Hartree equation becomes the cubic discrete nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\varphi_t = -\Delta\varphi_t + \lambda|\varphi_t|^2\varphi_t, \quad \text{on } \mathbb{Z}^d. \tag{5}$$

We prove a local enhancement of the mean-field approximation in the following sense: At a positive distance $\rho > 0$ from the initial BEC, the mean-field approximation error at time $t \le \rho/v$ is bounded as ρ^{-n} for arbitrarily large $n \ge 1$.

Specifically, we consider purely factorized initial states $\psi_{N,0} = \varphi_0^{\otimes N}$, with φ_0 supported away from the origin, i.e.

$$\varphi_0(x) = 0 \quad \text{for all} \quad |x| \le R.$$

We show that for smaller distances $r < R$, the mean-field approximation for $|x| \le r$ near the origin is significantly enhanced, for times short compared to $\rho = R - r$, the distance that has to be traversed.

To quantify this, take any bounded one-particle operator O with a kernel satisfying $O(x;y) = 1_{|x|\le r}O(x;y)1_{|y|\le r}$. Then, for any $n \ge 1$, there holds

$$\left|\,\mathrm{Tr}\left(\left(\gamma_{\psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\right)O\right)\right| \le \frac{C_n|O|_{\mathrm{op}}}{N}\frac{1}{\rho^n}, \qquad \text{for } t \le \frac{\rho}{v}. \tag{6}$$

Here, v can be chosen as any number that is strictly larger than the norm of the one-body group velocity operator $\|\mathrm{i}[-\Delta, |x|]\| \le 2d$. In particular, v is state-independent and of order 1.

The proof is based on new ballistic propagation bounds on the quantum fluctuations around the Hartree states, through a variant of the ASTLO (adiabatic spacetime localization observable) method, developed in our earlier works (see e.g. [3]), for the particle non-conserving generator of the fluctuation dynamics around Hartree states.

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The Heisenberg ferromagnet: a dilute Bose gas in disguise

Robert Seiringer

We revisit the quantum Heisenberg model of a ferromagnet. For $\Lambda \subset \mathbb{Z}^d$ a finite subset of $\mathbb{Z}^d$, and $\vec{S} = (S^1, S^2, S^3)$ an irreducible representation of $su(2)$ with dimension $2S+1$, $S \in \frac{1}{2}\mathbb{N}$, it is given by the Hamiltonian

$$H_\Lambda = \sum_{x \sim y \in \Lambda} \left(S^2 - \vec{S}_x \cdot \vec{S}_y \right)$$

where $x \sim y$ denotes nearest neighbours on Λ. We recall basic properties of this model, as well as the famous open problem concerning the appearance of long-range order at low temperature: for $d \geq 3$ and T small enough, it is expected that

$$\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \left\langle \vec{S}_x \cdot \vec{S}_y \right\rangle_{T,\Lambda} > 0$$

where $\langle \cdot \rangle_{T,\Lambda} = \operatorname{Tr} \cdot \mathrm{e}^{-H_\Lambda/T} / \operatorname{Tr} \mathrm{e}^{-H_\Lambda/T}$ denotes the finite volume Gibbs state at temperature $T > 0$. We then explain the spin-wave approximation, and sketch the proof of its validity in the form of a low-temperature expansion of the free energy: With

$$f(T) = -T \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \operatorname{Tr} \mathrm{e}^{-H_\Lambda/T}$$

the free energy per site in the thermodynamic limit, we have

$$\lim_{T \to 0} T^{-1-d/2} f(T) = S^{d/2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \ln(1 - \mathrm{e}^{-p^2})\, \mathrm{d}p \tag{1}$$

The leading term in this expansion is thus equal to the one of an ideal Bose gas at criticality, a manifestation of the fact that low-energy excitations form a dilute gas of bosons, known as "magnons". The validity of (1) for $d = 3$ was proved in [1], and that proof readily extends to $d \geq 3$. The case $d = 1$ was proved more recently in [2], while the case $d = 2$ is still open. (To be precise, it is open as a lower bound; an upper bound of the correct form was proved in [2].)

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The Gibbs state of the mean-field Bose gas

Phan Thành Nam

(joint work with Andreas Deuchert and Marcin Napiórkowski)

We consider a Bose gas in the torus $\Lambda = [0,1]^3$ with the mean-field interaction $N^{-1}v$, described by the grand-canonical Gibbs state

$$G_{\beta,N} = Z_{\beta,N}^{-1} \exp\left(-\beta\Big(\sum_{p\in\Lambda^*} (|p|^2 - \mu_{\beta,N}) a_p^* a_p + \frac{1}{2N} \sum_{k,p,q\in\Lambda^*} \hat{v}(k) a_{p+k}^* a_{q-k}^* a_q a_p \Big) \right)$$

on the Fock space $\mathfrak{F}(L^2(\Lambda))$. Here a_p^* and a_p are the standard creation and annihilation operators associated with momentum $p \in \Lambda^* = 2\pi\mathbb{Z}^3$. We focus on temperatures comparable to the critical temperature of Bose-Einstein condensation, namely,

$$\beta^{-1} \sim \beta_c^{-1} = 4\pi \left(\frac{N}{\zeta(3/2)} \right)^{2/3}.$$

The chemical potential $\mu_{\beta,N} \in \mathbb{R}$ is chosen such that the Gibbs state has N particles on average, i.e., $\mathrm{Tr}[\mathcal{N} G_{\beta,N}] = N$. We also assume that $\hat{v}$ is non-negative and decays sufficiently fast.

In this mean-field regime, the low-lying excitation spectrum was computed in Seiringer's seminal paper [9], which establishes the validity of Bogoliubov theory for the ground state and the Gibbs state at temperatures of order 1. In particular, at low temperatures, most particles condense into the zero-momentum mode, with only a finite number remaining in the thermal cloud.

In contrast, at temperatures of the order of the critical temperature β_c^{-1}, the number of thermally excited particles is always comparable to the total number of particles, a phenomenon already observed in the ideal gas. Consequently, the justification of temperature-dependent Bogoliubov theory becomes significantly more subtle. This relates to a question raised in the physics literature about the potential breakdown of Bogoliubov theory at high temperatures; see e.g. [1]. Our main result shows that, since the interaction is sufficiently weak in the mean-field regime, Bogoliubov theory remains valid for all temperatures of order β_c^{-1}.

Temperature-dependent Bogoliubov theory. A key idea proposed by Bogoliubov [3] is to replace a_0 in the Hamiltonian with a complex number z, where $|z|^2 \sim \mathrm{Tr}[a_0^* a_0 G_{\beta,N}] \gg 1$. This can be implemented using coherent states [8],

$$|z\rangle = \exp(z a_0^* - \overline{z} a_0)|\Omega_0\rangle, \quad a_0|z\rangle = z|z\rangle, \quad z \in \mathbb{C},$$

where Ω_0 is the vacuum of the zero-momentum Fock space $\mathfrak{F}_0$.

Formally, if we replace the Gibbs state $G_{\beta,N}$ with the ansatz $|z\rangle\langle z|\otimes G(z)$, then the c-number substitution suggests that $G(z)$ is close to the Gibbs state associated with the Bogoliubov Hamiltonian

$$H^{\mathrm{Bog}}(z) = \sum_{p\neq 0}(p^2-\mu_0)a_p^*a_p + \frac{N_0}{2N}\sum_{p\neq 0}\hat{v}(p)\left(2a_p^*a_p + (z/|z|)^2 a_p^*a_{-p}^* + (\overline{z}/|z|)^2 a_p a_{-p}\right)$$

on the excited Fock space $\mathfrak{F}_+$. Here, we take the chemical potential μ_0 of the ideal gas, chosen such that the non-interacting Gibbs state

$$G^{\mathrm{id}}_{\beta,N} = (Z^{\mathrm{id}}_{\beta,N})^{-1}\exp\Big(-\beta\sum_{p\in\Lambda^*}(|p|^2-\mu_0)a_p^*a_p\Big)$$

has N particles on average. We also follow an idea in [6] to take the simplification

$$|z|^2 \simeq N_0 = \mathrm{Tr}[a_0^*a_0 G^{\mathrm{id}}_{\beta,N}] = (\mathrm{e}^{-\beta\mu_0}-1)^{-1}.$$

All of this results in the Bogoliubov Gibbs state on $\mathfrak{F}_+$:

$$G^{\mathrm{Bog}}(z) = (Z^{\mathrm{Bog}})^{-1}\exp\left(-\beta H^{\mathrm{Bog}}(z)\right).$$

Note that $G^{\mathrm{Bog}}(z)$ is solvable since the spectrum of $H^{\mathrm{Bog}}(z)$ can be computed explicitly by Bogoliubov diagonalization method.

Φ^4-theory for the condensate. While Bogoliubov theory describes the thermally excited particles, the condensate is described by the probability distribution

$$g^{\mathrm{BEC}}(z) = (Z^{\mathrm{BEC}})^{-1}\exp\left(-\beta\left(\frac{\hat{v}(0)}{2N}|z|^4 - \mu^{\mathrm{BEC}}|z|^2\right)\right), \quad z\in\mathbb{C},$$

where the chemical potential μ^{BEC} is chosen such that

$$\int_{\mathbb{C}}|z|^2 g^{\mathrm{BEC}}(z)\,\mathrm{d}z + \mathrm{Tr}[\mathcal{N}G^{\mathrm{Bog}}(z)] = N,$$

with $\mathrm{d}z = \pi^{-1}\,\mathrm{d}(\Re z)\,\mathrm{d}(\Im z)$. This one-mode Φ^4-theory is related to the classical field theory derived in [7, 5], which concerns stronger interactions, and it was also used in [2] for a trial state in the context of the Gross-Pitaevskii regime.

Main result. Our main result is a justification of Bogoliubov theory as a norm approximation of the Gibbs state for all temperatures $\beta^{-1}\sim\beta_c^{-1}$:

(1) $$\mathrm{Tr}\,|G_{\beta,N}-\Gamma_{\beta,N}| \leq O(N^{-1/48})$$

with

$$\Gamma_{\beta,N} = \begin{cases}\int_{\mathbb{C}}|z\rangle\langle z|\otimes G^{\mathrm{Bog}}(z)g^{\mathrm{BEC}}(z)\,\mathrm{d}z, & \text{if}\quad N_0\geq N^{2/3}\\ G^{\mathrm{id}}_{\beta,N} & \text{if}\quad N_0\leq N^{2/3}.\end{cases}$$

Note that each state $|z\rangle\langle z|\otimes G^{\mathrm{Bog}}(z)$ is a quasi-free state, but averaging with respect to the Φ^4-measure $g^{\mathrm{BEC}}(z)\mathrm{d}z$ destroys this property. Thus, if $N_0\geq N^{2/3}$, our result goes beyond the standard quasi-free approximation. If $N_0\leq N^{2/3}$, the mean-field effect forces the Gibbs state to behave like a non-interacting one (when $N_0\sim N^{2/3}$, a nontrivial behaviour is visible if the interaction is stronger; see [7, 5]).

As a consequence of the norm approximation, we also obtain approximate expressions for the one- and two-particle density matrices of the Gibbs state, as well as a detailed classification of the limiting particle number distributions for the condensate.

An important ingredient in the proof of (1) is an asymptotic expansion of the free energy associated with the Gibbs state $G_{\beta,N}$, accurate up to $o(N^{2/3})$. Note that both the contributions of individual Bogoliubov modes and the particle number fluctuations in the condensate are of order $N^{2/3}$, and achieving higher accuracy is crucial for determining the structure of the Gibbs state.

Correlation inequality. In the first step of our proof we derive bounds for the free energy up to $O(N^{2/3})$. Using a Griffith (or Hellmann-Feynman) argument, we obtain $|\operatorname{Tr}[B_p G_{\beta,N}]| \le O(N^{2/3})$ with $B_p = \sum_{r,r+p\neq 0}(a^*_{r+p}a_r + \text{h.c.})$. But to validate Bogoliubov theory, we need $|\operatorname{Tr}[B_p^2 G_{\beta,N}]| \le O(N^{4/3})$.

The desired second moment bound is obtained by a new abstract correlation inequality. To be precise, we prove that if the perturbed Gibbs state $\Gamma_t = Z_t^{-1}\exp(-A+tB)$ satisfies $\sup_{t\in[-1,1]}|\operatorname{Tr}(B\Gamma_t)| \le a$, then

$$\operatorname{Tr}[B^2\Gamma_0] \le a\,\mathrm{e}^a + \frac{1}{4}\operatorname{Tr}([[B,A],B]\Gamma_0). \tag{2}$$

The idea of using a first-moment estimate for perturbed Gibbs states to deduce a second-moment estimate for the original Gibbs state is inspired by the recent work of Lewin, Nam, and Rougerie [7, Theorem 7.1]. Our proof of (2) is based on the key observation that Stahl's theorem [10], formerly known as the Bessis-Moussa-Villani (BMV) conjecture, implies an elegant convexity property of the Duhamel two-point function. We hope that this general result will find applications in other contexts. To access two-body density matrices, we also develop a higher-order version of (2), which is of independent interest. Further results and discussions can be found in our paper [4].

From a conceptual perspective, an interesting feature of our approach is that it does not rely entirely on the coercivity of the energy functional. We hope that this type of argument will be useful for understanding systems without a kinetic spectral gap, such as the Bose gas in the thermodynamic limit.

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Out-of-time ordered correlators for mean-field bosons

Simone Rademacher
(joint work with Marius Lemm)

Out-of-time ordered correlators (OTOCs) are quantifiers used in quantum many-body chaos to study the scrambling of quantum information among a large number of degrees of freedom. Quantum many-body chaos rests on the prediction that OTOCs can be derived as the limit of a classical dynamics, and, building on that, on the idea that exponential growth of OTOCs at a rate called butterfly velocity can be related to chaotic behaviour of the scrambling of quantum information. We study OTOCs to quantify correlations for the dynamics of mean-field bosons. Our first result is an asymptotic formula for the OTOCs in the large particle limit (Theorem 1) that we use in our second result to derive an upper bound on the butterfly velocity (Corollary 1).

To be more precise, we consider the dynamics of N bosons described on the symmetric Hilbert space $L^2_s(\mathbb{R}^{3N})$ by the mean-field Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\Delta_i) + \frac{1}{N}\sum_{1\le i<j\le N} v(x_i - x_j)$$

where we assume that the two-body interaction potential $v\colon \mathbb{R}^3 \to \mathbb{R}$ satisfies weak regularity assumptions $v^2 \le C(1-\Delta)$ for some $C > 0$. We study the OTOC for the Heisenberg dynamics w.r.t to the mean-field Hamiltonian of suitable centered observables

$$\mathcal{A}_t := \mathrm{e}^{\mathrm{i}tH_N}\big(A^{(1)} - \langle \varphi_t, A\varphi_t\rangle\big)\mathrm{e}^{-\mathrm{i}tH_N} \tag{1}$$

where $A^{(1)}$ denotes the N-particle operator that acts as the bounded one-particle $A\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ on the first, and as identity on the remaining $N-1$ particles and φ_t denotes the solution the Hartree equation

$$\mathrm{i}\partial_t \varphi_t = h_{\varphi_t}\varphi_t, \quad \text{with} \quad h_{\varphi_t} = -\Delta + (v * |\varphi_t|^2). \tag{2}$$

In fact, it is well known that for an initial pure condensate of the form

$$\psi_N = \varphi_0^{\otimes N} \tag{3}$$

for a suitable one-particle wave function $\varphi_0 \in L^2(\mathbb{R}^3)$ with $\|\varphi_0\|_2 = 1$, the operator $\mathcal{A}_t$ satisfies a law of large numbers, in the sense that for any bounded one-particle operator, it holds $\langle \psi_N, \mathcal{A}_t\psi_N\rangle \to 0$ as $N \to \infty$.

We are interested in analysing the quantum fluctuations around the expectation $\langle \varphi_t, A\varphi_t \rangle$ of the condensate φ_t using OTOCs. For this we study the commutator of $\mathcal{A}_t$ with the averaged operator

$$\mathcal{B}_0 := \sum_{j=1}^{N} \left(B^{(j)} - \langle \varphi_0, B\varphi_0 \rangle \right) \tag{4}$$

in the large particle limit and prove the following asymptotic formula.

Theorem 1 (Theorem 2.1 in [3]). *Let $v \le C(1-\Delta)$ and $\varphi_0 \in H^4(\mathbb{R}^3)$ with $\|\varphi_0\|_{L^2(\mathbb{R}^3)} = 1$. Let $t > 0$ and φ_t denote the solution to the Hartree equation (2) with initial data φ_0. Fix two self-adjoint, real operators A, B on $L^2(\mathbb{R}^3)$ satisfying the regularity assumptions*

$$\|(-\Delta+1)A(-\Delta+1)^{-1}\|,\ \|(-\Delta+1)B(-\Delta+1)^{-1}\| \le C.$$

Let ψ_N be given by (3), then we have for $\mathcal{A}_t$ and $\mathcal{B}_0$ given by (1) and (4), respectively,

$$\lim_{N\to\infty} \langle \psi_N, \left(\mathrm{i}\left[\mathcal{A}_t^{(1)}, \mathcal{B}_0\right]\right)^2 \psi_N \rangle = \frac{1}{4}\left(\langle (\varphi_0, \overline{\varphi}_0), \left[B, \widetilde{A}_{(t;0)}\right](\varphi_0, \overline{\varphi}_0) \rangle_S \right)^2 \tag{5}$$

where $\langle \cdot, \cdot \rangle_S$ denotes the symplectic form defined below in (7) and

$$\widetilde{A}_{(t;0)} := \Theta(t;0) A \Theta(t;0)^{-1}$$

with the symplectic dynamics $\Theta(t;0)$ given by (6) below.

The OTOC is asymptotically well described by the symplectic Bogoliubov dynamics $\Theta(t;s) \colon L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ satisfying

$$\mathrm{i}\partial_s \Theta(t;s) = \mathcal{T}_s \Theta(t;s), \quad \text{with} \quad \mathcal{T}_s = \begin{pmatrix} h_{\varphi_s} + \widetilde{K}_{1,s} & -\widetilde{K}_{2,s} \\ \overline{\widetilde{K}}_{2,s} & -h_{\varphi_s} - \widetilde{K}_{1,s} \end{pmatrix} \tag{6}$$

and $\Theta(t;t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\widetilde{K}_{1,s} = q_s K_{1,s} q_s$, $\overline{q}_s K_{2,s} q_s$ with $q_s = 1 - |\varphi_s\rangle\langle\varphi_s|$ and

$$K_{1,s} = v(x-y)\varphi_s(x)\overline{\varphi}_s(y), \quad K_{2,s} = v(x-y)\varphi_s(x)\varphi_s(y).$$

The symplectic Bogoliubov dynamics $\Theta(t;s)$ has been first observed in [1] in the context of central limit theorems for mean-field bosons and is well known to provide an effective description of the many-body mean-field dynamics. Moreover, $\Theta(t;s)$ is a unitary operator w.r.t. the symplectic form defined for $f_i, g_j \in L^2(\mathbb{R}^3)$ by

$$\langle (f_1, f_2), (g_1, g_2) \rangle_S := \langle (f_1, f_2), S(g_1, g_2) \rangle_S, \quad \text{with} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

As a consequence of the asymptotic formula in Theorem 1, we furthermore get an expansion of the OTOC for small times and an upper bound for the butterfly velocity for all times.

Corollary 1 (Corollary 2.2 in [3]). *Under the same assumptions as in Theorem 1, the following holds:*

(i) *There exists $T > 0$ such that for sufficiently small $|t| \leq T$*

$$\lim_{N\to\infty} \langle \psi_N, \big(\mathrm{i}\big[\mathcal{A}_t^{(1)}, \mathcal{B}_0\big]\big)^2 \psi_N\rangle$$
$$= -\frac{1}{2\mathrm{i}}\langle \varphi_0, [A,B]\varphi_0\rangle$$
$$\times \Big(1 - 2t\,\Re\big\langle \varphi_0, B\big[h_{\varphi_0} + \widetilde{K}_{1,0} - \widetilde{K}_{2,0}J, A\big]\varphi_0\big\rangle\Big) + O(t^2).$$

(ii) *We have for all $t \in \mathbb{R}$*

$$\lim_{N\to\infty} \langle \psi_N, \big(\mathrm{i}\big[\mathcal{A}_t^{(1)}, \mathcal{B}_0\big]\big)^2 \psi_N\rangle \leq C\,\mathrm{e}^{C|t|}$$

where the constant C depends on φ_0 through $\|\varphi_0\|_{H^1}$.

Corollary 1 (i) shows that the rate of initial scrambling of information for the many-body mean-field dynamics can be computed explicitly in terms of one-particle data. Such precise statements are usually not available for quantum many-body systems. For mean-field bosons so far only rough exponential bounds have been proven [2]. For larger times, we can only prove an upper bound on the butterfly velocity in Corollary 1 (ii). It is an interesting open question under which circumstances the OTOC actually grows exponentially since this corresponds to quantum many-body chaos. Theorem 1 turns this open problem into a question about non-linear dispersive PDEs.

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Lieb-Robinson bounds and the thermodynamic limit for continuous fermions

Oliver Siebert

(joint work with Benjamin Hinrichs and Marius Lemm)

For interacting non-relativistic fermions in $\mathbb{R}^d$ with a one-body potential V and a two-body interaction potential W, the standard Hamiltonian acting on the fermionic Fock space is given by

$$H = \int_{\mathbb{R}^d} (\nabla a_x^* \nabla a_x + V(x) a_x^* a_x)\,\mathrm{d}x + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y) a_x^* a_y^* a_y a_x\,\mathrm{d}x\,\mathrm{d}y. \tag{1}$$

For such a system, bounded velocities are not expected for general observables, and as in the few-body case, one must impose some kind of high-energy regularization. This has been implemented by Gebert, Nachtergaele, Reschke and Sims in [4] via a UV regularization in the two-body interaction. To this end, one replaces the pointwise creation and annihilation operators a^*, a in the second term in (1) by smeared-out operators, leading to the Hamiltonian

$$H_\Lambda = \int_{\mathbb{R}^d} (\nabla a_x^* \nabla a_x + V(x) a_x^* a_x)\, \mathrm{d}x + \int_\Lambda \int_\Lambda W(x-y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma)\, \mathrm{d}x\, \mathrm{d}y,$$

where φ_x^σ is a Gaussian function centered at point x with standard derivation σ and $\Lambda \subset \mathbb{R}^d$ is a bounded subset. Let $\tau_t^\Lambda(A) := \mathrm{e}^{\mathrm{i}tH_\Lambda^\sigma} A \mathrm{e}^{-\mathrm{i}tH_\Lambda^\sigma}$ denote the Heisenberg time evolution and τ_t^0 the free time evolution (where $W = 0$). They then obtain a many-body LRB that bounds the overlap (anticommutator) between the creation and annihilation operators in f and g for the *difference between* the interacting and the free time evolution,

$$\begin{aligned} &\|\{(\tau_t^\Lambda - \tau_t^0)(a(f)), a^*(g)\}\| + \|\{\tau_t^\Lambda(a(f)), a(g)\}\| \\ &\leq C_\sigma(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|}{4c_t}\right) |f(x)||g(y)|\, \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{2}$$

where $c_t \sim t^2 + 1$. The constant $C_\sigma(t)$ is independent of Λ, f, g, but grows exponentially in t, so the light cone grows exponentially as well. Furthermore, $C_\sigma(t)$ diverges for $\sigma \to 0$. A first step in their analysis is the proof of a one-body LRB for the Schrödinger operator with a cubic light cone $|x-y| \lesssim \langle t \rangle^3$ where the class of admissible one-body potentials V is rather restricted to those which can be expressed as a Fourier transform of a signed compactly supported measure. The many-body bound can then be derived from the one-body bound by a Gronwall type argument.

In [5] Hinrichs, Lemm and I apply the technique of ASTLOs to the model of regularized interacting continuous fermions from [4]. We first show a one-body maximum velocity propagation estimate for Schrödinger operators as in [1] with explicit tracking of the constants, in particular, the energy dependence. This allows us to derive a one-body LRB

$$\left|\left\langle \mathrm{e}^{-\mathrm{i}tH^1} f, \varphi_x^\sigma \right\rangle\right|^2 \leq C_{n,\sigma} \langle t \rangle^{1+2\delta} \int \left(1 \wedge \frac{\langle t \rangle}{|x-y|}\right)^n |f(y)|^2\, \mathrm{d}y, \tag{3}$$

where $\wedge$ denotes the minimum. In this bound, the energy and the spatial cutoff necessary for the norm propagation bound are now 'hidden' in the Gaussian, which is localized in both position and momentum space. The major improvement of this result over the similar one-body bound in [4] is the *almost linear* light cone for large n, namely given by $|x-y| \lesssim \langle t \rangle^{1+(1+2\delta)/n}$. Moreover, our result also holds for a more general class of one-body potentials V that are sufficiently regular and for which Kato-Rellich is applicable. One drawback is that the ASTLO method only yields a polynomial decay of power n, which can be chosen to be arbitrarily strong. With a similar Gronwall-type argument we get from (3) a many-body result as in [4]. The light cone is then exponentially large as well, but still applies to the more general class of one-body potentials.

By standard commutator and anticommutator expansions, we can extend (2) to observables that are polynomials in a^*, a operators (in contrast to single a^*, a operators). However, such bounds are not uniform in the operator norm and therefore do not allow an approximation argument for general observables of the CAR algebra.

We also discuss several applications of our result similar to those for lattice systems. We show a clustering result for ground states ψ_0 which are separated from the rest of the spectrum by a gap $\gamma > 0$ as in [6],

$$|\langle \psi_0, AB\psi_0 \rangle| \lesssim \frac{1}{\sqrt{\gamma} \wedge 1} \exp\left(-\frac{\gamma \wedge 1}{2} (n \log(\mathrm{dist}(A,B) + 1))^{1/4} \right), \tag{4}$$

where $\mathrm{dist}(A, B)$ is the distance between the supports of the observables A and B. Although A and B can be general polynomials in a^*, a, the constants are not uniform in the operator norm. In contrast to [6], the decay in (4) is only subpolynomial in $\mathrm{dist}(A, B)$ due to the polynomial decay and the exponentially growing prefactors in the many-body LRB. Finally, we construct a conditional expectation on the CAR algebra in the continuum to approximate quasi-local (time-evolved local) observables with *strictly* local ones. This was inspired by [7], where it was done for lattice fermions. However, the current LRB only allows for the approximation with a weaker form where finitely many modes for fixed regions are traced out in the complement. This is due to the fact that our LRB is not uniform in the number of a^*, a factors in the observables.

Moreover, analogously to [4], we show the existence of the thermodynamic limit for our class of one-body potentials. In [4] they show that the time evolution on the infinite volume system forms a strongly continuous group $(\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms on the CAR algebra $\mathfrak{A}$ and thus the tuple $(\mathfrak{A}, \tau_t)$ is a C^*-dynamical system. It should be noted that it is not difficult to show that (1) defines a self-adjoint Hamiltonian and hence a unitary group of $*$-automorphisms which are continuous in the (weaker) strong operator topology. Strong continuity, however, means that $t \mapsto \tau_t(A)$ is continuous *in norm*.

An open question is if the strong continuity still persists when the regularization gets removed, i.e., in the limit $\sigma \to 0$. This is actually true for the free dynamics when $W = 0$, which leads to the convenient setting of C^*-dynamical systems. However, this seems to be a peculiarity of the free case, and it is generally believed that including interactions between particles violates this property. For example, one could think of states that develop arbitrarily large local densities or energies within a finite time by filling the phase space with an increasing number of fermions at the same location with different momenta. Due to the non-zero repulsion or attraction at fixed distances, this would lead to arbitrarily strong attractions on individual fermions and thus to a discontinuity in the time evolution.

Instead, one can consider a weaker version of this question and try to construct a thermodynamic limit on a sufficiently big subalgebras of the CAR algebra. To this end, I showed in [8] how the resolvent algebra approach of Buchholz [2, 3] for bosons can also be applied to continuous fermions. In this way one can construct an extension of the CAR algebra over $L^2(\mathbb{R}^d)$, where the dynamics acts as a group of $*$-automorphisms which are continuous in time with respect to the norms of all n-particle sectors. By considering time averages, one then obtains a subalgebra in which the dynamics is even strongly continuous and which is dense in the extended CAR algebra with respect to all norms of the n-particle sectors. Therefore, one obtains a C^*-dynamical system, which provides a potential framework to discuss KMS states for interacting systems in infinite volume.

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Lieb-Robinson Bounds for Static Properties

Tom Wessel

Lieb-Robinson bounds (LRBs) are propagation bounds on the Heisenberg dynamics $\tau_t^H(A) = \mathrm{e}^{\mathrm{i}tH} A \,\mathrm{e}^{-\mathrm{i}tH}$. They say that $\tau_t^H(A)$ is mostly supported on the enlarged set $X_r = \{\, y \,|\, d(x,y) \leq r \,\}$ with $r \sim v\,t$ for all operators A supported on X. Here, v is the Lieb-Robinson velocity and depends on H, but not on A. This notion is made precise in the commutator version of the LRB, which states that $\|[\tau_t^H(A), B]\|$ is small whenever the supports X and Y of A and B, respectively, are far apart in the sense $d(X,Y) > v\,t$. And while these bounds are of dynamic nature, they can also be used to understand static properties via so-called spectral filter functions.

In the talk I consider a quantum spin system $\mathcal{H} = \bigotimes_{z\in\Lambda} \mathbb{C}^q$ on a finite lattice $\Lambda \Subset \mathbb{Z}^D$. The Hamiltonian $H = \sum_{Z\subset\Lambda} \Psi(Z)$ is given by an *interaction* Ψ with local terms $\Psi(Z) = \Psi(Z)^* \in \mathcal{A}_Z$, where $A \in \mathcal{A}_Z$ means that $A = A \otimes \mathbf{1}_{\Lambda\setminus Z}$. We implicitly assume good decay of $\|\Psi\|$ in $\mathrm{diam}(Z)$ the sense of some interaction norm, such that good LRBs for $H(s)$ are known. It is crucial to note that all results hold uniformly in Λ, but we only state them for a fixed Λ for simplicity.

1. Spectral Flow and Automorphic Equivalence

Hastings in 2004 [4] first used filter functions for what he called *quasi-adiabatic continuation*, which is now also known as *spectral flow*. Therefore, consider a smooth family of Hamiltonians $[0,1] =: I \ni s \mapsto H(s)$ with a uniformly gapped spectral patch in the following sense: There exist $g > \delta > 0$ and for each $s \in I$ there is a part of the spectrum $\sigma_*(s) \subset \sigma(H(s))$ which is gapped $d\big(\sigma_*(s), \sigma(H(s)) \setminus \sigma_*(s)\big) \geq g$ and not too big $\operatorname{diam}(\sigma_*(s)) \leq \delta$. And there exist smooth functions $f_\pm \colon I \to \mathbb{R}$ such that $f_\pm(s) \in \mathbb{R} \setminus \sigma(H(s))$ and $\sigma_*(s) = [f_-(s), f_+(s)] \cap \sigma(H(s))$. Now, let $P(s)$ be the spectral projection onto $\sigma_*(s)$. Then there exists a unitary $U(s)$ such that

$$P(s) = U(s)\, P(0)\, U(s)^*, \tag{1}$$

with generator $G(s)$, i.e. U solves

$$-\mathrm{i}\, \partial_s U(s) = G(s)\, U(s) \qquad U(0) = \mathbf{1}.$$

Differentiating (1) gives

$$-\mathrm{i}\, \partial_s P(s) = \big[G(s), P(s)\big]. \tag{2}$$

One easily checks that one can add any diagonal (w.r.t. P) term $\tilde{G} = P\, \tilde{G}\, P + P^\perp\, \tilde{G}\, P^\perp$ to G without changing (2). Hence, there are many generators $G(s)$ such that (1) is satisfied. One possible choice for $G(s)$ is the *Kato generator* $K(s) = -\mathrm{i}\, \big[\dot{P}(s), P(s)\big]$, however, it is not local in general.

Another choice for $G(s)$ is the *Hastings generator* $D(s) = \mathcal{I}_{H(s)}(\dot{H}(s)) := \mathcal{I}_{H(s),g,\delta}(\dot{H}(s))$ where

$$\mathcal{I}_{H,g,\delta}(A) := \int_{\mathbb{R}} W_{g,\delta}(t)\, \mathrm{e}^{\mathrm{i}tH}\, A\, \mathrm{e}^{-\mathrm{i}tH}\, \mathrm{d}t = \sqrt{2\pi} \sum_{n,m} \hat{W}_{g,\delta}(E_m - E_n)\, P_n\, A\, P_m \tag{3}$$

and $W_{g,\delta}$ is a function with Fourier transform $\hat{W}_{g,\delta}$ satisfying

$$\hat{W}_{g,\delta}(\omega) = \begin{cases} -\frac{\mathrm{i}}{\sqrt{2\pi}\,\omega} & |\omega| \geq g \\ 0 & |\omega| \leq \delta. \end{cases}$$

To obtain the equality in (3) we write $\mathrm{e}^{\mathrm{i}tH} = \sum_n \mathrm{e}^{\mathrm{i}tE_n} P_n$ in its energy eigenbasis and use the definition of the Fourier transformation. The first insight here is that $\mathcal{I}_H$ exactly inverts the Liouvillian on off-diagonal operators $A = P\, A\, P^\perp + P^\perp\, A\, P$, i.e. $-\mathrm{i}\, \big[H, \mathcal{I}_H(A)\big] = A$, which allows proving that $\dot{P} = -\mathrm{i}\, \big[\mathcal{I}_H(\dot{H}), P\big]$. The second insight is, that the integral representation in (3) enables us to use LRBs for times $|t| \leq T$, to approximate the integrand on an enlarged support. And for large times it turns out that one can construct a function $W_{g,\delta}$ with the above properties and stretched-exponential decay in t. This allows to bound the remaining integral for $|t| > T$.

One particular application of the spectral flow is the proof of a *local perturbations perturb locally* (LPPL) principle [2]: Assume, that $V \in \mathcal{A}_X$ and all $H(s) = H + s\, V$ have a uniformly gapped ground state $P(s)$ in the above sense. Then the ground states $P(0)$ and $P(1)$ agree away from X in the sense that the difference in expectation value $\left|\langle B\rangle_{P(1)} - \langle B\rangle_{P(0)}\right|$ decays in the distance $d(X,Y)$ for all $B \in \mathcal{A}_Y$. Other important applications are automorphic equivalence [2], response theory [1, 7, 8] and stability of the gap [6].

2. Quantum Belief Propagation

In 2007 Hastings [5] introduced a very similar approach, which he called quantum belief propagation (QBP), to relate Gibbs states at any positive temperature with another. Therefore, let $H(s) = H + s\,V$ and denote the Gibbs state at inverse temperature $\beta > 0$ by $\rho_\beta(s) = \mathrm{e}^{-\beta H(s)}/\operatorname{tr}(\mathrm{e}^{-\beta H(s)})$. It satisfies

$$\partial_s\, \rho_\beta(s) = -\frac{\beta}{2}\left\{\rho_\beta(s), \Phi_\beta^{H(s)}\big(V - \langle V\rangle_{\rho_\beta(s)}\big)\right\},$$

where $\langle V\rangle_{\rho_\beta(s)} = \operatorname{tr}\big(\rho_\beta(s)\,V\big)$,

$$\Phi_\beta^H(V) = \int_{\mathbb{R}} f_\beta(t)\,\mathrm{e}^{\mathrm{i}tH}\,A\,\mathrm{e}^{-\mathrm{i}tH}\,\mathrm{d}t, \tag{4}$$

and $f_\beta(t)$ is an explicit function, which has good decay in t. As for the spectral flow, we find an operator $\tilde{\eta}(s)$ such that $\rho_\beta(s) = \tilde{\eta}(s)\,\rho_\beta(0)\,\tilde{\eta}(s)^*$ and this operator can be approximated locally using LRBs and the decay of f_β. However, $\tilde{\eta}(s)$ is not unitary, which complicates the derivation of LPPL and requires assuming *decay of correlations* in the unperturbed Gibbs state $\rho_\beta(0)$. Moreover, the constants are not uniform in β and one cannot take the $\beta \to \infty$ limit. In [3] we rigorously discuss QBP and use it to prove an equivalence of decay of correlations, LPPL and *local indistinguishability* for Gibbs states, which I briefly explain in the talk.

3. Comments and Open Questions

There are more applications of LRBs for static properties, e.g. to prove decay of correlations for gapped ground states. And many of these have been generalized to broader classes of interactions, e.g. with only polynomial decay. For many of the results, improved LRBs, e.g. for special classes of Hamiltonians, also lead to improved results.

The main open question in view of this talk is to find some mapping between low temperature Gibbs states of gapped Hamiltonians (with constants uniform in β) under suitable additional assumptions.

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The Huang-Yang conjecture for the low-density Fermi gas

Christian Hainzl
(joint work with Emanuela Giacomelli, Phan Thành Nam, Robert Seiringer)

We investigate the ground state energy of a low-density repulsive Fermi gas in the thermodynamic limit, confirming a longstanding conjecture by Huang and Yang regarding its expansion in the small parameter $a\rho^{1/3}$, where the scattering length a is much smaller than the average interparticle distance.

The system is governed by the Hamiltonian

$$H_N := -\sum_{j=1}^{N} \Delta_{x_j} + \sum_{1\le i<j\le N} V(x_i - x_j),$$

acting on the antisymmetric Hilbert space $\bigwedge^N L^2(\Lambda, \mathbb{C}^2)$. As H_N is spin-independent, it preserves the subspace $\mathfrak{h}(N_\uparrow, N_\downarrow)$, consisting of wave functions with exactly N_σ particles of spin $\sigma \in \{\uparrow, \downarrow\}$.

The ground state energy is defined as

$$E_L(N_\uparrow, N_\downarrow) = \inf_{\psi\in\mathfrak{h}(N_\uparrow,N_\downarrow)} \frac{\langle\psi, H_N\psi\rangle}{\langle\psi,\psi\rangle},$$

and its density in the thermodynamic limit is given by

$$e(\rho_\uparrow, \rho_\downarrow) = \lim_{\substack{L\to\infty \\ N_\sigma/L^3\to\rho_\sigma, \sigma\in\{\uparrow,\downarrow\}}} \frac{E_L(N_\uparrow, N_\downarrow)}{L^3}.$$

Using a pseudopotential method, Huang and Yang predicted the following low-density expansion:

$$e(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(3\pi^2)^{2/3}\rho^{5/3} + 2\pi a\rho^2 + \frac{4}{35}(11 - 2\log 2)(3^2\pi)^{2/3} a^2 \rho^{7/3} + o(\rho^{7/3})_{\rho\to 0}.$$

The first term represents the kinetic energy of the filled Fermi sea, while the second, already highly nontrivial, was proven by Lieb, Seiringer, and Solovej (2005). Falconi, Giacomelli, Hainzl, and Porta provided an alternative proof using a method aligned with recent advances in bosonic systems by Schlein et al. Building on this, my talk focused on the final term, proportional to $\rho^{7/3}$, which we establish through distinct methods for the upper and lower bounds. The upper bound is derived using a trial state constructed via Bogoliubov-type unitary rotations, while the lower bound follows from a refined completion-of-squares argument.

Open Problem Session

The participants split into three subgroups: *Bose-Hubbard type Hamiltonians*, *dynamical bounds for continuum systems*, and *positive temperature systems.*

1. Bose-Hubbard Type Hamiltonians

The first group investigated bosonic systems on a d-dimensional lattice Λ with Hamiltonians of the form

$$H = \sum_{x,y\in\Lambda} J_{xy}\, b_x^*\, b_y + \text{h.c.} + U\big(\{n_x\}_{x\in\Lambda}\big),$$

where b_x^* and b_x are bosonic creation and annihilation operators satisfying the *canonical commutation relation* (CCR) $[b_x, b_y^*] = \delta_{xy}$, the first term is a general hopping term, and the second term is a density interaction, where $n_x = b_x^*\, b_x$ is the number operator.

For low-density initial states, there exist *Lieb-Robinson bounds* (LRBs) on information transport with Lieb-Robinson velocity $v \sim t^{d-1}\log^n(t)$ for some n (the expectation is that $n = 0$ should work) and macroscopic as well as microscopic particle propagation bounds for local J_{xy} with velocity $v \sim 1$. The used methods are *adiabatic spacetime localization observables* (ASTLOs), "quantum" walk techniques and projection onto few-particle sectors.

1.1. **Improved Lieb-Robinson Bounds.** The first discussed open question in this area is to improve the information and particle propagation bounds and unify the two. In particular, known protocols that would violate better LRBs use time-dependent Hamiltonians, special initial states and are not stable, i.e. they need to be implemented exactly. Hence, to obtain stronger bounds, one needs to restrict to a physically relevant class of special initial states. In particular, one seeks to obtain tighter bounds on the time-evolved local particle number under certain initial conditions, e.g., that $\langle n_x(t)^p\rangle \sim t^{dp}$ is improved to $\langle n_x(t)^p\rangle \sim c_p\, t^d$. These moment estimates are crucial because they can be used to truncate the unbounded operators to bounded ones via Markov's inequality. Physically, such improvements should come from tracking the dynamical growth of other local physical quantities than particle number, e.g., local energy.

Another direction is to additionally restrict to special Hamiltonians and initial states, where one might achieve Lieb-Robinson velocities $v \sim \log t$ or even $v \sim 1$. One example could be metastable states.

2. Dynamical Bounds for Continuum Systems

The idea for the second group was to combine Lieb-Robinson type estimates with continuum quantum systems.

2.1. **Local Estimates in Extended Boson Systems.** The first investigated problem concerned systems of N bosons at fixed (high) density ρ in an arbitrarily large domain $\Lambda \subset \mathbb{R}^3$ at zero temperature, with Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\Delta_i) + \frac{1}{\rho} \sum_{1\leq i<j\leq N} V(x_i - x_j).$$

Here $\rho = N/|\Lambda| \gg 1$ in the limit $|\Lambda|, N \to \infty$. The proposed approach is to adapt recent techniques developed for fermionic systems in order to demonstrate that the many-body evolution can be locally approximated by the corresponding Hartree dynamics. Through group discussions, a suitable class of initial data was identified, corresponding to zero-temperature states that satisfy certain local bounds, ensuring that the approximation remains effective over an appropriate time interval. The plenary discussion touched on propagation bounds for the nonlinear Hartree equation and on a potentially problematic terms in the Bogoliubov expansion.

2.2. **Lieb-Robinson Bounds for Fermions in the Continuum.** The second discussed question concerns fermions on $\mathbb{R}^d$ with Hamiltonian

$$H_\Lambda = \mathrm{d}\Gamma(-\Delta) + \iint_{\Lambda\times\Lambda} V(x-y)\, a_x^*\, a_x\, a_y^*\, a_y \,\mathrm{d}x\mathrm{d}y$$

for which Lieb-Robinson type estimates are proven if one replaces $a_x^\# = a^\#(\delta_x)$ with a smeared out version $a^\#(\phi_x)$. Denote the smeared out version with $\tilde{H}_\Lambda$. These LRBs then allow to take the thermodynamic limit and prove that

$$t \mapsto \mathrm{e}^{\mathrm{i}t\tilde{H}_{\mathbb{R}^d}}\, a(f)\, \mathrm{e}^{-\mathrm{i}t\tilde{H}_{\mathbb{R}^d}} \tag{1}$$

is strongly continuous. It is expected that strong continuity fails for the Hamiltonian H_Λ without the regularization. The open question is to find a particular interaction V and states to see that (1) for H_Λ is not strongly continuous.

3. Positive Temperature Systems

The third group studied expansions of thermodynamic functions and the validity of first-order perturbation theory at positive temperature for extended systems interacting via a pair potential λV, for a proper $\lambda \in \mathbb{R}$. The goal was to derive an expansion of the free energy in powers of λ. At first order, the free energy corresponds to the non-interacting case, while the second-order term should be given by perturbation theory, i.e.,

$$\mathrm{Tr}(H_N^{(0)}\, \rho) - T\, S(\rho) - F_0 = T\, S(\rho, \rho_0),$$

with the relative entropy $S(\rho, \sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\sigma \log \rho)$. It now remains to bound the relative entropy from above by $O(\lambda)$. Note that by Pinsker's inequality $S(\rho, \rho_0) \geq \frac{1}{2}\, \|\rho - \rho_0\|_1^2$, but this is not useful in the thermodynamic limit $\Lambda \nearrow \mathbb{R}^d$. The key challenge is to establish an upper bound of $\mathcal{O}(\lambda)$ for the relative entropy that is useful in the thermodynamic limit. Achieving this requires bounds that compare locally the Gibbs state of the interacting system with that of the free system. However, a direct approach to this remains unclear. Cluster expansion techniques provide a viable solution at high temperatures, but the primary interest lies in regimes near the Fermi temperature, where λ must be chosen independently of temperature. During the plenary discussion, it was commented that close to a classical phase transition, λ could lead to crossing the phase transition. Hence, extra conditions on ρ_0 might be necessary. Indeed, for ρ_0 satisfying decay of correlations, there might be relations to *quantum belief propagation* which is used to prove local indistinguishability from decay of correlations in lattice systems.

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