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Geometric Structures in Group Theory

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Abstract. The general subject of the conference was geometric group theory, that is the study of groups via suitable actions on spaces endowed with additional structures, for instance a distance, a measure (class), or both of them with natural compatibilities. The interactions with other mathematical fields such as algebraic topology (e.g. via homological methods), functional analysis (e.g. via the theory of operator algebras), probability theory (e.g. via random walks) or even representation theory and number theory, play an increasing role in the development of this part of group theory.

Mathematics Subject Classification (2020): 20Fxx, 57Mxx.

Introduction by the Organizers

The meeting reflected the diversity of the numerous facets of geometric group theory. For instance, some well-established subfields in their relationship with the nowadays traditional questions asked by Gromov in the early 90s, as well as some emerging questions such as the study of groups using their profinite completions, were addressed.

A traditional source of techniques and questions comes from the analogy between some finitely presented groups and lattices in hyperbolic spaces such as Kleinian or Fuchsian groups. The generalization of cocompact lattices in hyperbolic spaces leads to the notion of Gromov hyperbolic groups, while the generalization of non-uniform lattices in hyperbolic spaces results in various definitions of relatively hyperbolic groups. This wide class of groups shows many instances of interplays between algebraic properties of groups and metric properties of spaces on which these groups act. The first natural ground where these relationships can be observed is the Cayley graph of a finitely generated group: it is a prototype in the sense that the construction of the space is elementary but some deep results can be obtained following the idea that for these questions the large scale geometry of the groups is the most fruitful viewpoint. This is how Gromov proved the equivalence, for finitely generated groups, of being virtually nilpotent and having polynomial growth.

Hyperbolic groups can also be seen, via combinatorial approaches such as small cancellation theory, as generalizations of non-abelian free groups, and this leads to a natural interaction with mathematical logic. In addition to these connections, analytic techniques play an increasingly important role. Analysis happens to be precious when dealing with boundaries of groups, whose (quasi-)conformal structure allows to use geometric measure theory in order to obtain rigidity results in the context of negatively curved spaces. But it also turns out to be useful when dealing with quasi-isometries: roughly speaking, two metric spaces are said to be quasi-isometric with one another if they are equivalent for a relation obtained from bi-Lipschitz equivalence by allowing additive error terms in the definition. Some invariants, such as L^p-cohomology, have a strong analytic flavor and are useful when dealing with continuous families of groups (for instance solvable Lie groups). Last but not least, functional analysis plays a prominent role when dealing with problems related operator algebras and representation theory. Among its numerous equivalent formulations, Kazhdan's property (T) can be defined as a fixed point property for actions of Hilbert spaces; it turns out that using wider classes of topological vector spaces as coefficient modules is crucial for new problems in rigidity theory.

A meeting in geometric group theory often covers a wide range of classes of groups and of problems. The classical families of examples are free groups F_n and their groups of outer automorphisms $\mathrm{Out}(F_n)$, mapping class groups $\mathrm{MCG}(\Sigma)$ of surfaces, arithmetic groups of matrices seen as lattices in topological groups, braid groups, Artin groups and Coxeter groups. One new trend consists in studying Lie groups, or more generally locally compact groups, as geometric objects themselves, the class of totally disconnected groups being sometimes particularly interesting because it contains automorphism groups of many interesting discrete structures. These groups are intriguing and challenging because their study can no longer rely on techniques available for linear groups.

Among the new questions emerging in the field, one is of particular interest since it is, so to speak complementary to equivalence relations involving distances and/or measures: it is the problem of studying a (residually finite) finitely generated group via the collection of its finite quotients, or more conceptually (but equivalently) via its profinite completion. This is a more algebraic approach, but it is deeply rooted in geometric group theory in the sense that the class of groups for which the most striking results are available so far, for instance profinite rigidity, is provided by Fuchsian and Kleinian groups. Taking into account finiteness properties, in the most standard meaning from classical algebraic topology, is crucial to understand the subtleties of some rigidity results. In other words, this is a nice way to go back to topological techniques for group theory, and also to build another link with number theory when dealing with some arithmetic hyperbolic groups.

We had 52 participants (49 in person and 3 remote) from a wide range of countries, and 25 lectures.

The staff in Oberwolfach made a tremendous job as usual, from all viewpoints.

We think that this meeting has been a great scientific and human event, thanks to the official talks – all of great quality, and to the informal discussions. Once again, the unique environment at the MFO was one of the main reasons explaining this success.

Workshop: Geometric Structures in Group Theory

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Abstracts

Lattice envelopes of RAAGs

Pierre-Emmanuel Caprace
(joint work with Tom De Medts)

Given a countable discrete group Γ, a **(cocompact) lattice envelope** of Γ is a locally compact group G possessing a (cocompact) lattice isomorphic to Γ. Lattice envelopes were introduced by Furstenberg [Fur67], who focused on envelopes that are Lie groups. The Mostow rigidity theorem (and its generalizations by Margulis and Prasad) can be formulated as a theorem classifying the lattice envelopes (and the lattice embeddings) of certain discrete groups amongst semisimple Lie groups. Furman [Fur01] has obtained a striking generalization by classifying the lattice envelopes of higher rank lattices (e.g. $\mathrm{SL}_3(\mathbf{Z})$) amongst all locally compact groups. More recently, Bader–Furman–Sauer [BFS20] initiated the study of lattice envelopes on a general level, without no restriction on the source Γ and on the target G. Using their results, one can show the following.

Proposition 1 (See [CDM]). *Suppose that Γ satisfies the following conditions.*

(i) *Γ is acylindrically hyperbolic.*
(ii) *Γ is linear over a field (or Γ is finitely generated and residually finite).*
(iii) *Finite subgroups of Γ have a uniformly bounded order (e.g. Γ is virtually torsion-free).*
(iv) *Γ is not relatively hyperbolic with respect to virtually nilpotent subgroups.*

Then every lattice envelope of Γ is cocompact, and totally disconnected modulo a compact normal subgroup.

Given a finite simple graph M with vertex set S, the associated **right-angled Artin group** (or **RAAG**), defined by the presentation

$$\Gamma(M) = \langle S | [s,t] \text{ if } s \text{ is adjacent to } t \rangle,$$

satisfies the hypotheses of the proposition provided M is connected and the complement graph M^c is also connected. In particular, every lattice envelope of $\Gamma(M)$ is a compactly generated locally compact group that is quasi-isometric to $\Gamma(M)$. Relying on deep results by Huang–Kleiner [HK18] on the quasi-isometric rigidity of RAAGs, we establish the following.

Theorem 2 (See [CDM]). *Suppose that M satisfies the following conditions, where the symbol $s^\perp$ denotes the set of those $t \in S$ different from s that are adjacent to s in M.*

(R1) *For each $s \in S$, the induced graph on $S \setminus (s \cup s^\perp)$ is connected.*
(R2) *For all $s, t \in S$, if $s^\perp \subseteq t \cup t^\perp$, then $s = t$.*
(R3) *For each $s \in S$, the only automorphism of M fixing $s \cup s^\perp$ pointwise is the identity.*
(R4) *The complement graph M^c is connected*

Then every locally compact group quasi-isometric to $\Gamma(M)$ *has a discrete quotient commensurable with the right-angled Coxeter group* $W(M)$.

In particular, it follows that such a locally compact group cannot be virtually simple. Combining both results mentioned above, we infer that no lattice envelope of $\Gamma(M)$ is virtually simple. This is in sharp contrast with the case of free groups (i.e. RAAGs defined by empty graphs), or of right-angled Coxeter groups of type M (even if M satisfies all the conditions of the theorem).

Under the same conditions on M, we also show that the full automorphism group G of the Cayley graph of $\Gamma(M)$ with respect to its natural generating set, which is a cocompact lattice envelope of $\Gamma(M)$, possesses a smallest non-trivial normal subgroup G^+ which is simple, and such that the quotient G/G^+ is isomorphic to $W(M) \rtimes \mathrm{Aut}(M)$. In particular $\Gamma(M)$ has a lattice envelope that is almost simple (but not virtually simple).

Examples of graphs satisfying the conditions (R1)–(R4) include the n-cycles with $n \geq 5$. It can be shown that all finite simple graphs satisfy (R1)–(R4) asymptotically almost surely.

The theorem above can be combined with recent results by Horbez–Huang [HH23] on measure equivalence rigidity of RAAGs.

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Growth of automorphisms of virtually special groups

Elia Fioravanti

Breakthroughs of Rips and Sela in the 90s have led to a complete description of the outer automorphism group $\mathrm{Out}(G)$ for any Gromov-hyperbolic group G [6, 7, 8]. This relied on two fundamental properties of (torsion-free) hyperbolic groups:

- $\mathrm{Out}(G)$ is infinite if and only if G splits as an amalgamated product or HNN extension over a cyclic subgroup (either trivial or $\cong \mathbb{Z}$);
- if G is 1–ended, there is an $\mathrm{Out}(G)$–invariant JSJ decomposition of G over subgroups $\cong \mathbb{Z}$.

To this day, very little is known on the general behaviour of outer automorphisms beyond (relatively) hyperbolic groups. Already for non-positively curved groups (e.g. right-angled Artin groups), it is known that $\mathrm{Out}(G)$ can be infinite even if G does not split over any virtually abelian subgroups. Moreover, constructing $\mathrm{Out}(G)$–invariant decompositions becomes particularly delicate if edge groups are large or if G has flats with a complicated intersection pattern.

Rips recently asked about the structure of the group $\mathrm{Out}(G)$ at least when G is cubulated [9]. We address this problem in the slightly more restricted setting of (compact) special groups. A group G is *special* if it is the fundamental group of a compact special cube complex [4]; equivalently, G is a convex-cocompact subgroup of a right-angled Artin group.

Theorem 1 ([3]). *Let G be special and 1–ended. Then G admits an $\mathrm{Out}(G)$–invariant splitting as a graph of groups with the following properties:*

(1) edge groups are either $\cong \mathbb{Z}$, or equal to centralisers of subsets of G;
(2) vertex groups are either quadratically hanging (with trivial fibre), or "rigid" and convex-cocompact in G;
(3) if $H \leq G$ is a direct product of infinite groups, then H is elliptic in the splitting.

Moreover, there is a finite-index subgroup $\mathrm{Out}^0(G) \leq \mathrm{Out}(G)$ with a restriction homomorphism $\mathrm{res}_V : \mathrm{Out}^0(G) \to \mathrm{Out}(V)$ for each vertex group V. If V is "rigid", then the image of res_V virtually embeds in a finite direct product $\prod_i \mathrm{Out}(H_i)$, where the H_i are special groups of lower complexity than G.

Thus, rigid vertex groups can still have infinite outer automorphism groups, unlike in the hyperbolic setting. However, their automorphism groups are of lower "complexity" than the initial one. The notion of convex-cocompactness on G is not canonical, and it rather depends on the particular choice of the convex-cocompact embedding of G into a right-angled Artin group. Nevertheless, rigid vertex groups are convex-cocompact with respect to all such choices.

Using the fact that rigid vertex groups have "simpler" outer automorphism groups than G, one obtains the following general properties of $\mathrm{Out}(G)$ by induction on the complexity of G:

Corollary 2 ([3]). *For any virtually special group G:*

(1) $\mathrm{Out}(G)$ satisfies the Tits alternative: each subgroup of $\mathrm{Out}(G)$ is either virtually polycyclic or contains F_2;
(2) $\mathrm{Out}(G)$ is virtually torsion-free with finite cohomological dimension;
(3) $\mathrm{Out}(G)$ is boundary amenable.

One can also deduce results on growth of outer automorphisms. While the previous corollary was known when G is a right-angled Artin group [5, 2, 1], the following results appear to have been open even in this special case. We denote by $\|\cdot\|$ the conjugacy length of elements of G with respect to any finite generating set of G. The *stretch factor* of an outer automorphism ϕ is then the number

$$\mathrm{str}(\phi) := \sup_{g \in G} \limsup_{n \to +\infty} \|\phi^n(g)\|^{1/n}.$$

Corollary 3 ([3]). *For any virtually special group G and any $\phi \in \mathrm{Out}(G)$:*

(1) the stretch factor $\mathrm{str}(\phi)$ *is an algebraic integer and a weak Perron number;*
(2) if $\mathrm{str}(\phi) = 1$*, then ϕ grows at most polynomially;*
(3) for some $k \in \mathbb{N}$, the stretch factor $\mathrm{str}(\phi)$ *is realised on a ϕ^k–invariant subgroup of G that is either a surface group, a free product, or a group with a free abelian direct factor.*

More precise growth information can be obtained when ϕ coarsely preserves the coarse median on G induced by a convex-cocompact embedding in a right-angled Artin group. The simplest examples of automorphisms ϕ with this property are: all automorphisms of Gromov-hyperbolic groups, all automorphisms of right-angled Coxeter groups, untwisted automorphisms of right-angled Artin groups.

Corollary 4 ([3]). *Let G be virtually special and ϕ coarse-median preserving.*

(1) There are only finitely many growth rates $n \mapsto \|\phi^n(g)\|$ up to bi-Lipschitz equivalence. Each of these is either $\preceq n^p$ for some $p \in \mathbb{N}$, or it is $\sim n^p\lambda^n$ for some $p \in \mathbb{N}$ and some weak Perron number $\lambda > 1$.
(2) There is a "Nielsen–Thurston decomposition" for ϕ: for each growth rate $\mathfrak{o}$, there are only finitely many G–conjugacy classes of maximal subgroups of G all of whose elements grow at speed $\preceq \mathfrak{o}$ under ϕ. Moreover, all these subgroups are convex-cocompact in G.

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Free-by-cyclic groups are equationally Noetherian

Monika Kudlinska
(joint work with Motiejus Valiunas)

Let F_n denote the free group of rank n and let $\mathcal{G}$ be a collection of groups. Given a subset $S \subseteq F_n$, we let $V_{\mathcal{G}}(S)$ denote the set of all homomorphisms $\phi\colon F_n \to G$, for some $G \in \mathcal{G}$, such that $\phi(w) = 1_G$ for all $w \in S$. We say that $\mathcal{G}$ is an *equationally Noetherian family* if for every n and every subset $S \subseteq F_n$, there exists a finite subset $S_0 \subseteq S$ such that

$$V_{\mathcal{G}}(S) = V_{\mathcal{G}}(S_0).$$

A group G is *equationally Noetherian* if $\mathcal{G} = \{G\}$ is an equationally Noetherian family.

Whilst the property of being an equationally Noetherian group can be shown to hold in many cases, infinite equationally Noetherian families are rare. Indeed, every linear group is equationally Noetherian by Hilbert's basis theorem, however the family of all linear groups is not equationally Noetherian (see, e.g., [11, Corollary 1.2(iv)]). By the work of Z. Sela [10], every torsion-free word hyperbolic group is equationally Noetherian, however the family of all such groups is not an equationally Noetherian family [5, Example 3.15]. Remarkably,

Theorem 1 (Groves–Hull–Liang [6])**.** *The collection of all 3-manifold groups forms an equationally Noetherian family.*

A group G is *free-by-cyclic* if it contains a finitely generated free normal subgroup $F \trianglelefteq G$ such that $G/F \cong \mathbb{Z}$. Free-by-cyclic groups share many similarities with the family of 3-manifold groups. For instance, all free-by-cyclic groups are coherent by the work of Feighn–Handel [2]. Moreover, atoroidal free-by-cyclic groups are word-hyperbolic [1] and act geometrically on CAT(0) cube complexes [7], and thus are virtually special.

However, unlike in the case of finite-volume hyperbolic 3-manifold groups, there are examples of hyperbolic free-by-cyclic groups which are not LERF [8]. Furthermore, recent work of Munro–Petyt shows that there are free-by-cyclic groups which do not admit coarse median structures and thus are not hierarchically hyperbolic [9].

Our main result is the following.

Theorem 2. *Every free-by-cyclic group is equationally Noetherian.*

The key application is to the study of growth rates in free-by-cyclic groups. If G is a group and X a finite generating set for G, we write $\beta_n(G, X)$ denote the number of elements of G which can be expressed as a word of length at most n in $X^{\pm}$. The *growth rate* of G with respect to X is

$$e(G, X) := \lim_{n\to\infty} \beta_n(G, X)^{1/n}.$$

We let $\xi(G)$ denote the set of growth rates

$$\xi(G) := \{e(G, X) \mid X \text{ finite generating set of } G\}.$$

Fujiwara–Sela show that for any word-hyperbolic group G, the set of growth rates $\xi(G)$ is well ordered [4]. This was extended by K. Fujiwara to show that any equationally Noetherian group that admits a non-elementary acylindrical action on a hyperbolic space with uniformly short loxodromics has a well-ordered set of growth rates [3].

The geometry of a free-by-cyclic group $G = F_n \rtimes_\varphi \mathbb{Z}$ is determined by the dynamics of the automorphism $\varphi \in \mathrm{Aut}(F_n)$. If the word length of some element $g \in F_n$ grows exponentially under the iterations of φ, the free-by-cyclic group G admits a non-trivial relatively hyperbolic structure. We further show that any free-by-cyclic group $G = F_n \rtimes_\varphi \mathbb{Z}$ where $\varphi \in \mathrm{Aut}(F_n)$ acts polynomially on every $g \in F_n$, either virtually splits as a direct product $F_n \times \mathbb{Z}$, or acts 4-acylindrically on a simplicial tree. Combining these facts with the work of Fujiwara, we obtain the following corollary.

Corollary 3. *If G is a free-by-cyclic group then the set of exponential growth rates $\xi(G)$ is well ordered.*

An immediate question is the following

Problem 4. *Characterise the finite generating sets X of G which realise the minimum of $\xi(G)$.*

Motivated by the similarities between the family of free-by-cyclic groups and fundamental groups of 3-manifolds, it is tempting to ask:

Question 5. *Do free-by-cyclic groups form an equationally Noetherian family?*

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Hardy-Littlewood maximal inequality for hyperbolic groups

Koji Fujiwara
(joint work with Amos Nevo)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function. For $x \in \mathbb{R}^d$, define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball of radius r at x and $|B(x,r)|$ is its Lebesgue volume. This sublinear operator M is called *the Hardy-Littlewood maximal operator.*

Hardy-Littlewood (in the case $d = 1$ in 1930) and Wiener (in the case $d \geq 2$ in 1939) proved the following theorem called the *Hardy-Littlewood maximal inequality.*

Theorem 1. *For $d \geq 1$, there is a constant $C_d > 0$ such that for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^d)$, we have*

$$|\{Mf > \lambda\}| < \frac{C_d}{\lambda} \|f\|_{L^1}.$$

The Hardy-Littlewood maximal inequality has been generalized to locally symmetric spaces of non-compact type (Stromberg, 1981).

The Hardy-Littlewood maximal operator makes sense for a metric space with a measure. For a graph Γ with the counting measure on the set of vertices, define for $f \in \ell^1(\Gamma)$:

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \sum_{B(x,r)} |f(y)|.$$

The Hardy-Littlewood maximal inequality has been (essentially) known for regular trees by Rochberg-Taibleson (1991) using random walks. More recently, Naor-Tao [NT] gave a new proof of the inequality for k-regular trees. Namely, there exists a constant $C > 0$ such that for any ℓ^1-function f and a constant $\lambda > 0$, we have:

$$|\{Mf > \lambda\}| < \frac{C}{\lambda} \|f\|_1.$$

The constant C does not depend on k. One of the key ideas is geometric, which they call "expander estimates", which states the following:

Let T be the infinite $(k+1)$-regular tree with $k \geq 2$. Let E, F be finite subsets of T, and $r \geq 0$. Then

$$|\{(x,y) \in E \times F : d(x,y) = r\}| \leq 2|E|^{1/2}|F|^{1/2}k^{r/2},$$

where $|E|$ is the cardinality of E.

Genaralizing their expander estimates, we prove the Hardy-Littlewood maximal inequality for a hyperbolic group (in the sense of Gromov) as follows.

Theorem 2. [FN]. *Let G be a non-elementary hyperbolic group, and S a finite generating set. Let Γ be the Cayley graph for (G,S), with the counting measure* *on the set of vertices. Then the Hardy-Littlewood maximal inequality holds for* $f \in \ell^1(\Gamma)$.

In our theorem, the constant C depends on G and S. To prove the theorem, we follow the argument by Naor-Tao, showing an expander type estimate for Γ.

Lemma 3. *Let k be the exponential growth rate of Γ. Then there exists a constant D such that for any $r \geq 0$ and any finite sets $E, F \subset \Gamma$, we have*

$$|\{(x,y) \in E \times F : d(x,y) = r\}| \leq D|E|^{1/2}|F|^{1/2}k^{r/2}.$$

We also use the following estimate on growth due to Coornaert: there is a constant A such that for any $r > 0$

$$\frac{k^r}{A} \leq |B(r)| \leq Ak^r,$$

where $B(r)$ is the ball of radius r centered at the identity.

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Algebraic groups over the free groups and hyperbolic groups

Vincent Guirardel

(joint work with Chloé Perin)

The context of this talk is algebraic geometry over groups. We fix a base group Γ, thought as the group of constants. Here, Γ will be a fixed non-elementary torsion-free hyperbolic group. As a replacement of polynomials, we have words in formal variables such as $w(X_1, X_2, X_3) = X_1^2 a X_2 X_3^{-3} b X_1^{-1} c$ where $a, b, c \in \Gamma$ are constants. More generally, words are elements of the free product $\Gamma * \langle X_1, \ldots, X_n \rangle$

An algebraic variety is a subset $V \subset \Gamma^n$ defined by a collection word equations of the form $w(X_1, \ldots, X_n) = 1$. It is irreducible if it cannot be written as a union of two proper subvarieties. An algebraic map between algebraic varieties V, W is a map $F : V \to W$ given as the restriction of a tuple of word maps.

Definition 1. *An (affine) algebraic group over Γ is an algebraic variety $V \subset \Gamma^n$ together with a group law $V \times V \to V$ given by an algebraic map, and where $g \mapsto g^{-1}$ is also an algebraic map.*

The question we address is a classification of algebraic groups over a torsion free hyperbolic group Γ. We can rephrase this into the following provocative question: can we define $SL_N(\Gamma)$ for Γ a hyperbolic group ?

Here are easy examples of algebraic groups over Γ:

- $V = \Gamma$ with its standard multiplication
- Given $c \in \Gamma$, $V = Z(c)$ with its standard multiplication

- If V_1, V_2 are algebraic groups, then $V_1 \times V_2$ is an algebraic group too.
- Here is a complicated multiplication law on Γ^2:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1^2 x_2^{-1} y_1^2 y_2^{-1} x_2 x_1^{-1} y_2 y_1^{-1} \\ x_2 x_1^{-1} y_2 y_1^{-1} x_1^2 x_2^{-1} y_1^2 y_2^{-1} x_2 x_1^{-1} y_2 y_1^{-1} \end{pmatrix}$$

where the neutral element is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This looks complicated but this is just the standard law on Γ^2 twisted by the algebraic bijection $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 x_2 \\ x_2 x_1 x_2 \end{pmatrix}$.

To give more context we can look at the more general notion of definable or even interpretable groups over different groups Γ. For instance, if Γ is $SL_N(\mathbb{C})$ or the Heisenberg group $H_3(\mathbb{C})$, then any interpretable subgroup in Γ is in fact interpretable over the field $\mathbb{C}$. One can then apply a theorem by Hrushosvki and Weil saying that any interpretable group over $\mathbb{C}$ is an algebraic group (maybe not affine). Conversely, $(\mathbb{C}, +, .)$ is interpretable in $H_3(\mathbb{C})$ so every algebraic group over $\mathbb{C}$ is interpretable over the group $\Gamma = H_3(\mathbb{C})$.

For the same reason, $(\mathbb{Z}, +, .)$ is interpretable in the group $\Gamma = H_3(\mathbb{Z})$: arithmetic is interpretable in $H_3(\mathbb{Z})$. This implies for instance that any finitely presented group is interpretable over the group $H_3(\mathbb{Z})$.

Back to the case where Γ is a torsion-free hyperbolic group, the theorem we prove is the following:

Theorem 2. *Let Γ be a non-elementary, torsion-free hyperbolic group. Let V be an irreducible algebraic group over Γ.*

Then V is isomorphic as an algebraic group to $\Gamma^l \times Z(c_1) \times \dots Z(c_n)$ for some $l \in \mathbb{N}$ and $c_1, \dots, c_k \in \Gamma$.

The main tools we use for the proof are:

(1) The formalism for algebraic geometry over groups by Baumslag-Miasnikov-Remeslennikov [BMR99].
(2) The structure of Γ-limit groups by Sela and Kharlampovich-Miasnikov and their JSJ decompositions [Sel01, KM98].
(3) A result from the theory of automorphisms of free groups or free products by Handel-Mosher and G-Horbez [HM20, GH22].

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Hypergraph index, divergence and thickness for general Coxeter groups

Anne Thomas

(joint work with Pallavi Dani, Yusra Naqvi, Ignat Soroko)

The *divergence* of a pair of geodesic rays measures how fast they move away from each other. In symmetric spaces of noncompact type, divergence is either linear or exponential, and Gromov suggested in [18, 6.B_2(h), p. 133] that the same dichotomy should hold for CAT(0) spaces. In the 1990s, Gersten [16] used the idea of divergence of geodesic rays to define a quasi-isometry invariant, also called *divergence*.

For many important families of groups, it is now known that divergence can be linear, quadratic or exponential; these families include 3-manifold groups [17], mapping class groups [2, 13] and right-angled Artin groups [1, 3]. The first constructions of families of CAT(0) groups with divergence polynomial of any integer degree are due to Behrstock–Druţu [4] and independently Macura [23]. Brady and Tran [8, 9] constructed finitely presented groups (including subgroups of CAT(0) groups) with divergence neither polynomial of integer degree nor exponential.

The quasi-isometry invariant *thickness* was introduced by Behrstock, Druţu and Mosher [5] as a means to understand the geometry of non-relatively hyperbolic groups. A metric space X is thick of order 0 if none of its asymptotic cones has a cut-point (i.e. it is *wide*), and for integers $n \geq 1$, the space X is thick of order n if any two points in X can be connected by a "chain" of subsets, each of which is thick of order $n-1$. Behrstock and Druţu proved in [4, Corollary 4.17] that a group which is thick of order at most n has divergence at most r^{n+1}.

In the setting of right-angled Coxeter groups, the study of divergence was begun by Dani and Thomas [15], and continued by Behrstock, Falgas-Ravry, Hagen and Susse [6], Behrstock, Hagen and Sisto [7], and Levcovitz [20, 21, 22]. In [21], Levcovitz introduced the *hypergraph index* $h = h(W_\Gamma)$ of a right-angled Coxeter group W_Γ, a computable combinatorial invariant which takes values $h \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

Right-angled Coxeter groups are *rigid*, meaning that up to isomorphism they have unique defining graphs [24], but in general Coxeter groups are not rigid. Thus we refer to Coxeter systems rather than Coxeter groups. For arbitrary Coxeter systems (W, S), Caprace [10, 11] characterised the collections of special subgroups of W such that W is hyperbolic relative to such a collection, while Behrstock, Hagen, Sisto and Caprace proved in the Appendix to [7] that W admits a canonical minimal relatively hyperbolic structure, whose peripheral subgroups are in fact special subgroups. In the same Appendix, these four authors also proved that any

Coxeter group is either thick or is relatively hyperbolic, and provided an inductive construction of all Coxeter systems (W, S) such that W is thick. The idea of Levcovitz's hypergraph index is to compress into a single step certain stages of this construction which have the same effect upon the order of thickness.

Our first main result characterises linear and quadratic divergence for arbitrary Coxeter systems (W, S) (see Corollary 1.2 of [14]). The proof is brief and combines a Coxeter-theoretic result of Caprace and Fujiwara [12] with a general result of Kapovich and Leeb [19].

We introduce *hypergraph index* $h = h(W, S)$ for general Coxeter systems (W, S), generalising Levcovitz's definition in the right-angled case [21], and then relate hypergraph index, thickness and divergence as follows (see Theorem 1.4 of [14]). We prove:

(1) $h = 0$ if and only if W has linear divergence.
(2) If $h = 1$, then W has quadratic divergence.
(3) If h is finite, then W is thick of order at most h.
(4) $h = \infty$ if and only if W is relatively hyperbolic.

Our proofs make key use of the special subgroup structure of Coxeter groups.

We then conjecture that the following are equivalent:

(1) h is finite;
(2) W is thick of order h; and
(3) the divergence of W is polynomial of degree $h + 1$.

In the right-angled case, this conjecture was proved by Levcovitz as Theorem A of [22]. If our conjecture holds, there are many interesting corollaries, as in [22]. For instance, the quasi-isometry invariants divergence and thickness are difficult to compute, but hypergraph index provides an algorithmic way to detect them. It also follows that the divergence of any 1-ended Coxeter group is either polynomial of integer degree or exponential. In addition, our conjecture implies that hypergraph index is a (computable) quasi-isometry invariant of 1-ended Coxeter groups, which is not obvious from the definition. As evidence for our conjecture, we build on a family of examples from [15] to construct "many" examples of non-right-angled Coxeter systems for which this conjecture holds.

For our last main result, let $b_1(\Delta)$ be the first Betti number of the Dynkin diagram Δ for the Coxeter system (W, S). We prove that if h is finite, then $h \leq b_1(\Delta) + 1$ (see Theorem 1.7 of [14]). We do not know whether this bound is sharp.

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Actions of higher-rank lattices on uniformly convex Banach spaces

Mikael de la Salle
(joint work with Tim de Laat)

The talk, based on a joint work [3] with Tim de Laat and a previous breakthrough by Izhar Oppenheim [6], was devoted to group actions by isometries on real Banach spaces. Before entering further into the subject, let us start with a classical fact:

Theorem 1. *(Mazur-Ulam) Every isometry of a Banach space is affine.*

Therefore, the natural objects when considering group actions by isometries on Banach spaces are affine Banach spaces, that is Banach spaces where we forget the place of the origin. And a group Γ acting by isometries on an affine Banach space has a fixed point if and only if there is a choice of origin for which it becomes linear. When this holds for every action on a given space E, we say that Γ has the fixed point property with respect to E, and write Γ has FE.

The following examples are good to have in mind:

Example 2. *If Γ is a finite group, every affine action on a real vector space has a fixed point.*

Example 3. *If Γ is an infinite group, the natural (left-translation) action on $\{f \in \ell^1(\Gamma) \mid \sum_\gamma f(\gamma) = 1\}$ is a fixed-point free isometric action on an affine Banach space isometric to $\ell^1_0(\Gamma)$.*

Example 4. *If Γ is an infinite group, there is a proper function $f\colon \Gamma \to \mathbf{R}$ such that $\sup_g |f(\gamma^{-1}g) - f(g)| < \infty$ for every g (for example the word-length with respect to a finite generating set if Γ is finitely generated). Then the natural (left-translation) action on $f + \ell^\infty(\Gamma)$ is an isometric action with unbounded orbits on an affine Banach space isometric to $\ell^\infty(\Gamma)$.*

Example 5. *(Delorme-Guichardet, see [2]) A countable group has Kazdhan's property (T) (meaning that the trivial representation is isolated in the unitary dual, that is the space of irreducible unitary representations equippd with Fell's topology) if and only if every action by isometries on a Hilbert space has a fixed point.*

Therefore there are plenty of groups that have $F\ell^2$, but no countable infinite group has $F\ell^\infty$ or $F\ell^1_0$.

Probably motivated by these examples, results that they obtained by L_p spaces, and the general belief that higher-rank simple groups and lattices have the strongest forms of rigidity, Bader, Furman, Gelander and Monod [1] conjectured that if G is an almost simple connected linear algebraic group over a local field, and $\Gamma < G$ is a lattice, then Γ has FE for every uniformly convex spaces.

Lafforgue and Liao [4, 5] confirmed this conjecture for non-archimedean local fields (finite extension of $\mathbf{Q}_p$ of $\mathbf{F}_p((T))$). The proof for archimedean local fields ($\mathbf{R}$ or $\mathbf{C}$) took longer to be obtained: Oppenheim [6] proved the case when the Lie algebra of G contains a copy of $\mathfrak{sl}_4(\mathbf{R})$, and we extended it to all groups in [3].

Both proofs of Lafforgue-Liao, and ours, uses the same pattern: by a classical induction procedure, we can consider actions of G by isometries on E, and the idea is to construct, by hand, a sequence of probability measures μ_n on G such that, for every starting point $\xi \in E$, the sequence $\mu_n \cdot \xi := \int g \cdot \xi d\mu_n(g)$ is Cauchy and almost invariant. The difficult part is to prove the Cauchy criterion, and the proof has two aspects: an analytic one where one shows that certain moves $\mu \to \mu'$ in the space of measures give rise to very small moves $\mu \cdot \xi \mapsto \mu' \cdot \xi$, and a combinatorial one where one explores efficiently the set of measures along such moves in order. And here the proofs diverge. In the non-archimedean case, μ_n can essentially be taken as any sequence of probability measures that are invariant by left and right-multiplication by a maximal compact subgroup, so that the set of possible measures is parametrized by the Weyl chamber. In the real case, the measures are less natural: they are convolution of Gaussian probability measures in various 1-parameter subgroups of G (corresponding to the roots spaces associated to the root system). At the heart of the analytic part are some estimates for actions of nilpotent groups on uniformly convex Banach spaces. This provides in particular a new proof of property (T) for $\mathrm{SL}_3(\mathbf{R})$.

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C^*-rigidity for proper metric spaces

Federico Vigolo
(joint work with Diego Martínez)

This talk is centered around the interplay between coarse geometry and operator algebras. Starting with the former, recall that a function $f\colon X \to Y$ between metric spaces is called *controlled* if for every $r \geq 0$ there is some $R \geq 0$ such that if $d(x,x') \leq r$ then $d(f(x),f(x')) \leq R$. A mapping $g\colon Y \to X$ is a *coarse inverse* for f if $g \circ f$ is within bounded distance from the identity (*i.e.* $g(f(x))$ stays uniformly close to x when x ranges in X). The spaces X and Y are then *coarsely equivalent* if there are controlled maps $f\colon X \to Y$ and $g\colon Y \to X$ that are coarse inverses to one another. In other words, the only piece of information about X that is always retained up to coarse equivalence is which families of subsets of X have uniformly bounded diameter.

On the operator-algebraic side, the *Roe algebra* $C^*_{\mathrm{Roe}}(X)$ of a proper metric space X is a C^*-algebra consisting of limits of locally compact linear operators of finite propagation. The original interest in these algebras stemmed from index-theoretic considerations [7, 11, 12, 14], and they have since played a major role in the theory of operator algebras and noncommutative geometry.

If M is a complete Riemannian manifold of dimension greater than one, then $C^*_{\mathrm{Roe}}(X)$ can be constructed as a subalgebra of $\mathcal{B}(L^2(M))$. To do so, we say that an operator has *finite propagation* if there is some $r \geq 0$ such that

$$\mathbb{1}_A t \mathbb{1}_B = 0 \qquad (1)$$

for all $A, B \subset M$ measurable with $d(A, B) > r$, where $\mathbb{1}_{(\text{-})} \in \mathcal{B}(L^2(M))$ is the operator defined by multiplication with the indicator function. The operator t is *locally compact* if $t\mathbb{1}_A$ and $\mathbb{1}_A t$ are compact operators for every choice of measurable $A \subseteq X$ of finite diameter. The Roe algebra of M is defined as the norm-closure

$$C^*_{\mathrm{Roe}}(M) := \overline{\{t \in \mathcal{B}(L^2(M)) \mid t \text{ of finite propagation, locally compact}\}}.$$

For a discrete proper metric space X, $C^*_{\mathrm{Roe}}(X)$ is defined analogously, but in place of $L^2(M)$ one uses the space $\ell^2(X; \mathcal{H})$ of square-integrable functions with values in the infinite rank separable Hilbert spaces $\mathcal{H}$. More generally, one may give a definition of Roe algebra for arbitrary proper metric spaces as algebras of operators on geometric modules (see *e.g.* [8, 14] for modern references).

Considering algebras of operators of finite propagation can be viewed as a coarsening procedure (it is the noncommutative analogue of quotienting out uniformly bounded sets [6, 12]). From this perspective, it is unsurprising that $C^*_{\mathrm{Roe}}(X)$ depends solely on the coarse geometry of X. The *C^*-rigidity problem* asks whether the converse is true. Namely, if two metric spaces have isomorphic Roe algebras, do they have to be coarsely equivalent?

Starting with [13], the last decade saw an impressive amount of progress on this question (see *e.g.* [1, 2, 3, 4, 5] and references therein). I am pleased to report that we now have a complete solution to the C^*-rigidity problem:

Theorem 1. *If X and Y are proper metric spaces with $C^*_{\mathrm{Roe}}(X) \cong C^*_{\mathrm{Roe}}(Y)$ then X and Y are coarsely equivalent.*

This proves that there is a perfect correspondence between C^*-algebraic properties of Roe algebras and coarse geometric properties of proper metric spaces, with all the ensuing consequences. Moreover, a more refined result holds. Namely, let $\mathrm{CE}(X)$ denote the space of equivalence classes of coarse equivalence of X with itself, taken up to closeness. It is easy to see that $\mathrm{CE}(X)$ is a group. Examining the proof of the coarse invariance of roe algebras up to coarse equivalences, it is not hard to show that there is in fact a natural homomorphism from $\mathrm{CE}(X)$ to the group of outer automorphisms $\mathrm{Out}(C^*_{\mathrm{Roe}}(X))$ (an outer automorphism is an equivalence class of automorphisms up to conjugation by elements in the multiplier algebra of $C^*_{\mathrm{Roe}}(X)$). Our techniques then let us prove:

Theorem 2. *For every proper metric space X, the natural homomorphism*

$$\mathrm{CE}(X) \to \mathrm{Out}(C^*_{\mathrm{Roe}}(X))$$

is an isomorphism.

A short proof of both theorems in the restricted setting of metric spaces of bounded geometry is contained in [10]. This proof is not self contained, as it relies on [5, 13]. A long but self-contained proof of the theorem for arbitrary proper (extended) metric spaces is provided in [8, 9].

In the latter works we developed a general framework for studying C^*-rigidity problems, which applies simultaneously to several classes of C^*-algebras of geometry origin (e.g. the C^*-algebra of quasi-local operators), and shows that rigidity phenomena appears even under weaker assumptions, such as Morita equivalence/stable $*$-isomorphism.

Very briefly, the proof plan for the solution to the C^*-rigidity problem consists of three main steps:

(1) (Spatial Implementation): realize that an isomorphism of Roe algebras must be induced by conjugation by a unitary operator among the underlying Hilbert spaces.
(2) (Uniformization): prove that said unitary operator "uniformly almost preserves propagation", and use this to construct candidate maps between the spaces.
(3) (Concentration Inequality): prove that said maps are indeed well defined and give rise to a coarse equivalence.

Our main contribution is in the last step of this proof-plan.

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The handlebody group is a virtual duality group

Richard D. Wade
(joint work with Dan Petersen)

A (Bieri–Eckmann) *duality group of dimension* d is a group G which has a right G-module D (called the *dualising module*) and an element $C \in H_d(G; D)$ (called the *fundamental class*) such that the cap product by C induces isomorphisms

$$H^k(G; A) \cong H_{d-k}(G; D \otimes A)$$

for all left G-modules A and integers k. A group is a *virtual duality group* if it has a finite-index subgroup which is a duality group. Many discrete groups of interest in geometry and topology are virtual duality groups, including arithmetic lattices [4], mapping class groups of surfaces [6], and outer automorphism groups of free groups [2].

One reason to care about a group being a virtual duality group is that it provides a powerful tool for studying its cohomology in high degrees (close to the virtual cohomological dimension), provided one has a good understanding of the dualising module. In the case of arithmetic lattices, the dualising module is the *Steinberg module*, which is given by the homology of the associated *rational Tits building*. In the case of mapping class groups, the dualising module corresponds to the homology of the *complex of curves* of the surface. The dualising module of $\mathrm{Out}(F_N)$ is much less well-understood: see [8] for an overview of the difficulties involved.

A *handlebody group* is a mapping class group of a 3-dimensional handlebody V. If $S = \partial V$ is the boundary surface of V then the handlebody group $\mathrm{Mod}(V)$ embeds into the mapping class group $\mathrm{Mod}(S)$ of the boundary surface, via restriction of the mapping class to S. The disc complex $\mathcal{D}(V)$ is the subcomplex of the curve complex given by curves on S which bound discs in V. While one might initially guess that the homology of the disc complex is the dualising module of $\mathrm{Mod}(V)$, this is not the case as $\mathcal{D}(V)$ is a *contractible* subcomplex of the curve complex – its homology vanishes. However, there is a subcomplex $\mathcal{NS}(V)$ spanned by simplices corresponding to *non-simple* disc systems (these are sometimes called *non-filling* in the literature). These are the disc systems which *do not* cut V into a union of balls (some genus is left behind). Our main result is as follows:

Theorem 1 (Petersen-W, [7]). *Let $g \geq 2$ and V be a handlebody of genus g. The handlebody group* $\mathrm{Mod}(V)$ *is a virtual duality group of dimension $d = 4g - 5$. The complex $\mathcal{NS}(V)$ is homology equivalent to a wedge of spheres of dimension $2g - 3$, and its homology $H_{2g-3}(\mathcal{NS}(V))$ is the dualising module of* $\mathrm{Mod}(V)$.

The proof is obtained using the following theorem, which follows from combining work of Bieri and Eckmann with Poincaré–Lefschetz duality:

Theorem 2 (Bieri–Eckmann, [3]). *Let G be a virtually torsion-free group acting properly and cocompactly on a contractible n-manifold M. Then G is a virtual duality group if and only if ∂M is homology equivalent to a wedge of spheres of* *equal dimension q. When this holds, the dualising module is $\tilde{H}_q(\partial M, \mathbb{Z})$ and the vcd of G is $d = n - q - 1$.*

Our manifold M is obtained by intersecting a thick part of Teichmüller space with a *handlebody horoball* $\mathcal{H}_\epsilon$. These handlebody horoballs are subspaces of Teichmüller space introduced by Hainaut–Petersen [5], who showed that each $\mathcal{H}_\epsilon$ is contractible. Precisely one obtains M by choosing constants $0 < \eta < \epsilon$ which are smaller than the Margulis constant, and defines $M \subset \mathcal{T}$ to be the subspace of Teichmüller space consisting of marked surfaces σ such that

- Every essential curve on σ has length at least η, and
- the set of geodesic curves of length $\leq \epsilon$ on σ bound a *simple* disc system in V (the discs cut V into balls).

Surfaces in M don't contain any η-short curves, but contain enough ϵ-short curves bounding discs to cut V up into balls. Our main theorem is obtained from the following results:

- M is a contractible $6g-6$-dimensional manifold on which $\mathrm{Mod}(V)$ acts properly and cocompactly.
- The boundary of M is homotopy equivalent to the *suspension* of the complex of non-simple disc systems $\mathcal{NS}(V)$.
- The complex $\mathcal{NS}(V)$ is homology equivalent to a wedge of spheres of constant dimension $2g-3$.

These key tools/ingredients are:

- Methods from combinatorial algebraic topology, notably homotopy types of geometric realizations of posets and the tools developed to study such spaces (e.g. Quillen's *fiber lemma*).
- Stratifications of M and ∂M by posets closely related to the disc complex with contractible strata.
- An induction argument using relative methods (handlebodies with marked discs/points).

A more detailed outline of the proof is provided in the introduction of [7]. Our methods do not give a description of a generating set of $H_*(\mathcal{NS}(V))$ as a $\mathrm{Mod}(V)$-module, which would be desirable for using the above theorem to study the cohomology of $\mathrm{Mod}(V)$ close to the vcd. For more details on this and other problems on handlebody groups, see [1].

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On L^p-cohomology of semisimple groups

Antonio López Neumann

L^p-cohomology is a rather fine quasi-isometry invariant first introduced in [4] and popularized by Gromov in [5]. It can be defined in slightly different settings: one can talk about simplicial ℓ^p-cohomology of a simplicial complex, de Rham L^p-cohomology of a Riemannian manifold or continuous group L^p-cohomology of a locally compact second countable (lcsc) group. These different versions are continuously isomorphic under natural assumptions (e.g. a geometric action of a lcsc group on a contractible simplicial complex or manifold).

Here we deal with *continuous group* L^p*-cohomology* of lcsc groups for $1 < p < \infty$, which for a lcsc group G it can be defined as continuous cohomology with coefficients in the right regular representation in $L^p(G) := L^p(G, m_G)$ (where m_G is a left Haar measure) and we denote by $H^*_{\mathrm{ct}}(G, L^p(G))$.

Computing L^p-cohomology can be useful in problems around the quasi-isometric classification of solvable Lie groups [2]. Currently, the main drawback to use this invariant for this purpose is the lack of more systematic methods to compute it. Even in the classical situation of real semisimple groups we do not have a good description of this cohomology. Gromov predicted that for these groups (and also for their non-Archimedean counterparts), L^p-cohomology should exhibit a classical behaviour: vanishing below the rank and non-vanishing in degree equal to the rank.

Question 1. [5, p. 253] *Let F be a local field (Archimedean or not), let G be the F-points of a semisimple algebraic group defined over F and let $r = \mathrm{rk}_F(G)$.*
(1) *Do we have $H^k_{\mathrm{ct}}(G, L^p(G)) = \{0\}$ for $1 < p < \infty$ and $k < r$?*
(2) *Does there exist some $1 < p < \infty$ such that $H^r_{\mathrm{ct}}(G, L^p(G)) \neq \{0\}$?*

In this talk we review results pointing towards affirmative answers to these questions. We will mostly focus on advances towards question (1), as question (2) has now a rather satisfactory answer.

Theorem 2. [1, Theorem A] [6, 1.2] *Let F be a local field, let G be the F-points of a semisimple algebraic group defined over F and let $r = \mathrm{rk}_F(G)$.*
- *If $F = \mathbb{R}$, then for p large enough we have $H^r_{\mathrm{ct}}(G, L^p(G)) \neq \{0\}$.*
- *If F is non-Archimedean, then for all $p > 1$ we have $H^r_{\mathrm{ct}}(G, L^p(G)) \neq \{0\}$.*

Question (1) has an affirmative answer for degree 1 cohomology.

Theorem 3. [8, Théorème 1] [3, Theorem 1] *Let F be a local field, let G be the F-points of a semisimple algebraic group defined over F. Suppose that $\mathrm{rk}_F(G) \geq 2$. Then $H^1_{\mathrm{ct}}(G, L^p(G)) = \{0\}$ for all $1 < p < \infty$.*

We now list some results for higher degree L^p-cohomology of semisimple groups. In the non-Archimedean case, vanishing of L^2-cohomology (and cohomology with unitary coefficients) in degrees below the rank is implicit in the work of Garland. Dymara-Januszkiewicz and Oppenheim use interpolation to extend this result for values of p in certain closed intervals inside $(1, +\infty)$ containing 2. For real semisimple groups, vanishing of reduced L^2-cohomology is first due to Borel. Bourdon-Rémy and Stern showed vanishing of L^p-cohomology for p in closed intervals containing $(1, 2]$ relying on curvature pinching arguments by Pansu.

In all the results mentioned above, methods seem to encounter problems when dealing with large values of $p > 1$. The next result applies to a wider range of values of $p > 1$, but only for L^p-cohomology in degree 2.

Theorem 4. [7, Theorem 1] *Let F be a local field and let G be one of the following semisimple algebraic groups over F:*
- $\mathrm{SL}(4, D)$ *where D is any finite central division algebra over F,*
- *G is simple over F, has rank ≥ 4 and is not of type D_4 or of exceptional type,*
- *G is semisimple, non-simple over F of rank ≥ 3.*

Then
$$H^2_{\mathrm{ct}}(G, L^p(G)) = \{0\}$$
for all $p > 1$ if F is non-Archimedean and for large values of p if $F = \mathbb{R}$.

The proof of this result mixes homological and dynamical arguments and is closer in spirit to the proofs by Pansu and Cornulier-Tessera for vanishing in degree 1 than other proofs for higher degree cohomology. We will now explain some ideas of the proof in the case of $G = \mathrm{SL}_{n+1}(F)$ where $n \geq 3$.

The first step consists on invoking quasi-isometric invariance of L^p-cohomology in order to pass to a maximal parabolic subgroup, use its Levi decomposition to write it as a semi-direct product on which we apply the Hochschild-Serre spectral sequence. This gives an identification between $H^2_{\mathrm{ct}}(G, L^p(G))$ and a cohomology space of the form $H^1_{\mathrm{ct}}(R, L^p(R, V))$ where the group R is the group of upper triangular matrices in $\mathrm{SL}_n(F)$, V is some Banach space carrying an R-representation of exponential growth and the R-action on $L^p(R, V)$ is given by: $(\pi(g)F)(h) = g.F(hg)$ for $g, h \in R$ and $F \in L^p(R, V)$.

We then look at operator norms of elements $g \in R$, we have the equalities:
$$|||\pi(g)|||_{L^p(R,V)} = |||g|||_{L^p(R)}|||g|||_V = \Delta_R(g)^{-1/p}|||g|||_V$$
where Δ_R denotes the modular function of the group R. The key point here is that even though $|||g|||_V$ can grow exponentially fast in the length of g, if $\Delta_R(g) > 1$ then $n \mapsto |||g^n|||_{L^p(R)}$ decays exponentially fast, so we may hope to produce elements that contract the norm of π (i.e. $|||\pi(g)|||_{L^p(R,V)} < 1$). The rest of the proof consists in showing that such contracting elements do exist, and that there are enough of them to run a version of Mautner's phenomenon.

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The geometry of conjugation in Euclidean isometry groups

Petra Schwer

(joint work with Elizabeth Milićević and Anne Thomas)

Group theory has a long history of studying conjugacy classes and the conjugation problem. While studying non-emptiness of affine Deligne-Lusztig varieties we encountered a question which had not been asked before:

> Can one describe geometrically where in the Coxeter complex the elements conjugate to a given x are located?
> Can one also determine and geometrically describe the set of all conjugating elements?

It turns out the conjugacy classes of elements in affine Coxeter groups, and more generally those in the full isometry group G of n-dimensional Euclidean space $\mathbb{E}^n$, as well as in all split subgroups H of G, have a simple and beautiful geometric description. Their shape of is determined by the move-set of its linearization, while the coconjugation set is described in terms of the fix-set of the linearization.

We give formal statements of our characterizations from [1, 2] in Section 2, and illustrate them via examples in Section 3.

1. Move-spaces and Mod-sets

Let us first introduce the main players. The group G splits as a semidirect product $G = T \rtimes \mathrm{O}(n)$, where $T \cong \mathbb{R}^n$ is the translation subgroup of G and $\mathrm{O}(n)$ is the group of orthogonal transformations. We consider subgroups H of G which respect this splitting; that is, where $H = T_H \rtimes H_0$ for $T_H = T \cap H$ and $H_0 = H \cap \mathrm{O}(n)$. For any such $H \leq G$ and for all $h, h' \in H$, we write

$$[h]_H = \{khk^{-1} \mid k \in H\} \quad \text{for the } \textit{conjugacy class of } h \in H \text{ and}$$

$$\mathrm{C}_H(h, h') = \{k \in H \mid khk^{-1} = h'\} \quad \text{for the } \textit{coconjugation set (from } h \textit{ to } h'\textit{)}.$$

In particular, $C_H(h, h)$ is the centralizer of h in H, which we also denote by $C_H(h)$. For any $\lambda \in \mathbb{R}^n$, we write t^λ for the translation of $\mathbb{E}^n$ by the vector λ. For any split

$H \leq G$, we define $L_H = \{\lambda \in \mathbb{R}^n \mid t^\lambda \in T_H\}$, and observe that L_H is naturally a $\mathbb{Z}$-module. Then any $h \in H$ can be expressed uniquely as $h = t^\lambda h_0$, where $\lambda \in L_H$ and $h_0 \in H_0$. We call t^λ the *translation part* and h_0 the *spherical part* of h. For any $\lambda \in L_H$ and $h_0 \in H_0$, we have $h_0 t^\lambda h_0^{-1} = t^{h_0\lambda}$.

Recall that the *move-set* and *fix-set* of any isometry $g \in G$ are the affine subspaces of $\mathbb{R}^n$ given by, respectively,

$$\textsc{Mov}(g) = \{y \in \mathbb{R}^n \mid gx = x + y \text{ for some } x \in \mathbb{R}^n\} = Im(g - \mathrm{I})$$

and

$$\textsc{Fix}(g) = \{x \in \mathbb{R}^n \mid gx = x\} = \mathrm{Ker}(g - \mathrm{I}).$$

For example, if $r \in G$ is a reflection, then $\textsc{Mov}(r)$ is the line through the origin orthogonal to the affine hyperplane $\textsc{Fix}(r)$. If $g_0 \in \mathrm{O}(n)$, then $\textsc{Mov}(g_0)$ and $\textsc{Fix}(g_0)$ are both linear subspaces, and $\mathbb{R}^n$ has orthogonal decomposition $\mathbb{R}^n = \textsc{Mov}(g_0) \oplus \textsc{Fix}(g_0)$. The *mod-sets* are H-adapted versions of the move spaces, defined as follows. For any $h \in H$, the *mod-set (with respect to H) of h* is defined by:

$$\textsc{Mod}_H(h) = (h - \mathrm{I})L_H.$$

2. Main results

We address the first of the leading questions and provide a closed and geometric description of conjugacy classes.

Theorem 1 (Closed form of conjugacy classes)**.** *Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. Then the conjugacy class of h in H satisfies*

$$[h]_H = \bigcup_{u \in H_0} u \left(t^{\textsc{Mod}_H(h_0)} h \right) u^{-1} \tag{1}$$

and also

$$[h]_H = \bigcup_{u \in H_0} t^{u(\lambda + \textsc{Mod}_H(h_0))} u h_0 u^{-1} = \bigcup_{u \in H_0} t^{u\textsc{Mod}_H(h)} u h_0 u^{-1}. \tag{2}$$

In words, the two equalities of Theorem 1 tell us that $[h]_H$ is obtained by, respectively:

(1) first translating h by all elements of $\textsc{Mod}_H(h_0)$, and then conjugating the so-obtained collection $t^{\textsc{Mod}_H(h_0)} h$ by all elements of H_0; or
(2) for each $u \in H_0$, translating the u-conjugate of the spherical part h_0 of h by the set $t^{u(\lambda + \textsc{Mod}_H(h_0))} = t^{u\textsc{Mod}_H(h)}$.

Both descriptions give rise to algorithms to compute the conjugacy classes of elements in these groups.

In general mod sets are not necessarily equal to the intersection of the Movesapce with the coroot-lattice and one only has $\textsc{Mod}_H(h) \subseteq \textsc{Mov}(h) \cap L_H$. The next definition is motivated by this.

Definition 2 (Filling)**.** *Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. We say that $h \in H$* fills its move-set*, or that* filling occurs for h*, if*

$$\mathrm{Mod}_H(h) = \mathrm{Mov}(h) \cap L_H.$$

In [2], for certain split crystallographic groups we refine the relationships between mod-sets and move-sets observed above. We regard the lattice L_H as a free $\mathbb{Z}$-module of rank n, and prove in [2, Theorem 3.8] that for any split crystallographic $H = T_H \rtimes H_0$ which is contained in an affine Coxeter group, and all $h_0 \in H_0$:

(1) the rank of $\mathrm{Mod}_H(h_0)$ equals the dimension of the move-set $\mathrm{Mov}(h_0)$;
(2) $\mathrm{Mod}_H(h_0)$ is a finite-index submodule of $\mathrm{Mov}(h_0) \cap L_H$; and
(3) h_0 fills its move-set if and only if $L_H/\mathrm{Mod}_H(h_0)$ is torsion-free.

Our proofs of (1)–(3) in [2] use properties of affine Coxeter groups beyond their semidirect product structure, including their close relationship to finite Weyl groups. We do not know if (1)–(3) hold for split crystallographic groups which are not contained in affine Coxeter groups.

We now turn to the second question: which $k \in H$ conjugate a given $h \in H$ to some h' in its conjugacy class? We refer to this question as the *coconjugation problem*. The solution to the coconjugation problem in H crucially involves the fix-sets of elements of H_0.

For any $h' \in [h]_H$, the coconjugation set $\mathrm{C}_H(h, h')$ is equal to $kC_H(h)$ for any $k \in H$ such that $khk^{-1} = h'$. One could hence say that it is enough to consider centralizers to fully solve the coconjugation problem. However, in Theorem 3 below, we provide an intrinsic description of the coconjugation set that does not require prior knowledge of the centralizer, nor the determination of a conjugating element k as used above. Instead, the disjoint union in Theorem 3 can be parametrized by an explicitly described subset of the coconjugation set $\mathrm{C}_{H_0}(h_0, h_0')$. The *translation-compatible part* of $\mathrm{C}_{H_0}(h_0, h_0')$ is defined by:

$$\mathrm{C}^{\lambda,\lambda'}_{H_0}(h_0, h_0') = \{u \in \mathrm{C}_{H_0}(h_0, h_0') \mid \lambda' - u\lambda \in \mathrm{Mod}_H(h_0')\}. \tag{3}$$

Theorem 3 (Coconjugation)**.** *Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. Let $h = t^\lambda h_0$ and $h' = t^{\lambda'} h_0'$ be elements of H, where $\lambda, \lambda' \in L_H$ and $h_0, h_0' \in H_0$. Then*

$$\mathrm{C}_H(h, h') \neq \emptyset \iff \mathrm{C}^{\lambda,\lambda'}_{H_0}(h_0, h_0') \neq \emptyset. \tag{4}$$

Moreover, if these sets are nonempty, then

$$\mathrm{C}_H(h, h') = \bigsqcup_{u \in \mathrm{C}^{\lambda,\lambda'}_{H_0}(h_0, h_0')} t^{\eta_u + (\mathrm{Fix}(h_0') \cap L_H)} u \tag{5}$$

where for each u, the element $\eta_u \in L_H$ is a particular solution to the equation

$$\lambda' - u\lambda = (\mathrm{I} - h_0')\eta. \tag{6}$$

In the special case that $\text{Fix}(h_0) = \{0\}$, *we have that*

$$\eta_u = (\mathrm{I} - h_0')^{-1}(\lambda' - u\lambda)$$

is the unique solution to (6), *and* $\mathrm{C}_H(h, h')$ *is in bijection with* $\mathrm{C}_{H_0}^{\lambda,\lambda'}(h_0, h_0')$.

Geometrically, (5) means that the coconjugation set $\mathrm{C}_H(h, h')$ lies along translates of the fix-set $\text{Fix}(h_0')$, and so is orthogonal to $\text{Mov}(h_0')$. The reason for this appearance of the fix-set in our description of coconjugation sets is that we are solving Equation (6), and $\text{Fix}(h_0') = \text{Ker}(\mathrm{I} - h_0')$. In the special case that $h = h'$, Theorem 3 yields a new geometric description of the centralizer $C_H(h)$.

When nonemptiness of the set $\mathrm{C}_{H_0}^{\lambda,\lambda'}(h_0, h_0')$ can be determined, the equivalence (4) in Theorem 3 provides an algorithm to solve the conjugation problem in H. If, in addition, all elements of $\mathrm{C}_{H_0}^{\lambda,\lambda'}(h_0, h_0')$ and all solutions to Equation (6) can be computed, we obtain an algorithm which lists all elements of the coconjugation set.

3. Examples

3.1. **The wallpaper group *cmm*.** Let H be the wallpaper group ***cmm***, denoted 2^*22 in orbifold notation. Then H is split, H_0 is the Klein four group generated by two commuting reflections, say s_1 and s_2, and H is generated by s_1, s_2, and a 180° rotation, say ρ, about a point not on any reflection axis. The group H induces the tesselation of $\mathbb{E}^2$ by triangles depicted in Figure 1, and L_H is the lattice of heavy dots in these figures. There is a natural bijection between the elements of H and the tiles in these tesselations, and we identify each element of H with its corresponding tile. A few tiles are labeled in Figure 1.

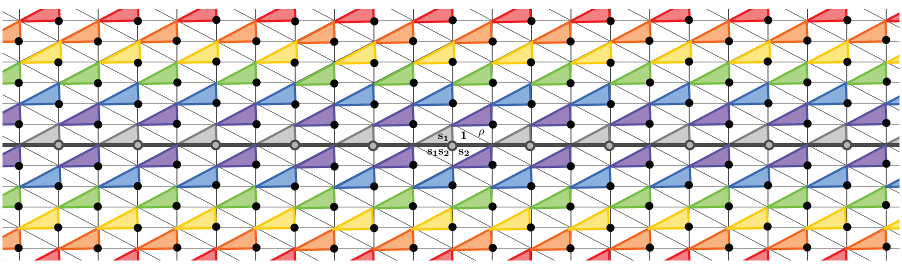

Figure 1. Conjugacy classes $[t^\lambda s_1]_H$ in the wallpaper group **cmm**.

The conjugacy classes in H are as follows. In Figure 1, each set of tiles of the same color is a conjugacy class $[t^\lambda s_1]_H$. The mod-set $\text{Mod}_H(s_1) \subseteq L_H$ is the set of large gray dots along the horizontal axis, and the move-set $\text{Mov}(s_1)$ is this horizontal axis. The horizontal lines in the figure are the sets $\text{Mov}(t^\lambda s_1)$. If $\lambda \in \text{Mov}(s_1)$ then $\text{Mov}(t^\lambda s_1) = \text{Mov}(s_1)$ is H_0-invariant, and the conjugacy class $[t^\lambda s_1]_H$ is the set of gray triangles along the horizontal axis. For $\lambda \notin \text{Mov}(s_1)$, the line $s_2\text{Mov}(t^\lambda s_1) = s_2 s_1 \text{Mov}(t^\lambda s_1)$ is distinct from $\text{Mov}(t^\lambda s_1)$, and so $[t^\lambda s_1]_H$ is a pair of horizontal "lines" of triangles (of the same color). The description of the conjugacy classes $[t^\lambda s_2]_H$ is similar, just involving vertical "lines" of triangles instead. Note that s_1 and s_2 both fill their move-sets.

3.2. **The full isometry group of $\mathbb{E}1^2$.** Let $H = G = \mathrm{Isom}(\mathbb{E}^2)$ be the full isometry group of the Euclidean plane. Let $\lambda \in \mathbb{R}^2$ be nonzero, let $r \in \mathrm{O}(2)$ be the unique linear reflection which fixes λ, and let g be the glide-reflection $g = t^\lambda r$. Then the conjugacy class $[g]_G$ is the disjoint union of all lines which are tangent to the circle of radius $\|\lambda\|$, as depicted in Figure 2.

More precisely, if ℓ is a line tangent to this circle, then the point p of ℓ corresponds to the element $t^p r_\ell$ of $[g]_G$, where $r_\ell \in \mathrm{O}(2)$ is the unique linear reflection preserving ℓ. Note that each same-color pair of "lines" of tiles in Figure 1 can be viewed as a "discrete shadow" of a pair of actual lines in Figure 2.

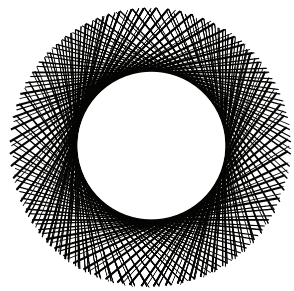

Figure 2. The conjugacy class of a glide-reflection in $\mathrm{Isom}(\mathbb{E}^2)$.

Now take any $\lambda \in \mathbb{R}^2$ and let $g_0 = -\mathrm{I}$. Then $g = t^\lambda g_0$ is the rotation by 180° about the point $\frac{1}{2}\lambda$, and all such rotations are conjugate in G.

For translations, given any $\lambda \in \mathbb{R}^2$ we can identify the conjugacy class $[t^\lambda]_G = \{t^{u\lambda} : u \in \mathrm{O}(2)\}$ with the circle of radius $\|\lambda\|$. Thus the finitely many conjugates of any translation in **cmm** are again just a discrete glimpse of its full conjugacy class in G.

4. Affine Coxeter group of type $\tilde{A}_2$

Let W be the affine Coxeter group of type $\tilde{A}_2$. Then $W = T \rtimes W_0$ where $W_0 = \langle s_1, s_2 \rangle$ is the Weyl group of type A_2, and the affine Weyl group is $W = \langle s_0, s_1, s_2 \rangle$. There is a natural bijection between the elements of W and the triangular tiles in the tessellation depicted in Figure 3. The coroot lattice $R^\vee$ is the set of heavy dots. On the left of Figure 3, each set of triangles shaded in the same color is a conjugacy class $[t^\lambda w]$, where $w \in \{s_1, s_2, w_0\}$. For each such reflection $w \in W_0$, the move-set $\mathrm{Mov}(w)$ is the heavy gray line orthogonal to its fixed hyperplane, and the mod-set $\mathrm{Mod}_W(w)$ is the set of coroot lattice elements on this gray line. In other words, each reflection $w \in W_0$ *fills its move-set.* The other, colored lines on the left of Figure 3 are move-sets $\mathrm{Mov}(t^\lambda w)$ for certain $\lambda \notin \mathrm{Mod}_W(w)$. Each conjugacy class is thus a triple of "lines" of triangles of the same color.

The right of Figure 3 depicts the coconjugation set $\mathrm{C}_W(x, x')$, where $x = t^\lambda s_1$ and $x' = t^{\lambda'} w_0$. To describe $\mathrm{C}_W(x, x')$, we first determine the $u \in W_0$ such that $us_1u^{-1} = w_0$ and $\lambda' - u\lambda \in \mathrm{Mod}_W(w_0)$; this gives $u \in \{s_2, s_2s_1\}$. Then for each such u, we translate the horizontal gray line $\mathrm{Fix}(w_0)$ by a particular solution $\eta_u \in R^\vee$ to the equation $\lambda' - u\lambda = (\mathrm{I} - w_0)\eta$. The elements of $\mathrm{C}_W(x, x')$ are the triangles $t^\mu u$ along these translates, as depicted in teal and aqua.

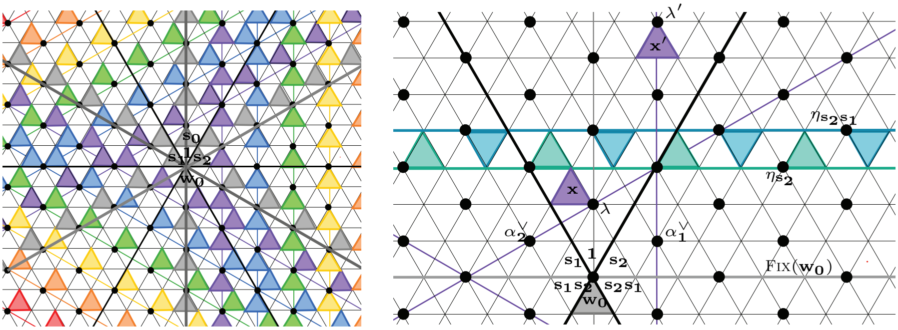

Figure 3. On the left, some conjugacy classes $[t^\lambda w]$ in type $\tilde{A}_2$, where w is a reflection. On the right, the coconjugation set $\mathrm{C}\, xx'$.

4.1. **Affine Coxeter group of type $\tilde{C}_2$.** To end this report we provide an example which illustrates the filling property. Let $W = T \rtimes W_0$ be of type $\tilde{C}_2$, so that $W_0 = \langle s_1, s_2 \rangle$ is of type C_2. In Figure 4, which depicts some of the conjugacy classes $[t^\lambda w]$ for $w \in W_0$ a reflection, we again see "lines" of conjugates. On the left of this figure, the reflections s_1 and $s_2 s_1 s_2$ *fill their move-sets.* However, on the right, the conjugacy classes leave "gaps" along the colored lines which are the move-sets. This is due to the fact that the reflections s_2 and $s_1 s_2 s_1$ *do not fill their move-sets*; rather, the mod-set $\mathrm{Mod}_W(w)$ is an index 2 submodule of $\mathrm{Mov}(w) \cap R^\vee$.

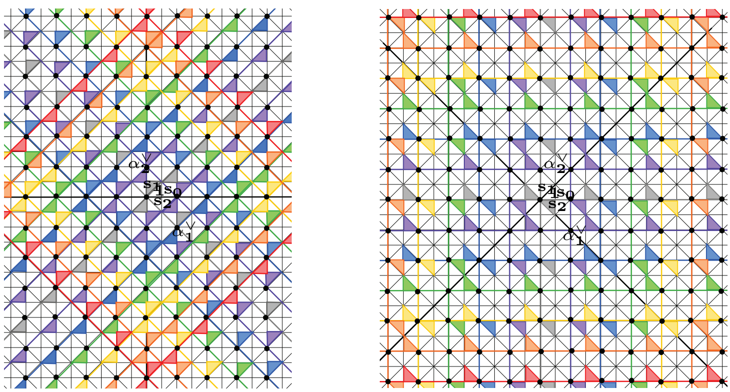

Figure 4. The conjugacy classes $[t^\lambda w]$ in type $\tilde{C}_2$ for $w \in \{s_1, s_2 s_1 s_2\}$ fill their move-sets (on the left), but do not fill their move-sets for $w \in \{s_2, s_1 s_2 s_1\}$ (on the right).

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On and around Anosov representations

Anna Wienhard

Anosov representations have been introduced by Franęois Labourie [8]. They give rise to interesting classes of discrete subgroups of Lie groups, and their study has become a very active area in mathematics. In the talk a gave a selective introduction to Anosov representations, highlighting some recent results of younger mathematicians.

1. Convex cocompact subgroups

A finitely generated discrete subgroup Γ of the isometry group of a symmetric space of non-compact type X is said to be convex cocompact if there exists a closed Γ-invariant convex set $C \subset X$ such that C/Γ is compact. Convex cocompact subgroups of symmetric spaces of rank one form an important class of discrete subgroups. Two of their key properties are

(1) **Convex cocompact subgroups are quasi-isometrically embedded**, i.e. there exist constants $K, C > 0$ such that for all $\gamma \in \Gamma$ we have

$$\frac{1}{K}|\gamma| - C \leq d_X(x_0, \gamma x_0) \leq K|\gamma| + C,$$

where $|\gamma|$ is the length function with respect to a finite generating set of Γ. In fact, also the converse holds, any quasi-isometrically embedded finitely generated group is convex cocompact.

(2) **Structural stability** A small deformation of a convex cocompact subgroup is still convex cocompact (this is made more precise later)

In higher rank, convex cocompactness is a very rigid notion. Independently, Kleiner-Leeb and Quint proved that any Zariski-dense convex cocompact discrete subgroup of a higher rank Lie group is already a cocompact lattice. On the other hand, being a quasi-isometric embedding is less rigid. In higher rank, quasi-isometric embeddings are not necessarily structurally stable. Anosov representations provide the right generalizations of convex cocompact subgroups in the higher rank setting. They satisfy the two key properties (and many others), and as we see in the definition below can be thought of as a strengthening of quasi-isometric embeddings.

2. Anosov representations

We will now change the point of view a bit and consider a finitely generated group Γ and consider homomorphisms of Γ into a semisimple Lie group G. Labourie's original definition of Anosov representations involved dynamics, here we present a characterization which is due to Kapovich-Leeb-Porti [9] and Bochi-Potrie-Sambarino [1].

2.1. **A strengthening of quasi-isometric embeddings.** In a higher rank symmetric space, there is a Weyl chamber valued distance function $d_{\mathfrak{a}} : X \times X \to \mathfrak{a}^+$. A simple root α is a special linear form on $\mathfrak{a}$. These are the ingredients we need to introduce Anosov representations

Definition 1. *Let Γ be a finitely generated group, G a semisimple Lie group with finite center and no compact factors and X its symmetric space. A homomorphism $\rho : \Gamma \to G$ is an α-Anosov representation for a simple root α if there exists constants $K, C > 0$ such that for all $\gamma \in \Gamma$ we have*

$$\frac{1}{K}|\gamma| - C \leq \alpha(d_{\mathfrak{a}}(x_0, \rho(\gamma)x_0) \leq K|\gamma| + C,$$

where $|\gamma|$ is the length function with respect to a finite generating set of Γ.

A consequence of the definition is that the finitely generated group Γ is hyperbolic [9, 1].

2.2. **Properties of Anosov representations.** Anosov representations satisfy many interesting properties

(1) Anosov representations are virtually faithful with discrete image.
(2) Anosov representations are quasi-isometrically embedded, in particular in rank one we have: Anosov = convex cocompact.
(3) Anosov representations are structurally stable, they form open subsets in the space of homomorphism $\mathrm{Hom}(\Gamma, G)$.

2.3. **Examples of Anosov representations.** There are many important examples of Anosov representations, which are established in [8, 7]

(1) Let H be a simple Lie group of rank one, $\Gamma < H$ a convex cocompact subgroup, and $H \to G$ an embedding into a semisimple Lie groups of higher rank. Then the composition $\iota : \Gamma \to G$ is an Anosov representations for some (explicitely computable) simple roots.
(2) For a surface group Γ, representations $\rho : \Gamma \to \mathrm{SL}(n, \mathbb{R})$ in the Hitchin component are Anosov with respect to any simple root.

For some Lie groups G and special roots α having an α-Anosov representation $\rho : \Gamma \to G$ places strong constraints on the groups Γ. Such constraints have first been established by Canary znc Tsouvalas [2], Tsouvalas [11], Dey [3]. When $G = \mathrm{Sp}(2n, \mathbb{R})$ there is a nice recent theorem of Dey, Greenberg, and Riestenberg [4], and independently Pozzetti and Tsouvalas [10]. Note that the set of simple roots for the symplectic group is $\{\alpha_1, \cdots, \alpha_n\}$.

Theorem 2. *If a representation $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ is α_k-Anosov for some k odd, then Γ is virtually a surface group or a free group.*

On the other hand, there are finitely generated hyperbolic groups that admit Anosov representations into $\mathrm{SL}(n, \mathbb{R})$, but do not admit any discrete embedding into any product of Lie groups of rank 1 [6]. In a recent theorem Douba, Flechelles, Weisman, and Zhu DoubaFlechellesWeismanZhu showed that there are many hyperbolic groups that admit Anosov representations.

Theorem 3. *Any finitely generated hyperbolic group Γ that acts properly and convex cocompactly on a* $\mathrm{CAT}(0)$*-cube complex admits an α_1-Anosov representation into* $\mathrm{SL}(n, \mathbb{R})$ *for some n.*

3. Beyond Anosov representations

One key property of Anosov representations is that they provide a structurally stable class of quasi-isometric embeddings into higher rank. Recently, Tsouvalas [12] constructed structurally stable quasi-isometric embeddings of a finitely generated hyperbolic group, which are not contained in the closure of the set of Anosov representations. Thus there is a whole world beyond Anosov representations still to be discovered.

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Computing certain invariants of topological spaces of dimension three

Andreas Thom

(joint work with Lukas Gohla)

For $d \geq 4$ and p a sufficiently large prime, we construct a lattice $\Gamma \leq \mathrm{PSp}_{2d}(\mathbb{Q}_p)$, such that its universal central extension cannot be sofic if Γ satisfies some weak form of stability in permutations. In the proof, we make use of high-dimensional expansion phenomena and, extending results of Lubotzky, we construct new examples of cosystolic expanders over arbitrary finite abelian groups. These results appeared in [7].

Ever since the influential paper of Gromov [8] and subsequent work of Weiss [11], the quest for a non-sofic group has inspired mathematicians. In order to show that a group is *sofic*, one has to provide sufficiently rich almost representations of the group in permutations. The competing notion of *stability* requires that every sufficiently accurate almost representation in permutations is close to an actual permutation representation. Thus, it is easy to see that any group which is both sofic and stable must be residually finite and, at least theoretically, this opens a route to the construction of non-sofic groups, see [6]. Unfortunately, stability is a rare phenomenon, even though there is some indication that lattices in algebraic groups of rank at least 3 share the right kind of rigidity in order to ensure at least some form of stability. This was first discovered in [3] and related to high-dimensional expansion, see also [10], but the techniques are currently not able to produce stability results for almost representations in permutations, see the discussion in [2].

In this talk we discuss particular torsion free lattices Γ in the algebraic group $\mathrm{PSp}_{2d}(\mathbb{Q}_p)$ for $d \geq 4$ and p large. These groups are residually finite and hence sofic, but admit finite central extensions $\tilde{\Gamma}$ which are not residually finite anymore. This phenomenon was first discovered by Deligne [4] in work on the congruence subgroup problem. Our main result says that these central extensions $\tilde{\Gamma}$ cannot be sofic if the group Γ is stable in a certain sense.

On a more conceptual level, which might be interesting in its own right, we introduce cohomological invariants that obstruct containment of p.m.p. actions of groups. These obstructions can be enhanced in the presence of cosystolic inequalities to obstruct also weak containment. This is exactly the route our proof takes: we show that for any sofic approximation of the finite central extension $\tilde{\Gamma}$, the induced sofic approximation of the lattice Γ admits a limit action which is not weakly contained in finite actions of the lattice. This contradicts a very weak form of stability that we introduce and discuss in [7]. We call a group *stable in finite actions* if any partition of a suffiently good sofic approximation can be modelled (in the spirit of Kechris' notion of weak containment) in a finite action. This seems much weaker than any other notion of stability that has been studied so far.

All this is formulated in our main result. We prove the following theorem:

Theorem 1. *Let $d \geq 4$ and p a large prime. There exists a torsionfree lattice Γ in $\mathrm{PSp}(\mathbb{Q}_p)$ such that Γ admits a finite and non-residually finite central extension $\tilde{\Gamma}$. If Γ is stable in finite actions, then $\tilde{\Gamma}$ is not sofic.*

The result should be compared with a result of Dogon in the hyperlinear setting, see [5], and also with results of Bowen–Burton [1]. Note however that the strength of our result lies in the fact that the notion of stability in finite actions is much weaker then any other previously introduced notion of stability. Currently, there is no residually finite group which is known not to be stable in finite actions.

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On the structure of varieties over free associative algebras

Zlil Sela

Sets of solutions to systems of equations (varieties) over free associative (non-commutative) algebras were studied from the 1960's by ring theorists (P.M. Cohn, G. Bergman, and others). Because of the strict failure of unique factorization over these free algebras, not much is known about these varieties, and no conjectures were ever made.

We presented our work in progress (partly joint with Agatha Atkarskaya) on the structure of some of these varieties. The structure that we found is based on our previous work on varieties over free groups and semigroups, and involves concepts from low dimensional topology with other concepts from commutative algebra.

We started by studying homogeneous solutions to homogeneous monomial systems of equations. For these we introduced a canonical Makanin-Razborov diagram, similar to the one that we constructed previously for varieties over free semigroups [1] that encodes all the homogeneous solutions to such system of equations.

The diagram enables us to filter general monomial systems of equations. First, we described all the solutions to monomial systems of equations with a single variable [2]. Then we studied monomial systems of equations with more than one variable that contains no quadratic parts. For these we introduced a non-commutative analogue of Hensel's lemma that enables to describe the solutions for these monomial systems.

Finally, we described sets of solutions to quadratic monomial systems of equations. For these we also have an analogue of Hensel's lemma that enables us to describe the top c homogeneous parts of the solutions for any given positive integer c.

Having understood all these type of monomial systems of equations, we use the Makanin-Razborov diagram that is associated with the top homogeneous part of the monomial system to analyze the solutions of a general monomial system of equations.

At this stage we are starting to apply our techniques and results on monomial systems of equations to analyze general (polynomial) systems of equations.

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A 4-dimensional pseudo-Anosov map

Bruno Martelli

A celebrated theorem of Thurston says that a 3-manifold M that fibers over the circle is hyperbolic if and only if the fiber S is a (finite type) surface with $\chi(S) < 0$ and the monodromy is *pseudo-Anosov*. It is natural to ask whether this result is confined to dimension 3, or if some version of it may holds in higher dimension.

We show that this theorem extends nicely in dimension 5, at least for the unique source of examples of fibering hyperbolic 5-manifolds that we have, recently built in [1]. As in dimension 3, the fiber of the hyperbolic 5-manifold built in [1] is a 4-manifold with a *Euclidean cone manifold* structure, with singular locus consisting of a codimension 2 geodesic surface with cone angle 3π, and the monodromy is *pseudo-Anosov* in the sense that it preserves the leaves of two orthogonal *horizontal and vertical singular foliations.* The leaves of the foliations are planes (that intersect the singular locus in lines), and the map stretches (contracts) the leaves of the horizontal (vertical) foliation by a factor λ (respectively, $1/\lambda$) where $\lambda = (1+\sqrt{5})/2$. This is completely analogous to the more familiar two-dimensional picture.

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Classifying group actions on hyperbolic spaces

Denis Osin
(joint work with Koichi Oyakawa)

For a given group G, it is natural to ask whether one can classify all isometric G-actions on Gromov hyperbolic spaces. We propose a formalization of this problem utilizing the complexity theory of Borel equivalence relations. In this paper [1], we focus on actions of general type, i.e., non-elementary actions without fixed points at infinity.

Our main result is the following dichotomy: for every countable group G, either all general type actions of G on hyperbolic spaces can be classified by an explicit invariant ranging in $\mathbb{RP}^\infty$ or they are unclassifiable in a very strong sense. Special linear groups $SL_2(F)$, where F is a countable field of characteristic 0, satisfy the former alternative, while non-elementary hyperbolic (and, more generally, acylindrically hyperbolic) groups satisfy the latter.

In terms of Borel complexity theory, we show that, for any countable weakly hyperbolic group G, the restriction of $\sim$ to $Hyp_{gt}(G)$ is either smooth or Borel bi-reducible to the orbit equivalence relation of the action of ℓ^∞ on $\mathbb{R}^\infty$.

In the course of proving our main theorem, we also obtain results of independent interest that offer new insights into algebraic and geometric properties of groups admitting general type actions on hyperbolic spaces.

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Profinite rigidity of Kähler groups

Claudio Llosa Isenrich
(joint work with Sam Hughes, Pierre Py, Matthew Stover, Stefano Vidussi)

A *smooth complex projective variety* is a manifold which is biholomorphic to a compact complex submanifold of some complex projective space. A classical problem in complex algebraic geometry is understanding the topology of smooth complex projective varieties. An important invariant of a smooth complex projective variety X is its algebraic fundamental group $\pi_1^{alg}(X)$. This raises the questions:

Question 1. *Let X be a smooth complex projective variety:*

(1) Can we recover X up to homeomorphism from $\pi_1^{alg}(X)$?
(2) Can we recover $\pi_1(X)$ up to isomorphism from $\pi_1^{alg}(X)$?

These fundamental questions arise naturally in the context of the seminal work of Grothendieck [7] in the 1970s and Question 1(2) has been a key motivation for much research in group theory, see e.g. [13]. This is because $\pi_1^{alg}(X)$ is naturally isomorphic to the profinite completion $\widehat{\pi_1(X)}$ of the topological fundamental group $\pi_1(X)$ of X. Here the profinite completion $\widehat{G}$ of a group G is the topological group defined as the closure of the image of the natural homomorphism $\phi : G \to \prod_{N \trianglelefteq_{fi} G} G/N$ in the direct product of its discrete finite quotient groups, with respect to the product topology. We call a group G *residually finite* if the morphism ϕ is injective, or, equivalently, if for every non-trivial element $g \in G \setminus \{1\}$, there is a homomorphism $\phi : G \to Q$ to some finite group Q with $\phi(g) \neq 1$.

Residual finiteness is a prerequisite for being able to recover G completely from its profinite completion. By work of Nikolov–Segal [12], two finitely generated groups have the same set of finite quotients if and only if their profinite completions are isomorphic as abstract groups. The problem of understanding a group from its set of finite quotients can thus be rephrased in terms of the *profinite genus*

$$\mathcal{G}(G) := \left\{ H \text{ finitely generated residually finite } \mid \widehat{H} \cong \widehat{G} \right\}.$$

For a class of groups $\mathcal{C}$, the *$\mathcal{C}$-profinite genus* of $G \in \mathcal{C}$ is then $\mathcal{G}_{\mathcal{C}}(G) := \mathcal{G}(G) \cap \mathcal{C}$. We say that a residually finite group G is *profinitely rigid* (resp. *$\mathcal{C}$-profinitely rigid*) if $\mathcal{G}(G) = \{G\}$ (resp. $\mathcal{G}_{\mathcal{C}}(G) = \{G\}$). Question 1(2) then becomes the question if groups in $\mathcal{P}$ are $\mathcal{P}$-profinitely rigid, where $\mathcal{P}$ denotes the class of fundamental groups of smooth complex projective varieties (*projective groups*). Since all of our results hold for the class $\mathcal{K}$ of fundamental groups of compact Kähler manifolds (*Kähler groups*), we state them in this more general context.

In 1993 Toledo constructed non-residually finite projective groups [16]. Thus, a restriction of the $\mathcal{K}$-profinite rigidity problem to residually finite groups is required. Moreover, in 1964 Serre proved that in general projective groups are not $\mathcal{P}$-profinitely rigid [14], using Galois conjugation. Since then many further examples of non-$\mathcal{P}$-profinitely rigid groups have been constructed, see e.g. [1, 2, 15]. In constrast, in [9], we prove the following profinite rigidity result for direct products of closed Riemann surfaces and a complex torus.

Theorem 2 ([9, Theorem A])**.** *Let $G = \pi_1(S_{g_1}) \times \ldots \times \pi_1(S_{g_r}) \times \pi_1(T^{2k})$ be a direct product of fundamental groups of closed Riemann surfaces S_{g_i} of genus $g_i \geq 2$ and a (real) $2k$-torus. Then G is $\mathcal{K}$-profinitely rigid. If, moreover, X is an aspherical compact Kähler manifold with $\widehat{\pi_1(X)} \cong \widehat{G}$, then X is homeomorphic to $S_{g_1} \times \ldots \times S_{g_r} \times T^{2k}$.*

The examples in [1] are finite extensions of a direct product of two surface groups, while the examples in [2] are finite index subgroups of a direct product of two surface groups. This shows that Theorem 2 is sharp. Note also that Theorem 2 solves a special case of the longstanding open problem, attributed to Remeslennikov, if free groups and surface groups are profinitely rigid.

Theorem 2 is a consequence of our more general result that the universal homomorphism of a Kähler group is a profinite invariant [9]. The universal homomorphism was introduced in [10] to rephrase results of Corlette–Simpson and Delzant [4, 5] in terms of a morphism to a direct product of *orbisurface groups* (orbifold fundamental groups of closed Riemann surfaces equipped with an orbifold structure with finitely many cone points):

Theorem 3 ([4, 5], see also [10, Theorem 9.1]). *Let G be a Kähler group. Then there is a* universal homomorphism $\phi = (\phi_1, \cdots, \phi_r) : G \to \Gamma_1 \times \ldots \times \Gamma_r$ *to a direct product of hyperbolic orbisurface groups Γ_i such that the ϕ_i are surjective with finitely generated kernel and every other morphism from G onto a hyperbolic orbisurface factors through a unique ϕ_i.*

The universal homomorphism is then a profinite invariant of Kähler groups in the following sense.

Theorem 4 ([9, Theorem D]). *Let G and H be Kähler groups and let $\phi = (\phi_1, \cdots, \phi_r) : G \to \Gamma_1 \times \ldots \times \Gamma_r$ and $\psi = (\psi_1, \cdots, \psi_s) : H \to \Lambda_1 \times \ldots \times \Lambda_s$ be their universal homomorphisms. Assume that there is an isomorphism $\Theta : \widehat{G} \to \widehat{H}$ of their profinite completions. Then, after reordering, $r = s$, $\Gamma_i \cong \Lambda_i$, and there are isomorphisms $\alpha_i : \widehat{\Gamma}_i \to \widehat{\Lambda}_i$ such that there is a commutative diagram*

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\widehat{\phi}} & \widehat{\Gamma}_1 \times \ldots \times \widehat{\Gamma}_r \\ \cong \big\downarrow \Theta & & \cong \big\downarrow (\alpha_1, \ldots, \alpha_r) \\ \widehat{H} & \xrightarrow{\widehat{\psi}} & \widehat{\Lambda}_1 \times \ldots \times \widehat{\Lambda}_r, \end{array}$$

where $\widehat{\phi}$ and $\widehat{\psi}$ denote the induced homomorphisms on profinite completions.

Key ingredients in our proof of Theorem 4 include the semicontinuity of the first Betti and ℓ^2-Betti number with respect to profinite completions, a factorization result for Kähler groups by Napier and Ramachandran [11], and Bridson, Conder and Reid's result that Fuchsian groups are distinguished by their profinite completions [3].

As a further consequence of Theorem 4 we prove the profinite invariance of the BNS-invariant $\Sigma^1(G) \subseteq H^1(G, \mathbb{R}) \setminus \{0\}$ of a Kähler group.

Theorem 5 ([9, Theorem E]). *Let G and H be Kähler groups with isomorphic profinite completions. Then there is a linear isomorphism $f : H^1(G, \mathbb{R}) \to H^1(H, \mathbb{R})$ defined over $\mathbb{Q}$ such that $f(\Sigma^1(G)) = \Sigma^1(H)$.*

Our proof of Theorem 5 relies on Delzant's characterisation of the BNS-invariant of a Kähler group in terms of its morphisms to surface groups [6], and results of Hughes and Kielak, relating the non-vanishing of twisted Alexander polynomials to virtual algebraic fibring of groups [8].

We end with the following related questions that to our knowledge remain open:

Question 6. *Is the $\mathcal{K}$-profinite genus of every residually finite Kähler group finite?*

Question 7. *Is there a simple Kähler group?*

Question 7 is a well-known variation of Serre's question whether there is a non-residually finite Kähler group; the latter was answered positively by Toledo [16].

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Profinite rigidity: Finitely presented versus finitely generated

ALAN W. REID

(joint work with Martin R. Bridson, Ryan Spitler)

The quest to understand the extent to which finitely generated groups are determined by their finite images has been greatly invigorated in recent years with input from low-dimensional geometry and topology. In our papers [4] and [5] with D. B. McReynolds, we provided the first examples of finitely generated, residually finite groups Γ (which are lattices in $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$) that are *profinitely rigid*: for finitely generated, residually finite groups Λ, if $\widehat{\Lambda} \cong \widehat{\Gamma}$ then $\Lambda \cong \Gamma$, where $\widehat{\Lambda}$ denotes the profinite completion of Λ.

This talk discussed ideas in the proof of the following theorem which provides the first examples of finitely presented groups that are profinitely rigid among finitely presented groups but not among finitely generated groups.

Theorem 1. *There exist finitely presented, residually finite groups* Γ *with the following properties:*

(1) $\Gamma \times \Gamma$ *is profinitely rigid among all finitely presented, residually finite groups.*

(2) There exist infinitely many non-isomorphic finitely generated groups Λ *such that* $\widehat{\Lambda} \cong \widehat{\Gamma} \times \widehat{\Gamma}$*.*

(3) If Λ *is as in (2), then there is an embedding* $\Lambda \hookrightarrow \Gamma \times \Gamma$ *that induces the isomorphism* $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$ *(in other words,* $\Lambda \hookrightarrow \Gamma \times \Gamma$ *is a Grothendieck pair).*

If M *is any Seifert fibred space with base orbifold* $S^2(3,3,4)$ *or* $S^2(3,3,6)$ *or* $S^2(2,5,5)$*, then* $\Gamma = \pi_1 M$ *has these properties.*

Underlying the proof of Theorem 1 are some of the ideas of [4] and [5], in particular the use of *Galois rigidity* in the context of $\Gamma = \pi_1(M)$ with M as in the statement of Theorem 1. Although Galois rigidity does not hold in the strict sense when applied to $\Gamma \times \Gamma$, enough control is gained to execute an endgame similar to that in [4] and [5].

The key structural parts of the proof are as follows:

Step 1: Prove that if M is as in the statement of Theorem 1, then $\pi_1(M)$ is profinitely rigid. This uses Galois rigidity to reduce to the "relative setting": namely if Λ is a finitely generated, residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma}$, then $\Lambda = \pi_1(N)$ where N is a Seifert fibred space. One can now use a result of Wilkes [8] to show that $\Lambda \cong \Gamma$.

Step 2: We show that if Λ is a finitely generated, residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$, then there is an embedding $\Lambda \hookrightarrow \Gamma \times \Gamma$ inducing the isomorphism at the level of profinite completion.

A crucial part of the proof, and where Galois rigidity is used, is in proving the following.

Proposition 2. *Let* Γ *be the fundamental group of a Seifert fibred space whose base orbifold* $S^2(p,q,r)$ *is one of those in Theorem 1 and let* $\Delta = \Delta(p,q,r)$ *be the orbifold fundamental group of this base. Let* Λ *be a finitely generated group with* $\widehat{\Lambda} = \widehat{\Gamma}_1 \times \widehat{\Gamma}_2$*, where* $\Gamma_1 \cong \Gamma_2 \cong \Gamma$*. If* Λ_i *is the projection of* Λ *to* $\widehat{\Gamma}_i$*, then there exist epimorphisms* $g_i : \Lambda_i \to \Delta$ *and hence a homomorphism*

$$g : \Lambda \hookrightarrow \Lambda_1 \times \Lambda_2 \overset{(g_1,g_2)}{\to} \Delta \times \Delta$$

with image a full subdirect product.

With this in hand, and with some additional argument, we can show that the epimorphsims $g_i : \Lambda_i \to \Delta$ fit into a short exact sequence: $1 \to Z \to \Lambda_i \to \Delta \to 1$ with Z central and infinite cyclic. As above, Λ_i can then be shown to be the fundamental group of a Seifert fibred space, and indeed $\Lambda_1 \cong \Lambda_2 \cong \Gamma$.

Step 3: We prove that there are no *finitely presented* Λ as in the statement of Theorem 1. To do this we prove the following result (that builds on arguments of [3])

Theorem 3. *Let M be a Seifert fibred space with hyperbolic base orbifold, let $\Gamma = \pi_1 M$, let D be the direct product of finitely many copies of Γ, and let $\Lambda < D$ be a subgroup such that the inclusion induces an isomorphism of profinite completions. If Λ is finitely presented, then $\Lambda = D$.*

Step 4: Finally we construct Grothendieck pairs $(\Gamma \times \Gamma, \Lambda)$ with Λ finitely generated. This is achieved using the following result (but we will not discuss in detail how it is implemented).

Theorem 4. *Let Δ be a non-elementary hyperbolic group and let Γ be a group with $H_2(\Gamma, \mathbb{Z}) = 0$ that maps onto Δ. Let G be a finitely generated group that maps onto a subgroup of finite index in $[\Gamma, \Gamma]$. Then,*

(1) there exists an infinite sequence of distinct finitely generated subgroups $P_n < G \times G$ such that each inclusion $u_n : P_n \hookrightarrow G \times G$ induces an isomorphism of profinite completions.
(2) If G is a central extension of a hyperbolic group and centralizers of elements in that hyperbolic group are virtually cyclic, then P_n is not abstractly isomorphic to P_m when $n \neq m$.

The subgroups P_i will not be finitely presented in general, even if G is finitely presented (cf. [1]).

In our setting we would like to apply Theorem 4 to the fundamental groups of the Seifert fibre spaces over the base orbifolds $S^2(p, q, r)$ listed in Theorem 1. More specifically, in the notation of Theorem 4, we would like to take $\Delta = \Delta(p, q, r)$ and $G = \Gamma = \pi_1 M$. But we cannot do this because $H_2(M, \mathbb{Z})$, although finite, is not trivial. Instead, we construct an auxiliary group B with finite abelianisation and with $H_2(B, \mathbb{Z}) = 0$ so that B maps onto a non-elementary hyperbolic group and $\pi_1 M$ maps onto a subgroup of finite index in $[B, B]$.

Theorem 4 fits into a well-established train of ideas for constructing Grothendieck pairs which we now briefly explain. Grothendieck [6] asked if there exist Grothendieck pairs of finitely presented groups. This problem was eventually solved by Bridson and Grunewald [2]. Their proof builds on an earlier argument of Platonov and Tavgen [7] who constructed the first Grothendieck pair of finitely generated groups. They did this by appealing to a special case of the following proposition, taking G to be a free group and Q to be Higman's famous example of a 4-generator, 4-relator group with $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$.

Proposition 5. *Let $f : G \to Q$ be an epimorphism of groups, with G finitely generated and Q finitely presented. Consider the fibre product*

$$P = \{(g, h) \mid f(g) = f(h)\} < G \times G.$$

Then,

(1) P is finitely generated;
(2) if $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$, then $P \hookrightarrow G \times G$ induces an isomorphism $\widehat{P} \overset{\cong}{\hookrightarrow} \widehat{G \times G}$.

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Toward Profinite Rigidity of Free and Surface Groups

Andrei Jaikin-Zapirain

(joint work with Ismael Morales)

Let Γ be a finitely generated residually finite group. We say that Γ is *profinitely rigid* if for every finitely generated residually finite group Λ having the same profinite completion as Γ, we have that $\Lambda \cong \Gamma$.

Already among virtually cyclic groups, there exist groups that are not profinitely rigid. Grunewald, Pickel and Segal [8] proved that virtually polycyclic groups Γ are *almost profinitely rigid*, i.e., there are only finitely many isomorphism classes of finitely generated residually finite groups Λ having the same profinite completion as Γ. The first examples of profinitely rigid groups containing free groups were found among Fuchsian and Kleinian groups [3, 4].

The main open problem in the area is to decide whether free and surface groups are profinitely rigid. Let F denote a finitely generated free group and S a hyperbolic surface group, and let Λ be a finitely generated residually finite group such that $\widehat{\Lambda} \cong \widehat{\Gamma}$ where $\Gamma = F$ or $\Gamma = S$. The strategy to prove that $\Lambda \cong \Gamma$ can be divided into two steps:

(a) Show that Λ is hyperbolic and virtually compact special.
(b) Assuming that Λ is hyperbolic and virtually compact special, prove that $\Lambda \cong \Gamma$.

The groups that are hyperbolic and virtually compact special play an important role in recent advances in geometric group theory. The most notable example is Agol's solution [2] to the virtually fibering conjecture for hyperbolic 3-manifolds. However, we want to note that in his proof [1], Agol used the property RFRS, which is weaker than being hyperbolic and compact special. A finitely generated group is *RFRS* (residually finite rationally solvable) if it is residually (virtually abelian and locally indicable). There is another example where one can substitute the condition of being hyperbolic and virtually compact special with virtually RFRS. Kielak and Linton [9] showed that a hyperbolic and virtually compact special group Λ with $\mathrm{cd}_{\mathbb{Q}}(\Lambda) = 2$ and $b_2^{(2)}(\Lambda) = 0$ is virtually free-by-cyclic. This implies, in particular, the solution of Baumslag's conjecture: one-relator groups with torsion are virtually free-by-cyclic. Fisher [5] showed that in the mentioned result of Kielak and Linton, one may substitute the condition of being hyperbolic and virtually compact special with the condition of being RFRS and obtain the same conclusion. These examples show the relevance of the next result in relation to Step (a).

Theorem 1 ([6]). *Let Λ be a finitely generated residually finite group such that $\widehat{\Lambda} \cong \widehat{\Gamma}$ where $\Gamma = F$ or $\Gamma = S$. Then Λ is RFRS.*

Now, if we are in the conditions of Step (b), i.e., we assume that Λ is hyperbolic and virtually compact special, we can use the hierarchy on these groups. The key step would be understanding when $\Lambda = F_1 *_H F_2$ is an amalgamated product or $\Lambda = F_1 *_H$ is an HNN extension of free groups. Wilton proved that $\Lambda \cong \Gamma$ if H is cyclic. The case when H is an arbitrary free group remains open.

A good test for our strategy would be to consider the case when Λ is a one-relator group. By a result of Linton [10], if $\widehat{\Lambda} \cong \widehat{\Gamma}$ where $\Gamma = F$ or $\Gamma = S$, then Λ is hyperbolic and virtually special. However, at this moment, we can prove that $\Lambda \cong \Gamma$ only in the case of $\Gamma = S$.

Theorem 2 ([7]). *Let Λ be a finitely generated residually finite group with $\mathrm{cd}(\Lambda) = 2$ and $b_2^{(2)}(\Lambda) = 0$ such that $\widehat{\Lambda} \cong \widehat{S}$. Then $\Lambda \cong S$.*

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The contraction space and its applications

Stefanie Zbinden

For almost 10 years, it has been known that if a group contains a strongly contracting element, then it is acylindrically hyperbolic. In fact, one can use the Projection Complex of Bestvina, Bromberg and Fujiwara [1] to construct a hyperbolic space where said element acts WPD. However, until recently, the following question remained unanswered: if Morse is equivalent to strongly contracting, does there exist a *universal WPD action*, that is, does there exists a space where all generalized loxodromics act WPD?

Both the contraction space [3] and [2] answer the above question positively by outlining procedures on how to associate hyperbolic spaces to a given starting space. The contraction space does so by coning off all geodesics which are not strongly-contracting "enough".

In work in progress with Cornelia Druţu and Davide Spriano, we generalize the construction of the contraction space to get the following application. If a group acts geometrically on a geodesic metric space which is injective or median, then it is either acylindrically hyperbolic or has linear divergence.

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Virtual homological torsion in graphs of free groups with cyclic edges

Jonathan Fruchter

(joint work with Dario Ascari)

In geometric group theory, it is common for insights from the study of 3-manifolds to find natural analogues in more combinatorial settings. One phenomenon in the theory of 3-manifolds is the following:

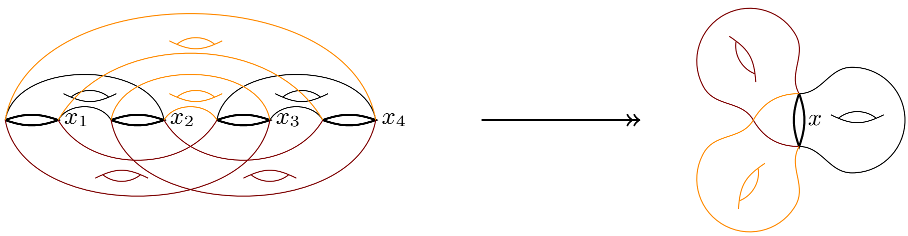
Theorem 1 ([5], *cf.* [1])**.** *Let M be a closed, hyperbolic 3-manifold. For any finite abelian group A, M admits a finite-sheeted cover $N \to M$ such that A is a direct factor of $H_1(M;\mathbb{Z})$.*

Recently, we proved that the same holds for hyperbolic graphs of free groups with $\mathbb{Z}$ edges:

Theorem 2. *Let G be a hyperbolic group which splits as a graph of free groups amalgamated along cyclic subgroups. Suppose that G is not isomorphic to a free product of free and surface groups. Then for every finite abelian group A, G admits a finite-index subgroup $G' \leq G$ such that A is a direct factor of $(G')^{ab}$.*

While the proofs of Sun and Chu-Groves rely on constructing immersed surfaces whose boundaries wrap geodesic loops multiple times (yielding homological torsion), our proof utilizes *branched surfaces*, namely, compact and orientable surfaces with boundary glued along their boundary components. As the following example illustrates, given such a branched surface B (which is not a surface), and a finite abelian group A, one can always construct a (pre-) cover B_A of B such that the "branching loops" of B_A generate a subgroup isomorphic to A in $(\pi_1(B_A))^{ab}$.

Example 3. *Let B be the branched surface which consists of three tori, each with a single boundary component, glued along their boundaries. Let $A = \mathbb{Z}/2$. Then B_A is a 4-fold cover of B with "branching loops" x_1, x_2, x_3 and x_4. In the abelianization, we have that $x_i + x_j = 0$ for every $1 \leq i, j \leq 4$, which implies $x_1 = x_2 = x_3 = x_4$ and $2x_1 = 0$.*



Wilton proved that if G is a non-free hyperbolic group which splits as a graph of free groups with $\mathbb{Z}$ edges, then G has a surface subgroup [6]. The assumption that G is not free can be translated into an assumption on the complexity of links in the geometric realization of a graph of spaces X_G with $\pi_1(X_G) = G$ (a graph of graphs whose edge spaces are cylinders). The links corresponding to a group G as in Theorem 2 are even more complicated, which allows us to construct maps from more complicated 2-complexes into X_G. In particular, we construct a branched surface B and a homomorphism $f : \pi_1(B) \to G$ such that for every finite abelian group A, the map $f_* : (\pi_1(B_A))^{ab} = A \oplus A' \to G^{ab}$ is injective on A. Using virtual retractions, we obtain a finite-index subgroup $G' \leq G$ such that A is a direct factor of $(G')^{ab}$.

Theorem 2 above has applications to the study of *profinite rigidity*. Since the abelianizations of all finite-index subgroups of a given group are a profinite invariant of that group, the theorem implies that free products of free and surface groups are profinitely rigid among hyperbolic graphs of free groups with cyclic edges. This solves additional cases of the following long-standing conjecture:

Conjecture 4. *Let F be a non-abelian free group of rank n and let $w \in F$. For every finite group G, w induces a word map $w : G^n \to G$ and a* word measure *given by $P_w(A \subset G) = \frac{|w^{-1}(A)|}{|G|^n}$. If two words $w, w' \in F$ induce the same word measure on every finite group, then there exists $f \in \mathrm{Aut}(F)$ such that $f(w) = w'$.*

This conjecture is known to hold only when w is a primitive element [4], [6] and [2], a commutator of non-primitive elements [3] or a (possibly non-orientable) surface word [7]. In addition, it is worth noting that:

(1) If the conjecture holds for some $w \in F$ then it also holds for every power $w^d \in F$ [3, Theorem 1.7].
(2) *A priori*, if (the $\mathrm{Aut}(F)$-orbit of) w is determined by its word measures when viewed as a word in F, it does not necessarily follow that w is determined by its word measures when viewed as a word in a larger free group $F * F'$.
(3) Hanany, Meiri and Puder [3, Theorem 2.2] showed that w is determined by its word measures if and only if the following holds: every w' which lies in the $\mathrm{Aut}(\hat{F})$-orbit of w, must lie in the $\mathrm{Aut}(F)$-orbit of w. This implies, in particular, that if the *double* $F *_w F$ is profinitely rigid among all doubles of F, then w is determined by its word measures.

Since doubles of free groups along (non-power) words are hyperbolic, we obtain the following corollary of Theorem 2:

Corollary 5. *Let F be a free group on $x_1, \dots, x_n$. Then Conjecture 4 holds for* partial surface words *in F (namely words of the form $[x_1, x_2] \cdots [x_{k-1}, x_k]$ for even $k \leq n$ or words of the form $x_1^2 \cdots x_k^2$ for $k \leq n$).*

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Stable cylinders for hyperbolic groups

Harry Petyt
(joint work with Davide Spriano, Abdul Zalloum)

Solving systems of equations over a group G is equivalent to the natural problem of finding homomorphisms to G [10]. The *equation problem* asks if it is decidable whether a given system of equations can be solved. This has been resolved in relatively few cases. When G is free, solution sets were completely described by Makanin and Razborov [5, 10]. More generally, Rips–Sela proved the following.

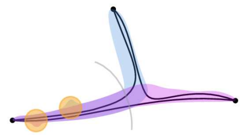
Theorem 1 ([11]). *Torsionfree hyperbolic groups have soluble equation problem.*

The “torsionfree” hypothesis has since been removed by Dahmani–Guirardel [2]. The principal geometric notion introduced by Rips–Sela in order to prove Theorem 1 is that of *stable cylinders.*

Definition 2. *Let G be a hyperbolic group. A θ–*cylinder *for $g, h \in G$ is a neighbourhood $C(g,h)$ of a geodesic γ from g to h with $\mathsf{d}_{Haus}(\gamma, C(x,y)) \leq \theta$. A collection $\{C(a,a') : a, a' \in A\}$ of cylinders is* (k,R)–stable *for $A \subset G$ if*

(1) $C(a,a') = C(a',a)$ for all $a, a' \in A$, and

(2) for all $a_1, a_2, a_3 \in A$, there are R–balls $B_1, \ldots, B_k$ in G such that, for any $x \in G$ with $\mathsf{d}(a_1, x) \leq \langle a_2, a_3 \rangle_{a_1}$, if $x \in C(a_1, a_2)$, then either $x \in C(a_1, a_3)$, or $x \in B_i$ for some i.

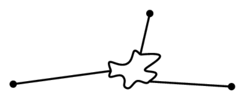


Condition (2)

A few comments on (2). Since G is hyperbolic, every geodesic triangle between a_1, a_2, a_3 has a coarse centre m, whose distance from a_1 is roughly the Gromov product $\langle a_2, a_3 \rangle_{a_1}$. Imagine a_2 as being very far away from a_1, and a_3 as being a small perturbation of a_2 at the same distance from a_1. The arm of $C(a_1, a_3)$ that joins a_1 to m will make up almost the entirety of $C(a_1, a_3)$, but (2) says that it is *equal* to the corresponding arm of $C(a_1, a_2)$, except for in a uniform number of small balls. If G is a surface group and we keep perturbing $a_i \in A$ to $a_{i+1} \in A$ along the perimeter of a circle centred on a_1, then we see that the positions of those small balls must be changing along the a_1 arm.

The main technical result of [11] is the following.

Theorem 3 ([11])**.** *Let G be a torsionfree hyperbolic group. For each n there exists R such that for any $A \subset G$ with $|A| \leq n$, there exist cylinders that are $(1,R)$–stable for A.*



Using the cylinders of Theorem 3, Rips–Sela construct “canonical representatives” for the elements of A. These satisfy a stability condition analogous to (2), and a triangle of them looks like the one depicted on the right.

Given a finite system E of equations over a torsionfree hyperbolic group G, canonical representatives enable one to lift E to a finite set of systems $\bar{E}_i$ of equations over a free group F. A solution in F of any $\bar{E}_i$ projects to a solution of E in G, and if E can be solved in G then at least one $\bar{E}_i$ must be soluble in F. In order to prove Theorem 1 from here, one simply applies the Makanin–Razborov algorithm to each of the finitely many $\bar{E}_i$.

The dependence on A in Theorem 3 introduces various difficulties in the above argument, which led Rips–Sela to ask the following.

Question 4 ([11])**.** *Do torsionfree hyperbolic groups admit globally stable cylinders (i.e. cylinders that are stable for $A = G$)?*

Globally stable cylinders yield globally defined canonical representatives. This can be thought of as strengthening Mineyev’s rational bicombing [6].

The above comments on (2) show that if G is one-ended then any globally stable collection of cylinders for G must have stability constant $k \geq 2$. Rips–Sela showed that $C'(\frac{1}{8})$ groups admit globally $(2, R)$–stable cylinders, but no further examples were found till recent work of Lazarovich–Sageev [4].

Theorem 5 ([4]). *If a hyperbolic group G acts properly and cocompactly on a CAT(0) cube complex, then G admits globally stable cylinders.*

The arguments of Lazarovich–Sageev work entirely by using the combinatorics of the hyperplanes of the CAT(0) cube complex, and finite-dimensionality plays an essential role. Our main result is the following.

Theorem 6 ([8]). *Residually finite hyperbolic groups admit globally stable cylinders.*

The key technical feature of our argument is the construction of a model space X that is sufficiently similar to a CAT(0) cube complex to be able to make hyperplane arguments. It cannot be *too* similar to a CAT(0) cube complex, though, because of groups with property (T). In particular, the hyperplane arguments involved will necessarily be different to those of [4].

In a sense, the model space X can be thought of as a kind of "acylindrical" version of a CAT(0) cube complex. In a proper group action, point-stabilisers must be finite. Acylindrical actions relax this by allowing infinite point-stabilisers but requiring that stabilisers of pairs of distant points are finite: the action is not proper, but if you move far enough then it starts to look proper. Analogously, the model space X is neither finite-dimensional nor a cell complex, but when you move far enough it starts to behave like a finite-dimensional CAT(0) cube complex.

Very briefly, X is built using a generalisation of Sageev's construction developed in [9]. The input is a collection W of walls on G (i.e. bipartitions $G = h^- \sqcup h^+$), and a family $\mathcal{D}$ of subsets of 2^W satisfying a couple of simple conditions that make $\mathcal{D}$ a *dualisable system* in the sense of [9]. The case $\mathcal{D} = 2^W$ is precisely Sageev's construction. The thing to have in mind for $\mathcal{D}$ is the family of all chains of walls (sequences $h_1, \dots, h_n$ such that h_i separates h_{i-1} from h_{i+1} for all i), though in the present case it is actually a refined family of chains.

To build the walls on G, we use a remarkable embedding result due to Bestvina–Bromberg–Fujiwara [1]. The additional *quasimedian* property, established in [3, 7], means that the embedding is not some kind of log-spiral with lots of backtracking.

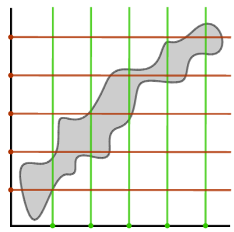
Theorem 7 ([1, 3, 7]). *Every residually finite hyperbolic group admits an action on a finite product of quasitrees such that orbit maps are quasimedian quasi-isometric embeddings. That is, the image of every quasigeodesic projects to an unparametrised quasigeodesic in each factor of the product.*

We first make walls on the quasitree factors. Those induce walls on the product of quasitrees, and intersecting with a G–orbit yields walls on G. The quasimedian property implies that they have quasiconvex halfspaces in G.

The choice of the dualisable system $\mathcal{D}$ is more subtle, but from there we feed W and $\mathcal{D}$ into the machinery of [9], and it spits out the desired model space X. Some of the properties of X are summarised below.

Theorem 8. *Let G be a residually finite hyperbolic group. There is a hyperbolic space X with the following properties.*

- *G acts properly and coboundedly on X.*
- *X has the structure of a median algebra.*
- *Metric balls in X are convex with respect to the median.*
- *X is determined by walls that come in finitely many colours.*
- *The walls of a given colour satisfy a strong separation property.*



These properties make it possible to construct globally stable cylinders in X using the combinatorics of walls, and to prove Theorem 6 one pulls these cylinders back to G. In fact, the construction works for any hyperbolic space satisfying the conclusions of Theorem 7. For instance, with some work one can use results of [1] and [7] for mapping class groups to show that they are satisfied by curve graphs of surfaces [8], and it follows that curve graphs also admit globally stable cylinders.

As a final comment, the stability constant k in Theorem 6 is equal to $2m + 1$, where m is the number of quasitrees in Theorem 7. By comparison, the value of k given by the proof of Theorem 5 is a tower of exponentials of height depending on the hyperbolicity constant of G.

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Profinite properties of Coxeter groups

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(joint work with Samuel M. Corson, Sam Hughes and Philip Möller)

As the main protagonists here are Coxeter groups, we start with a definition of these objects. Given a finite simplicial graph Γ with the vertex set $V(\Gamma)$, the edge set $E(\Gamma)$ and an edge-labeling $m\colon E(\Gamma) \to \mathbb{N}_{\geq 3} \cup \{\infty\}$, the associated *Coxeter group* W_Γ is given by the presentation

$$W_\Gamma := \left\langle V(\Gamma) \;\middle|\; \begin{array}{l} v^2 \text{ for all } v \in V(\Gamma), (vw)^2 \text{ if } \{v,w\} \notin E(\Gamma), \\ (vw)^{m(\{v,w\})} \text{ if } \{v,w\} \in E(\Gamma) \text{ and } m(\{v,w\}) < \infty \end{array} \right\rangle.$$

Since the edge-label 3 appears very often in the study of Coxeter groups it is convenient to omit this label in the graph Γ. Coxeter groups are a family of groups that generalize reflection groups and can be studied from various perspectives. Here, we focus on profinite properties of Coxeter groups.

Let G be a group and $\mathcal{N}$ be the set of all finite index normal subgroups of G. We equip each G/N, $N \in \mathcal{N}$ with the discrete topology and endow $\prod_{N\in\mathcal{N}} G/N$ with the product topology. We define a map

$$\iota\colon G \to \prod_{N\in\mathcal{N}} G/N \text{ by } g \mapsto (gN)_{N\in\mathcal{N}}.$$

The map ι is injective if and only if G is residually finite. The *profinite completion* of G, denoted by $\widehat{G}$, is defined as $\widehat{G} := \overline{\iota(G)}$. For a group G we denote by $\mathcal{F}(G)$ the set of isomorphism classes of finite quotients of G. We note that the set $\mathcal{F}(G)$ of a finitely generated residually finite group G encodes the same information as the profinite group $\widehat{G}$, see [4].

Let $\mathcal{C}$ be a class of finitely generated residually finite groups. A group G is called *$\mathcal{C}$-profinitely rigid* if $G \in \mathcal{C}$ and for any group H in the class $\mathcal{C}$ whenever $\widehat{G} \cong \widehat{H}$, then $G \cong H$. By definition, a finitely generated residually finite group G is called *profinitely rigid (in the absolute sense)* if G is profinitely rigid relative to the class consisting of all finitely generated residually finite groups.

Let $\mathcal{W}$ be the class consisting of all Coxeter groups. We raise the following question:

Question 1. *Are Coxeter groups $\mathcal{W}$-profinitely rigid?*

Before we give some partial results to the above question, we want to point out how $\mathcal{W}$-profinite rigidity is connected to the isomorphism problem of Coxeter groups. In 1911, Max Dehn formulated three decision problems for finitely presented groups: the word, conjugacy, and isomorphism problem. Whilst all three problems are unsolvable in full generality, for Coxeter groups, both the word and conjugacy problems have been solved. Despite much effort, the isomorphism problem amongst Coxeter groups remains open. The relevance of profinite rigidity to the isomorphism problem is the following well known fact which has appeared in [1] and [5, Proposition 8].

Lemma. *Let $\mathcal{C}$ be a class of finitely presented residually finite groups. If G is $\mathcal{C}$-profinitely rigid, then the isomorphism problem for G is solvable in $\mathcal{C}$.*

Virtually abelian Coxeter groups are characterized by their Coxeter graphs. A Coxeter group W_Γ is virtually abelian if and only if for every connected component Ω of Γ, the special parabolic subgroup W_Ω is finite or Ω is isomorphic to one of the graphs in Figure 1.

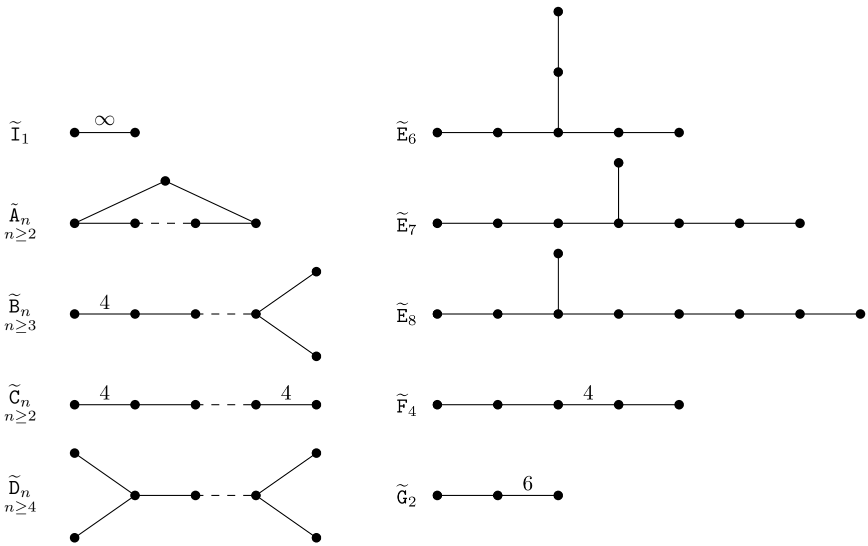

Figure 1. Coxeter graphs of irreducible affine type.

The Coxeter groups defined by the graphs in Figure 1 are precisely the irreducible affine Coxeter groups. In [3] we proved:

Theorem 2. *Finite products of irreducible affine Coxeter groups are profinitely rigid in the absolute sense.*

There are several interesting subclasses of Coxeter groups depending on the edge-labeling. For example, if $E(\Gamma) = \emptyset$ or $m(E(\Gamma)) = \{\infty\}$, then W_Γ is called *right-angled*. In [2] we showed:

Theorem 3. *Right-angled Coxeter groups are $\mathcal{W}$-profinitely rigid.*

Further, in [2] we showed that there exist right-angled Coxeter groups which are not profinitely rigid in the absolute sense. We denote by W_n the Coxeter group whose Coxeter graph Γ is a complete graph with n vertices and all edge-labels are ∞.

Theorem 4. *If $n \geq 4$, then the Coxeter group $W_n \times W_n$ is not profinitely rigid in the absolute sense.*

Using the recent result that all Higman-Thompson groups V_n can be generated by 3 involutions [7] one can upgrade the above result to:

Proposition 5. *The Coxeter group $W_n \times W_n$ is profinitely rigid in the absolute sense if and only if $n \leq 2$.*

Cohomological goodness in the sense of Serre is a crucial group property with many implications in the direction of profinite rigidity. By definition, a finitely generated residually finite group G is called *cohomologically good* if for every finite G-module M and every $q \geq 0$, the induced map on cohomology $H^q(\widehat{G}; M) \to H^q(G; M)$ is an isomorphism. In [6] we proved:

Theorem 6. *Coxeter groups are cohomologically good.*

The fact that Coxeter groups are cohomologically good makes them particularly well-suited to profinite methods. In particular, we have control about torsion elements in the profinite completion of a Coxeter group.

We showed that many properties of Coxeter groups that are characterized in terms of graphs are $\mathcal{W}$-profinite invariants. A property $\mathcal{P}$ is called *$\mathcal{W}$-profinite invariant* if it takes the same value on Coxeter groups whose profinite completions are isomorphic. In [6] we proved:

Theorem 7. *Let W_Γ and W_Λ be Coxeter groups and suppose $\widehat{W_\Gamma} \cong \widehat{W_\Lambda}$. Then,*

(1) W_Γ has Serre's fixed point property F$\mathcal{A}$ *if and only if W_Λ has* F$\mathcal{A}$*;*
(2) W_Γ is a hyperbolic group if and only if W_Λ is a hyperbolic group;
(3) W_Γ is virtually free if and only if W_Λ is virtually free;
(4) W_Γ is virtually surface if and only if W_Λ is virtually surface;
(5) The Euler characteristics are equal: $\chi(W_\Gamma) = \chi(W_\Lambda)$;
(6) $M(W_\Gamma) \cong M(W_\Lambda)$, where $M(-)$ denotes the Schur multiplier;
(7) W_Γ does not have non-trivial finite normal subgroups if and only if W_Λ does not have non-trivial finite normal subgroups.

A generalisation of the class of irreducible affine Coxeter groups is the class of minimal non-spherical Coxeter groups. Let W_Γ be an infinite Coxeter group. The group W_Γ is said to be *minimal non-spherical* if every proper special parabolic subgroup is finite. Using cohomological goodness of Coxeter groups and $\mathcal{W}$-profinite invariants we proved in [6]:

Theorem 8. *Minimal non-spherical Coxeter groups are $\mathcal{W}$-profinitely rigid.*

At the end we want to give a proof idea, that Coxeter groups where the defining graph is a triangle are $\mathcal{W}$-profinitely rigid.

Proposition 9. *Let $p, q, r \in \mathbb{N}_{\geq 3}$ and let $\Delta(p, q, r)$ denote a triangle graph with edge labels p, q, r. Then $W_{\Delta(p,q,r)}$ is $\mathcal{W}$-profinitely rigid.*

Proof. Let W_Ω be a Coxeter group such that $\widehat{W}_{\Delta(p,q,r)} \cong \widehat{W}_\Omega$. The Coxeter group $W_{\Delta(p,q,r)}$ does not have non-trivial finite normal subgroups. Since this property is a $\mathcal{W}$-profinite invariant, it follows that W_Ω does not have non-trivial finite normal subgroups. Further, $W_{\Delta(p,q,r)}$ is virtually a surface group and this property is a $\mathcal{W}$-profinite invariant, thus W_Ω is also virtually surface.

For a group G we denote by $\mathrm{CF}(G)$ the poset of conjugacy classes of finite subgroups in G. Let W be a virtually surface Coxeter group, then $\mathrm{CF}(W)$ is a $\mathcal{W}$-profinite invariant, see [6].

Now, we know that W_Ω is a virtually surface group without non-trivial finite normal subgroups and has 3 maximal finite subgroups D_p, D_q and D_r. By a characterisation of virtually surface Coxeter groups in terms of graphs, see [6] it follows that Ω is a cycle of length 3 with edge labels p, q, r. Hence $\Omega \cong \Delta(p,q,r)$ and therefore $W_{\Delta(p,q,r)} \cong W_\Omega$. □

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Waist inequalities and the Kazhdan property

Roman Sauer
(joint work with Uri Bader)

A family $\mathcal{F}$ of d-dimensional Riemannian manifolds of finite volume has a *uniform n-waist* if there is $\epsilon > 0$ such that for every $M \in \mathcal{F}$ and every smooth (or analytic) map $f\colon M \to \mathbb{R}^n$ there is a point $p \in \mathbb{R}^n$ such that

$$\mathrm{vol}_{d-n}\big(f^{-1}(\{p\})\big) \geq \epsilon \cdot \mathrm{vol}_d(M).$$

Let M be a closed Riemannian manifold with a Kazhdan fundamental group. Let $\mathcal{F}_M$ be the family of finite covers of M. It is an immediate consequence of the Buser-Cheeger inequality and the fact that the first positive eigenvalue of the Laplacian of any $\bar{M} \in \mathcal{F}_M$ is bounded away from zero that $\mathcal{F}_M$ has a uniform 1-waist. We prove the following theorem [2] that provides an extra dimension.

Theorem 1. *The family of finite covers of a closed Riemannian manifold with Kazhdan fundamental group has a uniform* 2*-waist.*

Recall that the vanishing of the first cohomology with arbitrary unitary coefficients characterizes the Kazhdan property for finitely generated groups. This characterization was extended in [1] from unitary coefficients to L^1-spaces. The new group-theoretic input that makes the theorem above possible is the following.

Theorem 2. *The second group cohomology of a finitely presented Kazhdan group with coefficients in every L^1-space is Hausdorff.*

This group-theoretic input leads to a uniform isoperimetric inequality in the real cellular cochain complex of $\bar{M} \in \mathcal{F}_M$ in low degrees. By Poincare duality this corresponds to a uniform isoperimetric inequality in the real cellular chain complex of $\bar{M} \in \mathcal{F}_M$ in low codegrees. Using tools from integer linear programming this can be improved to linear isoperimetric inequalities in the integer cellular chain complex. Finally, the Federer-Fleming deformation theorem and a general method by Gromov lead to a proof the first theorem.

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Drilling hyperbolic groups

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(joint work with Peter Haïssinsky, Jason Manning, Damian Osajda, Alessandro Sisto, Genevieve Walsh)

The *Cannon Conjecture* predicts that a word-hyperbolic group whose boundary at infinity is a two-sphere has a finite-index subgroup which is the fundamental group of a closed hyperbolic 3–manifold.

Motivated by this question, we introduce the concept of *group-theoretic drilling*:

Definition 1 (Drilling)**.** *Suppose that G is a hyperbolic group, and that g is an element so that $\langle g \rangle$ is a maximal cyclic subgroup. A* drilling of G along g *is a relatively hyperbolic pair $(\widehat{G}, P)$ along with a normal subgroup $N \trianglelefteq P$ with an identification $P/N \cong \langle g \rangle$ so that $\widehat{G}/\langle\langle N \rangle\rangle \cong G$, with the quotient map inducing the identification of P/N with $\langle g \rangle$.*

Drillings of hyperbolic groups exist in all dimensions, but we are motivated by three-dimensional topology. An example of drilling occurs in the following context: Let M be a closed hyperbolic 3–manifold, and let $\gamma \subset M$ be a simple geodesic. Then $M - \gamma$ admits a complete hyperbolic metric of finite-volume. On the level of π_1, we obtain a drilling in the sense of Definition 1. Our main result is a coarse analog of this phenomenon.

Theorem 2. *If G is a residually finite hyperbolic group with $\partial G \cong S^2$, then a finite-index subgroup G_0 of G admits a drilling $(\widehat{G}, P)$ where the Bowditch boundary of $(\widehat{G}, P)$ is a two-sphere.*

Theorem 2 allows us to reduce the Cannon Conjecture (in the residually finite case) to a relatively hyperbolic version. This relatively hyperbolic version should be more approachable, because the hypothesized 3–manifold should have non-empty boundary, and hence be Haken.

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