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# MFO-RIMS Tandem Workshop: Optimization, Theoretical Computer Science and Algebraic Geometry: Convexity and Beyond

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Abstract. The joint workshop brought together leading experts from real algebraic geometry, computational algebra, polynomial optimization, and theoretical computer science. The event fostered vibrant interdisciplinary exchange, with a particular emphasis on the role of computation–both as a practical tool for algorithm design and as a lens for uncovering deep mathematical structures.

*Mathematics Subject Classification (2020):* 14-XX, 49-XX, 68Qxx.



## Introduction by the Organizers

The MFO-RIMS Tandem Workshop "Optimization, Theoretical Computer Science and Algebraic Geometry: Convexity and Beyond", jointly organized by Yoshio Okamoto (Tokyo), Cordian Riener (Tromsø), Markus Schweighofer (Konstanz), and Yoshiyuki Sekiguchi (Kyoto), took place from February 16–21, 2025. The event was held in a hybrid format, with parallel in-person meetings at the Mathematisches Forschungsinstitut Oberwolfach (MFO) and the Research Institute for Mathematical Sciences (RIMS) in Kyoto, connected via a shared digital infrastructure.

The main goal of the workshop was to explore the rich interplay between convexity and its generalizations in optimization, algebraic geometry, and theoretical

computer science. Convexity has long served as a unifying concept in these disciplines, and recent developments have extended its scope to more general geometric and algebraic settings. The workshop brought together researchers from Europe the US and Asia working on these intersections, with the aim of facilitating new collaborations and strengthening existing links between mathematical communities across continents.

The scientific program featured a diverse collection of talks, spanning both foundational theory and recent advances. A prominent theme was optimization over non-classical domains, including polyhedral norms, non-Euclidean spaces, and singular semidefinite programs. Other key topics included the structure and complexity of symmetric and nonnegative polynomials, tensor methods in high-dimensional computation, algorithmic aspects of sum-of-squares proofs, and applications of algebraic and combinatorial methods in graph theory and discrete geometry.

Several talks focused on the algebraic geometry underlying matroid varieties, semistability in quiver representations, and D-module approaches to singularities. The workshop also highlighted holonomic and combinatorial techniques that bridge continuous and discrete models in optimization and complexity theory.

A central element of the workshop was a dedicated problem session held midweek, during which participants from both sites contributed open problems and research challenges. This structured interaction proved highly fruitful and led to several follow-up discussions. The time zone difference required careful scheduling, with some talks given live and others recorded for asynchronous viewing. Despite these logistical challenges, the hybrid format enabled broad participation and dynamic exchange between the Oberwolfach and Kyoto communities.

We believe the workshop was highly successful in creating an arena for scientific exchange. It provided both in-depth technical presentations and a platform for interdisciplinary engagement and contributed to building a stronger global network of researchers at the interface of optimization, algebra, and geometry. We thank the staff of MFO and RIMS for their excellent support in making this collaboration possible.

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## MFO-RIMS Tandem Workshop: Optimization, Theoretical Computer Science and Algebraic Geometry: Convexity and Beyond

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# Abstracts

## Matroid varieties, their defining equations, and their irreducibility

Fatemeh Mohammadi
(joint work with Emiliano Liwski)

The theory of point-line configurations can be seen as a geometric counterpart of matroids of rank 3. They arise in various areas of research, including the study of the connectivity of moduli spaces of hyperplane arrangements and the cohomology of their complement manifolds [1, 8], as well as the study of singularities and smoothness of their varieties [2]. They also arise in the study of determinantal varieties, as well as conditional independence models [3]. Of particular interest are $(v_r, b_k)$ configurations, which contain Steiner systems [4, 5].

**Definition 1.** A point-line configuration $M$ consists of a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ such that each line contains at least three distinct points, and any pair of points lies on at most one common line. A $(v_r, b_k)$ *configuration* is a point-line configuration such that:

- There are $v$ points and $b$ lines.
- Each line contains $k$ points, and each point lies on $r$ lines.

When $v = b$, and $r = k$, the configuration is called symmetric, denoted by $v_k$.

In this note, we focus on the associated varieties of point-line configurations $M$. Our primary goal is to determine the irreducible decomposition of these varieties. We describe a decomposition strategy from [11], based on identifying the set of minimally dependent matroids of $M$. We apply this method to the MacLane configuration, which cannot be handled using existing computer algebra systems.

We define two families of point-line configurations that play a key role in this note; see [9, 10]. In the following, we let $\mathcal{L}_p$ denote the set of lines containing $p$ for any point $p \in \mathcal{P}$. The *degree* of $p$ is defined as the size of $\mathcal{L}_p$, i.e., $\deg(p) = |\mathcal{L}_p|$.

**Definition 2.** For a point-line configuration $M$, we define $S_M = \{p \in \mathcal{P} : |\mathcal{L}_p| \geq 2\}$. We then consider the following chain of submatroids of $M$:

$$M_0 = M, \quad M_1 = S_M, \quad \text{and} \quad M_{j+1} = S_{M_j} \quad \text{for all } j \geq 1.$$

We say that $M$ is *nilpotent* if $M_j = \emptyset$ for some $j$.

**Definition 3.** For a point-line configuration $M$, we define $Q_M = \{p \in \mathcal{P} : |\mathcal{L}_p| \geq 3\}$. We then consider the following chain of submatroids of $M$:

$$M^0 = M, \quad M^1 = Q_M, \quad \text{and} \quad M^{j+1} = Q_{M^j} \quad \text{for all } j \geq 1.$$

We say that $M$ is *solvable* if $M^j = \emptyset$ for some $j$.

**Definition 4.** Let $M$ be a matroid of rank $n$ on the ground set $[d]$. A realization of $M$ is a collection of vectors $\gamma = \{\gamma_1, \ldots, \gamma_d\} \subset \mathbb{C}^n$ such that

$$\{\gamma_{i_1}, \ldots, \gamma_{i_p}\} \text{ is linearly dependent} \iff \{i_1, \ldots, i_p\} \text{ is a dependent set of } M.$$

The *realization space* of $M$ is defined as $\Gamma_M = \{\gamma \subset \mathbb{C}^n : \gamma \text{ is a realization of } M\}$. A matroid is called realizable if its realization space is non-empty. The *matroid variety* $V_M$ of $M$ is defined as the Zariski closure in $\mathbb{C}^{nd}$ of $\Gamma_M$. See [6, 7].

Similarly, we introduce the *circuit variety* of $M$ as follows:

$$V_{\mathcal{C}(M)} = V(I_{\mathcal{C}(M)}) = \{\gamma \subset \mathbb{C}^n : \gamma \text{ includes the dependencies of } M\}.$$

We recall the following result on the varieties of nilpotent and solvable matroids.

**Theorem 5** ([10, 9])**.** *Let $M$ be a point-line configuration on $[d]$.*

(i) *Assume $M$ has no points of degree above two. If $M$ is nilpotent, then $V_{\mathcal{C}(M)} = V_M$. If every proper submatroid of $M$ is nilpotent, then $V_{\mathcal{C}(M)} = V_M \cup V_{U_{2,d}}$, where $U_{2,d}$ is the uniform matroid of rank 2 with $d$ elements.*

(ii) *If $M$ is solvable, then $V_M$ is either irreducible or empty.*

**Decomposition strategy.** We begin by identifying the minimal matroids $\min(M)$. Next, the circuit variety is decomposed using Proposition 4.1 in [11]. In the recursive decomposition step, this process is applied to each circuit variety that appears in the decomposition. The decomposition continues iteratively for any new circuit varieties encountered, until each circuit variety falls into one of the following cases: if a circuit variety $V_{\mathcal{C}(N)}$ corresponds to a nilpotent configuration with no points of degree greater than two, it is replaced by $V_N$; if it corresponds to a configuration where all points have degree at most two, and all proper submatroids are nilpotent, it is replaced by $V_N \cup V_{U_{2,d}}$. After this step, we obtain a decomposition of $V_{\mathcal{C}(M)}$ as a union of matroid varieties. The irreducibility of each matroid variety is then verified by identifying the solvable matroids involved and applying Theorem 5.

We illustrate our algorithm from [11] for identifying minimal matroids of point-line configurations and computing their circuit varieties' irreducible decomposition.

**Example 6.** Let $M$ be the MacLane configuration $M$, which is the unique $8_3$ configuration. Then, $\min(M)$ consists of the following matroids; see Figure 1:

- The uniform matroid $U_{2,8}$.
- The matroids $M(i)$ for $i \in [8]$.
- A line of $M$ with the other 5 points coinciding at a single point outside it.
- A line with four points, three as double points, and a free point $x$ outside. The double points form pairs on a common line with $x$.

There are 8 matroids of the third type, each associated with a line of $M$, and 8 matroids of the fourth type, each determined by the choice of a free point. We label these matroids as $A_j, B_k$, for $j, k \in [8]$. This leads to the following decomposition:

$$V_{\mathcal{C}(M)} = V_M \cup V_{\mathcal{C}(U_{2,8})} \bigcup_{i=1}^{8} V_{\mathcal{C}(M(i))} \bigcup_{j=1}^{8} V_{\mathcal{C}(A_j)} \bigcup_{k=1}^{8} V_{\mathcal{C}(B_k)}. \tag{1}$$

Moreover, we have $\bigcup_{i=1}^{8} V_{\mathcal{C}(M(i))} = \bigcup_{i=1}^{8} V_{M(i)} \bigcup_{i=1}^{4} V_{D_i} \cup \bigcup_{i=1}^{8} V_{C_i}$, where

- The matroid $C_i$ is obtained by making one point of $M$ a loop, with the remaining points forming the uniform matroid $U_{2,7}$.
- The matroid $D_i$ is obtained from $M$ by making loops of two points that are not connected. The pairs of non-connected points are $\{1,3\},\{2,4\},\{5,7\},\{6,8\}$.

The matroid $V_M$ has two irreducible components, denoted $V_M^+$ and $V_M^-$. The matroids $U_{2,8}, A_j, B_k$ are nilpotent, hence their circuit and matroid varieties coincide. Moreover, $V_{A_j} \subset V_M$ and $V_{C_i} \subset V_{U_{2,9}}$, hence (1) equals:

$$V_{\mathcal{C}(M)} = V_M^+ \cup V_M^- \cup V_{U_{2,8}} \bigcup_{i=1}^{8} V_{M_M(i)} \bigcup_{i=1}^{4} V_{D_i} \bigcup_{k=1}^{8} V_{B_k}. \tag{2}$$

All the matroids in (2) are solvable, hence their varieties are irreducible. The decomposition is non-redundant, providing the irreducible decomposition of $V_{\mathcal{C}(M)}$.

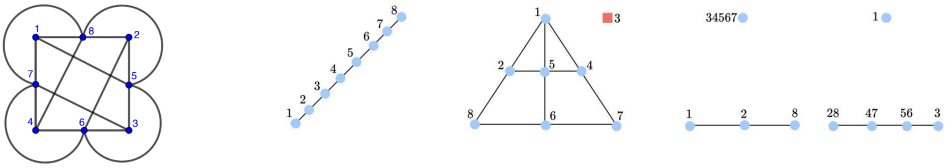


FIGURE 1. MacLane configuration $M$ and its minimal matroids

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## Lamination hulls and solving linear inequalities over convex sets

Saugata Basu
(joint work with Hamidreza A. Khorasgani, Hemanta K. Maji, Hai H. Nguyen)

**Definition 1** ($\Lambda$-hull)**.** Let $\Lambda \subset \mathbb{R}^d$ be some fixed subset. For any subset $S \subset \mathbb{R}^d$ we denote

$$G_\Lambda(S) = \{\alpha x + (1-\alpha)y \mid x, y \in S, x - y \in \Lambda, 0 < \alpha < 1\}, \tag{1}$$

and for each $i \in \mathbb{N}$, we denote $G_\Lambda^{(i)}(S) = \underbrace{G_\Lambda \circ \cdots \circ G_\Lambda}_{i}(S)$.

We call the subset $S^{(\Lambda,\infty)} = \bigcup_{i \geq 0} G_\Lambda^{(i)}(S)$ the $\Lambda$-*hull of* $S$.

**Remark 2.** If $\mathbf{0} \in \Lambda$, then taking $x = y \in S$ in (1), we get that $S \subset G_\Lambda(S)$.

**Remark 3.** If $\Lambda = \mathbb{R}^d$, then it follows from Carathéodory's theorem that for any subset $S \subset \mathbb{R}^d$, $S^{(\Lambda,\infty)} = G_\Lambda^{(\lceil \log_2(d+1) \rceil)}(S) = \mathrm{conv}(S)$, where $\mathrm{conv}(S)$ denotes the convex hull of $S$.

Characterizing the $\Lambda$-hull of subsets of $\mathbb{R}^d$ is an important problem arising in several unrelated areas of mathematics (such as in the study of certain partial differetial equations [4]) as well as in theoretical computer science (secure computation in cryptography [1]).

We prove the following theorem.

**Theorem 4.** [3] *Let $a, b, c \geq 0, d = a + b + c$. Let $\Lambda \subset \mathbb{R}^d$ be defined by*

$$\Lambda = \{\mathbf{0}\} \times \mathbb{R}^b \times \mathbb{R}^c \cup \mathbb{R}^a \times \{\mathbf{0}\} \times \mathbb{R}^c. \tag{2}$$

*Then, for each finite subset $S \subset \mathbb{R}^d$, $S^{(\Lambda,\infty)}$ is a semi-algebraic subset of $\mathbb{R}^d$. Moreover, there exists an algorithm that, given $S$ as input, produces a semi-algebraic description of $S^{(\Lambda,\infty)}$ as its output.*

**Remark 5.** The sequence $(G_\Lambda^{(i)})_{i \geq 0}$ need not stabilize. Let

$$S = \{(3/4, 1/4, 0), (1/4, 1/2, 0), (1/2, 1, 0), (1, 3/4, 0), (3/4, 1/2, 1)\}.$$

With this choice of $S$, $a = b = c = 1$, and $\Lambda \subset \mathbb{R}^3$ in (2), it is not difficult to see that the sequence $G_\Lambda^{(i)}(S)$ does not stabilize (this example is sometimes referred to as the "Tartar square" in the literature, see [2]).

**Remark 6.** Another interesting example also in $\mathbb{R}^3$ is the following one, which shows that even for finite sets $S$, $S^{(\Lambda,\infty)}$ need not always be a semi-linear set (though still semi-algebraic by Theorem 4). Take $\Lambda \subset \mathbb{R}^3$ as defined in (2) with $a = b = c = 1$ and let $S = \{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$. It is easy to verify that with these choices,

$$S^{(\Lambda,\infty)} = G_\Lambda^{(2)}(S) = \{(x, y, z) \mid z = xy, 0 \leq x, y \leq 1\}.$$

As an intermediate step in our proof of Theorem 4 we prove a result about convex subsets of $\mathbb{R}^d$ which might be of independent interest. Let $\Omega$ be a finite set and denote by $\mathrm{CL}(\Omega)$ (CL for "Convex Linear") the set of all formal convex linear combinations of elements of $\Omega$, i.e.

$$\mathrm{CL}(\Omega) = \{\sum_{\omega\in\Omega} a_\omega \cdot \omega \mid a_\omega \geq 0, \sum_{\omega} a_\omega = 1\}.$$

We also denote

$$M_\Omega = \{L_1 \overset{\circ}{\star} \cdots \overset{\circ}{\star} L_k \mid k \geq 1, L_i \in \mathrm{CL}(\Omega), 1 \leq i \leq k\},$$

and

$$P_\Omega = \{M_1 \oplus \cdots \oplus M_m \mid m > 0, M_j \in M_\Omega, 1 \leq j \leq m\}.$$

It might be useful to think of $M_\Omega$ as a set of "monomials" and $P_\Omega$ as a set of polynomials "polynomials" on $\mathrm{CL}(\Omega)$. There exists a evaluation map $\mathrm{eval}_\Omega$, that given, $\phi \in P_\omega$, and a tuple $\mathcal{S} = (S_\omega)_{\omega\in\Omega}$ of subsets of $\mathbb{R}^d$, produces a subset $\mathrm{eval}_\Omega(\phi)(\mathcal{S}) \subset \mathbb{R}^d$, defined by the formula obtained from $\phi$, by replacing $\omega$ by $S_\omega$ and interpreting $c\cdot X$ as dilation of $X$ by $c$, $+$ as Minkowski sum, $\oplus$ as union, and $\overset{\circ}{\star}$ as the "positive geometric join" operator defined by

$$X \overset{\circ}{\star} Y = \{\alpha x + (1-\alpha)y \mid x \in X, y \in Y, 0 < \alpha < 1\}.$$

For subsets $X, Y \subset \mathbb{R}^d$, we denote $X \geq Y$ iff $X$ contains the convex hull of $Y$.

**Theorem 7.** [3] *For any finite set $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 = \{X_1, \ldots, X_n\}$, $\Omega_2 = \{P_1, \ldots, P_m\}$, and a system of inequalities*

$$(X_i \geq \phi_i)_{1\leq i\leq n}, \tag{3}$$

*where $\phi_i \in P_\Omega, 1 \leq i \leq n$, there exists a tuple $(\psi_i)_{1\leq i\leq n}$, with $\psi_i \in P_{\Omega_2}$, such that for any tuple of sets $\mathcal{S} = (S_i)_{1\leq i\leq m}$, the tuple $(Y_i = \mathrm{conv}(\mathrm{eval}_{\Omega_2}(\psi_i)(\mathcal{S})))_{1\leq i\leq n}$, is the smallest (with respect to inclusion) tuple of convex subsets of $\mathbb{R}^d$, which satisfy the inequalities $(Y_i \geq \mathrm{eval}_\Omega(\phi_i)(\mathcal{S}'))_{1\leq i\leq n}$, where $\mathcal{S}' = (Y_1, \ldots, Y_n, S_1, \ldots, S_m)$.*

*If $S_1, \ldots, S_m$ are semi-algebraic subsets of $\mathbb{R}^d$, then so are $Y_1, \ldots, Y_n$. Moreover, if $S_1, \ldots, S_m$ are finite subsets, then each $Y_i$ is convex and is a union of the relative interiors of a finite set of polytopes in $\mathbb{R}^d$ (we call such sets* hemihedras*).*

**Example 8.** We give an illustration of Theorem 7. Consider the following system of inequalities with two variables with $\Omega_1 = \{X_1, X_2\}, \Omega_2 = \{P_1, P_2, P_3, P_4\}$.

$$X_1 \geq P_1 \oplus \left(\frac{1}{2}\cdot X_1 + \frac{1}{4}\cdot X_2 + \frac{1}{4}\cdot P_3\right),$$
$$X_2 \geq P_2 \oplus \left(\frac{1}{4}\cdot X_1 + \frac{1}{2}\cdot X_2 + \frac{1}{4}\cdot P_4\right).$$

Then the convex hulls of the following pair of sets

$$P_1 \oplus P_1 \overset{\circ}{\star} \left(\frac{1}{2}\cdot P_2 + \frac{1}{2}\cdot P_3\right) \oplus P_1 \overset{\circ}{\star} \left(\frac{1}{2}\cdot P_2 + \frac{1}{2}\cdot P_3\right) \overset{\circ}{\star} \left(\frac{2}{3}\cdot P_3 + \frac{1}{3}\cdot P_4\right),$$
$$P_2 \oplus P_2 \overset{\circ}{\star} \left(\frac{1}{2}\cdot P_1 + \frac{1}{2}\cdot P_4\right) \oplus P_2 \overset{\circ}{\star} \left(\frac{1}{2}\cdot P_1 + \frac{1}{2}\cdot P_4\right) \overset{\circ}{\star} \left(\frac{1}{3}\cdot P_3 + \frac{2}{3}\cdot P_4\right)$$

is the smallest convex solution to the given system of inequalities.

The connection between Theorems 4 and 7 is the following. We show that for $S$ a finite subset of $\mathbb{R}^d$ and $\Lambda$ as defined in (2), the set $S^{(\Lambda,\infty)}$ can be recovered (using a semi-algebraic procedure i.e. defined using a first-order formula in the theory of the reals) from a finite set of fibers $S^{(\Lambda,\infty)} \cap \pi^{-1}(x,y), (x,y) \in \mathbb{R}^a \times \mathbb{R}^b$, where $\pi : \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c \to \mathbb{R}^a \times \mathbb{R}^b$ denotes the projection map. Moreover, we prove that each fiber $S^{(\Lambda,\infty)} \cap \pi^{-1}(x,y), (x,y) \in \mathbb{R}^a \times \mathbb{R}^b$ can be characterized as the smallest convex solution to a system of inequalities (3), with $P_1, \ldots, P_m$ specialized to points in $\mathbb{R}^d$. Theorem 7 in conjunction with the semi-algebraic procedure mentioned above and the Tarski-Seidenberg theorem allows us to obtain effectively a semi-algebraic description of $S^{(\Lambda,\infty)}$. Note that while we only prove that $S^{(\Lambda,\infty)}$ is a semi-algebraic set, Theorem 7 implies that each of the fibers $S^{(\Lambda,\infty)} \cap \pi^{-1}(x,y), (x,y) \in \mathbb{R}^a \times \mathbb{R}^b$ are hemihedral. However, as illustrated in the example given in Remark 6, $S^{(\Lambda,\infty)}$ itself need not be hemihedral.

Theorem 4 is a key step in proving a result in cryptography. Suppose $X, Y, Z$ are finite subsets. A randomized function $f : X \times Y \times Z \to \mathbb{R}$, is a function that for each $(x,y) \in X \times Y$, induces a probability distribution function $f(x,y,\cdot) : Z \to \mathbb{R}$. In a recent work [1], the authors proved the decidability of the problem: given $f$ and $r > 0$, does there exist a $r$-bit secure randomized two-party protocol that computes $f$. The decision procedure reduced to testing whether a certain point $q_f \in \mathbb{R}^d$ belonged to $G_\Lambda^{(r)}(S_f)$ for a certain explicitly constructed finite subset $S_f \subset \mathbb{R}^d$ (where $a = \text{card}(X), b = \text{card}(Y), c = \text{card}(Z)$ and $d = a + b + c$). It was left open whether the problem of deciding whether there exists a secure $r$-bit protocol for $f$ for some $r > 0$ is decidable. Using the result in [1] mentioned above, this last decision problem reduces to testing whether the point $q_f$ belongs to the $S^{(\Lambda,\infty)}$. Theorem 4 implies that the problem is decidable.

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## Distance Optimisation in Polyhedral Norms

Nidhi Kaihnsa

(joint work with E. Duarte, J. Lindberg, A. Torres, M. Weinstein)

Given a subset $X$ in a metric space, a *Voronoi diagram* partitions the metric space by identifying the points closest to a given point $p$ on $X$. The region corresponding to the point $p$ is called the *Voronoi cell* of $p$. When the metric space is $\mathbb{R}^n$ endowed with Euclidean norm and $X$ is a real algebraic variety, the geometry of this decomposition was studied in [4]. It was shown that the Voronoi cell a point $p \in X$ is a convex semialgebraic set contained in the normal sapce of the variety at $p$. The authors also studied the algebraic boundary of the Voronoi cells by computing the locus of points that are closest to at least two points on $X$. This locus is called the medial axis.

We study this geometric problem for the case when $X$ is a co-dimension one variety in $\mathbb{R}^n$ endowed with the metric induced by a polyhedral norm. For a polyhedral norms the unit ball is given by a centrally symmetric (full-dimenstional) polytope. The main motivation to study this problem arises from algebraic statistics. Given a metric on a finite set of states, $\Omega$, it induces a Wasserstein distance on the space of probability distributions on the set $\Omega$, with a convex polytope as a unit ball. On the probability simplex, the problem of minimizing Wassertein distance between a probability distribution and an algebraic variety has been studied in great detail (cf. [2, 3]).

In [1] the authors compute the dimension of a Voronoi cell of a point on an algebraic variety for polyhedral norms. They also estimate the number of points on $X$ with full dimensional Voronoi cells. In the joint work with Duarte, Lindberg, Torres, and Weinstein, we are considering this problem for algebraic varieties with codimension one. Given a point $p \in X$, the Voronoi cell of $p$ and its dimension can be determined by the relative position of the normal space of the point and the normal fan of the unit ball. The dimension of Vornoi cell of each point decomposes the algebraic variety. We show that this decomposition infact gives a stratification of the variety. We also study the decomposition of the ambient space $\mathbb{R}^n$ by finding upper bounds on the degrees of medial axis.

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### A Universal Sequence of Tensors for the Asymptotic Rank Conjecture

Petteri Kaski
(joint work with Mateusz Michałek)

We present an explicit sequence of zero-one-valued three-tensors whose sequence of tensor ranks captures the worst-case tensor exponent of tensors of shape $d \times d \times d$. Joint work with Mateusz Michałek (Konstanz). Cf. ITCS 2025 Article No. 64 [1].

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### On the degree automatability of SoS proofs

Luis Felipe Vargas
(joint work with Alex Bortolotti, Monaldo Mastrolilli)

Semidefinite programming (SDP) relaxations have been a powerful technique for approximation algorithm design ever since the celebrated result of Goemans and Williamson. With the aim to construct stronger and stronger SDP relaxations, the Sum-of-Squares (SoS) hierarchy has emerged as a systematic and versatile method for approximating many combinatorial optimization problems. However, fundamental questions remain unanswered. For instance, it is still unknown under what conditions SoS can be *automated*, meaning whether one can find a degree-$d$ SoS proof in time $n^{O(d)}$, provided it exists. O'Donnell (2017) observed that the prevailing belief regarding the automatability of SoS using ellipsoid algorithms is not entirely accurate. Issues may arise when the only degree-$d$ proofs contain exceedingly large coefficients, thereby hindering the ellipsoid method from operating within polynomial time. In this talk we address this problem and provide sufficient conditions that guarantee polynomial time computability of SoS proofs.

**Polynomial optimization.** Polynomial optimization asks for minimizing a polynomial over a given set of polynomial constraints. That is, given polynomials $r, p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_n]$, the task is to find (or approximate) the infimum of the following problem:

$$\inf_{x \in S} r(x), \quad \text{where} \quad S = \{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_m(x) = 0\}. \tag{1}$$

Typically, $S$ is defined by a set of equality constraints, in this case $\mathcal{P} = \{p_1, \ldots, p_m\}$, as well as a set of inequality constraints, $\mathcal{Q}$. For the purpose of this talk we restrict to the case where $\mathcal{Q} = \emptyset$ and $S$ is finite. Nonetheless, we emphasize that our results readily extend to the semialgebraic setting, where $\mathcal{Q} \neq \emptyset$.

A common approach for solving (or approximating) a polynomial optimization problem is by means of sums of squares of polynomials, as we now explain. Let

$q \in \mathbb{R}[x_1, \ldots, x_n]$ be a polyomial. An SoS proof of "$q \geq 0$" (over $S$) from $\mathcal{P}$ is an identity of the form

$$q = \sum_{i=1}^{t_0} s_i^2 + \sum_{i=1}^{m} h_i p_i,$$

where $s_i, h_i \in \mathbb{R}[x_1, \ldots, x_n]$. Moreover, we say that the above SoS proof has *degree* at most $d$ if $\deg(s_i^2) \leq d$, for all $i \in [t_0]$, and $\deg(h_i p_i) \leq d$ for all $i \in [s]$. An SoS refutation of $\mathcal{P}$ is an SoS proof of "$-1 \geq 0$" from $\mathcal{P}$.

The SoS hierarchy is based on the following observation: if there exists an SoS proof of "$r - \theta \geq 0$" from $\mathcal{P}$, then we have that $\min_{x \in S} r(x) \geq \theta$. Moreover, the supremum of the values $\theta$ such that there is an SoS proof of "$r - \theta \geq 0$" from $\mathcal{P}$ of degree $d$, is called $d$-th SoS relaxation, also known as the $d$-th Lasserre relaxation of problem (1). It turns out that the $d$-th SoS relaxation can be formulated as an SDP of size $n^{O(d)}$.

SoS relaxations have gained increasing popularity and success; yet, they remain a relatively recent development. Fundamental questions about their properties and capabilities remain open. For example, we do not even know when SoS relaxations can be computed in polynomial time. Indeed, O'Donnell provided an example of a polynomial system $\mathcal{P}$ and a polynomial $r$ such that for all $\varepsilon > 0$ there is a degree-2 SoS proof of "$r + \varepsilon \geq 0$", but *every* proof has coefficients of magnitud $2^{2^n}$. This shows that the often repeated claim that, for any fixed degree $d$, the $d$-th SoS relaxation can be found (or approximated) in polynomial time is far from true.

O'Donnell (2017) posed the open problem of identifying meaningful conditions that ensure that "small" SoS proofs can be found. We will consider systems $\mathcal{P} = \{p_1 = 0, \ldots, p_m = 0\}$ of polynomials and an "input" polynomial $r$ of degree at most $d$, with the (mild) assumption that the bit complexity needed to represent $\mathcal{P}$ and $r$ is polynomial in $n$. Moreover, we assume that $\mathcal{P}$ is explicitly Archimedean, i.e. there is $N < 2^{poly(n^d)}$ such that there exists a "small" SoS proof of "$N - x_i^2 \geq 0$" from $\mathcal{P}$ for any variable $x_i$. We restate O'Donnell's question as follows: *Under what conditions on $\mathcal{P}$ does the following property hold?*

(P) Assume there exists an SoS proof of "$r \geq 0$" from $\mathcal{P}$ of degree $2d$. Then, for every $\varepsilon > 0$, there also exists an SoS proof of "$r + \varepsilon \geq 0$" from $\mathcal{P}$ with degree $O(d)$ and coefficients bounded by $2^{poly(n^d, \lg \frac{1}{\varepsilon})}$.

Since O'Donnell raised his question in 2017, very few papers have been published that address this issue. An initial elegant solution to this question is provided by Raghavendra and Weitz, which is based on the Nullstellensatz proof system. This criterion is applicable to various optimization problems, including Max-Clique, Matching, and Max-CSP. However, the criterion is subject to significant limitations. First, the criterion is sufficient but not necessary. Second, it is important to observe how the their criterion in is influenced by the complexity of a well-known problem known as the *Ideal Membership Problem* (IMP). This problem involves determining whether an input polynomial $r$ belongs to the ideal generated by $\{p_1, \ldots, p_m\}$.

In this talk, we present new criteria that guarantee property (P) holds, thus enabling the automation of Sum of Squares (SoS) methods. Specifically, we outline the main contributions and techniques:

(1) $\mathrm{SoS}_\varepsilon$ **criterion**. We start by presenting a general criterion, called the $\mathrm{SoS}_\varepsilon$ criterion, which guarantees that Property (P) is satisfied, and thus that SoS can be automated. This criterion, which is a generalization of the criterion by Raghavendra and Weitz, serves as tool for presenting our main contributions. The key distinction lies in the notion of completeness used in the criterion. Specifically, one of the conditions for applying citerior is the so-called $\mathrm{SoS}_\varepsilon$-completeness. We say, roughly, that the system $\mathcal{P}$ is $\mathrm{SoS}_\varepsilon$-complete if every degree-$d$ polynomial $q$ that vanishes on $S$ admits a "small approximate" SoS proof of "$q \geq 0$".

(2) SoS **approximability of polynomial systems**. A main difficulty for applying our new criterion is to prove that a system is $\mathrm{SoS}_\varepsilon$-complete. For this, we present the notion of SoS-approximation between polynomial systems, which turns out to be a powerful tool for showing $\mathrm{SoS}_\varepsilon$ completeness, and applying the $\mathrm{sos}_\varepsilon$ criterion.

(3) **SoS simulates PC and PC criterion** Polynomial Calculus (PC) is a dynamic proof system that permits to prove that an input polynomial belongs to an ideal. In general, PC and SoS are incomparable. However, in the presence of domain restrictions, this is not necessarily true. In a surprising result, Berkholz showed that over Boolean variables, SoS simulates PC for refutations, while for the converse there is a strict separation. In this talk we present an extension of Berkholz's result to general polynomial derivations in which the variable domains are finite rational sets. This result permits us to state a new criterion, called the PC-criterion for guaranteeing that property (P) holds, and thus SoS can be automated.

As a main application of the PC criterion, we examine constraint languages (and polynomial equations) that are closed under the semilattice and dual discriminator polymorphisms. Propositional formulas from HORN-SAT or 2-SAT can be easily translated into system of polynomial equations that are semilatice or dual discriminator closed, respectively. Moreover, these two classes extend HORN-SAT and 2-SAT formulas, respectively, to general finite domain cases and have held a significant role in the theory of Constraint Satisfaction Problems of the form CSP($\Gamma$);

**Theorem 1.** *For a system $\mathcal{P}$ of polynomial equations over $n$ variables that is closed under the semilattice (or dual discriminator) polymorphism, then the* PC *criterion applies.*

## Tensor networks, tensor cumulants, tensor PCA

Dmitriy (Tim) Kunisky
(joint work with Maximilian Jerdee, Cristopher Moore, Alexander S. Wein)

This work concerns hypothesis testing between probability distributions over tensors. It is easiest to think of such problems by analogy with a matrix-valued problem, which is often used as a model of *principal component analysis (PCA)* in the statistics literature.

In the general matrix-valued setting, we are given an observation $Y$ from one of two probability measures $\mathbb{Q}_n$ or $\mathbb{P}_n$ over $n \times n$ symmetric matrices, and must compute some test $t(Y) \in \{0, 1\}$ that attempts to output 0 if $Y \sim \mathbb{Q}_n$ and 1 if $Y \sim \mathbb{P}_n$ with as little error probability as possible. A specific widely studied model involves "noise" drawn from the so-called Gaussian orthogonal ensemble (GOE), a random matrix $W$ with $W_{ij} = W_{ji} \sim \mathcal{N}(0, 1 + \delta_{ij})$ independently. Under $\mathbb{Q}_n$ one observes $Y = W$, while under $\mathbb{P}_n$ one further draws a unit vector $x \sim \mathrm{Unif}(\mathbb{S}^{n-1})$ and observes $Y = W + \beta xx^\top$ for some $\beta > 0$.

A natural idea to perform hypothesis testing in such a model is to consider only *spectral statistics*: to first compute the eigenvalues $\lambda_1(Y) \geq \cdots \geq \lambda_n(Y)$, and compute a test $t(Y) = t(\lambda(Y))$. Indeed, it is known [1, 4] that in the above setting a simple such strategy is *information-theoretically optimal*:

**Theorem 1.** *The following hold in the above model:*

(1) *If $\beta > 1$, then:*
   (a) *If $Y \sim \mathbb{Q}_n$, then $\lambda_1(Y)/\sqrt{n} \to 2$ almost surely.*
   (b) *If $Y \sim \mathbb{P}_n$, then $\lambda_1(Y)/\sqrt{n} \to \beta + \beta^{-1}$ almost surely.*
   *In particular, there is a $t(Y)$ that distinguishes $Y \sim \mathbb{Q}_n$ from $Y \sim \mathbb{P}_n$ with high probability by computing and thresholding $\lambda_1(Y)$.*
(2) *If $0 < \beta < 1$, then there is no sequence of tests $t : \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \{0, 1\}$ such that $\lim_{n\to\infty} \mathbb{Q}_n[t(Y) = 0] = \lim_{n\to\infty} \mathbb{P}_n[t(Y) = 1] = 1$.*

Our work [2] is motivated by the analogous problem over tensors. Let $p \geq 3$. There is a certain natural Gaussian random tensor $W \in \mathrm{Sym}^p(\mathbb{R}^n)$ analogous to the GOE, and the model problem of *tensor PCA* [5] asks when it is possible to distinguish $Y \sim \mathbb{Q}_n$ drawn as $Y = W$ from $Y \sim \mathbb{P}_n$ drawn as $Y = W + \beta x^{\otimes p}$. While this problem has many interesting properties, we do not focus on this application in this talk, and instead describe some of the algebraic tools developed to try to understand analogous algorithms to the above over tensors.

At first, the task seems ill-posed: there are various different notions of tensor eigenvalues, but they are typically intractable to compute and are not generically associated to any analog of the spectral decomposition. We propose a different analogy by which the notion of "spectral algorithm" may be understood for tensors. In the matrix case, the orthogonal group $\mathcal{O}(n)$ acts on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ by $Q \cdot Y = QYQ^\top$. The symmetric functions of the eigenvalues $f(\lambda_1(Y), \ldots, \lambda_n(Y))$ are then precisely the functions $f(Y)$ that are invariant under this action, having $f(Q \cdot Y) = f(Y)$

for all $Q \in \mathcal{O}(n)$. Polynomial such $f$ are generated by the special polynomials

$$m_k(Y) = \mathrm{Tr}(Y^k) = \sum_{i=1}^{n} \lambda_i(Y)^k.$$

There is an analogous action of $\mathcal{O}(n)$ on symmetric $p$-tensors, where $Q$ acts on a tensor $Y$ by acting linearly on each "axis" of the tensor simultaneously. We propose that the natural spectral algorithms over tensors are invariant functions under this action. A generalization of the above fact about invariant polynomials is known (due to Weyl in the 1940's) in the tensor case: the tensor invariants are generated by the polynomials, for $p$-regular graphs $G$,

$$m_G(Y) = \sum_{i:E(G)\to[n]} \prod_{v\in V(G)} Y_{i(\partial v)},$$

where $\partial v$ is the set of edges incident to a vertex $v$. The matrix polynomials $m_k$ coincide with taking $G = C_k$ the $k$-cycle. Because of the graphical interpretation, in the general tensor case, these polynomials are often called *tensor networks*.

We thus arrive at the natural question: how well can tensor networks solve statistical problems like tensor PCA? (It turns out that our above descriptions of matrix and tensor PCA makes the distributions $\mathbb{Q}_n$ and $\mathbb{P}_n$ themselves invariant under the orthogonal action, in which case an optimal hypothesis testing algorithm may be taken to be an invariant function without loss of generality.) In particular, we follow a recently popular paradigm in theoretical computer science [3] of measuring whether low-degree polynomials (a proxy for efficient algorithms) can solve such a problem. Specifically, we consider whether a low-degree polynomial $f(Y)$ can *separate* the distributions $\mathbb{Q}_n$ and $\mathbb{P}_n$ by having

$$\mathbb{E}_{Y\sim\mathbb{P}_n}[f(Y)] - \mathbb{E}_{Y\sim\mathbb{Q}_n}[f(Y)] \gg \max\left\{\mathrm{Var}_{Y\sim\mathbb{P}_n}[f(Y)], \mathrm{Var}_{Y\sim\mathbb{Q}_n}[f(Y)]\right\}.$$

As this is an "$L^2$ notion" of separation, to understand it, it is useful to seek out an approximately orthogonal basis for the invariant polynomials under the inner product $\langle f, g\rangle = \mathbb{E}_{Y\sim\mathbb{Q}_n} f(Y)g(Y)$. One of our main results obtains such a basis, and shows that it has a surprising further property.

**Theorem 2** (Informal)**.** *For $D = D(n)$ not too large, there exist polynomials $\kappa_G(Y)$ indexed by $p$-regular graphs $G$ with $|V(G)| \le D$ (variants of the $m_G(Y)$ above) that satisfy the following:*

1. *The $\kappa_G(Y)$ for $|V(G)| \le D$ form a basis for the invariant polynomials of a $p$-tensor $Y$ of degree at most $D$.*
2. *This basis is approximately orthogonal: the matrix $(\langle \kappa_G, \kappa_H\rangle)_{G,H}$ is well-conditioned.*
3. *If $G$ is connected, then for $Q$ a uniformly random orthogonal matrix,*

$$\mathbb{E}_Q \kappa_G(A + Q\cdot B) = \kappa_G(A) + \kappa_G(B).$$

*A similar but slightly more complicated identity holds for general $G$.*

The surprising part of this result is that the third property is satisfied along with the first two, which merely state the properties of an approximately orthogonal

basis. This third property is both intrinsically interesting and useful for understanding problems like tensor PCA, since a random $Y \sim \mathbb{P}_n$ in tensor PCA has precisely the form $Y = A + Q \cdot B$.

Because of this third property, we call the $\kappa_G$ the *finite free cumulants* of a tensor. This is by analogy with the theory of free probability. In classical probability, the cumulants are statistics of a probability measure that are additive under summation of independent random variables. In free probability, there is a notion of *free independence* which, roughly speaking, asks that a matrix $B$ has its frame of eigenvectors rotated randomly relative to that of a matrix $A$. There is a corresponding notion of *free cumulant* that behaves like the classical cumulant but with respect to free independence over growing sequences of random matrices. More recently, a notion of *finite free cumulant* was developed that again behaves similarly, but is just a polynomial of a fixed finite-dimensional matrix. Finally, our notion generalizes this last one to tensors.

Based on this surprising result, we propose an intriguing direction for future research. We ask: in general, what do approximately orthogonal bases for invariant polynomials under various group actions look like? Can such general bases be chosen to also satisfy the additivity property above? And, what implications do these objects have in algorithmic statistics beyond the special case of tensor PCA?

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## Totally nonnegative matrices, chain enumeration and zeros of polynomials


Petter Brändén

(joint work with Leonardo Saud Maia Leite)


In this talk we provide a new view on, and new tools for, problems in algebraic combinatorics related to face numbers of complexes and zeros of polynomials. The key theorem relates totally nonnegative matrices to chain enumeration in partially ordered sets, and $f$-vectors of simplicial complexes and posets. It is used to develop a general theory for chain enumeration in posets and zeros of chain polynomials. The results obtained extend and unify results of the speaker, Brenti, Welker and Athanasiadis. In the process we define a notion of $h$-vectors for a large class of posets which generalize the notions of $h$-vectors associated to simplicial and cubical

complexes. We also use the methods developed to answer an open problem posed by Forgács and Tran on the real-rootedness of polynomials arising from certain bivariate rational functions. The talk is based on the preprint [1].

Recall that a matrix is totally nonnegative (TN) if all of its minors are nonnegative. Let $R = (r_{n,k})_{n,k=0}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix whose diagonal entries are all equal to one. Associate to $R$ a sequence $\{p_n(t)\}_{n=0}^{N}$ of polynomials defined recursively by $p_0(t) = 1$, and

$$p_n(t) = t \sum_{k=0}^{n-1} r_{n,k} \cdot p_k(t), \quad 0 < n \leq N. \tag{1}$$

We call these polynomials the *chain polynomials* associated to $R$. Our main theorem is

**Theorem 1.** *If $R$ is totally nonnegative, then all zeros of $p_n(t)$ are real and located in the interval $[-1, 0]$.*

**Example 2.** The matrix $R = \left(\binom{n}{k}\right)_{n,k=0}^{\infty}$ is TN. The chain polynomials associated to $R$ are

$$p_n(t) = \sum_{k=0}^{n} k! S(n,k) t^k,$$

where $S(n,k)$ is a Stirling number of the second kind. These polynomials are chain polynomials for intervals in boolean algebras.

We prove that in many cases the polynomials $p_n(t)$ are chain polynomials of posets, and use Theorem 1 as a tool to prove real-rootedness of chain polynomials of posets. The results obtained extend and unify results of the speaker, Brenti, Welker and Athanasiadis.

Recall that a sequence $\{a_i\}_{i=0}^{\infty}$ of real numbers is a *Pólya frequency sequence* if the Toeplitz matrix $(a_{i-j})_{i,j=0}^{\infty}$ is TN, where $a_k = 0$ if $k < 0$. Pólya frequency sequences were characterized by Aissen, Schoenberg, Whitney and Edrei as follows.

**Theorem 3.** *A sequence $\{a_i\}_{i=0}^{\infty}$ of real numbers is a Pólya frequency sequence if and only if its generating function is of the form*

$$\sum_{n=0}^{\infty} a_n x^n = C x^N e^{\gamma x} \prod_{i=1}^{\infty} \frac{1 + \alpha_i x}{1 - \beta_i x}, \tag{2}$$

*where $C, \gamma, \alpha_i, \beta_i$ are nonnegative real numbers, $N \in \mathbb{N}$, and $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) < \infty$.*

The special case of Theorem 1 when $R$ is a Toeplitz matrix translates as follows.

**Theorem 4.** *Let $f(x)$ be as in (2), and consider the formal power series*

$$\frac{1}{1 - t(f(x) - f(0))} = \sum_{n=0}^{\infty} r_n(t) x^n \in \mathbb{R}[t][[x]]. \tag{3}$$

*Then $r_n(t)$ is a real-rooted polynomial for each $n \in \mathbb{N}$.*

Theorem 3 solves an open problem of Forgács and Tran, namely the case of Theorem 3 when $f(x) = x^r / \prod_{i=1}^{m}(1 - \beta_i x)$, where $\beta_i \geq 0$ for each $i$.

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## Symmetric nonnegative functions and the Vandermonde map

Sebastian Debus

(joint work with Jose Acevedo, Grigoriy Blekherman, Cordian Riener)

The image of the Vandermonde map $(p_1, p_2, \ldots, p_d) : \mathbb{R}^n \to \mathbb{R}^d$, where $p_i(x) = \sum_{j=1}^n x_j^i$ denotes the $i$-th power sum polynomial, shows fascinating geometric properties. Let $\Pi_{n,d}$ denote the projection of the image of $\Delta_{n-1} = \{x \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\}$ onto its last $d-1$ coordinates. We call $\Pi_{n,d}$ a *Vandermonde cell*. In [2], we prove that for $n \geq d$ the set $\Pi_{n,d}$ has the combinatorial structure of the cyclic polytope $C(n, d-1)$ which is the convex hull of $n$ points on the real moment curve $(t, t^2, \ldots, t^{d-1})$. This was previously experimentally observed in [7].

Moreover, the Vandermonde cells $\Pi_{n,d} \subset \Pi_{n+1,d}$ form an increasing nested sequence in $\mathbb{R}^{d-1}$ and we prove that the Vandermonde cell at infinity $\Pi_d = \mathrm{cl} \bigcup_{n \in \mathbb{N}} \Pi_{n,d}$ has the combinatorial structure of an infinite cyclic polytope, i.e., of a convex body with infinitely many facets which are characterized by Gale's evenness condition. The methods used to prove the theorem rely on a description of the preimage of the boundary of $\Pi_{n,d}$ via multiplicity vectors of elements in $\Delta_{n-1}$ and are an adaptation of the work of Kostov [5] who studied the geometry of $(p_1, p_2, \ldots, p_d)(\mathbb{S}^{n-1})$.

For a real symmetric matrix $A$, all its eigenvalues are real. The trace of $A^i$ is given by evaluating $p_i$ in the eigenvalues of $A$. Thus, we have a natural relationship between inequalities in symmetric functions in several groups of variables and inequalities in traces of real symmetric matrices of all sizes. We prove that the problem of deciding nonnegativity of such inequalities is undecidable. This result is in sharp contrast to the case of bounded matrix sizes where the problem is decidable. It remains an open problem whether the problem is decidable in the setting of a single group of variables or, respectively, in the case of a single matrix.

Naturally, the study of the image of the Vandermonde map relates to the study of nonnegative symmetric functions. Normalized symmetric function inequalities have been previously investigated in [1, 4]. In [3], we investigate the relationship between the cones of homogeneous symmetric functions that are sums of squares and those that are nonnegative in all numbers of variables. Building on the study of Kostov [6] of the image of $(p_1, p_2, p_3, p_4)$ at infinity, we find that the symmetric function

$$4p_1^4 - 5p_2p_1^2 - \frac{139}{20}p_3p_1 + 4p_2^2 + 4p_4$$

is nonnegative in all numbers of variables and already not a sum of squares in the smallest possible case of 4 variables. In any larger degree, the cones are also distinct. This can be seen by the fact that the cones of sums of squares are semialgebraic while the cones of nonnegative symmetric functions are not semialgebraic

sets. We further investigate the differences between the cones in the even symmetric setting using tropicalization. Moreover, we find that the *superdominance order* on partitions encodes all inequalities of the form

$$p_{\lambda_1} \cdots p_{\lambda_l} \geq p_{\mu_1} \cdots p_{\mu_m}$$

valid on the nonnegative orthant independently of the numbers of variables.

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## Hard Graphs for Colour Refinement and Beyond


Sandra Kiefer

(joint work with Brendan D. McKay, Devini de Mel)


Colour Refinement is a combinatorial procedure that iteratively computes a coloring of its input graph, with the purpose of detecting (a)symmetries in the graph. The central complexity parameter of the algorithm is its number of iterations until termination, i.e., the number of colourings that it computes until the induced partition does not become finer anymore. The trivial upper bound on this number is $n - 1$, where $n$ is the number of vertices in the graph.

In my talk, I presented the construction and compact representation that Brendan D. McKay and I came up with to show that there are infinitely many long-refinement graphs, that is, infinitely many graphs with $n - 1$ iterations until termination. I also reported on recent progress with Devini de Mel concerning the uniqueness of the construction: with a single exception, the presented families are the only long-refinement graphs with minimal degrees.Via reverse-engineering partitions, we extend the classification to maximum degree 4.

I also spoke about generalizations of the algorithm and bounds on their expressive power concerning graph distinguishability. I leave as an open problem the systematic construction of long-refinement graphs (or non-existence proofs) for such generalizations.

## Graph Homomorphisms and Polynomials

Grigoriy Blekherman

Given two simple graphs $G, H$ a homomorphism $\varphi : G \to H$ is a map from the vertex set $V(G)$ of $G$ to the vertex set $V(H)$ of $H$ such that $\varphi$ preserves adjacency of vertices. The *homomorphism density* $t(H, G)$ is defined by

$$t(H,G) = \frac{\#\text{homomorphisms}\, G \to H}{\text{total \# of maps}\, V(G) \to V(H)}.$$

One can ask for complexity of deciding whether a polynomial expression in $t(G_i, H)$ is nonnegative for all target graphs $H$, and it was shown by Hatami and Norin that this problem is undecidable [3]. We now describe an alternative proof of this result (also giving more general information) using the geometry of polynomial maps.

For a graph $H$ let $A_H$ denote the adjacency matrix of $H$. For any $n \times n$ matrix $A$ recall that $\operatorname{tr}(A^d) = \lambda_1^d + \cdots + \lambda_n^d$, where $\lambda_i$ are the eigenvalues of $A$. Let $p_d$ to denote the $d$-th power sum polynomial: $p_d(x) = x_1^d + \cdots + x_n^d$. It is well known that for an even cycle $C_{2k}$ we have

$$t(C_{2k}, H) = \frac{\operatorname{tr} A_H^{2k}}{|V(H)|^{2k}} = \frac{p_{2k}(\lambda_1, \ldots, \lambda_n)}{|V(H)|^{2k}},$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $A_H$.

Therefore we have

$$(1) \quad \frac{t(C^6, H)}{t(C_2, H)^3} = \frac{\operatorname{tr} A_H^6}{(\operatorname{tr} A_H^2)^3} = \frac{p_6(\lambda_1, \ldots, \lambda_n)}{p_2^3(\lambda_1, \ldots, \lambda_n)}, \text{ and } \frac{t(C^4, H)}{t(C_2, H)^2} = \frac{p_4(\lambda_1, \ldots, \lambda_n)}{p_2^4(\lambda_1, \ldots, \lambda_n)}.$$

Define the *d-th Vandermonde map* $\nu_{n,d}$ by sending a point in $\mathbb{R}^n$ to its image under the first $d$ power sums:

$$\nu_{n,d}(x) = (p_1(x), \ldots, p_d(x)).$$

Let $\Delta_{n-1}$ be the probability simplex in $\mathbb{R}^n$: $\Delta_{n-1}$ consists of all vectors with nonnegative coordinates with the sum of coordinates equal to 1. The image $\nu_{n,d}(\Delta_{n-1})$ is called the *the* $(n, d)$*-Vandermonde cell* and is denoted by $\Pi_{n,d}$. Observe that the first coordinate of $\Pi_{n,d}$ is identically 1, and so we may project it out, and see $\Pi_{n,d}$ as the subset of $\mathbb{R}^{d-1}$, which is the image of $\Delta_{n-1}$ under $(p_2, \ldots, p_d)$.

The image of $\Pi_{n,3}$ looks as follows:

The Vandermonde cell $\Pi_{n,3}$ has $n$ special points of the form $(1/k, 1/k^2)$ for $k = 1, \ldots, n$. The lower boundary of the image consists of concave curves joining the special points $(1/k, 1/k^2)$. As $n$ goes to infinity we get infinitely many isolated points on the curve $y = x^2$. The upper boundary of the limit cell $\Pi_{\infty,3}$ is given by the curve $y^2 = x^3$ for $0 \leq x \leq 1$.

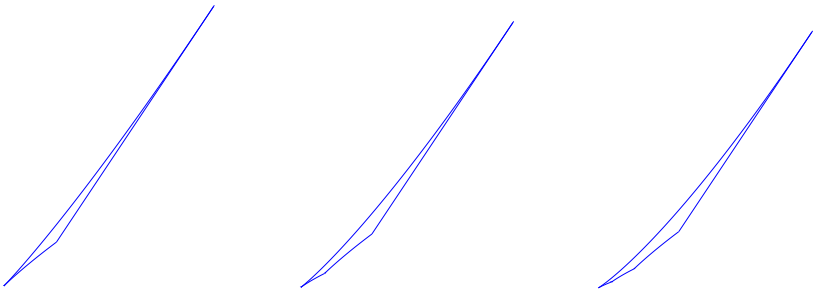

FIGURE 1. The sets $\Pi_{n,3}$ for $n = 3, 4, 5$

If we do not restrict to adjacency matrices of graphs, then the image of the map

$$(\lambda_1, \ldots, \lambda_n) \mapsto \left( \frac{p_6(\lambda_1, \ldots, \lambda_n)}{p_2^3(\lambda_1, \ldots, \lambda_n)}, \frac{p_4(\lambda_1, \ldots, \lambda_n)}{p_2^4(\lambda_1, \ldots, \lambda_n)} \right)$$

is simply the the Vandermonde cell $\Pi_{\infty,3}$. However, one can show that as we go over all graphs $H$ with any number of vertices, we actually get the full Vandermonde cell $\Pi_{\infty,3}$ in 1 [1].

For undecidability we need a product of independent copies of $\Pi_{\infty,3}$. We can obtain these independent copies by considering *necklace graphs*. The even cycles are $2k$ copies of complete graph $K_2$ glued together in a circular fashion. One can similarly take $2k$ copies of the triangle $K_3$ and glue them together in circular fashion, producing a necklace graph on $4k$ vertices. One can show that using necklace graphs for larger complete graphs $K_m$ we can obtain independent copies of $\Pi_{\infty,3}$ and use the same undecidability reduction [2][1]. Moreover, this reduction can be used to show undecidability for weighted homomorphism density inequalities, even with negative weights.

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## Combinatorial Shorest Paths on Combinatorial Polytopes

YOSHIO OKAMOTO

This talk is a biased overview of recent results and open problems on combinatorial shortest paths on combinatorial polytopes. In this talk, combinatorial polytopes mean bounded polyhedra that arise from combinatorics and discrete mathematics. We are mainly interested in two questions on the graphs (i.e., the 1-skeleta) of

such polytopes. The first question is to determine the graph distance between two vertices in a polytope. The second question is to determine the graph diameter of a polytope, where the graph diameter is defined as the maximum graph distance over all pairs of vertices.

In a perfect matching polytope of an undirected graph $G$, the vertices correspond to perfect matchings of $G$ and two vertices are joined by an edge if and only if the symmetric difference of the corresponding perfect matchings is composed of a single cycle. Then, the distance between two perfect matchings of $G$ is defined as the graph distance on the graph of the perfect matching polytope of $G$.

Aichholzer et al. [1] and Ito et al. [3] independently proved that the distance between two perfect matchings is NP-hard to compute. Moreover, Cardinal and Steiner [2] proved that the constant-factor approximation is also NP-hard. For the diameter computation, Nöbel and Steiner [5] recently prove that it is also NP-hard.

In a $d$-dimensional associahedron, the vertices correspond to binary tree with $d+2$ leaves, and two vertices are joined by an edge if and only if one of the corresponding binary trees can be obtained by a single rotation from the other. This is equivalently stated as the vertices correspond to triangulations of a convex polygon with $d+3$ vertices, and two vertices are joined by an edge if and only if one of the corresponding triangulations can be obtained by a single diagonal flip from the other.

The diameter of associahedra is well studied. Sleator et al. [7] proved that the diameter is at most $2d-4$ when $d \geq 10$, and this is tight for infinitely many $d$'s. Pournin [6] later proved that the bound of $2d-4$ is indeed tight for all $d \geq 10$. On the other hand, the distance computation in associahedra is a long-standing open problem.

A graph associahedron is a generalization of an associahedron. In a graph associahedron of an undirected graph $G$, the vertices correspond to elimination trees of $G$, and two vertices are joined by an edge if and only if one of the corresponding elimination trees can be obtained by a single flip from the other. When $G$ is a path, a graph associhedron of $G$ is identical to an associahedron.

Ito et al. [4] proved that the distance computation is NP-hard for graph associahedra. On the other hand, while several authors found the exact diameters for restricted classes of graphs and upper/lower bounds for diameters, less is known for the diameter computation of graph associahedra. In particular, we do not know whether the diameter computation can be done in polynomial time or it is NP-hard.

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## Identifying Generic Points from Non-generic Measurements

Shin-ichi Tanigawa

A $d$-dimensional framework is a pair $(G,p)$ of a graph $G=(V,E)$ and a vertex configuration $p:V\to\mathbb{R}^d$ in $d$-space. The framework $(G,p)$ is called *globally rigid* if, for any framework $(G,q)$ with the same underlying graph $G$,

$$\|p(i)-p(j)\| = \|q(i)-q(j)\| \qquad \forall ij\in E \tag{1}$$

implies that $q$ is the image of $p$ by a Euclidean isometry, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. A classical observation by Saxe shows that checking the global rigidity of a given framework is NP-hard even in one-dimensional space, implying that a concise characterization of globally rigid frameworks is unlikely to exist. However, Gortler-Healy-Thurston [4] proved the following surprising result.

**Theorem 1.** *For a graph $G=(V,E)$ and an integer $d$, let*

$$\mathcal{GR}(G,d) := \{p\in(\mathbb{R}^d)^V : (G,p) \text{ is globally rigid}\}.$$

*Then, $\mathcal{GR}(G,d)$ or its complement has measure zero. Moreover, there is a randomized polynomial time algorithm to decide which of the two cases holds for a given $G$.*

The theorem implies that, as long as we are concerned with generic $p$, global rigidity is determined by the underlying graph $G$.

In this project, we are interested in generalizing this result to a broader class of polynomial systems. In [3], we introduce the following generalized rigidity model: Let $g:(\mathbb{R}^d)^k\to\mathbb{R}$ be a homogeneous polynomial map. For a $k$-uniform hypergraph $G$ and a vertex configuration $p$, the corresponding framework $(G,p)$ is *globally $g$-rigid* if for any framework $(G,q)$ with the same underlying hypergraph $G$,

$$f(p(v_1),\dots,p(v_k)) = f(q(v_1),\dots,q(v_k)) \qquad \forall(v_1,\dots,v_k)\in E \tag{2}$$

implies that $q$ is the image of $p$ by an action of a stabilizer of $g$.

We relate the concept of global $g$-rigidity with the identifiability of secant varieties. In particular, we show how to extend the rank condition of Connelly–Gortler–Healy–Thurston [2, 4] for the Euclidean case to the general setting of global $g$-rigidity, using the tangential weak defectiveness of secant varieties due to Chiantini and Ottaviani [1]. As applications, we consider the following two special cases:

(i) When $k = 2$ and $g(x_1, x_2) = \|x_1 - x_2\|_q^q$ for $x_1, x_2 \in \mathbb{R}^d$, where $\|\cdot\|_q$ denotes the $\ell_q$-norm: In this case (when $q$ is even), the corresponding global $g$-rigidity problem has been well studied as the graph rigidity in $\ell_q$-normed space. In [6], we showed that the $\ell_q$-analogue of Theorem 1 is true if $d = 2$. It remains an open problem to extend this result to higher dimensions.

(ii) When $g(x_1, x_2, \ldots, x_k) = (x_1 \odot x_2 \odot \cdots \odot x_k) \cdot \mathbf{1}$, where $\odot$ denotes the Hadamard product, $\cdot$ denotes the dot product, and $\mathbf{1}$ denotes the all-one vector. In this case, the corresponding $g$-rigidity characterizes the unique identifiability of low-rank tensors from partial entries.

We show that global $g$-rigidity is not a generic property, i.e., there is a graph $G$ for which $(G, p_1)$ is globally $g$-rigid and $(G, p_2)$ is not globally $g$-rigid for some generic $p_1, p_2$. On the other hand, global $g$-rigidity remains a generic property if $G$ is an Erdös–Rényi random hypergraph. Concretely, in [5], we proved the following: Let $G_{n,t}$ be the random $k$-uniform hypergraph on $n$ vertices obtained by inserting each hyperedge of size $k$ with probability $t$, and let $p$ be a generic $d$-dimensional vertex configuration. If $t > \frac{(1+\varepsilon)\log n}{n^{k-1}}$, then a.a.s. $(G_{n,t}, p)$ is globally $g$-rigid. If $t < \frac{(1+\varepsilon)\log n}{n^{k-1}}$, then a.a.s. $(G_{n,t}, p)$ is not globally $g$-rigid. We believe that this phenomenon holds for other polynomial maps $g$ as well, which remains an open problem.

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## Gradient descent for unbounded convex functions on Hadamard manifolds and its applications to scaling problems


Keiya Sakabe

(joint work with Hiroshi Hirai)


We study the gradient descent for a lower-unbounded (geodesically-)convex function $f$ on a Hadamard manifold $M$, especially in $\inf_{x \in M} \|\nabla f(x)\| > 0$ case. A key mathematical tool for the analysis is $M^\infty$: the *boundary of* $M$, which is obtained by the visual compactification $M \sqcup M^\infty$ of $M$, and the resulting topology on $M \sqcup M^\infty$ is called the *cone topology*. The *recession function* [2] (also known as

*asymptotic slope* [4]) $f^\infty : M^\infty \to \mathbb{R} \sqcup \{\infty\}$ represents the asymptotic increasing ratio of $f$ at the infinity of $M$ in each direction. We show a duality theorem that the infimum of the gradient norm equals the supremum of minus the recession function, and that these infimum and supremum are attained at the limit of the gradient descent:

**Theorem 1** ([3, Theorem III.7])**.** *Let $M$ be a Hadamard manifold and $f : M \to \mathbb{R}$ be an $L$-smooth convex function. If $\inf_{x\in M} \|\nabla f(x)\| > 0$, then for the gradient descent sequence $(x_i)_{i\in\mathbb{N}}$ defined by $x_{i+1} := \exp_{x_i}(-\nabla f(x_i)/L)$, the following equation holds:*

$$\lim_{i\to\infty} \|\nabla f(x_i)\| = \inf_{x\in M} \|\nabla f(x)\| = \sup_{\xi\in M^\infty} -f^\infty(\xi) = -f^\infty\left(\lim_{i\to\infty} x_i\right),$$

*where the last limit exists in the sense of the cone topology.*

One motive for studying unbounded convex functions on Hadamard manifolds is *non-commutative optimization* [1], which minimizes the logarithm of the vector norm over an orbit of a rational representation of a reductive algebraic group acting on $V \cong \mathbb{C}^N$. Its objective function $f$ is unbounded and $\inf_{x\in M} \|\nabla f(x)\| > 0$ iff the closure of the orbit has the origin $0 \in V$; this case is called *unstable* in terms of geometric invariant theory. The (un)stability determination via the gradient descent is proposed in [1], and our result provides a theoretical foundation for the behavior of the gradient descent in the unstable case. Especially, $\lim_{i\to\infty} x_i \in M^\infty$ corresponds to the *maximum destabilizing 1-parameter subgroup* in this case, and one open problem is to construct an algorithm to compute it with some complexity bounds.

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## Algorithmic Aspects of Semistability of Quiver Representations

Tasuku Soma
(joint work with Yuni Iwamasa, Taihei Oki)

Let $V$ be a representation of an acyclic quiver $Q = (Q_0, Q_1)$ with dimension vector $\alpha$. Consider a natural action of $\mathrm{GL}(Q, \alpha) = \prod_{i \in Q_0} \mathrm{GL}(\alpha(i))$ given by $(g \cdot V)(a) = g_{ha} V(a) g_{ta}^{-1}$ for each arc $a$. For a weight $\sigma \in \mathbb{Z}^{Q_0}$, a representation $V$ is said to be $\sigma$-semistable if the orbit closure of $(V, 1)$ does not contain the origin, where the action on $\mathbb{C}$ is given by the multiplicative character $\chi_\sigma(g) = \prod_{i \in Q_0} \det(g_i)^{\sigma(i)}$. King [8] characterized the $\sigma$-semistability of $V$ with a finite linear system: $V$ is $\sigma$-semistable if and only if $\sigma(\underline{\dim} V) = 0$ and $\sigma(\underline{\dim} W) \leq 0$ for any subrepresentation $W \leq V$, where $\sigma(\underline{\dim} V) = \sum_{i \in Q_0} \sigma(i) \dim V(i)$. The $\sigma$-semistability of quiver representations captures operator scaling [7, 6, 4, 1], membership problem in the Brascamp–Lieb polytope [5], which have attracted much interest in theoretical computer science and combinatorial optimization in the last decade.

In this talk, we present efficient algorithms for several fundamental computational problems on the semistability of quiver representations: deciding the semistability and $\sigma$-semistability, finding the maximizers of King's criterion, and computing the Harder–Narasimhan filtration. We also investigate a class of polyhedral cones defined by the linear system in King's criterion, which we refer to as King cones. For rank-one representations, we demonstrate that these King cones can be encoded by submodular flow polytopes, enabling us to decide the $\sigma$-semistability in strongly polynomial time. Our approach employs submodularity in quiver representations, which may be of independent interest.

More broadly, quiver representations form a rich subclass of geometric invariant theory (GIT). [2] devised deterministic algorithms for the general setting of GIT, but they are not always polynomial-time algorithms. Our results show that the semistability of quiver representations forms a tractable class of general GIT problems due to the underlying submodular structure.

The full version of the paper is available at `https://arxiv.org/abs/2407.06493`.

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## Ehrhart theory on periodic graphs

Yusuke Nakamura

An *n-dimensional periodic graph* $(\Gamma, L)$ is a pair of a directed graph $\Gamma$ (that may have loops and multiple edges) and a free abelian group $L$ of rank $n$ such that $L$ freely acts on $\Gamma$ and its quotient graph $\Gamma/L$ is finite. For a vertex $x_0$ of $\Gamma$, the *growth sequence* $(s_{\Gamma,x_0,i})_{i\geq 0}$ (resp. *cumulative growth sequence* $(b_{\Gamma,x_0,i})_{i\geq 0}$) is defined as the number of vertices of $\Gamma$ whose distance from $x_0$ is $i$ (resp. at most $i$). Periodic graphs naturally appear in crystallography, as periodic tilings in combinatorics, and as Cayley graphs of virtually abelian groups in geometric group theory.

In [3], Grosse-Kunstleve, Brunner and Sloane conjectured that the growth sequences of periodic graphs are of quasi-polynomial type, i.e., there exist an integer $M$ and a quasi-polynomial $f_s : \mathbb{Z} \to \mathbb{Z}$ such that $s_{\Gamma,x_0,i} = f_s(i)$ holds for any $i \geq M$. In [7], the author, Sakamoto, Mase, and Nakagawa prove that this conjecture is true for any periodic graphs.

**Theorem 1** ([7])**.** *The growth sequences of periodic graphs are of quasi-polynomial type.*

Although it was proved to be of quasi-polynomial type, determining the explicit formulae of growth sequences is still difficult. In [5], Takuya Inoue and the author give an algorithm to compute the explicit formulae of growth sequences. Theoretically, our algorithm can be applied to arbitrary periodic graphs of any dimension. In the two-dimensional case, it is relatively simple to implement our algorithm into a computer program, and using the computer program, the growth sequences of some new examples can be computed. In higher dimensions, although it is possible to implement the algorithm in a computer program, it is sometimes not practical. Finding a better algorithm is one of the future work.

In [1], Benson proves that the growth sequences of virtually abelian groups are quasi-polynomial type. Theorem 1 can be seen as a generalization of Benson's theorem, since the Cayley graphs of virtually abelian groups are periodic graphs. In the proof of Theorem 1, we use the commutative monoid theory, and the proof is essentially different from Benson's original proof, where he uses his theory of "polyhedral sets". In [6], the author proves the rationality of the multivariate relative growth series for algebraic sets of virtually abelian groups. This result is a variant of Benson's theorem, and it had been conjectured by Evetts and Levine

[2]. We believe that the studies on periodic graphs still have the potential for further applications to similar studies of virtually abelian groups.

In Ehrhart theory, for a rational polytope $Q \subset \mathbb{R}^N$, it is proved that the function

$$h_Q : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}; \quad i \mapsto \#\big(iQ \cap \mathbb{Z}^N\big)$$

is a quasi-polynomial on $i \geq 0$. In [4], for a rational polytope $Q \subset \mathbb{R}^N$ with $0 \in Q$, we construct a periodic graph $(\Gamma_Q, \mathbb{Z}^N)$ such that its cumulative growth sequence $b_{\Gamma_Q,0,i}$ coincides with $h_Q(i)$. Therefore, we can say that the study of the growth sequences of periodic graphs essentially contains the Ehrhart theory of rational polytopes $Q$ satisfying $0 \in Q$. Since the cumulative growth sequence $(b_{\Gamma_Q,0,i})_i$ is a quasi-polynomial on $i \geq 0$, the following natural question arises.

**Question 1.** Find a reasonable class $\mathcal{P}$ of pairs $(\Gamma, x_0)$ that consist of a periodic graph $\Gamma$ and one of its vertices $x_0$ such that

- $\mathcal{P}$ contains the class $\{(\Gamma_Q, 0) \mid Q \text{ is a rational polytope with } 0 \in Q\}$, and
- for any $(\Gamma, x_0) \in \mathcal{P}$, the sequence $(b_{\Gamma,x_0,i})_i$ is an honest quasi-polynomial (i.e. a quasi-polynomial on $i \geq 0$).

Note that, unlike the case of Ehrhart theory, the growth sequences of periodic graphs in general are not necessarily quasi-polynomials, and they may have finite exceptional terms. However, it has been observed that for some highly symmetric periodic graphs, they are often honest quasi-polynomials. The intention of this question is to describe the properties of such good periodic graphs.

Another important topic of Ehrhart theory is the reciprocity law. When we think of the function $h_Q$ as a quasi-polynomial and substitute a negative value for it, we have $h_Q(-i) = (-1)^{\dim Q} \#\big(i \cdot \mathrm{relint}(Q) \cap \mathbb{Z}^N\big)$ for $i > 0$. In the growth sequences of some $n$-dimensional periodic graphs, it has been observed that they sometimes satisfy the equations

$$(\diamondsuit) \qquad f_b(-i) = (-1)^n f_b(i-1), \qquad f_s(-i) = (-1)^{n+1} f_s(i),$$

where $f_b$ and $f_s$ are the corresponding quasi-polynomials to the sequences $(b_{\Gamma,x_0,i})_i$ and $(s_{\Gamma,x_0,i})_i$. These equations in $(\diamondsuit)$ are consistent with the reciprocity laws of reflexive polytopes. Thus, the following natural question arises.

**Question 2.** Find a reasonable class $\mathcal{P}'$ of pairs $(\Gamma, x_0)$ such that

- $\mathcal{P}'$ contains the class $\{(\Gamma_Q, 0) \mid Q \text{ is a reflexive polytope}\}$, and
- for any $(\Gamma, x_0) \in \mathcal{P}'$, its growth sequence satisfies the reciprocity laws $(\diamondsuit)$.

In [4], Takuya Inoue and the author give answers to Questions 1 and 2.

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## Discrete Integrable Systems:A Matrix Decomposition Approach

Satoshi Tsujimoto

We will provide an overview of discrete integrable systems and their analysis based on matrix decomposition. In particular, the aim is to clarity the structure of discrete integrable systems through matrix decomposition using spectral-preserving deformations.

Discrete integrable systems include specific examples such as the KdV lattice and Lotka-Volterra lattice, which are nonlinear yet possess exact solutions with a large degree of freedom[1, 2]. These systems are closely related to the theory of orthogonal polynomials using Jacobi matrices, and they can be analyzed through spectral transformations and matrix decomposition[3, 4].

Furthermore, by performing ultra-discretization, one can obtain the box-ball system (a cellular automaton) as a limiting case of integrable systems, allowing for a simplified description of soliton dynamics[5, 6]. Such discretization is also beneficial for numerical computations, where subtraction-free computation schemes contribute to improved numerical stability.

This study utilizes matrix decomposition to analyze integrable systems and develops numerical computation methods that preserve their structure. In particular, d-qd and dlv algorithms for singular value computation have been shown to be effective in the analysis of integrable systems.

As highlighted above, research on discrete integrable systems enables the development of new analytical methods through orthogonal polynomials, matrix decomposition, and ultra-discretization, further contributing to applications in numerical computation and mathematical modeling[7, 8, 9].

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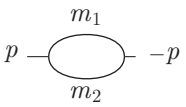
## A hypergeometric view on Landau singularity

Saiei-Jaeyeong Matsubara-Heo

Let $Z$ be a smooth affine complex variety, $f : (\mathbb{C}^*)^n \times Z \to \mathbb{C}$ be a regular function, and let $s, \nu_1, \ldots, \nu_n \in \mathbb{C}$ be generic parameters. In perturbative quantum field theory (pQFT), one often considers an integral of the following kind:

$$I_\Gamma(z; s, \nu) := \int_\Gamma f(x; z)^{-s} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \cdots x_n}, \tag{1}$$

known as the Lee-Pomeransky representation of a Feynman integral ([6]). Here, we do not specify the integration contour $\Gamma$. In pQFT, one considers a graph called a Feynman diagram and associate to it a graph polynomial $f$ called Lee-Pomeransky polynomial ([9]). A simple example of $f$ in physics could be the following one: consider the so-called *bubble graph*




where $m_1, m_2$ are scalar parameters and $p \in \mathbb{R}^D$ is a $D$-dimensional vector Let $M := p^2$ be the Lorenzian self inner product of $p$. The associated Lee-Pomeransky integral is $f(x_1, x_2; m_1, m_2, M) := (1 - m_1 x_1 - m_2 x_2)(x_1^{-1} + x_2^{-1}) + M$. Thus, we take $Z$ as $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[m_1, m_2, M]$ in this example.

**Problem.** Describe the locus where the integral (1) develops singularity.

Such a locus is called *Landau variety* since it dates back to the work [5]. In the case of bubble graph example, the following process captures a large part of singularity: consider a subvariety of $(\mathbb{C}^*)^2 \times \mathbb{C}^3$ defined by

$$D := \{(x, z) \in (\mathbb{C}^*)^2 \times Z \mid f(x, z) = 0,\ d_x f(x, z) = 0\}. \tag{2}$$

Let $\pi : (\mathbb{C}^*)^2 \times Z \to Z$ be a projection. The Zariski closure $\nabla_f$ of the projection $\pi(D)$ is an irreducible hypersurface defined by an identity $\{\lambda(m_1, m_2, M) = 0\}$, where $\lambda(m_1, m_2, M) = m_1^2 + m_2^2 + M^2 - 2m_1 m_2 - 2m_1 M - 2m_2 M$. In fact, the locus where the integral (1) diverges is precisely given by $\{m_1 m_2 M \lambda(m_1, m_2, M) = 0\}$.

For a general $f : (\mathbb{C}^*)^n \times Z \to \mathbb{C}$, one can still define $D$ by (2) and discuss $\nabla_f$. If $f$ is suitably non-degenerate with respect to its Newton polytope, $\nabla_f$ is known

as the *principal A-determinant* ([3]). However in physics, most of the examples are degenerate and quite often, the equality $\nabla_f = Z$ holds, which does not give rise to any component of Landau variety.

In order to treat such degenerate examples, an analytic point of view offers a new perspective. Namely, we consider a system of differential equations that annihilates (1) and extract Landau variety out of it. Let $D_Z$ be the ring of linear differential operators on $Z$. We define a left ideal of $D_Z$ by $I := \bigcap_{\Gamma} \mathrm{Ann}_{D_Z} I_\Gamma(z; s, \nu)$ and call it a *hypergeometric system*. The definition of the hypergeometric system looks abstract at a glance. However, if we enlarge the ring, it turns out that we obtain a complete description. To do so, we regard $s, \nu_i$ as symbols, rather than numbers. Let $\nu_0 := s$ and let $R^{\mathrm{pol}}$ denote the non-commutative ring generated by elements $\nu_j, \sigma_j^{\pm 1}$ $(j = 0, \dots, n)$ with relations

$$[\sigma_i, \nu_j] = \delta_{ij}\sigma_j, \quad [\sigma_i^{-1}, \nu_j] = -\delta_{ij}\sigma_j, \quad [\sigma_i, \sigma_j] = 0, \quad [\nu_i, \nu_j] = 0, \tag{3}$$

where $\delta_{ij}$ denotes Kronecker's delta. Each $\sigma_{\nu_i}$ corresponds to the forward shift $\nu_i \mapsto \nu_i + 1$. Let $K$ be the field of rational functions in $\nu_i$'s. We set $R := K \otimes_{\mathbb{C}[\nu_0, \dots, \nu_n]} R^{\mathrm{pol}}$, which also has a natural ring structure induced by that of $R^{\mathrm{pol}}$. Finally, we set $DD := D_Z \otimes_{\mathbb{C}} R$, which is again a ring. Now we set $J := \bigcap_{\Gamma} \mathrm{Ann}_{DD} I_\Gamma(z; s, \nu)$. A special case of the following theorem when $Z$ is a point is obtained in [8]. For simplicity, let us assume $Z = \mathbb{C}^N = \mathrm{Spec}\, \mathbb{C}[z_1, \dots, z_N]$ below.

**Theorem 1.** *$J$ is generated by the following operators:*

$$1 - \sigma_s f(\sigma_\nu; z), \nu_i - s\sigma_s\sigma_{\nu_i} \frac{\partial f}{\partial x_i}(\sigma_\nu; z), \partial_{z_j} + s\sigma_s \frac{\partial f}{\partial z_j}(\sigma_\nu; z). \tag{4}$$

Therefore, $I$ is obtained as a *non-commutative elimination* $I = J \cap D_Z$. A crucial point is that these generators give rise to a *commutative elimination* picture of Landau variety. The connection to commutative algebra tells us properties of Landau variety. For a fixed $z \in Z$, let us consider *likelihood equations*:

$$\left\{ (y, x) \in \mathbb{C}^{n+1} \mid 1 - yf(x; z) = \nu_i - syx_i \frac{\partial f}{\partial x_i}(x; z) = 0 \, (i = 1, \dots, n) \right\}. \tag{5}$$

Note that the defining equations of (5) is a commutative version of the first $n+1$ operators of (4). Namely, the former is obtained from the latter by by a replacement $\sigma_s \to y, \sigma_{\nu_i} \to x_i$. The cardinality of the set (5) is independent of a choice of generic $s, \nu_i$ and it coincides with the Euler characteristic of the vanishing locus $V(f(\cdot, z)) \subset (\mathbb{C}^*)^n$ of $f$ for a fixed $z$ ([4]). A locally constant function $\chi : Z \ni z \mapsto |\chi(V(f(\cdot, z)))| \in \mathbb{Z}_{\geq 0}$ defines the *Euler discriminant (locus)*. $\nabla_\chi(Z) := \{z \in Z \mid \chi_z < \max \chi_z\}$. Euler discriminant is a generalization of the principal $A$-determinant ([1]) and re-discovered in a study of Landau variety as a candidate for Landau variety ([2]). The following result proves that the Euler discriminant is indeed the Landau variety ([7]):

**Theorem 2.** *$\nabla_\chi(Z)$ is purely one-codimensional in $Z$ unless it is empty and coincides with Landau variety.*

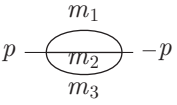
To state the actual elimination formula of Landau variety, we realize that we should work on a cotangent bundle $T^*Z = \mathbb{C}^N \times \mathbb{C}^N$ whose ring of regular functions is $\mathcal{O}_{T^*Z} = \mathbb{C}[z_1, \ldots, z_N, \xi_1, \ldots, \xi_N]$, where $\xi_i$ is the cotangent coordinates. Consider an ideal $J_0 \subset \mathbb{C}[y, x] \otimes_{\mathbb{C}} \mathcal{O}_{T^*Z}$ generated by

$$1 - yf(x; z), \nu_i - syx_i \frac{\partial f}{\partial x_i}(x), \xi_j + sy \frac{\partial f}{\partial z_j}(x; z).$$

The generators of $J_0$ are obtained from (4) by a replacement $\sigma_s \to y, \sigma_{\nu_i} \to x_i, \partial_j \to \xi_j$. Now we set $I_0 := J_0 \cap \mathcal{O}_{T^*Z}$. It is a technical part that one has to take the associated graded $\mathrm{gr}(I_0) \subset \mathcal{O}_{T^*Z}$ of $I_0$ with respect to the grading so that $z_i$ has degree 0 and $\xi_i$ has degree one. We set $\mathbb{C}^N \times \{0\} =: T^*_Z Z \subset T^*Z$. Let $\varpi : T^*Z = \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N = Z$ be the projection to the first $N$ coordinates.

**Theorem 3.** *The set $\varpi\left(V(\mathrm{gr}(I_0)) \setminus T^*_Z Z\right)$ is a closed algebraic subvariety of $Z$ and coincides with Landau variety.*

The following diagram is called a sunrise/sunset diagram.




The landau variety is given by $\{(m_1, m_2, m_3, M) \in \mathbb{C}^4 \mid m_1 m_2 m_3 M \Delta(m_1, m_2, m_3, M) = 0\}$, where $\Delta(m_1, m_2, m_3, M)$ is given by

$$\begin{aligned}(6)\quad & \Delta(m_1, m_2, m_3, M) \\ =& m_1^4 - 4\,m_1^3 m_2 + 6\,m_1^2 m_2^2 - 4\,m_1 m_2^3 + m_2^4 - 4\,m_1^3 m_3 + 4\,m_1^2 m_2 m_3 \\ & + 4\,m_1 m_2^2 m_3 - 4\,m_2^3 m_3 + 6\,m_1^2 m_3^2 + 4\,m_1 m_2 m_3^2 + 6\,m_2^2 m_3^2 - 4\,m_1 m_3^3 \\ & - 4\,m_2 m_3^3 + m_3^4 - 4\,m_1^3 M + 4\,m_1^2 m_2 M + 4\,m_1 m_2^2 M - 4\,m_2^3 M + 4\,m_1^2 m_3 M \\ & - 40\,m_1 m_2 m_3 M + 4\,m_2^2 m_3 M + 4\,m_1 m_3^2 M + 4\,m_2 m_3^2 M - 4\,m_3^3 M + 6\,m_1^2 M^2 \\ & + 4\,m_1 m_2 M^2 + 6\,m_2^2 M^2 + 4\,m_1 m_3 M^2 + 4\,m_2 m_3 M^2 + 6\,m_3^2 M^2 - 4\,m_1 M^3 \\ & - 4\,m_2 M^3 - 4\,m_3 M^3 + M^4.\end{aligned}$$

It is still open how to encode the combinatorics of a Feynman diagram to Landau variety. Does it contributes to the irreducible decomposition of the Landau variety as suggested by the method [2]?

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## Holonomic methods in optimization, statistics, and machine learning

Nobuki Takayama

**Definition 1.** (I.N.Bernstein=J.Bernstein, 1972) Let $I$ be a left ideal of the Weyl algebra $D_n = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$. Put $F_k = \bigoplus_{\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n\le k} \mathbb{C}x^\alpha\partial^\beta$ and $H(k) = \dim_{\mathbb{C}} \frac{F_k}{F_k\cap I}$. If the degree of the polynomial $H(k)$ is $n$, we call $I$ a holonomic ideal.

Any holonomic ideal $I$ contains ordinary differential operators for all directions of the form

$$\sum_{k=0}^{r_i} s_{ik}(x)\partial_i^k, \quad s_{ik}(x) \in \mathbb{C}[x],\ i = 1, \dots, n. \tag{1}$$

The opposite is not necessarily true, but we have the following theorem.

**Theorem 2.** (see, e.g., [7, Appendix]) *Let $R_n = \mathbb{C}(x)\langle\partial_1, \dots, \partial_n\rangle$ be the ring of differential operators with rational function coefficients and $J \neq R_n, \langle 0\rangle$ a left ideal generated by operators of the form (1) of $R_n$. Then, $J \cap D_n$ is a holonomic ideal of $D_n$.*

A classical function that is annihilated by a holonomic ideal is called a holonomic function. It follows from the Theorem above that a classical function annihilated by ordinary differential operators for all directions is a holonomic function. A distribution that is annihilated by a holonomic ideal is called a holonomic distribution.

**Theorem 3.** [1]

(1) *The degree of the Hilbert polynomial of a left ideal $I \subsetneq D_n$ of $D_n$ is equal to $n$ or more than $n$.*
(2) *If $I$ is a holonomic ideal in $D_n$, then $(I + x_n D_n) \cap D_{n-1}$ (restriction ideal) and $(I + \partial_n D_n) \cap D_{n-1}$ (integration ideal) are holonomic ideals in $D_{n-1}$.*

We have the following fact from the theorem; if a rapidly descreasing function $f$ is annihilated by a holonomic ideal $I \subset D_n$, then the $n-1$ variables $x'$ function $g(x') := \int_{-\infty}^{\infty} f(x)dx_n$ is annihilated by the integration ideal. Algorithms to construct integration ideals have been studied by several people (see, e.g., [3], [2], [5] and references therein). A lot of normalizing constants in statistics are expressed as definite integrals with parameters. Thus, we can apply Theorem 3 and

these algorithms to evaluate normalizing constants by solving numerically ordinary differential equations in integration ideals. This method is called the holonomic gradient method (HGM) (see, e.g., a list of papers on HGM for optimization, statistics, physics, and machine learning at [6]).

Now, we discuss on a statitical model with parameters. Let $m$ be the number of parameters $\theta$ and $n$ be the size of the data vector $x$. We consider a holonomic function $u(\theta, x)$ on $\mathbb{R}^{m+n}$ or a product $u(\theta, x)$ of a holonomic function and a delta function whose support is a real algebrac manifold or a Heaviside function whose support is a semi-algebraic set. We suppose that the holonomic function is smooth and non-negative on the support of the associated delta function or Heaviside function. We also suppose that the normalizing constant $\int_{\mathbb{R}^n} u(\theta, x)dx$ is annihilated by the integration ideal. We call such distribution $u(\theta, x)$ an *unnormalized holonomic distribution*.

**Theorem 4.** *For an unnormalized holonomic distribution $u(\theta, x)$, the maximal likelihood estimation with respect to $u(\theta, x)$ and data in the $x$ space can be described by a dynamical system.*

**Example 5.** The unnormalized Von-Mises distribution on $S^1 \ni x$ (which is the angle that represents a coordinate of $S^1 \subset \mathbb{R}^2$) is

$$u(\theta, x) = \exp(\theta_1 \cos x + \theta_2 \sin x) = \exp(\theta_1 x_1 + \theta_2 x_2)\delta(x_1^2 + x_2^2 - 1)$$

where $(x_1, x_2)$ is the coordinate of $\mathbb{R}^2$. The normalizing constant is

$$Z(\theta) = \int_0^{2\pi} \exp(\theta_1 \cos x + \theta_2 \sin x)dx.$$

Put $F = (Z, \partial_1 Z)^T$, $\partial_i = \partial/\partial\theta_i$. It satisfies

$$\begin{aligned} \frac{\partial F}{\partial \theta_1} &= \begin{pmatrix} 0 & 1 \\ \frac{\theta_1^2}{\theta_1^2+\theta_2^2} & \frac{\theta_2^2-\theta_1^2}{\theta_1(\theta_1^2+\theta_2^2)} \end{pmatrix} F =: P_1 F \\ \frac{\partial F}{\partial \theta_2} &= \begin{pmatrix} 0 & \theta_2/\theta_1 \\ \frac{\theta_1\theta_2}{\theta_1^2+\theta_2^2} & \frac{-2\theta_2}{\theta_1^2+\theta_2^2} \end{pmatrix} F =: P_2 F \end{aligned}$$

Let $X_i$, $i = 1, \ldots, N$ be observed data. The maximal likelihood estimation (MLE) is to find $\theta$ which maximizes the log likelihood

$$f = \log \ell(\theta; X) = \log \prod_{i=1}^{N} \frac{u(\theta, X_i)}{Z(\theta)}.$$

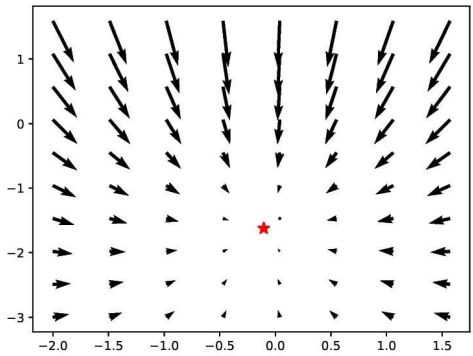


FIGURE 1. Vector field on $(\theta_1, \theta_2)$ space

It follows from [4, Prop 3.7] and our holonomic approach that the MLE is a point such that $\dot{\theta_1} = 0, \dot{\theta_2} = 0$ of the following system of ordinary differential equations

$$\begin{aligned}
\dot{\theta_1} &= = \sum_{i=1}^{N} \cos(X_i) - N\frac{(P_1F)_1}{F_1} \\
\dot{\theta_2} &= = \sum_{i=1}^{N} \sin(X_i) - N\frac{(P_2F)_1}{F_1} \\
\dot{F_i} &= \left(\sum_{i=1}^{N} \cos(X_i) - N\frac{(P_1F)_1}{F_1}\right)(P_1F)_i + \left(\sum_{i=1}^{N} \sin(X_i) - N\frac{(P_2F)_1}{F_1}\right)(P_2F)_i
\end{aligned}$$

where the subscript $_i$ denotes the $i$-th element of a vector. Figure 1 is a vector field for a wind direction dataset.

Open questions are

(1) When the target function $f$ is convex, develop an algorithm to find an orbit that converges to a point standing for the optimal point of the target function.
(2) Describe a global structure of the dynamical system of Theorem 4.

Note that the first question is not trivial when there is no closed form of the normalizing constant, because numerical errors of evaluating the normalizing constant may produce a wrong orbit.

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## Alternating Projections in Low Dimensions

Yoshiyuki Sekiguchi
(joint work with Hiroyuki Ochiai, Hayato Waki)

Alternating projections is an algorithm for finding a point in the intersection of two sets by projection a point onto each set alternatingly. We consider an affine subspace of the space of symmetric matrices and the cone $\mathbb{S}^n_+$ of positive semidefinite matrices. It is known that if the intersection of these two sets is transversal, the convergence rate is linear. If the intersection is non-transversal, then the rate is sublinear. [2] provided an upper bound on the rate using the singularity degree of an affine subspace and $\mathbb{S}^n_+$.

The singularity degree is an integer defined as the smallest number of steps in the facial reduction process, required to construct an exposing vector for the minimal face of $\mathbb{S}^n_+$ that contains the intersection of an affine subspace and $\mathbb{S}^n_+$. In addition to the convergence rate of alternating projections, singularity degree can be used to express the exponent for a separation inequality for an affine subspace and $\mathbb{S}^n_+$. A similar separation inequality holds for complex algebraic sets, where the exponent is determined by the degrees of the algebraic sets [1, 3]. This suggests a connection between singularity degree and algebraic geometry. Motivated by this observation, I propose the following problem.

**Problem.** Is there a quantity in algebraic geometry that can provide a dimension-free exponent for the separation inequality for an affine subspace and $\mathbb{S}^n_+$?

The exponent of the separation inequality is used to express the convergence rate of the alternating projections. Beyond the intrinsic importance of convergence analysis, we examine the tightness of the convergence rate given by singularity degree to obtain some insights on singularity degree. To this end, we analyze alternating projections in low dimensions, and aim to develop a deeper understanding of the algorithm. The following results are extracted from our papers [4, 5].

First, we derive a formula that expresses the new parameters in terms of the old parameters and eigenvalues of a parametric matrix that corresponds to the sequence generated by alternating projections. Applying this eigenvalue formula, we show that if alternating projections are applied to a 1-dimensional affine subspace and $\mathbb{S}^n_+$ whose intersection is a singleton, the convergence rate of alternating projections is bounded above by $O(k^{-1/2})$, independently of the singularity degree of the intersection. Furthermore, we use the eigenvalue formula to construct an example of 2-dimensional affine subspace where the singularity degree is 1, but the convergence rate remains linear.

Under additional assumptions, we derive a formula for alternating projections that allows us to construct the slowest curve for alternating projections. Applying the formula to a specific affine subspace and we explicitly construct the corresponding slowest curve. To extend this result to a general affine subspace, we parameterize a family of 3-planes whose intersection with $\mathbb{S}^3_+$ is a singleton. This parametrization provides geometric insights into the structure of such planes and their singularity degrees. Then we obtain a rational formula for the slowest curve for a subfamily of these planes. This rational formula also determines the exact image of one step of alternating projections up to degree 7. Moreover, if the intersection of a 3-plane and $\mathbb{S}^3_+$ is a singleton with singularity degree 2, we show that the convergence rate predicted by singularity degree is tight using this rational formula.

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## Closing nonzero duality gap of singular SDP by perturbation


Takashi Tsuchiya

(joint work with Bruno F. Lourenço, Masakazu Muramatsu, Takayuki Okuno)


Consider the standard form dual pair of semidefinite programs:

$$(\mathbf{P}) \qquad \min_X \ C \bullet X \ \text{ s.t. } A^i \bullet X = b_i, \ i = 1, \ldots, m, \ X \succeq 0$$

$$(\mathbf{D}) \qquad \max_{y,S} \ b^T y \quad \text{s.t. } C - \sum_{i=1}^m A^i y_i = S, \ S \succeq 0,$$

where $C$, $A^i, i = 1, \ldots, m$, $X$, and $S$ are real symmetric $n \times n$ matrices and $y \in \Re^m$. We denote the optimal values of $\mathbf{P}$ and $\mathbf{D}$ by $v(\mathbf{P})$ and $v(\mathbf{D})$, respectively.

Suppose that $\mathbf{P}$ and $\mathbf{D}$ have a nonzero finite duality gap. This implies that $\mathbf{P}$ and $\mathbf{D}$ are weakly feasible. It is hard to solve such SDP problems because the exsiting algorithms for SDP only work under the condition that both $\mathbf{P}$ and $\mathbf{D}$ are strongly feasible, i.e., Slater's condition is satisfied.

A simple practical idea to solve $\mathbf{P}$ and $\mathbf{D}$ under such a circumstance is to perturb the problems so that both perturbed primal and dual are strongly feasible. The perturbed pair are primal-dual strongly feasible and therefore share common a optimal value. Since the original unperturbed problems have different optimal

values, it is interesting to analyze the behavior of the common optimal value of the perturbed problems when perturbation is reduced to zero.

To implement this idea, we consider the following perturbed problems

**Regularized Primal-Dual Standard Form SDP (RPD-SDP)**

$$(1)\ \mathbf{P}(\varepsilon,\eta): \quad \min_{X} (C+\varepsilon I)\bullet X \ \text{ s.t. } A^i\bullet X = b_i+\eta A^i\bullet I,\ i=1,\ldots,m,\ X\succeq 0$$

and

$$(2)\quad \mathbf{D}(\varepsilon,\eta): \quad \max_{y,S} \sum_{i=1}^{m}(b_i+\eta A^i\bullet I)y_i \ \text{ s.t. } C-\sum_{i=1}^{m}A^i y_i+\varepsilon I = S,\quad S\succeq 0,$$

where $I$ denotes the $n\times n$ identity matrix. We call the pair (1) and (2) the *Regularized Primal-Dual Standard Form SDP or RPD-SDP for short.* $\mathbf{P}(\varepsilon,\eta)$ and $\mathbf{D}(\varepsilon,\eta)$ reduce to $\mathbf{P}$ and $\mathbf{D}$ when $\varepsilon$ and $\eta$ are set to zero. RPD-SDP is obtained by relaxing the semidefinite constraints $X\succeq 0$ of $\mathbf{P}$ and $S\succeq 0$ of $\mathbf{D}$ to $X\succeq -\eta I$ and $S\succeq -\varepsilon I$, respectively, and by redefining $X:=X+\eta I$ and $S:=S+\varepsilon I$.

Under the assumption that $\mathbf{P}$ and $\mathbf{D}$ are weakly feasible, the perturbed problems admit interior feasible solutions for any $\varepsilon>0$ and $\eta>0$, so, in this sense, the lack of interior solutions of $\mathbf{P}$ and $\mathbf{D}$ is fixed. However, this is only useful if something can be said about how the optimal values of $\mathbf{D}(\epsilon,\eta)$ and $\mathbf{P}(\epsilon,\eta)$ relate to the optimal values of the original $\mathbf{P}$ and $\mathbf{D}$.

For $\varepsilon>0$ and $\eta>0$, both (1) and (2) are strongly feasible, so they have optimal solutions and a common optimal value, which we denote by $v(\varepsilon,\eta)$. $v$ is referred to as *pd-regularized optimal value function.* We note that $v(\varepsilon,0)$ and $v(0,\eta)$ are also well-defined for any $\varepsilon>0$ and $\eta>0$, since $\mathbf{P}(\varepsilon,0)$ and $\mathbf{D}(0,\eta)$ have interior feasible solution for any $\varepsilon>0$ and $\eta>0$ and there is no duality gap in these cases according to the standard duality theory for convex programming.

But $v(0,0)$ is different since it is not well-defined when there exists a finite nonzero duality gap, and ironically, *the value of* $v(0,0)$ is what we really wish to compute.

We have recently analyzed the behavior of the pd-regularized optimal value function $v(\varepsilon,\eta)$ in the neighbourhood of $(\varepsilon,\eta)=(0,0)$ and demonstrated that $v(\varepsilon,\eta)$ have a directional limit when approaching to $(0,0)$. Let us define the directional limit

$$(3)\qquad v_a(\theta) := \lim_{t\downarrow 0} v(t\cos\theta, t\sin\theta)$$

as a function of the direction of approach. The function $v_a$ is referred to as *limiting pd-regularized optimal value function.* Then, the following theorem holds.

**Theorem 1.** [1] *If* $\mathbf{P}$ *and* $\mathbf{D}$ *are feasible, the limiting pd-regularized optimal value function* $v_a(\theta)$ *has the following properties.*

(1) $v_a(0)=v(\mathbf{P})$, $v_a(\pi/2)=v(\mathbf{D})$.
(2) $v_a(\theta)$ *is monotone decreasing in* $[0,\pi/2]$ *and is continuous on* $(0,\pi/2)$.

Theorem 1 shows that $v_a(\theta)$ is a monotone decreasing function whose value is between $v(\mathbf{P})$ and $v(\mathbf{D})$ and it is continuous in the open interval $(0,\pi/2)$ but may

be discontinuous at $\theta = 0$ and/or $\theta = \pi/2$. The following theorem in [2] shows that continuity holds at the both $\theta = 0$ and $\theta = \pi/2$ when the singularity degree of the both problems **P** and **D** is one.

**Theorem 2.** [2] *Suppose that* **P** *and* **D** *have a finite nonzero duality gap. If the singularity degree of both* **P** *and* **D** *is one, then* $v_a(\theta)$ *is continuous at* $\theta = 0$ *and* $\theta = \pi/2$. *Furthermore,* $v_a$ *is a monotonically decreasing continuous bijective function from* $[0, \pi/2]$ *to* $[v(\mathbf{D}), v(\mathbf{P})]$.

In [2] an instance is given for which $v_a$ is discontinuous at $\theta = \pi/2$. The singularity degree of the instance is 2.

In the talk, we introduced these results together with further open problems including extensions to general convex programs and possible variable transformations to recover continuity of $v_a$ for SDPs with higher singularity degree.

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## Open Problems

### Unsatisfiable Boolean formulas and the Komlós conjecture
Contributed by Dmitriy (Tim) Kunisky

The startlingly simple-looking Komlós conjecture is one of the main open problems in the field of discrepancy theory and has remained open for at least 30 years. The conjecture states that there exists an absolute constant $K > 0$ such that, for any $u_1, \dots, u_m \in \mathbb{R}^n$, there exist $x_1, \dots, x_m \in \{\pm 1\}$ having $\| \sum_{i=1}^m x_i u_i \|_\infty \leq K$. The conjecture proposes a dimension-free bound, while the best known bound due to Banaszczyk (1998) scales, in the case $m = n$, as $\sqrt{\log n}$. In the recent work [1], the author showed the lower bound $K \geq 1 + \sqrt{2}$, which remains the best known. This comes from a construction of "bad" vectors $u_1, \dots, u_m$ based on unsatisfiable Boolean formulas: roughly, each vector $u_i$ corresponds to a variable, each index $j \in \{1, \dots, n\}$ to a clause, and the unsatisfiability of the formula translates to stating that, for all $x_1, \dots, x_m \in \{\pm 1\}$, at least one entry of $\sum_{i=1}^m x_i u_i$ is large. The unsatisfiable formulas that would give the best lower bound on $K$ are those that are, in a certain quantitative sense, as close as possible to "regular," with each clause having roughly the same number of variables and each variable occurring in roughly the same number of clauses. Can such constructions of discrepancy instances ever exceed the lower bound $1 + \sqrt{2}$? More generally, how can we understand the extremal combinatorics of unsatisfiable Boolean formulas?

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## Zeros of Low-Degree, Low-Influence Polynomials on the Boolean Hypercube

Contributed by Per Austrin

For a degree-$d$ multilinear polynomial

$$f(x) = \sum_{\substack{S\subseteq[n]\\|S|\le d}} f_S \prod_{i\in S} x_i$$

in $n$ variables $x_1, \ldots, x_n$ with real-valued coefficients $\{f_S\}$, the (relative) influence of $x_i$ on $f$ is the squared $\ell_2$ mass of coefficients $f_S$ of terms containing $x_i$, relative to the total squared $\ell_2$ mass, i.e.,

$$\mathrm{RelInf}_i(f) = \sum_{\substack{S\subseteq[n]\\i\in S}} f_S^2 \Big/ \sum_{S\subseteq[n]} f_S^2 \,.$$

**Problem.** Let $f$ and $g$ be two degree-$d$ multilinear polynomials such that

$$\max_i(\mathrm{RelInf}_i(f), \mathrm{RelInf}_i(g)) \le \delta.$$

How small does $\delta$ need to be in order to guarantee that there is an $x \in \{-1,1\}^n$ where both $f(x) \neq 0$ and $g(x) \neq 0$?

The condition of bounded influences can be viewed as a kind of smoothness condition on $f$ and $g$, saying that no individual variable plays a large role. The condition of $f(x) \neq 0$ and $g(x) \neq 0$ is equivalent with saying that the union of the roots of $f$ and $g$ does not cover the entire Boolean hypercube.

It is known that it is sufficient to take $\delta \le 2^{-d}/d$, and that there are examples which shows that we need to take $\delta \le 1/d^2$. The main question is whether $\delta$ can be taken to be some polynomial in $d$. Further reading, including these known bounds, can be found in [1].

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## Number of connected components of the set of real zeros of two multi-affine polynomials?

Contributed by Saugata Basu (joint work with Daniel Perrucci)

It is a classical result originally due to Oleĭnik and I. G. Petrovskiĭ [1] that the number of connected components of a real algebraic set in $\mathbb{R}^n$ defined by any finite set of real polynomials of degrees at most $d$ is bounded by $d(2d-1)^{n-1}$. This

bound grows exponentially with $n$ and this exponential dependence is unavoidable. In [2] it is proved that if $P \in \mathbb{R}[X_1, \ldots, X_n]$ is a multi-affine polynomial (i.e. $\deg_{X_i}(P) \leq 1$ for $1 \leq i \leq n$), then the number of connected components of the real zeros of $P$ is bounded by $2^{d-1}$. This bound is independent of $n$ and is tight. It is also shown in the same paper that there exist for each $n, k, k \leq n$, three multi-affine polynomials of degrees bounded by 4, such that the number of connected components of their common real zeros is equal to $\binom{n}{k}$. Thus, it is not possible to obtain a bound on the number of connected components of the real zeros of a system of three real multi-affine polynomials in terms of their degrees only, independent of $n$.

**Problem.** Does there exist a bound on the number of connected components of the real zeros of a system of *two* multi-affine polynomials in terms of a bound on their degrees, independent of $n$ ?

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## Sum-of-Squares certificates for non-negativity of Quadratics
## Contributed by Grigoriy Blekherman

Let $A_1$ and $A_2$ be two $n \times n$ symmetric real matrices, and let $Q_1$ and $Q_2$ be the corresponding quadratic forms $Q_i(x) = x^t A_i x$. It is well known that $Q_1$ is nonnegative on the variety defined $V$ defined by vanishing of $Q_2$ if and only if there exists a scalar $\lambda$ such that the matrix $A_1 + \lambda A_2$ is positive semidefinite. This was rediscovered several times, and sometimes goes under the name of $S$-lemma. What happens if we instead ask for certificate of nonnegativity of $Q_1$ on the variety defined by *two* quadratics $Q_2$ and $Q_3$? If there exist two scalar $\lambda_2, \lambda_3$ such that $A_1 + \lambda_2 A_2 + \lambda_3 A_3$ is positive semidefinite, then this will certify nonnegativity of $Q_1$, but existence of $\lambda_i$ is no longer guaranteed. Instead we can look for *higher degree certificates:* find a sum of squares homogeneous polynomial $P$ and homogeneous polynomials $R_2$ and $R_3$ all of degree $2d$ such that $PQ_1 + R_2Q_2 + R_3Q_3$ is a sum of squares. The previously considered certificate had degree $2d$ equal to 0, and now we ask in what degree $2d$ is this certificate guaranteed to exist? Is the degree $2d$ of the certificate *independent of the number of variables* $n$?

## Separations of affine sets and psd matrices
## Contributed by Yoshiyuki Sekiguchi

**Problem.** Is there any quantity in algebraic geometry that can be used to express a dimension-free exponent for separation of an affine set and the set of positive semidefinite matrices? Is there a symbolic or a randomized algorithm to give such an exponent?

A separation inequality is an inequality in the following form:

$$\operatorname{dist}(z, X) + \operatorname{dist}(z, Y) \geq c \cdot \left( \frac{\operatorname{dist}(z, X \cap Y)}{1 + |z|^2} \right)^{\gamma}, \text{ for all } z \in K^n.$$

For complex algebraic sets $X, Y \subset \mathbb{C}^n$, Cygan [2] showed that this inequality holds with $\gamma = \deg X \cdot \deg Y$ for all $z \in C^n$. Let $n$ be the dimension of the ambient space, $d$ be the singularity degree of $E \cap \mathbb{S}^n_+$, $D$ be the maximum degree of the defining polynomials, and $\beta(s) = \binom{s}{\lfloor s/2 \rfloor}$. Variations of the separation inequality have been shown with the exponents in the table below.

| | |
|---|---|
| Complex Algebraic sets [2] | $\deg X \cdot \deg Y$ |
| Real Algebraic sets [5] | $D(6D-3)^{n-1}$ |
| Convex semialgebraic sets [1] | $\min\left\{ \frac{(2D-1)^n+1}{2}, \beta(n-1)D^n \right\}$ |
| Affine subsp. and $\mathbb{S}^n_+$ [6, 3] | $2^d$ |
| Polyhedra [4] | 1 |

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## Structure of Euler discriminants
## Contributed by Saiei-Jaeyeong Matsubara-Heo

*Discriminants* is a classical yet developing notion. We consider a family of complex varieties $\pi : X \to Z$. For a given $z \in Z$, the symbol $\chi_z$ denotes the absolute value of the topological Euler characteristic of the fiber of $\pi$ over $z$. The function $Z \ni z \mapsto \chi_z \in \mathbb{Z}$ takes a constant value at a generic point of $Z$. This value is denoted by $\chi^*$. The *Euler descriminant locus* $\nabla^{\pi}_{\chi}(Z)$ is the Zariski closure of the following set:

$$\{z \in Z \mid \chi_z < \chi^*\}.$$

Since the Euler characteristic of a finite set is its cardinality, this notion generalizes classical discriminant of a single polynomial. The definition of Euler discriminant is quite topological, but it turns out it admits an algebraic description as in [4]. It is also proved in [1] that Euler discriminant generalizes *principal A-determinant* of Gelfand-Kapranov-Zelevinsky [2].

Compared to principal $A$-determinant, the description of Euler discriminant is still cumbersome. In fact, principal $A$-determinant factorizes into a product

of *principal A-discriminants*, which admits an *elimination picture, i.e.,* one constructs an incident variety $I$ and a projection $p_I : I \to Z$ so that (the closure of) the image $p_I(I)$ is the principal $A$-determinant. **Can we find the same picture for Euler discriminant?** This is partially achieved in [4] for a family of very affine hypersurfaces with the aid of likelihood equation [3]. However, it relies on a certain compactification of the family and it is not intrinsic.

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