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Mini-Workshop: Alcoved Polytopes in Physics and Optimization

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Abstract. An alcoved polytope is a polytope whose facet normal are all in direction of type A roots. This fundamental class of polytopes has ample applications in for instance tropical geometry, statistics and algebra. The mini-workshop showcased various recent developments related to alcoved polytopes and their generalizations with a particular focus on their connections to physics and optimization.

Mathematics Subject Classification (2020): 52Bxx, 14Txx, 51Mxx, 81Uxx, 90Cxx.

Introduction by the Organizers

A polytope is alcoved if all its facet normals are all in the direction of (type A) roots. In other words, alcoved polytopes are given by inequalities of the form $x_i - x_j \leq a_{ij}$ where a_{ij} are real numbers. They appear in different fields under different names. For example, they are also known as polytropes which are tropical polytopes that are convex in the usual sense, and they are the Lipschitz polytopes of non-symmetric finite metric spaces. The aim of this mini-workshop was to connect experts working on different aspects of these polyhedra, including their applications in physics and optimization, and collaborate on open problems.

There were 16 participants, including the four organizers. The workshop schedule was structured to include dissemnation of new results as well as time for collaborative work. Generally we had two or three 45 minute talks in the morning, followed by an open problem session and working groups in the afternoons. The focus of the presented talks included these topics:

The structure of the type fan of alcoved polytopes: The *type fan* of alcoved polytopes is the space of alcoved polytopes organized by normal equivalence. This fan is connected to theoretical particle physics and the study of positroids. Further applications to physics were discussed in talks on the amplituhedron and positroidal subdivisions of hypersimplices.

Discrete Convex Analysis: Applications to optimization and economics were another cornerstone of this mini-workshop. Specifically, we would like to mention two talks on L-convex quotients and finding minimal representations of piecewise linear functions which is relevant for applications in machine learning.

Ehrhart theory: Another cluster of talks centered around lattice counting questions of alcoved polytopes such as order polytopes and the symmetric edge polytopes. Connecting with the first block of talks there were also two talks on recent developments concerning the Ehrhart theory of cosmological polytopes. These are lattice polytopes arising in physics that are closely related to symmetric edge polytopes.

Among the questions proposed during the problem sessions the following four were discussed the most:

- Analyze the structure of the type fan of alcoved polytopes further: Which positroid polytopes or order polytopes belong to the same cone of the type fan?
- Study the Ehrhart theory of a subclass of alcoved polytopes which are polar dual to symmetric edge polytopes. In particular, are their h^* vectors γ-positive?
- Alcoved polytopes could be described by flows on a directed graph in two ways: by support numbers or by facet volumes via the Minkowski existence-uniqueness theorem. What is the correspondence between these two presentations?
- How can we define quotients of L-convex sets and L-convex functions?

We formed four working groups that studied these questions during the mini-workshop. Each of these four groups decided to continue working on these questions remotely so we hope this will eventually lead to articles that were initiated at this mini-workshop.

Mini-Workshop: Alcoved Polytopes in Physics and Optimization

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Abstracts

When alcoved polytopes add

Leonid Monin
(joint work with Nick Early, Lukas Kühne)

A polytope in $\mathcal{H}_n = \{x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n$ is alcoved if all its facet normals are parallel to the roots $e_i - e_j$ for some $i \neq j \in [n]$. Equivalently, a polytope is alcoved if it is determined by the parameters $a_{i,j} \in \mathbb{R}$ for $1 \leq i, j, \leq n$ via the equation $x_1 + \cdots + x_n = 0$ and the inequalities

$$x_i - x_j \leq a_{i,j} \text{ for all } i, j \in [n], i \neq j. \tag{1}$$

Alcoved polytopes were introduced by Lam and Postnikov [4] and appeared in different fields under different names. They are known in the literature as *polytropes* as they are tropical polytopes which are convex in the usual sense [3]. Moreover, they are *Lipschitz polytopes* (for non-symmetric finite metric spaces) [2]. The class of alcoved polytopes includes order polytopes, hypersimplices, and the associahedron.

Unlike other families of polytopes, the class alcoved polytopes is not closed under Minkowski sums in general. This naturally raises the question when alcoved polytopes add.

Problem 1. *Let $P, Q \subseteq \mathcal{H}_n$ be alcoved polytopes. When is the Minkowski sum $P + Q$ alcoved?*

We call the alcoved polytopes P and Q compatible if their sum $P + Q$ is alcoved.

In this talk we present two results from [1] on the classification of compatible alcoved polytopes. First we show that the compatibility of alcoved polytopes can be checked on pairs:

Theorem 2. *Let $P_1, \ldots, P_k$ be alcoved polytopes in $\mathcal{H}_n$. Suppose P_i and P_j are pairwise compatible for all $i \neq j \in [n]$. Then the entire collection is compatible, i.e., $P_1 + \cdots + P_k$ is alcoved.*

Theorem 2 can be interpreted as a statement about the type fan of alcoved polytopes. It claims that the he combinatorial structure of the type fan is completely determined by its 2-dimensional cones.

As the second result, we give a characterization for the compatibility of alcoved simplices. Up to translation and scaling, every alcoved simplex in $\mathcal{H}_n$ is characterized by an *ordered set partition* of $[n]$.

Theorem 3. *Let S and T be two ordered set partitions of $[n]$ corresponding to the alcoved simplices Δ_S and Δ_T in $\mathcal{H}_n$. The simplices Δ_S and Δ_T are compatible if and only if the simplices corresponding to the restricted partitions $S|_I$ and $T|_I$ are compatible for all $I \subset [n]$ with $|I| \leq 6$.*

In the case when simplices Δ_S and Δ_T are full-dimensional, the pair of set partitions (S, T) defines a cyclic permutation $\pi_{S,T}$. In this case, Theorem 3 claims that two full-dimensional alcoved simplices are compatible Δ_S and Δ_T if and only if $\pi_{S,T}$ avoid three patterns, one of length four and two of length six:

$$1432, \quad 125634 \quad 145236.$$

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Adventures in Configuration Space

Nick Early

The positive moduli space $\mathcal{M}^+_{0,n}$ of n ordered points on $\mathbb{P}^1$ lies at the core of the Cachazo-He-Yuan (CHY) [1] scattering equations formalism for scattering amplitudes. The space $\mathcal{M}^+_{0,n}$ is an associahedron, and varying over all orderings leads to a tiling of $\mathcal{M}_{0,n}$ with associahedra. Each associahedron comes equipped with a canonical form, which has a logarithmic singularity on each face of $\mathcal{M}_{0,n}$; such canonical forms satisfy many interesting identities.

The CHY formula was extended by Cachazo, Early, Guevara and Mizera [2] to the moduli space $X^+(3, n)$ of polygons in $\mathbb{P}^2$. Tiling the full moduli space of points in $\mathbb{P}^2$ requires the theory of oriented matroids, which is known to be arbitrarily complicated.

On the other hand, the positive configuration space $X^+(k, n)$ and its tropicalization are relatively well-behaved; faces of $X^+(k, n)$ are in bijection with certain regular subdivisions of the hypersimplex $\Delta_{k,n}$ into positroid polytopes. In particular, such faces can be represented using matroidal blade arrangements [4] $\beta_{J_1}, \ldots, \beta_{J_d}$ on the vertices $e_{J_1}, \ldots, e_{J_d}$ of the hypersimplex $\Delta_{k,n}$.

Theorem 1. [4] *We have the following:*

- *The blade β_{J_1} is a tropical hypersurface. Set theoretically, it is, up to translation, the codimension one part of the normal fan to the alcoved simplex $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - 1$.*
- *The blade β_J induces a matroid subdivision of $\Delta_{k,n}$ into ℓ Schubert matroid polytopes, where ℓ is the number of cyclic intervals in J.*
- *The blade arrangement $\beta_{J_1}, \ldots, \beta_{J_d}$ induces a matroid subdivision of $\Delta_{k,n}$ if and only if $e_{J_a} - e_{J_b}$ alternates sign twice with respect to the cyclic order $(1, 2, \ldots, n)$.*

In recent joint work with Kühne and Monin [5], we considered the problem of a collection of blades labeled by arbitrary ordered set partitions, translated to a vertex e_J of $\Delta_{k,n}$.

Theorem 2 ([5])**.** *Suppose that $\mathbf{S}_1, \ldots, \mathbf{S}_d \in OSP(n)$ is a pairwise compatible collection of ordered set partitions. Then the blade arrangement $(\mathbf{S}_1)_v, \ldots (\mathbf{S}_d)_v$ induces a matroid subdivision of $\Delta_{k,n}$, and a cone in the Dressian $Dr(k,n)$.*

An interesting further question concerns tropical realizability.

Question 1. *When are the induced subdivisions realizable, i.e., when are they induced by a realizable tropical Plücker vector?*

Clearly, any single blade induces a regular matroid subdivision. But this is not the case in general.

For an example of a blade arrangement which induces a non-realizable tropical Plücker vector, recall that the Fano matroid polytope is a maximal face of a matroid subdivision that is induced by a non-realizable tropical Plücker vector in the Dressian $\mathrm{Dr}(3,7)$. It is induced by the arrangement of seven affine hyperplanes. In this case simply a collection of seven compatible two-splits of $\Delta_{3,7}$, induced by (the sum of) the seven tropical Plücker vectors

$$e^{123}, e^{145}, e^{167}, e^{246}, e^{257}, e^{347}, e^{356} \in \mathbb{R}^{\binom{7}{3}},$$

where e^{ijk} is the standard basis for $\mathbb{R}^{\binom{n}{3}}$.

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Ehrhart theory of alcoved polytopes and kinematic space

Elisabeth Bullock, Yuhan Jiang

(joint work with Nick Early)

We first describe a general method for computing the Ehrhart series of any alcoved polytope via a particular shelling order of its alcoves. In particular:

Theorem 1. *Fix an irreducible crystallographic root system $\Phi \subset V$, where* $\dim(V) = n$. *Let P be an alcoved polytope and let $\Gamma_P = (V, E)$ be the dual graph to the alcove triangulation of P. Pick some $v_0 \in V$ and orient the edges of Γ_P so that for all $\{u, w\} \in E$, $u \to w$ if and only if u appears before w in the breadth-first-search* *algorithm starting at* v_0. *There exists a weighting of the edges* E *and parameters* $\ell_1, \cdots, \ell_n$ *depending only on* Φ *such that the Ehrhart series of* P *is equal to*

$$Ehr(P, z) = \frac{\sum_{w \in V} z^{wt(w)}}{\prod_{i=0}^{n}(1 - z^{\ell_i})}$$

where $wt(w) = \sum_{u \to w} wt((u, w))$ *is the sum of the weights of the ingoing edges to* w.

A family of polytopes whose h^*-polynomial has received much attention is the hypersimplex $\Delta_{k,n}$. In [1], Nick Early conjectured that h^*_d is equal to the number of *decorated ordered set partitions of type* $\Delta_{(k,n)}$ with winding number d.

Definition 1 ([1])**.** *A* decorated ordered set partition $((S_1, \ldots, S_d), (r_1, \ldots, r_d))$ *of type* $\Delta_{(k,n)}$ *consists of an ordered set partition* $(S_1, \ldots, S_d)$ *of* $[n]$ *and a tuple of integers* $(r_1, \ldots, r_d)$ *such that* $\sum_{i=1}^{d} r_i = k$ *and* $1 \leq r_i \leq |S_i| - 1$. *We regard them up to cyclic rotation, so* $((S_1)_{r_1}, (S_2)_{r_2}, \ldots, (S_d)_{r_d})$ *is the same as* $((S_2)_{r_2}, \ldots, (S_d)_{r_d}, (S_1)_{r_1})$.

We place each S_i *on a circle in clockwise fashion then think of* r_i *as the clockwise distance between adjacent* S_i *and* S_{i+1}. *The* winding vector *of a decorated ordered set partition is an* n*-tuple of integers* $(l_1, \ldots, l_n)$ *such that* l_i *is the distance of the path starting from the block containing* i *to the block containing* $(i+1)$ *moving clockwise. If* i *and* $(i+1)$ *are in the same block then* $l_i = 0$. *If* $l_1 + \cdots + l_n = kw$, *then we define the* winding number *to be* w.

This conjecture was proved by [3] using enumerative methods. A natural question is whether there exists a nice bijection between alcoves in the hypersimplex and dOSPs, as well as a shelling order of the alcoves of the hypersimplex, so that the winding number of the dOSP in a given alcove is equal to number of adjacent alcoves occurring earlier in the shelling order. We present a conjectural answer to this question for the hypersimplex $\Delta_{2,n}$ using the breadth-first-search shelling order.

Decorated ordered set partitions also correspond with certain linear functions $X_{(\mathbf{S},\mathbf{r})}$ on the kinematic space $\mathcal{K}_{k,n}$, defined as follows:

$$\mathcal{K}_{k,n} = \left\{ (\mathfrak{s}_J) \in \mathbb{R}^{\binom{n}{k}} \ : \ \sum_{J : J \ni j} \mathfrak{s}_J = 0 \text{ for all } j \in [n] \right\}$$

It turns out that the set of $X_{(\mathbf{S},\mathbf{r})}$ where $(\mathbf{S}, \mathbf{r})$ has winding number one forms a basis of linear functions on $\mathcal{K}_{k,n}$ [2]. We present a "straightening" formula from our ongoing work with Early which can be used to expand a large class of the $X_{(\mathbf{S},\mathbf{r})}$ into this basis. As a corollary, one can show that the size of the support of these $X_{(\mathbf{S},\mathbf{r})}$ in this basis is roughly polynomial in the winding number of $(\mathbf{S}, \mathbf{r})$.

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Positroidal tilings of the hypersimplex and $m = 2$ amplituhedron

Melissa Sherman-Bennett
(joint work with Matteo Parisi, Ran Tessler, Lauren Williams)

In this talk, I discussed work connecting the hypersimplex and the $m = 2$ amplituhedron. The hypersimplex $\Delta_{k+1,n}$ is the image of the positive Grassmannian $Gr^{\geq 0}_{k+1,n}$ under the moment map. It is a polytope of dimension $n-1$ in $\mathbb{R}^n$. Meanwhile, the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ is the projection of the positive Grassmannian $Gr^{\geq 0}_{k,n}$ into the Grassmannian $Gr_{k,k+2}$ under a map $\tilde{Z}$ induced by a positive matrix $Z \in Mat^{>0}_{n,k+2}$. Introduced in the context of *scattering amplitudes*, it is not a polytope, and has full dimension $2k$ inside $Gr_{k,k+2}$. Nevertheless, there seem to be remarkable connections between these two objects via *T-duality*, as conjectured in [2]. In [4], joint with Parisi and Williams, we use ideas from oriented matroid theory, total positivity, and the geometry of the hypersimplex and positroid polytopes to obtain a deeper understanding of the amplituhedron. We show that the inequalities cutting out *positroid polytopes*—images of positroid cells of $Gr^{\geq 0}_{k+1,n}$ under the moment map—translate into sign conditions characterizing the T-dual *Grasstopes*—images of positroid cells of $Gr^{\geq 0}_{k,n}$ under $\tilde{Z}$. Moreover, we subdivide the amplituhedron into *chambers*, just as the hypersimplex can be subdivided into simplices, with both chambers and simplices enumerated by the Eulerian numbers. We use these properties to prove the main conjecture of [2]: a collection of positroid polytopes is a tiling of the hypersimplex if and only if the collection of T-dual Grasstopes is a tiling of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ for all Z.

In [3], joint with Parisi, Tessler and Williams, we utilize this correspondence to show that all tilings of $\mathcal{A}_{n,k,2}(Z)$ consist of the same number of tiles. This proves the "Magic Number Conjecture" of [1] in the case of $m = 2$. Along the way to this result, we also provide formulas for volumes of tree poisitroid polytopes in terms of circular extensions of certain cyclic partial orders.

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Taking the amplituhedron to the limit

Rainer Sinn
(joint work with Joris Koefler)

Similar to the talk given by Melissa Sherman-Bennett, we consider *amplituhedra* for $m = 2$. They are the image of the nonnegative Grassmannian by a map $\widetilde{Z}$ defined by a real $n \times (k+2)$ matrix Z whose maximal minors are positive (here, the 2 is m: in general, Z is a $n \times (k+m)$ matrix) . The *nonnegative Grassmannian* $\mathrm{Gr}(k,n)_{\geq 0}$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$ that can be written as the rowspan $[A]$ of a $k \times n$ matrix A of rank k with the property that all its maximal minors $p_I(A)$ are nonnegative (which are the Plücker coordinates of the subspace). In symbols, we get

$$\mathrm{Gr}(k,n)_{\geq 0} = \{[A] \in \mathrm{Gr}(k,n) \colon p_I(A) \geq 0 \text{ for all } I \subset [n],\ |I| = k\}\,.$$

A $n \times (k+2)$ matrix Z defines the rational map $\widetilde{Z} \colon \mathrm{Gr}(k,n) \dashrightarrow \mathrm{Gr}(k,k+2)$ by sending a the rowspan of A to the rowspan of $AZ \subset \mathbb{R}^{k+2}$. By our positivity assumption on Z and the Cauchy-Binet formula, the matrix AZ has rank k for all A representing a point in the nonnegative Grassmannian so that $\widetilde{Z}$ is well-defined on $\mathrm{Gr}(k,n)_{\geq 0}$. The amplituhedron $\mathcal{A}_{n,k,2}(Z)$ is the semi-algebraic set $\widetilde{Z}\,(\mathrm{Gr}(k,n)_{\geq 0}) \subset \mathrm{Gr}(k,k+2)$.

To take the limit of these objects for $n \to \infty$, we pick special matrices Z, namely those whose rows are vectors on the rational normal curve $C_{k+1} \subset \mathbb{P}^{k+1}$ parametrized by $\gamma_{k+1} \colon \mathbb{R} \to \mathbb{P}^{k+1}$, $t \mapsto (1 : t : t^2 : \ldots : t^{k+1})$. We pick n values $0 = t_1 < t_2 < \ldots < t_n = 1$ in the interval $[0,1]$ and call this a partition of $[0,1]$. We define the limit amplituhedron $\mathcal{A}_\infty$ as the union over all amplituhedra $\mathcal{A}_{n,k,2}(Z)$ for any n and all Z with $Z_{ij} = t_i^{j-1}$ for any partition of $[0,1]$ as above. This makes sense as subsets of $\mathrm{Gr}(k,k+2)$ and $\mathcal{A}_\infty$ is therefore a subset of $\mathrm{Gr}(k,k+2)$.

The main result discussed in the presentation is the following from [1].

Theorem 1 (Koefler, Sinn). *The limit amplituhedron $\mathcal{A}_\infty$ is a simplex-like positive geometry in $\mathrm{Gr}(k,k+2)$ for any $k \geq 1$.*

The article [2] gives an introduction to the notion of a positive geometry, which is recursively defined via a differential form and its residues along the (algebraic) boundary of the set. So to check this definition, we have to describe the Euclidean boundary $\partial\mathcal{A}_\infty$ of the set $\mathcal{A}_\infty \subset \mathrm{Gr}(k,k+2)$ and the Zariski closure $\partial_a\mathcal{A}_\infty = \overline{\partial\mathcal{A}_\infty}$ usually called the *algebraic boundary* of $\mathcal{A}_\infty$. This algebraic boundary has codimension 1 in $\mathrm{Gr}(k,k+2)$ and is the union of two Chow varieties. The *Chow variety* $\mathrm{CH}(X)$ of a curve $X \subset \mathbb{P}^{k+1}$ is the set of all subspaces $\Lambda \subset \mathbb{P}^{k+1}$ of codimension 2 that intersect X. It is a subset of $\mathrm{Gr}(k,k+2)$ of codimension 1.

Theorem 2 (Koefler, Sinn). *The algebraic boundary of $\mathcal{A}_\infty$ is the union of the Chow variety* $\mathrm{CH}(C_{k+1})$ *of the rational normal curve* $C_{k+1} \subset \mathbb{P}^{k+1}$ *and the Chow variety of the line* S_{01} *spanned by the two points* $\gamma_{k+1}(0) = (1:0:0:\ldots:0)$ *and* $\gamma_{k+1}(1) = (1:1:1:\ldots:1)$ *in* $\mathbb{P}^{k+1}$.

The Chow variety $\mathrm{CH}(S_{01})$ has a simple description in terms of linear algebra: it is the set of all $k \times (k+2)$ matrices A such that the $(k+2) \times (k+2)$ matrix

$$\begin{pmatrix} A \\ \gamma_{k+1}(0) \\ \gamma_{k+1}(1) \end{pmatrix}$$

has determinant 0. The expansion of this determinant is linear in the $k \times k$ minors of A. In other words, this is a linear condition on the Plücker coordinates of A; so geometrically, it is the intersection of $\mathrm{Gr}(k, k+2)$ with a hyperplane in $\mathbb{P}(\Lambda^k \mathbb{C}^{k+2})$. This is an example of a Schubert variety.

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Accordion lattices and the permutoassociahedron

Daria Poliakova

(joint work with Spencer Backman, Vincent Pilaud)

In this talk we described the orientation on the graph of Kapranov's permutoassociahedron, whose subgraphs corresponding to associahedral facets are Hasse diagrams of accordion lattices for all the reference n-gon triangulations. This oriented graph is conjecturally a Hasse diagram of a bigger lattice itself.

The Stasheff associahedron $K(n)$ is the polytope whose faces correspond to sets of non-crossing diagonals in an n-gon (alternatively, to bracketings of $n-1$ letters), with face lattice given by reverse inclusions. The edge graph of $K(n)$ is well-known to be the Hasse diagram of the Tamari lattice $T(n)$. A diagonal ij can be flipped positively to a diagonal $i'j'$ if $i + j < i' + j'$; in the language of bracketings this corresponds to directing associators as $(AB)C \to A(BC)$.

Accordion lattices $T(D)$ of [6], [4], [1] have alternative orientations of associahedra's edge graphs as Hasse diagrams, depending on *a reference triangulation D*. We draw D on a slightly rotated copy of our n-gon in blue, and the triangulations of interest in red. We say that a blue angle $\angle ijk$ is *closed* by a given red diagonal, if this diagonal is the last red diagonal intersecting ij and jk, counting from j. Now for a given red diagonal, draw arrows pointing to the blue angles closed by it (there are necessarily two such angles on different sides of the given diagonal). This diagonal can be flipped positively, if these arrows rotate it clockwise.

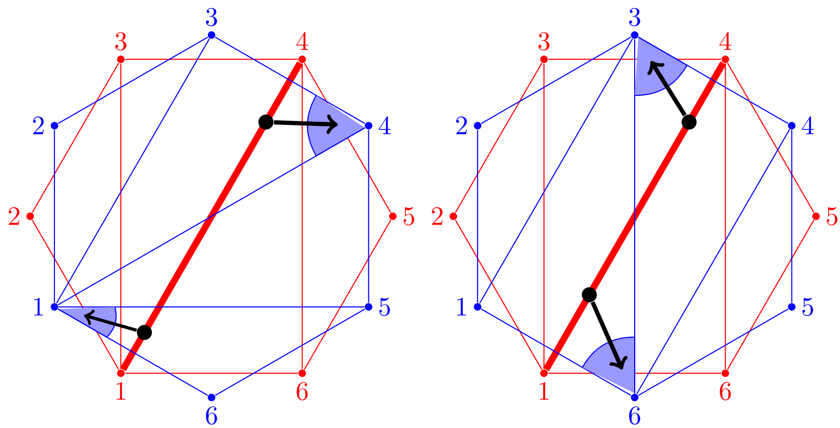

FIGURE 1. The same bold red diagonal can be positively flipped with respect to the reference triangulation on the left and can be negatively flipped with respect to reference triangulation on the right.

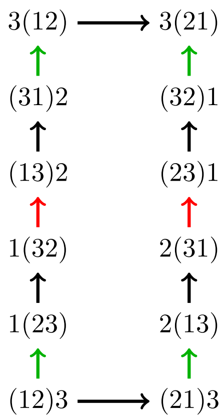

FIGURE 2. The graph of $PA(3)$

The permutoassociahedron $PA(n)$, a mixture of the permutahedron and the associahedron, was introduced as a combinatorial polytope in [5] with motivation in MacLane's categorical coherence and realized as convex polytope in several ways [7], [3], [2]. Its graph has binarily bracketed permutations as vertices, with edges given either by transpositions in minimal brackets (permutahedral edges) or by applications of the associativity low (associahedral edges). If one orients permutahedral edges as in the weak order and associahedral edges as in the Tamari order, the resulting graph fails to have a single source and single sink, thus it cannot be a Hasse diagram of a lattice. We therefore propose a different orientation for associahedral edges.

Definition 1. *Consider an edge corresponding to the associator of* $(A_1A_2)A_3$ *and* $A_1(A_2A_3)$*, where* A_i *are some consecutive subwords of the permutatation* $\sigma \in S_n$*. Call the subword* A_i erasable *if there exists an index* l *such that* $A_i \cap [l,n] = \emptyset$ *but* $A_j \cap [l,n] \neq \emptyset$ *for* $j \neq i$*. If the erasable subword is* A_1 *or* A_3*, orient the edge in the Tamari direction,* $(A_1A_2)A_3 \to A_1(A_2A_3)$*. If the erasable subword is* A_2*, orient the edge in the anti-Tamari direction,* $(A_1A_2)A_3 \leftarrow A_1(A_2A_3)$*.*

For example, $PA(6)$ has an anti-Tamari edge $((15)((64)2))3 \leftarrow (((15)(64))2)3$, because with $A_1 = 15$, $A_2 = 64$ and $A_3 = 2$, the erasable subword is A_2 for $l = 4$. Figure 2 shows permutahedral, Tamari and anti-Tamari edges in $PA(3)$ colored black, green and red respectively.

With orientations as above, fix σ and let $T(\sigma)$ denote the poset whose Hasse diagram is the corresponding subgraph.

Theorem 1. *$T(\sigma)$ is isomorphic to the accordion lattice $T(\min T(\sigma))$; in particular, $T(\sigma)$ is always a lattice.*

Theorem 2. *Every bracketing appears as $\min T(\sigma)$ for some σ; therefore, accordion lattices for all reference triangulations are realized this way.*

The following was verified in Sage for low dimensions.

Expectation 1. *The graph of $PA(n)$ with orientations as above is a Hasse diagram of a lattice.*

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Flowtroids

Alexander Postnikov

A *diagram* $G = (V, E, \{l_v\})$ is a directed graph on vertex set $V = V^{\text{obs}} \sqcup V^{\text{hidden}}$ with two types of vertices: *observable vertices* $u \in V^{\text{obs}}$ and *hidden vertices* $v \in V^{\text{hidden}}$, and certain integers, called *levels,* l_v assigned to hidden vertices $v \in V^{\text{hidden}}$. We assume that all observable vertices are sources.

For every diagram G, we define several geometrical objects:

- Flow polytope $P_G \subset \mathbb{R}^E$.
- Flowtroid polytope $Q_G \subset \mathbb{R}^{V^{\text{obs}}}$
- Flowtroid $F_G := Q_G \cap \mathbb{Z}^{V^{\text{obs}}}$
- Flowtroid variety $X_G \subset \mathbb{CP}^{|F_G|-1}$
- Positive flowtroid variety $X_G^{\geq 0} \subset \mathbb{RP}^{|F_G|-1}$
- Tropical positive flowtroid variety $\text{Trop}X_G^{\geq 0} \subset \mathbb{R}^{F_G}/(1, \ldots, 1)\mathbb{R}$

Flow polytope and flowtroid polytope are certain convex polytopes. Flowtroid is a polymatroid. Flowtroid variety is a complex projective variety. Positive flowtroid variety is a certain semi-algebraic set. Tropical positive flowtroid variety is a polyhedral fan.

We view elements $(f(e))_{e\in E} \in \mathbb{R}^E$ as flows on the graph G, where $f(e)$ is the flow through edge e. The *flow polytope* $P_G \subset \mathbb{R}^E$ is defined by requiring that $f(e) \geq 0$, for any edge $e \in E$, and fixing the *net flow* $\text{inflow}(v) - \text{outflow}(v)$ to be the level l_v, for any hidden vertex $v \in V^{\text{hidden}}$ (but allowing arbitrary net flows for observable vertices).

Let $\pi : \mathbb{R}^E \to \mathbb{R}^{V^{\text{obs}}}$ be the projection that assigns to a flow $(f(e))_{e\in E} \in \mathbb{R}^E$ the vector of net flows for all observable vertices $u \in V^{\text{obs}}$.

The *flowtroid polytope* is defined as the projection $Q_G = \pi(P_G)$ of the flow polytope, and the flowtroid F_G is the set of integer lattice points of the flowtroid polytope Q_G.

Moreover, we define the *boundary measurement map* $M_G : \mathbb{R}^E_{\geq 0} \to \mathbb{RP}^{|F_G|-1}$. The *positive flowtroid variety* $X_G^{\geq 0} \subset \mathbb{RP}^{|F_G|-1}$ is defined as the image of this map. The *flowtroid variety* $X_G \subset \mathbb{CP}^{|F_G|-1}$ is the Zariski closure of $X_G^{\geq 0}$, and the tropical positive flowtroid variety is its tropicalization.

We define the notion of a *reduced diagram* G. We show that, for a reduced diagram, the tropical positive flowtroid variety is piecewise-linearly isomorphic to the normal fan of the *fiber polytope* $\text{Fiber}(P_G \xrightarrow{\pi} Q_G)$.

This setup generalizes and unifies several other constructions studied earlier.

We say that a diagram G is a *plabic graph* if G is a planar graph embedded into a disk such that all observable vertices are on the boundary of the disk and have degree 1, all internal vertices are either sources of sinks, and all levels $l_e \in \{1, -1\}$.

In case of plabic graphs, the above general setup specializes to the construction of the positive Grassmannian given in [P06]. In this case, flowtroids are positroids. Elements of flowtroids correspond to Plücker coordinates. Flowtroid varieties are positroid varieties. Positive flowtroid varieties are the positroid cells in the Grassmannian. The boundary measurement map M_G is exactly the boundary measurement map from [P06]. The flow polytope P_G is exactly the matching polytope from [PSW09]. Tropical positive flowtroid varieties are tropical positroid varieties, and, in particular, the tropical positive Grassmannian.

More generally, if diagrams G are arbitrary planar graphs embedded into a disk with observable vertices arranged on the boundary of the disk, then flowtroids F_G are exactly polypositroids studied in [LP24].

In case when G are arbitrary bipartite graphs (not necessarily planar) such that observable vertices consist of all vertices in one part of G, then the class of flowtroid polytopes Q_G is exactly the class of generalized permutohedra obtained as Minkowski sum of the coordinate simplices $\Delta_I := \text{conv}(e_i, i \in I)$, $I \subset [n]$. This class of polytopes was studied in [P09]. It includes the usual permutohedron, the associahedron, graph-associahedra, and many other interesting polytopes. In this case, the construction is closely related to the study of triangulations of products of simplices $\Delta^{m-1} \times \Delta^{n-1}$ (when $G = K_{m,n}$) and root polytopes, see [P09].

The class of flowtroids F_G which are matroids is exactly the class of *gammoids,* studied in the matroid literature.

In general, the class of flowtroids F_G is closed under the following operations on polymatroids:

- Minkowski sums
- direct products
- projections $(x_1, \dots, x_n) \to (x_1 + x_2, \dots, x_n)$
- intersections with hyperplanes $\{x_i = \text{Const}\}$
- S_n-action
- parallel translation

Moreover, the class of flowtroids is the minimal class of polymatroids that contains uniform matroids of rank 1 and is closed under these operations.

We also discuss the dual construction. In the dual setting, instead of flow polytopes, we use alcoved polytopes, and instead of flowtroid polytopes we use projections of alcoved polytopes. The dual story is completely parallel to the above discussion. All results and constructions mentioned above have dual analogues.

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On cosmological polytopes

Martina Juhnke

(joint work with Torben Donzelmann, Benedikt Rednoß, Liam Solus, Christoph Thäle, Lorenzo Venturello)

The cosmological polytope of a graph G was recently introduced to give a geometric approach to the computation of wavefunctions for cosmological models with associated Feynman diagram G [1]. Basic results in the theory of positive geometries dictate that this wavefunction may be computed as a sum of rational functions associated to the facets in a triangulation of the cosmological polytope. Given a graph $G = (V, E)$, the *cosmological polytope* of G is

$$\mathcal{C}_G = \text{conv}(\mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_f, \mathbf{e}_i - \mathbf{e}_j + \mathbf{e}_f, -\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_f \ : \ f = ij \in E) \subset \mathbb{R}^{V \cup E}.$$

We describe basic properties of cosmological polytopes and, in particular, we are interested in triangulations of these polytopes. One of the main results from [2] computes a Gröbner basis of the toric ideal for this purpose:

Theorem 1. *Given a graph $G = (V, E)$ the toric ideal has a* squarefree *Gröbner basis. The elements of this Gröbner basis can be completely described in terms of the graph.*

In particular, it is shown that there are certain generators, forced to be in the Gröbner basis, for every single edge of the graph (referred to as *fundamental binomials*) and other generators that can be constructed from (directed) paths and cycles in the graph. As a consequence, the cosmological polytope has a regular unimodular triangulation. We explicitly describe such a triangulation if the underlying graph is a tree or cycle and compute the normalized volume of the cosmological polytope in these cases.

In the second part of this talk, we consider cosmological polytopes for Erdős-Rényi graphs $G(n,p)$. In this model, given n vertices each edge (of the complete graph) is drawn independently with probability p. We are interested in the number of edges of the cosmological poytopes in this case. The main result, which is unpublished joint work with Torben Donzelmann, Benedikt Rednoß and Christoph Thäle is the following:

Theorem 2. *Let $G \sim G(n,p)$ with $p = n^{-\alpha}$ for $\alpha < 1$. Let f_1 denote the number of edges of $\mathcal{C}_G$. Then f_1 obeys a central limit theorem, i.e.,*

$$d_K\left(\frac{f_1 - \mathbb{E}(f_1)}{\mathbb{V}(f_1)}, N\right) \to 0,$$

as n goes to ∞, where $N \sim \mathcal{N}(0,1)$.

In the previous theorem d_K denotes the Kolmogorov distance and N follows a standard normal distribution. It is further shown that an analogous statement holds for the number of edges in a regular unimodular triangulation of the cosmological polytope.

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New Results on h^*-polynomials of Cosmological Polytopes

Benjamin Schröter
(joint work with Aenne Benjes, Kamillo Ferry)

There are various polyhedral objects one may associate with a graph G, e.g., alcoved polytopes or their duals which include (symmetric) edge polytopes, but also the matroid base polytope, the independence complex or broken circuit complex of the graphical matroid of G. Yet another of these objects is the *cosmological polytope* $\mathcal{C}_G$ of the graph G. Simplified this is the convex hull of the vectors $e_u + e_v - e_f$, $e_u - e_v + e_f$ and $-e_u + e_v + e_f$ for all edges $f = \{u, v\}$ of the graph G. They have been introduced in [1] by Arkani-Hamed, Benincasa and Postnikov as a geometric tool to study the physics of cosmological time evolution and the wavefunction of the universe in a model whose Feynman diagram is given by the graph G. A central role in this theory is played by positive geometries and canonical forms that a cosmological polytope defines, and hence a better understanding of their facet structure and triangulations is required. Therefore, Benincasa [2] as well as Kühne and Monin [6] investigated the face structure of cosmological polytopes combinatorially, while Juhnke, Solus and Venturello showed in [5] that these polytopes pose a unimodular triangulation by applying methods from toric geometry and Gröbner bases to construct so-called good triangulations. Using these results Bruckamp, Gotermann, Juhnke, Ladin and Solus [4] shifted the focus towards the Ehrhart theory and in particular the h^*-polynomials of cosmological polytopes. They found explicit formulas for these polynomials whenever the graph belongs to the families of multitrees or multicycles.

In [3] Benjes, Ferry and I completed this story on which I reported in my presentation. We found a way to enumerate all maximal simplices in a good triangulations of any cosmological polytope as they are in bijection to certain decorations of the graph G. Furthermore, we provide a method to turn such a triangulation into a half-open decomposition from which we deduce that the h^*-polynomial of a cosmological polytope agrees with the following specialization of the Tutte polynomial $\mathrm{T}_G(x, y) \in \mathbb{Z}[x, y]$ of the graph G

$$h^*(\mathcal{C}_G; z) \;=\; (1+z)^{m-r}\,(2z)^r\,\mathrm{T}_G\left(\frac{1+3z}{2z}, 1\right)$$

where m denotes the number of edges and loops of G and r its rank. From this result we derive further equivalent formulas, reprove results of Bruckamp et al. and solve several open problems and conjectures.

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Quotients in Discrete Convex Analysis

Georg Loho

(joint work with Marie Brandenburg, Ben Smith)

Given a vector space V and a collection $(L_i)_{i\in E}$ of subspaces of V indexed by a finite set E, the function

$$f\colon 2^E \to \mathbb{R}, \quad f(S) = \dim\left(\sum_{i\in S} L_i\right) \;\forall S \subseteq E$$

turns out to be a *submodular* function [2]. More generally a set function is submodular if

$$f(A\cup B) + f(A\cap B) \le f(A) + f(B) \quad \text{for all } A, B \subseteq E \;.$$

Let $\phi\colon V \to W$ be a linear map. This gives rise to another submodular function derived from the images of the subspaces $(L_i)_{i\in E}$,

$$g\colon 2^E \to \mathbb{R}, \quad g(S) = \dim\left(\sum_{i\in S} \phi(L_i)\right) \;\forall S \subseteq E \;.$$

It turns out that f and g are in an intriguing relation, namely

$$f(X) - g(X) \le f(Y) - g(Y) \quad \forall X \subseteq Y \subseteq E \;.$$

If two submodular functions $f, g\colon 2^E \to \mathbb{R}$ fulfill this property, they are said to form a *quotient*. To a submodular function f taking only integral values, one can associate an *M-convex set* [3], that is, the set $P \cap \mathbb{Z}^E$ with

$$P := \left\{x \in \mathbb{R}^E \mid x(S) \le f(S) \forall S \subseteq E, x(E) = f(E)\right\} \;;$$

this is the set of integer points in the *(integral) generalized permutahedron* P associated with f. An M-convex set is a *matroid* if and only if it is a subset of the unit cube in $\mathbb{R}^E$, and then the integer points are the characteristic vectors of bases of the matroid. The notion of a *matroid quotient* is well-studied and several equivalent definitions are known for this. It turns out that one can generalize many of them also to the level of submodular functions and M-convex sets, respectively. Given submodular functions f, g with corresponding M-convex sets $M, N \subseteq \mathbb{Z}^E$, one obtains the following equivalent conditions (among others) [1]:

(A) For all $X \subseteq Y \subseteq E$ holds

$$g(Y) - g(X) \le f(Y) - f(X).$$

(B) For all $x \in N, y \in M, i \in E$ with $x_i - y_i > 0$ there is a $j \in E$ with $y_i - x_i > 0$ such that

$$x - e_i + e_j \in N, y + e_i - e_j \in M.$$

(C) There is an M-convex set $R \subseteq \mathbb{Z}^{E\cup e}$ and numbers $s < t \in \mathbb{Z}$ such that

$$R \cap \{x_e = s\} = N \times \{s\} \text{ and } R \cap \{x_e = t\} = M \times \{t\}.$$

Another equivalent characterization is based on the generalization of *linking systems* or *bimatroids* which originated in the construction of matroids by induction through a bipartite graph. For disjoint sets V and U, an M-convex set $\Gamma \subseteq \mathbb{Z}^V \times \mathbb{Z}^U$ is called a *linking set* from V to U. Then for an M-convex set $T \subseteq \mathbb{Z}^U$, the *induction of T through* Γ is the M-convex set

$$\mathrm{ind}_\Gamma(T) = \left\{x \in \mathbb{Z}^V \mid \exists y \in T \text{ such that } (x, -y) \in \Gamma\right\} .$$

This is in general a versatile tool to construct M-convex sets in various specific contexts.

While the story of quotients allows for many different equivalent characterizations, it becomes more subtle when one considers the more general *M-convex functions*. These are functions generalizing *valuated matroids* to more general M-convex domains beyond the set of bases of a matroid. Writing analogous expressions to define quotients of M-convex functions does not necessarily result in a set of equivalent characterizations. Indeed, considering definitions inspired by questions from optimization or tropical flag varieties yields different notions. The exact status of the equivalence of these notion is still subject to ongoing work.

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Support cones of alcoved polytopes and simple games

Raman Sanyal

(joint work with Aenne Benjes and Benjamin Schröter)

Let $A = (a_\rho)_{\rho \in E}$ be a finite collection of distinct unit vectors that positively span $\mathbb{R}^d$. Any $b = (b_\rho)_{\rho \in E} \in \mathbb{R}^E$ defines a (possibly empty) polytope

$$P_A(b) \ := \ \{x \in \mathbb{R}^d : \langle a_\rho, x\rangle \le b_\rho \text{ for } \rho \in E\},$$

called an **A-polytope**. If $P_A(b) \neq \emptyset$, then b is called a **support vector** if the intersection of $P_A(b)$ with every hyperplane $H_\rho(b) := \{x \in \mathbb{R}^d : \langle a_\rho, x\rangle = b_\rho\}$ for $\rho \in E$ is non-empty. The **support cone** $\mathcal{S}_A \subseteq \mathbb{R}^E$ is the set of all support vectors. The support cone is a full-dimensional polyhedral cone with lineality space $\mathrm{im}(A)$. Most importantly for any $b, b' \in \mathcal{S}_A$

$$P_A(b) \ = \ P_A(b') \quad \Longleftrightarrow \quad b \ = \ b' .$$

Support vectors were introduced by Aleksandrov [1]. McMullen [4] studied A-polytopes modulo translation and called $\mathcal{S}_A/\mathrm{im}(A)$ the *closed inner region.* Support cones as well as their subdivisions given by type and discriminental fans are ubiquitous objects in discrete and convex geometry, algebraic geometry, and optimization that are notoriously complicated. In this talk we focus on the class of alcoved polytopes, for which the notions above can be interpreted and treated in terms of graph. For the case of *double cycles*, the subdivisions can be related to *simple games.*

Type fans and discriminental fans. The **normal fan** of a polytope $P \subset \mathbb{R}^d$ with vertex set V is the pure d-dimensional complex generated by the cones $N_v(P) = \{c \in \mathbb{R}^d : \langle c, v\rangle \geq \langle c, u\rangle \text{ for all } u \in V\}$. Two polytopes are **strongly isomorphic**[1] if they have the same normal fan. The classes of strongly isomorphic A-polytopes partition $\mathcal{S}_A$ into relatively open cones. Taking closures yields a fan structure on $\mathcal{S}_A$ called the **type fan** $\mathcal{T}_A$. The maximal cones of $\mathcal{T}_A$ are precisely the equivalence classes of simple A-polytopes for which all a_ρ are facet-defining.

The collection of hyperplanes $H_\rho(b)$ for $b \in \mathbb{R}^E$ defines an affine hyperplane arrangement $\mathcal{H}_A(b)$ with affine oriented matroid $\mathcal{O}_A(b)$. The decomposition of $\mathbb{R}^E$ with respect to $b \mapsto \mathcal{O}_A(b)$ is induced by an arrangement of linear hyperplanes, called the **discriminental arrangement** D_A by Manin and Shekhtman [3]. The connected components $\mathbb{R}^E \setminus \bigcup \mathrm{D}_A$ are in bijection to the uniform matroids among $\{\mathcal{O}_A(b) : b \in \mathbb{R}^E\}$. The interior of the support cone $\mathcal{S}_A$ is the union of connected components whose closure induce the **discriminental fan** $\mathcal{D}_A$. If $\mathcal{O}_A(b)$ is uniform, then the closure of every region of $\mathbb{R}^d \setminus \mathcal{H}_A(b)$ is a simple polytope. This implies that $\mathcal{D}_A$ refines the type fan $\mathcal{T}_A$ and the refinement is proper in general.

Alcoved polytopes. Let D be a loop-less directed graph D on nodes $[d] = \{1, \ldots, d\}$ and edges $E \subseteq [d] \times [d]$. We assume that D is strongly connected. For $A_D = (e_i - e_j)_{ij \in E}$ and edge weights $b \in \mathbb{R}^E$, $P_A(b)$ is a polytope in $\mathbb{R}^d/(1, \ldots, 1)\mathbb{R}$. We call $P_{A_D}(b)$ an **alcoved polytope** [2] and write $P_D(b)$.

A **circuit** of D is a subset $C = C_+ \uplus C_- \subseteq E$ such that C_+ together with the reorientation $-C_- = \{vu : uv \in C_-\}$ is a directed cycle. The hyperplanes of the discriminental arrangement for A_D are precisely the circuit hyperplanes

$$\sum_{uv \in C_+} b_{uv} = \sum_{uv \in C_-} b_{uv} \,.$$

The support cone of A_D is bounded by the hyperplanes of **almost positive** circuits, that is, circuits C with $|C_-| \leq 1$.

Double cycles and simple games. for $|u - v| \leq 1$ as well as $1d$ and $d1$. In addition to pairs of anti-parallel edges, the circuits of size > 2 can be naturally identified with the subsets of $[d]$. For double cycles, the type and discriminental fans coincide and the maximal cones can be interpreted as weighted simple games.

Simple games model basic voting scenarios [5]. Consider $[d]$ as a set of voters and $I \subseteq [d]$ as a coalition. A **simple game** is partition of $2^{[d]}$ into the set of loosing coalitions $\mathcal{L}$ and winning coalitions $\mathcal{W} = 2^{[d]} \setminus \mathcal{L}$ such that $J \in \mathcal{L}$ and $I \subseteq J$ implies $I \in \mathcal{L}$. We also require $\{i\} \in \mathcal{L}$ and $[d] \setminus \{i\} \notin \mathcal{L}$ for all i. A simple game is **weighted** if there are weights $\omega \in \mathbb{R}^d_{\geq 0}$ and a quota $Q \geq 0$ such that $I \in \mathcal{L}$

[1] also called *normally equivalent*, *analogous*, or *locally similar*.

if and only if $\sum_I \omega_i > Q$. Simple games are also known as *Boolean functions* and weighted simple games correspond to *threshold functions*. The fundamental question is to combinatorially characterize weighted simple games among all simple games.

Theorem 1. *Weighted simple games on d voters are in bijection to the maximal cones of the type fan of the double cycle of length d.*

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Decomposition Polyhedra of Piecewise Linear Functions

Marie-Charlotte Brandenburg

(joint work with Moritz Grillo and Christoph Hertrich)

Continuous piecewise linear (CPWL) functions play a crucial role many fields of mathematics and its applications. While they have traditionally been used to describe problems in geometry, discrete and submodular optimization, or statistical regression, they recently gained significant interest as functions represented by neural networks with rectified linear unit (ReLU) activations. Extensive research in this context has been put into understanding complexity questions, where a major source of complexity in all the aforementioned fields is nonconvexity. On the other hand, it is a well-known folklore fact that every (potentially nonconvex)

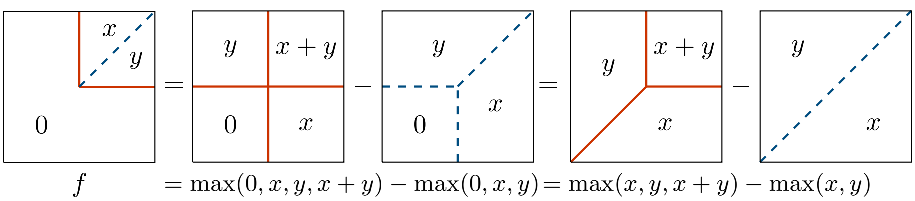

Figure 1. A piecewise linear function f (left), a non-minimal decomposition $g - h$ (middle) and a minimal decomposition $g' - h'$ (right) according to [3].

CWPL function $f\colon \mathbb{R}^n \to \mathbb{R}$ can be written as the difference $f = g - h$ of two *convex* CPWL functions [2, 4]. Consequently, a natural idea to circumvent the challenges induced by nonconvexity is to use such a decomposition $f = g - h$ and solve the desired problem separately for g and h. However, the crucial question arising from this strategy is: *how much more complex are g and h compared to f?* A well-established measure for the complexity of a CPWL function is the number of its linear pieces. Therefore, the main question we study is the following.

Problem 1. *How to decompose a CPWL function f into a difference $f = g - h$ of two convex CPWL functions with as few pieces as possible?*

There exist many ways in the literature to obtain such a decomposition, but none of them guarantees minimality or at least a useful bound on the number of pieces of g and h depending on those of f. In fact, no finite procedure is known that guarantees to find a minimal decomposition, with the exception of a recent solution for special cases in dimension 2 [3].

We propose a novel perspective on Problem 1 making use of polyhedral geometry. Instead of aiming for a globally optimal decomposition, we restrict to solutions that are *compatible with a given regular polyhedral complex $\mathcal{P}$*. In short, this means fixing where the functions g and h may have breakpoints, that is, points where they are not locally linear.

Theorem 2 ([1, Theorem 3.5]). *The set of decompositions of f which are compatible with $\mathcal{P}$ is a polyhedron $\mathcal{D}_\mathcal{P}(f)$ that arises as the intersection of two shifted polyhedral cones,*

$$\mathcal{D}_\mathcal{P}(f) = \mathcal{V}_\mathcal{P}^+ \cap (f + \mathcal{V}_\mathcal{P}^+).$$

We call this polyhedron the decomposition polyhedron of f *with respect to $\mathcal{P}$.*

When $\mathcal{P}$ is a polyhedral fan, then the polyhedral cone $\mathcal{V}_\mathcal{P}^+$ is known as the *deformation cone* or *type cone* of the polyhedral fan. Given a decomposition polyhedron, we can characterize its faces as follows.

Theorem 3 ([1, Section 3]). *Let $(g, h) \in \mathcal{D}_\mathcal{P}(f)$ be a decomposition $f = g - h$ which is compatible with $\mathcal{P}$. Then*

- *(g, h) is contained in a bounded face of $\mathcal{D}_\mathcal{P}(f)$ if and only if (g, h) is* reduced, *i.e., there exists no nonzero convex function ϕ such that $g - \phi$ and $h - \phi$ are convex,*
- *(g, h) is a vertex of $\mathcal{D}_\mathcal{P}(f)$ if and only if there is no nontrivial coarsening of the polyhedral complexes underlying g and h among the underlying complexes of any $(g', h') \in \mathcal{D}_\mathcal{P}(f) \setminus \{(g, h)\}$,*
- *any minimal decomposition that is compatible with $\mathcal{P}$ is a vertex of $\mathcal{D}_\mathcal{P}(f)$.*

The last statement implies a finite procedure to find a minimal decomposition among those decompositions that are compatible with $\mathcal{P}$, by simply enumerating the (potentially many) vertices of $\mathcal{D}_\mathcal{P}(f)$. However, the solution provided by this procedure is not guaranteed to be minimal over all possible compatible polyhedral complexes. It remains an open question to construct a minimal decomposition in general dimensions, or to describe a finite procedure to compute it.

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Finite metric spaces and fundamental polytopes

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(joint work with Emanuele Delucchi, Leonie Mühlherr)

This talk was centered around the study of finite metric spaces through the lens of polytopes and hyperplane arrangements. Finite metric spaces arise in several applied contexts. Let us mention for instance mathematical biology, where finite metrics model genetic dissimilarities between different species [3]. In this setting, a main research direction is to identify suitable classes of metric spaces and study the combinatorics and geometry of the associated subset of the metric cone, e.g., for geometric statistics. A combinatorial invariant of metric spaces that is widely used in applications as well as for theoretical considerations is their *injective hull*, introduced by Isbell and rediscovered by Dress under the name *tight span*.

Motivated by the theory of optimal transport, Vershik described a correspondence between finite metric spaces and a class of symmetric convex polytopes, the so-called *Kantorovich-Rubinstein-Wasserstein (KRW) polytopes* or *fundamental polytopes* [5]. The KRW polytope of an n-metric ρ is a polytope in $\mathbb{R}^n$ defined as the convex hull:

$$KRW(\rho) = \operatorname{conv}\left\{ \frac{e_i - e_j}{\rho_{ij}} \,\middle|\, 1 \leq i, j \leq n \right\},$$

where e_i is the i-th standard basis vector of $\mathbb{R}^n$. The metric is called *generic* if it is strict and the KRW polytope is simplicial.

If ρ is the n-metric with $\rho_{ij} = 1$ for all $i \neq j$, then $KRW(\rho)$ is the type A_n *root polytope*. If ρ is a graph-metric (i.e., there is a graph G with vertex set $[n]$ such that ρ_{ij} is the number of edges of a shortest path from i to j in G), then $KRW(\rho)$ is the *symmetric edge polytope* of G.

The dual of a KRW polytope is the so-called *Lipschitz polytope*. This is a symmetric *alcoved polytope*.

The problem of understanding the combinatorial structure (e.g., computing face numbers) of KRW and Lipschitz polytopes is open and significant for applications. The focus of the talk was to study and classify finite metric spaces according to the combinatorial properties of their KRW polytopes.

The first progress on this question was achieved by Gordon and Petrov who gave a description of the face poset of KRW polytopes via linear inequalities on the values of the metric [2]. This is the starting point for an in-depth study of KRW polytopes using the theory of hyperplane arrangements.

In this talk we introduced the so-called *Wasserstein arrangement* whose cells correspond to the combinatorial types of KRW polytopes. This arrangement builds upon the Gordon–Petrov description of the faces of KRW polytopes and could be of independent interest. Moreover, via the above connection to alcoved polytopes, intersecting this arrangement with the metric cone naturally yields the type-fan of symmetric alcoved polytopes.

Using the computer algebra system `OSCAR` together with the Julia package `CountingChambers.jl` [1] and the software `TOPCOM` [4] we obtain an enumeration of the combinatorial types of KRW polytopes of generic metrics on $n = 4, 5, 6$ points. These numbers already demonstrate that the subdivision of the metric cone by combinatorial types of KRW polytopes is much finer than the subdivision by tight spans.

Lastly, we clarified the relation between KRW polytopes and tight spans by providing examples of five-point metrics with isomorphic tight spans and combinatorially different KRW polytopes and vice versa. Hence, the two fan structures of the metric fan induced by the tight spans or the KRW polytopes are not refinements of each other.

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Preservation of Inequalities under Hadamard Products

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(joint work with Petter Brändén and Luis Ferroni)

The Hadamard product of two formal power series $a(x) = \sum_{n\geq 0} a_n x^n$ and $b(x) = \sum_{n\geq 0} b_n x^n$ is defined by

$$(a \star b)(x) = \sum_{n\geq 0} a_n b_n x^n \,.$$

In many instances, in combinatorics and otherwise, the coefficients agree with evaluations of a polynomial function, i.e., there is a polynomial p such that $a_n = p(n)$ for all $n \geq 0$. It is a well-known fact that this is the case if and only if

$$\sum_{n\geq 0} a_n x^n \;=\; \frac{\mathcal{W}(p)}{(1-t)^{\deg p+1}}\,,$$

where $\mathcal{W}(p)$ is a polydnomial of degree at most $\deg p$. An example is the Ehrhart series of a lattice polytope whose coefficients are given by the number of lattice points in the n-th dilate of the polytope; in this case p is the Ehrhart polynomial of the polytope [2] and $\mathcal{W}(p)$ plays the role of the h^*-polynomial.

We study the preservation of combinatorially relevant properties of $\mathcal{W}(p)$ under the Hadamard product, i.e., we consider questions of the form *Given that $\mathcal{W}(p)$ and $\mathcal{W}(q)$ have a certain property, does also $\mathcal{W}(pq)$ have that property?* In this direction, a theorem by Wagner [3] asserts that having only nonpositive, real zeros is a property that is preserved by the Hadamard product. This property furthermore implies inequalities of particular interest in combinatorics (see, e.g., [4]): Given a polynomial $a(x) = \sum_{j=0}^{d} a_j x^j$ with nonnegative coefficients and only real zeros, then $a(x)$ is

- *unimodal*, i.e. $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some $0 \leq k \leq d$.
- *log-concave*, i.e. $a_j^2 \geq a_{j-1}a_{j+1}$ for $0 \leq j \leq d$.
- *ultra log-concave of order d*, i.e. $\{a_j/\binom{d}{j}\}_{j=0}^{d}$ is log-concave.

Furthermore, $a(x)$ has *no internal zeros*, i.e. if $i < j$ and $a_i, a_j \neq 0$ then also $a_k \neq 0$ for all $i < k < j$. Given a polynomial with no internal zeros, ultra log-concavity implies log-concavity implies unimodality. It can be seen that the weakest property in this hierarchy, unimodality, is generally not preserved under the Hadamard product [5, 6]. We prove that if $\mathcal{W}(p)$ and $\mathcal{W}(q)$ are ultra log-concave with no internal zeros, then so is $\mathcal{W}(pq)$ [1, Theorem 1.2]. For the proof we use the theory of Lorentzian polynomials developed in [7]. Whether the Hadamard products preserves log-concavity remains an open question at this point [1, Question 6.1].

A polynomial $a(x)$ is *symmetric* (or palindromic) with center of symmetry $d/2$ if $a(x) = t^d a(1/x)$. A property of symmetric polynomials that is intensively studied in geometric combinatorics is γ-positivity. A symmetric polynomial $a(x)$ is γ-positive if $a(x) = \sum_{i=0}^{\lfloor d/2\rfloor} \gamma_i x^i (1+x)^{d-2i}$ for $\gamma_i \geq 0$. The polynomial $\sum_{i=0}^{\lfloor d/2\rfloor} \gamma_i x^i$ is called the *γ-polynomial*.

Observe, that γ-positivity implies unimodality. We show that γ-positivity is preserved under the Hadamard product [1, Theorem 1.3]. Further, we show that if the γ-polynomial of a γ-positive polynomial is ultra log-concave then so is the polynomial itself [1, Theorem 3.3].

We also study symmetric decompositions of polynomials. Given a polynomial h of degree at most d, by basic linear algebra considerations, there exist uniquely determined symmetric polynomials $a(x) = t^d a(1/x)$ and $b(x) = x^{d-1}b(1/x)$ such that $h = a + xb$. The pair (a, b) is called the *symmetric decomposition* of h. This decomposition is called γ-positive if both a and b are γ-positive. We show that the property of having a γ-positive symmetric decomposition is preserved under the Hadamard product [1, Theorem 1.4].

Further, we prove the preservation of having an interlacing decomposition, a property that implies both real-rootedness and unimodality. We also disprove a conjecture by Fischer and Kubitze [8] regarding the real-rootedness of polynomials under Hadamard powers.

For further details, we refer to the full article this talk was based on [1].

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