Mathematisches Forschungsinstitut Oberwolfach

Report No. 13/2025

DOI: 10.4171/OWR/2025/13

Mini-Workshop: Recent Results on Loop Spaces

Organized by
Sergio Cacciatori, Como
Batu Güneysu, Chemnitz
Eva Kopfer, Bonn

2 March – 7 March 2025

Abstract. This Oberwolfach mini-workshop gathered mathematicians with an expertise in loop spaces, in order to discuss recent breakthrough results in this field, and the implications of these. A particular focus was given on recent results that rigorously connect loop spaces to index theory.

Mathematics Subject Classification (2020): 55P35.

Introduction by the Organizers

Loop spaces have been a constant source of deep and interesting mathematical problems since the 1980's, both as a tool that provides new insights in existing theories, or as an independent object that leads to new results and that serves as a guiding example for abstract (analytic) theories of infinite dimensional spaces. The workshop mainly focused on the following three subtopics:

I: Localization Formulae on Loop Spaces, and the Atiyah-Singer Index Theorem. Being considered to be one of the most important mathematical results of the 20th century, the Atiyah-Singer Index Theorem (A-S-I) states that the index of a possibly twisted Dirac operator (a geometric object) on a closed even dimensional Riemannian spin manifold M can be calculated topologically. While classic proofs of this result use either cobordism theory or heat kernel methods, it was noticed by Atiyah [1] and Witten [22] in the 1980's, that one can derive the A-S-I also as follows: firstly, one assumes that one can integrate differential forms on the loop space $\mathscr{L}M$ (which, although the spin structure on M determines an orientation on $\mathscr{L}M$, is analytically not possible in any strict sense). Then, one formally calculates the index of the Dirac operator as the integral of a differential form on $\mathscr{L}M$. Finally, as $\mathscr{L}M$ carries a natural S^1-action, a formal application of the Duistermaat-Hackman localization formula reduces the latter integral on $\mathscr{L}M$ to an integral on M, and this number miraculously [1] turns out to be the right hand side of the A-S-I. These considerations have been extended to the setting of the twisted A-S-I by Bismut [3], who also noticed that there is an intimate connection between these formal loop space calculations to the hypoelliptic Laplacian [4]. Ultimately, it took over 35 years to implement Atiyah's and Witten's ideas in a mathematically well-defined way. This was achieved in [8] (see also [18, 5, 2]), using techniques from noncommutative geometry, specifically cyclic homology. In this context, methods from *supergeometry* that have been originally developed by physicist have traditionally played a decisive role. Particularly concrete results can be obtained in the situation where $M = G$ is a compact Lie group.

II: Dirac Operators on Loop Spaces. The study of loop spaces as an independent object has most prominently lead to the discovery of what is nowadays called the *Witten genus* - which plays the role of an universal elliptic genus. Here, the starting point is that on a closed Riemannian spin manifold M which carries an action of a compact Lie group G, there is a natural way to twist the Dirac operator on M by an element $g \in G$. The index of this twisted operator again admits a topological representation, and this equivariant A-S-I (also called the *Character Valued A-S-I*) is intimately connected to Weyl-type character formulae. Now, it turns out that if M admits a higher spin structure - a so called *string structure*, then the loop space $\mathscr{L}M$ admits a spin structure. While the mathematically rigorous construction of the actual spin bundle over $\mathscr{L}M$ as well as a Hilbert space of spinors is not obvious at all, one can nevertheless formally calculate the index of an S^1-twisted Dirac operator on $\mathscr{L}M$, which leads to the Witten genus of M [23]. While a complete mathematical realization of a Dirac operator on $\mathscr{L}M$ has remained open until today, a recent breakthrough has been made in [16, 17], where the spin bundle on $\mathscr{L}M$ has been constructed rigorously for the first time.

III: Curvature and Quasi-Regular Dirichlet Forms on Loop Spaces. Let X be a Polish space with a Borel measure μ defined on it. A *Dirichlet form* in $L^2(X, \mu)$ is a densely defined, closed and symmetric bilinear form $\mathscr{E} \geq 0$ in $L^2(X, \mu)$, which is assumed to be stable under normal contractions. It follows from Kato's classical theory that $\mathscr{E}$ canonically induces a self-adjoint operator $H \geq 0$ in $L^2(X, \mu)$, and the semigroup $P_t := \exp(-tH)$ induces a contraction semigroup in $L^q(X, \mu)$, $q \in [1, \infty]$. Under an additional topological/analytic assumption on $\mathscr{E}$, its so called *quasi-regularity*, it turns out that there exists a Markov process on X whose transition probabilities are given by P_t. Now if M is a closed Riemannian manifold and μ is the Brownian loop measure on the space of continuous loops $\mathscr{L}_c M$, then [7] there exists a canonically given quasi-regular Dirichlet form on $L^2(\mathscr{L}_c M, \mu)$. While this examples serves as one of the most important genuinely infinite dimensional examples of the theory of regular Dirichlet forms, recent results [10, 13, 6, 9] suggest a very surprising new connection: the conjunction of upper and lower bounds on the Ricci curvature of M is equivalent to a functional inequality on $\mathscr{L}_c M$, namely a gradient estimate for the semigroup of this infinite dimensional space. Current research focuses on the case where X is a singular space, such as e.g. an Alexandrov surface.

The mini-workshop has brought together 14 scientists from the following countries:

- Canada
- France
- Germany
- Iran
- Italy
- Luxembourg
- United States of America
- United Kingdom

A total of 11 talks has been given and each talk has lead to a fruitful discussion among the participants, which we believe has lead to new insights in deep open problems in the context of the three topics above.

Concerning topic I:

I.1. Generalization of the localization formula to the setting of the equivariant index theorem.

I.2. Examination of possible rigorous connections between the constructions from [8, 18, 5, 2], which rely on cyclic homology, and the hypoelliptic Laplacian (see also [12]).

Concerning topic II:

II.1. Comparison of the C^*-algebraic approach to spinors on loop space from [16, 17] to the approach from [14] on loop groups, which relies on deep results from representation theory.

II.2. First steps in the construction of an Hilbert space of sections of the spinor bundle over a loop space.

Concerning topic III:

III.1. Generalization of the characterization of curvature bounds by functional inequalities on path/loop spaces to singular spaces, such as e.g. metric measure spaces satisfying synthetic curvature bounds in the spirit of Sturm and Lott/Villani [20, 21, 15], see also [19].

III.2. Examination of the possibility of defining a spin structure on a singular (metric measure) space N through an orientation on the space of continuous loops $\mathscr{L}_c N$ (e.g. by a proper infinite dimensional variant of the machinery from [11]).

We thank the MFO for creating a very stimulating and inspiring atmosphere.

Sergio Cacciatori, Batu Güneysu, and Eva Kopfer

References

[1] Atiyah, M. F.: Circular symmetry and stationary-phase approximation. Colloq. Honneur L. Schwartz, Éc. Polytech. 1983, Vol. 1, Astérisque 131, 43–59 (1985).

[2] Boldt, S. & Cacciatori, S. & Güneysu, B.: A Chern-Simons transgression formula for supersymmetric path integrals on spin manifolds. Journal of Geometry and Physics, Volume 195 (2024) 105041.

[3] Bismut, J.-M.: Index theorem and equivariant cohomology on the loop space. Commun. Math. Phys. 98, No. 2, 213–237 (1985).

[4] Bismut, J.-M.: *The hypoelliptic Laplacian on the cotangent bundle.* J. Am. Math. Soc. 18, No. 2, 379–476 (2005).

[5] Cacciatori, S. & Güneysu, B.: Odd characteristic classes in entire cyclic homology and equivariant loop space homology. J. Noncommut. Geom. 15, No. 2, 615–642 (2021).

[6] Cheng, L.J. & Grong, G. & Thalmaier, A.: *Functional inequalities on path space of sub-Riemannian manifolds and applications.* Nonlinear Analysis 210 (2021), 112387.

[7] Driver B. K. & Röckner M.: *Construction of diffusions on path and loop spaces of compact Riemannian manifolds.* C. R. Acad. Sci. Paris Series I 315 (1992), 603–608.

[8] Güneysu, B. & Ludewig, M.: The Chern character of ϑ-summable Fredholm modules over dg algebras and localization on loop space. Adv. Math. 395, Article ID 108143, 52 p. (2022).

[9] Haslhofer, R. & Naber, A.: *Ricci curvature and Bochner formulas for martingales.* Commun. Pure Appl. Math. 71, No. 6, 1074–1108 (2018).

[10] Haslhofer, R. & Kopfer, E. & Naber, A.: *Differential Harnack inequalities on path space.* Adv. Math. 410, Part A, Article ID 108714, 47 p. (2022).

[11] Honda, S.: *Ricci curvature and orientability.* Calc. Var. Partial Differ. Equ. 56, No. 6, Paper No. 174, 47 p. (2017).

[12] Jeffrey, L. C. & Mracek, J. A.: *Hyperfunctions, the Duistermaat–Heckman Theorem and Loop Groups'*, in Andrew Dancer, Jørgen Ellegaard Andersen, and Oscar García-Prada (eds), Geometry and Physics: Volume I: A Festschrift in honour of Nigel Hitchin (Oxford, 2018; online edn, Oxford Academic, 20 Dec. 2018).

[13] Kopfer, E. & Streets, J.: *Bochner formulas, functional inequalities and generalized Ricci flow.* J. Funct. Anal. 284, No. 10, Article ID 109901, 42 p. (2023).

[14] Landweber, G.: *Dirac operators on loop space PhD thesis*, Harvard 1999.

[15] Lott, J. & Villani, C.: *Ricci curvature for metric-measure spaces via optimal transport.* Ann. Math. (2) 169, No. 3, 903–991 (2009).

[16] Ludewig, M: *The Clifford algebra bundle on loop space.* SIGMA, Symmetry Integrability Geom. Methods Appl. 20, Paper 020, 27 p. (2024).

[17] Ludewig, M.: *The spinor bundle on loop space.* arXiv:2305.12521.

[18] Miehe, J.: *The Chern character of ϑ-summable C_q-Fredholm modules over locally convex differential graded algebras.* ArXiv:2312.01106.

[19] Naber, A.: *Characterizations of Bounded Ricci Curvature on Smooth and NonSmooth Spaces.* ArXiv:1306.6512.

[20] Sturm, K.-T.: *On the geometry of metric measure spaces. I.* Acta Math. 196, No. 1, 65-131 (2006); corrigendum ibid. 231, No. 2, 387–390 (2023).

[21] Sturm, K.-T.: *On the geometry of metric measure spaces. II.* Acta Math. 196, No. 1, 133–177 (2006).

[22] Witten, E.: Supersymmetry and Morse theory. J. Differ. Geom. 17, 661–692 (1982).

[23] Witten, E: *The Index Of The Dirac Operator In Loop Space* Elliptic Curves and Modular Forms in Algebraic Topology, Lecture Notes in Mathematics 1326, Springer (1988) 161–181.

Mini-Workshop: Recent Results on Loop Spaces

Table of Contents

Abstracts

Forms on families of supermanifolds: cohomology and duality

Simone Noja

In this talk, I explore the geometry of differential and integral forms on families of supermanifolds. I begin by introducing the concepts of differential and integral forms, along with their associated cohomology, on a fixed supermanifold. I then extend this framework to families of supermanifolds, understood as submersive maps between supermanifolds, highlighting the subtleties and new structures that arise in this setting. After defining an appropriate version of de Rham and Spencer cohomology for families of supermanifolds, I discuss a supergeometric analog of Poincaré-Verdier duality.

References

[1] S.L. Cacciatori, S. Noja, R. Re, *The Universal de Rham / Spencer Double Complex on a Supermanifold*, Doc. Math. **27**, 489–518 (2022)
[2] R. Catenacci, P.A. Grassi, S. Noja, *Superstring Field Theory, Superforms and Supergeometry*, J. Geom. Phys., **148** (2020) 103559
[3] P. Deligne *et al.*, *Quantum Field Theory and Strings: a Course for Mathematicians*, Vol 1, AMS (1999)
[4] K. Eder, *Super Fiber Bundles, Connection Forms and Parallel Transport*, J. Math. Phys. **62** (2021) 063506
[5] K. Eder, J. Huerta, S. Noja, *Poincaré Duality for Supermanifolds, Higher Cartan Structures and Geometric Supergravity*, arXiv:2312.05224v3 (2025)
[6] D. Hernández Ruipérez, J. Muñoz Masqué, *Construction Intrinsèque du faisceau de Berezin d'une varieté graduée*, C. R. Acad. Sc. Paris, **301**, 915-918 (1985)
[7] E. Kessler, *Supergeometry, Super Riemann Surfaces and the Superconformal Action Principle*, Springer - Lecture Notes in Mathematics (2019)
[8] Yu. I. Manin, *Gauge Fields and Complex Geometry*, Springer-Verlag, (1988)
[9] S. Noja, R. Re, *A Note on Super Koszul Complex and the Berezinian*, Ann. Mat. Pura Appl. **201**, 403–421 (2022)
[10] S. Noja, *On the Geometry of Forms on Supermanifolds*, Diff. Geom. Appl. **88** 101999 (2023)
[11] O.V. Ogievetskii, I.B. Penkov, *Serre Duality for Projective Supermanifolds*, Funct. Anal. its Appl. **18** 68–70 (1984)
[12] I. B. Penkov, *$\mathscr{D}$-Modules on Supermanifolds*, Invent. Math. **71**, 501-512, (1983)
[13] A. Polishchuk, *De Rham Cohomology for Supervarieties*, Eur. J. Math. **10**, 24 (2024)

Fermionic Dyson expansions and stochastic Duistermaat-Heckman localization on loop spaces

Jonas Miehe
(joint work with Batu Güneysu)

Given a self-adjoint operator $H \geq 0$ and (appropriate) densely defined and closed operators $P_1, \ldots, P_n$ in a Hilbert space $\mathscr{H}$, we provide a systematic study of bounded operators given by iterated integrals

(1)
$$\int_{\{0\leq s_1\leq\cdots\leq s_n\leq t\}} \mathrm{e}^{-s_1H} P_1 \mathrm{e}^{-(s_2-s_1)H} P_2 \cdots \mathrm{e}^{-(s_n-s_{n-1})H} P_n \mathrm{e}^{-(t-s_n)H}\, \mathrm{d}s_1 \ldots \mathrm{d}s_n.$$

for $t > 0$. These operators arise naturally in noncommutative geometry and the geometry of loop spaces. Using Fermionic calculus, we give a natural construction of an enlarged Hilbert space $\mathscr{H}^{(n)}$ and an analytic semigroup $\mathrm{e}^{-t(H^{(n)}+P^{(n)})}$ thereon, such that $\mathrm{e}^{-t(H^{(n)}+P^{(n)})}$ composed from the left with (essentially) a Fermionic integration gives precisely the iterated operator integral (1). This formula allows to establish important regularity results for the latter, and to derive a stochastic representation for it, in case H is a covariant Laplacian and the P_j's are first-order differential operators. Finally, with H given as the square of the Dirac operator on a spin manifold, this representation is used to derive a stochastic refinement of the Duistermaat-Heckman localization formula on the loop space of a spin manifold.

References

[1] Güneysu, B. & Miehe, J. Fermionic Dyson expansions and stochastic Duistermaat-Heckman localization on loop spaces. (2024), https://arxiv.org/abs/2410.14034

[2] Boldt, S., Güneysu, B. Feynman–Kac formula for perturbations of order ≤ 1, and noncommutative geometry. Stoch. PDE: Anal. Comp. 11, 1519–1552 (2023).

[3] Güneysu, B. & Ludewig, M.: *The Chern character of ϑ-summable Fredholm modules over dg algebras and localization on loop space.* Adv. Math. 395, Article ID 108143, 52 p. (2022).

[4] Ludewig, M.: Strong short-time asymptotics and convolution approximation of the heat kernel. Ann. Global Anal. Geom. 55, No. 2, 371-394 (2019).

Real loci and the based loop group

Lisa Jeffrey

Let ΩG denote the based loop group for a compact simply connected Lie group G with maximal torus T. It is an infinite-dimensional symplectic manifold equipped with a Hamiltonian action of $T\times S^1$ (studied by Atiyah and Pressley [1]). The space ΩG is also equipped with an antisymplectic involution τ which is associated to an involutive automorphism σ of G, and which is compatible with the Hamiltonian action of $T \times S^1$.

Duistermaat [2] studied this situation for a symplectic manifold M equipped with a Hamiltonian group action and an antisymplectic involution τ, and showed that the image of the moment map of M is the same as the image of the moment map of the fixed point set of the involution. Hausmann, Holm and Puppe [3] also exhibited a degree-halving isomorphism between the cohomology groups of M and M^τ. We show (in joint work with Augustin-Liviu Mare [4]) that both Duistermaat's result and the result of Hausmann, Holm and Puppe extend to ΩG.

References

[1] M.F. Atiyah, A.N. Pressley, Convexity and loop groups, in: *Arithmetic and Geometry: Papers dedicated to I.R. Shafarevich on the occasion of his sixtieth birthday,* Vol. II: Geometry, Birkhäuser, Boston, 1983.
[2] J. J. Duistermaat, Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution. *Trans. Amer. Math. Soc.* **275** (1983), 417–429.
[3] J.-C. Hausmann, T. Holm, V. Puppe, Conjugation spaces. *Alg. Geom. Top.* **5** (2005), 923–954.
[4] L. Jeffrey, A.-L. Mare, Real loci of based loop groups. *Transformation Groups* vol. 15, no. 1, 2010, pp. 134–153

The stringor bundle and the spinor bundle on loop space

Matthias Ludewig
(joint work with Peter Kristel and Konrad Waldorf)

Motivated by Witten's construction of the index of the Dirac operator on loop space [1] and Stolz' conjecture that a compact string manifold of dimension at least 5 with ric > 0 should have vanishing Witten genus [6], it has been a long-standing open problem to characterize string manifolds via geometric structures on their loop space. It was conjectured by Stolz and Teichner (2005) that a compact spin manifold M of dimension $n \geq 5$ is string if and only if the loop space LM is spin, with the loop space spinor bundle admitting a certain geometric structure called the fusion product.

In the talk, we gave an overview of the ingredients going into the proof. To arrive at a reasonable formulation of the conjecture, one has to work in the setting of von Neumann algebra bundles, and the fusion product must be formulated in terms of the corresponding tensor product of bimodule bundles (the *Connes fusion product*). Such a framework was recently developed in joint work with Konrad Waldorf and Peter Kristel.

Recently, Kristel and Waldorf [4] showed that a fusive loop space spin structure on LM (which is equivalent to a string structure on M by earlier work of Waldorf [3]) gives rise to a spinor bundle on loop space together with a fusion product. On the other hand, the author showed in [7] that the fusion product's existence is then obstructed by a certain bundle 2-gerbe, the *fusion 2-gerbe*. It turns out that the characteristic 4-class of this gerbe is precisely $\frac{1}{2}p_1(M))$, which characterizes the existence of a string structure; this solves the conjecture.

Using the fusion product, one may construct a 2-Hilbert bundle on M, which we call the *stringer bundle* [2]. This uses a newly developed theory of 2-Hilbert bundles, which are higher vector bundles locally modeled on the bicategory of von

Neumann algebras, bimodules and intertwiners. In the finite-dimensional setting, this was carried out in [5].

References

[1] Witten, E. The index of the Dirac operator in loop space. *Elliptic Curves And Modular Forms In Algebraic Topology (Princeton, NJ, 1986).* **1326** pp. 161–181 (1988)

[2] Kristel, P., Ludewig, M. & Waldorf, K. The stringor bundle. (2024), https://arxiv.org/abs/2206.09797

[3] Waldorf, K. String geometry vs. spin geometry on loop spaces. *J. Geom. Phys..* **97** pp. 190–226 (2015)

[4] Kristel, P. & Waldorf, K. Connes fusion of spinors on loop space. *Compositio Mathematica.* **160**, 1596–1650 (2024)

[5] Kristel, P., Ludewig, M. & Waldorf, K. 2-vector bundles. (2022), https://arxiv.org/abs/2106.12198

[6] Stolz, S. A conjecture concerning positive Ricci curvature and the Witten genus. *Mathematische Annalen.* **304** pp. 785–800 (1996)

[7] Ludewig, M. The spinor bundle on loop space. (2024), https://arxiv.org/abs/2305.12521

[8] Stolz, S. & Teichner, P. The spinor bundle on loop space., https://people.mpim-bonn.mpg.de/teichner/Math/ewExternalFiles/MPI.pdf

Homotopy theory of Polyhedral Products

Jelena Grbic

Having a family of objects with rich and complex structures that contain non-trivial features, yet are still amenable to study or computation, is a goal for many mathematicians. To construct such topological spaces with intrinsic symmetries – spaces that allow for the identification of various topological invariants in combinatorial and algebraic terms, while simultaneously addressing problems in tangential areas of mathematics and mathematical physics - polyhedral products are introduced. For a simplicial complex K on $[m]$ vertices and m pointed CW pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}$, the polyhedral product $(\underline{X}, \underline{A})^K$ is constructed as the colimit over the face category of a simplicial complex K

$$(\underline{X}, \underline{A})^K = \mathrm{colim}_{\sigma \in K} (\underline{X}, \underline{A})^\sigma$$

where $(\underline{X}, \underline{A})^\sigma = \{(x_1, \dots, x_m) \in \prod_i X_i \mid x_i \in A_i i \notin \sigma\}$.

In this talk, we focus on the homotopy theory of moment-angle complexes $\mathcal{Z}_K = (D^2, S^1)^K$ and Davis-Januszkiewicz spaces $DJ_K = (\mathbb{C}P^\infty, *)^K$. These spaces can be seen as topological analogues of smooth projective toric varieties and to them associated spaces. To highlight the significance of these spaces in mathematics, we demonstrate they are related to various concepts, including the complement of coordinate subspace arrangements, quasi-toric manifolds, complex cobordisms and intersections of quadrics in complex geometry.

First, we identify the homology of the loop space ΩDJ_K as $H * (\Omega DJ_K, k) \cong \mathrm{Tor}_{k[K]}(k, k)$, the homology of the Stanley-Reisner ring $k[K]$ associated with the simplicial complex K. This identification allows us to approach the Kaplansky-Serre conjecture in the case of $k[K]$. The conjecture suggests that the Poincaré series of $\mathrm{Tor}_{k[K]}(k, k)$ is a rational function. It turns out that this series is a particular rational function when $k[K]$ is a Golod ring, or equivalently, when all cup products and higher Massey products in $H^*(\mathcal{Z}_K; k)$ vanish for every field k. This motivates the study of Golod rings in homotopy theoretical terms. We establish new methods for calculating Massey products in terms of the combinatorics of K.

Second, we briefly summarise the study of the loop space $\Omega\mathcal{Z}_K$ by modelling it in terms of configuration spaces with labels in $(D^1, S^0)^K$. The results of this approach are applied to describe a new family of simplicial complexes K for which $k[K]$ are Golod rings.

References

[1] *V. M. Buchstaber* and *T. E. Panov*, Toric topology. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society (AMS) (2015)

[2] *P. Beben* and *J. Grbić*, Configuration spaces and polyhedral products. Adv. Math. 314, 378–425 (2017)

[3] *J. Grbić* and *A. Linton*, Non-trivial higher Massey products in moment-angle complexes. Adv. Math. 387, Article ID 107837, 52 p. (2021)

[4] *J. Grbić* et al., The homotopy types of moment-angle complexes for flag complexes. Trans. Am. Math. Soc. 368, No. 9, 6663–6682 (2016)

The Chern-Simons functional integral, Kauffman's bracket polynomial, and deformation quantization

Jonathan Weitsman

We study Chern-Simons Gauge Theory in axial gauge on $\mathbb{R}^3$. This theory has a quadratic Lagrangian and therefore expectations can be computed nonperturbatively by explicit formulas, giving an (unbounded) linear functional on a space of polynomial functions in the gauge fields, as a mathematically well-defined avatar of the formal functional integral. We use differential-geometric methods to extend the definition of this linear functional to expectations of products of Wilson loops corresponding to oriented links in $\mathbb{R}^3$, and derive skein relations for them. In the case $G = SU(2)$ we show that these skein relations are closely related to those of the Kauffman bracket polynomial, which is closely related to the Jones polynomial. We also study the case of groups of higher rank. We note that in the absence of a cubic term in the action, there is no quantization condition on the coupling λ, which can be any complex number. This is in line with the fact that the Jones polynomial, in contrast to the manifold invariants of Witten and Reshetikhin-Turaev, is defined for any value of the coupling. The appearance of the parameter $e^{\frac{1}{2\lambda}}$ in the expectations and skein relations is also natural. Likewise, the extension of the theory to noncompact groups presents no difficulties. Finally we show how computations similar to ours, but for gauge fields in two dimensions, yield the Goldman bracket.

References

[1] J. Weitsman, The Chern-Simons Functional Integral, Kauffman's Bracket Polynomial, and other link invariants, Preprint, arXiv:2405.16775 [math.DG] (2024)

A Chern-Simons transgression formula for supersymmetric path integrals on spin manifolds

Sebastian Boldt
(joint work with Sergio Cacciatori, Batu Güneysu)

Earlier results show that the $N = 1/2$ supersymmetric path integral $\mathfrak{J}^g$ on a closed even dimensional Riemannian spin manifold (X, g) can be constructed in a mathematically rigorous way via Chen differential forms and techniques from noncommutative geometry, if one considers $\mathfrak{J}^g$ as a current on the loop space LX, that is, as a linear form on differential forms on LX. This construction admits a Duistermaat-Heckman localization formula. In this note, fixing a topological spin structure on X, we prove that any smooth family $g_\bullet = (g_t)_{t\in[0,1]}$ of Riemannian metrics on X canonically induces a Chern-Simons current $\mathfrak{C}^{g_\bullet}$ which fits into a transgression formula for the supersymmetric path integral. In particular, this result entails that the supersymmetric path integral induces a differential topological invariant on X, which essentially stems from the $\hat{A}$-genus of X.

References

[1] L. Álvarez-Gaumé: Supersymmetry and the Atiyah-Singer index theorem. Comm. Math. Phys. 90 (1983), no. 2, 161–173.
[2] M. Atiyah: Circular symmetry and stationary-phase approximation. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). Astérisque No. 131 (1985), 43–59.
[3] J.-M.Bismut: Index theorem and equivariant cohomology on the loop space. Comm. Math. Phys. 98 (1985), no. 2, 213–237.
[4] J.-P. Bourguignon & P. Gauduchon: Spineurs, opérateurs de Dirac et variations de métriques. Comm. Math. Phys. 144 (1992), no. 3, 581–599.
[5] S. Boldt & B. Güneysu: Feynman-Kac formula for perturbations of order ≤ 1 and noncommutative geometry. To appear in Stochastics and Partial Differential Equations: Analysis and Computations, 2022.
[6] S. Cacciatori & B. Güneysu: Odd characteristic classes in entire cyclic homology and equivariant loop space homology. J. Noncommut. Geom. 15 (2021), no. 2, 615–642.
[7] B. Güneysu & M. Ludewig: The Chern Character of θ-summable Fredholm Modules over dg Algebras and Localization on Loop Space. Adv. Math. 395 (2022).
[8] B. Güneysu & S. Pigola: The Calderón-Zygmund inequality and Sobolev spaces on noncompact Riemannian manifolds. Adv. Math. 281 (2015), 353–393.
[9] F. Hanisch & M. Ludewig: The Fermionic integral on loop space and the Pfaffian line bundle, preprint 2021, arXiv:1709.10028.
[10] F. Hanisch & M. Ludewig: A rigorous construction of the supersymmetric path integral associated to a compact spin manifold. Comm. Math. Phys. 391 (2022), no. 3, 1209–1239.
[11] H. Hess & R. Schrader & D.A. Uhlenbrock: Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifolds. J. Differential Geometry 15 (1980), no. 1, 27–37 (1981).
[12] A. Kriegl & Peter W. Michor: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs. 53. Providence, RI: American Mathematical Society (1997).
[13] Free loop spaces in geometry and topology. Including the monograph Symplectic cohomology and Viterbo's theorem by Mohammed Abouzaid. Edited by Janko Latschev and Alexandru Oancea. IRMA Lectures in Mathematics and Theoretical Physics, 24. European Mathematical Society (EMS), Zürich, 2015.
[14] D. B. Ray & I. M. Singer: R-torsion and the Laplacian on Riemannian manifolds. Advances in Math. 7 (1971), 145–210.

[15] R. T. Seeley: Complex powers of an elliptic operator. 1967 Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966) pp. 288–307 Amer. Math. Soc., Providence, R.I.
[16] P. Topping: Lectures on the Ricci flow. London Mathematical Society Lecture Note Series, 325. Cambridge University Press, Cambridge, 2006.
[17] K. Waldorf: Transgression to loop spaces and its inverse, II: Gerbes and fusion bundles with connection. Asian J. Math. 20 (2016), no. 1, 59–115.

Analytic torsion and Anosov flow

Shu Shen
(joint work with Jean-Michel Bismut)

Let X be a closed manifold, and let F be a flat vector bundle on X. Denote by $H(X,F)$ the cohomology of F.

Given metrics g^{TX} and g^F on TX and F, one can define the analytic torsion T_F, a positive real number, using the regularized determinant of the associated Hodge Laplacian. Under certain assumptions, such as g^F being flat and $H(X,F)=0$, the analytic torsion T_F is independent of g^{TX} and of the flat g^F, becoming a topological invariant of F.

On the other hand, given a dynamical flow $(\phi_t)_{t\in\mathbf{R}}$ on X, one can define formally a dynamical zeta function $R_F(\sigma)$, using the closed orbits of the flow. Under suitable condition, such as $(\phi_t)_{t\in\mathbf{R}}$ being Anosov without 0-resonance, the value $R_F(0)$ is a well-defined non zero complex number.

Conjecture 1 (Fried [4]). *Under certain conditions, we have*

$$T_F = |R_F(0)|.$$

It is classical that for general flat vector bundle F, the Ray-Singer metric $\|\cdot\|^{\mathrm{RS}}$ on $\det H(X,F)$ is a natural generalization of the analytic torsion T_F. This leads to the following questions:

(1) What is the dynamical counterpart of the Ray-Singer metric ?
(2) How can one formulate a generalized version of Fried's conjecture ?
(3) How can the generalized Fried conjecture be established ?

The main result of [1] provides answers to the first two questions.

Theorem 1. *If $(\phi_t)_{t\in\mathbf{R}}$ is Anosov, there exists a canonical non zero element $\tau\in\det H(X,F)$ such that*

(1) τ generalises the $(R_F(0))^{-1}$ for Anosov flow without 0-resonance,
(2) τ is compatible with the Poincaré duality,
(3) In a family setting, τ is flat with respect to the Gauss-Manin connection.

This result extends the previous works of Dang–Guillarmou–Rivière–Shen [3] and Chaubet–Dang [2]. With the element τ in hand, we can formulate a generalized version of Fried's conjecture.

Conjecture 2 (Generalized Fried). *If $(\phi_t)_{t\in\mathbf{R}}$ is Anosov and if F is unimodular, then*

$$\|\tau\|^{\mathrm{RS}} = 1.$$

References

[1] J.-M. Bismut and S. Shen. Anosov vector fields and Fried sections. *arXiv e-prints*, page arXiv:2405.14583, May 2024.

[2] Y. Chaubet and N. V. Dang. Dynamical torsion for contact Anosov flows. *Anal. PDE*, 17(8):2619–2681, 2024.

[3] N. V. Dang, C. Guillarmou, G. Rivière, and S. Shen. The Fried conjecture in small dimensions. *Invent. Math.*, 220(2):525–579, 2020.

[4] D. Fried, *Lefschetz formulas for flows*, The Lefschetz centennial conference, Part III (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1987, pp. 19–69.

Brownian loops and Riemann's zeta function

Anton Thalmaier

We discuss various probabilistic representations of the Riemann ξ-function where $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ for $s \in \mathbb{C}$. Here $\zeta(s)$ denotes the meromorphic extension of the classical Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \quad \operatorname{Re} s > 1.$$

The ξ-function is analytic in the whole complex plane. In this talk we explain how to represent $\xi(s)$ as an expectation in terms of the maximal diameter R of Brownian loops in $\mathbb{R}^3$. Itô's excursion theory of Brownian motion on the real line then gives an equivalent representation in terms of the variable $T = T_1 + T_2$ where T_i are the first hitting times of height 1 of two independent BES(3) processes, starting at 0 at time 0. It is shown that the functional equation $\xi(s) = \xi(1-s)$ is a direct consequence of Itô's excursion theory. The Ray-Knight theorem for the BES(3) process allows to represent the variable $T = T_1 + T_2$ in law as a quadratic form of an independent sequence of standard $N(0,1)$ Gaussians variables $(G_i)_{i\in\mathbb{N}}$. Connections to ferromagnetic spin systems are noted.

References

[1] Philippe Biane, Jim Pitman, and Marc Yor, *Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 4, 435–465. MR 1848256

[2] Philippe Biane, *La fonction zêta de Riemann et les probabilités*, La fonction zêta, Ed. Éc. Polytech., Palaiseau, 2003, pp. 165–193. MR 1989224

Heat kernel analysis on Alexandrov-Dynkin surfaces

Maxime Marot
(joint work with Batu Güneysu)

We say that (S, d) is an *Alexandrov surface (with locally bounded integral curvature)* if S is a 2-dimensional smooth manifold, d in an intrinsic finite distance and for any compact K there exists a constant $C \in \mathbb{R}$, depending only on K, such that the sum of angle excesses of any family of non-overlapping triangles contained in K is bounded above by C. Let h be a Riemannian metric on S. We define $\mathcal{V}(S, h)$ the set of functions $u \in L^1_{loc}(S)$ such that $\Delta_h u$ is a signed Radon measure. Due to the theorem of Reshetnyak-Huber it is possible to see d as a *subharmonic distance*, that is to say, there exists h a Riemannian metric and $u \in \mathcal{V}(S, h)$ such that

$$d(x, y) = d_{h,u}(x, y) = \inf_{\gamma} \int_0^1 e^{u(\gamma(t))} |\gamma'(t)|_h \, dt.$$

From now on we set (S, d, μ) to be an Alexandrov surface equipped with the 2-dimensional Hausdorff measure μ. The Sobolev spaces are to be understood in the sens of Ambrosio-Gigli-Savaré's theory. We also write $\mathcal{E}$ the Cheeger form and $-\Delta$ for the non-negative self-adjoint operator associated. Our first result is this theorem:

Theorem 1. *The following statements hold:*

(i) The intrinsic distance associated to $\mathcal{E}$ coincide with d, i.e. $d_{\mathcal{E}} = d$ where

$$d_{\mathcal{E}}(x, y) := \sup\{f(y) - f(x) \mid f \in W^{1,2}_{loc}(S, d, \mu) \cap C(S) \text{ s.t. } |Df|^2 \leq 1 \ \mu\text{-a.e.}\}.$$

Let ε, λ be two positive constants.

(ii) $\mathcal{E}$ supports a local weak $(1, 2)$-Poincaré inequality i.e. for any compact K there exist $C, R_P, k > 0$, depending only on K, ε and λ such that

$$\int_{B(z,r)} |u - u_B|^2 \, d\mu \leq Cr^2 \int_{B(z,kr)} |Du|^2 \, d\mu$$

holds for any $u \in W^{1,2}(S, d, \mu)$, $z \in K$, $r < R_P$ and with $u_B := \fint_{B(z,r)} u \, d\mu$.

(iii) μ is locally doubling i.e. for any compact K there exists $D, R_D > 0$, depending only on K, ε and λ, such that

$$\mu(B(z, 2r)) \leq D\mu(B(z, r))$$

for any $z \in K$, $0 < r < R_D$.

Thus with these three ingredients there exists a parabolic Harnack inequality and so then there exists a pointwise uniquely defined heat kernel. This heat kernel satisfies the usual properties of symmetry, semigroup identity and is Hölder continuous. It is also positive. Directly from the existence of the heat kernel we can deduce the existence of a family of Wiener measures on $C([0, \infty), S)$ [2, Lemma 3.5].

The Dynkin class $\mathrm{Dyn}(S)$ is defined to be the space of Radon measures $\mathfrak{m} \geq 0$ with $\mathfrak{m}(N) = 0$ for all Borel $N \subset S$ with $\mathrm{Cap}_{\mathcal{E}}(N) = 0$, such that for all $\lambda > 0$ one has

$$\sup \left\{ \left| \int (\Delta + \lambda)^{-1} f \, d\mathfrak{m} \right| : f \in \mathcal{C}_c^\infty(S), \int |f| \, d\mu \leq 1 \right\} < \infty.$$

Moreover, an Alexandrov surface (S, d) is called an *Alexandrov-Dynkin surface* if the curvature measure ω admits a decomposition $\omega = \omega_1 - \omega_2$, $\omega_i \geq 0$, with $\omega_2 \in \mathrm{Dyn}(S)$. By deforming conformally an Alexandrov-Dynkin surface we can obtain a new one with curvature bounded below.

Theorem 2. *Assume that (S, d) is an Alexandrov-Dynkin surface. Pick a Riemannian metric h on S and $u \in \mathcal{V}(S, h)$ with $d = d_{h,u}$. Then there exists a non-negative function $\Phi \in L^\infty(S) \cap \mathcal{V}(S, h)$, such that $(S, d_{h,u+\Phi})$ has curvature measure bounded below by a constant $\kappa \in \mathbb{R}$ i.e. $\omega \geq \kappa\mu$.*

It is well known that any $\mathrm{CBB}(\kappa)$ surface is $\mathrm{CD}(2, \kappa)$, so from the bilipschitz equivalence we obtain Gaussian bounds for the heat kernel.

References

[1] F. Fillastre, D. Slutskiy, eds., Reshetnyak's Theory of Subharmonic Metrics, Springer International Publishing, Cham, 2023.

[2] B. Güneysu, $RCD^\star(K, N)$ spaces and the geometry of multi-particle Schrödinger semigroups., Int. Math. Res. Not. 2022, No. 4, 3144–3169 (2022).

[3] K.-T. Sturm, Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and Lp-Liouville properties., Journal Für Die Reine Und Angewandte Mathematik (Crelles Journal) 1994 (1994) 173–196.

[4] K.-T. Sturm, Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations, Osaka Journal of Mathematics 32 (1995) 275–312.

[5] K.T. Sturm, Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality, J. Math. Pures Appl. (9) 75 (1996) 273–297.

[6] A. Petrunin, Alexandrov meets Lott-Villani-Sturm, Münster J. of Math. 4 (2011), 53–64.

Smooth Unitary Representation of every Pre-Representation $\mathbb{Z}_2^n$-Lie Supergroup

Fatemeh Nikzad Pasikhani

(joint work with Mohammad Mohammadi and Saad Varsaie)

In this talk, we show that any pre-representation of a $\mathbb{Z}_2^n$-Lie Supergroup can be uniquely extended to a smooth unitary representation. By a $\mathbb{Z}_2^n$-Lie Supergroup, we refer to a Harish-Chandra pair $(G_0, g_{\mathbb{C}})$ where G_0 is a common finite-dimensional Lie Group and $g_{\mathbb{C}}$ is a $\mathbb{Z}_2^n$-graded Lie superalgebra such that there exists an action $\mathrm{Ad} : G_0 \times g_{\mathbb{C}} \to g_{\mathbb{C}}$ preserves the $\mathbb{Z}_2^n$-grading and $\mathrm{Ad}|_{g_0} : G_0 \times g_0 \to g_0$ is the adjoint action of G_0 on $g_0 \cong \mathrm{Lie}(G_0)$.

References

[1] T. Covolo,J. Grabowski and N. Poncin, *The category of* $\mathbb{Z}_2^n$*-supermanifolds*, J. Math. Phys. 57 (2016), no. 7, 073503, 16 pp.

[2] M. Mohammadi and H. Salmasian, *The Gelfand-Naimark-Segal construction for unitary representatins of* $\mathbb{Z}_2^n$*-graded Lie supergroups*, Banach Center Publications., 113 (2017), 263–274.

[3] K.H. Neeb and H. Salmasian, *Differentiable vectors and unitary representations of Frechet-Lie supergroups*, Math. Z., 275(1-2): 419–451, 2013.

[4] Nikzad Pasikhani, Fatemeh, Mohammad Mohammadi, and Saad Varsaie, *Stability Theorem for* $\mathbb{Z}_2^n$*-Lie Supergroups*, Workshop on Geometric Methods in Physics. Cham: Springer Nature Switzerland, 2022.

[5] Pasikhani, Fatemeh Nikzad, Mohammad Mohammadi, and Saad Varsaie, *Extension of local positive definite Z2n-superfunctions*, Journal of Mathematical Analysis and Applications 529.1 (2024): 127554.

[6] M. Stephane, K.H. Neeb and H. Salmasian, *Categories of unitary representations of Banach—Lie supergroups and restriction functors*, Pacific Journal of Mathematics 257.2 (2012): 431–469.

Reporter: Maxime Marot

Participants

Dr. Sebastian Boldt
Fakultät für Mathematik
TU Chemnitz
Postfach 964
09009 Chemnitz
GERMANY

Prof. Dr. Sergio Cacciatori
Dip. di Scienza ed Alta Tecnologia
Univ. dell'Insubria
Via Valleggio 11
22100 Como 22100
ITALY

Prof. Dr. Jelena Grbic
School of Mathematics
University of Southampton
Southampton SO17 1BJ
UNITED KINGDOM

Prof. Dr. Batu Güneysu
Fakultät für Mathematik
Technische Universität Chemnitz
09126 Chemnitz
GERMANY

Prof. Dr. Lisa Claire Jeffrey
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Dr. Eva Kopfer
Institut für angewandte Mathematik
(IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Matthias Ludewig
Institut für Mathematik und Informatik
Universität Greifswald
Walther-Rathenau-Straße 47
17489 Greifswald
GERMANY

Maxime Marot
Fakultät für Mathematik
TU Chemnitz
Postfach 964
09009 Chemnitz
GERMANY

Dr. Jonas Miehe
Fakultät für Mathematik
TU Chemnitz
Postfach 964
09009 Chemnitz
GERMANY

Dr. Fatemeh Nikzad Pasikhani
Department of Mathematics
Institute for Advanced Studies in Basic
Sciences (IASBS)
Zanjan 45137-66731
IRAN, ISLAMIC REPUBLIC OF

Dr. Simone Noja
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Shu Shen
Institut de Mathématiques de Jussieu -
Paris Rive Gauche
Sorbonne Université
Case Courrier 247
4 place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Anton Thalmaier
Department of Mathematics
University of Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Prof. Dr. Jonathan Weitsman
Northeastern University
Department of Mathematics
567 Lake Hall
360 Huntington Ave.
Boston MA 02115-5000
UNITED STATES