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MATRIX-MFO Tandem Workshop: Nonlinear Geometric Diffusion Equations

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Abstract. This tandem workshop with MATRIX in Creswick, Australia, brought together leading experts from the fields of geometric partial differential equations and geometric analysis in general. The focus of the workshop was on recent developments and directions in non-linear geometric diffusion equations. The main flows considered were mean curvature flow, inverse mean curvature flow, Ricci flow, Willmore flow, as well as related flows. A number of the results and methods were in the setting of general relativity, where flows have been very successful in helping solve major problems (for example the resolution of the Penrose conjecture by Huisken/Ilmanen using the inverse mean curvature flow). For four days of the workshop there were combined (with MATRIX) morning talks and extensive discussions.

Mathematics Subject Classification (2020): 53E10, 53E20, 53E40.

Introduction by the Organizers

This tandem workshop jointly organised with MATRIX in Creswick, Australia, brought together leading experts on geometric partial differential equations and on geometric analysis in general. Two full conferences were held, one at MFO and one at MATRIX, accompanied by joint online streamed sessions on four of the days. The joint sessions consisted of a talk (50 min.) followed by an extended Q&A session (45 min.) and further in depth discussion. The joint online talks and conversations were particularly well received by the participants of MATRIX and Oberwolfach. One could follow the talks well online and one saw not only the talk but had the chance to see/interact with colleagues as well as the speaker at the other end during the conversation time. This lead to further interaction outside the talks, which one hopes will have lead to progress in current joint projects, or helped defining and working on new projects. The slight prior uncertainty of whether the online interaction would be engaging enough quickly proved unjustified and we had a very inspiring time across the hemispheres. In particular, the time plan was very much appreciated and praised by the participants. There were enough online talks, to spark interaction and exchange between the continents, but not too much, so that direct interaction did not suffer and enough time was kept free for working directly with one another on projects. A general feeling among participants seems to be that it was a good decision not to record the offline lectures. The danger of recording talks is that participants at the other end would feel obliged to try and watch the online talks, thus detracting from the time needed to work with others on projects: It would have been an overload. Thematically, the scientific part of the MFO talks can roughly be grouped into a few themes.

- Inverse mean curvature flow, flows related to mathematical relativity, capacity (Cabezas-Rivas, Fogagnolo, Pluda, Wolff)
- Intrinsic flows, like Ricci- and Yamabe flow (Chen, Reiris, Wang)
- Mean curvature flow (Buzano, Klingenberg, Litzinger, Lynch, Mramor, Stolarski, Vogiatzi)
- Higher-order problems (Lamm, Mäder-Baumdicker, Rupp).

The theme complex around inverse mean curvature flow (IMCF) provided a good overview over very recent exciting developments in this field. Esther Cabezas-Rivas spoke about her work on how to define a suitable notion of weak IMCF of crystalline structures related to Wulff shapes with respect to a norm in a quite weak regularity regime. The talks by Mattia Fogagnolo and Alessandra Pluda focused on a recent series of groundbreaking results by various groups/researchers on level set approximations of IMCF by p-harmonic functions. The idea is, very roughly, that the p-parameter family of p-harmonic functions somewhat interpolate between the case $p = 1$, which resembles IMCF, and $p = \infty$, which resembles the equidistance flow. This relatively new development has the potential to deepen the ties between geometric evolution problems and pure PDEs even more firmly. The talk by Markus Wolff focused on a whole new family of flows of mean curvature type, which live in degenerate ambient spaces, i.e. null hypersurfaces. Exciting relations to the Yamabe flows were discussed.

In the intrinsic flow section, we saw two Ricci flow talks. Eric Chen talked on recent work with Richard Bamler, where a degree theory for solutions to Ricci flows coming out of cones is developed. This theory is used to show existence of a Ricci flow coming out of cones with non-negative scalar curvature, thus generalising results of Deruelle, Schulze, Simon on flows coming out of non-negatively curved cones. An interesting insight was provided by Martín Reiris, who explained how the entropy quantity of Perelman can be obtained as a limit of Colding's reduced volume. Guofang Wang showed us that an isoperimetric inequality between the total Q-curvature and the total scalar curvature holds, and then explained how this leads to a new type of optimal Sobolev inequality. His tools were based on an intrinsic fully nonlinear Yamabe-type flow.

In the higher order section, we saw in a talk by Tobias Lamm, that earlier results by Simon, Koch/Lamm on flowing out of weak initial C^0 (or almost C^0 data) can be generalised to data with small BMO norms, and systems with small BMO norms. Elena Mäder-Baumdicker explained how one can obtain a monotonicity quantity for fourth order flows, analogous to the one found by Struwe for the harmonic meap heat flow. Fabian Rupp introduced us to a *conformal class preserving* gradient flow for the Willmore functional and showed us how it can be used to find new examples of conformally constrained Willmore tori. An interesting connection to an elastic energy gradient flow in hyperbolic space was involved in his arguments.

The mean curvature flow section focused on many aspects of the field. Maxwell Stolarski showed us that singularities of mean curvature flows with bounded curvature have some structure, thus complimenting his results, that in high dimensions, such singularities do occur. Alexander Mramor looked at mean curvature flow in three dimensions, and showed how new results and insights can be used to understand possible singularity formulation, and construct minimal surfaces. Stephen Lynch introduced a new generalisation of CMC hypersurfaces to higher co-dimensions and used it to give a simple, globally defined, canonical description of bubblesheet regions. Staying in higher co-dimension, Florian Litzinger showed us that the curve shortening flow of a curve becomes planar, and round (after scaling) if one assumes a bound on a suitably defined entropy of the initial curve. In the talk of Reto Buzano the existence of mean curvature flow non-compact self-shrinkers was shown using mini-max methods, and we saw that these have one asymptotically conical end in the case of large genus. Wilhelm Klingenberg showed us that space-like graphical rotationally symmetric line congruence evolving under mean curvature flow with respect to the neutral Kähler metric in the space of oriented lines of the Euclidean 3-space, namely in TS^2, subject to suitable Dirichlet and Neumann boundary conditions, converges to a holomorphic disc. Artemis Vogiatzi considered mean curvature flow of sub-manifolds in spheres. She showed that if a certain pinching condition depending on the dimension and co-dimension is initially satisfied, that then the flow will converge to a totally geodesic limit.

Acknowledgements: We thank the participant Nikita Cernomazov for taking care of the technical side, in particular for setting up the Zoom meetings. The organizers would like to thank Florian Litzinger for his assistance in collecting the extended abstracts and putting this report together.

MATRIX-MFO Tandem Workshop: Nonlinear Geometric Diffusion Equations

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Abstracts

Noncompact self-shrinkers with arbitrary genus

Reto Buzano
(joint work with Huy The Nguyen, Mario Schulz)

A mean curvature flow starting from a closed embedded surface in Euclidean space $\mathbb{R}^3$ must necessarily form a singularity in finite time. By work of Huisken, Ilmanen, and White, such singularities are always modelled on surfaces that shrink self-similarly along the flow, so called self-shrinkers. Noncompact self-shrinkers model local singularities along the mean curvature flow, i. e. situations where the flow does not become extinct at the singular time. Therefore, a very natural objective in this setting is to understand and construct examples of complete embedded (two-dimensional) self-shrinkers.

The simplest self-shrinkers are flat planes, the sphere of radius 2 and cylinders of radius $\sqrt{2}$, all centred at the origin. In fact, Brendle [1] proved that there are no other embedded self-shrinkers of genus zero. Angenent constructed a rotationally symmetric, closed self-shrinker of genus one. Based on numerical simulations, Ilmanen [3] conjectured the existence of noncompact embedded self-shrinkers with dihedral symmetry and *arbitrary genus.* For high genus, these surfaces resemble the union of a sphere and a plane desingularised along the line of intersection. Kapouleas, Kleene and Møller [4] as well as X. Nguyen [7] were able to formalise the desingularisation procedure and prove the existence of such self-shrinkers if the genus is sufficiently large. However, the nature of the desingularisation method does not allow the construction of low genus examples. Our main result establishes the existence of self-shrinkers with arbitrary genus, resolving Ilmanen's conjecture.

Theorem 1 (Theorem 1.2 of [2])**.** *For each $g \in \mathbb{N}$ there exists a complete, embedded, noncompact self-shrinker $\Theta_g \subset \mathbb{R}^3$ for mean curvature flow which has genus g and is invariant under the action of the dihedral group $\mathbb{D}_{g+1}$.*

This theorem provides the first example of a *noncompact* self-shrinker of genus one as well as the very first example of a self-shrinker of genus two.

A self-shrinker Σ is a critical point for the *Gaußian area functional*

$$F(\Sigma) = \frac{1}{4\pi} \int_\Sigma e^{-\frac{1}{4}|x|^2} \, d\mathcal{H}^2(x) \tag{1}$$

where $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. As such, Σ is a minimal surface in the (conformally flat) Riemannian manifold $\big(\mathbb{R}^3, e^{-\frac{1}{4}|x|^2} g_{\mathbb{R}^3}\big)$ called *Gaußian space.* Since all self-shrinkers are unstable, a promising approach to prove our theorem is via min-max constructions in Gaußian space. Such a min-max theory was developed by Ketover and Zhou [6] based on the pioneering work by Almgren and Pitts, and later refinements by Simon–Smith and Colding–De Lellis. The equivariant version of min-max theory was introduced by Ketover [5].

In this talk, we explain how we pick an effective sweepout with the right symmetry and topology as well as good control on the Gaußian area (in fact, we can

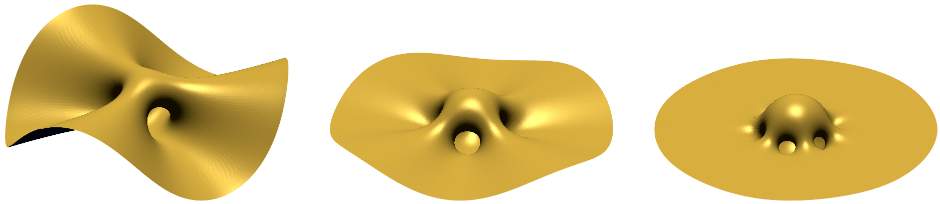

Figure 1. Numerical simulations of the self-shrinkers Θ_g for genus 2, 3, and 5.

show that the Gaußian area along our model sweepout always stays below 2.5). We also explain why every sweepout in the equivariant saturation obtained from this model sweepout satisfies the necessary mountain pass condition to apply the above mentioned min-max theory. A key difficulty is then that the convergence of a min-max sequence is only obtained in the sense of varifolds and additional work is required to control the topology of the limit surface. Using Brendle's classification result mentioned above, we can rule out that our limit self-shrinker has genus zero. Indeed, the dihedral symmetry forces the multiplicity of the convergence to be odd, and the Gaußian area bound of 2.5 rules out multiplicity $m \geq 3$. On the other hand, by the mountain pass condition, it is not possible to converge to a multiplicity one plane. It is also known that the genus cannot increase in the min-max limit, hence our self-shrinkers have genus between 1 and g. This is when the symmetry group comes to save us. Indeed, we can prove a quite general structure theorem for surfaces with dihedral symmetry group $\mathbb{D}_{g+1}$ and genus between 1 and g: they are either disjoint from the vertical symmetry axis or they must exactly have genus g. As our self-shrinkers contain the origin by construction, we are in the latter case.

In the second part of the talk, we are interested in the number of ends of our self-shrinkers Θ_g. Conjecturally they all have precisely one end (clearly by construction they have *at least* one asymptotically conical end), but again, the convergence in the min-max procedure is too weak to give any useful control. By analysing the behaviour for $g \to \infty$ we are able to rule out any additional ends for Θ_g if the genus g is sufficiently large. More precisely, we have the following result.

Theorem 2 (Theorem 1.3 of [2]). *The sequence of self-shrinkers $\{\Theta_g\}_{g\in\mathbb{N}}$ constructed in Theorem 1 converges to the union of the horizontal plane and the self-shrinking sphere in the sense of varifolds as $g \to \infty$. The convergence is locally smooth away from the intersection circle. In particular, if g is sufficiently large, then Θ_g has exactly one asymptotically conical end.*

Let us briefly sketch the main steps in the proof of this result. First, due to the Gaußian area bound $F(\Theta_g) \leq 2.5$ we can apply Allard's compactness theorem to extract a subsequence converging in the varifold sense to a stationary integer varifold Θ_∞. It is easy to see that this varifold must contain the horizontal plane P and be rotationally symmetric. By standard regularity theory, the convergence is smooth unless the second fundamental form A blows up in a suitable sense. This last condition is equivalent to the genus going to infinity near points of non-smooth convergence by Ilmanen's localised Gauß-Bonnet estimate for self-shrinkers. All of this lets us conclude that the convergence is smooth away from a set of circles where the genus concentrates.

We then employ again our structure theorem for surfaces with dihedral symmetry group mentioned above to conclude that there is in fact only one such circle contained in the plane P where *all* the genus concentrates. Moreover, this circle must have positive diameter and be the intersection of precisely two smooth self-shrinkers that both have genus zero. One of them is the plane P, as previously established. By Brendle's classification of genus zero self-shrinkers, the other one must be a sphere or a cylinder. But by our Gaußian area bound we can exclude the cylinder, since

$$F(P) + F(\mathbb{S}^2_2) < 2.5 < F(P) + F(\mathbb{R} \times \mathbb{S}^1_{\sqrt{2}}). \tag{2}$$

We finish the talk with open questions, conjectures, and numerical results.

References

[1] S. Brendle, *Embedded self-similar shrinkers of genus* 0, Ann. of Math. (2) **183** (2016), no. 2, 715–728.

[2] R. Buzano, H. T. Nguyen, and M. B. Schulz, *Noncompact self-shrinkers for mean curvature flow with arbitrary genus*, J. reine angew. Math. **818** (2025), 35–52.

[3] T. Ilmanen, *Lectures on mean curvature flow and related equations*, preprint (1998).

[4] N. Kapouleas, S. J. Kleene, and N. M. Møller, *Mean curvature self-shrinkers of high genus: Non-compact examples*, J. reine angew. Math. **739** (2018), 1–39.

[5] D. Ketover, *Equivariant min-max theory*, preprint (2016).

[6] D. Ketover and X. Zhou, *Entropy of closed surfaces and min-max theory*, J. Differential Geom. **110** (2018), no. 1, 31–71.

[7] X. H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow, Part III*, Duke Math. J. **163** (2014), no. 11, 2023–2056.

Inverse Mean Curvature Flow coming out of crystals

Esther Cabezas-Rivas

(joint work with Salvador Moll, Marcos Solera)

We manage to define weak solutions for the Inverse Mean Curvature Flow (IMCF) to model crystal growth, and hence we give sense to the evolution equation both in an anisotropic setting and with extremely mild regularity assumptions.

To put our results into context, we quickly recall the classical theory: given $M = \partial\Omega$ a smooth hypersurface of $\mathbb{R}^n$, we say that the family of immersions $X : M \times [0, T) \to \mathbb{R}^n$ is a solution of the IMCF if each point moves in the outward normal direction with velocity given by the inverse of the mean curvature, that is, $\partial_t X = -\frac{1}{H}\nu$. Early results [4, 7] ensure that if $\partial\Omega$ is smooth, mean convex ($H > 0$) and starshaped, then smooth solutions exist for all times and converge (after suitable rescaling) to a round sphere.

For the anisotropic version, we need to consider a norm F on $\mathbb{R}^n$ and the evolution becomes $\partial_t X = -\frac{1}{H_F}\nu_F$ with $\nu_F = \nabla F(\nu)$ the anisotropic normal, and

H_F its corresponding divergence. By assuming additionally that $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ joint with the ellipticity condition $\nabla^2 F > 0$, Xia [8] achieved the anisotropic counterpart of the aforementioned result.

Unfortunately, classical smooth solutions are very restrictive, as shown by the example of a very thin torus, which would puff out until developing zero curvature along the central inner ring, and hence a singularity forms because the flow speed explodes. To preclude this behaviour, Huisken and Ilmanen [5] introduced a level set version of the flow by considering the evolving hypersurfaces as the level sets of a function v, that is, $M_t = X(M,t) = \{v = t\}$. In this way, the original parabolic initial value problem transforms into the following elliptic boundary value one:

$$\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = |\nabla v| \qquad \text{in} \quad \Omega^e = \mathbb{R}^n \setminus \Omega,$$

with $v = 0$ on $\partial\Omega$ and $v \to \infty$ as $|x| \to \infty$. In this setting, the singularities are replaced by a jump mechanism allowing instantaneous replacement of the solution by its outward minimising hull. In this setting, removing the geometric constraints of mean convexity and starshapedness, one can still find a unique solution $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^e)$, as shown in [5]. The anisotropic version of this outcome, with the same smoothness and ellipticity assumptions for F as before, can be found in [3].

Again, this framework is too confining to let evolve a crystal shape (think about the easiest one: a square). To start with, we can only assume that $\partial\Omega$ is Lipschitz and no regularity requirements have to be imposed to F (as the actual growth of a crystal in nature does not occur by rounding off the corners). By adapting to this scenario an approximation technique by Moser [6], which obtains level set solutions of IMCF as limits of p-harmonic functions after a change of variable, our main result reads as follows:

Theorem 1. [2] *Consider $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. Then there exists a unique function $v \in BV_{\mathrm{loc}}(\Omega^e) \cap L^\infty_{\mathrm{loc}}(\Omega^e)$ which is a weak solution of the crystalline IMCF, which formally means that it solves*

$$\begin{cases} \operatorname{div}\big(\partial F(Dv)\big) = F(Dv) & \text{in } \Omega^e, \\ v = 0 & \text{on } \partial\Omega, \\ v \to \infty & \text{as } |x| \to \infty, \end{cases} \tag{1}$$

where Dv represents the distributional derivative of a BV-function, which makes sense as a Radon measure. Moreover, the discontinuity or singular set of v has measure zero with respect to the $(n-1)$-dimensional Hausdorff measure, and v lies between two explicit barriers. Additionally, if $\partial\Omega$ satisfies a uniform interior ball condition (with a Wulff shape playing the role of a ball), v is continuous and satisfies a Harnack inequality.

Let us give a brief sketch of the strategy of the proof.

Step 1. For each $p > 1$ we look for minimizers $u = u_p$ of the p-capacity, i.e.,

$$\mathrm{Cap}_p^F(\overline{\Omega}) := \inf \left\{ \int_{\Omega^e} F^p(\nabla u)\,\mathrm{d}x \,:\, u \in \dot{W}^{1,p}(\Omega^e), u|_{\partial\Omega} = 1 \right\},$$

where $\dot{W}^{1,p}(\Omega^e)$ stands for the homogeneous Sobolev space, which for $p^* = \frac{np}{n-p}$ can be identified with the following space: $\{u \in L^{p^*}(\Omega^e) \,:\, \nabla u \in L^p(\Omega^e; \mathbb{R}^n)\}$. With this aim, we consider the energy

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{\Omega^e} F^p(\nabla u), & \text{if } u \in \dot{W}^{1,p}(\mathbb{R}^n), \quad \text{with} \quad u|_\Omega = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Since $\mathcal{F}$ is a proper and convex functional, it is well-known that u is a minimizer of $\mathcal{F}$ if and only if $0 \in \partial\mathcal{F}(u)$, and it remains to characterize the latter. Indeed,

Theorem 2. [1] *Let $1 < p < n$, $\Omega \subset \mathbb{R}^n$ an open bounded set with Lipschitz boundary. If $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ with $u|_\Omega = 1$ and $\mathrm{w} \in L^{(p^*)'}(\mathbb{R}^n)$, then*

$$\mathrm{w} \in \partial\mathcal{F}(u) \;\Leftrightarrow\; \left[\begin{array}{c} \exists\, z \in L^\infty(\Omega^e; \mathbb{R}^n), \textit{with } z \in \partial F(\nabla u), \textit{ such that} \\ \mathrm{w} = -\mathrm{div}(pF^{p-1}(\nabla u) z) \textit{ in the weak sense.} \end{array} \right]$$

Recall that, by 1-homogeneity of F, it holds $z \in \partial F(\nabla u)$ if and only if $F^\circ(z) \le 1$ and $z \cdot \nabla u = F(\nabla u)$, where F° denotes the polar norm. In particular, we conclude that minimizers of the p-capacity are solutions to

$$\begin{cases} \mathrm{div}\big(F^{p-1}(\nabla u)\partial F(\nabla u)\big) = 0 & \text{in } \Omega^e, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{2}$$

Step 2. We derive a comparison principle which yields uniqueness of solutions of (2), and explicit barriers for the solution $u_p \in \dot{W}^{1,p}(\Omega^e)$ for each $p > 1$.

Step 3. Set $v_p = (1-p)\log u_p$, and derive estimates to find a sequence $p_k \searrow 1$ and a function $v \in BV_{\mathrm{loc}}(\Omega^e) \cap L^\infty_{\mathrm{loc}}(\Omega^e)$ such that $v_{p_k} \to v$ in $L^q_{\mathrm{loc}}(\Omega^e)$ for every $1 \le q < \frac{n}{n-1}$, where the limit v is the unique weak solution of (1) (see [2, Definition 3.1] for technicalities).

Additionally, despite of our milder regularity hypotheses and our a priori different notion of solutions, we recover all the *classical* **geometric properties of the sublevel sets**

$$E_t = \{x \,:\, v(x) < t\} \quad \text{and} \quad G_t = \{x \,:\, v(x) \le t\},$$

that is, those features that one would expect from [5]. In fact, we have a variational definition for the solution v, and its corresponding sublevel sets also minimize a suitable functional; E_t and G_t have finite perimeter for all t; the anisotropic perimeter P_F of G_t and E_t coincide for all $t > 0$; $e^{-t}P_F(E_t)$ is constant for $t > 0$; E_t is outward F-minimizing for any $t > 0$, that is, it minimizes the anisotropic perimeter when compared with all bounded subsets that contain it, while enclosing Ω; G_t is strictly outward F-minimizing for any $t \ge 0$. Moreover, among the anisotropic perimeter minimising envelopes of Ω, G_0 is the one that maximises the Lebesgue measure.

A further advantage of our approach is that it allows to construct some explicit examples of weak solutions, like the expected evolution of Wulff shapes, a rectangle whose vertices do not grow linearly, as well as a concrete case of three simultaneously growing squares where *fattening* occurs.

Finally, our work opens up new challenges and research directions within this milder anisotropic framework:

- replace $\mathbb{R}^n$ by a suitable manifold;
- consider norms F depending on the point and not just on the direction;
- obtain monotonicity formulas under the flow leading to new geometric inequalities in anisotropic settings, under weaker regularity assumptions;
- address polycrystalline growth;
- flows driven by other velocities, and Alexandrov-type theorems.

References

[1] E. Cabezas-Rivas, S. Moll and M. Solera, *Characterization of the subdifferential and minimizers for the anisotropic p-capacity*, Adv. Calc. Var. **18** (2025), no. 1, 25–48

[2] E. Cabezas-Rivas, S. Moll and M. Solera, *Weak solutions of anisotropic (and crystalline) inverse mean curvature flow as limits of p-capacitary potentials*, J. Funct. Anal. **287** (2024), no. 11, Paper No. 110642, 56 pp.

[3] F. Della Pietra, N. Gavitone and C. Xia, *Motion of level sets by inverse anisotropic mean curvature*, Commun. Anal. Geom. **31**, (2023), no. 1, 97–118.

[4] C. Gerhardt, *Flow of Nonconvex Hypersurfaces into Spheres*, J. Differential Geom. **32** (1990), 299–314.

[5] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality.* J. Differential Geom. **59** (2001), no. 3, 353–437.

[6] R. Moser, *The inverse mean curvature flow and p-harmonic functions.* J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 77–83.

[7] J. Urbas, *On the Expansion of Star-shaped Hypersurfaces by Symmetric Functions of Their Principal Curvatures*, Math. Z. 205 (1990) 355–372.

[8] C. Xia, *Inverse anisotropic mean curvature flow and a Minkowski type inequality*, Advances in Mathematics, **315** (2017), 102–129.

Expanding Ricci solitons asymptotic to cones with non-negative scalar curvature

Eric Chen

(joint work with Richard Bamler)

The Ricci flow starting from a compact Riemannian manifold (M^n, g_0) consists of a family of metrics $g(t)$ on M^n which satisfies $\frac{\partial}{\partial t} g = -2\mathrm{Ric}_g$ and $g(0) = g_0$, for $t \in [0, T)$ and some maximal time $T \in (0, \infty]$. When $n = 2$, after normalizing to fix the volume, the flow always converges to a metric of constant Gauss curvature, providing another proof of the uniformization theorem [4]. When $n = 3$, a complete understanding of singularity formation as $t \to T$ makes possible Ricci flow with surgery and its subsequent applications to three-manifold topology [7, 9, 8].

To develop a theory of Ricci flow with surgery in higher dimensions for potential new topological applications, especially in dimension $n = 4$, an understanding of singularity formation would again be essential [2]. One family of singularities arising in higher dimensions are the conical singularities; in fact, any asymptotically conical gradient shrinking soliton arises as a singularity model for some compact Ricci flow [10]. Recent work of Bamler further indicates the importance of this class—any four-dimensional compact Ricci flow with a finite-time singularity has a blowup model which is either a smooth compact gradient shrinking soliton, $S^2 \times \mathbb{R}^2$, $S^3/\Gamma \times \mathbb{R}$, or a cone $(\mathbb{R}_+ \times N^3, dr^2 + r^2 h)$ with non-negative scalar curvature [1].

The appearance of such conical singularities along the flow might be resolved by cutting and gluing in an expanding Ricci soliton, if one asymptotic to the same cone exists. When restricting to cones with positive curvature operator, the existence of such expanders is known [5], and using these it is indeed possible to resolve such isolated conical singularities [6]. Relaxing to cones of non-negative scalar curvature, with Richard Bamler we obtain the existence of expanding Ricci solitons asymptotic to such cones over S^3/Γ [3].

Theorem 1. *For any cone metric $dr^2 + r^2 h$ on $\mathbb{R}_+ \times S^3/\Gamma$ with non-negative scalar curvature, there is a gradient expanding soliton metric g on $\mathbb{R}^4/\Gamma$ with non-negative scalar curvature that is asymptotic to it.*

This result arises from a degree theory we establish for the natural projection map from the space of asymptotically conical expanding solitons on a fixed smooth orbifold with boundary X^4 to the space of cone metrics with non-negative scalar curvature. Let $\mathcal{M}_{\mathrm{grad},R\geq 0}(X)$ denote the space of isometry classes $[(g, \nabla f, \gamma)]$ of expanding gradient solitons with non-negative scalar curvature on the interior of X with metric g and potential field ∇f, asymptotic to the conical metric γ on a tubular neighborhood of ∂X via a fixed set of coordinates at infinity. Also let $\mathsf{Cone}_{R\geq 0}(\partial X)$ denote the space of conical metrics γ on $\mathbb{R}_+ \times \partial X$. Then when X has isolated singularities, a regular boundary, and satisfies certain additional topological assumptions, we have (roughly):

Theorem 2. *The projection map $\Pi : \mathcal{M}_{\mathrm{grad},R\geq 0}(X) \to \mathsf{Cone}_{R\geq 0}(\partial X)$ is proper in a suitable topology and has a well-defined, integer valued degree,* $\deg_{\exp}(X) \in \mathbb{Z}$*. This degree is an invariant of the smooth structure of X.*

If the expander degree $\deg_{\exp}(X) \in \mathbb{Z}$ is nonzero, then Π is surjective, and therefore any member of $\mathsf{Cone}_{R\geq 0}(\partial X)$ is indeed the asymptotic cone of some expanding soliton on X. When $X \approx D^4$ or D^4/Γ, an argument showing that the Gaussian expander is the unique expander asymptotic to the flat cone over S^3 implies that $\deg_{\exp}(X) = 1$, from which the existence result stated earlier follows.

One issue that arises during our construction is that $\mathcal{M}_{\mathrm{grad},R\geq 0}(X)$ may not have a local Banach manifold structure, due to analytical properties of the linear operator $L_f := \triangle - \nabla_{\nabla f} + 2\mathrm{Rm}$ associated with the expanding soliton equation. We must therefore work with the space $\mathsf{GenCone}(\partial X)$ of generalized cone metrics $\gamma = dr^2 + r(dr \otimes \beta + \beta \otimes dr) + r^2 h$ on $\mathbb{R}_+ \times \partial X$ together with the larger space of expanding solitons $\mathcal{M}(X)$ asymptotic to these. This adds additional complications, but also yields deformations to infinitely many non-gradient expanding solitons near any asymptotically conical gradient expander with non-negative scalar curvature; all gradient expanders must be asymptotic to elements of $\mathsf{Cone}(\partial X)$.

For possible applications to Ricci flow with surgery on four-manifolds, it would be desirable to extend our existence result to cover all cones with non-negative scalar curvature. The link of such a cone must be diffeomorphic to a connected sum $(S^3/\Gamma_1)\#\cdots\#(S^3/\Gamma_k)\#(\#^\ell S^2\times S^1)$, and therefore in light of our degree theory it is natural to ask:

Question. *Is there any relation between the degrees* $\deg_{\exp}(X_1)$, $\deg_{\exp}(X_2)$, *and* $\deg_{\exp}(X_1\#_\partial X_2)$*? What is* $\deg_{\exp}(D^3\times S^1)$*?*

Above, the connected sum involves cutting along half-balls intersecting the boundaries ∂X_i to study expanders asymptotic to cones over $\partial X_1\#\partial X_2$. If these degrees can be shown to be nonzero, this would yield natural candidates for resolving the conical singularities encountered along the Ricci flow in four dimensions.

References

[1] Richard H Bamler. Structure theory of non-collapsed limits of Ricci flows. *arXiv e-prints*, arXiv:2009.03243, September 2020.

[2] Richard H Bamler. Recent developments in Ricci flows. *arXiv e-prints*, arXiv:2102.12615, February 2021.

[3] Richard H. Bamler and Eric Chen. Degree theory for 4-dimensional asymptotically conical gradient expanding solitons. *arXiv e-prints*, arXiv:2305.03154, May 2023.

[4] Xiuxiong Chen, Peng Lu, and Gang Tian. A note on uniformization of Riemann surfaces by Ricci flow. *Proc. Amer. Math. Soc.*, 134(11):3391–3393, 2006.

[5] Alix Deruelle. Smoothing out positively curved metric cones by Ricci expanders. *Geom. Funct. Anal.*, 26(1):188–249, 2016.

[6] Panagiotis Gianniotis and Felix Schulze. Ricci flow from spaces with isolated conical singularities. *Geom. Topol.*, 22(7):3925–3977, 2018.

[7] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv Mathematics e-prints*, math/0211159, November 2002.

[8] Grisha Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. *arXiv Mathematics e-prints*, math/0307245, July 2003.

[9] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv Mathematics e-prints*, math/0303109, March 2003.

[10] Maxwell Stolarski. Closed Ricci Flows with Singularities Modeled on Asymptotically Conical Shrinkers. *arXiv e-prints*, arXiv:2202.03386, February 2022.

Penrose inequalities in low regularity regimes

Mattia Fogagnolo

(joint work with Luca Benatti, Lorenzo Mazzieri)

The Riemannian Penrose inequality (RPI) takes place in an asymptotically flat Riemannian 3-manifold (M,g) with nonnegative scalar curvature R, and endowed with a minimal, closed and outermost boundary ∂M. In this report, outermost just means that no other closed minimal surface S encloses ∂M. These Riemannian manifolds arise as spacelike slices of time-symmetric Lorentzian spacetimes obeying the dominant energy condition. Moreover, we say that (M, g) is C^k_τ-asymptotically flat if $M = \mathbb{R}^3$ with a compact set set removed and

$$g = \delta + \eta$$

with η decaying by

$$\sum_{|\beta|=0}^{k} |\partial_\beta \eta| \leq \frac{\mathrm{C}}{|x|^{|\beta|+\tau}}.$$

In C^1_τ-asymptotically flat 3-manifolds with $\tau > 1/2$, with nonnegative scalar curvature, Bartnik and Chrusciél proved that the following quantity

$$m_{ADM} = \lim_{r\to+\infty} \frac{1}{16\pi} \int_{\{|x|=r\}} g^{ij}(\partial_i g_{jk} - \partial_k g_{ij})\nu^k d\sigma, \tag{1}$$

where $\{x^1, x^2, x^3\}$ are the classical coordinates of $\mathbb{R}^3$, is well-defined (and possibly infinite), and thus not depending on the chosen chart. This quantity has been conceived in [2] as a notion of relativistic total mass. The restriction to $\tau > 1/2$ is in fact fully sharp, since already for $\tau = 1/2$ one can choose charts giving different values of the quantity on the RHS of (1).

The RPI states that

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq m_{ADM}, \tag{2}$$

and has been first conjectured by Penrose [8] as a test for the cosmic censorship conjecture.

The geometric inequality (2) has been proved by Huisken-Ilmanen [5] for connected boundaries in C^1_1 -asymptotically flat manifolds also satisfying $Ric \geq -1/|x|^2$, and by Bray [4] for possibly disconnected boundaries in C^2_τ-asymptotically flat manifolds with $\tau > 1/2$. In [3] we provide (2) in the sharp C^1_τ-asymptotically flat regime with $\tau > 1/2$, for connected boundaries. The proof follows the general strategy as Huisken-Ilmanen's, relying on the monotonicity of the Hawking mass

$$m_H(\Sigma_t) = \frac{|\Sigma_t|^{\frac{1}{2}}}{16\pi^{\frac{3}{2}}} \left(4\pi - \int_{\Sigma_t} \frac{H^2}{4} d\sigma\right) \tag{3}$$

along the level sets $\Sigma_t = \partial\{w \leq t\}$ evolving by weak Inverse Mean Curvature Flow (IMCF), that is a suitable weak notion for

$$\begin{cases} div\left(\dfrac{\nabla w}{|\nabla w|}\right) = |\nabla w| & \text{on } M \setminus \partial M, \\ w = 0 & \text{on } \partial M, \\ w \to +\infty & \text{as } dist(x, o) \to +\infty. \end{cases}$$

In fact, if one can also show

$$\lim_{t+\infty} m_H(\Sigma_t) \leq m_{ADM}, \tag{4}$$

then coupling it with the monotonicity of (3) yields (2). In the talk, I discuss how in [3] we modify Huisken-Ilmanen's strategy for this last step by first highlighting the centrality of an priori nonsharp boundedness of $m_H(\Sigma_t)$ for any t, and then explaining how to obtain it via the following novel method. Let Σ_t^s be the level sets $\{w_2 = s\}$ where w_2 solves

$$\begin{cases} div\,(\nabla w_2) = |\nabla w_2|^2 & \text{on } M \setminus \partial M, \\ w_2 = 0 & \text{on } \partial M, \\ w_2 \to +\infty & \text{as } dist(x, o) \to +\infty. \end{cases}$$

Then, by [1] the quantity

$$m_H^{(2)}(\Sigma_t^s) = \mathrm{Cap}(\Sigma_t^s) \left[4\pi + \int_{\Sigma_t^s} |\nabla w_2|^2 - \int_{\Sigma_t^s} |\nabla w_2| H \right]$$

is monotone nondecreasing in s. Moreover, building on the fact that e^{-w_2} is harmonic, Green's representation formula yields that $\lim_{s \to +\infty} m_H^{(2)}(\Sigma_t^s) \leq m_{ADM}$, and since by Hölder's and Sobolev inequality $m_H(\Sigma) \leq C m_H^{(2)}(\Sigma)$ for any surface Σ, we obtain the claimed uniform bound on the Hawking mass along the IMCF. A question raised by M. Simon concerns the viability of the above scheme and consequently the validity of (2) for metrics of $C^0 \cap W_\tau^{1,3}$-regularity with nonnegative distributional scalar curvature, a setting where the positive mass theorem has been proved by Lee-LeFloch [7].

I then pass to consider Huisken's notion of isoperimetric mass, reading

$$m_{iso} = \sup_{(\Omega_j)_{j \in \mathbb{N}}} \limsup_{j \to +\infty} \frac{2}{|\partial \Omega_j|} \left(|\Omega_j| - \frac{|\partial \Omega_j|^{\frac{3}{2}}}{6\sqrt{\pi}} \right).$$

I explain how in [3] we obtain (2) with m_{iso} in place of m_{ADM} without any assumption of asymptotic flatness, but only requiring existence of the weak IMCF. By recent work of Xu [9], this is granted e.g. if g merely satisfies a Euclidean-like isoperimetric inequality. The proof of such Isoperimetric RPI stems from the asymptotic comparison

$$\lim_{t \to +\infty} (\Sigma_t) \leq \limsup_{t \to +\infty} \frac{2}{|\partial \Omega_t|} \left(|\Omega_t| - \frac{|\partial \Omega_t|^{\frac{3}{2}}}{6\sqrt{\pi}} \right)$$

where $\Omega_t = \{w \leq t\}$, that happens to be a simple application of De l'Hopital theorem.

During the talk, I also point out that m_{iso} has been recently observed [6] to be nonnegative in any asymptotically flat setting, regardless of any curvature assumption. A natural question discussed with Huisken concerns then how to fine-tune the definition of isoperimetric mass in order to make it more sensitive to nonnegative scalar curvature and minimality of the boundary.

Finally, I mention some preparatory work with Gatti and Pluda towards an Isoperimetric RPI for continuous metrics with nonnegative scalar curvature in an approximated sense.

References

[1] V. Agostiniani, L. Mazzieri and F. Oronzio, *A Green's function proof of the positive mass theorem*, Comm. Math. Phys. **405** (2024), no. 2, Paper No. 54, 23 pp.

[2] R. Arnowitt, S. Deser and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. (2) **122** (1961), 997–1006; MR0127946

[3] L. Benatti, M. Fogagnolo and L. Mazzieri, *On the isoperimetric Riemannian Penrose inequality*, Comm. Pure Appl. Math. **78** (2025), no. 5, 1042–1085

[4] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267

[5] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437;

[6] J. L. Jauregui, D. A. Lee and R. Unger, *A note on Huisken's isoperimetric mass*, Lett. Math. Phys. **114** (2024), no. 6, Paper No. 134, 7 pp.

[7] D. A. Lee and P. G. LeFloch, *The positive mass theorem for manifolds with distributional curvature*, Comm. Math. Phys. **339** (2015), no. 1, 99–120

[8] R. Penrose, *Naked Singularities*, Annals of the New York Academy of Sciences, **224** (1973), 125–134.

[9] K. Xu, *Isoperimetry and the properness of weak inverse mean curvature flow*, Calc. Var. Partial Differential Equations **63** (2024), no. 8, Paper No. 216, 19 pp.

Parabolic evolution to the Bishop family of holomorphic discs

Wilhelm Klingenberg

(joint work with Brendan Guilfoyle)

We prove that a space-like graphical rotationally symmetric line congruence evolving under mean curvature flow with respect to the neutral Kähler metric in the space of oriented lines of the Euclidean 3-space, namely in TS^2, see [1], subject to suitable Dirichlet and Neumann boundary conditions, converges to a holomorphic disc. This disc comes in a one parameter family which forms the Bishop family of discs associated to the complex point in the Lagrangian boundary condition. This result is a special case of independent interest of the results that appeared in [2] and [3]. We ended the talk with an application to the Critical Catenoid, a minimal surface of the type of the annulus. Namely we prove that the normal line congruence of this surface bounds a holomorphic disc in TS^2.

References

[1] B. Guilfoyle and W. Klingenberg, An indefinite Kähler metric on the space of oriented lines, J. London Math. Soc. 72.2 (2005) 497–509.

[2] B. Guilfoyle and W. Klingenberg, Higher codimensional mean curvature flow of compact spacelike submanifolds, Trans. Amer. Math. Soc. 372.9 (2019) 6263–6281.

[3] B. Guilfoyle and W. Klingenberg, Proof of the Toponogov Conjecture on complete surfaces, J. Gökova Geom. Topol. GGT 17 (2024) 1–50.

[4] B. Guilfoyle and W. Klingenberg, Parabolic evolution with boundary to the Bishop family of holomorphic discs, 2023, arxiv https://arxiv.org/pdf/2304.05702

Parabolic equations with rough data

Tobias Lamm
(joint work with Herbert Koch)

The study of parabolic equations with rough data is motivated by some geometric problems such as defining lower bounds for the scalar curvature for irregular metrics. In this talk I showed how irregular initial data can be smoothed out with the help of parabolic equations such as the harmonic map flow or the Ricci–DeTurck flow.

Curve shortening flow in high codimension

Florian Litzinger

We consider the flow of closed curves immersed in Euclidean space of any dimension by their curvature. More precisely, let $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^n$ be a one-parameter family of smooth immersions satisfying

$$\text{(CSF)} \qquad \begin{aligned} \frac{\partial \gamma}{\partial t} &= \kappa N, \\ \gamma(\cdot, 0) &= \gamma_0 \end{aligned}$$

for some initial smooth closed curve $\gamma_0 : S^1 \to \mathbb{R}$, where $\kappa(\cdot, t)$ and $N(\cdot, t)$ denote the curvature and normal vector of $\gamma_t := \gamma(\cdot, t)$, respectively. It is known that solutions exist at least for a short time [4]. On the other hand, the final time of existence ω of the curve shortening flow of a compact initial curve must be finite, which prompts the study of the behaviour of the flow when approaching the final time.

The case of evolving embedded curves in the plane, $n = 2$, is well studied. Most prominently, Gage and Hamilton [4, 5] showed that convex curves shrink to a point, becoming asymptotically circular in the process. The final time of existence of the flow is characterised by the curvature tending to infinity. Meanwhile, Grayson [6, 7] proved that any embedded curve continues to be embedded under the flow and eventually becomes convex without developing singularities. Due to its impact on the literature, the resulting *Gage–Hamilton-Grayson theorem* can, to some degree, be considered a model result for the analysis of singularities of the flow.

The flow of curves in higher codimension exhibits some notable differences to the planar case. In particular, embeddedness need not be preserved and the avoidance principle, i. e., the fact that two initially disjoint planar curves will remain so throughout the evolution, no longer holds. The main tool for the analysis of the flow in high codimension is Huisken's monotonicity formula [9], which holds in any codimension.

For the flow of curves immersed in $\mathbb{R}^3$, Altschuler [2] carried out a careful analysis of the blow-up limits of the flow, proving that singularity formation is an essentially planar phenomenon. He showed that for any blow-up sequence of a type-I singularity (that is, the curvature does not grow faster than $(\omega - t)^{\frac{1}{2}}$), there exists a subsequence such that a rescaling of the curve along it converges to a planar self-similarly shrinking solution, while for a type-II singularity (that is, not of type-I) there exists a blow-up sequence such that a sequence of rescalings along it converges to the translating Grim Reaper solution.

In the case of arbitrary codimension, we can show that Altschuler's results continue to hold. That is, the emergence of singularities is still governed by the curvature becoming unbounded, and evolving curves in fact become asymptotically planar in the vicinity of a singularity. We give a simple proof of the latter fact using the monotonicity formula. Moreover, close to a type-I singularity, a sequence of rescalings along a blow-up sequence converges to a planar self-similarly shrinking solution, while for a type-II singularity, we show the existence of an essential blow-up sequence converging to the Grim Reaper. Some of these results already appeared in work of Yang–Jiao [12].

We then show, combined with a bound on a suitably defined entropy of the inital curve, that such curves do become circular and thus shrink to a point. In order to obtain such a result in the spirit of the Gage–Hamilton-Grayson theorem, inspired by the general strategy employed in Huisken's proof [10] of Grayson's theorem, we first seek to rule out the emergence of type-II singularities. In our case, this is done by means of the aforementioned initial entropy bound. We then study the possible type-I singularities using the classification of self-similarly shrinking solutions in the plane [1], enabling us to exclude all but the circle.

The entropy functional, introduced by Colding–Minicozzi [3] and several other works in the context of mean curvature flow, can be seen as a Gaussian-weighted length functional and thus a measure of geometric complexity. For a curve γ, its entropy $\lambda(\gamma)$ is defined by

$$\lambda(\gamma) = \sup_{x_0 \in \mathbb{R}^n,\, t_0 > 0} (4\pi t_0)^{-\frac{1}{2}} \int_\gamma \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}} \,\mathrm{d}\mu.$$

Among other favourable properties, the monotonicity formula implies that entropy is monotone non-increasing under curve shortening flow, whereby an upper bound for the initial curve propagates with the flow. Notably, Guang [8] showed that the entropy of the Grim Reaper solution is equal to 2.

Then, combining these arguments, we obtain our main theorem [11]:

Theorem. *Suppose that $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^n$ is a smooth solution of (CSF) and assume that the entropy of γ_0 satisfies*

$$\lambda(\gamma_0) \leq 2.$$

Then ω is finite, and the rescaled flow converges to the round circle.

Possible directions for future research include the study of the flow of non-compact curves, or of closed curves in a Riemannian manifold with appropriate conditions on the curvature, both with a bound on the entropy of the initial curve.

References

[1] U. Abresch and J. Langer. "The normalized curve shortening flow and homothetic solutions". In: *J. Differential Geom.* 23.2 (1986), pp. 175–196. DOI: 10.4310/jdg/1214440025.

[2] S. J. Altschuler. "Singularities of the curve shrinking flow for space curves". In: *J. Differential Geom.* 34.2 (1991), pp. 491–514. DOI: 10.4310/jdg/1214447218.
[3] T. H. Colding and W. P. Minicozzi II. "Generic mean curvature flow I: generic singularities". In: *Ann. of Math. (2)* 175.2 (2012), pp. 755–833. DOI: 10.4007/annals.2012.175.2.7.
[4] M. Gage and R. S. Hamilton. "The heat equation shrinking convex plane curves". In: *J. Differential Geom.* 23.1 (1986), pp. 69–96. DOI: 10.4310/jdg/1214439902.
[5] M. E. Gage. "Curve shortening makes convex curves circular". In: *Invent. Math.* 76.2 (1984), pp. 357–364. DOI: 10.1007/bf01388602.
[6] M. A. Grayson. "The heat equation shrinks embedded plane curves to round points". In: *J. Differential Geom.* 26.2 (1987), pp. 285–314. DOI: 10.4310/jdg/1214441371.
[7] M. A. Grayson. "The shape of a figure-eight under the curve shortening flow". In: *Invent. Math.* 96.1 (1989), pp. 177–180. DOI: 10.1007/bf01393973.
[8] Q. Guang. "Volume growth, entropy and stability for translating solitons". In: *Comm. Anal. Geom.* 27.1 (2019), pp. 47–72. DOI: 10.4310/cag.2019.v27.n1.a2.
[9] G. Huisken. "Asymptotic behavior for singularities of the mean curvature flow". In: *J. Differential Geom.* 31.1 (1990), pp. 285–299. DOI: 10.4310/jdg/1214444099.
[10] G. Huisken. "A distance comparison principle for evolving curves". In: *Asian J. Math.* 2.1 (1998), pp. 127–133. DOI: 10.4310/ajm.1998.v2.n1.a2.
[11] F. Litzinger. *Singularities of low entropy high codimension curve shortening flow.* Apr. 5, 2023. arXiv: 2304.02487 [math].
[12] Y. Y. Yang and X. X. Jiao. "Curve shortening flow in arbitrary dimensional Euclidian space". In: *Acta Math. Sin. (Engl. Ser.)* 21.4 (2005), pp. 715–722. DOI: 10.1007/s10114-004-0426-z.

Canonical foliation of bubblesheets

Stephen Lynch
(joint work with Jean Lagacé)

Foliations by constant mean curvature (CMC) hypersurfaces play an important role in Riemannian geometry and general relativity [1]. In the context of the Ricci flow, R. Hamilton constructed a canonical CMC foliation for necks. These are Riemannian manifolds which, after scaling, locally resemble the standard cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ around each point. Hamilton employed his foliation as a technical tool in the construction of a Ricci flow with surgeries on 4-manifolds with PIC [2]. It has also been used in the proofs of classification results for ancient solutions [3].

Motivated by similar applications to geometric flows, we generalise Hamilton's construction to bubblesheets. These are Riemannian manifolds which, after scaling, locally resemble the cylinder $\mathbb{R}^k \times \mathbb{S}^{n-k}$. Bubblesheets occur much more frequently than necks near singularities of both the Ricci and mean curvature flows (in the absence of a dimension restriction or curvature condition which rules them out). A key new difficulty arises in this more general setting: whereas Hamilton's foliation consisted of embedded spheres of codimension one, we must instead work with spheres of codimension k.

The obvious generalisation of CMC to higher codimensions is to impose that the mean curvature vector be parallel (with respect to the connection induced on the normal bundle). It turns out that there are bubblesheets which do not admit a foliation by leaves with parallel mean curvature—examples can be obtained from very collapsed Berger metrics on the 3-sphere after an arbitrarily small perturbation. The issue seems to be that, in general, high-codimension submanifolds may not support any nonzero parallel normal sections. This means that in a certain sense submanifolds with parallel mean curvature are too scarce. Compare this with the situation in codimension one: every compact 2-sided hypersurface has parallelisable normal bundle, with a global parallel frame being given by any unit normal.

To overcome these issues we introduce a new curvature condition for submanifolds of higher codimension, which generalises both CMC and parallel mean curvature vector. We call this condition quasi-parallel mean curvature (QPMC). To see what we mean by QPMC, fix a Riemannian manifold (M, g) of dimension n and consider a submanifold $\Sigma \subset M$ of codimension k. The Levi-Civita connection of (M, g) induces a connection $\nabla^\perp$ and hence a Laplacian $\Delta^\perp$ acting on sections of the normal bundle $N\Sigma$. The operator $-\Delta^\perp$ admits a complete set of eigensections with nonnegative eigenvalues $\lambda_m = \lambda_m(\Sigma, g)$, labelled in nondecreasing order, repeating according to multiplicity. We write P_λ for the L^2-orthogonal projection onto the eigenspace associated with $\lambda \in \mathrm{spec}(-\Delta^\perp)$ and define

$$Q := \sum_{\substack{\lambda \in \mathrm{spec}(-\Delta^\perp) \\ \lambda < \lambda_{k+1}}} P_\lambda.$$

A section of $N\Sigma$ is called quasi-parallel if it lies in the image of Q, and so the submanifold Σ is said to have QPMC if its mean curvature vector H satisfies

$$(1 - Q)(H) = 0. \tag{1}$$

We note that $1 - Q$ is a nonlocal pseudodifferential operator acting on sections of $N\Sigma$. Expressed as $H = Q(H)$, (1) is a weakly elliptic quasilinear system for the position of the submanifold, with nonlocal right-hand side.

On the cylinder $\mathbb{R}^k \times \mathbb{S}^{n-k}$, for each of the slices $\{z\} \times \mathbb{S}^{n-k}$, the image of Q is precisely the space of parallel vector fields obtained from restricting the coordinate vectors on $\mathbb{R}^k$. Moreover, since on any slice we have the spectral gap $\lambda_k = 0 < n - k = \lambda_{k+1}$, for any nearby submanifold Q still has rank equal to k (even though the kernel of $\Delta^\perp$ might now be $\{0\}$). Exploiting these facts, and using the implicit function theorem, we are able to show that every bubblesheet admits a canonical foliation by embedded copies of $\mathbb{S}^{n-k}$ with QPMC. After pulling back to $\mathbb{R}^k \times \mathbb{S}^{n-k}$ by any map which is almost a homothety, the QPMC leaves are small perturbations of the round slices $\{z\} \times \mathbb{S}^{n-k}$.

It remains to be seen whether the QPMC condition is also useful in other contexts. Since it appears to be a natural generalisation of the notion of parallel mean curvature, one is also led to wonder which properties of that class of submanifolds carry over. For example, what can be said about a QPMC sphere embedded in a Euclidean space? Must it be round?

References

[1] G. Huisken, S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Inventiones Mathematicae **124** (1996), 281–311.

[2] R. Hamilton, *Four-manifolds with positive isotropic curvature*, Communications in Analysis and Geometry **5** (1997) 1–92.

[3] S. Brendle, *Ancient solutions to the Ricci flow in dimension 3*, Acta Mathematica **225** (2020), 1–102.

A parabolic monotonicity formula for the biharmonic map heat flow

Elena Mäder-Baumdicker

(joint work with Casey Lynn Kelleher, Nils Neumann)

Parabolic monotonicity formulas such as Huisken's monotonicity formula [2] for the Mean Curvature Flow or Struwe's monotonicity formula for the harmonic map heat flow [6, 1] are an essential tool for second order geometric flows, even with a global constraint such as the volume preserving Mean Curvature Flow [4]. In order to look for a parabolic monotonicity formula, several properties should be satisfied by the monotone object Ψ (which we call a *pre-entropy*): Positivity of Ψ for general solutions of the geometric flow, scaling invariance under parabolic rescaling, shrinking solitons should be critical points of Ψ and a monotonicity property of Ψ should be satisfied that can have correction terms involving the initial energy of the evolving object and constants independent of the flow.

In the talk we explained that a parabolic monotonicity formula in the sense formulated above can also be proven for fourth order flows in some cases [5]. For the linear case of the biheat equation $\partial_t u = -\Delta^2 u$ on $\mathbb{R}^n \times [0,\infty)$ we consider $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$ and horizontal layers

$$T_R(t_0) := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t_0 - 16R^4 < t < t_0 - R^4\}$$

and define

$$\Psi(u, R) := \frac{1}{2}\int_{T_R(t_0)} |\Delta u|^2 B - \frac{1}{2}\int_{T_R(t_0)} |\nabla u|^2 \Delta B,$$

where B is the rotational symmetric backwards biheat kernel on $\mathbb{R}^n$ concentrating around (x_0, t_0). Note that $-\Delta B(\cdot, t) < 0$ only around x_0 for all $t > 0$ so that the second term is positive for u with support around x_0. Due to the oscillating nature of B one needs to overcome several difficulties in estimating the derivative of Ψ. We do this by including an appropriate cut-off function which we find by deriving novel properties of B. As a consequence, the monotonicity formula for $n \le 3$ has the form $(R_1 < R_2)$

$$\Psi(u, R_1, \varphi) \le \Psi(u, R_2, \varphi) + C(R_2 - R_1)E_0, \quad \text{where}$$

$$E_0 := \int |\Delta u_0|^2 + \int |\nabla u_0|^2 + \int |\nabla\Delta u_0|^2.$$

for a solution of the equation $\partial_t u = -\Delta^2 u$ on $\mathbb{R}^n$ and constants C that do not depend on u nor on R_1, R_2.

We further explained that these results can be carried over to the extrinsic biharmonic map heat flow for $n \leq 3$, which is the gradient descent flow of the functional $\int |\Delta u|^2$ for $u : \mathbb{R}^n \to N \hookrightarrow \mathbb{R}^k$, where N is a smooth closed manifold and Δ is the extrinsic Laplacian (seeing u as a map to $\mathbb{R}^k$). For these dimensions and on a closed manifold M, it was shown by Lamm that no singularities develop and the flow subconverges to a smooth biharmonic map [3]. His techniques and results enable us to bound the terms similar to E_0 appearing in the non-linear case for $n \leq 3$. For $n = 4$ we mentioned a weaker result which also needs the additional assumption that an appropriate initial energy is small. We also emphasized the structural differences of the second and fourth order case in general and how the growth of the biheat kernel B at t_0 is involved in the computations.

References

[1] R. Hamilton, *Monotonicity formulas for parabolic flows on manifolds*, Comm. Anal. Geom. **1** (1993), no. 1, 127–137.

[2] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geometry **31** (1990), no. 1, 285–299.

[3] T. Lamm, *Heat flow for extrinsic biharmonic maps with small initial energy*, Ann. Global Anal. Geom. **26** (2004), no. 4, 369–384.

[4] B. Lambert, E. Mäder-Baumdicker, *Nonlocal estimates for the volume preserving mean curvature flow and applications*, Calc. Var. Partial Differential Equations **62** (2023), no. 7, Paper No. 202, 32 pp.

[5] E. Mäder-Baumdicker, N. Neumann, *A Monotonicity formula for the extrinsic biharmonic map heat flow*, preprint, in preparation.

[6] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geometry **28** (1988), no. 3, 485–502.

On the long term limit of the mean curvature flow in 3-manifolds

Alexander Mramor

(joint work with Ao Sun)

Given the recent spectacular advances in the understanding of singularities the mean curvature flow of closed embedded surfaces in $\mathbb{R}^3$, particularly advancements on the generic mean curvature flow [6, 3, 4, 14, 15], the mean convex neighborhood conjecture [5], and the multiplicity one conjecture [1], a natural question one may ask is how these results may be applied to show interesting results in closed 3-manifolds. Indeed, even in closed 3-manifolds the singularity analysis, done by rescalings, reduces one roughly speaking to the study of flows in $\mathbb{R}^3$ and one can check these results apply in this more general setting.

On the large scale though flows in $\mathbb{R}^3$ can behave quite differently from flows in a given closed Riemannian manifold. Most relevant here is that in $\mathbb{R}^3$ compact flows must go extinct in finite time by the comparison principle, which for topological applications can be quite useful because as a consequence all regions of the initial data must eventually correspond to a high curvature region which in a fair number of cases can be categorized up to diffeomorphism. On the other hand, a simple closed curve in (S^2, g_{round}) which bisects the area will flow to a great circle. In general, because the MCF is the gradient flow of the area functional, one expects that a flow M_t which does not go extinct in finite time will converge to a minimal surface as $t \to \infty$. Our first result, from [12] (as are the following ones), sharpens this basic principle in 3-manifolds:

Theorem 1. *Let M be a 2-sided properly embedded surface in a closed Riemannian 3-manifold (N, g), and let M_t be an almost regular flow from M. Then either M_t goes extinct in finite time or there exists a sequence $t_i \to \infty$ for which M_{t_i} converges to $m_1\Sigma_1 + \cdots + m_k\Sigma_k$ in the sense of varifolds, where each Σ_j is a smoothly embedded minimal surface and each m_j is a positive integer. Moreover, all the Σ_j's are disjoint.*

To show the limit surfaces Σ_i are smooth we were heavily inspired by Ilmanen's analysis of singularities for surface flows [9] which uses a number of 2-dimensional facts, such as Gauss–Bonnet. The notion of almost regular flows is taken from [1], which are well behaved weak flows for which the multiplicity one conjecture is resolved – of course given the ubiquity of singularities in the mean curvature flow its preferable for any results of this flavor to allow for weak flows. As one may expect one can also gain some control on the multiplicity and genus of the Σ_i in terms of M as well as study the uniqueness of them as discussed further in [12]. It is interesting to note that conceivably some of the Σ_i may be one sided even if M is not, considering for instance taking M to be the normal bundle over an embedded $\mathbb{R}P^2$ in an appropriate background space (N, g).

It's a fact that for any such initial data above there is at least one almost regular flow emanating from it, and one may also suppose it can be approximated by generic flows (and hence surgery flows, by [7]) which will be useful in some of the applications listed below. One first result is that we can prove via the flow a classical result in minmax of Simon and Smith:

Corollary 1. *Suppose N is diffeomorphic to S^3. Then N contains a minimal embedded S^2.*

The idea is itself inspired by the original proof; in short one may consider a foliation of N by embedded spheres and flow the leaves simultaneously and without too much work show that the flow of one of the leaves must never go extinct. Of course, as the gradient flow of the area functional in a generic sense the flow should converge to only stable minimal surfaces. Indeed, in the toy case of simple closed curves in S^2 the flow of any slight upwards perturbation of a great circle will go extinct in a round point. With this in mind we showed the following:

Theorem 2. *Let M^2 be a 2-sided properly embedded surface of a closed compact Riemannian 3-manifold (N^3, g). Then there exists a piecewise almost regular flow emanating from it which either goes extinct in finite time, or there exist $t_i \to \infty$, such that M_{t_i} converges possibly with multiplicity to a (potentially disconnected) stable minimal surface.*

In the above the number of jumps one potentially takes is finite and can be arranged as small as one wishes, so the flow is generic in the sense of Colding and Minicozzi [6]. From a geometric perspective it would be preferable if one need only to perturb the initial data and one can show this is the case if the multiplicities m_i for the unstable Σ_i were equal to one. In analogy to the multiplicity one conjecture (replacing the MCF with RMCF and minimal surfaces with shrinkers, which are all unstable minimal surfaces in the Gaussian metric) one may guess this is the case but there are examples of flows which converge to stable minimal surfaces with high multiplicity constructed in [2]. Still, the result above provides a way to obtain stable minimal surfaces with the flow more or less and this can conceivably be useful because stable minimal surfaces often enjoy special geometric and topological properties, such as in PSC ambient manifolds N.

With some topological assumptions on M and N one can rule out a piecewise almost regular flow of M from going extinct, so using the result above we continue on to give some existence results which as in the application above are mainly novel from the perspective that the flow is used to construct them. On the flip side the flow method is roughly speaking quite explicit in comparison to compactness methods; consider that Brakke's surface evolver is used to create many of the pictures of minimal surfaces one may find online. We highlight here two applications from [12] here, the first can be compared to the classical results [13, 11]:

Corollary 2. *Suppose $\pi_2(N)$ is nontrivial. Then N contains an embedded stable minimal sphere or projective plane.*

We point out that in the proof of this we invoke the 3D sphere theorem (of topology) to give embedded initial data to flow: a deficiency of the mean curvature flow compared to [11] in this context is that we cannot start with merely immersed spheres and later on find a minimal embedded one. We also point out that the result uses the approximating surgery flows discussed above to rule out the case the flow involved in the proof does not go extinct. The next result, on the existence of higher genus stable minimal surfaces, can be compared to [8]:

Corollary 3. *Suppose $\iota : \Sigma \to N$ is an orientable properly embedded surface of N of genus $g \geq 1$ for which $\iota_* \pi_1(\Sigma) \to \pi_1(N)$ is injective, and that N has trivial second homotopy group. Then either $M = \iota(\Sigma)$ is homotopic to an orientable stable minimal surface or a double cover of a stable minimally embedded connect sum of $g + 1$ projective planes, where g is the genus of Σ.*

The condition here that $\pi_2(N)$ is trivial is of course implied by N being aspherical; its easy to see considering universal covers that hyperbolic 3-manifolds are aspherical and these are generic in a sense by [10] – its use is to rule out the case that the flow of Σ converges to a stable minimal sphere with multiplicity and to deal with spherical components pinching off along the flow. Again the approximating generic flows and subsequent surgery flows to Σ_t are useful, in this case for instance showing that if the topology of Σ_t drops or splits into two nontrivial pieces then Σ was not incompressible to begin with.

References

[1] Richard Bamler and Bruce Kleiner. *On the multiplicity one conjecture for mean curvature flows of surfaces.* Preprint, arXiv:2312.02106, 2023.
[2] Jingwen Chen and Ao Sun. *Mean curvature flow with multiplicity 2 convergence in closed manifolds.* Preprint, arXiv:2402.04521.
[3] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. *Mean curvature flow with generic initial data.* Invent. Math., 237(1):121–220, 2024.
[4] Otis Chodosh, Kyeongsu Choi, and Felix Schulze. *Mean curvature flow with generic initial data ii.* arXiv preprint arXiv:2302.08409, 2023.
[5] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits. *Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness.* Acta Math., 228(2):217–301, 2022.
[6] Tobias Colding and William Minicozzi. *Generic mean curvature flow i; generic singularities.* Ann. of Math, pages 755–833, 2012.
[7] Joshua Daniels-Holgate. *Approximation of mean curvature flow with generic singularities by smooth flows with surgery.* Adv. Math., 410:Paper No. 108715, 42, 2022.
[8] Michael Freedman, Joel Hass, and Peter Scott. *Least area incompressible surfaces in 3-manifolds.* Invent. Math., 71(3):609–642, 1983.
[9] Tom Ilmanen. *Singularities of mean curvature flow of surfaces.* preprint, 1995.
[10] Joseph Maher. *Random Heegaard splittings.* J. Topol., 3(4):997–1025, 2010.
[11] William Meeks and S.T. Yau. *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory.* Ann. of Math. (2), 112(3):441–484, 1980.
[12] Alexander Mramor and Ao Sun, *On the long-time limit of the mean curvature flow in closed manifolds.* Preprint, arXiv:2412.03475.
[13] Jonathan Sacks and Karen Uhlenbeck. *The existence of minimal immersions of 2-spheres.* Ann. of Math. (2), 113(1):1–24, 1981.
[14] Ao Sun and Jinxin Xue. *Initial Perturbation of the Mean Curvature Flow for Asymptotical Conical Limit Shrinker.* To appear in Science China Math.
[15] Ao Sun and Jinxin Xue. *Initial Perturbation of the Mean Curvature Flow for closed limit shrinker.* Preprint, arXiv: 2104.03101

Flows by curvature and nonlinear potentials. Monotonicity formulas and functional inequalities.

Alessandra Pluda

(joint work with Luca Benatti, Marco Pozzetta)

Let (M, g) be a complete, noncompact Riemannian manifold without boundary of dimension $n \geq 3$ with nonnegative Ricci curvature $\mathrm{Ric} \geq 0$. For $p \in [1, 2]$, given $\Omega \subseteq M$ closed and bounded we define w_p as the solution to

$$(1) \qquad \begin{cases} \Delta_p w_p = |\nabla w_p|^p & \text{on } M \setminus \Omega, \\ w_p = 0 & \text{on } \partial\Omega, \\ w_p \to +\infty & \text{as } \mathrm{d}(x, \Omega) \to +\infty, \end{cases}$$

where $\Delta_p f = \mathrm{div}(|\nabla f|^{p-2} \nabla f)$. We denote $\Omega_t^p = \{w_p \leq t\}$, possibly omitting p.

If $p = 1$, w_1 is a solution to the weak inverse mean curvature flow (IMCF) given by the equation $\mathrm{H} = |\nabla w_1|$ and for $p > 1$ the function $u_p = \exp(-w_p/(p-1))$ is p-harmonic. For every $p \in [1, 2]$ there exists a unique solution w_p to (1) and, as p approaches 1^+, $w_p \to w_1$ locally uniformly.

The literature contains numerous instances of monotonicity formulas related to solutions to (1), starting from the ones in the seemingly distinct frameworks of IMCF and linear potential theory by Huisken-Ilmanen and Colding-Minicozzi [6, 7, 9]. More often than not, such monotonicity formulas are used to prove geometric inequalities [1, 2, 13].

In [5], we propose a comprehensive description of monotone geometric quantities along the weak inverse mean or the level sets flows of (1). Our result includes the monotonicity of the Hawking mass and its nonlinear potential theoretic counterparts in 3-manifolds with nonnegative scalar curvature. We show rigorously that Geroch's monotonicity along the Inverse Mean Curvature Flow is the limit case of corresponding ones along the level sets of p-capacitary potentials. We also recover Willmore and Minkowski-type inequalities on Riemannian manifolds with nonnegative Ricci curvature.

Our unified perspective is based on a new "geometric" regularity result: we prove that almost every level set $\partial\Omega_t^p$ is a curvature varifold. Thus, the known regularity of the level sets of the weak inverse mean curvature flow is shared with the solutions w_p for every $p \in [1, 2]$ and it is not a peculiarity of IMCF due to its geometric nature. Note that almost every level set $\partial\Omega_t^p$ was known to be regular around $\mathcal{H}^{n-1}$-almost every point, but in general, there is no control on the critical set $\{|\nabla w_p| = 0\}$, that may have positive $\mathcal{H}^n$-measure. This basic regularity alone is not enough to globally define the notions of second fundamental form or mean curvature of the level sets and to infer their topological properties (specifically the Gauss-Bonnet theorem and the Euler characteristic). We can finally give a (weak) geometric meaning to the mean curvature and the second fundamental form of almost every $\partial\Omega_t^p$. Consequently, we present a weak version of the Gauss-Bonnet theorem: let (M, g) be a complete, noncompact, 3-dimensional Riemannian manifold, let $p \in (1, 2)$, and w_p be the solution to (1). Then, for almost every $t \in [0, +\infty)$ it holds

$$\int_{\partial\Omega^p t} \mathrm{R}^\top \, \mathrm{d}\mathcal{H}^{n-1} \in 8\pi\mathbb{Z}.$$

In a sense, in [5] we have shown that equation (1) for $p > 1$ is a more analytical counterpart to the weak inverse mean curvature flow both at the level of the convergence of solutions and of their gradients: $\partial\Omega_t^p$ converges (up to subsequence) to $\partial\Omega_t^1$ as curvature varifolds for almost every $t > 0$ and $\nabla w_p \to \nabla w_1$ in L^q_{loc}.

A clear example illustrating that the case $p > 1$ is the mirror image of the more degenerate, yet geometrically rich, case $p = 1$ also at the level of monotonicity formulas and their consequences is is the proof of the Ricci-pinching conjecture.

A Riemannian manifold (M, g) is said to be *Ricci-pinched* if $\mathrm{Ric} \geq 0$ and there exists a constant $\varepsilon > 0$ such that $\mathrm{Ric} \geq \varepsilon \mathrm{R} g$, where Ric and R are the Ricci and scalar curvature of (M, g), respectively. Hamilton conjectured that a complete, connected, Ricci-pinched Riemannian 3-manifold is flat or compact. This conjecture has been proven to be a theorem through the use of Ricci flow by Lott [12], Deruelle-Schulze-Simon [8] (with certain curvature hypotheses), and by Lee-Topping [11] (in full generality).

If we additionally suppose that M has superquadratic volume growth, the result can be shown with a simple proof based on the monotonicity of the Willmore functional along the inverse mean curvature flow [10]. For simplicity, we stick with the more restrictive case of (M, g) with strictly positive *asymptotic volume ratio*

$$\mathrm{AVR}(g) = \frac{3}{4\pi} \lim_{r \to +\infty} \frac{|B_r(p)|}{r^3} > 0, \qquad \text{with } p \in M.$$

Note that this condition is independent of the point $p \in M$.

Let M be noncompact and suppose by contradiction that M is not flat. Then there must exist a point $o \in M$ with $\mathrm{R}(o) > 0$ and a radius $r \ll 1$ such that $\partial B_r(o)$ is a smooth surface with

$$\mathcal{F}_1(B_r(o)) = \int_{\partial B_r(o)} \mathrm{H}^2 \, \mathrm{d}\sigma < 16\pi.$$

Let w_1 be the weak inverse mean curvature flow starting from $\Omega = B_r(o)$. Combining Gauss equation with Gauss-Bonnet theorem one gets

$$2 \int_{\partial \Omega_t} \mathrm{Ric}(\nu, \nu) \, \mathrm{d}\sigma \geq \varepsilon \left(16\pi - \int_{\partial \Omega_t} \mathrm{H}^2 \, \mathrm{d}\sigma \right) \qquad \text{if genus}(\partial \Omega_t) = 0,$$

$$2 \int_{\partial \Omega_t} \mathrm{Ric}(\nu, \nu) + |\mathring{\mathrm{h}}|^2 \, \mathrm{d}\sigma \geq \int_{\partial \Omega_t} \mathrm{H}^2 \, \mathrm{d}\sigma \qquad \text{if genus}(\partial \Omega_t) \geq 1.$$

Then

$$\mathcal{F}_1{}'(\Omega_t) \leq -2 \int_{\partial \Omega_t} |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu, \nu) \, \mathrm{d}\sigma \leq \max\{\varepsilon \left(\mathcal{F}_1(\Omega_t) - 16\pi \right), -\mathcal{F}_1(\Omega_t)\},$$

thus $\mathcal{F}_1(\Omega_t)$ tends to zero as $t \to +\infty$, contradicting the Willmore inequality

$$\int_{\partial \Omega} \mathrm{H}^2 \, \mathrm{d}\sigma \geq 4\mathrm{AVR}(g) |\mathbb{S}|^2 \quad \text{for every } \Omega \subset M.$$

It is possible to prove the Ricci-pinching theorem along the level set flow of (1) for every $p \in [1, 2]$. However, one must replace the Willmore functional with a suitable quantity [3, 4]. Such a quantity $\mathcal{F}_p$ must be monotone along (1) and coincide with $\mathcal{F}_1$ in the limit case $p \to 1^+$. To find it, one considers as a model case $M = \mathbb{R}^3$, $\Omega = B_r(o)$. Call w_p and w_1 the solutions to (1) for $p > 1$ and $p = 1$ respectively. Then, it holds $w_p = \frac{3-p}{2} w_1$. Hence $|\nabla w_p| = \left(\frac{3-p}{2}\right) |\nabla w_1| = \left(\frac{3-p}{2}\right) \mathrm{H}$, and we can interpret the term $\left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2$ as a sort of deficit from the inverse mean curvature flow in the model case.

It turns out that

$$\mathcal{F}_p(\Sigma) = \int_{\Sigma} \mathrm{H}^2 - 4 \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p} \right)^2 \mathrm{d}\sigma.$$

is a good replacement for $\mathcal{F}_1$. In [3, 4] the contraction argument is slightly more tricky and uses the p-iso-capacitary inequality. Worth noticing that, as in [10], also in [3, 4] some topological arguments must enter into play, we cannot expect a purely analytical proof. Our weak version of the Gauss-Bonnet theorem presented above turned out to be crucial.

References

[1] V. Agostiniani, M. Fogagnolo, L. Mazzieri, *Sharp Geometric Inequalities for Closed Hypersurfaces in Manifolds with Nonnegative Ricci Curvature* Invent. math. **222** (2020), 1033–1101.
[2] L. Benatti, M. Fogagnolo, L. Mazzieri, *Minkowski inequality on complete Riemannian manifolds with nonnegative Ricci curvature*, Anal. PDE **17** (2024), 3039–3077.
[3] L. Benatti, A. León Quirós, F. Oronzio, A. Pluda, *Nonlinear potential theory and Ricci-pinched* 3*-manifolds*, preprint arXiv:2412.20281.
[4] L. Benatti, F. Oronzio, C. Mantegazza, A. Pluda, *A note on Ricci-pinched* 3*-manifolds*, preprint arXiv:2409.05078.
[5] L. Benatti, A. Pluda, M. Pozzetta, *Fine Properties of Nonlinear Potentials and a Unified Perspective on Monotonicity Formulas*, preprint arXiv:2411.06462.
[6] T. H. Colding, *New Monotonicity Formulas for Ricci Curvature and Applications. I.* Acta Math. **209** (2012), 229–263.
[7] T. H. Colding, W. P. Minicozzi, *On Uniqueness of Tangent Cones for Einstein Manifolds*, Invent. Math. **196** (2014), 515–588.
[8] A. Deruelle, F. Schulze, M. Simon, *Initial stability estimates for Ricci flow and three dimensional Ricci-pinched manifolds*, preprint arXiv:2203.15313.
[9] G. Huisken, T. Ilmanen, *The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality*, J. Differential Geom. **59** (2001), 353–437.
[10] G. Huisken, T. Körber, *Inverse mean curvature flow and Ricci-pinched three-manifolds*, J. Reine Angew. Math. **814** (2024), 1–8.
[11] M. C. Lee, P. M. Topping, *Three-manifolds with non-negatively pinched Ricci curvature*, J. Differential Geom. (to appear) preprint: arXiv.2204.00504.
[12] J. Lott, *On 3-manifolds with pointwise pinched nonnegative Ricci curvature*, Math. Ann. **388** (2024) 2787–2806.
[13] C. Xia, J. Yin, X. Zhou, *New monotonicity for p-capacitary functions in 3-manifolds with nonnegative scalar curvature.* Adv. Math. **440** (2024), Paper No. 109526.

Deriving Perelman's Entropy from Colding's Monotonic Volume

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(joint work with Javier Peraza)

Monotonic quantities play a fundamental role in elliptic and parabolic PDEs, particularly in the study of singularities, regularity of solutions, and asymptotics [5]. For instance, Huisken's monotonicity formula for the mean curvature flow [4] and Struwe's monotonicity formula for harmonic map heat flows [8] (see also Hamilton's generalization of both formulas to general manifolds [7]), are well-known examples of monotonic quantities for parabolic equations, which have been widely applied in geometric analysis. For the 'elliptic counterparts' of these parabolic cases, the minimal surface equation and the harmonic map equation, there are analogous, though distinct, quantities, such as Allard's monotonicity for minimal surfaces [11] and the well known monotonicity for harmonic maps [20], respectively. Additional examples include Hamilton's monotonic formula for the Yang-Mills heat flow [7] and Price's monotonic formula for the elliptic Yang-Mills equation [6], as well as

Almgren's frequency for harmonic functions [17] and Poon's parabolic frequency for the heat equation [18] (see also the recent generalization of the parabolic frequency to manifolds by Colding and Minicozzi [19]).

Parabolic monotonic formulas are often intriguing and challenging to derive and typically rely on a peculiar use of backward solutions to heat-type equations, whereas elliptic quantities do not. Despite appearing unrelated, Davey [2] (see also [1]) demonstrated that, at least in a handful of examples that include some of the ones mentioned earlier, the parabolic monotonicity can be derived from a subtle use of elliptic ones. This is achieved by applying elliptic monotonicity to associated equations on suitably constructed N-dimensional spaces, and taking the limit as $N \to \infty$. These equations, which are equivalent to the parabolic equation (they hold if and only if the parabolic does) are formally identical to the elliptic one up to terms that vanish as $N \to \infty$. Put differently, the parabolic equation can be viewed as the elliptic one in an N-space plus negligible additional terms, allowing the monotonic parabolic formula to be derived as a limit of an elliptic one (see the related discussions by Tao [9] and Šverák [10]).

For the Ricci-flat equation,

$$Ric = 0, \tag{1}$$

the best known monotonic quantity is the Bishop-Gromov relative volume, widely applied across a variety of contexts in differential geometry. Another example, which will be central to this article, was introduced by Colding in [12]. This new quantity, which we will refer to as 'monotonic volume', is defined along the level sets of positive Green functions and was used, for instance, to study asymptotic cones on Ricci flat non-parabolic manifolds [12, 13]. Generalizations of Colding's monotonic volume were later given by Colding and Minicozzi in [16] and applications to General Relativity were explored by Agostiniani, Mazzieri and Oronzio in [14].

For the Ricci flow equation,

$$\partial_t g = -2Ric, \tag{2}$$

the 'parabolic counterpart' of the Ricci flat equation, Perelman [3] introduced two fundamental monotonic quantities: the reduced volume and the entropy. Additionally, he demonstrated that the reduced volume can be derived from the Bishop-Gromov relative volume on a carefully constructed N-space that becomes Ricci-flat as the dimension N goes to infinity. However, no analogous justification on the origin of the entropy was provided.

In this talk, we demonstrate that Perelman's entropy arises as the limit as $N \to \infty$ of Colding's monotonic volume when appropriately applied to Perelman's N-space. This proves that, as in the previously discussed parabolic examples, both the reduced volume and the entropy can be understood as originating from a unified framework using monotonic formulas from the 'counterpart' Ricci flat equation.

References

[1] B. Davey and M. Smit Vega Garcia, Variable-coefficient parabolic theory as a high-dimensional limit of elliptic theory, *Calc. Var.*, 63(40), 2024.

[2] B. Davey, Parabolic Theory as a High-Dimensional Limit of Elliptic Theory, *Arch. Rational Mech. Anal.*, 228, 2018.

[3] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, *arXiv preprint* arXiv:math/0211159, 2002.

[4] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, *J. Differential Geom.*, 31(1):285–299, 1990.

[5] T.H. Colding and W.P. Minicozzi, Monotonicity and its analytic and geometric implications, *Proc. Natl. Acad. Sci. USA*, 110(48), 2012.

[6] P. Price, A monotonicity formula for Yang-Mills fields, *Manuscripta Math.*, 43(2–3), 1983.

[7] R.S. Hamilton, Monotonicity formulas for parabolic flows on manifolds, *Commun. Anal. Geom.*, 1(1), 1993.

[8] M. Struwe, On the evolution of harmonic maps in higher dimensions, *J. Differential Geom.*, 28(3), 1988.

[9] T. Tao, *Poincaré's Legacies: Pages from Year Two of a Mathematical Blog*, American Mathematical Society, Providence, RI, 2009.

[10] V. Šverák, Theory of PDE, course notes, `https://www-users.cse.umn.edu/~sverak/course-notes.pdf`, 2011.

[11] W.K. Allard, On the First Variation of a Varifold, *Ann. Math.*, 95(3):417, 1972.

[12] T.H. Colding, New monotonicity formulas for Ricci curvature and applications. I, *Acta Math.*, 209(2):229–263, 2012.

[13] T.H. Colding and W.P. Minicozzi, On uniqueness of tangent cones for Einstein manifolds, *Invent. Math.*, 196(3):515–588, 2013.

[14] V. Agostiniani, L. Mazzieri, and F. Oronzio, A Green's Function Proof of the Positive Mass Theorem, *Commun. Math. Phys.*, 405(2), 2024.

[15] H.-D. Cao and X.-P. Zhu, Hamilton–Perelman's Proof of the Poincaré Conjecture and the Geometrization Conjecture, `https://arxiv.org/abs/math/0612069`.

[16] T.H. Colding and W.P. Minicozzi, Ricci curvature and monotonicity for harmonic functions, *Calc. Var. Partial Differ. Equ.*, 49(3–4):1045–1059, 2013.

[17] F.J. Almgren Jr., Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two, *Bull. Am. Math. Soc. (N.S.)*, 8(2):327–328, 1983.

[18] C. Poon, Unique Continuation for parabolic equations, *Commun. Partial Differ. Equ.*, 21(3–4):521–539, 1996.

[19] T.H. Colding and W.P. Minicozzi II, Parabolic Frequency on Manifolds, *Int. Math. Res. Not.*, 2022(15):11878–11890, 2021.

[20] R. Schoen and K. Uhlenbeck, A regularity theory for harmonic maps, *J. Differential Geom.*, 17(2), 1982

The Conformally Constrained Willmore Flow and Figure-8 Curves

Fabian Rupp

(joint work with Anna Dall'Acqua, Marius Müller, Manuel Schlierf)

The Willmore functional is a conformally invariant energy that measures the total bending of an immersed closed surface $f\colon \Sigma \to \mathbb{R}^3$ by

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |H|^2 \mathrm{d}\mu,$$

where H is the mean curvature vector and μ is the area measure of the induced metric $f^*g_{\mathbb{R}^3}$. We are interested in minimizing $\mathcal{W}$ in a dynamic way among tori with prescribed *conformal class* which we identify with some complex number $\omega = x + iy \in \mathbb{C}$ with $0 \leq x \leq \frac{1}{2}$, $y > 0$, and $|\omega| \geq 1$, see [3] and [5] for the static case. This constraint is analytically inconvenient to handle since the Teichmüller projection degenerates at isothermic immersions, such as CMC surfaces and surfaces of revolution. Formally, the constrained gradient flow equation takes the form

$$\partial_t f = -(\Delta H + |A^0|^2 H - \langle q, A^0 \rangle), \tag{1}$$

where A^0 is the trace free part of the second fundamental form and q is a transverse traceless symmetric 2-covariant tensor, however this equation is difficult to implement directly due to the degeneracy of the Teichmüller projection.

For rotationally symmetric tori $f_\gamma \colon \mathbb{T}^2 \to \mathbb{R}^3$, we can fully describe the flow as a weighted gradient flow of the profile curves $\gamma \colon \mathbb{S}^1 \to (\mathbb{H}^2, g_{\mathbb{H}^2})$ in hyperbolic geometry. Indeed, the Willmore energy and the conformal class of f_γ can be expressed in terms of the profile curves. We have

$$\mathcal{W}(f_\gamma) = \frac{\pi}{2}\mathcal{E}(\gamma), \qquad \omega = i \cdot \max\left\{\frac{\mathcal{L}(\gamma)}{2\pi}, \frac{2\pi}{\mathcal{L}(\gamma)}\right\},$$

where $\mathcal{E}, \mathcal{L}$ denote the (hyperbolic) elastic energy and length, given by

$$\mathcal{E}(\gamma) = \int_\gamma |\kappa|^2 \mathrm{d}s, \qquad \mathcal{L}(\gamma) = \int_\gamma \mathrm{d}s,$$

with κ the curvature vector and $\mathrm{d}s$ the arc-length element of $\gamma \colon \mathbb{S}^1 \to (\mathbb{H}^2, g_{\mathbb{H}^2})$. Viewing $\mathbb{H}^2 = \mathbb{R} \times (0, \infty)$, (1) may be equivalently expressed in terms of the evolving profile curves by

$$\partial_t \gamma = -\frac{1}{4\gamma_2^4}\Big(2\nabla_s^2 \kappa + |\kappa|^2 \kappa - 2\kappa - \lambda(\gamma)\kappa\Big), \tag{2}$$

where ∇_s is the connection in the normal bundle along γ and $\lambda(\gamma)$ is a suitable critical Lagrange multiplier which is chosen such that $\partial_t \mathcal{L}(\gamma) = 0$ along a solution. This can be seen as the constrained gradient flow of $\mathcal{E}$ fixing $\mathcal{L}$ with respect to a metric that degenerates near the axis of rotation.

Our main result yields that singularities can only form if the total hyperbolic curvature of the profile curve vanishes and provides a criterion to exclude this, ensuring global existence and full convergence without the usual 8π-energy bound.

Theorem 1 ([2]). *Let $f_0 \colon \mathbb{T}^2 \to \mathbb{R}^3$ be a rotationally symmetric torus. Then there exists a maximal solution $f \colon [0, T) \times \mathbb{T}^2 \to \mathbb{R}^3$ to the conformally constrained Willmore flow* (1) *such that at least one of the following is true.*

- *(i) There exists $t_j \nearrow T$ along which the hyperbolic total curvature of the profile curves converges to zero.*
- *(ii) We have $T = \infty$ and, after conformal reparametrization, the flow smoothly converges to a conformally constrained Willmore torus.*

Moreover, if $\mathcal{W}(f_0) \leq 8\pi$ or if the initial profile curve has hyperbolic length L and turning number $m \in \mathbb{Z}$ such that $L < 2\pi|m|$, then case (ii) *occurs.*

We use our flow to find new examples of conformally constrained Willmore tori with energy exceeding 8π. Case (i) in the theorem does not necessarily imply the formation of a singularity. Indeed, there exist stationary solutions γ_*, the so-called figure-8-elastica, which have vanishing hyperbolic total curvature. In fact, we even conjecture the conformally constrained Willmore flow of tori of revolution cannot develop any singularities, neither in finite or in infinite time, regardless of the behavior of the hyperbolic total curvature.

The figure-8-elastica γ_* also plays an important role in the classical (unconstrained) Willmore flow

$$\partial_t^\perp f = -(\Delta H + |A^0|^2 H), \tag{3}$$

along which it necessarily forms a singularity. As in the singular example for spheres, it is not known if this occurs in finite or infinite time, although there is some numerical evidence [1] for the latter. Assuming this and non-degenerating area, we can show that the singular limit is a drop-shaped non-smooth torus of revolution, the inverted catenoid. We construct a parametrization of this surface as a $C^{1,\alpha}$-graph over a sphere and use a local well-posedness result in this low regularity regime [4] to flow it to a round sphere.

Theorem 2 ([2]). *There exists a parametrization f_0 of an inverted catenoid with $f_0 \in C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$ for all $\alpha \in (0,1)$ and a family of immersions $f \in C^\infty((0,\infty) \times \mathbb{S}^2, \mathbb{R}^3)$ with*

$$\|f(t) - f_0\|_{C^{1,\alpha}(\mathbb{S}^2)} \to 0 \quad \text{for } t \searrow 0, \text{ and}$$
$$\mathcal{W}(f(t)) < 8\pi \text{ for } t > 0, \quad \mathcal{W}(f(t)) \nearrow 8\pi \quad \text{for } t \searrow 0,$$

satisfying (3) *such that $f(t)$ smoothly converges to a round sphere for $t \to \infty$ up to reparametrization.*

Our result suggests that the singular flow of γ_*, if it exists globally and has non-degenerating area, may be extended past its singularity in a way that changes topology and does not cause jumps in the energy. We believe that this provides an essential example to keep in mind for further investigations of singularities of the Willmore flow, particularly in the context of developing *surgery theory* or concepts of *weak solutions.*

References

[1] J. W. Barrett, H. Garcke and R. Nürnberg, *Stable approximations for axisymmetric Willmore flow for closed and open surfaces*, ESAIM Math. Model. Numer. Anal. **55** (2021), no. 3, 833–885.

[2] A. Dall'Acqua, M. Müller, F. Rupp, and M. Schlierf, *Dimension reduction for Willmore flows of tori: fixed conformal class and analysis of singularities*, preprint (2025).

[3] E. C. Kuwert and R. M. Schätzle, *Minimizers of the Willmore functional under fixed conformal class*, J. Differential Geom. **93** (2013), no. 3, 471–530.

[4] J. S. LeCrone, Y. Shao and G. Simonett, *The surface diffusion and the Willmore flow for uniformly regular hypersurfaces*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), no. 12, 3503–3524.

[5] T. Rivière, *Variational principles for immersed surfaces with L^2-bounded second fundamental form*, J. Reine Angew. Math. **695** (2014), 41–98.

On the Structure of Singularities of Mean Curvature Flows with Mean Curvature Bounds

Maxwell Stolarski

A family of hypersurfaces $\{M_t^n \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ evolving by mean curvature flow generally encounters singularities in finite time. At such singularities, the second fundamental form of the hypersurface always blows up [3], but its trace, the mean curvature, can remain bounded [7]. After reviewing examples of this pathological singularity formation [1, 7, 9], we demonstrate how to incorporate the theory of varifolds with bounded mean curvature to study the general structure of singularities of mean curvature flows with uniform mean curvature bounds [8].

In particular, we can combine the compactness theorems and monotonicity formulas for integer rectifiable n-varifolds with bounded first variation together with the compactness theorems and monotonicity formula for mean curvature flows [4] to study mean curvature flows with uniform mean curvature bounds. This implies the existence and uniqueness of a final time slice of the flow, in a weak sense. Spatial blow-ups of the final time-slice yield tangent cones. Analogously, spacetime blow-ups of the flow yield tangent flows, which, for flows with bounded mean curvature, are necessarily static flows of stationary cones. In fact, the tangent cones occuring as spatial blow-ups of the final time-slice exactly correspond to the cones arising as tangent *flows*. Consequently, the tangent flow is unique if and only if the tangent cone of the final time-slice is unique.

Uniqueness of tangent flows is generally an open problem with major implications for the regularity of the flow. For mean curvature flows with bounded mean curvature, we show that if the tangent flow based at a point (x_0, T) in the final time-slice is given by the static flow of a multiplicity one stationary cone $\mathcal{C}$ such that $\mathcal{C} \setminus \{\mathbf{0}\}$ is smooth, then the tangent flow is unique. Here, uniqueness of the tangent flow at (x_0, T) reduces to the uniqueness of the tangent *cone* at x_0 in the final time-slice. We show graphicality (over $\mathcal{C}$) of a time-slice propagates outward in space and backward in time. This propagation of graphicality combined with interior regularity of the flow implies that time-slices satisfy an additional interior regularity condition. With that, one can then apply Simon's uniqueness result for tangent cones of integer n-rectifiable varifolds with bounded first variation [6] to prove this uniqueness of tangent flows result.

Because the compactness theorems and monotonicity formulas generalize to varifolds and Brakke flows, all the results above generalize to Brakke flows of arbitrary co-dimension with integral mean curvature bounds in an open subset of Euclidean space. When the flow is smooth, if additionally the tangent flow is given by the Simons cone in dimensions $n \geq 7$, then we characterize the smooth minimal surfaces that pinch off at smaller scales around a singularity. These are precisely given by the minimal surfaces constructed in [2] which are asymptotic to the Simons cone.

The work discussed here prompts several questions and directions for future research. Recall that for mean curvature flows with uniformly bounded mean curvature, uniqueness of tangent flows at (x_0, T) is equivalent to uniqueness of tangent cones of the $t = T$ time-slice at x_0. These time-slices are generally integer n-rectifiable varifolds with bounded first variation. While there are many uniqueness results for tangent cones of stationary integer n-rectifiable varifolds, there very few such uniqueness results for tangent cones of integer n-rectifiable varifolds with bounded first variation. Potential future work generalizing the uniqueness results for stationary varifolds to varifolds with bounded first variation, perhaps under some additional assumptions, would immediately imply uniqueness of tangent flows for mean curvature flows with bounded mean curvature.

Furthermore, the examples of finite-time mean curvature flow singularities with bounded mean curvature [7] are only known to exist dimensions $n \geq 7$. [5] proved that the mean curvature must blow-up at a finite-time singularity when $n = 2$. However, it is unknown whether the mean curvature must blow-up at a finite-time singularity when $2 < n < 7$. [5] relies on the Gauss-Bonnet formula, and so does not generalize to higher dimensions $n > 2$. On the other hand, [7] and the foundational work [9] use stability of the Simons cone in dimensions $n \geq 7$, and so does not generalize to dimensions $n < 7$. It therefore remains an open question whether the mean curvature always blows up at a finite-time mean curvature flow singularity in dimensions $3 < n < 7$.

References

[1] S. Angenent, P. Daskalopoulos, and N. Sesum, *Type II smoothing in mean curvature flow*, arXiv pre-prints 2108.08725 (2021).

[2] R. Hardt and L. Simon, *Area minimizing hypersurfaces with isolated singularities*, Journal für die reine und angewandte Mathematik **362** (1985), 102–129.

[3] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, Journal of Differential Geometry **20** (1984), 237–266.

[4] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, Journal of Differential Geometry **31** (1990), 285–299.

[5] H. Li and B. Wang, *The extension problem of the mean curvature flow (I)*, Inventiones mathematicae **218** (2019), 721–777.

[6] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Annals of Mathematics **118** (1983), 525–571.

[7] M. Stolarski, *Existence of mean curvature flow singularities with bounded mean curvature*, Duke Mathematical Journal **172** (2023), 1235–1292.

[8] M. Stolarski, *On the Structure of Singularities of Weak Mean Curvature Flows with Mean Curvature Bounds*, arXiv pre-prints 2311.16262 (2023).

[9] J. J. L. Velázquez, *Curvatuer blow-up in perturbations of minimal cones evolving by mean curvature flow*, Annali della Scuola Normale Superiore di Pisa **21** (1994), 595–628.

Quartically Pinched Submanifolds for the Mean Curvature Flow in the Sphere

Artemis Aikaterini Vogiatzi

Mean curvature flow is a geometric evolution equation that describes how a submanifold embedded in a higher-dimensional space evolves over time by deforming in the direction of its mean curvature vector. In this work, we introduce a new sharp curvature pinching condition for the mean curvature flow in the sphere, that is of fourth order in the second fundamental form. More precisely, we consider

$$|A|^2 \leq \sqrt{\left(\frac{|H|^2}{n-2} + 4K\right)^2 + (4n-16)K^2},$$

where K is the constant sectional curvature of $\mathbb{S}^{n+m}$, $n \geq 8$ is the dimension of the submanifold $\mathcal{M}^n$ and $m \geq 1$ is the codimension. Of course, this quartic pinching condition is preserved by the flow. Using a blow up argument, we prove a codimension and a cylindrical estimate, where in regions of high curvature, the submanifold becomes approximately codimension one, quantitatively, and is weakly convex and moves by translation or is a self shrinker. With a decay estimate, the rescaling converges smoothly to a totally geodesic limit in infinite time, without using Stampacchia iteration or integral analysis.

While the quadratic pinching condition plays a fundamental role in controlling the singularity formation, we show that a more refined quartic pinching condition provides significantly stronger control over the geometry of the flow in the sphere. The quartic pinching condition improves upon the quadratic one in several ways. First, it incorporates the ambient curvature K, allowing for a more precise treatment of mean curvature flow in spaces of nonzero sectional curvature. This is particularly important in extending results beyond Euclidean space and understanding the behavior of submanifolds in more general ambient geometries. Secondly, this condition is sharp, for $n \geq 8$ in a topological sense. By considering the submanifold $\mathcal{M} = \mathbb{S}^2(r) \times \mathbb{S}^{n-2}(s) \subset \mathbb{S}^{n+1}(1)$, with $r^2 + s^2 = 1$, we have

$$|A|^4 - \left(\frac{|H|^2}{n-2} + 4\right)^2 - (4n-16) \;=\; \frac{4((n-2)^2-1)}{(n-2)^2} \cdot \frac{s^4}{r^4} \geq 0,$$

for any $n \geq 3$. Also, it satisfies

$$-|A|^2 + \sqrt{\left(\frac{|H|^2}{n-2} + 4K\right)^2 + 4(n-4)K^2} \;>\; -|A|^2 + \frac{|H|^2}{n-2} + 4K,$$

meaning that it improves the quadratic pinching condition in the sphere. This quartic pinching condition also satisfies

$$\frac{|H|^2}{n-2} + 4K \;<\; \sqrt{\left(\frac{|H|^2}{n-2} + 4K\right)^2 + 4(n-4)K^2} \;<\; \frac{|H|^2}{n-2} + 2\sqrt{n}K,$$

where the upper bound of the quartic pinching is not preserved by the flow, but it simplifies our calculations down to second order.

This result improves rigidity of singularity models and leads to sharper classification results for possible singularity formations. This leads to stronger regularity results and enables us to obtain refined long-time convergence theorems for the mean curvature flow in the sphere. Overall, the introduction of the quartic pinching condition provides a substantial refinement of our analysis, leading to stronger curvature control, a more precise classification of singularity models, and improved long-time behaviour results for the flow.

Our approach fundamentally relies on the preservation of a curvature pinching condition along the flow and pointwise gradient estimates that provides precise control over the mean curvature in regions of high curvature. Specifically, the pointwise gradient estimates for the mean curvature flow stem directly from a quartic curvature bound, which in turn plays a key role in our analysis. The significance of these gradient estimates lies in their ability to provide control over the mean curvature and consequently the entire second fundamental form within a fixed-size neighbourhood. Unlike general parabolic-type derivative estimates, which depend on global curvature maxima, our estimates rely solely on pointwise mean curvature. This distinction is critical, as it allows us to maintain curvature control throughout the blow-up process leading to singularities. Using these gradient estimates, we obtain a codimension estimate that establishes that in regions of high curvature, under a quadratic pinching assumption, singularity models must always be codimension one, regardless of the original codimension of the evolving submanifold. This insight leads to the conclusion that at a singular time, there exists a rescaling of the flow that converges to a smooth codimension one limiting flow in Euclidean space. Additionally, we establish a cylindrical estimate that provides a refinement of the curvature pinching as the flow approaches a singularity. These estimates demonstrate that under a cylindrical-type pinching assumption, the limiting flow must be weakly convex and evolves by either translation or is self-shrinking. Our classification results show that singularity models for this pinched flow can be described, up to homothety, as either shrinking round spheres, shrinking round cylinders, or translating bowl solitons.

Lastly, due to the spherical background space, with a decay estimate, the rescaling converges smoothly to a totally geodesic limit in infinite time. This result is obtained without the use of Stampacchia iteration or integral analysis, as it was vastly used in papers so far. Similar work has been done using the quadratic bound in $\mathbb{C}P^n$, in [14].

References

[1] Ben Andrews and Charles Baker. Mean curvature flow of pinched submanifolds to spheres. *J. Differential Geom.*, 85(3):357–395, 2010.

[2] Charles Baker. The mean curvature flow of submanifolds of high codimension. Australian National University, 2011. Thesis (Ph.D.)–Australian National University.

[3] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[4] Gerhard Huisken. Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. *Invent. Math.*, 84(3):463–480, 1986.

[5] Gerhard Huisken. Deforming hypersurfaces of the sphere by their mean curvature. *Mathematische Zeitschrift*, 195, 205–219, 1987.

[6] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow singularities for mean convex surfaces. *Calc. Var. PDE*, 8(1):1–14, 1999.

[7] Stephen Lynch and Huy The Nguyen. Convexity estimates for high codimension mean curvature flow. Preprint, arXiv:2006.05227 [math.DG].

[8] Keaton Naff. A planarity estimate for pinched solutions of mean curvature flow. *Duke Math. J.*, 171(2):443–482, 2022.

[9] Keaton Naff. Singularity models of pinched solutions of mean curvature flow in higher codimension. *Journal für die reine und angewandte Mathematik*, 2023(794), 2023.

[10] Huy The Nguyen. Cylindrical Estimates for High Codimension Mean Curvature Flow. arXiv:1805.11808v1, 2018.

[11] Huy The Nguyen. High Codimension Mean Curvature Flow with Surgery. Preprint, arXiv:2004.07163v2 [math.DG].

[12] Huy T. Nguyen and Artemis A. Vogiatzi. Singularity Models for High Codimension Mean Curvature Flow in Riemannian Manifolds. Preprint, arXiv:2303.00414v1 [math.DG].

[13] Giuseppe Pipoli and Carlo Sinestrari. Mean curvature flow of pinched submanifolds of $\mathbb{CP}^n$. *Comm. Anal. Geom.*, 25 (4): 799 - 846, 2017.

[14] Artemis A. Vogiatzi. Mean Curvature Flow of High Codimension in Complex Projective Space. Preprint, arXiv:2311.01407v2 [math.DG].

[15] Artemis A. Vogiatzi. Sharp quartic pinching for the mean curvature flow in the sphere. Preprint, arXiv:2408.08022v1 [math.DG].

σ_k-Yamabe problem and optimal Sobolev inequalities

Guofang Wang

(joint work with Yuxin Ge, Wei Wei)

In this talk we are interested in two objects, the σ_k-Yamabe problem and various optimal Sobolev inqualities on $\mathbb{S}^n$.

1. σ_k-Yamabe problem, revisited

Let (M, g_0) be a compact Riemannian manifold with metric g_0 and $[g_0]$ the conformal class of g_0. Let S_g be the Schouten tensor of the metric g defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

Here Ric_g and R_g are the Ricci tensor and scalar curvature of g respectively. The importance of the Schouten tensor in conformal geometry can be viewed in the following decomposition of the Riemann curvature tensor

$$Riem_g = W_g + S_g \wedge g,$$

where $\wedge$ is the Kulkani-Nomizu product. Note that $g^{-1} \cdot W_g$ is invariant in a given conformal class.

Define $\sigma_k(g)$ be the σ_k*-scalar curvature* or k*-scalar curvature* by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$ and σ_k is the kth elementary symmetric function. Here for an $n \times n$ symmetric matrix A we define $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \cdots, \lambda_n)$ is the set of eigenvalues of A. It is clear that $\sigma_1(g)$ is a constant multiple of the scalar curvature R_g. The k-scalar curvature $\sigma_k(g)$, which was first studied by Viaclovsky, is a natural generalization of the scalar curvature.

We are interested in finding "good" metrics in a conformal class. A constant σ_k-curvature metric is such a good one. When $k = 1$, the problem becomes to find metrics with constant scalar. This is the well-known Yamabe problem and solved finally by Schoen. When $k \geq 2$, the "standard" σ_k-Yamabe problem asks, if there is a conformal metric g in a class $\mathcal{C}_k([g_0])$ such that

$$\sigma_k(g) = const., \tag{1}$$

if

$$\mathcal{C}_k([g_0]) \neq \emptyset. \tag{2}$$

Here $\mathcal{C}_k([g_0]) = \Gamma_k^+ \cap [g_0]$ and

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \leq k\} \tag{3}$$

are Garding's cones. Since for $k \geq 2$, (1) is fully nonlinear equation, one needs to require (2) to guarantee the ellipticity of (1). The problem was solved either (i) when $k = 2$ or when (M, g_0) is locally conformally flat, or (ii) $k \geq n/2$.

Here we are interested in

Problem. *Find a solution of* (1), *i.e.* $\sigma_k(g) = c$, *in the following larger cone*

$$\mathcal{C}_{k-1}([g_0]) \neq \emptyset. \tag{4}$$

The problem was asked by Jeffrey Case to us. We gave an affirmative answer to this question.

2. Optimal Sobolev inequalities

The Sobolev inequality on $\mathbb{R}^n$, $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq c(n) \left\{ \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \right\}^{\frac{n-2}{n}} \tag{5}$$

plays an important role in analysis and differential geometry. Here $c(n)$ or $c(k, n)$ and $c(s, n)$ is optimal constant, depending only on n or/and k or s, such that the inequality holds with that constant on the right side. There are higher order generalizations of the Sobolev inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \geq c(s, n) \left\{ \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2s}} \right\}^{\frac{n-2s}{2}}, \tag{6}$$

for $0 < s < n/2$. Here $(-\Delta)^{s/2}$ is the fractional Laplacian. When $s = k$ for some natural number k, it is clear that

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 = \int_{\mathbb{R}^n} |\nabla^k u|^2.$$

Hence (6) in this case is the k-th order Sobolev inequality. Recall the GJMS operator on $\mathbb{S}^n$, defined by

$$L_{2k} := \Pi_{j=1}^{k}\left(-\Delta + \frac{(n-2j)(n+2j-2)}{4}\right). \tag{7}$$

Therefore, in this case,i.e., $s = k$, inequality (6) is equivalent to

$$\mathcal{E}_{2k}(u) := \int_{\mathbb{S}^n} uL_{2k}u \geq \frac{\Gamma(n/2+k)}{\Gamma(n/2-k)}\omega_n^{\frac{2k}{n}}\left(\int_{\mathbb{S}^n}|u|^{\frac{2n}{n-2k}}\right)^{\frac{n-2k}{n}}, \tag{8}$$

with equality if and only if

$$u(\xi) = a(1+b\cdot\xi)^{-\frac{n-2k}{2}}, \quad \text{for } a \in \mathbb{R}, b \in \mathbb{B}^{n+1}. \tag{9}$$

If u is positive and let $g = u^{\frac{4}{n-2k}}g_{\mathbb{S}^n}$, then $\mathcal{E}_{2k}(g) := \mathcal{E}_{2k}u$ is the so-called total k-th Q curvature, while the right hand side is a suitable power of the volume of g. Hence (8) is an isoperimetric type inequality:

The total k-th Q curvature achieves its minimum at the round sphere metric $g_{\mathbb{S}^n}$ among conformal metrics g with a volume constraint, i.e., $vol(g) = vol(g_{\mathbb{S}^n})$.

It is therefore natural to ask

Is there an isoperimetric inequality between the total k-th Q curvature and the total l-th Q curvature for $0 < l < k \leq n/2$?

We propose the following conjecture

Conjecture. *Let $0 < l < k < n/2$. The following inequality*

$$\int_{\mathbb{S}^n} uL_{2k}u \geq \frac{\Gamma(n/2+k)}{\Gamma(n/2-k)}\left(\frac{\Gamma(n/2+l)}{\Gamma(n/2-l)}\right)^{-\frac{n-2k}{n-2l}}\omega_n^{\frac{2(k-l)}{n-2l}}\left(\int_{\mathbb{S}^n} u^{\frac{n-2l}{n-2k}}L_{2l}u^{\frac{n-2l}{n-2k}}\right)^{\frac{n-2k}{n-2l}}, \tag{10}$$

holds for any smooth positive function u with a property that $g = u^{\frac{4}{n-2k}}g_{\mathbb{S}^n}$ has a positive k-th Q curvature. Moreover, equality holds if and only if

$$u(\xi) = a(1+b\cdot\xi)^{-\frac{n-2k}{2}}, \quad \textit{for } a \in \mathbb{R}, b \in \mathbb{B}^{n+1}. \tag{11}$$

We prove that the Conjecture is true for $k = 2$ and $l = 1$ under a weaker condition.

Theorem. *Let $n \geq 5$. For any smooth positive function u with a property that $g = u^{\frac{4}{n-4}}g_{\mathbb{S}^n}$ has a positive scalar curvature, i.e., $L_2u^{\frac{n-2}{n-4}} > 0$, it holds*

$$\int_{\mathbb{S}^n} uL_4u \geq (\frac{n}{2}-2)(\frac{n}{2}+1)\{(\frac{n}{2})(\frac{n}{2}-1)\}^{\frac{2}{n-2}}\omega_n^{\frac{2}{n-2}}\left(\int_{\mathbb{S}^n} u^{\frac{n-2}{n-4}}L_2u^{\frac{n-2}{n-4}}\right)^{\frac{n-4}{n-2}}, \tag{12}$$

with equality if and only if

$$u(\xi) = a(1+b\cdot\xi)^{-\frac{n-4}{2}}, \quad \textit{for } a \in \mathbb{R}, b \in \mathbb{B}^{n+1}. \tag{13}$$

The proof follows from our previous results with collaborators, [3, 4, 1, 2].

The result leads to the study of the following new Yamabe problem: finding a conformal metric with a constant quotient of two different total Q curvatures. For example

$$\frac{Q_2(g)}{Q_1(g)} = const.,$$

where $Q_1(g) = R_g$ the scalar curvature. We proved an Obata type theorem for this problem. The existence will be carried out in our joint program.

References

[1] Ge, Y., Lin, C.-S., Wang, G., On the σ_2-scalar curvature, J. Differential Geom. 84 (2010), no. 1, 45–86.

[2] Ge, Y., Wang, G., On a conformal quotient equation. II Ge, Y. Wang, G., Comm. Anal. Geom. 21 (2013), no. 1, 1–38.

[3] Guan, P., Viaclovsky, J., Wang, G., Some properties of the Schouten tensor and applications to conformal geometry, Trans. Amer. Math. Soc. 355 (2003), no. 3, 925–933.

[4] Guan, P., Wang, G., Geometric inequalities on locally conformally flat manifolds, Duke Math. J. 124 (2004), no. 1, 177–212.

Area preserving null mean curvature flow along asymptotically Schwarzschildean lightcones

Markus Wolff

(joint work with Klaus Kröncke)

A version of mean curvature flow along a null hypersurface $\mathcal{N}$ was first studied by Roesch–Scheuer [5], and defined there as the projection of the codimension-2 flow (tangentially) onto $\mathcal{N}$. Recall that a null hypersurface $\mathcal{N}$ in an ambient spacetime $(\overline{M}, \overline{g})$ is an oriented hypersurface with a degenerate induced metric. In particular, there exists a *null generator* $\underline{L} \in \Gamma(T\mathcal{N})$ such that $\underline{L}$ is orthogonal to all tangent directions on $\mathcal{N}$ (including itself). Hence, $\underline{L}$ is orthogonal to any spacelike cross section $\Sigma \subseteq \mathcal{N}$, i.e., a spacelike submanifold Σ such that all integral curves of $\underline{L}$ intersect Σ precisely once. In particular, for a fixed choice of $\underline{L}$ and spacelike cross section S_0, any spacelike cross section Σ can be written as a graph of a function ω over S_0.

In addition, $\underline{L}$ gives rise to a null frame $\{\underline{L}, L\}$ of the normal bundle $\Gamma(T^\perp\Sigma)$ of Σ, where $L \in \Gamma(T^\perp\Sigma)$ is the unique null vector field such that $\overline{g}(\underline{L}, L) = 2$. With respect to this null frame, the vector-valued mean second fundamental form $\vec{\mathrm{II}}$ and codimension-2 mean curvature vector $\vec{\mathcal{H}}$ of Σ in the ambient spacetime decompose as

$$\vec{\mathrm{II}} = -\frac{1}{2}\chi\underline{L} - \frac{1}{2}\underline{\chi}L,$$
$$\vec{\mathcal{H}} = -\frac{1}{2}\theta\underline{L} - \frac{1}{2}\underline{\theta}L,$$

where $\underline{\chi}$, χ, and $\underline{\theta}$, θ denote the null second fundamental forms and null expansions with respect to $\{\underline{L}, L\}$, respectively. The spacetime mean curvature $\mathcal{H}^2$ of Σ is defined as the Lorentzian length of $\vec{\mathcal{H}}$, cf. [1], i.e.,

$$\mathcal{H}^2 := \overline{g}(\vec{\mathcal{H}}, \vec{\mathcal{H}}) = \underline{\theta}\theta.$$

Finally, we note that $\underline{\theta}$ extends to a well-defined, smooth function along a null hypersurface, i.e., $\underline{\theta} \in C^\infty(\mathcal{N})$, and so for any spacelike cross sections Σ_1, Σ_2 we find that $\underline{\theta}_1(p) = \underline{\theta}_2(p)$ if $p \in \Sigma_1 \cap \Sigma_2$.

As in [5], a family of spacelike cross sections $x\colon \Sigma \times [0, T) \to \mathcal{N}$ is said to evolve under mean curvature flow along $\mathcal{N}$ if

$$\frac{d}{dt}x = \frac{1}{2}\overline{g}(\vec{\mathcal{H}}, L)\underline{L} = -\frac{1}{2}\theta\underline{L}. \tag{1}$$

We note that (1) is independent of the choice of $\underline{L}$ and (locally) equivalent to a scalar parabolic equation for ω. If barriers exist and a suitable gauge condition is satisfied, Roesch–Scheuer [5] have shown that the flow (smoothly) converges to a marginally outer trapped surface (MOTS) with $\theta \equiv 0$, which are the stationary points of the flow. Note that if the null hypersurface $\mathcal{N}$ is a Null Cone, i.e., $\underline{\theta} > 0$ on $\mathcal{N}$, we can rewrite (1) as

$$\frac{d}{dt}x = -\frac{1}{2\underline{\theta}}\mathcal{H}^2\underline{L}.$$

As $\underline{\theta}$ determines the first variation of area, *area preserving null mean curvature flow* is now given by

$$\frac{d}{dt}x = -\frac{1}{2\underline{\theta}}\left(\mathcal{H}^2 - \fint \mathcal{H}^2\right)\underline{L}, \tag{2}$$

which is mean curvature flow modified to preserve area. It is straightforward to see that the stationary points of the flow are surfaces of constant spacetime mean curvature (STCMC), cf. [1]. We have proven the following stability result:

Theorem 1 (Kröncke–W. '24 [7])**.** *Let $\mathcal{N}$ be an asymptotically Schwarzschildean lightcone of mass $m > 0$. For $B_1^0, B_2^0, B_3^0 > 0$ there exists positive constants B_1, B_2, B_3 and $\sigma_0(B_1, B_2, B_3)$ such that the following holds for all $\sigma \geq \sigma_0$: If $\Sigma_0 \in B_\sigma(B_1^0, B_2^0, B_3^0)$, then the solution of area preserving null mean curvature flow starting from Σ_0 remains in $B_\sigma(B_1, B_2, B_3)$, exists for all times and (smoothly) converges (exponentially fast) to a surface of constant spacetime mean curvature.*

Here, we say a null hypersurface $\mathcal{N}$ is asymptotically Schwarzschildean if it admits an asymptotically flat background foliation, cf. [4], that asymptotes to the foliation by centered spheres of the standard Schwarzschild lightcone. In particular, we have $\underline{\theta} > 0$ in the asymptotic region, so (2) is well-defined. With respect to this background foliation of $\mathcal{N}$, we define the a-priori class $B_\sigma(B_1, B_2, B_3)$ of spacelike cross sections by

$$B_\sigma(B_1, B_2, B_3) := \left\{\Sigma \subseteq \mathcal{N}\colon |\sigma - \omega| \leq B_1, |\mathring{A}| \leq \frac{B_2}{\sigma^4}, |\nabla\mathring{A}| \leq \frac{B_3}{\sigma^5}\right\},$$

where $\mathring{A}$ denotes the trace-free part of the scalar second fundamental form $A := \underline{\theta}\chi$. We note that in the Minkowski lightcone, $\mathring{A} \equiv 0$ if and only if $\mathcal{H}^2$ is constant, cf. [6]. As $\mathcal{H}^2 = 2\,\mathrm{R}$ in this case, where R denotes the scalar curvature of Σ, $\mathcal{H}^2$ is constant if and only if Σ is an intrinsically round sphere. In particular, a-priori estimates show that any $\Sigma \in B_\sigma(B_1, B_2, B_3)$ is $C^{2,\alpha}$-close to such a round sphere. While the C^0-estimate within the a-priori class imposes a rather strong restriction on the position of these round spheres, we note that it is trivially satisfied for STCMC surfaces in the Schwarzschild lightcone, as $\omega = \sigma$ in this case, cf. [2]. This restriction is indeed crucial to establish the desired gradient estimate which shows that the a-priori class remains preserved under the flow and that the solution exists for all times. In addition, the a-priori estimates imply a suitably defined stability of the surfaces and we find

$$\frac{d}{dt}\int\left(\mathcal{H}^2 - \fint \mathcal{H}^2\right)^2 \leq -\frac{3m}{\sigma^3}\int\left(\mathcal{H}^2 - \fint \mathcal{H}^2\right)^2,$$

leading to the exponential convergence.

Using a family of coordinate spheres as initial data, we show that the limiting STCMC surfaces form a smooth foliation of $\mathcal{N}$:

Theorem 2 (Kröncke–W. '24 [7])**.** *Let $\mathcal{N}$ be an asymptotically Schwarzschildean lightcone of mass $m > 0$. Then there exists an asymptotic foliation of $\mathcal{N}$ by surfaces of constant spacetime mean curvature. Moreover, the leaves are unique within the a-priori class.*

In analogy to the seminal work of Huisken–Yau [3] in the Riemannian setting, we propose our result as a natural, geometric choice of background foliation near null infinity and define a geometric notion of center of mass in the context of general relativity.

References

[1] C. Cederbaum and A. Sakovich, *On the center of mass and foliations by constant spacetime mean curvature surfaces for isolated systems in General Relativity*, Calc. Var. **60**(214) (2021).

[2] P.N. Chen and Y.K. Wang, *Two rigidity results for surfaces in Schwarzschild spacetimes*, arXiv:2306.07477 (2023).

[3] G. Huisken and S.T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Inv. math. **124** (1996), 281–311.

[4] M. Mars, A. Soria, *The asymptotic behaviour of the Hawking energy along null asymptotically flat hypersurfaces*, Class. Quant. Grav. **32**(18) (2015), 185020.

[5] H. Roesch and J. Scheuer, *Mean Curvature Flow in Null Hypersurfaces and the Detection of MOTS*, Comm. Math. Phys. **390** (2022), 1–25.

[6] M. Wolff, *Ricci flow on surfaces along the standard lightcone in the $3+1$-Minkowski spacetime*, Calc. Var. **62**(90) (2023).

[7] K. Kröncke and M. Wolff, *Foliations of asymptotically Schwarzschildean lightcones by surfaces of constant spacetime mean curvature*, arXiv:2412.17563 (2024).

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