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Mini-Workshop: Hardy Inequalities in Discrete and Continuum Settings

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Abstract. Hardy inequalities play a fundamental role in various fields, such as mathematical physics, operator theory, the analysis of partial differential equations, potential theory, and probability theory. The goal of this workshop was to bring together leading experts to foster cross-interactions between them, with special focus on the following three topics, which naturally overlap: optimal Hardy weights on manifolds and graphs; fractional Laplacians; Sub-Riemannian analogues, magnetic Laplacians and Dirac operators.

Mathematics Subject Classification (2020): Primary: 35J10, 35R02, 43A80, 53C23, 47A10. Secondary: 26D15, 47D07, 53C21, 39A12, 31C25.

Introduction by the Organizers

The *Mini-Workshop: Hardy Inequalities in Discrete and Continuum Settings* (02 March – 07 March 2025) was organized by E. Berchio (Torino), M. Keller (Potsdam), Y. Pinchover (Haifa) and L. Roncal (Bilbao). It took place in a hybrid format, with a total of 14 invited researchers (including 3 who participated online), 1 master's student and 4 organizers, with a broad geographic and academic age representation.

On the technical side, there were 12 onsite talks and 3 online talks broadcast via Zoom. Onsite talks were approximately 75 minutes long, while online talks lasted about 50 minutes, with ample time allotted for discussion in both formats. The participants came from various mathematical backgrounds, spanning the theory of partial differential equations, mathematical physics, graph theory, harmonic analysis and spectral theory. Despite their diverse expertise, they shared a common interest: *Hardy inequalities.* Generally, the talks reported on recent advancements in the field, by the researchers and their collaborators. In order to address the different mathematical backgrounds, the participants made special effort to present talks containing the latest results in the respective fields, in a manner accessible to all participants. The talks were scheduled from Monday to Friday, with additional discussion sessions held on Tuesday afternoon and Friday morning, following the final presentations.

The topics discussed in the workshop can be summarized as follows.

- *Optimal Hardy weights on manifolds and graphs.* In the last decade, a new focus came into play, which was set on optimality of the Hardy weight itself going back to a question posed by Agmon. Specifically, optimality has both sharpness of the constant and non-existence of a minimizer as a consequence. Revisiting the original discrete roots of the problem has led to the surprising insight that, while the constant is optimal, the original weight can be improved. In $\mathbb{Z}^d$, $d \geq 3$, a precise form of an optimal Hardy weight can be proven; however, the higher order terms seem to be non-definite. Beyond $p = 2$, the optimality theory for graphs and general p exists by now, but a specific inequality for $\mathbb{Z}^d$ is still not known. Moreover, generalizations to Riemannian manifolds have been intensively pursued and analogous questions arise. During the workshop, these issues were addressed. Asymptotic behaviour of the optimal constant as the dimension $d \to \infty$ was discussed, via interesting connections between discrete Hardy inequalities on lattices with continuous Hardy-type inequalities on the torus. Additionally, new approaches to obtain optimal Poincaré–Hardy inequalities on the hyperbolic spaces were presented, and generalizations to certain classes of Riemannian manifolds were discussed, including transference methods to the discrete setting. Finally, it was shown how recent improvements in optimality have contributed to progress on Landis' conjecture for graphs.

- *Fractional Laplacians.* Hardy inequalities for the fractional Laplacian on $\mathbb{R}^d$ have been investigated by several authors, and interest was later regained with the seminal work using a non-local ground state representation approach. Despite recent advances in more general settings, there are several contexts in which fractional versions of the Hardy inequality are yet to be fully understood. In particular, there is a probabilistic interpretation of powers of the Laplacian in terms of an anomalous diffusion. From this perspective, the powers $(-\Delta)^\alpha$ in $L^2((0,\infty))$ with $\alpha \in \mathbb{N}$ are subcritical and there is no criticality transition in powers. It was discussed in the workshop the surprising fact that the situation is very different in the discrete setting, since it was shown that the integer powers of the discrete Laplacian $(-\Delta)^\alpha$ on $\ell^2(\mathbb{N})$ are subcritical if and only if $\alpha = 1$.

- *Sub-Riemannian analogues, magnetic Laplacians and Dirac operators.* Beyond the Euclidean space, lattices, and Riemannian manifolds, the study of Hardy and related inequalities has been extended to the framework of sub-Riemannian geometry, particularly in the Heisenberg group. Another vibrant branch of research concerns the study of Schrödinger-type operators in the presence of magnetic fields. Introducing non-trivial magnetic perturbations of Hamiltonian operators induces repulsive effects in quantum mechanics which have been quantified by Hardy-type inequalities. Some of the talks in these directions addressed Hardy, Hardy–Rellich, and Rellich identities and inequalities with sharp constants for Grushin vector fields, quantitative Hardy-type inequalities for the magnetic Laplacian by proving a similar inequality first for the Pauli operator.

The interaction between the different communities turned out to be very lively, featuring broad discussions on the latest open problems in the field and an exchange of state-of-the-art tools. Among the topics that sparked significant activity and exchange of knowledge were: extended Dirichlet spaces and criticality theory for nonlinear Dirichlet forms; affirmative answers to the Landis conjecture on $\mathbb{R}^d$ under a positivity assumption on the operator involved; connections of superharmonic functions for Hardy-type inequalities for Sobolev-Bregman forms and trade-offs in fractional Hardy inequalities. The mini-workshop was overall highly productive and fostered stimulating discussions among researchers from diverse backgrounds, which will, hopefully, lead to future scientific collaborations.

Mini-Workshop: Hardy Inequalities in Discrete and Continuum Settings

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Abstracts

Hardy (in)equalities for semigroups

Krzysztof Bogdan

The purpose of the talk is to point out the connection of superharmonic functions for Hardy-type inequalities for Sobolev-Bregman forms, or p-forms, where $1 < p < \infty$. The guiding principle is that for a Dirichlet form $\mathcal{E}$ on $L^2(X, dx)$ with generator $\mathcal{L}$ and (nonnegative, superharmonic) function $h > 0$, $\mathcal{L}h \le 0$, we have

$$\mathcal{E}(u,u) \ge \int_X \frac{-\mathcal{L}h}{h} u^2 dx$$

for functions u in the domain of $\mathcal{E}$, see [1]. More specifically, we consider a (symmetric) transition probability density

$$p_t(x,y) = p_t(y,x), \quad x, y \in X,\ t > 0,$$

(nonnegative) increasing function $f : [0,\infty) \to [0,\infty)$, a measure μ on X, and define

$$h(x) := \int_0^\infty f(s) P_s\mu(x) ds, \quad x \in X.$$

This function is superharmonic, in fact, excessive for the semigroup $P_t u(x) := \int_X p_t(x,y)u(y)dy$. If we denote

$$q(x) := \frac{1}{h(x)} \int_0^\infty f'(s) P_s\mu(x) ds,$$

then, guided by the above principle, we get the Hardy inequality

$$\mathcal{E}(u,u) \ge \int_X u(x)^2 q(x) dx,$$

We will use the function h to *condition* in a sense of Doob the following Sobolev-Bregman form of the fractional Laplacian

$$\mathcal{E}_p[u] \quad := \quad \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x)) \left(u(y)^{\langle p-1 \rangle} - u(x)^{\langle p-1 \rangle} \right) \nu(x,y)\, dxdy,$$

where

$$\nu(x,y) := 2^\alpha \Gamma\big((d+\alpha)/2\big) \pi^{-d/2} / |\Gamma(-\alpha/2)| |y-x|^{-d-\alpha}, \qquad x, y \in \mathbb{R}^d,$$

In [4], we give a ground-state representation for $\mathcal{E}_p[u]$, which extends the Hardy identities from [2] and [3, Proposition 4.1]. We then get the following optimal Hardy inequality.

Theorem 1. *Let $0 < \alpha < d \wedge 2$, $0 \le \beta \le d - \alpha$, and $1 < p < \infty$. Then,*

$$\mathcal{E}_p[u] \ge \kappa_{(d-\alpha)/p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} dx \quad \text{for all} \quad u \in L^p(\mathbb{R}^d),$$

where

$$\kappa_\beta := \frac{2^\alpha \Gamma\big(\frac{\beta+\alpha}{2}\big) \Gamma\big(\frac{d-\beta}{2}\big)}{\Gamma\big(\frac{\beta}{2}\big) \Gamma\big(\frac{d-\beta-\alpha}{2}\big)}.$$

It is important to notice that this improves the Hardy inequality *transferred* from L^2, namely for $p \neq 2$, we have

$$\kappa_{(d-\alpha)/p} > \frac{4(p-1)}{p^2} \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{4}\right)^2}{\Gamma\left(\frac{d-\alpha}{4}\right)^2}.$$

We also give applications to contractivity of Schrödinger perturbations of the fractional Laplacian. The above results are given in [4]. See also [5].

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On the Landis Conjecture

Ujjal Das

(joint work with Yehuda Pinchover)

In this workshop, I presented our partial affirmative answer to the Landis conjecture. The conjecture concerns the fastest speed at which a solution of a Schrödinger equation in $\mathbb{R}^N$ can decay at infinity. More precisely, if a solution of the Schrödinger equation

$$\mathcal{H}[u] := (-\Delta + V)[u] = 0 \text{ in } \mathbb{R}^N, \quad \|V\|_{L^\infty(\mathbb{R}^N)} \leq 1\,,$$

decays as fast as $\mathrm{e}^{-k|x|}$ with $k > 1$, then Landis claimed that $u \equiv 0$ in $\mathbb{R}^N$. Under a *positivity* assumption on the operator $\mathcal{H}$, we provide a sharp decay criterion that ensures when a solution of the above equation in $\mathbb{R}^N$ is trivial. It turns out that our decay criterion is in fact weaker than what Landis has proposed in the conjecture.

We define that $\mathcal{H} := -\Delta + V \geq 0$ in $\mathbb{R}^N$ if the $\mathcal{H}[\varphi] = 0$ admits a positive supersolution in $\mathbb{R}^N$. We assume the Schrödinger operator $\mathcal{H} \geq 0$ in $\mathbb{R}^N$, where $N \geq 1$, $V \leq 1$, and $V \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $q > N/2$. Then our precise result is as follows:

If $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ solves $\mathcal{H}[u] = 0$ in $\mathbb{R}^N$ satisfying

$$|u(x)| = \begin{cases} O(1) & N = 1, \\ O\left(x|^{(2-N)/2}\right) & N \geq 2, \end{cases} \quad \text{as } |x| \to \infty, \tag{1}$$

$$\text{and} \quad \liminf_{|x|\to\infty} \frac{|u(x)|}{\left(\frac{e^{-|x|}}{|x|^{(N-1)/2}}\right)} = 0\,, \tag{2}$$

then $u \equiv 0$.

Here we enlist some of the key highlights of our result.

- We only assume that the potential V is bounded from above.
- Our decay assumption on u is weaker than Landis'.
- We do not assume the exponential decay of u at infinity in $\mathbb{R}^N$; it is enough to have a sequence (x_n) with $|x_n| \to \infty$ such that $(u(x_n))$ decays faster than $e^{-|x_n|}/|x_n|^{\frac{N-1}{2}}$.
- Our result is sharp i.e., there are potentials V s.t.

  $$\mathcal{H}[\varphi] = (-\Delta + V)[\varphi] = 0 \quad \text{in} \quad \mathbb{R}^N$$

  admits a positive solution u which satisfies $u(x) \asymp e^{-|x|}/|x|^{\frac{N-1}{2}}$.

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Sharp Hardy-Poincaré-Sobolev inequalities in the Caffarelli-Kohn-Nirenberg hyperbolic space

Baptiste Devyver

(joint work with Louis Dupaigne, Pierre-Damien Thizy)

Consider the Sobolev inequality in the Eculidean space $\mathbb{R}^d$ with its best constant, $d \geq 3$:

$$C_{Sob}(\mathbb{R}^d)\left(\int_{\mathbb{R}^d} |u|^{\frac{2d}{d-2}}\,dx\right)^{\frac{d-2}{d}} \leq \int_{\mathbb{R}^d} |\nabla u|^2\,dx \tag{1}$$

The best constant is known and optimizers are classified: they are multiples, dilations and translations of the celebrated Aubin-Talenti bubble $\left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$. Remembering that the hyperbolic space, seen as the unit ball in $\mathbb{R}^d$ endowed with the hyperbolic metric $\left(\frac{2}{1-|x|^2}\right)^2 (dx_1^2 + \cdots + dx_d^2)$, is conformal to the unit ball of the Euclidean space, and using the conformal invariance of Sobolev inequalities, the inequality (1) can be equivalently reformulated as a Poincaré-Sobolev inequality on the hyperbolic space $\mathbb{H}^d$ (with the same best constant!):

$$(2)\qquad C_{Sob}(\mathbb{R}^d)\left(\int_{\mathbb{H}^d}|f|^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}}\leq\int_{\mathbb{H}^d}|\nabla f|^2\,dx-\frac{d(d-2)}{4}\int_{\mathbb{H}^d}f^2\,dx$$

Since the Aubin-Talenti bubble is not compactly supported, the best constant $C_{Sob}(\mathbb{R}^d)$ in inequality (2) is not achieved. On the other hand, it is well-known that the bottom of the spectrum of $-\Delta$ on $\mathbb{H}^d$ is equal to $\frac{(d-1)^2}{4}$, and it is not achieved.

A natural question arises: can one improve the constant $\frac{d(d-2)}{4}$ on the right hand side of (2), and replace it by the bottom of the spectrum $\frac{(d-1)^2}{4}$, *while keeping the same constant $C_{Sob}(\mathbb{R}^d)$ on the left hand side*? Equivalently, considering the following Poincaré-Sobolev inequality with its best constant:

$$(3)\qquad C_d\left(\int_{\mathbb{H}^d}|f|^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}}\leq\int_{\mathbb{H}^d}|\nabla f|^2\,dx-\frac{(d-1)^2)}{4}\int_{\mathbb{H}^d}f^2\,dx$$

then the question is: does one have $C_d = C_{Sob}(\mathbb{R}^d)$? Let us mention at this point another equivalent form of the Poincaré-Sobolev inequality (3), namely a Hardy-Sobolev inequality in the upper-half space $\mathbb{R}^d = \{(x_1,\cdots,x_d)\,;\,x_d>0\}$:

$$(4)\qquad C_d\left(\int_{\mathbb{R}^d_+}|f|^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}}\leq\int_{\mathbb{R}^d_+}\left(|\nabla f|^2-\frac{f^2}{4x_d^2}\right)dx$$

The above question has been addressed in a series of paper; the answer depends on the dimension in a delicate way:

(i) If $d\geq 4$, then $0<C_d<C_{Sob}(\mathbb{R}^d)$ and there exists optimizers.
(ii) In dimension 3, $C_3 = C_{Sob}(\mathbb{R}^3)$ and there is no optimizer.

Point (i) is due to V. Maz'ya [6] for the fact that $C_d>0$, and A. Terktikas, K. Tintarev [7] for the remaining part, while point (ii) appears independantly in the work of R. Benguria, R. Frank, M. Loss [1] as well as G. Mancini, K. Sandeep [5].

In our work, we study a generalization of this result. Let us present the setting: consider the *Caffarelli-Kohn-Nirenberg inequality* in dimension $d\geq 3$; it writes:

$$(5)\qquad C_{a,b}\left(\int_{\mathbb{R}^d}\frac{|f|^p}{|x|^{bp}}\,dx\right)^{2/p}\leq\int_{\mathbb{R}^d}\frac{|\nabla f|^2}{|x|^{2a}}\,dx,$$

where $a,b\in\mathbb{R}$ are two parameters satisfying $a<\frac{d-2}{2}$ and $0\leq b-a\leq 1$. It contains both the Hardy inequality and the Sobolev inequality as special cases. This inequality can be recast as a standard Sobolev inequality on a weighted manifold $(\mathbf{E},g_{\mathbf{E}},\mu_{\mathbf{E}})$: indeed, define a space $\mathbf{E}=\mathbb{R}^d\setminus\{0\}$, a measure $d\mu_{\mathbf{E}}(x) = |x|^{-bp}\,dx$, a metric $(g_{\mathbf{E}})_{i,j} = |x|^{-bp+2a}\delta_{i,j}$, and a dimension n by $p=\frac{2n}{n-2}$, then the Caffarelli-Kohn-Nirenberg inequality (5) writes equivalently:

$$C_{a,b}\left(\int_{\mathbf{E}}|u|^{\frac{2n}{n-2}}\,d\mu_{\mathbf{E}}\right)^{\frac{n-2}{n}}\leq\int_{\mathbf{E}}|\nabla u|^2\,d\mu_{\mathbf{E}} \tag{6}$$

In the range of parameters $0 < b - a < 1$, optimizers are known to exists, but according to F. Catrina and Z. Q. Wang [2], they are not always radially symmetric nor explicit, and as a consequence, the best constant $C_{a,b}$ is not known for all possible values of the parameters a, b. Introduce $\alpha = 1 + a - \frac{bp}{2}$. Then, according to a celebrated result by J. Dolbeault, M. Esteban and M. Loss [3], the optimal range of parameters for which optimizers are radially symmetric, is characterized in terms of the parameter α as:

$$\alpha^2 \leq \frac{d-1}{n-1}$$

In any case, let us denote by $C_{n,\alpha}(\mathbf{E}) = C_{a,b}$ the best constant and hence emphasize its dependance upon the two parameters α and n. By conformally changing the metric and restricting oneself to the unit ball, one can define a *hyperbolic* version $(\mathbf{H}, g_{\mathbf{H}}, \mu_{\mathbf{H}})$ of the weighted manifold $(\mathbf{E}, g_{\mathbf{E}}, \mu_{\mathbf{E}})$: letting $\psi(x) = \frac{1-|x|^{2\alpha}}{2}$, one defines $\mathbf{H} = \mathbb{B} \setminus \{0\}$ (the punctured unit ball), $g_{\mathbf{H}} = \psi^{-2} g_{\mathbf{E}}$, and $\mu_{\mathbf{H}} = \psi^{-n}\mu_{\mathbf{E}}$. This weighted Riemannian manifold has been introduced by L. Dupaigne, I. Gentil and S. Zugmeyer [4]; it is a reasonnable to call it "hyperbolic", since it was computed in this paper that it has constant negative weighted scalar curvature. Using the conformal invariance properties of Sobolev inequalities, one can rewrite equivalently the Caffarelli-Kohn-Nirenberg inequality as a Poincaré-Sobolev inequality in the weighted space $(\mathbf{H}, g_{\mathbf{H}}, \mu_{\mathbf{H}})$:

$$C_{n,\alpha}(\mathbf{E})\left(\int_{\mathbf{H}}|u|^{\frac{2n}{n-2}}\,d\mu_{\mathbf{H}}\right)^{\frac{n-2}{n}}\leq\int_{\mathbf{H}}|\nabla u|^2\,d\mu_{\mathbf{H}}-\frac{n(n-2)}{4}\alpha^2\int_{\mathbf{H}}u^2\,d\mu_{\mathbf{H}} \tag{7}$$

In a joint forthcoming work with L. Dupaigne and P-D.Thizy, we study the possibility of extending the results of Benguria, Frank, Loss and Mancini, Sandeep to this more general setting. More precisely, we consider the following:

Question: *can we improve* (7) *by removing on the right-hand side a greater constant than* $\frac{n(n-2)}{4}$ *– possibly even removing the bottom of the spectrum of the weigthed Laplacian – while keeping the same constant* $C_{n,\alpha}(\mathbf{E})$ *on the left-hand side?*

In our work, we compute the bottom of the weighted Laplacian (it is equal to $\frac{(d-1)^2}{4}\alpha^2$), and show among other results that:

(i) if $n \geq 4$, then the constant $\frac{n(n-2)}{4}\alpha^2$ in (7) is best and cannot be improved.
(ii) for $d = 3 \leq n < 4$, one can always improve the constant $\frac{n(n-2)}{4}\alpha^2$ in (7) into $\frac{n(n-2)}{4}\alpha^2 + \varepsilon$ for some small $0 < \varepsilon = \varepsilon(n, \alpha)$, while keeping the same constant $C_{n,\alpha}(\mathbf{E})$ on the left-hand side. However, at least if n is close enough to 4, one *cannot* replace $\frac{n(n-2)}{4}\alpha^2$ by the bottom of the spectrum

$\frac{(d-1)^2}{4}\alpha^2$. This is in sharp contrast with the unweighted situation studied by Benguria, Frank, Loss and Mancini, Sandeep.

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Logarithmic Hardy inequality. Fractional Sobolev–Hardy–Maz'ya inequality for $1 < p < 2$

Bartłomiej Dyda

(joint work with Sven Jarohs, Firoj Sk; Michał Kijaczko)

The talk consisted of two separate parts.

I. Let us consider the fractional p-Laplacian

$$(-\Delta_p)^s u(x) = C_{N,s,p}\, p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\, dy,$$

where $p \in (1,\infty)$, $s \in (0,1)$, p.v. stands for the Cauchy principal value, and u is suitably regular at $x \in \mathbb{R}^N$ and integrable at infinity with respect to the kernel $z \mapsto |z|^{-N-sp}$. The constant $C_{N,s,p}$ here is chosen so that

$$\lim_{s\to 0^+} (-\Delta_p)^s u(x) = |u(x)|^{p-2}u(x) \quad \text{and} \quad \lim_{s\to 1^-} (-\Delta_p)^s u(x) = -\Delta_p u(x)$$

hold for smooth compactly supported functions. Then $C_{N,s,p} \asymp s$ as $s \to 0^+$ and $C_{N,s,p} \asymp (1-s)$ as $s \to 1^-$. The goal of our work [4] was to find a suitable operator L_{Δ_p} to improve the understanding at the limit $s \to 0^+$. To be precise, we define an operator L_{Δ_p}, which we call the *logarithmic p-Laplace* operator, such that the expansion

$$(-\Delta_p)^s u(x) = |u(x)|^{p-2}u(x) + sL_{\Delta_p}u(x) + o(s) \quad \text{for } s \to 0^+ \tag{1}$$

holds for a suitable class of functions u. For $p = 2$, this problem was studied in [2]. In [4], we prove the following theorem

Theorem 1. *Let* $0 < s < 1$ *and* $1 < p < \infty$. *Suppose* $u \in C_c^\alpha(\mathbb{R}^N)$ *for some* $\alpha > 0$. *Then for* $x \in \mathbb{R}^N$

(2)

$$\begin{aligned}L_{\Delta_p} u(x) &:= \frac{d}{ds}\Big|_{s=0} (-\Delta_p)^s\, u(x) \\ &= C_{N,p} \int_{B_1(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^N} dy \\ &\quad + C_{N,p} \int_{\mathbb{R}^N \setminus B_1(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) - |u(x)|^{p-2}u(x)}{|x - y|^N} dy \\ &\quad + \rho_N\, |u(x)|^{p-2}u(x)\end{aligned}$$

where $C_{N,p}$ *and* ρ_N *are some explicit constants. Moreover, for any* $1 < q \leq \infty$, *the difference quotients are convergent in* $L^q(\mathbb{R}^N)$, *i.e.,*

$$\frac{(-\Delta_p)^s\, u - |u|^{p-2}u}{s} \xrightarrow{s \to 0^+} L_{\Delta_p} u \quad \text{in } L^q(\mathbb{R}^N).$$

In the talk, we have shown the proof of (2).

In [4], we setup a variational framework for the operator L_{Δ_p}. To this end we needed some kind of Hardy inequality, more specifically, one described in the following theorem.

Theorem 2. *Suppose that* $\Omega \subset \mathbb{R}^N$, $\Omega \neq \mathbb{R}^N$ *is open and locally plump (which means that* $\exists \kappa \in (0,1)\, \forall x \in \Omega\, \exists B_{\kappa r}(z) \subset \Omega$, *where* $B_{\kappa r}(z)$ *denotes a ball centred at* z *with radius* r*). Then there exists a constant* $C < \infty$ *such that for every* $u \in L^p(\Omega)$ *the following inequality holds,*

$$\int_\Omega |u(x)|^p \ln^+\left(\frac{1}{\delta_x}\right) dx \leq c \left(\iint_{\substack{\Omega \times \Omega \\ |x-y|<1}} \frac{|u(x) - u(y)|^p}{|x - y|^N}\, dy\, dx + \int_\Omega |u(x)|^p\, dx \right),$$

where $\delta_x = \operatorname{dist}(x, \Omega^c)$.

Another, but different form of Hardy inequality with a logarithm has appeared in [1]. While the inequality itself is different, we could still use some of the ideas of [1] in our proof.

In the talk, we have proven Theorem 2 in the special case of the half-space $\Omega = \mathbb{R}^{N-1} \times (0, \infty)$ and briefly mentioned what has to be done in the general case.

II. Let us consider a half-space $\Omega = \mathbb{R}^{N-1} \times (0, \infty)$. Let $1 < p < 2$, $0 < s < 1$ and $1 < sp < N$. On the one hand, it is known [7] that the following Hardy inequality holds

$$\mathcal{E}[u] := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dy\, dx - \mathcal{D}_{N,s,p} \int_\Omega \frac{|u(x)|}{x_N^{sp}}\, dx \geq 0\,,$$

with the optimal (and explicitely given) constant $\mathcal{D}_{N,s,p}$. Here and later we consider $u \in C_c^\infty(\Omega)$. On the other hand, the following Sobolev inequality holds with

$p^* = pN/(N - sp)$ and some constant $c > 0$,

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dy\, dx \geq c \left(\int_\Omega |u(x)|^{p^*}\, dx \right)^{p/p^*}.$$

As it turns out, also the following Hardy-Sobolev-Maz'ya inequality holds,

$$\mathcal{E}[u] \geq c' \left(\int_\Omega |u(x)|^{p^*}\, dx \right)^{p/p^*}, \tag{3}$$

which was proved in [3] in the case when $p \geq 2$ (and in [8] in case $p = 2$). The reason why the assumption $p \geq 2$ was needed was the fact that in the ground state representation of Frank and Seiringer [7], only in such a case the remainder term was present. In [5], we proved that one can have a remainder (albeit in another, weaker form) also for $1 < p < 2$, and this weaker form still suffices to prove (3). We note that a more complicated, but optimal form of a remainder was also obtained by Fischer [6].

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Uncertainty Principles: from Hardy to Resolvent Inequalities

Luca Fanelli

In Classical Mechanics, an action is the area enclosed within a closed curve in the phase-space, which describes the periodic motion of a particle. The Uncertainty Principle, usually attributed to Max Planck, states that any action J need to be an integer multiple of the elementary action $\hbar$, namely

$$J = n\hbar, \qquad \hbar \leq\simeq 6.6 \times 10^{-34}\mathrm{j} \cdot \mathrm{s}.$$

Passing from Classical to Quantum Mechanics, position and momentum are replace by the following operators on L^2

$$x \mapsto \text{multiplication operator by } x \qquad p \mapsto -i\hbar\nabla$$

and the well known formula for the Uncertainty Principle states the bound from below

$$\Delta x \Delta p \geq \hbar.$$

In order to see mathematical manifestations of the Uncertainty Principle, we set the following functional framework:

$$\begin{aligned} &\mathcal{S} \text{ symmetric on Hilbert space} && \langle \mathcal{S}f, f\rangle = \langle f, \mathcal{S}f\rangle \\ &\mathcal{A} \text{ skewsymmetric on Hilbert space} && \langle \mathcal{A}f, f\rangle = -\langle f, \mathcal{A}f\rangle \end{aligned}$$

$$\|(\mathcal{S}+\mathcal{A})\, f\|^2 = \|\mathcal{S}f\|^2 + \|\mathcal{A}f\|^2 + \langle (\mathcal{S}\mathcal{A}-\mathcal{A}\mathcal{S})f, f\rangle$$

$$\Rightarrow \qquad \langle (\mathcal{A}\mathcal{S}-\mathcal{S}\mathcal{A})f, f\rangle \leq \|\mathcal{S}f\|^2 + \|\mathcal{A}f\|^2.$$

Notice that

$$\langle (\mathcal{A}\mathcal{S}-\mathcal{S}\mathcal{A})f, f\rangle = 2\Re\, \langle \mathcal{A}f, \mathcal{S}f\rangle$$

This gives the (stronger) inequality

$$|\langle (\mathcal{A}\mathcal{S}-\mathcal{S}\mathcal{A})f, f\rangle| \leq 2\|\mathcal{S}f\| \cdot \|\mathcal{A}f\| \qquad (\star)$$

The main examples are the Harmonic Oscillator (Heisenberg Uncertainty Principle), for which the choices are $S = x$, $A = \nabla$, the Coulomb potential $S = x/|x|$, $A = \nabla$, the Hardy Inequality $S = x/|x|^2$, $A = \nabla$. Another important manifestation of the uncertainty if the Virial Theorem, which states that if $[H, A]$ is strictly positive, then the point spectrum of H is empty. In this seminar, we show the connection between the above mentioned examples and the dispersive feature of the Schrödinger equation, which is related to the Kato-Smoothing phenomenon, and we introduce some recent results concerned with the Kato-Yajima inequality. Some interesting open problems concerning the best constant in an inequality by Kato and Yajima will also be presented.

Optimal Poincaré-Hardy Inequalities on Manifolds and Graphs

Florian Fischer
(joint work with Christian Rose)

The talk is separated into two parts and is based on [4]. In the first part, we review a method to obtain optimal Poincaré-Hardy inequalities on the hyperbolic spaces, and discuss briefly generalisations to certain classes of Riemannian manifolds. Thereafter in the second part, we show how to transfer the method to the discrete setting.

To be more specific, in the first part we recall the following result by Berchio, Ganguly and Grillo [1]:

Theorem 1 ([1]). *On the hyperbolic space $\mathbb{H}^d$, $d \geq 3$, an optimal Poincaré-Hardy weight is given by the radial function*

$$W(r) := \lambda_0(\mathbb{H}) + \frac{1}{4r^2} + \frac{(d-1)(d-3)}{4\sinh^2(r)}, \qquad r > 0,$$

where $r = d(x,o)$ for $x \in S_o(r)$, $d(\cdot, o)$ is the distance to a fixed point $o \in \mathbb{H}^d$, and $\lambda_0(\mathbb{H})$ is the bottom of the spectrum of the Dirichlet Laplacian $-\Delta$.

The main idea in the proof is to consider the radial positive superharmonic function

$$u(r) := \frac{r}{\sinh^{d-1}(r)}, \quad r > 0,$$

and to see that W is nothing but the Fitzsimmons ratio of the square root of u, i.e.,

$$W = \frac{-\Delta\sqrt{u}}{\sqrt{u}}.$$

This basic idea has been adapted to find optimal Poincaré-Hardy inequalities on other manifolds such as Damek-Ricci spaces [3]. Here the main observation is to interpret the denominator of u, $\sinh^{d-1}$, as the volume density of $\mathbb{H}^d$, and to do the corresponding substitution for Damek-Ricci spaces.

In the second part, we show how to transfer the method to locally finite weighted graphs over the discrete measure space (X, m). We overcome the main difficulties by considering model graphs with respect to the combinatorial metric d_c and taking the area function as the new denominator, that is, the function that sums over all edge weights between two spheres with respect to some fixed vertex $o \in X$. Let k_- and k_+ be the radial inner and outer curvatures of the model graph with respect to o, and define the curvature ratio function via $\kappa = k_+/k_-$. With this in hand, we have the following.

Theorem 2 ([4]). *On a model graph over X with respect to some fixed vertex $o \in X$, assume that κ is bounded $\kappa(1) \geq 1$, and $\kappa(r) \geq 1/r + (1 - 1/r)\kappa(r-1)$ for $r := d_c(x,o) \geq 2, x \in S_o(r)$. Then an optimal Hardy weight is given by the radial function*

$$w(r) := k_-(r)\left(\left(\sqrt{\kappa(r)} - 1\right)^2 + \sqrt{\kappa(r)}\left(2 - \sqrt{1 + \frac{1}{r}}\right) - \sqrt{\kappa(r-1)}\sqrt{1 - \frac{1}{r}}\right),$$

for $r \geq 1$. Hence, for $r \geq 2$,

$$w(r) \geq k_-(r)\left(\left(\sqrt{\kappa(r)} - 1\right)^2 + \frac{\sqrt{\kappa(r)}}{4r^2}\right).$$

Moreover, if k_- and κ are constant, then $\lambda_0(X) \geq k_-\left(\sqrt{\kappa} - 1\right)^2$. Furthermore, if κ is constant, then

$$w > \frac{-\Delta\sqrt{G_o}}{\sqrt{G_o}}, \qquad r \geq 1,$$

where G_o is the minimal positive Green's function of the model graph at o.

We explain the proof and observe that the general result about optimal Hardy weight of Keller, Pinchover and Pogorzelski [7] cannot be applied. Moreover, we note that the known optimal inequalities on the natural numbers [6] and on homogeneous regular trees [2] are included, confer also with [5].

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Some remarks on one-dimensional Hardy inequalities

Rupert L. Frank

(joint work with Ari Laptev, Timo Weidl)

The talk discusses doubly-weighted Hardy inequalities on the halfline. The following theorem is due to Tomaselli (1969).

Theorem 1. *Let $1 < p < \infty$ and let V, W be nonnegative, a.e.-finite, measurable functions on $(0, \infty)$ such that*

$$\int_0^s V(t)^{-\frac{1}{p-1}}\, dt < \infty \qquad \text{for all } s \in (0,\infty)\,.$$

Then for any locally absolutely continuous function u on $(0,\infty)$ satisfying $\liminf_{r\to 0} |u(r)| = 0$ we have

$$\int_0^\infty W(r)|u(r)|^p\, dr \le \frac{p^p}{(p-1)^{p-1}} \overline{B} \int_0^\infty V(r)|u'(r)|^p\, dr$$

and

$$\int_0^\infty W(r)|u(r)|^p\, dr \le \left(\frac{p}{p-1}\right)^p \underline{B} \int_0^\infty V(r)|u'(r)|^p\, dr$$

with

$$\overline{B} := \sup_{s>0} \left(\int_0^s V(t)^{-\frac{1}{p-1}}\, dt\right)^{p-1} \left(\int_s^\infty W(t)\, dt\right)$$

and

$$\underline{B} := \sup_{s>0} \left(\int_0^s V(t)^{-\frac{1}{p-1}} dt\right)^{-1} \int_0^s W(t) \left(\int_0^t V(t')^{-\frac{1}{p-1}} dt'\right)^p dt\,.$$

Here are some remarks:

- It is easy to see that if for some $1 < p < \infty$ there is a constant C such that $\int_0^\infty W|u|^p\,dr \le C\int_0^\infty V|u'|^p\,dr$ for all u as in the theorem, then $\overline{B} \le \overline{c_p}C$ and $\underline{B} \le \underline{c_p}C$ for (explicit) constants $\overline{c_p}$ and $\underline{c_p}$. In particular, $\overline{B}$ is finite if and only if $\underline{B}$ is finite, and this finiteness is equivalent to the validity of the doubly-weighted Hardy inequality.
- The theorem contains *two* inequalities. For our application *both* of them are needed. The first one seems to be much better known than the second.
- The first inequality is due to Kac and Krein (1958) in the special case $V \equiv 1$ and $p = 2$. A simple and elegant proof of the first inequality was given by Muckenhoupt (1972).
- There is a similar theorem for functions satisfying the vanishing condition $\liminf_{r\to\infty}|u(r)| = 0$ at infinity. This can be obtained from the above theorem by the change of variables $r \mapsto r^{-1}$.

Tomaselli's proof uses ODE theory. In [1, 2] we provide an alternative proof, which is based on the following improved Hardy inequality.

Theorem 2. *Let* $1<p<\infty$*. Then for any locally absolutely continuous function* u *on* $(0,\infty)$ *satisfying* $\liminf_{r\to 0}|u(r)|=0$ *we have*

$$\int_0^\infty \max\left\{\sup_{0<s\le r}\frac{|u(s)|^p}{r^p},\ \sup_{r\le s<\infty}\frac{|u(s)|^p}{s^p}\right\}dr \le \left(\frac{p}{p-1}\right)^p\int_0^\infty |u'(r)|^p\,dr\,.$$

Interestingly, this theorem can be deduced from the basic Hardy inequality

$$\int_0^\infty \frac{|u(r)|^p}{r^p}\,dr \le \left(\frac{p}{p-1}\right)^p\int_0^\infty |u'(r)|^p\,dr\,,$$

upon which it improves.

The doubly-weighted Hardy inequality with $p = 2$ can be used to give a sufficient condition for Schrödinger operators in $L^2(\mathbb{R}^d)$ to have only finitely many negative eigenvalues; see [1, 2]. The constant in this condition is optimal.

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Fractional nonlinear diffusions on manifolds: well-posedness and smoothing effects

Gabriele Grillo
(joint work with Elvise Berchio, Dario D. Monticelli, Matteo Muratori, Fabio Punzo)

We consider an N-dimensional manifold M such that $\mathrm{Ric}(M) \geq -(N-1)k$ for some $k > 0$ and that there $\exists c > 0$ s.t. the Faber-Krahn inequality holds:

$$\lambda_1(\Omega) \geq c\, m(\Omega)^{-\frac{2}{N}} \tag{1}$$

for any Ω is open, relatively compact, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_M$ with homogeneous Dirichlet b.c.. This is equivalent to assuming a Euclidean-type Nash inequality or, if $N \geq 3$, a Euclidean-type Sobolev inequality. We consider the following Cauchy problem, that we refer to as fractional porous medium equation:

$$\begin{cases} u_t = -(-\Delta_M)^s\,(u^m) & \text{in } M \times (0,\infty)\,, \\ u = u_0 & \text{on } M \times \{0\}\,, \end{cases} \tag{2}$$

where $s \in (0,1), m > 1$. Here, M is a complete, connected, noncompact Riemannian manifold, and Δ_M the Laplace-Beltrami operator. Our goal is to prove basic well–posedness results for solutions, in a suitable sense, provided M satisfies appropriate geometric assumptions, and to prove smoothing effects for data in a suitable class, larger than $L^1(M)$. When $M = \mathbb{R}^N$, equation (2) have been introduced and thoroughly studied by Bonforte, de Pablo, Quiros, Rodriguez, Vázquez. The fractional Laplacian in our setting will be defined by the spectral theorem. This can be written explicitly e.g. through the semigroup: for the Laplacian on a stochastically complete manifold, and for a suitable class of functions f:

$$\begin{aligned}(-\Delta_M)^s f(x) &= c\int_0^{+\infty} [T_t f(x) - f(x)]\frac{\mathrm{d}t}{t^{1+s}} \\ &= c\int_0^{+\infty}\left(\int_M k_M(t,x,y)\,(f(y)-f(x))\;\mathrm{d}m(y)\right)\frac{\mathrm{d}t}{t^{1+s}},\end{aligned}$$

where m is the Riemannian measure, T_t is the heat semigroup and k_M the heat kernel on M. Our definition of solution to (2) is the *weak dual* one, which can be informally understood by applying $(-\Delta)^{-s}$ to both sides yielding, formally,

$$\partial_t\big[(-\Delta_M)^{-s}u\big] + u^m = 0\,.$$

and can be made rigorous by testing the above equation appropriately. We introduce a *weighted* L^1 space, in which the weight is the fractional Green function G^s_M, which exists and tends to zero at infinity due to the running assumption. Such a space is indicated by $L^1_{G^s_M}(M)$, so that $L^1(M) \subset L^1_{G^s_M}(M)$, $L^1(M) \neq L^1_{G^s_M}(M)$. We prove what follows (see [1] for details):

- Existence of a WDS for data in $L^1_{G^s_M}(M)$.
- Smoothing effects for solutions, namely quantitative bounds on the L^∞ norm of the solution $u(t)$ in terms of some norm of the initial datum u_0. For example:

$$\|u(t)\|_\infty \leq C \left(\frac{\|u(t)\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_1 \right) \leq C \left(\frac{\|u_0\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_1 \right) \quad \forall t > 0\,.$$

  and, under additional assumption, similar long-time bounds, which take into account the geometry of M.

Similar results, which depend crucially on the volume growth of balls in M, have been proved in [2] on manifolds satisfying $\mathrm{Ric} \geq 0$.

Uniqueness issues are *not* dealt with, and remain an open problem. Among several other open problems, we mention the proof of existence of *fundamental solutions*, i.e. solutions taking a Dirac delta as initial datum, as well as the proof of *pointwise bounds* for general solutions.

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Hardy inequalities on integer lattices

Shubham Gupta

It turns out that Hardy-type inequalities on discrete spaces are much more complicated and involved as compared to their continuous counterparts. Indeed, even for several simple discrete models, many fundamental question remain unanswered. In the talk, we review some recent developments that happened in the last few years, mostly focusing on an important discrete model, namely, the integer lattices $\mathbb{Z}^d$. The first discrete Hardy inequality proved in [2], states that

$$\frac{1}{4} \sum_{n=1}^{\infty} \frac{|u(n)|^2}{n^2} \leq \sum_{n=1}^{\infty} |u(n) - u(n-1)|^2, \tag{1}$$

for all finitely supported functions $u : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfying $u(0) = 0$. The constant $1/4$ is the best possible. It is well known that the inverse square weight is *critical* (meaning, it cannot be improved pointwise) in the continuous setting. One could ask if this happens to be the case in the discrete setting as well. This was settled by Keller-Pinchover-Pogorzelski in the negative [4], by giving a pointwise improvement of weight in (1):

Theorem 1. *Let* $u : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ *be a finitely supported function with* $u(0) = 0$*. Then*

$$\sum_{n=1}^{\infty} w^{\mathrm{KPP}}(n)|u(n)|^2 \leq \sum_{n=1}^{\infty} |u(n) - u(n-1)|^2,$$

where

$$w^{\mathrm{KPP}}(n) := 2 - \sqrt{1 - \frac{1}{n}} - \sqrt{1 + \frac{1}{n}} = \frac{1}{4n^2} + \frac{5}{64n^4} + \cdots > \frac{1}{4n^2}.$$

The weight w^{KPP} is critical. This work led to the quest of finding "large" Hardy weights in more general discrete settings: on integers [5], for higher order operators [7], on infinite graphs [4]. Unlike dimension one, the situation in higher dimensional lattices $\mathbb{Z}^d$, $d \geq 3$ is much more different. Particularly, one cannot deduce the classical analogue of (1) for $d \geq 3$, from the critical Hardy weights obtained in [4]. The case $d \geq 3$ was first studied by Rozenblum and Solomyak [6], and later by Kapitanski and Laptev [3]. However, the constants obtained in these works are not optimal, and determining them remains an open question till date. To that end, we proved the following result, concerning the asymptotic behaviour of the optimal constant, as the dimension $d \to \infty$ [1]:

Theorem 2. *Let* u *be a finitely supported function on* $\mathbb{Z}^d$ *with* $u(0) = 0$*. Let* $C_H(d)$ *be the optimal constant in the discrete Hardy inequality*

$$C_H(d) \sum_{n \in \mathbb{Z}^d} \frac{|u(n)|^2}{|n|^2} \leq \sum_{n \in \mathbb{Z}^d} |Du(n)|^2.$$

Then $C_H(d) \sim d$*, as* $d \to \infty$*, that is, there exist positive constants* c_1, c_2, N *(independent of* d*) such that* $c_1 d \leq C_H(d) \leq c_2 d$ *for all* $d \geq N$*.*

The result was obtained by connecting discrete Hardy inequalities on lattices with continuous Hardy-type inequalities on the torus. This provides a fruitful bridge between two areas.

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Quantitative Hardy inequalities for magnetic Hamiltonians

Hynek Kovařík
(joint work with Luca Fanelli)

The history of Hardy inequalities for magnetic Dirichlet forms goes back to the pioneering works by Laptev and Weidl [2, 4, 5]. The latter show, roughly speaking, that introducing a magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ turns a critical operator, i.e. two-dimensional Laplacian, into a subcritical operator, i.e. two-dimensional magnetic Laplacian $(i\nabla + A)^2$, which can be understood as a mathematical manifestation of the diamagnetic effect. Here $A \in L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ is a vector potential such that $\nabla \times A = B$ in the sense of distributions. The fields A and B have the physical interpretation of the magnetic potential and magnetic field, respectively. For magnetic fields with finite normalized flux

$$\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^2} B\, dx \tag{1}$$

the above phenomenon is expressed by a lower bound on the associated quadratic form in the following way. If $B \neq 0$, then there exists a constant $C(B) > 0$ such that

$$\int_{\mathbb{R}^2} |(i\nabla + A)u|^2\, dx \;\geq\; C(B) \begin{cases} \displaystyle\int_{\mathbb{R}^2} \frac{|u|^2}{1+|x|^2}\, dx & \text{if } \alpha \notin \mathbb{Z}, \\[2ex] \displaystyle\int_{\mathbb{R}^2} \frac{|u|^2}{1+|x|^2 \log^2 |x|}\, dx & \text{if } \alpha \in \mathbb{Z} \end{cases} \tag{2}$$

for all $u \in C_0^\infty(\mathbb{R}^2)$. It is important to recall that no lower bound with a non-negative and nonzero integral weight as above holds if $B = 0$.

In this talk I present the result obtained in [1] in which we adopt a new approach. Namely, we prove a quantitative Hardy-type inequality for the magnetic Laplacian by proving a similar inequality first for the Pauli operator

$$P(A) = \begin{pmatrix} H_+(A) & 0 \\ 0 & H_-(A) \end{pmatrix}, \qquad H_\pm(A) = (i\nabla + A)^2 \pm B$$

in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Indeed, working directly with the Pauli operator allows us to develop a new method of proof in which the problem is reduced to a family of one-dimensional weighted inequalities. With the help of well-known results established in the literature, [3], we then calculate the sharp constants in these inequalities. The advantage of this approach is that it gives us much more information about the constant in the resulting Hardy inequality. In particular, in the case of non-integer flux we prove that

$$\int_{\mathbb{R}^2} |(i\nabla + A)u|^2\, dx \;\geq\; \mu_\alpha^2\, \beta(B;\rho) \int_{\mathbb{R}^2} \frac{|u(x)|^2}{\rho^2 + |x|^2}\, dx, \qquad u \in C_0^\infty(\mathbb{R}^2), \tag{3}$$

where $\rho > 0$ is arbitrary, $\beta(B;\rho)$ is a constant, and where

$$\mu_\alpha := \min_{m\in\mathbb{Z}} |m-\alpha|$$

denotes the distance between α and the set of integers.

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Criticality transition for powers of the discrete Laplacian

David Krejčiřík

(joint work with Borbala Gerhat, František Štampach)

The uniqueness of the world we live in consists in that $\mathbb{R}^3$ is the lowest dimensional Euclidean space for which the Brownian motion is *transient*. Indeed, it is well known that the Brownian particle in $\mathbb{R}^d$ will escape from any bounded set after some time forever if $d \geq 3$, while the opposite holds true in low dimensions, i.e. the Brownian motion is *recurrent* in $\mathbb{R}^1$ and $\mathbb{R}^2$ (see Figure 1). This is a well known criticality transition in dimensions.

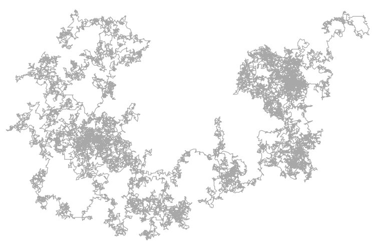

Figure 1. The Brownian motion in $\mathbb{R}^2$.

Since the Brownian motion is mathematically introduced via the heat equation, it is not surprising that the transiency is closely related to spectral properties of the Laplacian. Indeed, the self-adjoint realisation $-\Delta$ in $L^2(\mathbb{R}^d)$ is *subcritical* if and only if $d \geq 3$, meaning that there exists a non-trivial non-negative function ρ (Hardy weight) such that the Hardy-type inequality $-\Delta \geq \rho$ holds in the sense of quadratic forms. On the other hand, $-\Delta$ is critical if $d = 1, 2$ in the sense that $\inf \sigma(-\Delta + V) < 0$ for every non-trivial non-positive function V.

The case of Brownian particles dying on massive subsets of $\mathbb{R}^d$ is less interesting in the sense that the Dirichlet Laplacian $-\Delta$ in $\mathbb{R}^d \setminus \overline{\Omega}$ with any Ω non-empty and open is always subcritical. In particular, the Brownian motion in the half-space $\mathbb{R}^{d-1} \times (0, \infty)$ is transient for every $d \geq 1$, so no criticality transition in dimensions occurs.

There is a probabilistic interpretation of powers of the Laplacian in terms of an anomalous diffusion. From this perspective, the case of the half-line is equally uninteresting because all the powers $(-\Delta)^\alpha$ in $L^2((0,\infty))$ with $\alpha \in \mathbb{N}$ are subcritical [1] (see also [2] for $\alpha \in (0,1)$). There is no criticality transition in powers.

The objective of our paper [5] is to disclose the surprising fact that the situation is very different in the discrete setting. Indeed, we demonstrate that the integer powers of the discrete Laplacian $(-\Delta)^\alpha$ on $\ell^2(\mathbb{N})$ are subcritical if and only if $\alpha = 1$. What is more curious in fine properties of this transition, we consider possibly non-integer powers and reveal the following precise threshold in all positive powers.

Theorem 1 ([5])**.** *Let $\alpha > 0$. Then*

$$(-\Delta)^\alpha \text{ on } \ell^2(\mathbb{N}) \text{ is subcritical} \qquad \Longleftrightarrow \qquad \alpha < 3/2.$$

Here the implication $\Longleftarrow$ for $\alpha = 1$ was known since the classical work of Hardy's [7] (for the recently discovered optimal Hardy weight, see [9, 10] and our related contributions [12, 11, 4]). The other values of α's and the opposite implication $\Longrightarrow$ are new. The proof of the theorem is based on the Birman–Schwinger principle [6].

As for the case of the discrete Laplacian on the full line, let us mention the following transition in powers:

$$(-\Delta)^\alpha \text{ on } \ell^2(\mathbb{Z}) \text{ is subcritical} \qquad \Longleftrightarrow \qquad \alpha < 1/2.$$

Here the implication $\Longleftarrow$ is due to [3] (see also [8] for optimal Hardy inequalities). The opposite implication $\Longrightarrow$ follows by the method of proof of our paper [5].

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Discrete optimal Hardy type inequalities and lattices in H-type groups

Felix Pogorzelski
(joint work with Philipp Hake, Matthias Keller, Yehuda Pinchover)

The development of discrete Hardy inequalities has progressed significantly throughout the past decade. The first part of the talk is devoted to a general criterion for graph Laplacians to obtain optimal, i.e. null-critical Hardy type weights. We then apply the proposed method to discrete Laplacians over lattices in certain stratified Lie groups. This leads to the second part of the lecture, where we explain the relevant connections for H-type groups. In particular, we present some new optimal Hardy type weights with explicit first order term.

The results in the main part of the talk are based on the doctoral project [6] of Philipp Hake supervised by Felix Pogorzelski within the research project "Hardy inequalities on graphs and Dirichlet spaces" (joint with Matthias Keller, Yehuda Pinchover) funded by the DFG. The support is gratefully acknowledged.

The setting. A *graph* is a pair (X, b), where X is a countably infinite set endowed with the discrete topology, and $b : X \times X \to [0, \infty)$ satisfies $b(x, x) = 0$, $b(x, y) = b(y, x)$ and $\sum_{z \in X} b(x, z) < \infty$ for all $x, y \in X$. The elements of X are called *vertices*. Two vertices $x, y \in X$ share an *edge* and we write $x \sim y$ if $b(x, y) > 0$. We will always assume that graphs are *connected*, i.e. for every $x, y \in X$ one finds $x_i \in X$, $1 \leq i \leq k$ such that $x \sim x_1 \sim \cdots \sim x_k \sim y$. A graph (X, b) is *locally finite* if for all $x \in X$, the number of $y \in X$ sharing an edge with x is finite. We write $C(X)$ for the real vector space of all mappings $f : X \to \mathbb{R}$ and denote by $C_c(X)$ all elements in $C(X)$ with finite support. The *formal domain* for a graph (X, b) is given as

$$\mathcal{F} = \big\{ f \in C(X) : \sum_{y \in X} b(x, y)|f(y)| < \infty \text{ for all } x \in X \big\}.$$

Then the *graph Laplacian* or just *Laplacian* is defined as

$$\Delta : \mathcal{F} \to C(X), \quad \Delta u(x) = \sum_{y \in X} b(x, y)\big(u(x) - u(y)\big).$$

An element $u \in \mathcal{F}$ is callled *superharmonic* if $\Delta u \geq 0$ and *harmonic* if $\Delta u = 0$. Via the Green formula, the Laplacian is connected to the *quadratic form* associated with (X, b), given by $h(u) = \frac{1}{2} \sum_{x, y \in X} b(x, y)\big(u(x) - u(y)\big)^2$, where $u \in \mathcal{F}$. The *extended space* is given as

$$\mathcal{D}_0 = \big\{ u \in \mathcal{F} : h(u) < \infty, \exists\, (u_n) \in C_c(X) \text{ s.t. } u_n \xrightarrow{n} u \text{ ptw., and } h(u_n) \xrightarrow{n} h(u) \big\}.$$

Hardy (type) weights. A map $w : X \to \mathbb{R}$ is called

- *Hardy type weight* if $h(\varphi) \geq w(\varphi) = \sum_{x\in X} w(x)\varphi(x)^2$ for all $\varphi \in C_c(X)$,
- *Hardy weight* if w is a Hardy type weight, $w \geq 0$ and $w \neq 0$,
- *critical* if there is no $\tilde{w} \geq w$, $\tilde{w} \neq w$ s.t. $h(\varphi) \geq \tilde{w}(\varphi)$ for all $\varphi \in C_c(X)$.

A critical Hardy (type) weight is said to be *null-critical* if the ground state (i.e. the up to multiplication by constants unique strictly positive harmonic function) for the operator $\Delta - w$ is not in $\ell^2(X, |w|) = \{f \in C(X) : \sum_{x\in X} f(x)^2 |w(x)| < \infty\}$.

In [8], the authors used the supersolution construction developed in the continuum in [2] to obtain null-critical Hardy weights.

Theorem 1 ([8])**.** *Let (X, b) be locally finite and let $u \in \mathcal{F}$, $u > 0$ be given. Assume additionally that*

- *$\Delta u \geq 0$ and $\Delta u = 0$ outside of a finite set,*
- *u is proper, i.e. $u^{-1}(K)$ is finite for every compact $K \subseteq (0, \infty)$,*
- *u is of bounded oscillation, i.e. $\sup_{x\sim y} \frac{u(x)}{u(y)} < \infty$.*

Then $w = \frac{\Delta u^{1/2}}{u^{1/2}}$ is a null-critical Hardy weight.

The theorem can be used to apply Hardy weights for concrete graphs.

Example. Consider the standard (Cayley) graph over $\mathbb{Z}^d$ with $b(x, y) \in \{0, 1\}$, where $d \geq 3$. Considering $u = G(0, \cdot)$ as the Green function for the simple random walk, one obtains the null critical Hardy weight

$$w(x) = \frac{\Delta G(0, \cdot)^{1/2}(x)}{G(0, x)^{1/2}} = \frac{(d-2)^2}{4} \frac{1}{|x|^2} + \mathcal{O}\big(|x|^{-3}\big), \quad x \to \infty.$$

The leading term corresponds to the weight obtained for the Laplacian on $\mathbb{R}^d$, but in contrast to the continuum setting, there are higher order terms appearing. Moreover, it is known that the best constant for the global Hardy inequality for $\mathbb{Z}^d$ lattice graphs grows linearly in dimension, cf. [7, 5]. Consequently, some higher order terms in the above asymptotic expansion have to come with a negative sign.

We present a new criterion to obtain Hardy type weights which will be contained in [6]. It complements the previous theorem from [8].

Theorem 2. *Let (X, b) be a graph (not necessarily locally finite), and assume that for $u \in \mathcal{F}$, $u > 0$ we additionally have $\Delta u \in \ell^1(X)$ and $u \in \mathcal{D}_0$.*
Then $w = \frac{\Delta u^{1/2}}{u^{1/2}}$ is a critical Hardy type weight. If additionally $\sum_{x\in X} \Delta u(x) \neq 0$, then w is null-critical.

We make some remarks about the theorem. As always, take $u \in \mathcal{F}$ with $u > 0$.

- one can replace $u \in \mathcal{D}_0$ by $u \in C_0(X)$, i.e. u vanishes at infinity.
- if $\Delta u \geq 0$, one only has to assume $\Delta u \in \ell^1(X)$ and that u is a potential (i.e. a convolution with the Green kernel) to obtain a null-critical weight.
- the condition $\Delta u \in \ell^1(X)$ is necessary for critical weights which are not null-critical (positive critical). It does not seem to be necessary for null-critical weights, as can be seen from the weights obtained for trees in [3].

H-type groups. H-type groups are a subclass of homogeneous Lie groups that come along with sub-Riemannian structures. Most results presented here can be extended to the more general realm of stratified (homogeneous) Lie groups. As a set, an *H-type group* is of the form $G = \mathbb{R}^m \times \mathbb{R}^n$ where $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The group law is given as

$$(v,t)(w,s) = \left(v + w,\, t_1 + s_1 + \frac{1}{2}v^T\Omega_1 w,\, \ldots,\, t_n + s_n + \frac{1}{2}v^T\Omega_n w\right),$$

where $\Omega_i \in \mathbb{R}^{m\times m}$ are skew-symmetric and orthogonal matrices that anti-commute, i.e. $\Omega_i\Omega_j = -\Omega_j\Omega_i$ for $1 \le i < j \le n$. The *Cygan-Korányi gauge* is given by $\|(v,t)\|_{CK} = \big(|v|^4 + 16|t|^2\big)^{1/4}$. We consider lattices in G, by which we mean a discrete, cocompact subgroup $X \le G$. Given a symmetric set $0 \notin S = S^{-1} \subseteq G$ that generates X as a group, and a weight function $\theta : S \to (0,\infty)$ satisfying $\theta(s) = \theta(s^{-1})$ and $\sum_{s\in S}\theta(s) < \infty$, a graph (X,b) is obtained by setting $b(x,y) = \theta(s)$ if $y = xs$ and zero otherwise. Clearly this graph is locally finite if and only if S is finite. If S is such that $\sum_{s\in S}\theta(s)\|s\|_{CK}^2 < \infty$, we set $A_S := \sum_{s\in S}\theta(s)v_s v_s^T \in \mathbb{R}^{m\times m}$, where $s = (v_s, t_s)$. The *sub-Laplacian $\mathcal{L}_S$* is the differential operator on $C^\infty(G)$ acting as

$$\mathcal{L}_S = \sum_{i,j=1}^{m}(A_S)_{ij}Z_iZ_j,$$

where $Z = \{Z_i : 1 \le i \le m\}$ are the invariant vector fields that correspond to the first stratum in the Jacobian basis for the Lie algebra $\mathrm{Lie}(G)$. It is known that any fundamental solution $\Gamma_S : G\setminus\{0\} \to \mathbb{R}$ to $\mathcal{L}_S$ is of the form $\Gamma_S(x) = \|x\|^{2-Q}$, where $Q = m + 2n$ is the homogeneous dimension of G and $\|\cdot\|$ is a homogeneous quasi norm (necessarily equivalent to $\|\cdot\|_{CK}$).

We present the following theorem which will be contained in [6].

Theorem 3. *Suppose that $Q \ge 3$ and $\sum_{s\in S}\theta(s)\|s\|_{CK}^{Q+1} < \infty$ and that $u = \|\cdot\|^{2-Q}$ is a fundamental solution for $\mathcal{L}_S$. Then $w = \frac{\Delta u^{1/2}}{u^{1/2}}$ is a null-critial Hardy type weight. Moreover,*

$$w(x) = \frac{(Q-2)^2}{8}\,\frac{\big|\big(\nabla_Z\|\cdot\|\big)(x)A_S^{1/2}\big|^2}{\|x\|^2} + \mathcal{O}\big(\|x\|_{CK}^{-3}\big), \quad x \to \infty, \quad x \in X.$$

We make some remarks on the theorem.

- For analogous results in the continuum, see e.g. [4, 1].
- The proof is an application of the above criterion. To verify the assumptions $\Delta u \in \ell^1(X)$ and $\sum_{x\in X}\Delta u(x) \ne 0$, for certain homogeneous functions f one can approximate the graph Laplacian $\Delta f_{|X}$ by the sub-Laplacian $\mathcal{L}_S f$. Here the moment condition on S is needed.
- To determine the sign of w is a delicate issue. In more special situations, one can find criteria on S and θ to guarantee that w is negative in at most finitely many points.

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Hardy and Rellich identities for Grushin operators

Prasun Roychowdhury

(joint work with Debdip Ganguly, Jotsaroop Kaur)

In this talk, I will present Hardy, Hardy-Rellich, and Rellich identities and inequalities with sharp constants for Grushin vector fields, inspired by the work of Ghoussoub-Moradifam [2] in Euclidean space. Our explicit remainder terms provide significant improvements over existing results in the literature. The approach builds on abstract Hardy-Rellich identities involving the Bessel pair and clever uses of spherical harmonics techniques developed by Garofalo-Shen [4]. Additionally, following the framework of Bez-Machihara-Ozawa [1], we construct spherical vector fields corresponding to Grushin vector fields and establish optimal Rellich identities by comparing the Grushin operator with its radial and spherical components.

We will also see some alternative proofs of Hardy identities and inequalities with improved Hardy constants in specific Sobolev subspaces. Furthermore, we compute the deficit involving the L^2-norm of the Laplacian and its radial components for the Grushin operator by computing spherical vector fields that correspond to the Grushin vector fields. As a consequence, some new second-order Heisenberg-Pauli-Weyl and Hydrogen uncertainty principles, along with certain symmetrization principles in the Grushin space, will be discussed.

If (V, W) is a Bessel pair, then we can prove certain abstract Hardy-type or abstract Hardy-Rellich-type inequalities (see [3, Corollary 3.1, Theorem 1.3]). However, we can ask the converse: if these types of inequalities hold, can we conclude that (V, W) is a Bessel pair? The equivalence of this result was established by Ghoussoub and Moradifam in the Euclidean setting (see [2, Theorem 2.1, Theorem 3.3]). In the same spirit, it would be interesting to study this problem in the Grushin setting.

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The extended Dirichlet space and criticality theory for nonlinear Dirichlet forms

Marcel Schmidt

(joint work with Ian Zimmermann)

Nonlinear Dirichlet forms were introduced by Cipriani and Grillo [3] as those lower semicontinuous convex functionals $\mathcal{E}\colon L^2(X,\mu) \to [0,\infty]$ whose induced (in general nonlinear) semigroup is order preserving and $L^\infty(X,\mu)$ - contractive. They show that - as in the case of classical Dirichlet forms - these properties are related to the compatibility of $\mathcal{E}$ with certain normal contractions. Recently, in [4, 2], this characterization was extended to more general normal contractions and in [8] the following very symmetric version of the compatibility with normal contractions was obtained: $\mathcal{E}$ is a nonlinear Dirichlet form if and only if for all normal contractions $C\colon \mathbb{R} \to \mathbb{R}$ and all $f, g \in L^2(X,\mu)$ it satisfies

$$\mathcal{E}(f + Cg) + \mathcal{E}(f - Cg) \leq \mathcal{E}(f + g) + \mathcal{E}(f - g).$$

Typical examples are energy functionals of p-Laplacians, where p need not be a constant but can be a function. More precisely, for some open domain Ω and measurable $p\colon \Omega \to [1,\infty)$ the functional (or rather its lower semicontinuous relaxation)

$$\mathcal{D}_p\colon L^2(\Omega) \to [0,\infty], \quad \mathcal{D}_p(f) = \begin{cases} \int_\Omega \frac{1}{p(x)} |\nabla f(x)|^{p(x)} dx & \text{if } f \in W^{1,1}_{\mathrm{loc}}(\Omega) \\ \infty & \text{else} \end{cases}$$

is a nonlinear Dirichlet form (as are its restrictions to certain smaller effective domains). Other examples are Cheeger energies on metric measure spaces, which are not necessarily assumed to be infinitesimally Hilbertian. Many more examples are described in [3, 4].

While nonlinear Dirichlet forms have been applied successfully, a lot of the basic theory available for classical Dirichlet forms is missing in their context. Motivated by this lack and by possible applications to optimal Hardy inequalities, we develop criticality theory for a large class of nonlinear Dirichlet forms. A main tool for these considerations is the extended Dirichlet space, whose existence we show along the way.

To do so we need three standing assumptions. The least restrictive is the *symmetry* of $\mathcal{E}$, i.e., we assume $\mathcal{E}(f) = \mathcal{E}(-f)$ for all $f \in L^2(X,\mu)$. It is satisfied by most examples. As a very weak form of linearity we assume the Δ_2-*condition* (this name is borrowed from the theory of Orlicz spaces), namely that there exists $K > 0$ such that $\mathcal{E}(2f) \le K\mathcal{E}(f)$ for all $f \in L^2(X,\mu)$. Under this condition the effective domain $D(\mathcal{E}) = \{f \in L^2(X,\mu) \mid \mathcal{E}(f) < \infty\}$ is a vector space. For stating the next assumption we introduce the *Luxemburg seminorm* of $\mathcal{E}$ given by

$$\|\cdot\|_L \colon D(\mathcal{E}) \to [0,\infty), \quad \|f\|_L = \inf\{\lambda > 0 \mid \mathcal{E}(\lambda^{-1} f) \le 1\}.$$

In order to transfer some of the Hilbert space arguments available for classical Dirichlet forms to the nonlinear situation our third standing assumption is that $(D(\mathcal{E}), \|\cdot\|_L)$ is *reflexive* in the sense that $D(\mathcal{E})$ is norm dense in the bidual of $(D(\mathcal{E}), \|\cdot\|_L)$. For the functional $\mathcal{D}_p$ all three assumptions are satisfied if $\inf_{x\in\Omega} p(x) > 1$ and $\sup_{x\in\Omega} p(x) < \infty$ (in which case it is lower semicontinuous and hence equals its lower semicontinuous relaxation).

As for classical Dirichlet forms we define the *extended Dirichlet space* $D(\mathcal{E}_e)$ as the set of those functions $f \in L^0(X,\mu)$ for which there exists an $\mathcal{E}$-Cauchy sequence (f_n) in $D(\mathcal{E})$ with $f_n \to f$ locally in measure. Such a sequence is called *approximating sequence* for f. Our first main result is the following.

Theorem 1 (Existence of the extended Dirichlet form)**.** *Under our three standing assumptions the functional* $\mathcal{E}_e \colon L^0(X,\mu) \to [0,\infty]$

$$\mathcal{E}_e(f) = \begin{cases} \lim\limits_{n\to\infty} \mathcal{E}(f_n) & \text{if } (f_n) \text{ is an approximating sequence for } f \\ \infty & \text{if } f \text{ has no approximating sequence} \end{cases}$$

is well-defined and lower semicontinuous with respect to local convergence in measure.

This extends classical results by Silverstein [11] and Schmuland [10] to the nonlinear setting but the proof requires new ideas. Even the existence of $\lim_{n\to\infty} \mathcal{E}(f_n)$ for approximating sequences is non-trivial in the nonlinear situation.

The Δ_2-condition extends to $\mathcal{E}_e$ and hence $D(\mathcal{E}_e)$ is a vector space and the Luxemburg seminorm $\|\cdot\|_{L,e}$ of $\mathcal{E}_e$ is well-defined on $D(\mathcal{E}_e)$. Moreover, we denote by $G_\alpha = (\alpha + \partial\mathcal{E})^{-1}$, $\alpha > 0$, the resolvent induced by the subgradient $\partial\mathcal{E}$ of $\mathcal{E}$.

Theorem 2 (Characterization subcriticality (transience))**.** *Under our four standing assumptions the following assertions are equivalent.*

(i) *There exists* $g \colon X \to (0,\infty)$ *such that* $Gg = \lim_{\alpha\to 0+} G_\alpha g < \infty$ *a.e.*

(ii) *There exists* $h \colon X \to (0,\infty)$ *such that*

$$\int_X |f| h\, d\mu \le \|f\|_{L,e}, \quad f \in D(\mathcal{E}_e).$$

(iii) $\ker \mathcal{E}_e = \{0\}$.

(iv) $(D(\mathcal{E}_e), \|\cdot\|_{L,e})$ *is a reflexive Banach space.*

(v) *For one/all* $1 \le p < \infty$, *one/all integrable* $w \colon X \to (0,\infty)$ *there exists a decreasing* $\alpha \colon (0,\infty) \to (0,\infty)$ *such that*

$$\left(\int_X |f|^p w\, d\mu\right)^{1/p} \le \alpha(r)\|f\|_{L,e} + r\|f\|_\infty, \quad r > 0,\ f \in D(\mathcal{E}_e) \cap L^\infty(X,\mu).$$

Assertions (i) - (iv) appear in the textbook characterization of transience for classical Dirichlet forms, cf. [6]. The main observation here is that the inequalities in (ii) and (v) have to be formulated with respect to the Luxemburg seminorm (instead of powers of $\mathcal{E}_e$) and that other than in the classical situation $Gg < \infty$ need not hold for all $g \in L^1(X, \mu)_+$. Assertion (v) is a weak Hardy inequality and its relation to subcriticality was first observed in [9]. As a direct application of this theorem we obtain the existence of equilibrium potentials and hence a potential theory recovering recent results of [5, 1, 7]. For criticality (recurrence) the following characterization is very similar to the one for classical Dirichlet forms.

Theorem 3 (Characterization criticality (recurrence)). *Additionally to our standing assumptions assume that $\partial\mathcal{E}(0) = \{0\}$. The following assertions are equivalent.*

(i) $\mathcal{E}_e(1) = 0$.
(ii) There exists (f_n) in $D(\mathcal{E})$ with $f_n \to 1$ locally in measure and $\mathcal{E}(f_n) \to 0$.
(iii) For all $g\colon X \to [0, \infty)$ and $Gg = \lim_{\alpha\to 0+} G_\alpha g$ we have

$$\mu(\{g > 0\} \cap \{Gg < \infty\}) = 0.$$

A measurable set $A \subseteq X$ is called *invariant* if $\mathcal{E}(1_A f) \leq \mathcal{E}(f)$ for all $f \in L^2(X, \mu)$. Moreover, $\mathcal{E}$ is called *irreducible* if every invariant set is either null or co-null. Irreducibility implies $\ker \mathcal{E}_e \subseteq \mathbb{R} \cdot 1$ and we obtain:

Theorem 4 (Dichotomy of criticality and subcriticality). *Under our four standing assumptions and $\partial\mathcal{E}(0) = \{0\}$ the nonlinear Dirichlet form $\mathcal{E}$ is either critical or subcritical.*

In summary, we obtain criticality theory for a relatively large class of nonlinear Dirichlet forms that is quite similar to the one for classical Dirichlet forms. However, due to the lack of linearity and the lack of representation theorems there are subtle differences in the statements and many proofs need different ideas compared to the classical situation.

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Trade-off in Hardy inequalities

Antoni Szczukiewicz
(joint work with Krzysztof Bogdan and Bartłomiej Dyda)

We will talk about trade-off in Hardy inequalities on $\mathbb{R}$. Our primary focus is on the trade-off in fractional Hardy inequality, which extends and refines the classical fractional Hardy inequality.

We will start by introducing the well-known one-dimensional fractional Hardy inequality. For $\alpha \in (0,2)$, $p \in (-1,\alpha)$, and any function $u \in L^2(|x|^{-\alpha}dx)$, the following inequality holds:

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+1}}\,dx\,dy \geq C(p)\int_{\mathbb{R}} \frac{u^2(x)}{|x|^\alpha}\,dx,$$

where $C(p)$ is an explicit constant given by

$$C(p) = B(p+1,\alpha-p)\left(\frac{\sin\left(\pi\left(\frac{\alpha}{2}-p\right)\right)}{\sin\left(\frac{\pi\alpha}{2}\right)} - 1\right),$$

and this constant attains its optimal value for $p = \frac{\alpha-1}{2}$.

As a key result, we will derive the balanced fractional Hardy inequality:

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+1}}\,dx\,dy \geq D^+_{\alpha,s}\int_0^\infty \frac{u^2(x)}{|x|^\alpha}\,dx + D^-_{\alpha,s}\int_{-\infty}^0 \frac{u^2(x)}{|x|^\alpha}\,dx,$$

where the constants $D^+_{\alpha,s}$ and $D^-_{\alpha,s}$ are optimal and depend on the new parameter s, which controls the trade-off between them. These constants satisfy

$$D^+_{\alpha,s} = B\left(\frac{\alpha+1}{2},\frac{\alpha+1}{2}\right)\left(\frac{1}{\sin(\frac{\pi\alpha}{2})} - s\right),$$

$$D^-_{\alpha,s} = B\left(\frac{\alpha+1}{2},\frac{\alpha+1}{2}\right)\left(\frac{1}{\sin(\frac{\pi\alpha}{2})} - \frac{1}{s}\right).$$

Additionally, we will refine the fractional Hardy inequality by incorporating an extra nonnegative term:

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+1}}\,dx\,dy \geq D^+_{\alpha}\int_0^\infty \frac{u^2(x)}{|x|^\alpha}\,dx + D^-_{\alpha}\int_{-\infty}^0 \frac{u^2(x)}{|x|^\alpha}\,dx +$$
$$B\left(\frac{\alpha+1}{2},\frac{\alpha+1}{2}\right)\left(\sqrt{\int_0^\infty \frac{u^2(x)}{|x|^\alpha}\,dx} - \sqrt{\int_{-\infty}^0 \frac{u^2(x)}{|x|^\alpha}\,dx}\right)^2.$$

In particular, for $\alpha = 1$, we obtain the improved fractional Hardy inequality:

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}\,dx\,dy \geq \left(\sqrt{\int_0^{\infty} \frac{u^2(x)}{|x|}\,dx} - \sqrt{\int_{-\infty}^{0} \frac{u^2(x)}{|x|}\,dx}\right)^2.$$

Moreover, we will present similar results where the kernel $|x-y|^{-\alpha-1}$ is replaced by a more general nonnegative, symmetric, and homogeneous kernel $k(x,y)$.

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