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Finite Groups, Fusion Systems and Applications

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Abstract. The Classification of Finite Simple Groups (CFSG) is considered to be one of the most important results of modern mathematics, and has led to many applications both inside and outside group theory. The theory of fusion systems, although originating in topology and modular representation theory, has quite recently grown into a new field with the potential for very strong impact in finite group theory, and in particular on one of the programmes for a new proof of part of the CFSG. The workshop focussed on some of these developments, as well as recent applications of finite group theory both within group theory and in other areas such as algebraic topology and algebraic combinatorics.

Mathematics Subject Classification (2020): 20Dxx, 20Exx, 20Gxx, 20Jxx, 20Pxx.

Introduction by the Organizers

The workshop "Finite groups, fusion systems and applications" was attended by 52 participants (4 online) with a total of 13 female participants. There were 22 presentations – six of 1 hour, fourteen of 40 minutes, and two 25 minute lectures from PhD students.

There were two major lectures on the theory underpinning the Classification of Finite Simple Groups (CFSG). The original proof of the CFSG was spread over hundreds of articles over several decades, and the goal of the ongoing Gorenstein-Lyons-Solomon (GLS) project is to produce a new unified proof of the classification and is now nearing its completion. This project benefits from a variety of new ideas, and also new tools such as the recent geometric methods for identifying finite groups of Lie type and sporadic simple groups, and from the clearer understanding of the local structure of the known finite simple groups which has accumulated since the completion of the original proof. Stroth gave a lecture on his work on the so-called uniqueness case for groups of restricted even type, an extremely difficult problem that is a part of the endgame of the classification. Stroth announced his solution of this case, a proof of some 900 pages which is projected to form two of the final volumes in the GLS project. Another lecture was given by Stellmacher on an alternate approach to the proof of a large section of the CFSG. At the heart of this programme, which was initiated by Meierfrankenfeld, is the amalgam method pioneered by Goldschmidt and Stellmacher. This work leads naturally into the study of saturated fusion systems, another main topic of the workshop.

The theory of fusion systems generalizes important aspects of finite group theory, since each finite group leads to a saturated fusion system which encodes the conjugacy relations between p-subgroups (traditionally referred to as fusion). It is an important objective to understand the prevalence and behaviour of exotic fusion systems, i.e. fusion systems which are not realized by finite groups. The theory of saturated fusion systems connects previously independent developments in local finite group theory, modular representation theory and homotopy theory. As a result, the different methods in these topics merge and complement each other, often in surprising ways. For example, building on earlier work of Broto, Levi and Oliver that was motivated by applications in homotopy theory, Chermak introduced group-like structures called localities, which are also important for applications in finite group theory. Lectures on fusion systems and localities were given by Chermak, Grazian, Lynd, Oliver, Salati and Serwene. Oliver presented a ground-breaking result (proved together with his coauthors Broto, Møller and Ruiz) which reduces the realizability of a fusion system by a finite group to the realizability of components of the fusion system (or more generally the realizability of a normal subsystem containing every component). This underlines the importance of understanding the exotic behavior of simple fusion systems through classification results. Grazian presented her classification of fusion systems with certain elementary abelian self-centralizing subgroups whose presence often cause fusion systems to be exotic. Salati on the other hand reported on his effort to severely limit the structure of fusion systems having abstract properties resembling the structure of fusion systems of many groups of Lie type in defining characteristic p. This is achieved by using localities to translate concepts and results from the classification programme of Meierfrankenfeld and others. Localities are special cases of "partial groups". Lynd presented his work with Hackney which aims to understand the structure of partial groups through higher Segal conditions. Serwene lectured on the connection between fusion systems and modular representation theory, mentioning in particular results on weight conjectures and the first Hochschild cohomology group of a block algebra. Chermak suggested in his talk a new line of research introducing a notion of finite p-dimension of locally finite countable groups. This is motivated by his effort to extend results from finite localities to infinite localities with certain finiteness properties.

The third focus of the workshop was on properties and applications of finite group theory, particularly simple groups. Here there were several spectacular lectures containing announcements of the resolution of longstanding conjectures. First, a major problem at the interface of group theory and topology is Quillen's conjecture – that for any finite group G and prime p, the poset of nontrivial p-subgroups of G is contractible if and only if G has a nontrivial normal p-subgroup. Quillen himself proved the 'if' part of this, but the much harder converse has remained open for over 40 years. Piterman gave a survey of the problem, and presented the recent resolution of the conjecture for all primes p apart from 2 – the first real progress since a pioneeering 1993 paper of Aschbacher and Smith. Tiep lectured on the use of character-theoretic methods in attacking the 40-year old Thompson conjecture – that every finite simple group G possesses a conjugacy class C such that $C^2 = G$ – and announced the solution of the conjecture for all sufficiently large simple groups. Lifshitz showed how he has used his recently developed theory of hypercontractivity to resolve the Liebeck-Nikolov-Shalev conjecture on the growth of subsets in simple groups. And Roney-Dougal announced the solution of a 30-year old conjecture of Pyber on the number of subgroups of symmetric groups. In his lecture, Dona gave a new perspective on a classical theorem of Jordan on finite linear groups. Gill presented some new techniques for studying binary actions of simple groups, an area motivated by Cherlin's theory of homogeneous combinatorial structures. Praeger lectured on her classification of the finite arc-transitive graphs that admit the action of a simple group and embed in a surface as a regular map, and van Bon presented his and Stellmacher's use of amalgam methods to study locally s-arc transitive graphs. Finally, two of the PhD students del Valle and Huang lectured on their work in permutation groups.

In summary, the workshop brought together researchers across a broad range of fields, many of them early in their careers; almost half of the talks were given by young speakers, and all of them gave excellent lectures. The schedule left plenty of time for scientific exchange, and new collaborations were started, as well as ongoing ones continued.

Workshop: Finite Groups, Fusion Systems and Applications

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Abstracts

Almost Finite Groups

Andrew Chermak

Let G be a locally finite group, of countable (possibly finite) cardinality. Equivalently, G is the union of a tower $\gamma = (G_n)_{n\in\mathbb{N}}$ of finite subgroups G_n; and we say that γ is a *framing* of G.

Let p be a prime, let $Sub_p(G)$ be the set of all p-subgroups of G (partially ordered by inclusion), and let $Max_p(G) \subseteq Sub_p(G)$ be the set of all maximal p-subgroups of G. Define $\Omega_p(G)$ to be the set of all subgroups X of G such that $X = \cap\mathcal{M}$ for some non-empty subset $\mathcal{M}$ of $Max_p(G)$. Then $\Omega_p(G)$ is partially ordered by inclusion, and we say that G is *p-finite* if there exists $d \in \mathbb{N}$ such that there exists no tower in $\Omega_p(G)$ of length greater than d. If such a number d exists, then the least such d is the *p-dimension* of G, denoted by $dim_p(G)$. If G is p-finite for all primes p then G is *almost finite.* For example, if F is the algebraic closure of a finite field, and V is a finite-dimensional vector space over F, then $GL(V)$ is almost finite. On the other hand, the direct limit of the symmetric groups $Sym(n)$ (i.e. the set of all permutations of $\mathbb{N}$ having finite support) is locally finite and countable, but is not p-finite for any prime p.

Subgroups and homomorphic images of p-finite groups are again p-finite (and so similarly for almost finite groups), and if G is p-finite then G acts transitively on $Max_p(G)$ by conjugation. Our concern is with *simple* p-finite groups G such that G contains an element of order p.

For any finite group H, a subgroup X of H is a *p-component* of H if $H = [H, H]$ and $H/O_{p'}(H)$ is simple. In this lecture we outline a proof of the following result.

Theorem A. *Let G be a simple p-finite group such that G contains an element of order p, and assume that G is not of order p. Then there exists a framing $(G_n)_{n\in\mathbb{N}}$ of G such that:*

(1) For each n, G_n is perfect and $G_n/O_{p'}(G_n)$ is simple.
(2) Both $dim_p(G_n)$ and $dim_p(G_n/O_{p'}(G_n))$ are constant functions of n.

The proof of Theorem A relies only on elementary group theory. If one now deploys the CFSG (the classification of the finite simple groups) then one obtains the following result, as a corollary to a Theorem of Hartley and Shute [HS].

Theorem B. *Let G be a simple p-finite group. Then G is almost finite, and if G is infinite then G is a group of Lie type ${}^d\S(E)$, where E is an infinite subfield of the algebraic closure of a finite field.*

The motivation for these results arose from an effort by the speaker, together with Alex Gonzales, to gain insight into the "p-local compact groups" introduced by Carles Broto, Ran Levi, and Bob Oliver [BLO]. These have an equivalent formulation as "compact localities" at the prime p. In the language of localities [CG], one has the following theorem (due to Gonzales [G]).

Theorem (Gonzales). *Let $\mathcal{L}$ be a compact locality at the prime p. Then there exists a sequence $\gamma = (\mathcal{L}_n \to \mathcal{L}_{n+1})_{n\in\mathbb{N}}$ of injective homomorphisms of finite localities at p such that $\mathcal{L}$ is the direct limit of γ in the category of partial groups.*

By definition, localities (compact or otherwise) are countable, and are "finite-dimensional" in a sense analogous to the notion of finite p-dimension for groups. What Gonzales' result adds to this is that compact localites are locally finite. The category of finite localities (like the category of finite saturated fusion systems) provides a framework in which to carry out elementary finite-group-theoretic arguments, and for that reason we conjecture:

Theorem A'. *Let $\mathcal{L}$ be a simple, locally finite locality at the prime p. Then there exists a framing γ of $\mathcal{L}$ (as in the above theorem of Gonzales), such that*

(1) Each $\mathcal{L}_n$ is a simple locality at p, and
(2) $dim_p(\mathcal{L}_n)$ is a constant function of n.

The goal here is to obtain a classification of the simple, locally finite, proper localities - especially at $p = 2$. For this one requires the classification (initiated by Aschbacher) of the simple, saturated fusion systems at $p = 2$ (the CFSF2). The following formulation would suffice.

CFSF2. *Let e be a natural number and let $\mathbb{F}_e$ be the set of isomorphism classes of simple saturated fusion systems $\mathcal{F}$ over a finite 2-group, such that the locality associated with $\mathcal{F}$ has dimension e. Let $\mathbb{L}_e$ be the set of all $\mathcal{F} \in \mathbb{F}_e$ such that $\mathcal{F}$ is the fusion system at the prime 2 of a simple group of Lie type. Then, for all but one value e^* of e, the complement of $\mathbb{L}_e$ in $\mathbb{F}_e$ is finite; while $\mathbb{F}_{e^*}$ is the union of $\mathbb{L}_e$, a set of Benson-Solomon fusion fusion systems, and a finite set.*

We end with a question: Does there exist a simple, proper locality which is not locally finite ?

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Greedy bases for primitive groups

Coen del Valle
(joint work with Sofia Brenner, Colva Roney-Dougal)

Let $G \leq \mathrm{Sym}(\Omega)$ be a finite permutation group. A *base* for G is a subset $\mathcal{B} \subseteq \Omega$ such that the pointwise stabiliser $G_{(\mathcal{B})}$ is trivial. Integral in computational group theory (see e.g. [10]), bases and related group invariants have been a subject of significant interest over the past half-century. Finding small bases can be critical for computational purposes — the size of a smallest base for G is the *base size*, denoted $b(G)$.

Much work has been done to determine the base size of primitive groups. Suppose that G is an almost simple primitive group with socle G_0 and point stabiliser H. We say that G has *standard action* if, up to equivalence, at least one of the following holds

(i) G_0 is alternating and Ω is an orbit of either uniform subsets or uniform partitions;
(ii) G_0 is classical with natural module V and Ω is an orbit of either subspaces or pairs of subspaces of V; or
(iii) $G_0 = \mathrm{Sp}_n(q)$ where q is even and $H \cap G_0 = \mathrm{O}_n^{\pm}(q)$.

Otherwise, the action of G is *non-standard.* A famous conjecture of Cameron [7] asserted that the base size of all almost simple primitive groups with non-standard action should be at most 7. Remarkably, this conjecture was resolved in the affirmative [3, 4, 5, 6].

The problem of finding a smallest base for arbitrary finite G is NP-hard [1], so it has been of interest to find polynomial time algorithms which compute bases of size as close to minimum as possible. One such algorithm is the so-called greedy algorithm: start with $\mathcal{B}_0 = \emptyset$, and for $i \geq 1$ we set $\mathcal{B}_i = \mathcal{B}_{i-1} \cup \{\beta_i\}$, where β_i is an element in a longest orbit of the pointwise stabiliser $G_{(\mathcal{B}_{i-1})}$. Since orbits and stabilisers may be computed in polynomial time, so too can a greedy base. We call the size of a largest greedy base for G the *greedy base size*, and denote it $\mathcal{G}(G)$.

In 1992, Blaha [1] showed that $\mathcal{G}(G) \leq \lceil b(G) \log\log |\Omega| \rceil + b(G)$. Additionally, Blaha demonstrated that — up to a constant — this bound is best possible. In 1999, Peter Cameron [7] suggested that for primitive groups the situation appears different.

Conjecture 1 (Cameron's Greedy Conjecture)**.** *There exists an absolute constant c such that if G is any finite primitive permutation group, then $\mathcal{G}(G) \leq cb(G)$.*

Recently, a program has begun to settle Cameron's Greedy Conjecture. Thus far, the focus has been primarily on the almost simple primitive groups. The greedy base sizes for primitive G with sporadic socle have been determined in a recent paper [8], showing that for such groups the greedy base size coincides with the base size. Similarly, if G is any primitive group of odd order, then $b(G) = \mathcal{G}(G)$ [2].

The situation when G is almost simple primitive with alternating socle is more complicated. When such G is equipped with a standard action on uniform partitions we have that $\mathcal{G}(G) \leq 11b(G)$ [9, Theorem 3.18]. A similar bound has also been determined in most cases for the standard actions of G on uniform subsets, but the problem remains open when the size of the ground set is less than $4r^2$, where r is the subset size [9, Theorem 2.9]. On the other hand, when such G is equipped with a non-standard action, $b(G) = \mathcal{G}(G)$ [9]. Thus, given the situation for sporadic groups, and non-standard groups with alternating socle, the following conjecture is natural.

Conjecture 2. *Let G be a finite almost simple primitive group with non-standard action. Then $\mathcal{G}(G) \leq 7$.*

To prove the conjecture, it remains only to consider those groups with socle of Lie type.

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Jordan's theorem: explicit, CFSG-free, over arbitrary field

Daniele Dona

(joint work with Jitendra Bajpai)

In 1878, Jordan proved the following (see [5, Thm. 40]).

Theorem 1. *Let $\Gamma \leq \mathrm{GL}_n(\mathbb{C})$ be finite. Then there is a constant $J(n)$, depending only on n, and there an abelian normal subgroup $A \trianglelefteq \Gamma$ with $|\Gamma/A| \leq J(n)$.*

The result has been improved and generalized several times. Different proofs have different features, as they may: (1) feature explicit constants, (2) not rely on CFSG, or (3) work for any field K. Theorem 1 satisfies (2). A network of proofs from the 1900s-1910s by Blichfeldt, Bieberbach, and Frobenius satisfies (1)(2). In 1984 Weisfeiler gave a proof satisfying (1)(3), improved 15-20 years later by Guralnick and Collins. In 1998 Larsen and Pink proved a version satisfying (2)(3).

With Bajpai, we gave a proof that satisfies (1)(2)(3) (see [1, Thm. 1.2]).

Theorem 2. *Let K be any field, and let $\Gamma \leq \mathrm{GL}_n(\overline{K})$ be finite. Then there are $\Gamma_3 \trianglelefteq \Gamma_2 \trianglelefteq \Gamma_1 \trianglelefteq \Gamma$, each of them normal inside Γ, such that*

(a) $|\Gamma/\Gamma_1| \leq J'(n) := n^{n^{2^{23}n^{10}}}$;
(b) either $\Gamma_1 = \Gamma_2$, *or* $\mathrm{char}(K) = p > 0$ *and* Γ_1/Γ_2 *is a product of finite simple groups of Lie type of characteristic p;*
(c) Γ_2/Γ_3 *is abelian of size not divisible by* $\mathrm{char}(K)$*;*
(d) either $\Gamma_3 = \{e\}$, *or* $\mathrm{char}(K) = p > 0$ *and* Γ_3 *is a p-group.*

The strategy follows closely [7], with explicit computations handled via techniques that the authors developed with Helfgott [2] [3]. The main intermediate result is the following (see [1, Thm. 6.7], and compare it to [7, Thm. 0.5]).

Theorem 3. *Let $G \leq \mathrm{GL}_n$ be a connected almost simple adjoint group over K, with $(r, d, D, \iota) = (\mathrm{rk}(G), \dim(G), \deg(G), \mathrm{mdeg}(^{-1}))$. Let $\Gamma \leq G(\overline{K})$ be finite with*

(i) $|\Gamma| > (2dDrn\iota)^{(2dDr\iota)^{11d^4}}$, *and*
(ii) $\Gamma \not\leq H(\overline{K})$ *for any* $H \lneq G$ *with* $\dim(H) < d$ *and* $\deg(H) \leq (2dDr)^{4d^2}$.

Then $\mathrm{char}(K) = p > 0$, *there is some $q = p^e$ such that $\mathbb{F}_q$ is an $\mathbb{F}_p$-subalgebra of K or K^2, and there is a Steinberg endomorphism $F : G \to G$ such that F or F^2 is the $\mathbb{F}_q$-Frobenius map, with $[G^F, G^F] \leq \Gamma \leq G^F$ and with $[G^F, G^F]$ simple.*

Going from Theorem 3 to Theorem 2 essentially works as follows. If (i)(ii) hold, throw the finite simple group $[G^F, G^F]$ into (b) and its index in Γ into (a). If (i) fails, throw everything into (a). If (ii) fails, the dimension goes down and we enter an induction, as long as we can make H into a connected almost simple adjoint group as well: passing to H^o makes it connected, throwing the index into (a); the quotient by $R_u(H)$ makes it reductive, and the unipotent radical is thrown into (d); the quotient by $Z(H)$ makes it semisimple adjoint, and the centre is thrown into (c); a semisimple is a product of almost simple parts, and the induction kicks in. The base case $\dim(G) = 0$ is handled via (a) and the degree bound in (ii).

To achieve Theorem 3, let us sketch the strategy in [7]. Assume that (i)(ii) hold. First, we need to find the "correct" $\mathbb{F}_q$. One finds a regular unipotent $u \in \Gamma$ (implying already $p > 0$). Using u, one can determine some $V \leq Z(U)$ abelian unipotent such that $|\Gamma \cap V(\overline{K})| > 1$: then $\Gamma \cap V(\overline{K})$ is group-isomorphic to some $\mathbb{F}_q$, in turn a subalgebra of K or K^2 depending on $\dim(V) \in \{1, 2\}$.

Secondly, we spread $\mathbb{F}_q$ everywhere. For $\gamma \in \Gamma$ in a dense open set of G, the group $H_{(\gamma)} = \langle V, \gamma V \gamma^{-1} \rangle$ is almost simple of the form A_1, B_2, G_2. One builds first a Steinberg endomorphism $F : H_{(\gamma)} \to H_{(\gamma)}$, yielding either $A_1(q)$, ${}^2B_2(2^{2f+1})$, or ${}^2G_2(3^{2f+1})$. Furthermore $H_{(\gamma)}$ is the same for all γ, and γ runs in an open set, so we similarly build $F : G \to G$ in a way that gives $\Gamma \leq G^F$. The index $[G^F : \Gamma]$ is bounded explicitly, and so is $|G^F/[G^F, G^F]|$: considering the normal core of $[G^F, G^F] \cap \Gamma$ in the simple group $[G^F, G^F]$, we conclude that $[G^F, G^F] \leq \Gamma$ as long as Γ is large enough.

Many of the steps above are non-explicit in [7]. In [1], we treat them using techniques that were used in [2] [3] to study the growth of sets inside classical groups. We illustrate one example, related to determining dimensional estimates: it is used for instance (but not exclusively) to achieve the first steps of the strategy, namely finding $u \in \Gamma$ regular unipotent and V intersecting Γ nontrivially.

A *dimensional estimate* is for us an upper bound of the form $|\Gamma \cap V(\overline{K})| \ll |\Gamma|^{\dim(V)/\dim(G)}$ for varieties $V \subseteq G$, where the implicit constant depends only on the data of V, G (dimension, degree). The first step leading to explicit estimates of this form is an upper bound on the degree of intersections of varieties.

Lemma 4. *Let $\{Z_i\}_{i \in I}$ be a collection of varieties in $\mathbb{A}^N$. Assume that $\dim(Z_i) \leq d$ and $\deg(Z_i) \leq D$ for all $i \in I$. Then $\deg\left(\bigcap_{i\in I} Z_i\right) \leq D^{d+1}$.*

The idea is to take intersections and keep track of degree changes for every irreducible component X: if X is intersected non-trivially, $\dim(X)$ goes down by ≥ 1, and $\deg(X)$ is multiplied by $\leq D$ by Bézout. Hence, the worst-case degree is bounded by the value achieved by descending d times until we have $\dim(X) = 0$ everywhere, which would mean $D \cdot D^d = D^{d+1}$.

Through Lemma 4 we prove a second result related to *escape from subvarieties.* In its classic form of [6], an escape result asserts that if a set A is such that $\langle A\rangle \not\subseteq V(\overline{K})$ then $A^k \not\subseteq V(\overline{K})$ for k bounded in terms of the data of V. This fact is useless to us, since $A = \Gamma$ implies $A^k = \langle A\rangle = \Gamma$ too. However, we may rephrase it to show that *escaping from subgroups* implies escaping from subvarieties.

Lemma 5. *Let $\Gamma \leq G(\overline{K})$. If $\Gamma \subseteq V(\overline{K})$ for some $V \subsetneq G$, then either $|\Gamma| \leq \deg(V)^{\dim(V)+1}$, or $\Gamma \leq H(\overline{K})$ for some $H \lneq G$ with $\deg(H) \leq \deg(V)^{\dim(V)+1}$.*

The idea is to build a sequence of V_i and S_i with $S_i \subseteq \Gamma \subseteq V_i(\overline{K})$, $V_i = \bigcap_{\gamma \in S_i} V\gamma$, and $V_i \subsetneq V_{i-1}$, starting with $V_0 := V$ and $S_0 := \{e\}$, and stop when either $\Gamma \leq \mathrm{Stab}(V_i)(\overline{K})$ or $\dim(V_i) = 0$. To go from i to $i+1$, take $\gamma_{i+1} \in \Gamma$ not in $\mathrm{Stab}(V_i)(\overline{K})$ and set $V_{i+1} := V_i \cap V_i\gamma_{i+1}$ and $S_{i+1} := S_i \cup S_i\gamma_{i+1}$, which satisfy all requirements. When we stop, we are in one of two cases. If $\Gamma \leq H := \mathrm{Stab}(V_i)(\overline{K})$, write $H = \bigcap_{v_i,\gamma} v_i^{-1} V\gamma$ and bound $\deg(H)$ by Lemma 4. If $\dim(V_i) = 0$, V_i is made of $\deg(V_i)$ points, and Lemma 4 yields the bound on $|\Gamma| \leq \deg(V_i)$.

A third lemma bounds the degree of the *exceptional locus* of a variety V through a map f. The non-explicit version dates back to works of Chevalley (see [4]).

Lemma 6. *Let $V \subseteq \mathbb{A}^N$ be an irreducible variety, and let $f : V \to \mathbb{A}^M$ be a morphism. Then there is a variety*

$$E \supseteq \{v \in V \mid \dim(f^{-1}(f(v))) > \dim(V) - \dim(\overline{f(V)})\}$$

with $\dim(E) < \dim(V)$ and $\deg(E) \leq \deg(f)^{2\dim(V)} \deg(V)^2$.

(Different, more refined versions are available and used in the actual proof.)

With Lemmas 5–6 in hand, we can prove the dimensional estimate we need.

Proposition 7. *Let $G \leq \mathrm{GL}_n$ be connected almost simple, with $d = \dim(G)$ and $D = \deg(G)$. Let $\Gamma \leq G(\overline{K})$ be finite. Then*

- *either* $|\Gamma| \leq (2dD)^{d+1}$,
- *or* $\Gamma \leq H(\overline{K})$ *with* $\dim(H) < d$ *and* $\deg(H) \leq (2dD)^{d+1}$,
- *or* $|\Gamma \cap V(\overline{K})| \leq (2d\deg(V))^{d^{\dim(V)}} |\Gamma|^{\dim(V)/d}$ *for any variety* V.

The idea is to assume that the first two points do not hold, and use maps to create an induction process on $\dim(V)$ to prove the third point. For a variety $W \subseteq \mathbb{A}^N$ and a "nice" map $f : \mathbb{A}^N \to \mathbb{A}^N$ (i.e., with $f(\Gamma) \subseteq \Gamma$), we can bound $|\Gamma \cap W(\overline{K})|$ by counting the points of Γ on the image $f(W)$, on a generic "good" fibre F (i.e., of dimension $\dim(W) - \dim(\overline{f(W)})$), and on the exceptional locus E:

$$|\Gamma \cap W(\overline{K})| \leq |\Gamma \cap \overline{f(W)}(\overline{K})| \cdot |\Gamma \cap F(\overline{K})| + |\Gamma \cap E(\overline{K})|.$$

In practice the maps are more complicated, involving multiple copies of V: for instance, we may have $f : V \times V \to V\gamma V$ for $\gamma \in \Gamma$ such that $\dim(V\gamma V) > \dim(V)$. Then there are two base cases $V = G$ and $\dim(V) = 0$ for the induction. Lemma 5 is used to guarantee the existence of γ, and Lemma 6 to guarantee the descent $\dim(E) < \dim(V)$; the rest is patient bookkeeping.

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On the binary actions of the alternating and symmetric groups

Nick Gill

(joint work with Pierre Guillot)

Throughout we study a group G acting on a finite set Ω (on the right). We call the action *binary* if, roughly speaking, we can deduce the orbits of G on Ω^k (for any positive integer k) from knowledge of the orbits of G on Ω^2.

Our aim is to classify all such actions. We report on progress in two specific cases.

A precise definition. We start by giving a precise definition of a binary action. To do this we need to introduce two equivalence relations on the sets Ω^k where k is any positive integer k. These equivalence relations are defined with respect to the action of G on Ω.

So, let $I, J \in \Omega^k$ be k-tuples of elements of Ω, for some $k \geq 1$, written $I = (I_1, \dots, I_k)$ and $J = (J_1, \dots, J_k)$.

(1) We say that I and J are *2-related* if, for each choice of indices $1 \leq \ell_1 < \ell_2 \leq k$, there exists $g \in G$ such that $I_{\ell_i}^g = J_{\ell_i}$ for all $i \in \{1, 2\}$.
(2) We say that I and J are *k-related* if there exists $g \in G$ such that $I_\ell^g = J_\ell$ for all $\ell \in \{1, 2, \dots, k\}$.

Now we say that the action of G on Ω is *binary* if, for all $k \geq 2$ and all $I, J \in \Omega^k$, I and J are 2-related if and only if they are k-related.

Some context and the motivating question. The notion of a binary action is due to the model theorist Gregory Cherlin. It is a particular instance of the more general notion of the *relational complexity* of a finite group action. Cherlin studied the "stratification" induced by the notion of relational complexity on the universe of finite group actions. He was able to demonstrate some very interesting properties of this stratification, particularly with regards to the existence of sporadic behaviour in this universe [1].

It is easy to find examples of binary actions: first, if the action of G on Ω is regular, then it is binary; second, the natural action of $G = \mathrm{Sym}(\Omega)$ on Ω is binary.

On the other hand it has hitherto proved rather difficult to classify the binary actions of even very well understood families of groups. One reason is that the notion of a binary action behaves rather badly with respect to basic group operations like taking subgroups and quotients.

Nonetheless, we do at least have a full understanding of the *primitive* binary actions. The following theorem was originally a conjecture of Cherlin and is now a theorem, thanks to the work of various authors including Cherlin, Dalla Volta, Gill, Hunt, Liebeck, Spiga and Wiscons [2, 3, 5, 6].

Theorem 1. *A finite primitive binary permutation group must be one of the following:*

(1) a symmetric group $\mathrm{Sym}(n)$ *acting naturally on* n *elements;*
(2) a cyclic group of prime order acting regularly on itself;
(3) an affine orthogonal group $V \rtimes \mathrm{O}(V)$ *with* V *a vector space over a finite field equipped with a non-degenerate anisotropic quadratic form, acting on itself by translation, with complement the full orthogonal group* $\mathrm{O}(V)$*.*

The task of extending this theorem to cover all *transitive* binary permutation groups is a daunting one. So, for now, our focus is on the following question:

Question 2. *What are the transitive binary actions of the almost simple groups?*

The alternating and symmetric groups. We present two results dealing with particular cases of Question 2. The first has appeared on the arXiv [4]; the second will appear soon.

Theorem 3. *Let $G = A_n$ with $n \geq 2$ and let H be a subgroup of G and suppose that the action of G on the set of right cosets of H in G is binary if and only if one of the following holds:*

(1) H is normal in G, i.e. H is either $\{1\}$ or A_n itself, or else $n = 4$ and $H = \langle (1,2)(3,4),(1,3)(2,4) \rangle$.
(2) $n = 5$ and H is conjugate to $\langle (1,2)(3,4),(1,3)(2,4) \rangle$.

Theorem 4. *Let $G = S_n$ with $n \geq 2$ and let H be a subgroup of G. The action of G on the set of right cosets of H in G is binary if and only if one of the following holds:*

(1) H is normal in G, i.e. H is either $\{1\}$, A_n or S_n itself, or else $n = 4$ and $H = \langle (1,2)(3,4),(1,3)(2,4) \rangle$.
(2) $H \cong S_k$ for some $k \in \{2, \dots, n-1\}$ and the action is permutation isomorphic to the natural action of G on Σ^{n-k} where $\Sigma = \{1, \dots, n\}$.
(3) $H = \langle g \rangle$ where g is an odd involution.
(4) $n = 4$ and H is a Sylow 2-subgroup of G.
(5) $n = 5$ and H is conjugate to $\langle (1,2)(3,4),(1,3)(2,4) \rangle$.

It is noticeable that A_n and S_n exhibit relatively few transitive binary actions. To answer Question 2 we must extend these classifications to cover the almost simple groups of Lie type and the almost simple sporadic groups.

A useful graph. The proofs of Theorems 3 and 4 involved extensive study of a particular family of graphs, defined as follows: Let $\mathcal{C}$ be a conjugacy class of involutions in a group G. We define $\Gamma(\mathcal{C})$ to be the graph whose vertices are the elements of $\mathcal{C}$, with an edge between $x, y \in \mathcal{C}$ if and only if $xy \in \mathcal{C}$.

When $g \in \mathcal{C}$, we define the *component group of g in* $\Gamma(\mathcal{C})$ to be the subgroup of G generated by all the elements in the connected component of $\Gamma(\mathcal{C})$ containing g. The connection to binary actions is given by the next lemma.

Lemma 5. [4] *Let G act on the set of cosets of a subgroup H of even order, and assume that the action is binary. Let $\mathcal{C}$ be a conjugacy class of involutions of G of maximal fixity. Then, for any $g \in \mathcal{C} \cap H$, the component group of g in $\Gamma(\mathcal{C})$ is contained in H.*

It turns out that, for $n \geq 6$, the component group of an involution in A_n is always isomorphic to either A_{n-1} or A_n. This fact, together with the lemma above, allows one to immediately conclude that if $G = A_n$ and H is a subgroup of even order such that the action of G on the set of cosets of H is binary, then $H = A_n$ and the action is trivial. This is a significant first step in proving Theorem 3.

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Fusion systems with soft subgroups

Valentina Grazian
(joint work with Chris Parker, Martin van Beek)

Fusion systems are structures that encode the properties of conjugation between p-subgroups of a group, for p any prime number. A saturated fusion system $\mathcal{F}$ on a finite p-group S is a category where the objects are the subgroups of S and the morphisms are certain injective group homomorphisms. As an example, it is always possible to define the saturated fusion system realized by a finite group G on one of its Sylow p-subgroups S: in this case the morphisms are the restrictions of conjugation maps induced by the elements of G.

In order to characterize a saturated fusion system $\mathcal{F}$ on the p-group S, it is not necessary to describe the automorphism group $\mathrm{Aut}_{\mathcal{F}}(\mathrm{P})$ in $\mathcal{F}$ of every subgroup P of S. Indeed, the Alperin-Goldschmidt fusion theorem guarantees that every morphism in $\mathcal{F}$ is the composition of restrictions of morphisms belonging to $\mathrm{Aut}_{\mathcal{F}}(\mathrm{S})$ and to the automorphism groups of the so-called *$\mathcal{F}$-essential subgroups* of S:

Definition 1. *A proper subgroups E of S is $\mathcal{F}$-*essential *if*

- *$C_S(E\alpha) \leq E\alpha$ for every morphism α in $\mathcal{F}$ appliable to E (where $C_S(P)$ is the centraliser of P in S),*
- *$|N_S(E)| \geq |N_S(E\alpha)|$ for every morphism α in $\mathcal{F}$ appliable to E (where $N_S(P)$ is the normaliser of P in S),*
- *the outer automorphism group $\mathrm{Out}_{\mathcal{F}}(\mathrm{E}) \cong \mathrm{Aut}_{\mathcal{F}}(\mathrm{E})/\mathrm{Inn}(\mathrm{E})$ of E in $\mathcal{F}$ contains a strongly p-embedded subgroup.*

For this reason, the first step to study saturated fusion systems is to determine the $\mathcal{F}$-essential subgroups.

An active research direction in the theory of fusion systems consists in the understanding of the ones that are *not* realized by a finite group, called *exotic*, especially for odd primes (this question was suggested by Oliver in [1, Part III, 7.4]). The recent classification of saturated fusion systems defined on p-groups of maximal nilpotency class [5] revealed that, in many cases, exoticity is caused by the presence of *abelian $\mathcal{F}$-pearls*: $\mathcal{F}$-essential abelian subgroups isomorphic to the direct product $C_p \times C_p$ of two cyclic groups of order p. The concept of *$\mathcal{F}$-pearls* in fusion systems was first introduced in [3] and the name highlights the fact that these are "small" subgroups enriching the structure of the fusion system.

Abelian $\mathcal{F}$-pearls are the smallest examples of *soft subgroups*, defined by Héthelyi in [7] as self-centralizing abelian subgroups with index p in their normalizer in the p-group. This motivates the study of fusion systems on p-groups containing soft subgroups as a way to generalize the work on $\mathcal{F}$-pearls. We present our recent results in the subject, not published yet.

Our first result characterizes the $\mathcal{F}$-essential subgroups containing a soft subgroup:

Theorem 1. *Let $\mathcal{F}$ be a saturated fusion system on the p-group S, p odd, and let A be a soft subgroup of S. If E is an $\mathcal{F}$-essential subgroup of S containing A, then one of the following holds*

(1) $E = A$;
(2) $E = N_S(A)$ *and A is not $\mathcal{F}$-essential;*
(3) $p = 3$, $[S\colon A] \geq 3^3$, *E is the unique maximal subgroup of S containing A and S has sectional rank at least 4 (that is, at least one subgroup of S cannot be generated by less than 4 elements).*

We remark that there are examples of saturated fusion systems on 3-groups of sectional rank 4 with $\mathcal{F}$-essential subgroups as in case (3) and the list of candidates for $\mathcal{F}$-essential subgroups in Theorem 1 is best possible.

Next, we prove that if the soft subgroup A is $\mathcal{F}$-essential, then we can construct a saturated fusion subsystem $\mathcal{G}$ of $\mathcal{F}$ defined on a subgroup $M(A)$ of S having maximal nilpotency class. In particular $\mathcal{G}$ is known thanks to the classification achieved in [5].

Theorem 2. *Let $\mathcal{F}$ be a saturated fusion system on the p-group S, p odd, and let A be a soft subgroup of S. Suppose A is $\mathcal{F}$-essential with $[S\colon A] \geq p^2$. Let N be the unique maximal subgroup of S containing A. Then $\mathcal{F}$ and A uniquely determine a maximal class subgroup $M(A)$ of S such that*

(1) *$M(A)$ and N have the same nilpotency class;*
(2) $A_0 = M(A) \cap A \cong C_p \times C_p$;
(3) *there exists a unique involution* $\tau_A \in Z(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(\mathrm{A})))$ *s.t.* $A_0 = [A, \tau_A]$;
(4) *if $\mathcal{G}$ is the fusion system on $M(A)$ determined by*

$$\tau_A,\ \mathrm{Inn}(\mathrm{M(A)})\ \textit{and}\ \{\theta|_{\mathrm{A_0}} \mid \theta \in \mathrm{O^{p'}}(\mathrm{Aut}_{\mathcal{F}}(\mathrm{A}))\}$$

then $\mathcal{G}$ is a saturated fusion subsystem of $\mathcal{F}$, $O_p(\mathcal{G}) = 1$ and A_0 is an abelian $\mathcal{G}$-pearl.

As an application of Theorems 1 and 2, we believe we can characterize the reduced fusion systems on 3-groups of sectional rank 3 (the only missing case in the classification of saturated fusion systems on p-groups of sectional rank 3 presented in [4]), obtaining what follows:

Conjecture 1. *Let p be an odd prime, let S be a p-group of sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S with $O_p(\mathcal{F}) = 1$ and $\mathcal{F} = O^p(\mathcal{F})$. Then either S has maximal nilpotency class or S has a maximal subgroup that is abelian.*

Note that in the first case $\mathcal{F}$ is known tanks to the work in [5].

The reduced fusion systems on p-groups possessing a maximal subgroup that is abelian are also known ([8, 2, 9]) and form another important family of saturated fusion systems containing many examples of exotic ones. As a future project, in a joint work with J. Lynd, B. Oliver, C. Parker, J. Semeraro and M. van Beek, we intend to study a larger family, considering reduced fusion systems $\mathcal{F}$ on p-groups possessing a normal subgroup that is abelian and $\mathcal{F}$-essential. We believe this larger collection of fusion systems will contain only one previously unkown family of exotic ones, described in the preprint [6].

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Stronger diameter bounds for classical groups

Harald Helfgott

(joint work with Jitendra Bajpai, Daniele Dona)

Let G be a finite group and A a subset of G generating G. We would like to bound the diameter of the Cayley graph $\Gamma(G, A)$; this diameter equals the smallest k such that every element of G can be written as a product $a_1 a_2 \cdots a_\ell$, $\ell \leq k$, $a_i \in A \cup A^{-1}$, where $A^{-1} = \{g^{-1} : g \in A\}$. (We will assume $A = A^{-1}$ from now on, to make the graph symmetric.) We aim for bounds independent of A. We will assume G is simple (as the problem can be essentially reduced to that case) and non-abelian (as otherwise $\Gamma(G, A)$ can be very large for some A: consider $G = \mathbb{Z}/2025\mathbb{Z}$, $A = \{-1, 1\}$). Babai's conjecture states that, for G finite, simple and non-abelian, and A arbitrary,

$$\operatorname{diam}(\Gamma(G, A)) \leq (\log |G|)^C \tag{1}$$

where C is an absolute constant.

This conjecture was first proved for the family $\mathrm{SL}_2(\mathbb{F}_p)$ [H08] and then generalized in a series of results. While there were generalizations to groups of arbitrary rank ([BGT11, PS16]), the dependence of the exponent of C on the rank was very poor: C was an exponential tower $e^{e^{e^{\cdots}}}$ of height depending on the rank r, in fact of height proportional to r^2. What we have done is introduce new methods [BDH21, BDH24, BDH25] that make C polynomial on r:

$$C \leq 408r^4. \tag{2}$$

Strategy. Let us go over the main ideas that have led to (2). First, let us review ideas common to all proofs of diameter bounds for groups of Lie type.

1) Dimensional estimates. Let G be a simple linear algebraic group over a field K. A *dimensional estimate* is a bound of the following form: for any $A \subset G(K)$ generating $G(K)$, and any variety $V \subset G$,

$$|A \cap V(\overline{K})| \leq c|A^k|^{\frac{\dim V}{\dim G}}, \tag{3}$$

where $A^k = \{a_1 \cdots a_k : a_i \in A\}$, and c and k depend only on $\dim V$, $\deg V$ and $\dim G$. In other words, either A^k is much larger than A (in which case all is going well: we replace A by A^k and iterate) or the intersection of A with any variety is at most roughly what we would expect from $\dim V$. The first results of this kind were in [LP11] (available since 1998), which considered the case of A a (non-algebraic) subgroup, and in [H08], [H11], where A is a set but the proofs are case-by-case for different V.

2) Pivoting. There is an idea (the term "pivot" comes from [H11]) that is implicit in the proof of sum-product theorems (e.g., [BKT04]) but is now carried out directly in the group. In brief, we can do induction in G (or rather G/T, for T a maximal torus) given a generating set A; there is no need for a natural ordering.

Now let us list the kinds of improvements in [BDH21], [BDH24], [BDH25]. (Here [BDH21] was essentially a first draft for what became [BDH24] and [BDH25].) Let us first go over what goes into [BDH24].

a) Optimized quantitative algebraic geometry. Given a morphism $f : V \to \mathbb{A}^n$ from a variety $V \subset \mathbb{A}^m$, how do we bound $\deg(\overline{f(V)})$? If $W \subset \mathbb{A}^n$ is a variety, how do we bound $\deg f^{-1}(W)$? It is hard to find satisfactory answers in the literature even to such basic questions – or to answer them oneself. Part of what we do is give good explicit versions of much of the first chapter of [M99]. This is enough to get rid of exponential towers; the bound on C in (1) becomes doubly exponential on the rank r.

b) Escape from varieties, first improvement. An *escape argument* tells us that, given an action of a group $G = \langle A \rangle$ on $\mathbb{A}^n$ and a point x on a variety $V \subset \mathbb{A}^n$, if $\langle A \rangle x$ is not contained in V (i.e., there is a way out of the variety), there is a k bounded only in terms of $\deg V$ and $\dim V$ such that $A^k x$ is not contained in V (i.e., there is a short way out of the variety). This is a known kind of argument in group theory. We gave clear improvements on the existing bounds on k by defining and keeping track of the right quantity in the inductive argument in the proof. These improvements (together with (a)) are enough to give a bound

$$\operatorname{diam}(\Gamma(G, A)) \leq C_1 (\log |G|)^{C_2} \tag{4}$$

with C_1 doubly exponential on r and C_2 polynomial on r.

c) Improved inductive procedure. Both [BGT11] and [PS16] use non-trivial inductive procedures to prove (3) (in [BGT11]'s case, the procedure comes from [LP11]) in full generality. We show one can do better by using a different inductive procedure. The result is better bounds on C_1 and C_2 in (4), with better powers.

d) Improved pivoting. This gives us a quantitative improvement.

All of this [BDH24] is enough to give (1) with C exponential on r (or, somewhat better, a bound like (1) with C polynomial on r, but with a doubly exponential factor in front). Now, what is in [BDH25] but not in [BDH24]?

Rethinking escape from varieties. Consequences. We need not improve on (3) for general V; it is enough to do so for V a conjugacy class and for V a torus. Conjugacy classes are orbits and tori are groups; should we not use these facts?

The crucial step is to ask for more from escape. "Old" escape showed that there is a short way out of a variety. "New" escape, which we prove, shows that there exists a recipe book ("Ariadne's cookbook"), listing a small number of short recipes $a = a_1 \cdots a_k \in A^k$; for any x in V, one of those recipes will take x out of V, that is, $ax \notin V$. It is applying this idea that we manage to prove (2).

Zukunftsmusik. The fact remains that we should aim to show that the exponent C in (1) satisfies

$$C \ll (\log r)^{O(1)}.$$

While this bound would still not be quite as strong as Babai's conjecture, it would match what is already known for permutation groups: for $G = \mathrm{Alt}_n$ and an arbitrary set A generating G,

$$\text{(5)} \qquad \mathrm{diam}(\Gamma(G, A)) \leq (\log |G|)^{O((\log n)^3 \log\log n)} \qquad \text{(see [HS14])}$$

The theory (or theories) of groups over the field with one element (which is not a field and has two elements) strengthens this parallel.

It is bemusing that we have stronger results for Alt_n than for algebraic groups: surely algebraic groups have more structure? What is helpful in Alt_n is that it acts non-trivially on sets much smaller than itself; in those sets, precisely because they are small, a random walk must equidistribute quickly. We can try to get a "blurrier picture" of the action of an algebraic group; the theory of buildings can be helpful here.

Let us end with a very simple and concrete example. Say we have a point x on a hyperplane V in $\mathbb{A}^n$, and a set A of linear transformations such that $\langle A\rangle x \not\subset V(K)$. Then it is easy to see that $A^{n-1}x \not\subset V(K)$ (try it! this is the simplest escape example) yet it could happen that $A^{n-2}x \subset V(K)$ (how? look at the permutation $(12\ldots n)$ for inspiration). All the same, that is not generic behavior; what we expect for A "typical" is that $A^k x \not\subset V(K)$ for every x when k is in the order of $\log n$. (In the case of permutation groups, this is essentially the same as "random graphs have small diameter".) We can also ask ourselves whether a small k always suffices for arbitrary A and the specific varieties V we are looking at; would that be typical or atypical behavior?

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Transitive subgroups of primitive groups

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(joint work with Lei Wang)

Determining the transitive subgroups of primitive groups is a long-standing problem dating back to the 1900s and it is closely related to the group factorisation problem. More precisely, if G is a transitive permutation group with point stabiliser H, then a subgroup K of G is transitive if and only if $G = HK$, which we call a *factorisation* of G. Group factorisations have been investigated for more than a century, and it remains a very active area. For example, the factorisations of all the finite almost simple groups have very recently been determined in [1, 2].

In this talk we focus on transitive subgroups of finite primitive permutation groups (or equivalently, the factorisations of finite groups $G = HK$ such that H is a core-free maximal subgroup in G).

Problem. *Determine all the transitive subgroups of primitive groups, up to conjugacy.*

One of the key tools for studying the finite primitive groups is the O'Nan-Scott theorem from the 1980s, which describes the finite primitive groups in terms of the structure and action of the socle of the group. Following [4], this theorem divides the primitive groups into five families:

affine, almost simple, diagonal type, product type, twisted wreath products.

We are mainly interested in diagonal type groups in this talk. In particular, there is a special interest in determining the regular subgroups of these groups because they arise naturally in the study of Cayley graphs of finite groups (recall that a simple graph is a Cayley graph if and only if its automorphism group has a regular subgroup on the vertex set). Another goal is to better understand the soluble transitive subgroups of diagonal type groups.

Let $k \geq 2$ be an integer and let T be a non-abelian finite simple group. Then $D = \{(t, \dots, t) : t \in T\}$ is a core-free subgroup of T^k, so $T^k \leq \mathrm{Sym}(\Omega)$ is a transitive permutation group with $\Omega = T^k/D$. A group $G \leq \mathrm{Sym}(\Omega)$ is of *diagonal type* if

$$T^k \lhd G \leq N_{\mathrm{Sym}(\Omega)}(T^k) \cong T^k.(\mathrm{Out}(T) \times S_k).$$

Moreover, a group G of this form is primitive if and only if its induced permutation group P on the set of k factors of T^k either primitive, or $k = 2$ and $P = 1$. Building on earlier work of Liebeck, Praeger and Saxl [3], as well as some more recent results of Morris and Spiga [5], we are able to classify the regular subgroups and the soluble transitive subgroups of these groups.

Theorem. *The regular subgroups and the soluble transitive subgroup of diagonal type primitive groups are classified, up to conjugacy.*

Let us highlight the special case where $k = 2$ and $P = 1$, which turns out to be the central part of the proof of the above theorem and finds some interesting applications. Here $G \leq \mathrm{Hol}(T) = T{:}\mathrm{Aut}(T)$, which is the *holomorph* of T. As observed in [3], if B is a transitive subgroup of G, then there exist $H, K \leq \mathrm{Aut}(T)$ isomorphic to some quotient groups of B satisfying the condition

$$T \lhd HK = HT = KT \leq \mathrm{Aut}(T).$$

This naturally brings into play the factorisations of almost simple groups, which have been recently classified, as noted above.

The study of regular subgroups of holomorphs is of independent interest. For example, for finite groups X, Y of the same order, the following are equivalent:

(a) Y is isomorphic to a regular subgroup of $\mathrm{Hol}(X)$ with respect to its action on X;
(b) there exists a *Hopf-Galois structure* of type Y on any Galois extension of fields with Galois group isomorphic to X;
(c) there exists a *skew brace* with additive group isomorphic to X and multiplicative group isomorphic to Y.

Corollary. *The types of Hopf-Galois structures are determined on any Galois extension of fields whose Galois group is finite simple.*

Corollary. *The skew braces with finite simple additive groups are classified, up to isomorphism.*

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On The Liebeck–Nikolov–Shalev Conjecture

Noam Lifshitz

(joint work with Lásló Pyber, Endre Szabó, and Nick Gill)

For subsetsof a group $A, B \subseteq G$ We denote $AB = \{ab : a \in A, b \in B\}$ and for $\sigma \in G$ we write $A^\sigma = \{\sigma^{-1} a \sigma : a \in A\}$ Liebeck and Shalev [5] proved that for every finite simple group G and every conjugacy class A, $A^\ell = G$, for $\ell \leq C\frac{\log |G|}{\log |A|}$, where C is an absolute constant. The Liebeck–Nikolov–Conjecture is a stronger variant for sets that are not necessarily normal. It states that for every A there exist $\sigma_1, \ldots, \sigma_\ell \in G$, such that $A^{\sigma_1} \cdots A^{\sigma_\ell} = G$, where again $\ell \leq C\frac{\log |G|}{\log |A|}$ for an absolute constant $C > 0$. Dona [1] proved a variant with $\ell = \left(\frac{\log |G|}{\log |A|}\right)^{1+o(1)}$, and in a recent pair of papers [2, 6] Gill, Pyber, Szabo and I proved the conjecture.

The above conjectures and results are part of a subfield of group theory known as growth in finite groups. It is worth noting that one of the central remaining open problems is Babai's conjecture, which concerns showing that for every generating subset A of a finite simple group G we have $A^\ell = G$ for $\ell \leq \log |G|^C$ for an absolute constant $C > 0$.

The main tool for showing such growth results in finite groups is character bounds. These are upper bounds on the values $\chi(\sigma)$ in terms of $\chi(1)$ and σ. Character bounds are then used in conjunction with the Frobenius formula, which I'm going to give here a probabistic version of. Given functions $f, g \colon G \to \mathbb{C}$ we write $f * g(x)$ for $\mathbb{E}_{y \sim G}[f(y) g(y^{-1} x)]$. The Frobenius formula then states that if f and g are class functions and $f = \sum_\chi \hat{f}(\chi)\chi, g = \sum_\chi \hat{g}(\chi)$ are expressed as a linear combinations of the characters. Then $f * g = \sum_\chi \frac{\hat{f}(\chi)\hat{g}(\chi)}{\chi(1)}\chi$. Now if f, g are supported on A, B respectively, then $f * g$ is supported on AB so estimates on the characters can then be used to obtain growth.

The characters χ are an orthonormal basis with respect to the L^2 inner product $\langle f, g\rangle = \mathbb{E}_{x \sim G}[f(x) g(x)]$. Since the characters are othonormal it is easy to show that for all class functions $f_1, \dots, f_\ell$ of expectation 1 we have

$$\|f_1 * f_2 * \cdots * f_\ell - 1\|_2^2 = \sum_{\chi \neq 1} \frac{\hat{f}_1(\chi)^2 \cdots \hat{f}_\ell(\chi)^2}{\chi(1)^{2\ell-2}}.$$

And in the special case where $f = \frac{|G| \cdot 1_A}{|A|}$ for a conjugacy class $A = \sigma^G$ we obtain that the right hand side takes the form $\sum_{\chi \neq 1} \frac{\chi(\sigma)^{2\ell}}{\chi(1)^{2\ell-2}}$ when the right hand side is smaller than 0.1 this implies an L_2 mixing time, which in turn implies that $A^{2\ell} = G$.

One of the crucial ideas in [2, 6] is to generalize the Frobenius formula to functions that are not necessarily class functions. Let $\mathbb{C}[G]$ be the group algebra, namely the space of complex valued functions on the group. The group algebra $\mathbb{C}[G]$ can be decomposed as an orthogonal direct sum $\bigoplus_{\rho \in \mathrm{Irr}(G)} W_\rho$, where W_ρ is the space of matrix spanned by the matrix coefficients of ρ. Recall that for given an irrep (V, ρ) a *matrix coefficient* corresponds to $v \in V, \varphi \in V^*$ and is the function in $\mathbb{C}[G]$ given by $g \mapsto \varphi(gv)$. Given a function f we can orthogonally decompose it as a sum of 'pieces' f^ρ with $f^\rho \in W_\rho$. The idea now is that upper bounds on $\|f^\rho\|_2$ can play a similar role to character bounds when f is not necessarily a class function.

Write f^σ for the function with $f^\sigma(\tau) = f(\tau^{-1}\sigma\tau)$.

We proved that

$$\mathbb{E}_{\sigma_1,\dots,\sigma_\ell \sim G}[\|f^{\sigma_1} * \cdots f^{\sigma_\ell}\|_2^2] = \sum_{\chi} \frac{\|f^{=\chi}\|_2^{2\ell}}{\chi(1)^{2\ell-2}}.$$

We then went on to give upper bounds on $\|f^{=\chi}\|_2^2$ by proving that when $f = \frac{|G|1_A}{|A|}$ we have $\|f^{=\chi}\|_2^2 = \chi(1)\mathbb{E}_{a,b\sim A}[\chi(ab^{-1})]$.

We then went on to apply character bounds due to Guralnick, Larsen, and Tiep [3] and due to Larsen and Tiep [4]. We also proved new character bounds for the symmetric group, and combined them with probabilistic arguments to prove the LNS conjecture.

The above technique is suitable for covering a group by products of conjugates of a set. For such type of result one is in the lookout for upper bounds of the form $\|f^\chi\|_2 < \chi(1)^{1-1/\ell}$ to obtain a mixing time of $O(\ell)$. When one desires to show that $A^\ell = G$, such as in Babai's conjecture it turns out that such bounds are still useful. There one needs an upper bound of the form $\|f^\chi\|_2 < \chi(1)^{1/2-1/\ell}$ due to a method by Sarnak–Xue–Gowers.

Unfortunately the above character theoretic method never gives bounds better that $\chi(1)^{1/2}$. However, there are other methods of getting those knows as hypercontractive bounds.

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Partial groups and higher Segal conditions

Justin Lynd
(joint work with Philip Hackney)

In Chermak's original group theoretic formulation [3], a partial group is a set together with a multivariable product that is only defined on a subset of multipliable words in the underlying set, together with an inversion for that product. Here we report on work understanding the higher Segal conditions of Dyckerhoff and Kapranov [5] in the context of partial groups. We define a new invariant of a partial group, its *degree*, develop the discrete geometry of partial group actions as a tool for computing this invariant, and through this we show that partial groups form a rich class of higher Segal sets of finite group theoretic significance.

Partial groups as symmetric sets. Partial groups seem to be best regarded as certain types of symmetric simplicial sets [9]. The simplex category Δ has objects the totally ordered sets $[n] = \{0, 1, \dots, n\}$ $(n \geq 0)$ and morphisms monotone maps $[m] \to [n]$. The symmetric simplex category $\Upsilon \supset \Delta$ has the same objects, but has morphisms all functions. A symmetric simplicial set is a presheaf on Υ, i.e., a functor $\Upsilon^{\mathrm{op}} \to \mathsf{Set}$. It amounts to a simplicial set X together with compatible actions of the symmetric groups $\Sigma_{[n]}$ on the sets $X_n := X([n])$ of n-simplices for each n.

The standard example of a simplicial set is the nerve BC of a category. If the category is a groupoid G, then its nerve BG enjoys the structure of a symmetric simplicial set. Grothendieck's Nerve Theorem characterizes nerves of categories (resp. groupoids) as those simplicial sets (resp. symmetric sets) X for which the Segal maps

$$\mathscr{E}_n \colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are bijections for all $n \geq 2$, a stipulation on X called the *Segal condition.* The map $\mathscr{E}_n$ sends an n-simplex x to its standard spine $(\epsilon_{0,1}^* x, \dots, \epsilon_{n-1,n}^* x)$, where $\epsilon_{i,j}$ is the map from $[1]$ to $[n]$ that sends 0 to i and 1 to j.

We consider two different weakenings of the Segal condition and their interaction. The first leads to one model for a *partial category*, but we will focus on the symmetric version, a model for *partial groupoid.*

Definition 1. A symmetric set is *spiny* if the Segal maps are all injections. A *partial groupoid* is a spiny symmetric set. A *partial group* is a reduced partial groupoid, i.e., one with a single 0-simplex.

Spininess, originally an insight of González [8], allows one to write an n-simplex f in a partial groupoid as $f = [f_1|\cdots|f_n]$ where $f_i = \epsilon^*_{i-1,i}f$ is the i-th principal edge. Chermak's partial product is given by the span $X_1 \times_{X_0} \cdots \times_{X_0} X_1 \leftarrow X_n \rightarrow X_1$, where the right map is $\epsilon^*_{0,n}$. The inversion $f \mapsto f^{-1}$ is the action of the longest element $i \mapsto n-i$ in $\Sigma_{[n]}$. Many partial groups of interest fall under the umbrella of the next example, but not all.

Example 1. Let G be a group and S a set equipped with a partial action of G in the sense of Exel [6] (see also [10]). This gives rise to a transporter groupoid $T_S(G)$ having object set S and morphisms $s \xrightarrow{g} g\cdot s$ whenever g acts on s, as well as a functor $T_S(G) \to G$. Let $L_S(G) \subseteq BG$ be the image of the corresponding map on nerves. Then $L_S(G)$ is a partial group whose n-simplices are those $[g_1|\cdots|g_n]$ such that there is $s \in S$ with $g_1 \cdot s$ defined, $g_2 \cdot (g_1 \cdot s)$ defined, and so forth.

Higher Segal conditions. The second weakening is a family of associativity conditions on simplicial objects, the higher Segal conditions of Dyckerhoff and Kapranov [5]. They come in lower and upper versions, one for each $d \geq 1$; the ordinary Segal condition corresponds to lower 1-Segality. Satisfaction of either the lower or the upper $(d-1)$-Segal condition implies lower *and* upper d-Segality. The 2-Segal conditions were independently introduced by Gálvez-Carrillo–Kock–Tonks [7] and have been studied intensely in recent years. We explain, based on a distillation of [11], the meaning of the higher Segal conditions here only for partial groupoids X, where they become concrete and combinatorial, and where they collapse to the lower odd conditions. (For example a partial group is lower or upper 2-Segal if and only if it is lower 1-Segal, i.e., a group.)

If X is a partial groupoid, then X is *lower* $(2k-1)$*-Segal* if, given $n \geq 1$ and a *gapped sequence*

$$0 \leq i_0 \lll i_1 \lll \cdots \lll i_k \leq n,$$

of length $k+1$ (i.e., adjacent terms are at distance at least two), and given a potentially multipliable word $w = (f_1, \ldots, f_n) \in X_1 \times_{X_0} \cdots \times_{X_0} X_1$ of length n,

$$d_{i_0}w,\ d_{i_1}w,\ \ldots,\ d_{i_k}w \in X_{n-1} \implies w \in X_n.$$

Here, if $i_0 = 2$ say, the expression d_2w carries the tacit assumption that $[f_2|f_3] \in X_2$, so that one can form the word $d_2w = (f_1, d_1[f_2|f_3], f_4, \ldots, f_n)$.

Definition 2. The *degree* $\deg(X)$ of a partial groupoid X is the smallest k such that X is lower $(2k-1)$-Segal.

Example 2. If G is a group, the classifying space for commutativity $B_{\mathrm{com}}(G) \subseteq BG$ is the partial group with n-simplices $[g_1|\cdots|g_n] \in BG_n$ whenever $g_1, \ldots, g_n$ pairwise commute, cf. [1]. It is lower 3-Segal. The first of the lower 3-Segal conditions ($n = 4$) says that if $g_1, \ldots, g_4$ are elements of G such that g_1, g_2, g_3 commute, g_1, g_2g_3, g_4 commute, and g_1, g_2, g_3 commute, then all four elements commute. $B_{\mathrm{com}}(G)$ is lower 1-Segal only if G is abelian, in which case it is BG. So $\deg(B_{\mathrm{com}}(G)) = 1$ or 2 depending on whether G is or is not abelian.

Degree as Helly number. We now describe tools for computing the degree of a partial group. As motivation, we start with the following subexample of Example 1. Let Φ be a finite root system. A fixed set Φ^+ of positive roots admits a partial action by the Weyl group W, and we form the partial group $L = L_{\Phi^+}(W)$. It has elements/1-simplices $L_1 = W \backslash \{w_0\}$, where w_0 is the longest element.

Theorem 1 (Hackney–L.)**.** *The degree of $L_{\Phi^+}(W)$ is the Helly number of Φ^+ with respect to convex subsets.*

The classical Helly number is the smallest h such that whenever each collection of h members of a family of convex sets has nonempty intersection, then the entire family has nonempty intersection. Helly's Theorem from 1913 says that the Helly number for convex sets in $\mathbb{R}^d$ is $d+1$. In the situation of the theorem, a subset A of Φ^+ is convex if it coincides with its convex cone $\mathbb{R}_{\geq 0}A \cap \Phi^+$. We compute the Helly number of Φ^+ explicitly by showing that it is closely related to the maximal dimension of an abelian subalgebra of the associated semisimple Lie algebra, as was computed by Malcev. In fact it coincides with this number in simply laced types. For example, $\deg(L_{\Phi^+}(W)) = 16, 27, 36$ in cases E_6, E_7, E_8. In other words, $L_{\Phi^+}(W)$ is 71-Segal when $W = W(E_8)$, but not 69-Segal.

It turns out the above story persists for arbitrary partial groups in almost full generality.

Definition 3. An *action* of a partial groupoid L is a map $\rho\colon E \to L$ of partial groupoids with the following partial lifting condition: for each n-simplex $g \in L_n$ and each $x \in E_0$ mapping to the source of g, there is at most one $e \in E_n$ such that $\rho(e) = f$.

We interpret this through a Grothendieck correspondence for partial actions, giving a way for a simplex g of L to act on an element $x \in E_0$, by choosing a lift e of g with source x and setting $g \cdot x = y$, the target of e. The *domain* $\mathrm{D}_\rho(g)$ of g is the set of x's for which there is such a lift. This is of course modeled on the map $BT_S(G) \xrightarrow{\rho} L_S(G) \subseteq BG$ of Example 1 and on domains of partial maps. In that case, $E = BT_S(G)$ is (the nerve of) a groupoid, and ρ is surjective, a situation we will temporarily call "nice" here.

Any action gives rise to a closure operator cl on E_0, sending a subset A of E_0 to the intersection of those domains of simplicies that contain A. An indexed family of subsets $\{A_1, \ldots, A_k\}$ of such a closure space is *Helly independent* if

$$\bigcap_{i=1}^{k} \mathrm{cl}\left(\bigcup_{j \neq i} A_j\right) = \varnothing,$$

and the *Helly number* $h(\rho)$ is the maximal size of an independent set. (This is an equivalent definition, dual to the one given before.)

Theorem 2 (Hackney–L.)**.** *If L is a partial group that is not a group and ρ is a nice action of L, then $\deg(L) \leq h(\rho)$. Also, $h(\rho) \leq \deg(L)$ if domains of simplices satisfy the descending chain condition.*

Every partial group has an explicit, nice action, so Theorem 2 gives effective means for computing the degree. It applies, for example, to the discrete localities [4] associated with p-local compact groups [2].

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Irreducible restrictions of spin representations of symmetric and alternating groups

Lucia Morotti
(joint work with Alexander Kleshchev, Pham Huu Tiep)

Let G be a group, F an algebraically closed field and V an irreducible FG-representation. Given $H < G$, one can ask whether the restriction $V{\downarrow_H}$ of V to H is still irreducible. This question is a natural question that can be asked on its own, but it also has applications (when both G and H are almost quasi-simple) to the Aschbacher-Scott program on maximal subgroups of finite classical groups.

Double covers of symmetric groups are specific groups $\widetilde{S}_n$ and $\widetilde{A}_n$ with a central element z of order 2 such that $\widetilde{S}_n/\langle z\rangle \cong S_n$ and $\widetilde{A}_n/\langle z\rangle \cong A_n$.

Since z is central of order 2, if V is an irreducible representation of $\widetilde{S}_n$ or $\widetilde{A}_n$ then either z acts trivially on V, in which case V is also a representation of S_n or A_n, or $p \neq 2$ and z acts as $-1 \neq 1$, in which case V is called a spin representation.

Both representations and spin representations of symmetric and alternating groups are labeled by certain subsets of partitions.

For representations of symmetric and alternating groups, the question of characterising when $V{\downarrow_H}$ is irreducible was essentially answered in [1, 3, 4, 5, 7, 8, 10].

In particular, excluding the basic spin case for $p = 2$, it is shown that $V{\downarrow_H}$ is irreducible if and only if (V, H) are in a certain finite list of infinite families or 'small' exceptional cases.

Previous papers [2, 9] on irreducible restrictions of spin representations of symmetric and alternating groups covered the cases of H being almost quasi-simple or $\pi(H)$ being primitive in S_n. For subgroups H with $\pi(H)$ imprimitive, only partial results were known (in particular, in positive characteristic not even the case of $\pi(H)$ maximal imprimitive was fully treated).

In [6], joint with Kleshchev and Tiep, we started working on the still open cases. In particular we are now able to completely describe when restrictions to lifts of maximal imprimitive subgroups of S_n or A_n are irreducible.

This allows us to formulate the following result:

Theorem. [2, 6, 9] *Let $p \neq 2$, $G \in \{\widetilde{S}_n, \widetilde{A}_n\}$, $H < G$ and V be an irreducible spin representation of G. If $V{\downarrow_H}$ is irreducible then one of the following holds:*

- *(i)* $\widetilde{A}_{n-2} \leq H$,
- *(ii)* *V is basic spin and $H \leq \widetilde{S}_{n-k,k}$ for some k or $H \leq \widetilde{S_a \wr S_b}$ for some a, b with $ab = n$,*
- *(iii)* *V is second basic spin, n is even, $p \mid (n-1)$ and $H \leq K$ for some* $(G, K) \in \{(\widetilde{S}_n, \widetilde{S_{n/2} \wr S_2}), (\widetilde{S}_n, \widetilde{S_2 \wr S_{n/2}}), (\widetilde{A}_n, \widetilde{S_{n/2} \wr S_2} \cap \widetilde{A}_n)\}$,
- *(iv)* $n \leq 14$.

More about Cases (i)-(iv) in the above theorem.

In Case (i) H is in a finite list of large natural subgroups of $\widetilde{S}_n$ or $\widetilde{A}_n$. Using branching rules by Brundan and Kleshchev it is possible to explicitly describe when $V{\downarrow_H}$ is irreducible. These descriptions only use combinatorial conditions that the corresponding labeling partitition must satisfy.

In Case (iv) computations will allow us to obtain an explicit list of small exceptional cases.

This leaves only Cases (ii) and (iii) open. In particular only irreducible restrictions of first and second basic spin modules are not fully understood yet (after the above mentioned computations for $n \leq 14$).

For basic spin representations, irreducible restrictions to maximal subgroups with $\pi(H)$ imprimitive have been completely described in [2, 9]. As there is a large number of such maximal subgroups to which basic spin representations restrict irreducibly, it seems out of reach, at least for now, to completely understand when basic spin representations restrict irreducibly to arbitrary subgroups. The cases of H almost quasi-simple or of $\pi(H)$ primitive have however been completely covered in [9].

For second basic spins, restrictions $V{\downarrow_K}$ with (G, K) one of the three pairs in Case (iii) are irreducibles. More work is currently planned to understand the non-maximal case. As a first step toward this, it is shown in [6] that if V is second basic spin, $V{\downarrow_H}$ is irreducible and $H \leq \widetilde{L \wr S_2}$ or $H \leq \widetilde{S_2 \wr L}$ for some $L \leq S_{n/2}$, then L is primitive or Cases (i) or (iv) hold.

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Realizability and tameness of fusion systems

Bob Oliver

(joint work with Carles Broto, Jesper Møller, and Albert Ruiz)

In this talk, I described recent work [4], where we show among other results that a saturated fusion system $\mathcal{F}$ is realizable if there is a normal, realizable subsystem $\mathcal{E}$ in $\mathcal{F}$ that is centric in $\mathcal{F}$ (i.e., contains its centralizer). Another result is that every realizable fusion system $\mathcal{F}$ is tame, meaning very roughly that it is realized by a finite group that has "just as many" automorphisms as $\mathcal{F}$ has. Stated in this way, the results depend on the classification of finite simple groups (CFSG), but we also prove more precise statements formulated in such a way that their proofs are independent of the classification.

The starting point for this work is the concept of fusion systems over finite p-groups, originally defined by Puig. For each group G and each $P, Q \leq G$, let $\mathrm{Hom}_G(P,Q)$ be the set of homomorphisms from P to Q induced by conjugation by an element of G. For a given prime p and a finite p-group S, a *fusion system* over S is a category $\mathcal{F}$ whose objects are the subgroups of S, whose morphisms are injective homomorphisms between the subgroups, and where for each $P, Q \leq S$,

- $\mathrm{Hom}_{\mathcal{F}}(P,Q) \supseteq \mathrm{Hom}_S(P,Q)$, and
- for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, we have $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P),P)$.

A fusion system is *saturated* if it satisfies certain additional axioms motivated by the Sylow theorems and other properties of finite groups (see [2, § I.2] for details).

For each finite group G, each prime p, and each $S \in \mathrm{Syl}_p(G)$, let $\mathcal{F}_S(G)$ (the fusion system of G over S) be the category whose objects are the subgroups of S, and where for all $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$. Thus this category encodes the G-conjugacy relations among subgroups and elements of a fixed Sylow p-subgroup. By a theorem of Puig, $\mathcal{F}_S(G)$ is a saturated fusion system for each finite group G and each $S \in \mathrm{Syl}_p(G)$. A saturated fusion system $\mathcal{F}$ over a finite p-group S is called *realizable* if it is isomorphic to $\mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$.

A *fusion subsystem* $\mathcal{E} \leq \mathcal{F}$ is a subcategory that is itself a fusion system over a subgroup $T \leq S$. The fusion subsystem $\mathcal{E}$ is *normal in* $\mathcal{F}$ ($\mathcal{E} \trianglelefteq \mathcal{F}$) if $\mathcal{E}$ is saturated, T is strongly closed in $\mathcal{F}$ (i.e., $g \in T$ implies $\varphi(g) \in T$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g \rangle, S)$), and certain other conditions are satisfied motivated by analogies with finite normal subgroups. For example, if $H \trianglelefteq G$ are finite groups and $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_{S\cap H}(H) \trianglelefteq \mathcal{F}_S(G)$ (where $S \cap H \in \mathrm{Syl}_p(H)$).

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Two fusion subsystems $\mathcal{E}_1, \mathcal{E}_2$ in a saturated fusion system $\mathcal{F}$ over $T_1, T_2 \leq S$ *commute in* $\mathcal{F}$ if there is a morphism of fusion systems $\mathcal{E}_1 \times \mathcal{E}_2 \longrightarrow \mathcal{F}$ extending the inclusions $\mathcal{E}_i \longrightarrow \mathcal{F}$. The *centralizer fusion subsystem* $C_{\mathcal{F}}(\mathcal{E})$ of a fusion subsystem $\mathcal{E} \leq \mathcal{F}$ over $T \leq S$ (if it exists) is the largest fusion subsystem of $\mathcal{F}$ that commutes with $\mathcal{E}$.

If $\mathcal{E} \trianglelefteq \mathcal{F}$, then $C_{\mathcal{F}}(\mathcal{E})$ always exists and is saturated and normal by a theorem of Henke. (The subsystem $C_{\mathcal{F}}(\mathcal{E})$ was first defined by Aschbacher, but he didn't prove all of these properties.)

A normal fusion subsystem $\mathcal{E} \trianglelefteq \mathcal{F}$ is called *centric in* $\mathcal{F}$ if it contains its centralizer; i.e., if $C_{\mathcal{F}}(\mathcal{E}) \leq \mathcal{E}$. Thus $\mathcal{E}$ is centric in $\mathcal{F}$ if each fusion subsystem of $\mathcal{F}$ that commutes with $\mathcal{E}$ is contained in $\mathcal{E}$.

We can now state a first version of our main theorem.

Theorem 1 ([4, Theorem A]). *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over finite p-groups. Assume that $\mathcal{E}$ is centric in $\mathcal{F}$, and that $\mathcal{E}$ is realizable. Then $\mathcal{F}$ is also realizable.*

The proof of the theorem stated in this form requires the classification of finite simple groups. A second version of the theorem, formulated so as not to require CFSG, will be given later. But before doing this, we need to define tameness and components of fusion systems.

If $\mathcal{F}$ is a saturated fusion system over a finite p-group S, then $\mathrm{Aut}(\mathcal{F})$ is the group of all $\alpha \in \mathrm{Aut}(S)$ that extend to an automorphism of $\mathcal{F}$ as a category, and $\mathrm{Out}(\mathcal{F}) = \mathrm{Aut}(\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)$. If $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group G with $S \in \mathrm{Syl}_p(G)$, then there is a natural homomorphism

$$\widetilde{\kappa}_G\colon \mathrm{Out}(G) = \frac{\mathrm{Aut}(G)}{\mathrm{Inn}(G)} \cong \frac{N_{\mathrm{Aut}(G)}(S)}{\mathrm{Aut}_{N_G(S)}(G)} \xrightarrow{\qquad} \frac{\mathrm{Aut}(\mathcal{F})}{\mathrm{Aut}_{\mathcal{F}}(S)} = \mathrm{Out}(\mathcal{F})$$

that sends the class of $\beta \in N_{\mathrm{Aut}(G)}(S)$ to the class of $\beta|_S \in \mathrm{Aut}(\mathcal{F})$. In this situation, there are natural homomorphisms κ_G and μ_G such that

$$\widetilde{\kappa}_G = \mu_G \circ \kappa_G \colon \mathrm{Out}(G) \xrightarrow{\kappa_G} \mathrm{Out}(\mathcal{L}_S^c(G)) \xrightarrow{\mu_G} \mathrm{Out}(\mathcal{F}_S(G)),$$

where $\mathcal{L}_S^c(G)$ is the centric linking system of G (see [2, Definition III.3.1]).

Definition 2 ([1, Definition 2.5]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then $\mathcal{F}$ is* tamely realized *by a finite group G if $S \in \mathrm{Syl}_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, and $\kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(\mathcal{L}_S^c(G))$ is split surjective. The fusion system $\mathcal{F}$ is* tame *if it is tamely realized by some finite group G.*

In fact, tameness — the question of whether all realizable fusion systems are tame — was our original motivation for beginning this project. In earlier work by several different authors [1, 3, 5], it had already been shown that all fusion systems of known finite simple groups are tame, and it was natural to ask whether all fusion systems of all finite groups are tame (assuming CFSG).

Tameness is motivated by the problem: given fusion systems $\mathcal{E} \trianglelefteq \mathcal{F}$ over $T \trianglelefteq S$, where $\mathcal{E} = \mathcal{F}_T(H)$ for some finite group H with $T \in \mathrm{Syl}_p(H)$. Under what conditions can one find G such that $H \trianglelefteq G$, $S \in \mathrm{Syl}_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$? This requires understanding the relation between $\mathrm{Out}(G)$ and $\mathrm{Out}(\mathcal{L}_S^c(G))$, as encoded by the homomorphism κ_G.

If p is odd, then μ_G is an isomorphism by theorems of Chermak, Glauberman, and Lynd, and so $\mathrm{Out}(\mathcal{L}_S^c(G)) \cong \mathrm{Out}(\mathcal{F}_S(G))$. Thus in this case, $\mathcal{F}_S(G)$ is tamely realized by G if the natural map from $\mathrm{Out}(G)$ to $\mathrm{Out}(\mathcal{F}_S(G))$ is split surjective. When $p = 2$, μ_G is always surjective, but not always injective.

The *components* of a fusion system $\mathcal{F}$ are defined as for groups: those subnormal fusion subsystems that are quasisimple. Here, "subnormal" is defined in the obvious way (obvious once one has defined "normal"), and a fusion system $\mathcal{F}$ is *quasisimple* if $O^p(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}/Z(\mathcal{F})$ is simple. The basic properties of the components of $\mathcal{F}$ were shown by Aschbacher. For example, they commute with each other and with $O_p(\mathcal{F})$, and hence generate a central product.

We can now reformulate our main theorem in such a way that its proof does not use CFSG. Let $\mathrm{Comp}(\mathcal{F})$ denote the sets of components of $\mathcal{F}$. A finite group G is *p'-reduced* if $O_{p'}(G) = 1$. Note that if a fusion system $\mathcal{F}$ is realized by a finite group G, then it is also realized by the p'-reduced group $G/O_{p'}(G)$.

Theorem 3 ([4, Theorem 5.4]). *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over finite p-groups such that $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$. If $\mathcal{E}$ is realized by a finite p'-reduced group all of whose components are known quasisimple groups, then $\mathcal{F}$ is tamely realized by a finite p'-reduced group all of whose components are known quasisimple groups.*

Just as for groups, one can show that $\mathrm{Comp}(\mathcal{E}) \subseteq \mathrm{Comp}(\mathcal{F})$ whenever $\mathcal{E} \trianglelefteq \mathcal{F}$. So the important condition on components is that all components of $\mathcal{F}$ are also components of $\mathcal{E}$. Also, if $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{E}$ is centric in $\mathcal{F}$, then $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$ by results of Aschbacher. So Theorem 1 is a special case of Theorem 3 (together with CFSG).

The following two consequences of Theorem 3 are also proven without using CFSG (see [4, Theorems 5.5, 5.6]).

Corollary 4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group.*

- *If all components of $\mathcal{F}$ are realized by known finite quasisimple groups, then $\mathcal{F}$ is tamely realized by a finite p'-reduced group all of whose components are known quasisimple groups.*
- *If $\mathcal{F}$ is realized by a finite p'-reduced group all of whose components are known quasisimple groups, then $\mathcal{F}$ is tamely realized by a finite p'-reduced group all of whose components are known quasisimple groups. (The case $\mathcal{E} = \mathcal{F}$ of Theorem 3.)*

Thus upon assuming CFSG, we have shown that all realizable fusion systems over finite p-groups are tame.

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On Quillen's conjecture

Kevin I. Piterman

(joint work with Stephen D. Smith)

In the 1970s, Kenneth Brown and Daniel Quillen initiated the study of the topological and combinatorial properties of p-group posets motivated by algebraic and cohomological questions concerning finite groups. Let G be a finite group and p a prime number. In [2], Brown analysed the poset $\mathcal{S}_p(G)$ of non-trivial p-subgroups of G, ordered by inclusion, and showed that its (reduced) Euler characteristic is divisible by $|G|_p$, the largest power of p dividing the order of G. This result is commonly known as the "Homological Sylow Theorem", and it is related to the computation of the p-rational part of the Euler characteristic of (non-necessarily finite) groups satisfying certain finiteness conditions. In 1978, Quillen introduced the poset $\mathcal{A}_p(G)$ of non-trivial elementary abelian p-subgroups of G, and investigated homotopical properties of these posets. Recall that a poset can be regarded as a topological space by considering the geometric realization of its associated order complex, allowing us to study its homotopy type. In [6], Quillen proved that the inclusion $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a homotopy equivalence. Furthermore, when G is a finite group of Lie type in characteristic p, he showed that $\mathcal{S}_p(G)$ has the homotopy type of the building of G, and thus is homotopy equivalent to a non-trivial wedge of spheres whose dimension equals that of the building.

In general, the topology of the posets $\mathcal{A}_p(G)$ remains poorly understood, and they might not have the homotopy type of a wedge of spheres (this fails for $p = 3$ and G the alternating group on 13 letters, see [7]). A key question concerns the contractibility or acyclicity of these posets. In this direction, Quillen proved that if the largest normal p-subgroup of G, denoted by $O_p(G)$, is non-trivial, then $\mathcal{A}_p(G)$ is contractible. He further conjectured that the reciprocal should hold and proved some cases supporting his claim. In fact, for these cases, he established a stronger statement:

$$\text{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \mathcal{A}_p(G) \text{ is not } \mathbb{Q}\text{-acyclic.}$$

For instance, groups of Lie type in defining characteristic p and solvable groups satisfy (H-QC). Moreover, for solvable groups, an even stronger property holds:

$(\mathcal{QD})_p$ If $O_p(G) = 1$ then $\mathcal{A}_p(G)$ has non-zero homology in the largest possible degree.

Recall that the largest possible degree corresponds to the dimension of the order complex, which in the case of $\mathcal{A}_p(G)$ is the p-rank of G minus one. Later, various authors extended Quillen's proof of the solvable case to p-solvable groups using the Classification of Finite Simple Groups (CFSG), also proving that $(\mathcal{QD})_p$ holds for these groups.

A major breakthrough in resolving the conjecture was made by Michael Aschbacher and Stephen D. Smith in [1]. They showed that (H-QC) holds for G if $p > 5$, provided that whenever $\mathrm{PSU}_n(q)$ is a component of G with $p \mid q + 1$ and q odd, then p-extensions of $\mathrm{PSU}_m(q^{p^e})$ satisfy $(\mathcal{QD})_p$ for all $m \leq n$ and $e \in \mathbb{Z}$. Here, a p-extension of L is a semidirect product $L \rtimes B$ where B is an elementary abelian p-group inducing outer automorphism on L. Their proof relies on a series of reductions on a minimal counterexample to their main theorem, using the CFSG to analyse p-extension of simple groups for odd primes p. In particular, they show that for such a minimal counterexample G, every component L must have a p-extension failing $(\mathcal{QD})_p$, and then provide a "$(\mathcal{QD})$-list" of potential candidates for L when p is odd (see Theorem 3.1 in [1]). Notably, simple groups of Lie type in characteristic p naturally appear on this list since their p-group posets have the homotopy type of the building, whose dimension is strictly less than that of the $\mathcal{A}_p$-poset. Unitary groups $\mathrm{PSU}_n(q)$ with $p \mid q + 1$ and q odd are also contained in this list at first, but Aschbacher–Smith conjectured that they shouldn't. The assumption $p > 5$ is necessary in their proof since, for $p = 3, 5$, components of type $\mathrm{Sz}(2^5)$ $(p = 5)$, $\mathrm{PSL}_2(2^3)$ $(p = 3)$ and $\mathrm{PSU}_3(2^3)$ $(p = 3)$ present an obstruction to key arguments used in their reductions. The next step involves constructing "Robinson" subgroups for each simple group in the $(\mathcal{QD})$-list for $p > 5$. The final contradiction is obtained by using these Robinson subgroups to show that the Lefschetz module for $\mathcal{A}_p(G)$ is non-trivial, which in turn implies that $\mathcal{A}_p(G)$ cannot be $\mathbb{Q}$-acyclic. For unitary groups, such Robinson subgroups do not always exist, but the main hypothesis says that they cannot be components in the first place.

In a joint work with S.D. Smith [5], we extended the reductions of [1] to any prime p by using more topological and combinatorial methods while significantly reducing the use of the CFSG. For instance, our approach allows us to eliminate components of type $\mathrm{Sz}(2^5)$ ($p = 5$), $\mathrm{PSL}_2(2^3)$ ($p = 3$) and $\mathrm{PSU}_3(2^3)$ ($p = 3$) in a minimal counterexample. As a result, the Aschbacher–Smith original strategy can now be extended to all odd primes, provided that Robinson subgroups can be constructed in the final step for the simple groups in the $(\mathcal{QD})$-list for $p = 3, 5$ (recall that unitary groups are still excluded by hypothesis). While this construction is straightforward for $p = 5$, the components of Ree type don't have Robinson subgroups for $p = 3$. To handle this case, we propose an alternative argument. If a group G satisfies certain inductive hypotheses on (H-QC) and L is a component of Lie type in characteristic p such that G contains no order-p graph automorphism for L, then G satisfies (H-QC). In particular, in a minimal counterexample G, this situation cannot occur, so for $p = 3$, $L \cong \mathrm{Ree}(3^a)$ cannot be a component. Consequently, we can extend the Aschbacher–Smith theorem to all odd primes p. Finally, a recent result by Antonio Diaz Ramos [3] shows that p-extensions of unitary groups satisfy $(\mathcal{QD})_p$ (notably when $p \mid q + 1$ with p, q both odd), thus completing the proof of Quillen's conjecture for odd primes.

Since [5] makes these reductions possible for $p = 2$, we conclude that if G is a minimal counterexample to (H-QC) for $p = 2$, then every component of G must have some 2-extension failing $(\mathcal{QD})_2$. However, unlike the case for odd primes, we don't have a list of simple groups failing $(\mathcal{QD})_2$ for some 2-extension. In [4], it is shown that, with at most eight possible exceptions, 2-extensions of exceptional groups of Lie type in odd characteristic satisfy $(\mathcal{QD})_2$. On the other hand, alternating and sporadic components either admit suitable Robinson subgroups or can be eliminated using some of our methods. In summary, if G is a minimal counterexample to (H-QC) for $p = 2$, then [4, 5] imply that every component L of G has a 2-extension failing $(\mathcal{QD})_2$, $\mathcal{A}_p(L) \to \mathcal{A}_p(\mathrm{Aut}_G(L))$ is the zero map in homology, and every component of G that is neither alternating, sporadic, nor a group of Lie type in characteristic 3, must be either a classical group in odd characteristic, or of type A_n, D_n or E_6 in characteristic 2 with G containing order-2 graph and field automorphisms for such components. The CFSG is only used at this final step to make this list complete.

Finally, for $p = 2$, the group $\mathrm{PSL}_3(4)$ remains resistant to all the methods and reductions proposed in our works, showing that we still need to develop new techniques to deal with these components.

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Maps, Simple Groups, and Arc-Transitive Graphs

Cheryl E. Praeger
(joint work with Martin Liebeck (and also Cai Heng Li and Shu Jiao Song))

By a *map* M, we mean an embedding of a (simple) graph $\Gamma = (V, E)$ in a surface $\mathcal{S}$, such that the faces (connected components of $\mathcal{S} \setminus \Gamma$) are simply connected, that is, homeomorphic to an open unit disc. We assume that $\Gamma, \mathcal{S}$ are connected (but $\mathcal{S}$ may or may not be orientable, or compact, and may be with or without boundary).

Symmetries of maps. An *automorphism* of a map M is an automorphism of the graph Γ that extends to a self-homeomorphism of $\mathcal{S}$. The set of all map automorphisms forms the *map group* $G = \mathrm{Aut}(M)$ which is a subgroup of $\mathrm{Aut}(\Gamma)$, often a proper subgroup. We are interested in maps with lots of symmetry.

We measure the symmetry of a map M via the action of the map group G on the *flags* of M, that is, triples (v, e, f) consisting of a pairwise incident vertex v, edge e, and face f. The only map automorphism fixing a fag is the identity, and the most symmetrical maps are called *flag-regular*; they are the ones with G transitive (hence regular) on flags. We are concerned with *arc-transitive maps* where G is transitive on arcs, that is, on incident pairs (v, e), but not necessarily transitive on flags. For such maps the number k of edges incident with a vertex is constant and the stabiliser G_v of a vertex v is one of C_k, or D_k (with k even), or D_{2k} (and in this last case the map is flag-regular); and in all cases $|G_{v,e}| \leq 2$.

The study of flag-regular embeddings goes back at least to the early 1970s with Biggs' classification [1] of orientably regular embeddings of complete graphs. Topological methods, together with studying monodromy groups, led Graver and Watkins [3] to subdivide the family of edge-transitive maps into 14 distinct subfamilies, of which 5 subfamilies were arc-transitive. One of the five arc-transitive families comprises the flag-regular maps, while for the other four subfamilies, two are vertex-rotary (with $G_v = C_k$) and the other two are vertex-reversing (with $G_v = D_k$). A group theoretic framework for studying flag-regular embeddings was introduced by Gardiner et al [2] in 1999, and in 2005 I heard Gareth Jones speak at a mini-conference in Oxford about his then recent classification [4] of flag-regular embeddings of merged Johnson graphs. Such graphs admit arc-transitive, vertex-primitive actions of finite symmetric groups, and this prompted me to ask if a similar classification might be possible for all primitive actions of almost simple groups.

Recently, in joint work with Cai Heng Li and Shu Jiao Song [6, 7], we gave a new characterisation of the five subfamilies of arc-regular maps including five group theoretic constructions which between them produce all locally finite arc-transitive maps. In particular, these constructions lead to the following characterisation in

[11], with Martin Liebeck, of finite graphs which admit an almost simple subgroup of automorphisms and can be embedded as an arc-transitive map.

Theorem 1. [11, Proposition 2.1] *Let $\Gamma = (V, E)$ be a finite connected graph, admitting an almost simple subgroup $X \leq Aut(\Gamma)$ acting arc-transitively. Let $e = \{u, v\}$ be an edge of Γ and let $G \leq X$. Then Γ has a G-arc-transitive embedding if and only if the following three conditions hold.*

(a) $X = GX_v$ and $X_v = G_v X_{v,u}$;
(b) G_v is cyclic or dihedral and $G_{v,u} = 1$ or C_2;
(c) G contains an involution g such that $g \in N_G(X_{v,u})$ and $\langle X_v, g\rangle$.

Our objective became more ambitious, namely: determine all finite graphs admitting an almost simple arc-transitive group X of automorphisms that can be embedded as G-arc-transitive maps with map group $G \leq X$.

Cyclic/dihedral factorisations. Martin Liebeck and I extracted the first two criteria from those given in Theorem 1 concerning factorisations of almost simple groups: *Let X be a finite almost simple group with socle X_0, and let A, B be subgroups not containing X_0 such that $X = AB$ and $A \cap B$ is cyclic or dihedral.* We call such a factorisation a *cyclic/dihedral factorisation*, and our first objective was to determine all of them.

Theorem 2. [11, Theorems 1.1, 1.2, and 1.4] *All cyclic/dihedral factorisations of finite almost simple groups are known.*

The examples are given explicitly in a number of tables [11, Tables 10.1, 10.3–10.8] (which occupy six pages), together with those satisfying the following:

(Exceptional) $X_0 = A_n$ and (interchanging A and B if necessary), $A \cap X_0 = A_{n-1}$ and B is transitive in the natural permutation action of degree n with point stabiliser $A \cap B$ cyclic or dihedral.

The analysis was extremely delicate. It relied on known results about factorisations of almost simple groups, as explained in detail in [11], and in particular using new results from [8, 9, 10].

Arc-transitive embeddings. As a main application, we classified the graphs Γ admitting an almost simple arc-transitive group X of automorphisms, such that Γ has a 2-cell embedding admitting an arc-transitive map group $G \leq X$. This involved considering each cyclic/dihedral factorisation $X = AB$ from Theorem 2, and choosing one of A, B to be the map group G, say $G = A$, and the other subgroup B to be a vertex-stabiliser $X_v = B$. This choice determines the vertex set of an as yet unknown graph Γ as the coset space $[X : B]$ with vertex $v = B$, the trivial coset. We must determine (up to isomorphism) all involutions g satisfying condition (c) of Theorem 1. Each such involution determines a graph Γ by taking the arc set as the X-orbit on ordered pairs containing the coset pair (B, Bg). The graph Γ is arc-transitive by construction, and is connected since $\langle X_v, g\rangle = X$. Moreover Γ admits an arc-transitive embedding by Theorem 1. (We must of course consider both choices of A, B as the map group.)

Theorem 3. *If a connected graph Γ admits an almost simple arc-transitive subgroup X of automorphisms, and has an embedding with arc-transitive map group $G \leq X$, then either*

(a) Γ is a complete graph K_n or a Johnson graph $J(n, 2)$; or
(b) Γ is one of fourteen explicitly known graphs; or
(c) X has socle $X_0 = A_n$; $G \cap X_0 = A_{n-1}$, and $G \cap X_v$ is cyclic or dihedral.

Necessary and sufficient conditions on parameters for embeddability of complete graphs and Johnson graphs are known, see [4, 5].

In the exceptional case (c) of Theorem 3 we construct infinitely many graphs which have arc-transitive embeddings [11, Section 4, Theorem 4.4]: we take $n = (p-1)!/2 = |A_{p-1}|$, $X = A_n$, and $X_v = A_p$ acting transitively in its coset action on $[A_p : C_p]$ of degree n so $G \cap X_v = C_p$. The difficult part of the construction is to prove existence of an involution g satisfying Theorem 1(c). It would be interesting to know more about examples in this case.

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Counting subgroups of the symmetric group

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(joint work with Gareth Tracey)

One of the most elementary, but difficult, questions we can ask about a finite group G is how many subgroups it has, i.e. to determine $|\mathrm{Sub}(G)|$. For G the symmetric group S_n, an elementary argument shows that $|\mathrm{Sub}(G)| \geq 2^{n^2/16}$.

Pyber showed in 1993 [2] that $|\mathrm{Sub}(\mathrm{S}_n)| \leq 24^{n^2/6+o(n^2)}$, and conjectured that in fact $|\mathrm{Sub}(\mathrm{S}_n)| \leq 2^{n^2/16+o(n^2)}$. This talk presents a proof of this conjecture. As part of the proof we give asymptotically tight bounds on the number of p-subgroups of S_n for each prime p, and on the number of nilpotent subgroups of S_n.

One motivation for enumerating these subgroups is to determine properties of random subgroups of S_n, selected uniformly amongst all subgroups, or amost all conjugacy classes of subgroups. Erdős conjectured that if $m \leq 2^a$ then the number of groups of order m is bounded above by the number of groups of order 2^a. Building upon this, Pyber conjectured in 1993 [2] that as $n \to \infty$, the probability that a random group of order at most m is nilpotent tends to 1, and Kantor conjectured [1], also in 1993, that the probability that a random subgroup of S_n is nilpotent tends to 1. We prove that the probability that a random nilpotent subgroup of S_n is a 2-group tends to 1 as $n \to \infty$, and hence show that Kantor's conjecture is false.

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Fusion systems of characteristic p-type – an approach via localities

Edoardo Salati

Within the frame of the classification of the finite simple groups (CFSG) the study of the p-local structure of a finite group G (especially at the prime $p = 2$) represents the major tool of investigation. A p-local subgroup of G is a subgroup of the form $N_G(P)$ for some non-trivial p-subgroup P of G. Among a number of results making evident the power of the 2-local analysis we report the following.

Theorem (Gorenstein-Walter Dichotomy Theorem)**.** *Let G be a finite simple group with $\mathrm{rk}_2(G) \geq 3$. Then G is either of component type or of characteristic 2-type.*

Both the properties of being of component type (see [Asc:15, Section 2]) or of characteristic 2-type depend only on the 2-local structure of G. In particular, a group H is of characteristic 2 if $C_H(O_2(H)) \leq O_2(H)$, where $O_2(H)$ denotes the largest, normal 2-subgroup of H; G is of characteristic 2-type if all the 2-local subgroups of G are of characteristic 2 (note that the above definitions obviously generalize to all primes).

The p-local theory of finite groups spawned a number of tools, fusion systems being one of them. A *saturated fusion system* on the p-group S is a category $\mathcal{F}$ satisfying:

– the set of objects is the set of subgroups of S,
– for any $P, Q \leq S$, $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}_{Grp}(P, Q)$,

together with certain additional axioms modeled on the properties of the p-fusion categories of finite groups, obtained as above by taking a Sylow p-subgroup of a finite group G as S and G-conjugations as morphism, see [AKO:11, Definitions 1.1, 2.1 and 2.2].

Many ideas of group theory have a translation in terms of fusion systems. For example, the theorem of Gorenstein and Walter translates to the following.

Theorem (Dichotomy Theorem for fusion systems)**.** *Let $\mathcal{F}$ be a saturated fusion system on the p-group S. Then either $\mathcal{F}$ is of component type or of characteristic p-type.*

Parallel to the study of fusion systems, in the past fifteen years researchers have worked on closely related structures, namely *localities*, developed in 2013 by Chermak for the solution of a conjecture about fusion systems in connection with algebraic topology, see [Ch:13]. A *linking locality* $\mathcal{L}$ on the p-group S is a partial group in the sense of [Ch:13, Definition 2.1] (think of a non-empty set with a unit and inverses of each element just like in a group, but with a weakened product, not necessarily defined on all words of finite length) with the following features:

– S is embedded in $\mathcal{L}$ as a subgroup and it is a maximal p-subgroup of $\mathcal{L}$,
– there is a set $\Delta \subseteq \{\textit{subgroups of } S\}$, overgroup-closed and "not too small", such that for every $P \in \Delta$, $N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$ of characteristic p,
– $\mathcal{L}$ has an associated fusion system $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ such that for all $P \in \Delta$ the map $\pi_P : N_{\mathcal{L}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P)$ is an epi with kernel $C_{\mathcal{L}}(P)$.

The local structure theorem for groups. Paying attention back to the simple groups of characteristic 2-type, a currently ongoing programme of Meierfrankenfeld, see [MSS:03], aims at describing the p-local structure of the finite groups of characteristic p-type (whose prototypes are the finite simple groups of Lie type in their defining characteristic), promising a new classification of the simple ones at the prime 2. There is then a natural associated question: is a similar programme possible for fusion systems of characteristic p-type?

We approach the problem by looking at a single, far-reaching result: the local structure theorem for finite groups with a large p-subgroup [MSS:16]. A p-subgroup Q of G is *large* if $C_G(Q) = Z(Q)$ and $N_G(U) \leq N_G(Q)$ for any non-trivial subgroup $U \leq Z(Q)$; even though groups with a large p-subgroup may fail being of characteristic p-type, they are of *parabolic characteristic* p: all the p-local subgroups containing a p-Sylow are of characteristic p.

The theorem describes (and almost classifies) the structure of such p-local subgroups of G. The proof works as follows: first, one reduces the problem to the study of certain maximal p-local subgroups M (having some "nice" properties) containing a chosen p-Sylow S; such an M contains a canonical, normal, elementary abelian p-subgroup Y_M, giving rise to a canonical $\mathbb{F}_pM$-module. Afterwards one shows that any p-local subgroup L of G containing S has an analogous canonical $\mathbb{F}_pL$-module Y_L which is isomorphic to a module embedded in the pair (M, Y_M).

One is left then to analyze, and possibly classify, the pairs (M, Y_M), accomplished by studying occurrence of offenders (see [MPS:18, Section 1] for the definition) on Y_M or related subgroups, yielding the major case-history subdivision made of five cases.

A local structure theorem for localities. Extending the definition of large subgroups to fusion systems and localities in a compatible way is a relatively easy task; in particular, after choosing the set Δ for the locality $\mathcal{L}$ as the largest possible (that is, the set of *subcentric* subgroups as in [He:15, Definition 1]), it turns out that the quest for an analogous of the local structure theorem can be equivalently pursued for fusion systems or for localities.

Note that localities easily detect fusion systems of characteristic p-type: in particular, if $\mathcal{F}$ is the fusion system on S associated to the linking locality $\mathcal{L}$ and Δ is the set of subcentric subgroups, then

$$\mathcal{F} \text{ is of characteristic } p\text{-type} \iff \Delta = \{1 \neq P \leq S\},$$
$$\mathcal{F} \text{ is of parabolic char. } p \iff \Delta \supseteq \{1 \neq P \trianglelefteq S\},$$

where the existence of a large subgroup implies that $\mathcal{F}$ is of parabolic characteristic p in the sense of [He:15, Remark 10.9].

However, in the process of translating arguments from the proof in [MSS:16] important obstructions appear when working with fusion systems, but not when working in a locality, making for an obvious choice.

Already at the very beginning, in a locality $\mathcal{L}$ with a large subgroup Q one can identify pairs (M, Y_M), with M a subgroup of $\mathcal{L}$ and Y_M a canonical $\mathbb{F}_pM$-module, analogous to those in groups (such an argument doesn't work for fusion systems). This allows to reduce the analysis to a classification of modules based on offenders and on a case-history closely resembling that in [MSS:16]. When an analogous offender can be found, one obtains the same list of modules to go through; it turns out that, for the studied cases, the elimination process of modules that are incompatible with the overall structure of $\mathcal{L}$ yields the same results as in [MSS:16], where a whole group G is present in place of $\mathcal{L}$. This is somehow surprising, since within G such elimination process heavily relies on Q^G, which is not completely translatable in terms of the locality (more generally, conjugation in localities may be poorly behaved). The trick here consists in restricting the problem on a smaller set of conjugates of Q, over which one retains enough control in $\mathcal{L}$.

This happens in at least two of the five cases, and is expected in a third one as well; we report an example, to be compared with [MSS:16, Theorem G].

Theorem (Theorem G). *Let $(\mathcal{L}, \Delta, S)$ be a $\mathcal{K}_p$-locality, $Q \leq S$ a large subgroup and assume that $\mathcal{L}$ is Q-replete. Suppose that $M \in \mathfrak{M}_{\mathcal{L}}(S)$ is such that*

– *Y_M is asymmetric and char p-tall;*
– *Y_M is Q-short and $Q \ntrianglelefteq M$.*

Then one of the following holds for q a power of p and $\overline{M^\dagger} := M^\dagger / C_{M^\dagger}(Y_M)$.

(1) $\overline{M^\circ} \cong \mathrm{SL}_n(q)$ for $n \geq 3$ and Y_M is a corresponding natural module.

(2) $p = 2$, $\overline{M} \in \{\mathrm{O}_4^-(2), \mathrm{Sp}_4(2)', \mathrm{Sp}_4(2)\}$, $Y_M = O_2(M)$ *is a corresponding natural module and* $N_{\mathcal{L}}(Q) \leq M^\dagger$*. If* $\overline{M} \cong \mathrm{O}_4^-(2)$*, then for every non-singular vector* $x \in Y_M$, $\langle x \rangle \notin \Delta$.
(3) There exists a unique $\overline{M}$*-invariant set* $\mathcal{K}$ *of subgroup of* $\overline{M}$ *such that* Y_M *is a natural* $\mathrm{SL}_2(q)$*-wreath product module for* $\overline{M}$ *with respect to* $\mathcal{K}$*. Then also the following hold.*
  (a) $Y_M = O_p(M)$ *and* $N_{\mathcal{L}}(Q) \leq M^\dagger$.
  (b) $\overline{M^\circ} = O^p(\langle \mathcal{K} \rangle)\overline{Q}$.
  (c) Q *is transitive over the set* $\mathcal{K}$.
  (d) If $|\mathcal{K}| \geq 2$*, then* $q \in \{2, 4\}$ *and for any* $K \in \mathcal{K}$, $\langle [Y_M, F] \mid K \neq F \in \mathcal{K} \rangle \notin \Delta$.

A fourth case was dealt by different means; in particular, we show that the main obstruction to a full classification of the occurring modules in [MSS:16], namely the so-called *tall, char p-short case* [MSS:16, Chapter 6], vanishes in localities.

Finally, one last case seems to resist any argument borrowed from [MSS:16]; here restriction to fusion systems of characteristic p-type may remove the issue, but a result encompassing the more general setting of fusion systems with a large subgroup requires additional work and possibly additional tools to manipulate partial groups and localities, yet to be developed.

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The length of mixed identities for finite groups

Jakob Schneider

(joint work with Henry Bradford and Andreas Thom)

The study of word maps has seen a lot of progress in recent years. Here a word w is just an element of the free group $\mathbf{F}_r = \langle x_1, \ldots, x_r \rangle$ of rank r. Given such a w and a group G, the corresponding word map, also denoted by $w\colon G^r \to G$, is the map that sends the r-tuple $(g_1, \ldots, g_r) \in G^r$ to the value $\varphi(w)$ of w under the homomorphism $\varphi\colon \mathbf{F}_r \to G$ which maps $x_i \mapsto g_i$ $(i = 1, \ldots, r)$. There are two extremal behaviors of the map w: Either its image $w(G^r)$ is just the neutral element, or w is close to random, meaning $\mathbf{P}_{g_1,\ldots,g_r}(w(g_1, \ldots, g_r) = g) \sim \frac{1}{|G|}$ for all $g \in G$ and $g_1, \ldots, g_r \in G$ equidistributed. A weaker formulation of the latter case is that the mentioned probability is non-zero, i.e. w is surjective. In [3] it was shown that the word image is ε-dense in natural nomalized metrics of the groups S_n, $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{GO}_n^\bullet(q)$ and $\mathrm{GU}_n(q)$ for n resp. m large enough (and $\varepsilon > 0$ chosen arbitrarily). Another (deeper) result in this direction is the celebrated solution of the Ore conjecture [5], establishing that every element of a non-abelian finite simple group is a commutator.

In this talk we are interested in words with constants, i.e. elements of the free product $w = c_0 x_{i(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{i(l)}^{\varepsilon(l)} c_l \in G * \mathbf{F}_r$ $(c_0, \ldots, c_l \in G)$ for a group G, which is finite in most cases. Here the word map $w\colon G^r \to G$ is defined analogously to the case of a word without constants, by substitution. The word w with constants is called a *mixed identity* for G if $w(G^r) = 1_G$. We are interested in the following question:

Question 1. *Let G be a finite group. What is the length of a shortest mixed identity $w \in G * \mathbf{F}_r$ for G?*

As a partial answer, we have the following theorem (see [1]) which reduces the interesting groups G to almost simple ones:

Theorem 1. *Either the finite group G has a mixed identity of length at most* 8*, or it is almost simple.*

In the latter case, there is a constant $c > 0$ such that if G has no mixed identity of length $< c$, then G is almost simple with a simple group of Lie type as its socle. This holds, since the alternating groups A_n satisfy a mixed identity $w(x) = [x, \sigma]^{30}$ for a 3-cycle σ, which is of length 60, and the sporadic groups can be ignored as they are finitely many.

Hence in the following we focus on finite groups G with $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for a non-abelian simple group S. We present some new results from [1] using the Landau notation:

Theorem 2. *Let G be an almost simple group with socle $\mathrm{PSL}_n(q)$ and $q = p^e$ for a prime number p. Then, G has a mixed identity of length $O(q)$. Moreover, if $G \leq \mathrm{PGL}_n(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$, F is the Frobenius automorphism $x \mapsto x^p$ acting coordinate-wise, and $f \mid e$ is the smallest positive integer such that $F^f \in G$, then any mixed identity of G is of length $\Omega(\frac{e}{f} p^f)$.*

In contrast to the projective special linear groups, the projective symplectic and orthogonal groups do have short mixed identities. This observation is due to Tomanov [6].

Theorem 3 (Tomanov)**.** *The group $\mathrm{PSp}_{2m}(q)$ for $m \geq 2$ satisfies a mixed identity of length* 8*.*

Theorem 4 (Tomanov)**.** *There exists a mixed identity for* $\mathrm{P}\Omega_{2m+1}(q)$ *for* $m \geq 2$ *of length* 8*.*

We list here a couple of new results from [1] in this area. The essence of these theorems for symplectic and orthogonal groups is that short mixed identities do not exist when there is no critical constant (these are constants $c \in G$ such that $x^{-1}cx$ is a subword of w, with x a variable or its inverse) which is *small*. Here the notion of smallness needs to be specified for every Lie type.

Theorem 5. *Let* q *be a prime power and* $m \geq 2$*. For* q *odd, a shortest mixed identity for* $\mathrm{PSp}_{2m}(q)$ *without critical constants, which lift to involutions (the* small *elements) in* $\mathrm{Sp}_{2m}(q)$*, is of length* $\Theta(q)$*. For* q *even it is* $\Omega(q)$*.*

Theorem 6. *Let* $p \neq 2$*. If there is a mixed identity* $\overline{w} = \overline{c}_0 x_{i(1)}^{\varepsilon(1)} \cdots x_{i(l)}^{\varepsilon(l)} \overline{c}_l \in \mathrm{P}\Omega_n^{\bullet} * \mathbf{F}_r$ *for* $\mathrm{P}\Omega_n^{\bullet}$ *of length* $\Omega(q)$ *which lifts to* w*. Then one of its* critical constants c_j *must be* small*.*

Here for orthogonal groups, in Theorem 6, an element is small if and only if it satisfies a quadratic polynomial $p(x) = x^2 - \lambda x + 1$.

For unitary groups there are no non-central critical constants, and we have the following result.

Theorem 7. *Let* G *be an almost simple group with socle* $\mathrm{PSU}_n(q)$*. Then,* G *has a mixed identity of length* $O(q^2)$*. Moreover, any mixed identity for* $\mathrm{PSU}_n(q)$*, even with constants from* $\mathrm{PGL}_n(q^2)$*, is of length* $\Omega(q)$*.*

The following questions are currently studied, but so far we have no complete solution to them:

Question 2. *What about the remaining families of finite groups of Lie type: How long is a shortest mixed identity in this case?*

Question 3. *All mixed identities of bounded length constructed for the groups considered are* singular*, i.e.* $\varepsilon(w) = 1$*, where* $\varepsilon\colon G * \mathbf{F}_r \to \mathbf{F}_r$ *is the unique homomorphism such that* $G \ni g \mapsto 1_{\mathbf{F}_r}$ *and which fixes* $\mathbf{F}_r$ *elementwise (the* augmentation map*). What about the shortest non-singular identities for these groups?*

Question 2 we are currently working on. Indeed, Tomanov [6] proves:

Theorem 8. *Let* G *be a simple algebraic group of Lie type and* K *a locally compact field that is non-discrete (i.e.* K *is non-locally finite). Then there is a Zariski null set* $S \subseteq G(K)$ *of small constants such that there there is no mixed identity for* $G(K)$ *with all critical constants non-small.*

The notion of *small* here is more sophisticated and we omit the definition. Only note that it agrees with the definitions of small in the other cases. The proof happens by the idea of Tits of attracting and repulsing points.

For Question 3 we have new results from [4] and [2].

Theorem 9. *A shortest non-singular mixed identity for* A_n *and* $\mathrm{PSL}_n(q)$ *is of length* $\Omega(\log(n)/\log\log(n))$*.*

Similar results hold for $\mathrm{PSU}_n(q)$ but here the bound depends on q.

With these two facts, we establish:

Corollary 1. *Let $l \in \mathbb{N}$. Then there are only finitely many non-abelian simple groups of Type A_n, $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$ that satisfy a non-singular mixed identity of length $\leq l$.*

The corollary can be seen as a first step to settle the conjecture, that its statements holds for all families of finite simple groups. This can be seen as an improvement of Jones's result in [7].

On the size of word images, we have the following lower bound. Let K be an algebraically closed field:

Theorem 10. *Let $w = c_0 x_{i(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{i(l)}^{\varepsilon(l)} c_l \in \mathrm{GL}_n(K) * \mathbf{F}_r$ a reduced word. Then the word image $w(\mathrm{SL}_n(K)^r)$ has dimension at least $n-1$.*

We expect that similar results hold for other classical quasi-simple groups. Also note that the bound in the last theorem is sharp up to factor two: The dimension of the image of the word $w(x) = c^x = x^{-1}cx$, for a transvection c, is $2(n-1)$.

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Applications of Fusion Systems in Block Theory

Patrick Serwene

Fix G to be a finite group, $p \in \mathbb{P}$, k a field with $\mathrm{char} k = p$ and $k = \overline{k}$.

We first recall the concept of the **fusion category** of a finite group: $P \in \mathrm{Syl}_p(G)$, define $\mathcal{F}_P(G)$ to be the category with objects subgroups of P (say "category on P") and if $Q, R \leq P$ then $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \{c_g \mid g \in G, {}^g Q \leq R\}$.

Fusion systems generalise this concept and are categories on p-groups with applications in Group Theory, Representation Theory and Topology.

The fusion category is a fusion system, but not every fusion system is of this form. If it is we call it **realisable**, otherwise **exotic**.

Blocks also induce fusion systems: If b is a block of kG, then the fusion system $\mathcal{F}_P(G,b)$ is defined on a defect group P of b. Again, not every fusion system is of this form. If it is we call it **block-realisable**, otherwise **block-exotic**. We conjecture that the two mentioned ways for fusion systems to arise coincide:

Conjecture 1. *Let $\mathcal{F}$ be a fusion system. Then $\mathcal{F}$ is exotic if and only if $\mathcal{F}$ is block-exotic.*

One direction is a consequence of Brauer's Third Main Theorem: Every realisable fusion system is also realisable as the fusion system of a principal block.

Applications to Weight Conjectures: Statements from Block Theory can be expressed in the language of fusion systems. This is done for many known results and open conjectures in [1], where the link is made by either Alperin's or the Ordinary Weight Conjecture. For realisable fusion systems, these are true. We proved several of these conjectures for 27 exotic fusion systems on a Sylow 7-subgroup of $G_2(7)$, classified by Parker and Semeraro in [3]:

Theorem 2. [2] *Six weight conjectures from* [1] *hold for the exotic Parker–Semeraro systems.*

Applications to Linckelmann's Conjecture: For an algebra A it is known that $\mathrm{HH}^1(kG) \neq 0$ when $p \mid |G|$. Linckelmann conjectured that also $\mathrm{HH}^1(kGb) \neq 0$ when b is a block of kG with non-trivial defect group. In recent work we try to tackle this conjecture with the help of fusion systems. First, recall that $\mathrm{hch.dim}(A) = \sup\{n \in \mathbb{N} \mid \mathrm{HH}^n(A) \neq 0\}$ and we say that an algebra A has **Happel's property**, if $\mathrm{hch.dim}(A) < \infty \Rightarrow \mathrm{gl.dim}(A) < \infty$.

Theorem 3. [4] *Let $\mathcal{F}$ be the fusion system of a block b. If $\mathcal{F}$ is realisable, then kGb has Happel's property.*

Linckelmann's conjecture holds for J_1, J_2 and for A_n when $p \geq 3$ and we have certain assumptions on the respective block.

The idea for future work is to find a reduction of Linckelmann's conjecture to (quasi)simple groups with the help of fusion systems.

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My thoughts on the Meierfrankenfeld program

BERND STELLMACHER

About 20 years ago, a program began whose original goal was to classify the finite groups of local characteristic p (LCp-Hypothesis). Over the years the program has undergone an evolution that made large parts of it independent of the LCp-Hypothesis.

In the literature this program is often called the MSS project. In my talk I briefly explained why, from a personal perspective, I prefer the term Meierfrankenfeld program; and I also identified those aspects of the program I personally consider essential and characteristic for the program.

In the following G is a finite group and p is a prime divisor of $|G|$ such that $O_p(G) = 1$. Also let $S \in Syl_p(G)$ and M be a maximal p-local subgroup of G with $N_G(\Omega_1(Z(S))) \leq M$. We also fix the notation:

$$Q := O_p(M) \quad \text{and} \quad E := F_p^*(M) \cap C_M(Y_M).$$

Here $F_p^*(M)$ is the inverse image of $F^*(M/Q)$ and Y_M is the largest p-reduced elementary abelian normal subgroup of M.

The following elementary observation allows the program to be partitioned into two parts, which can be treated completely independent from each other.

Basic Lemma. Suppose that M is the only maximal p-local subgroup containing E. Then Q is a large p-subgroup of G.

Here Q is a large p-subgroup of G if

$$C_G(Q) \leq Q \quad \text{and} \quad N_G(A) \leq N_G(Q) \text{ for all } 1 \neq A \leq C_G(Q).$$

This allows the subdivision:

Part I. The Q!-case: Q is a large p-subgroup of G. (Standard examples: groups of Lie type in characteristic p where the center of a Sylow p-subgroup is a root subgroup)

Part II. The Non-E!-case: E is contained in at least two maximal p-local subgroups of G. (Standard examples: groups of Lie type in characteristic p where the center of a Sylow p-subgroup is not a root subgroup)

Comments on Part I. In this case, large parts of the p-local analysis have already been carried out, and the global analysis of the groups in question have already begun. Cornerstones of these investigations are [3] and [4]. Both results do not use the LCp-Hypothesis.

The first result provides input that is then used for the global analysis. More precisely, for every p-local subgroup L of G with $S \leq L$ and $L \not\leq M$ the Local Structure Theorem in [3] gives the structure of $L/C_L(Y_L)$ and the structure of the $\mathbb{F}_p L$-module Y_L.

In addition to this information, the global analysis by means of [4] also requires the structure of M/Q and a meaningful case subdivision that distinguishes between the "large bulk" of the groups under consideration and "small cases".

The Small World Theorem provides this subdivision and further information for the global analysis. The small cases are:

– The rank 1 case, where M is the unique maximal p-local subgroup containing S.
– The rank 2 case, where there exist two p-minimal subgroups P_1 and P_2 with

$$S \leq P_1 \cap P_2,\ O_p(P_i) \neq 1 \text{ and } O_p(\langle P_1, P_2\rangle) = 1.$$

– An exceptional rank 3 case.

Right now Meierfrankenfeld and Stellmacher are working on a version of this theorem that does not require the LCp-Hypothesis. The exceptional rank 3 case was already treated in [5] (but only in the version given by the LCp-Hypothesis). Meierfrankenfeld, Parker and Stroth are working on the global analysis of those groups that do not have small rank.

Comments on Part II. This part is much further away from completion than Part I. There is an old and outdated preprint by Meierfrankenfeld, Stroth and Stellmacher that subdivides this part into three cases and uses the LCp-Hypothesis, and there exists another preprint by Meierfrankenfeld and Stroth that does the global analysis in one of these cases.

In my talk I will use the following hypothesis which is more general than the LCp-Hypothesis but still may allow further generalizations.

Hypothesis for Part II. As in Part I G is a finite group with $O_p(G) = 1$, $S \in Syl_p(G)$, M is a maximal p-local subgroup of G of characteristic p with $N_G(\Omega_1(Z(S)))$, $Q = O_p(M)$ and $E = F_p^*(M) \cap C_M(Y_M)$. Moreover, in this part E is contained in at least two maximal p-local subgroups and

$$N_G(A) \text{ has characteristic } p \text{ for all } 1 \neq A \leq C_G(Q).$$

In this Part II new features are used for the p-local analysis:

– Point stabilizers. Let H be a finite group and $T \in Syl_p(H)$. Then $C_H(\Omega_1(Z(T)))$ is the (full) point stabilizer of H with respect to T.

– Uniqueness subgroups (for M). Let $X \leq M$. Then X is a uniqueness subgroup for M if M is the unique maximal p-local subgroup of G containing X.

Cornerstones of the p-local analysis is the $W(B)$-Theorem from [1] and the Point Stabilizer Theorem from [2]. Both results are independent of the LCp-Hypothesis. The first result shows that for any finite group H of characteristic p and P a point stabilizer of H either $W(P) \trianglelefteq H$ or there exist Baumann blocks in H and two distinct Baumann blocks of H centralize each other.

Here $W(P)$ is a non-trivial characteristic subgroup of $O_p(P)$ whose definition only depends on $O_p(P)$ but not on P or H; and Baumann blocks are subnormal subgroups of H with a very restricted structure, see [1] for the definition and structure.

One objective is now to discuss those groups that do not possess Baumann blocks in any maximal p-local subgroup. This leads to the following working hypothesis.

Additional Hypothesis. No maximal p-local subgroup of G containing E possesses a Baumann block.

This hypothesis allows to impose a partial order the set of maximal p-local subgroups of G different from M but containing E according to the size of their point stabilizers. Let $\mathfrak{M}^\bullet(E)$ be the set of maximal elements with respect to this partial order. The following theorem gives some useful uniqueness subgroups.

Theorem. Let $H \in \mathfrak{M}^\bullet(E)$ and $T \in Syl_p(M \cap H)$. The following subgroups are uniqueness subgroups for M.

(1) $C_G(\Omega_1(Z(S)))$ and $C_G(\Omega_1(Z(T)))$.
(2) P, where P is a point stabilizer of $F_p^*(M)$.

Combing this information gives:

Theorem. There exists $H \in \mathfrak{M}^\bullet(E)$, a point stabilizer P of $M \cap H$ and a P-invariant normal subgroup $Q \leq F \trianglelefteq F_p^*(M)$ such that for $L := FP$ the following hold:

(1) F/Q is product of components of M/Q.
(2) Y_L is an FF-module for L.
(3) P is point stabilizer of H.
(4) $O_p(\langle L, H\rangle) = 1$.

The investigation of the amalgam $\langle L, H\rangle$ by means of suitable uniqueness subgroups has yet to be carried out.

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The Uniqueness Case, The Component Type

Gernot Stroth

In the classification of the finite simple groups there was a subdivison in groups of component type ($H/O(H)$ possesses a subnormal quasisimple group for some 2-local subgroup H of G) and groups of local characteristic two ($F^*(H) = O_2(H)$ for all 2-local subgoups H of G). In dealing with the latter one tries to move to odd primes p and tries to prove that for this prime one has a component type. More precisely, let M be a maximal 2-local subgroup then one searchs for elements $x \in M$, $o(x) = p$, p odd, such that $C_G(x) \not\leq M$ and $C_G(x)/O_{p'}(C_G(x))$ has a component. For this one has to deal with $O_{p'}(C_G(x))$. There are signalizer functor methods which can be applied. They start to work if $m_p(C_M(x)) \geq 3$. The uniqueness case as a last step in the classification of the finite simple groups is the case where all this does not work. We will describe this in more details.

Definition 1. For an odd prime p set

$$m_{2,p}(G) = \max\{m_p(H) \mid H \text{ a 2-local subgroup of } G\}$$

$$e(G) = \max\{m_{2,p}(G) \mid p \text{ odd}\}$$

Definition 2. For odd primes p we define

- $\sigma_1(G) = \{p > 2, m_{2,p}(G) > 3\}$
- $\sigma_2(G) = \{p > 3, m_{2,p}(G) = 3\}$
- $\sigma_3(G) = \{p \geq 3, m_{2,p}(G) = 3\}$

Let $i_0 = \min\{i, \sigma_i(G) \neq \emptyset\}$. Then set

$$\sigma(G) = \sigma_{i_0}(G)$$

Definition 3. Let P be a p-group, then define

$$\Gamma_{P,k} = \langle N_G(X) \mid X \leq P, m(X) \geq k\rangle$$

Definition 4. Almost strongly p-embedded Let p be an odd prime and M a maximal 2-local subgroup of G. We call M almost strongly p-embedded in G if $m_p(M) > 1$, $\Gamma_{P,2}(G) \leq M$ for $P \in \mathrm{Syl}_p(M)$ and one of the following holds:

- $\Gamma_{P,1}(G) \leq M$ (M is strongly p-embedded)
- $p > 3$, M is solvable, and there is a subgroup P_0 of P of order p, weakly closed in P with respect to G, such that $L = E(C_G(P_0)) \cong \mathrm{PSL}_2(p^n)$, $n \geq 2$ and $N_G(X) \leq M$ for all $P_0 \neq X \leq P$, $|X| = p$.
- $p = 3$, M is solvable $P \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and $\Gamma_{J(P),1}(G) \leq M$.

Definition 5. Uniqueness case. We say that G is in the uniqueness case provided $e(G) \geq 3$, G is of local characteristic 2, and for each $p \in \sigma(G)$ G possesses an almost strongly p-embedded maximal 2-local subgroup.

Theorem 1. *(M. Aschbacher* [1]*) There are no simple ($\mathcal{K}$)-groups in the uniqueness case.*

In the revision of the classification of the finite simple groups (GLS for short) the groups of local characteristic 2 have changed to groups of even type or even groups of restricted even type. For a definition see [2, Definition 8.8]. Nevertheless one needs a similar kind of uniqueness result for groups of (restricted) even type.

The work on this version of the uniqueness case started in 1993 [6] and in the talk at the workshop we present the (final) version from 2023. As this is part of the GLS project and there the proof of the $e(G) = 3$ case still is not done, there might be some changes in the assumptions as it has also been from 1993 on.

We start with the new definitions.

Definition 6. (**Uniqueness group**) We assume $e(G) \geq 3$ and $p \in \sigma(G)$. A subgroup $M = M_p$ is called a uniqueness group for the prime p if $p \in \sigma(M)$, $|G : M|$ is odd and one of the following holds:

- one of the following holds
  (1) (2-local type) M is a maximal 2-local subgroup of G with $F^*(M) = O_2(M)$; or
  (2) (component type) $F^*(M) = O_2(M)K$, $Z(K) = O_2(K)$ and K is a quasisimple group of Lie type in characteristic two (some are excluded), $m_p(K) \geq 2$ and $m_p(C_M(K)) \leq 1$.

In both cases for $P \in \mathrm{Syl}_p(M)$, we have : If $x \in P$, $o(x) = p$, $m_p(C_M(x)) \geq 3$, then $N_G(\langle x\rangle) \leq M$. Further $\Gamma_{P,2}(G) \leq M$ or $p = 3$ and $P \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$.

- $F^*(M) = O_2(M)$. Set $M/O_2(M) = \overline{M}$. Then $\overline{Q} = O_p(\overline{M})$ is elementary abelian of order p^n. We have $C_{\overline{M}}(\overline{Q}) = \overline{Q} \times \overline{X}$. Further if P is a Sylow p-subgroup of M, then $\overline{P} = (\overline{P} \cap \overline{X}) \times \overline{Q}$, $m_p(\overline{X}) = 1$ and $\overline{M}$ induces on $\overline{Q}$ a Borel subgroup of an automorphism group of $\mathrm{PSL}_2(p^n)$, containing the Borel subgroup of $\mathrm{PSL}_2(p^n)$. Let Q be a preimage of $\overline{Q}$ in P. Then $\Gamma_{Q,1}(G) \leq M$. Further if $\omega \in P$ is a nontrivial element with $\overline{\omega} \in \overline{X}$, then the following holds:
  (a) $\langle\omega\rangle$ is strongly closed in P with respect to G.
  (b) $C_G(\omega) \not\leq M$ but $O_{p'}(C_G(\omega)) \leq M$.

Definition 7. (**Uniqueness case**) We say that a simple group G is in the uniqueness case if G is $\mathcal{K}_2$-simple and the following holds.

(1) $e(G) \geq 3$
(2) For every $p \in \sigma(G)$ there is a uniqueness subgroup M_p with $p \in \sigma(M_p)$.
(3) Let M_p be a uniqueness subgroup of G with $p \in \sigma(M_p)$. If H is any 2-local subgroup of G such that $H \cap M_p \geq E, E \cong E_{p^2}, \Gamma_{E,1}(G) \leq M_p$, then $H \leq M_p$.

Theorem 2. *If G is a simple group of restricted even type in the uniqueness case and M is a uniqueness group of G for some $p \in \sigma(G)$, $S \in \mathrm{Syl}_2(M)$, then M contains any 2-local subgroup H of G with $S \leq H$.*

If $O_2(M) \neq 1$, then by [2, page 90, Theorem $\mathcal{M}(S)$] there are no such groups. If $O_2(M) = 1$ a similar theorem, which also states that there are no groups in the uniqueness case still waits to be proved.

In the talk we give a survey of the proof of Theorem 2. We mainly concentrate us on the case that the uniqueness group is of component type.

Suppose the theorem is false. The main idea is to establish an amalgam (G_1, G_2) which then eventually leads to a contradiction. Let M be some uniqueness group for the prime p, and S be a Sylow 2-subgroup of M. Then G_1 is a certain subgroup of M with $S \leq G_1$. In case of the component type there is a root subgroup of K in $Z(S \cap K)$. We then choose $G_1 = N_M(R)$. The structure of M tells us that $F^*(G_1) = O_2(G_1)$. Then choose G_2 with $S \leq G_2$, $O_2(G_2) \neq 1$ and minimal with respect to $G_2 \not\leq M$. This exists by the assumption that the theorem is false. Then we prove that we may even choose G_2 that additionally $F^*(G_2) = O_(G_2)$. Using the structure of M, in particular K, we prove that there is $E \leq G_1$, E elementary abelian of order p^2 and $\Gamma_{E,1}(G) \leq M$. This then implies by Definition 7(3) that $O_2(\langle G_1, G_2 \rangle) = 1$. Hence (G_1, G_2) is an amalgam. The amalgam method then yields that $Z_2 = \langle Z(S)^{G_2} \rangle$ is a $2F$-module. The classification of the $2F$-modules [3, 4, 5] and the minimality of G_2 then provides us with the structure of $G_2/C_{G_2}(Z_2)$. This finally yields the statement that $m_2(G_2/C_{G_2}(Z_2)) \leq 3$. Set $Q = O_2(G_1 \cap K)$. Then we prove hat $Q \not\leq C_{G_2}(Z_2)$. As $m_2(G_2/C_{G_2}(Z_2)) \leq 3$ we get $|Q : C_Q(Z_2)| \leq 8$. Now we may use the structure of groups of Lie type (i.e. K), in particular the action of the normalizer of a root group on the radical of this normalizer. As $|Q : C_Q(Z_2)| \leq 8$, this shows that $K \cong \mathrm{PSL}_n(2)$ or $\mathrm{SU}_n(2)$. In both cases we receive a contradition applying Holt's theorem [2, page 89,Theorem SF]. This then implies that Definition 6(2) cannot hold.

In case of Definition 6(1) we first prove that G possesses a unique uniqueness group M, which is a uniqueness group for all $p \in \sigma(G)$. Then we prove that G is of parabolic charactreristic 2, i.e. $F^*(H) = O_2(H)$ for all 2-local subgroups of G with $|G : H|$ odd. Then again we set up amalgams, in this case there are eight different ones to be treated. In fact G_2 is again minimal with respect not to be contained in M and a list of further properties. To each G_2 we then choose an appropriate $G_1 \leq M$. The study of these amalgams then in any case eventually leads to a contradiction.

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Character Estimates for Finite Classical Groups and Thompson's Conjecture

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(joint work with Michael Larsen)

A finite group G is called *almost quasisisimple* if $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for some finite non-abelian simple group G. An important problem in complex representation theory of finite almost quasisimple groups is the following

Problem 1. *Let G be a finite almost quasisimple group, $g \in G \setminus \mathbf{Z}(G)$. Find an explicit, and as small as possible, $0 \leq \alpha = \alpha(g) < 1$, such that*

$$|\chi(g)| \leq \chi(1)^{\alpha},\ \forall\ \chi \in \mathrm{Irr}(G).$$

Results on Problem 1 have proved to be useful in a number of applications, which usually involve using *Frobenius character formula.* Recently, building on [2, 4, 5, 8], we have proved the following uniform exponential character bound, which works for all elements in all finite quasisimple groups of Lie type:

Theorem 2. [6] *There exists an explicit constant $c > 0$ such that for all finite quasisimple groups G of Lie type, all $\chi \in \mathrm{Irr}(G)$, and all $g \in G$, we have*

$$|\chi(g)| \leq \chi(1)^{1-c\frac{\log|g^G|}{\log|G|}}.$$

Our main focus will be on *finite classical groups* $G = \mathrm{Cl}(V) = \mathrm{Cl}_n(q)$, where $V = \mathbb{F}_q^n$. For any $g \in \mathrm{Cl}(V)$, the *support* $\mathsf{supp}(g)$ is defined to be

$$\mathsf{supp}(g) = \inf_{\lambda \in \overline{\mathbb{F}_q}} \mathrm{codimKer}(g - \lambda \cdot 1_{\tilde{V}}),$$

where $\tilde{V} = V \otimes \overline{\mathbb{F}_q}$. Note that the ratios $\mathsf{supp}(g)/n$ and $(\log|g^G|)/(\log|G|)$ are of the same magnitude for all $g \in G = \mathrm{Cl}_n(q)$.

Recall that the notion of *level* of an irreducible character χ of a finite classical group $G = \mathrm{Cl}_n(q)$ was introduced in [4, 5]. Our next result gives a sharp estimate for the character ratio $\chi(g)/\chi(1)$ when both the level of χ and the support of g are not large compared to n:

Theorem 3. [7] *Let $n, m, j \in \mathbb{Z}_{\geq 1}$, q any prime power, and let $\varepsilon = \pm$. Suppose that*

$$n \geq 8j^2 + 4j + 4mj + 3$$

and χ is an irreducible complex character of G, where either $G = \mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, or $2 \not| q$ and $G = \mathrm{SO}_n^{\varepsilon}(q)$ of level j. Then there exists some root of unity $\beta \in \mathbb{C}$ such that when $g \in G$ has support m, we have

$$\Big|\frac{q^{mj}\chi(g)}{\chi(1)} - \beta\Big| < q^{-n/4}.$$

As an application of Theorems 2 and 3, we prove the asymptotic version of Thompson's Conjecture, which asserts that every finite non-abelian simple group G contains a conjugacy class $C \subset G$ such that $C^2 = G$. According to a private communication by Khukhro, the conjecture first appeared in Kourovka's Notebook as Problem 9.24 in 1984. It was communicated to Mazurov by Thompson in Oberwolfach in 1982, and although Thompson did not want to claim authorship, he consented to Mazurov describing it as "Thompson's problem". Within a year [1] it had achieved its modern name. In 1998, Ellers and Gordeev [3] reduced the conjecture to the case of finite simple groups of Lie type over fields of order ≤ 8.

Theorem 4. [7] *There exists a constant N such that if G is a finite non-abelian simple group of order $\geq N$, then G contains a conjugacy class C satisfying $C^2 = G$.*

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Vertex stabilizer amalgams of locally s-arc transitive graphs

John van Bon

Let Δ be a connected undirected graph, without loops or multiple edges. A *G-graph* is a graph Δ together with a subgroup $G \leq \mathrm{Aut}(\Delta)$. The vertex set of a G-graph Δ will be denoted by $V\Delta$, and the stabilizer in G of a vertex $z \in V\Delta$ by G_z. An *s-arc* emanating from a vertex $x_0 \in V\Delta$ is a path $(x_0, x_1, \ldots, x_s)$ with $x_{i-1} \neq x_{i+1}$ for $1 \leq i \leq s-1$. A G-graph Δ is called

- *thick* if the valency at each vertex is at least 3;
- *locally finite* if, for each $z \in V\Delta$, G_z is a finite group;
- *locally s-arc transitive* if, for every $z \in V\Delta$, G_z is transitive on the set of s-arcs emanating from z.

For a 1-arc (x_1, x_2) in Δ, the triple $(G_{x_1}, G_{x_2}; G_{x_1} \cap G_{x_2})$ is called the *vertex stabilizer amalgam* of Δ with respect to the 1-arc (x_1, x_2). It is easy to see that, for a locally s-arc transitive G-graph Δ with $s \geq 1$, the group G is transitive on edges. Therefore, when $s \geq 1$, the vertex stabilizer amalgam does not depend on the choice of the edge $\{x_1, x_2\}$ and is uniquely determined up to the order of the vertices in a 1-arc.

The main problem in the theory of locally s-arc transitive graphs is to bound s and to classify the vertex stabilizer amalgams for large s. The study of locally

s-arc transitive graphs was initiated by Tutte in [10, 11], where he showed that if Δ is a finite trivalent locally s-arc transitive graph with G transitive on the vertex set of Δ, then $s \leq 5$ and $|G_z| = 2^{s-1} \cdot 3$ for each $z \in V\Delta$ (here s is chosen as large as possible). This result was generalized by Goldschmidt [6], who proved that if Δ is a locally finite trivalent G-graph with $s \geq 1$, then $s \leq 7$ and the vertex stabilizer amalgam belongs to one of 15 isomorphism types.

About ten years ago Van Bon and Stellmacher [7] showed that for $s \geq 6$ the vertex stabilizer amalgam of a thick, locally finite and locally s-arc transitive G-graph must be a weak (B, N)-pair, hence is known by the classification of weak (B, N)-pairs obtained by Delgado and Stellmacher [9]. In particular, this yields $s \leq 9$. However, this is not the end of the story. For $4 \leq s \leq 5$ the structure of the vertex stabilizer amalgam also seems to be very restricted, and a classification of the vertex stabilizer amalgams might be feasible. Before going into some detail, we need to introduce some more notation and definitions. For a vertex $z \in V\Delta$, G_z acts on the set of its neighbors $\Delta(z)$. The kernel of this action is denoted by $G_z^{[1]}$, and the group induced by G_z on $\Delta(z)$ is denoted by $G_z^{\Delta(z)}$. The Thompson-Wielandt theorem ensures that, under some mild conditions, there exist a prime p and a vertex $z \in V\Delta$ for which $O_p(G_z^{[1]}) \neq 1$. This motivates the following definitions. A locally finite G-graph Δ is called

- *local characteristic* p, if there exists a prime p such that
$$C_{G_z}(O_p(G_z^{[1]})) \leq O_p(G_z^{[1]}), \text{ for all } z \in V\Delta;$$
- *pushing up type* with respect to the 1-arc (x, y) and the prime p, if Δ is of local characteristic p and
$$O_p(G_x^{[1]}) \leq O_p(G_y^{[1]}).$$

We divide the set of G-graphs into three distinct classes:

- Δ is of local characteristic p but not of pushing up type;
- Δ is of pushing up type;
- Δ is not of local characteristic p.

The beginning of the proof of [7] establishes that a thick, locally finite and locally s-arc transitive G-graph with $s \geq 6$, has to be of local characteristic p, but cannot be of pushing up type. Hence, the second and third class are empty in this case. This is no longer true when $s = 5$. However, the vertex stabilizer amalgams for thick, locally finite and locally s-transitive G-graphs with $s \geq 4$, which are not of local characteristic p were determined in [8]. Except for 2 vertex stabilizer amalgams, they all belong to one infinite family. In all cases, the corresponding G-graphs have $s = 5$. We expect that the vertex stabilizer amalgams of thick, locally finite and locally s-arc transitive G-graphs that are of local characteristic p, but not of pushing up type, are weak BN-pairs when $s \geq 4$.

The classification of vertex stabilizer amalgams of thick, locally finite and locally s-arc transitive G-graphs of pushing up type with $s \geq 4$ is in progress. At the start of this project no such G-graphs were known. In [5] it was shown that a G-graph of pushing up type with $s \geq 4$ will have a vertex z with $|\Delta(z)| = q + 1$

and $\mathrm{PSL}_n(q) \trianglelefteq G_x^{\Delta(z)} \leq \mathrm{P\Gamma L}_2(q)$, where q is an odd prime power. Furthermore, in [1] it was shown that exactly four vertex stabilizer amalgams are possible when $q = 3$, with corresponding G-graphs constructed in [2, 3, 4]. Additional G-graphs have been constructed for various other values of q in [3, 4], yielding new shapes of vertex stabilizer amalgams. The current belief is that the amalgams found so far are the only ones that can exist.

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Pseudo-quadratic forms over simple artinian rings with involution

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(joint work with Bernhard Mühlherr)

1. Introduction

The notion a pseudo-quadratic form was introduced independently in the late 1960s by Bak and Tits. In this talk we give a uniform description of the pseudo-maximal parabolic subgroups of an arbitrary classical group in terms of pseudo-quadratic modules defined over simple artinian rings with involution. Proofs of the assertions we make will appear in [2].

2. A result of Jacobson

We call a ring with involution (k, τ) a *Jacobson pair* if either k is a skew field and the involution τ is arbitrary or $k = \ell^{\mathrm{op}} \oplus \ell$ for some skew field ℓ and $\tau(a, b) = (b, a)$ for all $(a, b) \in k$. In this second case, we set $\mathrm{J}(\ell) = (k, \tau)$. Suppose that (F, ρ) is a ring with involution and that z is an element of $F^\times$ such that $z^\rho = \pm z$. We set $\rho_z(a) = za^\tau z^{-1}$ for all $a \in F$. Then ρ_z is also an involution. We call (F, ρ_z) an *isotope* of (F, ρ). A ring with involution (K, σ) is *standard* if for some $m \geq 1$ and some Jacobson pair (k,τ), K is the matrix ring $M_m(k)$ and σ is the conjugate-transpose map, where conjugation means "apply τ to every entry." In [1], Jacobson showed that if (F,ρ) is a simple ring with involution and F is artinian, then there exist $m \geq 1$, a Jacobson pair (k,τ) and an isomorphism from an isotope of (F,ρ) to the standard ring with involution (K,σ) determined by m and (k,τ).

3. Pseudo-quadratic modules

Let Ω be a *pseudo-quadratic module*. This means that Ω is a 7-tuple

$$(K, K_0, \sigma, \varepsilon, L, Q, H),$$

where (K,σ) is a ring with involution, $\varepsilon = \pm 1 \in K$, $K_0 = \{a - \varepsilon a^\sigma \mid a \in K\}$, L is a right K-module, H is a ε-hermitian form on L and Q is a map from L to K satisfying

(i) $Q(at) \equiv t^\sigma Q(a)t \pmod{K_0}$ and
(ii) $Q(a+b) \equiv Q(a) + Q(b) + H(a,b) \pmod{K_0}$

for all $a, b \in L$ and all $t \in K$. If $z \in K^\times$ and $z^\sigma = \delta z$ for $\delta = \pm 1$, then

$$\Omega_z = (K, zK_0, \sigma_z, \delta\varepsilon, L, zQ, zH)$$

is also a pseudo-quadratic module. We call Ω_z an *isotope* of Ω and define two pseudo-quadratic modules to be *similar* if one is isomorphic to an isotope of the other.

4. The group P_Θ

Let $\Theta = (K, K_0, \sigma, \varepsilon, L, Q, H)$ be a pseudo-quadratic module. We set $U_\Theta = \{(a,t) \in L \times K \mid q(a) - t \in K_0\}$ and make U_Θ into a group by setting

$$(a,t) + (b,s) = \big(a+b, s+t+H(a,b)\big)$$

for all $(a,t),(b,s) \in U_\Theta$. We write the group U_Θ additively even though it is not, in general, abelian. Let M denote the subgroup of $\mathrm{Aut}(U_\Theta)$ consisting of the maps $(a,t) \mapsto (as, s^\sigma ts)$ for all $s \in K^\times$ and let N denote the subgroup of $\mathrm{Aut}(U_\Theta)$ consisting of the maps $(a,t) \mapsto (\varphi(a), t)$ for all $\varphi \in \mathrm{Isom}(\Theta)$. The subgroups M and N commute elementwise. We denote by P_Θ the semi-direct product $U_\Theta \cdot MN$. If Θ' is similar to Θ, then $U_{\Theta'} \cong U_\Theta$.

5. Tensor products

Let $\Omega = (K, K_0, \sigma, \varepsilon, L, Q, H)$ be a pseudo-quadratic module, let (k,τ) be a Jacobson pair and let $m \geq 1$. Suppose that (K,σ) is the standard ring with involution determined by m and (k,τ). Then there exists a pseudo-quadratic module

$$\omega = (k, k_0, \tau, \varepsilon, V, q, h)$$

defined over (k,τ) from which Ω can be reconstructed. In particular,

$$L \cong V \otimes_k M_{1,m}(k),$$

where $M_{1,m}(k)$ is the space of $1 \times m$ matrices over k, so as an additive group, we can identify L with $\{(v_1, \dots, v_m) \mid v_i \in L \text{ for all } i\}$, and

$$H\big((v_1, \dots, v_m), (v'_1, \dots, v'_m)\big) = [h(v_i, v'_j)] \in M_m(k) = K$$

for all $(v_1, \dots, v_m), (v'_1, \dots, v'_m) \in L$. We can also start with an arbitrary pseudo-quadratic module ω defined over a Jacobson pair (k, τ) and an integer $m \geq 1$ and apply our construction to obtain a unique pseudo-quadratic module defined over the standard ring with involution (K, σ) determined by m and (k, τ). We denote this pseudo-quadratic module by $\omega \otimes_k K$.

6. Classical groups

Let G be a group. We say that G is a *classical group* if $G \cong \mathrm{Isom}(\omega)$, where ω is a non-degenerate pseudo-quadratic module defined over a Jacobson pair (k, τ) of finite Witt index n. If we are in the case that $(k, \tau) = \mathrm{J}(\ell)$ for some skew field ℓ, then $G \cong GL(Z)$, where Z is a right vector space over ℓ of dimension n.

7. Pseudo-maximal parabolic subgroups

Let $\omega = (k, k_0, \tau, \varepsilon, V, f, h)$ and n be as in the previous section. We assume that $G = \mathrm{Isom}(\omega)$ if k is a skew field and $G = GL(Z)$ if $(k, \tau) = \mathrm{J}(\ell)$ for some ℓ. Let P be a subgroup of G. We say that P is a *pseudo-maximal parabolic subgroup* of G if either k is a skew field and P is the stabilizer of a totally isotropic subspace of V of dimension m for some positive integer $m \leq n$ or $(k, \tau) = \mathrm{J}(\ell)$ and P is the stabilizer of a flag $X \subset Y$ such that $m := \dim_\ell X = \dim_\ell(Z/Y)$ for some positive integer $m < n/2$. In both cases, we call m the *degree* of the pseudo-maximal parabolic subgroup P.

If k is a skew field, every maximal parabolic subgroup is pseudo-maximal. If either $(k, \tau) = \mathrm{J}(\ell)$ for some ℓ or k is a skew field, ω is hyperbolic, $k_0 = 0$, $\varepsilon = 1$ and $m = n - 1$, however, the corresponding pseudo-maximal subgroup is the intersection of two maximal parabolic subgroups.

8. Main theorem

Let $G = \mathrm{Isom}(\omega)$ for some non-degenerate pseudo-quadratic module

$$\omega = (k, k_0, \tau, \varepsilon, V, q, h)$$

defined over a Jacobson pair (k, τ) of finite Witt index. Let P be a pseudo-maximal subgroup of degree m of G, let $K = M_m(k)$, let W be a hyperbolic submodule of V of Witt index m and let θ denote the restriction of ω to $W^\perp$. We let Θ denote the pseudo-quadratic module $\theta \otimes_k K$ obtained from θ and m by our tensor product construction. Then Θ is non-degenerate, its Witt index is finite, and

$$P \cong P_\Theta,$$

where P_Θ is the group defined in Section 4. Furthermore, for every non-degenerate pseudo-quadratic pseudo-quadratic module Ω defined over a simple artinian ring with involution of finite Witt index, the group P_Ω is isomorphic to a pseudo-maximal parabolic subgroup of some classical group.

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