

Pfister’s local-global principle for Azumaya algebras with involution

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Abstract. We prove Pfister’s local-global principle for hermitian forms over Azumaya algebras with involution over semilocal rings, and show in particular that the Witt group of nonsingular hermitian forms is 2-primary torsion. Our proof relies on a hermitian version of Sylvester’s law of inertia, which is obtained from an investigation of the connections between a pairing of hermitian forms extensively studied by Garrel, signatures of hermitian forms, and positive semidefinite quadratic forms.

1. Introduction

Pfister’s local-global principle is a fundamental result in the algebraic theory of quadratic forms over fields. It states that the torsion in the Witt ring is 2-primary, and that a nonsingular quadratic form represents a torsion element in this ring if and only if its signature (the difference between the number of positive and the number of negative entries in any diagonalisation of the form according to Sylvester’s “law of inertia”) is zero at all orderings of the field. The above facts are of course well known, and can easily be found in standard references such as [28] or [19].

The main result of this paper (Theorem 6.6) is Pfister’s local-global principle for nonsingular hermitian forms over Azumaya algebras with involution over semilocal commutative rings in which 2 is a unit. (The special case of central simple algebras with involution was treated in [21, 27]; see also [7].) The assumption that the base ring is semilocal is minimal in the sense that Pfister’s local-global principle is known to hold for nonsingular quadratic forms over such rings, but not in general. We refer to [5] for more details.

Our version of Sylvester’s law of inertia (Theorem 6.1) is used in the proof of the main result, which is inspired by Marshall’s proof of Pfister’s local-global principle in the context of abstract Witt rings, cf. [23]. A crucial ingredient in our proof is a certain pairing of forms investigated by Garrel in his 2023 paper [10]. We were also very fortunate that we could put many results from the recent papers by First [8] and Bayer–Flückiger, First and Parimala [6] to good use.

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2. Preliminaries

In this paper all rings are assumed unital and associative with 2 invertible. We identify quadratic forms over commutative rings with symmetric bilinear forms, and assume that all fields are of characteristic different from 2. Rings are not assumed to be commutative unless explicitly indicated. Our main references for rings with involution and hermitian forms are [16] and [8].

2.1. Hermitian forms over rings with involution

Let (A, σ) be a ring with involution and let $\varepsilon \in Z(A)$ be such that $\varepsilon\sigma(\varepsilon) = 1$. We denote the category of ε -hermitian modules over (A, σ) by $\mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$. The objects of $\mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$ are pairs (M, h) , where M is a finitely generated projective right A -module and

$$h: M \times M \rightarrow A$$

is an ε -hermitian form. Since we always assume that $2 \in A^\times$, all hermitian modules are even. We denote the category of nonsingular ε -hermitian modules over (A, σ) (also known as ε -hermitian spaces) by $\mathfrak{S}^\varepsilon(A, \sigma)$. The morphisms of $\mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$ and $\mathfrak{S}^\varepsilon(A, \sigma)$ are the isometries. We denote isometry by \simeq . See [16, I, Sections 2 and 3] for more details. If $\varepsilon = 1$, we simply write $\mathfrak{S}\text{erm}(A, \sigma)$ and $\mathfrak{S}(A, \sigma)$. It is common to say hermitian form instead of hermitian module.

We denote the Witt group of nonsingular ε -hermitian forms over (A, σ) by $W^\varepsilon(A, \sigma)$ and note that since $2 \in A^\times$, metabolic forms are hyperbolic, cf. [8, Section 2.2] for a succinct presentation.

For $a \in A^\times$, we denote the inner automorphism $A \rightarrow A$, $x \mapsto axa^{-1}$ by $\text{Int}(a)$. We define $\text{Sym}^\varepsilon(A, \sigma) := \{x \in A \mid \sigma(x) = \varepsilon x\}$ and $\text{Sym}^\varepsilon(A^\times, \sigma) := \text{Sym}^\varepsilon(A, \sigma) \cap A^\times$. We also write $\text{Sym}(A, \sigma)$ instead of $\text{Sym}^1(A, \sigma)$ and $\text{Skew}(A, \sigma)$ instead of $\text{Sym}^{-1}(A, \sigma)$. For $a_1, \dots, a_\ell \in \text{Sym}^\varepsilon(A, \sigma)$ we denote by $\langle a_1, \dots, a_\ell \rangle_\sigma$ the diagonal ε -hermitian form

$$A^\ell \times A^\ell \rightarrow A, \quad (x, y) \mapsto \sum_{i=1}^{\ell} \sigma(x_i) a_i y_i.$$

Let $(M, h) \in \mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$. We denote by $D_{(A, \sigma)}(h) := \{h(x, x) \mid x \in M\}$ the set of elements of A represented by h .

If $(M, h) \in \mathfrak{S}^\varepsilon(A, \sigma)$, the *adjoint involution of h* is the involution ad_h on the ring $\text{End}_A(M)$ implicitly defined by

$$h(x, \text{ad}_h(f)(y)) = h(f(x), y) \quad (2.1)$$

for all $x, y \in M$ and all $f \in \text{End}_A(M)$, cf. [8, Section 2.4].

Consider a second ring with involution (B, τ) and let (S, ι) be a commutative ring with involution such that (A, σ) and (B, τ) are (S, ι) -algebras with involution in the sense of [16, I, (1.1)], i.e., A and B are both S -algebras and the involutions σ and τ are compatible with ι :

$$\sigma(sa) = \iota(s)\sigma(a), \quad \tau(sb) = \iota(s)\tau(b), \quad \forall a \in A, b \in B, s \in S.$$

Then $(A \otimes_S B, \sigma \otimes \tau)$ is an (S, ι) -algebra with involution and, if $(M_1, h_1) \in \mathfrak{S}\text{erm}^{\varepsilon_1}(A, \sigma)$ and $(M_2, h_2) \in \mathfrak{S}\text{erm}^{\varepsilon_2}(B, \tau)$, then

$$(M_1 \otimes_S M_2, h_1 \otimes h_2) \in \mathfrak{S}\text{erm}^{\varepsilon_1 \varepsilon_2}(A \otimes_S B, \sigma \otimes \tau).$$

If $(M_1, h_1) \simeq (M'_1, h'_1)$ in $\mathfrak{S}\text{erm}^{\varepsilon_1}(A, \sigma)$ and $(M_2, h_2) \simeq (M'_2, h'_2)$ in $\mathfrak{S}\text{erm}^{\varepsilon_2}(B, \tau)$, then

$$(M_1 \otimes_S M_2, h_1 \otimes h_2) \simeq (M'_1 \otimes_S M'_2, h'_1 \otimes h'_2).$$

Let $(M, h) \in \mathfrak{S}\text{erm}^{\varepsilon}(A, \sigma)$ and $(N, \varphi) \in \mathfrak{S}\text{erm}^{\mu}(S, \iota)$. Since $(A \otimes_S S, \sigma \otimes \iota) \cong (A, \sigma) \cong (S \otimes_S A, \iota \otimes \sigma)$ as rings with involution, it is not difficult to see that upon identifying $A \otimes_S S$ with A ,

$$h \otimes_S \varphi(m_1 \otimes n_1, m_2 \otimes n_2) = h(m_1, m_2)\varphi(n_1, n_2)$$

for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$, and

$$(M \otimes_S N, h \otimes_S \varphi) \simeq (N \otimes_S M, \varphi \otimes_S h)$$

in $\mathfrak{S}\text{erm}^{\varepsilon\mu}(A, \sigma)$.

Lemma 2.1. *Let R be a commutative ring, assume that A is an R -algebra and that σ is an R -linear involution on A . Let $\iota = \sigma|_{Z(A)}$. Let $u_1, \dots, u_k \in R$ and $(M, h) \in \mathfrak{S}\text{erm}^{\varepsilon}(A, \sigma)$. Then*

$$\langle u_1, \dots, u_k \rangle_{\iota} \otimes_{Z(A)} h \simeq \langle u_1, \dots, u_k \rangle \otimes_R h$$

(under the canonical identifications $Z(A) \otimes_{Z(A)} A \cong R \otimes_R A \cong A$).

Proof. It suffices to show that $\langle u \rangle_{\iota} \otimes_{Z(A)} h \simeq \langle u \rangle \otimes_R h$ for $u \in R$. This follows from the isometries $\langle u \rangle_{\iota} \simeq \langle u \rangle \otimes_R \langle 1 \rangle_{\iota}$, $\langle 1 \rangle_{\iota} \otimes_{Z(A)} h \simeq h$ (which are straightforward, using the observations preceding the lemma) and associativity of the tensor product. ■

We finish this section with a well-known result for which we could not find a reference.

Lemma 2.2. *Let R be a commutative ring, assume that A and B are R -algebras and that σ and τ are R -linear involutions on A and B , respectively. Let $(M, \varphi) \in \mathfrak{S}^{\varepsilon_1}(A, \sigma)$ and $(N, \psi) \in \mathfrak{S}^{\varepsilon_2}(B, \tau)$. Then the map*

$$\xi: \text{End}_A(M) \otimes_R \text{End}_B(N) \rightarrow \text{End}_{A \otimes_R B}(M \otimes_R N)$$

induced by $\xi(f \otimes g) = [x \otimes y \mapsto f(x) \otimes g(y)]$ yields an isomorphism of R -algebras with involution

$$(\text{End}_A(M) \otimes_R \text{End}_B(N), \text{ad}_{\varphi} \otimes \text{ad}_{\psi}) \cong (\text{End}_{A \otimes_R B}(M \otimes_R N), \text{ad}_{\varphi \otimes \psi}).$$

Proof. The map ξ is an isomorphism of R -algebras by [9, Theorem 1.3.26 and Corollary 1.3.27]. To finish the proof it suffices to check that

$$\text{ad}_{\varphi \otimes \psi}(\xi(f \otimes g)) = \xi(\text{ad}_{\varphi}(f) \otimes \text{ad}_{\psi}(g))$$

using the definition of adjoint involution (2.1), which is a straightforward computation. ■

2.2. Quadratic étale algebras

Let R be a commutative ring and let S be a quadratic étale R -algebra. We recall some results from [8, Section 1.3] and [16, I, (1.3.6)]. The algebra S has a unique standard involution ϑ , and the trace $\text{Tr}_{S/R}$ satisfies

$$\text{Tr}_{S/R}(x) = \vartheta(x) + x \quad \text{for all } x \in S. \quad (2.2)$$

Furthermore, $\text{Tr}_{S/R}$ is an involution trace for ϑ (cf. [16, I, Proposition 7.3.6]) and thus if $h \in \mathfrak{S}\text{erm}(S, \vartheta)$ is nonsingular, then $\text{Tr}_{S/R}(h)$ is nonsingular by [16, I, Proposition 7.2.4]. Furthermore, if h is hyperbolic, then so is the quadratic form $\text{Tr}_{S/R}(h)$ by the first paragraph of [16, p. 41]. The converse holds if R is semilocal by [8, Corollary 8.3].

If R is connected and S is not connected, then $S \cong R \times R$ as R -algebras, and

$$\vartheta: (x, y) \mapsto (y, x)$$

is the exchange involution.

If R is semilocal, then there exists $\lambda \in S$ such that $\lambda^2 \in R^\times$, $\vartheta(\lambda) = -\lambda$, and $\{1, \lambda\}$ is an R -basis of S , cf. [8, Lemma 1.19].

2.3. Azumaya algebras with involution

Let R be a commutative ring. Recall from [16, III, (5.1)] that an R -algebra A is an *Azumaya R -algebra* if A is a faithful finitely generated projective R -module and the *sandwich map*

$$\text{sw}: A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A), \quad a \otimes b^{\text{op}} \mapsto [x \mapsto axb] \quad (2.3)$$

is an isomorphism of R -algebras. Here A^{op} denotes the *opposite algebra* of A , which coincides with A as an R -module, but with twisted multiplication $a^{\text{op}}b^{\text{op}} = (ba)^{\text{op}}$. It is clear that A^{op} is also an Azumaya R -algebra.

The centre $Z(A)$ is equal to R and $\text{End}_R(A)$ is again an Azumaya R -algebra. More generally, if M is a faithful finitely generated projective right R -module, then $\text{End}_R(M)$ is an Azumaya R -algebra. If A and B are Azumaya R -algebras, their tensor product $A \otimes_R B$ is again an Azumaya R -algebra.

First's paper [8] contains a wealth of information about Azumaya algebras, with and without involution, and we refer to it for a number of definitions and results that we recall in the remainder of this section.

Proposition 2.3 ([8, Proposition 1.1]). *A is Azumaya over $Z(A)$ and $Z(A)$ is finite étale over R if and only if A is projective as an R -module and separable as an R -algebra.* ■

Since the behaviour of the involution on the centre plays an important role in the study of algebras with involution, this result helps motivate the following.

Definition 2.4 ([8, Section 1.4]). We say that (A, σ) is an *Azumaya algebra with involution over R* if the following conditions hold:

- A is an R -algebra with R -linear involution σ ;

- A is separable projective over R ;
- the homomorphism $R \rightarrow A, r \mapsto r \cdot 1_A$ identifies R with $\text{Sym}(Z(A), \sigma)$.

Let (A, σ) be an Azumaya algebra with involution over R . Note that A is Azumaya over $Z(A)$ by Proposition 2.3, but may not be Azumaya over R . Indeed, “Azumaya algebra with involution” means “Azumaya algebra-with-involution” rather than “Azumaya-algebra with involution”.

The following lemma makes the connection between Definition 2.4 and a different definition of Azumaya algebra with involution (the first sentence of Lemma 2.5) that is introduced in [25, Section 4].

Lemma 2.5. *Let A be an R -algebra with R -linear involution such that A is an Azumaya algebra over $Z(A)$, $Z(A)$ is R or a quadratic étale extension of R , and $R = \text{Sym}(Z(A), \sigma)$. Then (A, σ) is an Azumaya algebra with involution over R .*

The converse holds if R is connected.

Proof. The first statement is a direct consequence of Proposition 2.3. For the converse, assume that R is connected and that (A, σ) is an Azumaya algebra with involution over R . Then A is Azumaya over $Z(A)$ by Proposition 2.3, and $Z(A)$ is R or a quadratic étale extension of R by [8, Proposition 1.21]. ■

Remark 2.6. If $R = F$ is actually a field, then (A, σ) is an Azumaya algebra with involution over R if and only if it is a *central simple F -algebra with involution* in the sense of [17, Sections 2.A and 2.B].

We recall the following, proved in [8, second paragraph of Section 1.4].

Proposition 2.7 (Change of base ring). *Let T be a commutative R -algebra. Then $Z(A \otimes_R T) = Z(A) \otimes_R T$ and $(A \otimes_R T, \sigma \otimes \text{id})$ is an Azumaya algebra with involution over T (and in particular $Z(A \otimes_R T) \cap \text{Sym}(A \otimes_R T, \sigma \otimes \text{id}) = T$).* ■

Corollary 2.8. *Let T be a commutative R -algebra which is also a field. Then either $Z(A \otimes_R T) = T$ or $Z(A \otimes_R T)$ is a quadratic étale extension of T , and $(A \otimes_R T, \sigma \otimes \text{id})$ is a central simple T -algebra with involution.* ■

Remark 2.9. Let S be a quadratic étale R -algebra with standard involution ϑ . If T is a commutative R -algebra, then $S \otimes_R T$ is a quadratic étale T -algebra, cf. [9, Propositions 2.3.4 and 9.2.5].

Moreover, (S, ϑ) is an Azumaya algebra with involution over R in the sense of [25, Section 4] (i.e., it has the properties listed in the first sentence of Lemma 2.5), and therefore an Azumaya algebra with involution over R in the sense of [8] (i.e., in the sense of this paper), by Lemma 2.5. Therefore $T = \text{Sym}(S \otimes_R T, \vartheta \otimes \text{id})$ by Proposition 2.7, and it follows that $\vartheta \otimes \text{id}$ is the standard involution of $S \otimes_R T$. Then (2.2) applies and yields, for all $x \in S$,

$$\text{Tr}_{S \otimes_R T/T}(x \otimes 1) = \text{Tr}_{S/R}(x) \otimes 1.$$

We collect some results about ideals of A .

Proposition 2.10. *The following properties hold:*

- (1) *If I is an ideal of R , then $A \otimes_R (R/I) \cong A/AI$;*
- (2) *$J(A) = A \cdot J(R)$, where J denotes the Jacobson radical;*
- (3) *If $a \in A$ is such that $a \in (A/\mathfrak{m})^\times$ for every maximal ideal \mathfrak{m} of R , then $a \in A^\times$.*

Proof. (1) This is well known. Item (2) is [8, Lemma 1.5], since A is a separable projective R -algebra. Item (3) is [8, Lemma 1.6], using item (1). ■

Definition 2.11 (See [8, Section 1.4]). We say that the involution σ on A is of *orthogonal, symplectic or unitary type* at $\mathfrak{p} \in \text{Spec } R$ if $(A \otimes_R \text{qf}(R/\mathfrak{p}), \sigma \otimes \text{id})$ is a central simple algebra with involution of orthogonal, symplectic or unitary type, respectively, over the quotient field $\text{qf}(R/\mathfrak{p})$, and that σ is of *orthogonal, symplectic or unitary type* if it is of orthogonal, symplectic or unitary type, respectively, at all $\mathfrak{p} \in \text{Spec } R$.

We often simply say that σ is orthogonal, symplectic or unitary (at \mathfrak{p}). We refer to [17] for the definitions of orthogonal, symplectic and unitary involutions on central simple algebras.

Proposition 2.12 (See [8, Proposition 1.21] for a more detailed statement). *Assume that R is connected. Then precisely one of the following holds:*

- (1) *σ is of orthogonal type, $Z(A) = R$, and $\sigma|_{Z(A)} = \text{id}_{Z(A)}$;*
- (2) *σ is of symplectic type, $Z(A) = R$, and $\sigma|_{Z(A)} = \text{id}_{Z(A)}$;*
- (3) *σ is of unitary type, $Z(A)$ is a quadratic étale R -algebra, and $\sigma|_{Z(A)}$ is the standard involution on $Z(A)$.* ■

Remark 2.13. If (A, σ) is a central simple algebra over a field, the involution σ is called *of the first kind* if $\sigma|_{Z(A)} = \text{id}_{Z(A)}$ and *of the second kind* otherwise (so that “second kind” is the same as “unitary type”), cf. [17].

Lemma 2.14. *Let R be semilocal connected, let T be a quadratic étale R -algebra, and let τ be an R -linear involution on T . If T is connected, then every nonsingular hermitian form over (T, τ) is diagonalizable.*

Proof. The algebra T is semilocal by [16, VI, Proposition 1.1.1]. Since T is also connected, every projective T -module is free by [11]. Furthermore, $2 \in T^\times$.

If $\tau = \text{id}$, the result follows since every quadratic form over T is diagonalizable (see [5, Proposition 3.4]).

If $\tau \neq \text{id}$. Then τ is the standard involution on T by [8, Lemma 1.17], and (T, τ) is an Azumaya algebra with involution over R , cf. Remark 2.9. Since $Z(T) \neq R$, and in particular τ is not symplectic, we may conclude by [8, Proposition 2.13]. ■

Theorem 2.15 (Hermitian Morita equivalence). *Let R be semilocal connected and let (B, τ) be an Azumaya algebra with involution over R such that A and B are Brauer*

equivalent and $\sigma|_S = \tau|_S$, where $S := Z(A) = Z(B)$. Assume that S is connected. Then there exists $\delta \in \{-1, 1\}$ such that the categories $\mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$ and $\mathfrak{S}\text{erm}^{\delta\varepsilon}(B, \tau)$ are equivalent, and this equivalence induces an isomorphism of Witt groups

$$W^\varepsilon(A, \sigma) \cong W^{\delta\varepsilon}(B, \tau).$$

Specifically, if $\sigma|_S = \text{id}_S$, then $\delta = 1$ if σ and τ are both orthogonal or both symplectic and $\delta = -1$ otherwise; if $\sigma|_S \neq \text{id}_S$, then δ can be chosen freely in $\{-1, 1\}$.

Proof. If $\sigma|_S \neq \text{id}_S$, then S is a quadratic étale R -algebra by Proposition 2.12 and is thus semilocal by [16, VI, Proposition 1.1.1].

Since A and B are Brauer equivalent, there exist faithful finitely generated projective S -modules P and Q such that $A \otimes_S \text{End}_S(P) \cong B \otimes_S \text{End}_S(Q)$, cf. [16, III, (5.3)]. Since S is semilocal connected, P and Q are free by [11]. It follows that there exist $k, \ell \in \mathbb{N}$ such that

$$\text{End}_A(A^k) \cong \text{End}_B(B^\ell). \quad (2.4)$$

Consider the nonsingular hermitian forms $(A^k, \varphi := k \times \langle 1 \rangle_\sigma)$ and $(B^\ell, \psi := \ell \times \langle 1 \rangle_\tau)$ over (A, σ) and (B, τ) , respectively. Since φ and ψ are hermitian, the involutions σ and ad_φ are of the same type and the same is true for the involutions τ and ad_ψ , cf. [8, Proposition 2.11]. In particular, $\sigma|_S = \text{ad}_\varphi|_S$ and $\tau|_S = \text{ad}_\psi|_S$.

The categories $\mathfrak{S}\text{erm}^\varepsilon(\text{End}_A(A^k), \text{ad}_\varphi)$ and $\mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$ are equivalent by hermitian Morita theory. The same is true for the categories $\mathfrak{S}\text{erm}^\varepsilon(\text{End}_B(B^\ell), \text{ad}_\psi)$ and $\mathfrak{S}\text{erm}^\varepsilon(B, \tau)$. Furthermore, these equivalences respect orthogonal sums and send hyperbolic spaces to hyperbolic spaces, cf. [16, I, Theorem 9.3.5].

The involution ad_φ induces an involution ω on $\text{End}_B(B^\ell)$ of the same type via the isomorphism (2.4), and thus $\omega|_S = \text{ad}_\psi|_S$ since $\sigma|_S = \tau|_S$. By the Skolem–Noether theorem [6, Theorem 8.6], ω differs from ad_ψ by an inner automorphism: there exists $\delta \in \{-1, 1\}$ and a unit $u \in \text{End}_B(B^\ell)$ with $\text{ad}_\psi(u) = \delta u$ such that $\omega = \text{Int}(u) \circ \text{ad}_\psi$, where δ can be freely chosen in $\{-1, 1\}$ if $\sigma|_S \neq \text{id}_S$. Moreover, δ is unique if $\sigma|_S = \text{id}_S (= \text{ad}_\psi|_S)$: if $\text{Int}(u) = \text{Int}(u')$, then $u = u's$ for some $s \in S^\times$, and it follows that $\text{ad}_\psi(u) = \delta u$ if and only if $\text{ad}_\psi(u') = \delta u'$.

By [16, I, (5.8)] (see also [8, Section 2.7]) there is an equivalence between the categories $\mathfrak{S}\text{erm}^\varepsilon(\text{End}_B(B^\ell), \omega)$ and $\mathfrak{S}\text{erm}^{\delta\varepsilon}(\text{End}_B(B^\ell), \text{ad}_\psi)$ (and an induced equivalence between the categories $\mathfrak{S}^\varepsilon(\text{End}_B(B^\ell), \omega)$ and $\mathfrak{S}^{\delta\varepsilon}(\text{End}_B(B^\ell), \text{ad}_\psi)$) which respects orthogonal sums and hyperbolic spaces. The equivalence of the categories $\mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$ and $\mathfrak{S}\text{erm}^{\delta\varepsilon}(B, \tau)$ and the isomorphism of the Witt groups $W^\varepsilon(A, \sigma)$ and $W^{\delta\varepsilon}(B, \tau)$ follows.

It remains to show the claim about δ when $\sigma|_S = \text{id}_S$. Since δ is unique, it suffices to check its value at any $\mathfrak{p} \in \text{Spec } R$ (i.e., after tensoring over R by $\text{qf}(R/\mathfrak{p})$). By [17, Proposition 2.7] we know that $\delta = 1$ if σ and τ are both orthogonal or both symplectic at \mathfrak{p} and $\delta = -1$ otherwise. The statement then follows, using Proposition 2.12. ■

Remark 2.16. Let R be connected and let (A, σ) be an Azumaya algebra with involution over R . Note that when $Z(A)$ is not connected, then $Z(A) \cong R \times R$ by Section 2.2 and Lemma 2.5. Hence, $W^\varepsilon(A, \sigma) = 0$, cf. [8, Example 2.4].

2.4. Orderings, spaces of signatures

Let R be a commutative ring, and let (A, σ) be an Azumaya algebra with involution over R . The real spectrum of R , $\text{Sper } R$, is the set of all orderings on R , cf. [15, Definitions 3.3.1 (a) and 3.3.4, and Proposition 3.3.5]. It is equipped with the Harrison topology, with subbasis given by the sets of the form

$$\mathring{H}(r) := \{\alpha \in \text{Sper } R \mid r > 0 \text{ at } \alpha\} = \{\alpha \in \text{Sper } R \mid r \in \alpha \setminus -\alpha\},$$

for all $r \in R$.

Let $\alpha \in \text{Sper } R$. We often write $x \geq_\alpha y$ for $x - y \in \alpha$. The support of α is the prime ideal $\text{Supp}(\alpha) := \alpha \cap -\alpha \in \text{Spec } R$, and we denote by $\bar{\alpha}$ the ordering induced by α on the quotient field $\kappa(\alpha) := \text{qf}(R/\text{Supp}(\alpha))$, and by $k(\alpha)$ a real closure of $\kappa(\alpha)$ at $\bar{\alpha}$. Observe that $\kappa(\bar{\alpha}) := \text{qf}(\kappa(\alpha)/\text{Supp}(\bar{\alpha})) = \text{qf}(\kappa(\alpha)/\{0\}) = \kappa(\alpha)$, and that we can take $k(\bar{\alpha}) = k(\alpha)$.

We also define

$$(A(\alpha), \sigma(\alpha)) := (A \otimes_R \kappa(\alpha), \sigma \otimes \text{id}),$$

which is a central simple $\kappa(\alpha)$ -algebra with involution by Corollary 2.8.

Remark 2.17. Let M be an R -module and let $\alpha \in \text{Sper } R$. Then, if $x \in M \otimes_R \kappa(\alpha)$, there are $t \in \mathbb{N}$, $m_i \in M$, $s_i \in R$ and $r_i \in R \setminus \text{Supp}(\alpha)$ such that

$$x = \sum_{i=1}^t m_i \otimes \frac{\bar{s}_i}{\bar{r}_i} = \sum_{i=1}^t m_i \otimes \frac{\bar{s}'_i}{\bar{r}'_i} = \left(\sum_{i=1}^t m_i s'_i \right) \otimes \frac{1}{\bar{r}'},$$

for some $s'_i \in R$, $r' \in R \setminus \text{Supp}(\alpha)$. Therefore

$$M \otimes_R \kappa(\alpha) = \left\{ m \otimes \frac{1}{\bar{r}} \mid m \in M, r \in R \setminus \text{Supp}(\alpha) \right\}.$$

We denote by $\text{Sign } R$ the set of signatures of R , i.e., the space of all morphisms of rings from the Witt ring $W(R)$ to \mathbb{Z} . We recall some facts from [14, Section 5 up to p. 89]:

- The set $\text{Sign } R$ is equipped with the coarsest topology that makes all maps

$$\text{sign}_\bullet \varphi: \text{Sper } R \rightarrow \mathbb{Z}, \quad \alpha \mapsto \text{sign}_\alpha \varphi$$

continuous, for $\varphi \in W(R)$. When R is semilocal, a basis for this topology is given by the sets

$$H(u_1, \dots, u_k) := \{\tau \in \text{Sign } R \mid \tau(u_1) = \dots = \tau(u_k) = 1\},$$

for $k \in \mathbb{N}$ and $u_1, \dots, u_k \in R^\times$.

- Denoting by $\text{Sper}^{\max} R$ the space of all elements of $\text{Sper } R$ that are maximal for inclusion (equipped with the induced topology), the natural map

$$\text{Sper}^{\max} R \rightarrow \text{Sign } R, \quad \alpha \mapsto \text{sign}_\alpha,$$

is continuous and surjective. If R is semilocal, this map is a homeomorphism and we identify $\text{Sign } R$ and $\text{Sper}^{\max} R$. In general, we have continuous maps

$$\text{Sper}^{\max} R \subseteq \text{Sper } R \xrightarrow{\xi} \text{Spec } R,$$

where ξ is defined by $\xi(\alpha) := \text{Supp}(\alpha)$. (Note that if $R = F$ is a field, then $\text{Sper}^{\max} R = \text{Sper } R = X_F$, the space of orderings of F .)

The following theorems, due to Knebusch, explain how to obtain the maximal ordering associated to a signature on a semilocal ring. These results can be found in [13, Theorem 4.8] and [14, pp. 87–88]. The connection with maximal orderings is given in the second reference, but is presented for connected rings. It is pointed out that this assumption can be made without loss of generality, but we quickly present an argument in the proof below.

Theorem 2.18. *Assume that R is semilocal and let s be a signature on R . Define*

$$Q(s) := \left\{ r_1^2 u_1 + \cdots + r_k^2 u_k \mid k \in \mathbb{N}, u_i \in R^\times, s(\langle u_i \rangle) = 1, \sum_{i=1}^k R r_i = R \right\}.$$

Then

- (1) $Q(s)$ is closed under sum, $Q(s) \cap -Q(s) = \emptyset$ and $p(s) := R \setminus (Q(s) \cup -Q(s))$ is a prime ideal of R .
- (2) $\alpha(s) := Q(s) \cup p(s)$ is a maximal ordering on R with support $p(s)$ such that $\text{sign}_{\alpha(s)} = s$.

Proof. For (1), see [13, Theorem 4.8]. For (2), it follows immediately from the properties of $Q(s)$ that $\alpha(s)$ is an ordering on R with support $p(s)$ and that $\text{sign}_{\alpha(s)} = s$. Suppose that $\alpha(s)$ is not maximal, so that there is $\beta \in \text{Sper } R$ such that $\alpha(s) \subsetneq \beta$. Take $x \in \beta \setminus \alpha(s)$. Since $x \notin \alpha(s)$ we have $x \in -\alpha(s) \setminus \alpha(s) = -Q(s)$ and thus $x = -(r_1^2 u_1 + \cdots + r_k^2 u_k)$ as described above. In $\text{qf}(R/\text{Supp } \beta)$ we have $\bar{x} = -\sum_{i=1}^k \bar{r}_i^2 \bar{u}_i$ with $\bar{u}_i \in \bar{\beta}$, $\bar{u}_i \neq 0$, so that $\bar{u}_i >_{\bar{\beta}} 0$. In particular, $\bar{x} \in -\bar{\beta}$. Since $\bar{x} \in \bar{\beta}$ by choice of x we have $\bar{x} = 0$, which implies that $\bar{r}_i = 0$ for every i , i.e., $r_1, \dots, r_k \in \text{Supp } \beta$, contradicting $\sum_{i=1}^k R r_i = R$. ■

Applying this result to $s = \text{sign}_\alpha$ for $\alpha \in \text{Sper}^{\max} R$, we obtain the following theorem.

Theorem 2.19. *Assume that R is semilocal and that $\alpha \in \text{Sper}^{\max} R$. Then*

$$\alpha \setminus \text{Supp}(\alpha) = \left\{ r_1^2 u_1 + \cdots + r_k^2 u_k \mid k \in \mathbb{N}, u_i \in \alpha \cap R^\times, \sum_{i=1}^k R r_i = R \right\}.$$

The special case of R local easily follows, but is worth noting and was obtained earlier (without the link to $\text{Sper } R$, which was introduced later), first in [12] if $2 \in R^\times$, and then in general in [13]:

$$\alpha \setminus \text{Supp}(\alpha) = \{u_1 + \cdots + u_k \mid k \in \mathbb{N}, u_i \in \alpha \cap R^\times\}. \quad \blacksquare$$

Finally, we mention the following simple fact for future use.

Lemma 2.20. *Let F be a field with space of orderings X_F . Let $P \in X_F$, and denote by F_P any real closure of F at P . Then F is cofinal in F_P , i.e., for every $a \in F_P$ there exists $b \in F$ such that $a \leq b$. In particular, for every $a \in F_P$, if $a > 0$, then there exists $b \in F$ such that $0 < b \leq a$.*

Proof. Let $p(X) = a_0 + a_1X + \cdots + a_{k-1}X^{k-1} + X^k \in F[X]$ be such that $p(a) = 0$. Then $a \leq \max\{1, |a_0| + \cdots + |a_{k-1}|\}$ by [15, Proposition 1.7.1]. The second statement follows by taking inverses. ■

2.5. Positive definite matrices

The results in this section are well known, but we could not find a reference for the quaternion case. In this section we assume that F is a real closed field and that $(D, \vartheta) \in \{(F, \text{id}), (F(\sqrt{-1}), \gamma), ((-1, -1)_F, \gamma)\}$, where γ denotes the canonical involution in each case. We consider norms with values in F . For vectors in D^k we consider the euclidean norm, i.e., for $X = (x_1, \dots, x_k)^t \in D^k$ we have

$$\|X\| = \sqrt{\sum_{i=1}^k n(x_i)},$$

where $n(x) := \vartheta(x)x$. On $M_k(D)$ we use the induced operator norm, i.e., $\|M\|_{\text{op}} := \sup_{\|X\|=1} \|MX\|$. Tarski's transfer principle [24, Corollary 11.5.4] ensures that this supremum exists and that the operator norm is equivalent to the maximum norm determined by the unique ordering of F on the F -vector space $M_k(D)$ (both properties can be expressed by first-order formulas in the language of ordered fields, that are true in \mathbb{R}). Therefore, $\|\cdot\|_{\text{op}}$ defines the same topology as the one induced by the ordering of F .

Lemma 2.21. *Let $k \in \mathbb{N}$. Then*

$$\text{PD}_k(D, \vartheta) := \{B \in \text{Sym}(M_k(D), \vartheta^t) \mid \vartheta(X)^t BX > 0 \text{ for every } X \in D^k \setminus \{0\}\}$$

is an open subset of $\text{Sym}(M_k(D), \vartheta^t)$ for the topology induced by the unique ordering of F .

Proof. Since this property can be expressed by a first-order formula in the language of fields in each of the three cases $(D, \vartheta) \in \{(F, \text{id}), (F(\sqrt{-1}), \gamma), ((-1, -1)_F, \gamma)\}$, it suffices to prove it for $F = \mathbb{R}$ by Tarski's transfer principle [24, Corollary 11.5.4]. The well-known proof below works in all three cases, and we give the details in order to point out that it also works in the quaternion case. Observe that for every $X, Y \in D^k$, we have $\|\vartheta(X)^t Y\| \leq \|X\| \cdot \|Y\|$ (the verification is direct in the quaternion case).

We reformulate $M \in \text{PD}_k(D, \vartheta)$ as: there is $\delta > 0$ such that $\vartheta(X)^t MX \geq \delta$ for every $X \in D^k$, $\|X\| = 1$. Let $M \in \text{PD}_k(D, \vartheta)$ and let δ be as described above. Let $B \in \text{Sym}(M_k(D), \vartheta^t)$ be such that $\|B - M\|_{\text{op}} \leq \varepsilon$. Then

$$\vartheta(X)^t BX = \vartheta(X)^t MX + (\vartheta(X)^t BX - \vartheta(X)^t MX),$$

and, if $\|X\| = 1$,

$$\begin{aligned} |\vartheta(X)^t BX - \vartheta(X)^t MX| &= |\vartheta(X)^t (B - M)X| = \|\vartheta(X)^t (B - M)X\| \\ &\leq \|X\| \cdot \|(B - M)X\| \leq \|B - M\|_{\text{op}} \leq \varepsilon \end{aligned}$$

from which the result follows if we take $\varepsilon = \delta/2$. ■

3. \mathcal{M} -signatures of hermitian forms

If (A, σ) is a central simple F -algebra with involution over a field F , h is a hermitian form over (A, σ) , and P is an ordering on F , we defined in [1, Section 3.2] the signature of h at P with respect to a particular Morita equivalence \mathcal{M}_P (which determines the sign of the signature), denoted $\text{sign}_P^{\mathcal{M}_P} h$. We only considered nonsingular forms in [1, Section 3.2], which was unnecessarily restrictive since the method we used (reduction to the Sylvester signature via scalar extension to a real closure of F at P and hermitian Morita theory) applies in fact to forms that may be singular.

We will use the notation and results from [1]. Note that this signature is also presented in [2, second half of p. 499], where the omission in [1] of one (irrelevant) case for (D_P, ϑ_P) has been rectified.

Let R be a commutative ring (with $2 \in R^\times$), let (A, σ) be an Azumaya algebra with involution over R , let $S = Z(A)$ and let $\iota := \sigma|_S$.

Definition 3.1. Let h be a hermitian form over (A, σ) and let $\alpha \in \text{Sper } R$. Then $h \otimes \kappa(\alpha)$ is a hermitian form over the central simple algebra with involution $(A(\alpha), \sigma(\alpha))$ and we define the \mathcal{M} -signature of h at α by

$$\text{sign}_\alpha^{\mathcal{M}} h := \text{sign}_\alpha^{\mathcal{M}_{\bar{\alpha}}} (h \otimes \kappa(\alpha)), \quad (3.1)$$

where $\mathcal{M}_{\bar{\alpha}}$ is a Morita equivalence as in [1, Section 3.2].

Note that the superscript \mathcal{M} on the left-hand side of (3.1) signifies that each computation of $\text{sign}_\alpha^{\mathcal{M}}$ depends on a choice of Morita equivalence $\mathcal{M}_{\bar{\alpha}}$. The use of a different Morita equivalence can result at most in a change of sign, cf. [1, Proposition 3.4]. Therefore, the notation $\text{sign}_\alpha^{\mathcal{M}}$ should specify what Morita equivalence $\mathcal{M}_{\bar{\alpha}}$ is used but, in order not to overload the notation, we assume that for a given α we always use the same Morita equivalence $\mathcal{M}_{\bar{\alpha}}$.

In fact, the main drawback of the \mathcal{M} -signature is that the sign of the signature of a form can be changed arbitrarily at each ordering by taking a different Morita equivalence. This is in particular a problem if we hope to consider the total signature of a form as a continuous function on $\text{Sper } R$. We solved this problem in the case of central simple algebras with involution by introducing a “reference form”, that determines the sign of the signature at each ordering, cf. [1, Section 6] and [2, Section 3]. We will show in a forthcoming publication that the same can be done in the case of Azumaya algebras with involution.

Remark 3.2. Let (B, τ) be a central simple F -algebra with involution, $P \in X_F$, and F_P a real closure of F at P . Note that $F = \text{Sym}(Z(B), \tau)$. Then,

$$B \cong M_n(D) \quad \text{and} \quad B \otimes_F F_P \cong M_{n_P}(D_P),$$

where D and D_P are division algebras with involution over F and F_P , respectively. (We do not need to name the involutions on D and D_P here.) Clearly, $n \leq n_P \leq \sqrt{\dim_F B}$. Let $b \in \text{Sym}(B, \tau)$. By [3, Proposition 4.4], we have $\text{sign}_P^{\mathcal{M}_P} \langle b \rangle_\tau \leq n_P \leq \sqrt{\dim_F B}$. Applying this to $(A(\alpha), \sigma(\alpha))$ in Definition 3.1, let \mathcal{S} be a set of generators of A as an R -module, and let $a \in \text{Sym}(A, \sigma)$. We have

$$\text{sign}_\alpha^{\mathcal{M}} \langle a \rangle_\sigma \leq \sqrt{\dim_{\kappa(\alpha)} A(\alpha)} \leq |\mathcal{S}|,$$

which provides a bound on $\text{sign}_\alpha^{\mathcal{M}} \langle a \rangle_\sigma$ which is independent of a and α .

Proposition 3.3. *Let h be a hermitian form over (A, σ) , let q be a quadratic form over R and let $\alpha \in \text{Sper } R$. Then*

$$\text{sign}_\alpha^{\mathcal{M}}(q \otimes h) = (\text{sign}_\alpha q) \cdot (\text{sign}_\alpha^{\mathcal{M}} h),$$

where the same Morita equivalence is used in the computation of the signature on both sides of the equality.

Proof. By definition $\text{sign}_\alpha^{\mathcal{M}}(q \otimes h)$ is equal to $\text{sign}_\alpha^{\mathcal{M}_{\bar{\alpha}}}((q \otimes h) \otimes \kappa(\alpha))$, where $(q \otimes h) \otimes \kappa(\alpha)$ is considered as a form over $(A(\alpha), \sigma(\alpha))$. But

$$\text{sign}_\alpha^{\mathcal{M}_{\bar{\alpha}}}((q \otimes h) \otimes \kappa(\alpha)) = (\text{sign}_{\bar{\alpha}} q \otimes \kappa(\alpha)) \cdot (\text{sign}_\alpha^{\mathcal{M}_{\bar{\alpha}}} h \otimes \kappa(\alpha))$$

by [1, Proposition 3.6]. The result follows. \blacksquare

Lemma 3.4. *Let T be a quadratic étale R -algebra with standard involution ϑ . Let h be a hermitian form over (T, ϑ) . Then, for every $\alpha \in \text{Sper } R$, $\text{sign}_\alpha^{\mathcal{M}} h = 0$ implies*

$$\text{sign}_\alpha \text{Tr}_{T/R}(h) = 0.$$

Proof. Note that (T, ϑ) is an Azumaya algebra with involution over R and that $\vartheta \otimes \text{id}$ is the standard involution on $T \otimes_R k(\alpha)$ by Remark 2.9. In particular, signatures are defined, and by the definition of signatures for central simple algebras with involution, the form $h \otimes_R k(\alpha)$ has signature 0 at the unique ordering on $k(\alpha)$. Writing $h \otimes_R k(\alpha) \simeq \varphi \perp 0$, where φ is nonsingular and 0 is the zero form of appropriate rank, cf. [3, Proposition A.3], it follows that φ has signature 0 at the unique ordering on $k(\alpha)$, and thus is weakly hyperbolic (i.e., $\ell \times \varphi$ is hyperbolic for some $\ell \in \mathbb{N}$) by [21, Theorem 4.1] (or [7, Theorem 6.5]).

As recalled in Section 2.2, $\text{Tr}_{T/R}$ is R -linear and $\text{Tr}_{T/R}(h)$ is a quadratic form over R . Also, $\text{Tr}_{T \otimes_R k(\alpha)/k(\alpha)}$ is an involution trace for $\vartheta \otimes \text{id}$, and it follows that $\text{Tr}_{T \otimes_R k(\alpha)/k(\alpha)}(\varphi)$ is weakly hyperbolic by Section 2.2, and thus has signature 0. Since $\text{Tr}_{T/R}(h) \otimes k(\alpha) = \text{Tr}_{T \otimes_R k(\alpha)/k(\alpha)}(h \otimes k(\alpha))$ by Remark 2.9, and observing that

$$\text{Tr}_{T \otimes_R k(\alpha)/k(\alpha)}(h \otimes k(\alpha)) \simeq \text{Tr}_{T \otimes_R k(\alpha)/k(\alpha)}(\varphi) \perp \text{Tr}_{T \otimes_R k(\alpha)/k(\alpha)}(0),$$

it follows that $\text{sign}_\alpha \text{Tr}_{T/R}(h) = 0$. \blacksquare

Definition 3.5. We call

$$\text{Nil}[A, \sigma] := \{\alpha \in \text{Sper } R \mid \text{sign}_\alpha^{\mathcal{M}} = 0\}$$

the set of *nil orderings* of (A, σ) . By the observation after Definition 3.1, $\text{Nil}[A, \sigma]$ is independent of the choice of Morita equivalence at each α .

Remark 3.6. If (A, σ) is a central simple F -algebra with involution, this definition is equivalent to our original one ([1, Definition 3.7] by [1, Theorem 6.4]), from which it follows that if σ is orthogonal and τ is any symplectic involution on A , then $\text{Nil}[A, \sigma] = X_F \setminus \text{Nil}[A, \tau]$.

Also note that if $P \in X_F \setminus \text{Nil}[A, \sigma]$ and F_P is a real closure of F at P , then

$$(A \otimes_F F_P, \sigma \otimes \text{id}) \cong (M_{n_P}(D_P), \text{Int}(\Phi_P) \circ \vartheta_P^t),$$

where $D_P \in \{F_P, F_P(\sqrt{-1}), (-1, -1)_{F_P}\}$, ϑ_P is the canonical involution on D_P , and $\Phi_P \in \text{Sym}(M_{n_P}(D_P)^\times, \vartheta_P^t)$, cf. [4, p. 4 and Remark 6.2].

Lemma 3.7. *Let $\alpha \in \text{Sper } R$. Then statements (1) and (2) below are equivalent:*

- (1) $\alpha \in \text{Nil}[A, \sigma]$;
- (2) $\bar{\alpha} \in \text{Nil}[A(\alpha), \sigma(\alpha)]$.

Assume in addition that σ is of unitary type at $\text{Supp}(\alpha)$. Then (1) and (2) and the following statements are equivalent:

- (3) $Z(A(\alpha)) \otimes_{\kappa(\alpha)} k(\bar{\alpha}) \cong k(\bar{\alpha}) \times k(\bar{\alpha})$, i.e., $Z(A(\alpha)) \otimes_{\kappa(\alpha)} k(\alpha) \cong k(\alpha) \times k(\alpha)$ since we can take $k(\bar{\alpha}) = k(\alpha)$ as observed before;
- (4) $Z(A) \otimes_R k(\alpha) \cong k(\alpha) \times k(\alpha)$;
- (5) $\alpha \in \text{Nil}[S, \iota]$.

Proof. (1) \Rightarrow (2): Assume $\bar{\alpha} \notin \text{Nil}[A(\alpha), \sigma(\alpha)]$. Then there exists $z \in \text{Sym}(A(\alpha), \sigma(\alpha))$ such that $\text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(z)_{\sigma(\alpha)} \neq 0$ (see [1, Theorem 6.4]). Using Remark 2.17, write $z = a \otimes \frac{1}{r}$, for some $r \in R \setminus \text{Supp}(\alpha)$. Since \bar{r} is invertible in $\kappa(\alpha)$, we have

$$0 \neq \text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(z)_{\sigma(\alpha)} = \text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(z\bar{r}^2)_{\sigma(\alpha)} = \text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(ar \otimes 1)_{\sigma \otimes \text{id}_{\kappa(\alpha)}} = \text{sign}_{\bar{\alpha}}^{\mathcal{M}}(ar)_{\sigma},$$

contradicting that $\alpha \in \text{Nil}[A, \sigma]$.

(2) \Rightarrow (1): This follows from the fact that $\text{sign}_{\alpha}^{\mathcal{M}} h = \text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}} h \otimes \kappa(\alpha) = 0$ for any hermitian form h over (A, σ) .

For the remaining equivalences, recall that if (B, τ) is a central simple algebra with involution of the second kind over a field F , then $P \in \text{Nil}[B, \tau]$ if and only if $Z(B \otimes F_P) \cong F_P \times F_P$, cf. [2, bottom of p. 499 and Definition 2.1].

The equivalence (2) \Leftrightarrow (3) follows from this observation, and we have (3) \Leftrightarrow (4) since $k(\alpha)$ is a real closure of $\kappa(\alpha)$ at $\bar{\alpha}$ and $Z(A(\alpha)) = Z(A) \otimes_R \kappa(\alpha)$ (by Proposition 2.7). Finally, the equivalence (4) \Leftrightarrow (5) follows from (1) \Leftrightarrow (4) applied to (S, ι) since $S = Z(S) = Z(A)$. ■

Corollary 3.8. *Assume that R is semilocal and that σ is of unitary type. Then there is $d \in R^\times$ such that $\text{Nil}[A, \sigma] = \mathring{H}(d)$.*

Proof. By [8, Proposition 1.21], $S = Z(A)$ is a quadratic étale R -algebra, and for every $\alpha \in \text{Sper } R$, $\sigma(\alpha)$ is of unitary type at the prime ideal $\text{Supp}(\alpha)$. Furthermore, as recalled in Section 2.2, $S = R \oplus \lambda R$ for some $\lambda \in S$ such that $d := \lambda^2 \in R^\times$. In particular, for any $\alpha \in \text{Sper } R$,

$$Z(A(\alpha)) = S \otimes_R \kappa(\alpha) = \kappa(\alpha) \oplus (\lambda \otimes 1)\kappa(\alpha) = \kappa(\alpha)(\sqrt{\bar{d}}),$$

where \bar{d} is the image of d in $R/\text{Supp}(\alpha)$, and where the first equality follows from Proposition 2.7. Therefore, for $\alpha \in \text{Sper } R$:

$$\begin{aligned} \alpha \in \text{Nil}[A, \sigma] &\iff Z(A(\alpha)) \otimes_{\kappa(\alpha)} k(\bar{\alpha}) \cong k(\bar{\alpha}) \times k(\bar{\alpha}) \text{ by Lemma 3.7} \\ &\iff \kappa(\alpha)(\sqrt{\bar{d}}) \otimes_{\kappa(\alpha)} k(\bar{\alpha}) \cong k(\bar{\alpha}) \times k(\bar{\alpha}) \\ &\iff \sqrt{\bar{d}} \in k(\bar{\alpha}) \iff \bar{d} \in \bar{\alpha} \iff d \in \alpha \\ &\iff \alpha \in \mathring{H}(d) \text{ since } d \text{ is invertible.} \quad \blacksquare \end{aligned}$$

Remark 3.9. Recall from Section 2.2 that when R is connected, then either S is connected or $S \cong R \times R$. Moreover, if $\text{Sper } R \setminus \text{Nil}[A, \sigma] \neq \emptyset$, we cannot have $S \cong R \times R$ by Lemma 3.7. Therefore, if R is connected and $\text{Sper } R \setminus \text{Nil}[A, \sigma] \neq \emptyset$, then S is also connected.

Lemma 3.10. *Let F be a field and let (B, τ) be a central simple F -algebra with involution of the first kind. If $\text{Nil}[B, \tau] \neq \emptyset$, then $\deg B$ is even.*

Proof. Assume that $\deg B$ is odd. Then B is split and τ is orthogonal by [17, Corollary 2.8(1)]. It follows that $\text{Nil}[B, \tau] = \emptyset$ by [1, Definition 3.7]. \blacksquare

Lemma 3.11. *Assume that R is semilocal connected and that σ is of orthogonal or symplectic type. If $\text{Nil}[A, \sigma] \neq \emptyset$, then $\text{Skew}(A^\times, \sigma) \neq \emptyset$.*

Proof. Let $\alpha \in \text{Nil}[A, \sigma]$. Then $\bar{\alpha} \in \text{Nil}[A(\alpha), \sigma(\alpha)]$ by Lemma 3.7. Hence, $\deg A \otimes_R \kappa(\alpha)$ is even by Lemma 3.10. Since R is connected, the rank of A is constant (since A is a projective R -module, see [26, p. 12]). It follows that $\deg A \otimes_R \text{qf}(R/\mathfrak{p})$ is even for every $\mathfrak{p} \in \text{Spec } R$. We can then apply [8, Lemma 1.26] with $\varepsilon = -1$. \blacksquare

3.1. Elements of maximal signature

In [4], working with central simple algebras with involution, we investigated the maximal value that the signature at $P \in X_F$ can take (when it is non-zero) when applied to one-dimensional nonsingular forms. We found that this maximal value is the matrix size of the algebra over its skew-field part after scalar extension to F_P (and linked it to the existence of positive involutions), cf. [4, Proposition 6.7, Theorem 6.8].

We are interested in the same question when (A, σ) is an Azumaya algebra with involution. More precisely, we will show in Corollary 3.19 that if $\alpha \notin \text{Nil}[A, \sigma]$, then this maximal value is the matrix size $n_{\bar{\alpha}}$ of $A \otimes_R k(\alpha)$ over its skew-field part (i.e., $A \otimes_R k(\alpha) \cong M_{n_{\bar{\alpha}}}(D_{\bar{\alpha}})$ using the notation from Remark 3.2). We first introduce some notation.

Definition 3.12. If (B, τ) is an Azumaya algebra with involution over R , and $\alpha \in \text{Sper } R$, we define:

$$m_{\alpha}(B, \tau) := \max \{ \text{sign}_{\alpha}^{\mathcal{M}} \langle b \rangle_{\tau} \mid b \in \text{Sym}(B^{\times}, \tau) \},$$

$$M_{\alpha}^{\mathcal{M}}(B, \tau) := \{ b \in \text{Sym}(B^{\times}, \tau) \mid \text{sign}_{\alpha}^{\mathcal{M}} \langle b \rangle_{\tau} = m_{\alpha}(B, \tau) \}.$$

Observe that $m_{\alpha}(B, \tau)$ is independent of the choice of the Morita equivalence $\mathcal{M}_{\bar{\alpha}}$ (cf. Definition 3.1), and is finite by Remark 3.2.

We introduce some notation that will be used in the next four results. For $\alpha \in \text{Sper } R$, define

$$\mathcal{S}_{\alpha}(A, \sigma) := \bigcup \{ D_{(A, \sigma)} \langle a_1, \dots, a_k \rangle_{\sigma} \mid k \in \mathbb{N}, a_i \in \text{Sym}(A, \sigma), a_i \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha)) \}.$$

Furthermore, for $\mathfrak{p} \in \text{Spec } R$, we denote by $\pi_{\mathfrak{p}}$ the canonical projection from R to R/\mathfrak{p} and by $\pi_{A\mathfrak{p}}$ the canonical projection from A to $A/A\mathfrak{p}$. Then, denoting by $\bar{\sigma}$ the involution induced by σ on $A/A\mathfrak{p}$, we define

$$\mathcal{S}'_{\mathfrak{p}} := \bigcup \{ D_{(A/A\mathfrak{p}, \bar{\sigma})} \langle \pi_{A\mathfrak{p}}(a_1), \dots, \pi_{A\mathfrak{p}}(a_k) \rangle_{\bar{\sigma}} \mid k \in \mathbb{N}, a_i \in \text{Sym}(A, \sigma),$$

$$a_i \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha)) \}.$$

Lemma 3.13. *Let $\alpha \in \text{Sper } R$. Then there is an element $b \in \text{Sym}(A, \sigma)$ such that $b \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha))$. In particular, $b \in \mathcal{S}_{\alpha}(A, \sigma)$.*

Proof. Let $c \in \text{Sym}(A(\alpha)^{\times}, \sigma(\alpha))$ be such that $\text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}} \langle c \rangle_{\sigma(\alpha)} = m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$. By Remark 2.17, there are $b_0 \in A$ and $r \in R \setminus \text{Supp}(\alpha)$ such that $c = b_0 \otimes \frac{1}{\bar{r}}$. Let $b_1 := \frac{1}{2}(b_0 + \sigma(b_0))$. Then $b_1 \in \text{Sym}(A, \sigma)$ and

$$b_1 \otimes \frac{1}{\bar{r}} = \frac{1}{2} \left(b_0 \otimes \frac{1}{\bar{r}} + \sigma(b_0) \otimes \frac{1}{\bar{r}} \right) = \frac{1}{2} (c + \sigma(\alpha)(c)) = c.$$

We take $b := r b_1$. Then $b \in \text{Sym}(A, \sigma)$ and, since $\bar{r} \in \kappa(\alpha)^{\times}$, we have $b \otimes 1 = \bar{r}^2 c \in A(\alpha)^{\times}$, and

$$\text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}} \langle b \otimes 1 \rangle_{\sigma(\alpha)} = \text{sign}_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}} \langle \bar{r}^2 c \rangle_{\sigma(\alpha)} = m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha)),$$

so that $b \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha))$. ■

Lemma 3.14. *Assume that $R = F$ is a field and let $P \in X_F \setminus \text{Nil}[A, \sigma]$. Let $a \in \text{Sym}(A, \sigma)$ be such that $\text{sign}_P^{\mathcal{M}} \langle a \rangle_{\sigma} = m_P(A, \sigma)$. Then:*

- (1) *a is invertible in A ;*
- (2) *There is $\mu \in P \setminus \{0\}$ such that $\text{sign}_P^{\mathcal{M}} \langle a - r \rangle_{\sigma} = m_P(A, \sigma)$ for every $r \in P$ such that $r \leq_P \mu$.*

Proof. By [4, Proposition 6.7], we have

$$m_P(A, \sigma) = n_P.$$

We use the notation from Remark 3.6. Since $P \notin \text{Nil}[A, \sigma]$, it follows from the computation of \mathcal{M} -signatures (cf. the beginning of Section 3) that $\text{sign}_P^{\mathcal{M}} \langle a \rangle_\sigma$ is equal to the Sylvester signature of the form $\langle \Phi_P^{-1}(a \otimes 1) \rangle_{\vartheta_P^t}$, where $\Phi_P^{-1}(a \otimes 1) \in \text{Sym}(M_{n_P}(D_P), \vartheta_P^t)$. Since $D_P \in \{F_P, F_P(\sqrt{-1}), (-1, -1)_{F_P}\}$ and ϑ_P is the canonical involution on D_P , the matrix $\Phi_P^{-1}(a \otimes 1)$ can be diagonalized by congruences (which does not change the Sylvester signature), so we can assume that $\Phi_P^{-1}(a \otimes 1)$ is diagonal. Since it has Sylvester signature n_P , its diagonal elements are all positive, i.e., $\Phi_P^{-1}(a \otimes 1) \in \text{PD}_{n_P}(D_P, \vartheta_P)$. In particular, $\Phi_P^{-1}(a \otimes 1)$ is invertible. Therefore $a \otimes 1$ is not a zero divisor, and a is not a zero divisor in A . It follows that a is invertible since A is Artinian. This proves (1).

For (2): The element $a \otimes 1$ is in $\Phi_P \cdot \text{PD}_{n_P}(D_P, \vartheta_P)$, which is an open subset of $\text{Sym}(M_{n_P}(D_P), \vartheta_P^t)$ by Lemma 2.21. It follows that there is $\varepsilon > 0$ in F_P such that for all $M \in \text{Sym}(M_{n_P}(D_P), \vartheta_P^t)$ that satisfy $\|M\|_{\text{op}} < \varepsilon$ we have

$$a \otimes 1 - M \in \Phi_P \cdot \text{PD}_{n_P}(D_P, \vartheta_P).$$

Taking $\mu \in F_P$ such that $0 < \mu < \varepsilon$, we obtain $\|\mu I_{n_P}\|_{\text{op}} < \varepsilon$. Hence,

$$a \otimes 1 - \mu I_{n_P} \in \Phi_P \cdot \text{PD}_{n_P}(D_P, \vartheta_P).$$

In particular, the signature of the form $\langle a \otimes 1 - \mu I_{n_P} \rangle_{\sigma \otimes \text{id}}$ equals n_P .

However, F is cofinal in F_P by Lemma 2.20, so we can find such a μ in F . The choice of μ guarantees that $\text{sign}_P^{\mathcal{M}} \langle a - r \rangle_\sigma = n_P = m_P(A, \sigma)$ whenever $r \in P, r \leq_P \mu$. ■

Lemma 3.15. *Let \mathfrak{m} be a maximal ideal of R and let $\alpha \in \text{Sper } R$ be such that $\text{Supp } \alpha \subseteq \mathfrak{m}$. Assume that*

$$\forall a \in \text{Sym}(A, \sigma) \quad a \otimes_R 1_{\kappa(\alpha)} \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}} (A(\alpha), \sigma(\alpha)) \text{ implies } a \in A\mathfrak{m}. \quad (3.2)$$

Then property (3.2) is preserved under quotients by $\text{Supp } \alpha$ and, if $\text{Supp } \alpha = \{0\}$, under localisation at \mathfrak{m} . More precisely:

- (1) *Let $R_1 := R / \text{Supp } \alpha$. Then property (3.2) holds for $(A_1, \sigma_1) := (A \otimes_R R_1, \sigma \otimes_R \text{id}_{R_1})$ and the ordering α_1 induced by α on R_1 together with the maximal ideal $\mathfrak{m}_1 := \mathfrak{m} / \text{Supp } \alpha$ of R_1 , i.e.,*

$$\forall a \in \text{Sym}(A_1, \sigma_1) \quad a \otimes_{R_1} 1_{\kappa(\alpha_1)} \in M_{\bar{\alpha}_1}^{\mathcal{M}_{\bar{\alpha}_1}} (A_1(\alpha_1), \sigma_1(\alpha_1)) \text{ implies } a \in A_1 \mathfrak{m}_1.$$

Furthermore, if $\alpha \in \text{Sper}^{\max} R$, then $\alpha_1 \in \text{Sper}^{\max} R_1$.

- (2) *Assume that $\text{Supp } \alpha = \{0\}$. Then property (3.2) holds for $(A_2, \sigma_2) := (A \otimes_R R_{\mathfrak{m}}, \sigma \otimes_R \text{id}_{R_{\mathfrak{m}}})$ and the ordering α_2 induced by α on $R_{\mathfrak{m}}$ together with the unique maximal ideal \mathfrak{m}_2 of $R_{\mathfrak{m}}$, i.e.,*

$$\forall a \in \text{Sym}(A_2, \sigma_2) \quad a \otimes_{R_{\mathfrak{m}}} 1_{\kappa(\alpha_2)} \in M_{\bar{\alpha}_2}^{\mathcal{M}_{\bar{\alpha}_2}} (A_2(\alpha_2), \sigma_2(\alpha_2)) \text{ implies } a \in A_2 \mathfrak{m}_2.$$

Furthermore, if $\alpha \in \text{Sper}^{\max} R$, then $\alpha_2 \in \text{Sper}^{\max} R_{\mathfrak{m}}$.

Proof. (1) We have natural maps

$$R \rightarrow R_1 = R / \text{Supp } \alpha \rightarrow \text{qf}(R_1) = \kappa(\alpha) = \kappa(\alpha_1)$$

with $\bar{\alpha} = \bar{\alpha}_1$, and thus

$$A \rightarrow A_1 = A \otimes_R R_1 \rightarrow A \otimes_R \text{qf}(R_1) = A(\alpha) = A_1(\alpha_1),$$

while

$$\sigma_1(\alpha_1) := \sigma_1 \otimes_{R_1} \text{id}_{\kappa(\alpha_1)} = (\sigma \otimes_R \text{id}_{R_1}) \otimes_{R_1} \text{id}_{\kappa(\alpha_1)} = \sigma \otimes_R \text{id}_{\kappa(\alpha_1)} = \sigma(\alpha).$$

Let $b \in \text{Sym}(A_1, \sigma_1)$ be such that

$$b \otimes_{R_1} 1_{\kappa(\alpha_1)} \in M_{\bar{\alpha}_1}^{\mathcal{M}_{\bar{\alpha}_1}}(A_1(\alpha_1), \sigma_1(\alpha_1)) = M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha)).$$

Then $b = c \otimes_R 1_{R_1}$ for some $c \in A$ (the argument is similar to Remark 2.17) and

$$b \otimes_{R_1} 1_{\kappa(\alpha_1)} = c \otimes_R 1_{R_1} \otimes_{R_1} 1_{\kappa(\alpha_1)} = c \otimes_R 1_{\kappa(\alpha)}.$$

Therefore $c \otimes_R 1_{\kappa(\alpha)} \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha))$ and thus $c \in A_{\mathfrak{m}}$ by (3.2). It follows that $b = c \otimes_R 1_{R_1} \in A_{\mathfrak{m}} \otimes_R 1_{R_1} \subseteq (A \otimes_R R_1)(\mathfrak{m} / \text{Supp } \alpha) = A_1 \mathfrak{m}_1$ (the inclusion follows from $am \otimes 1 = (a \otimes 1)(m \otimes 1) = (a \otimes 1)(1 \otimes (m + \text{Supp } \alpha))$).

The statement about the maximality of α_1 follows from the fact that the homeomorphism in [15, Proposition 3.3.11] clearly preserves inclusions.

(2) Note that α_2 is indeed an ordering on $R_{\mathfrak{m}}$ since $(R \setminus \mathfrak{m}) \cap \text{Supp } \alpha = \emptyset$, cf. [15, Proposition 3.3.10]. We also have

$$\alpha_2 = \left\{ \frac{r}{s^2} \mid r \in \alpha, s \in R \setminus \mathfrak{m} \right\}$$

(by [15, Proof of Proposition 3.3.10]) and $\text{Supp } \alpha_2 = \{0\}$, so that $\kappa(\alpha_2) = \text{qf}(R_{\mathfrak{m}})$. Observe that the map $R \rightarrow \text{qf}(R)$, which is the first step in the computation of signatures of elements of $\text{Sym}(A, \sigma)$ (since $\text{Supp } \alpha = \{0\}$, cf. Definition 3.1) factors through $R_{\mathfrak{m}}$, giving $R \rightarrow R_{\mathfrak{m}} \rightarrow \text{qf}(R)$, with $\text{qf}(R) = \text{qf}(R_{\mathfrak{m}})$, i.e., $\kappa(\alpha) = \kappa(\alpha_2)$. Finally, a direct verification shows that $\bar{\alpha} = \bar{\alpha}_2$.

Let $b \in \text{Sym}(A_2, \sigma_2)$ be such that $b \otimes_{R_{\mathfrak{m}}} 1_{\kappa(\alpha_2)} \in M_{\bar{\alpha}_2}^{\mathcal{M}_{\bar{\alpha}_2}}(A_2(\alpha_2), \sigma_2(\alpha_2))$. Then $b = c \otimes_R \frac{1}{s}$ for some $s \in R \setminus \mathfrak{m}$ (the argument is again similar to Remark 2.17). Since s is invertible in $R_{\mathfrak{m}}$, b has the same signature at α_2 as $bs^2 = cs \otimes_R 1_{R_{\mathfrak{m}}}$, so that

$$(cs \otimes_R 1_{R_{\mathfrak{m}}}) \otimes_{R_{\mathfrak{m}}} 1_{\kappa(\alpha_2)} \in M_{\bar{\alpha}_2}^{\mathcal{M}_{\bar{\alpha}_2}}(A_2(\alpha_2), \sigma_2(\alpha_2)).$$

We have $(cs \otimes_R 1_{R_{\mathfrak{m}}}) \otimes_{R_{\mathfrak{m}}} 1_{\kappa(\alpha_2)} = cs \otimes_R 1_{\kappa(\alpha)}$, and thus

$$\begin{aligned} cs \otimes_R 1_{\kappa(\alpha)} &\in M_{\bar{\alpha}_2}^{\mathcal{M}_{\bar{\alpha}_2}}(A_2(\alpha_2), \sigma_2(\alpha_2)) \\ &= M_{\bar{\alpha}_2}^{\mathcal{M}_{\bar{\alpha}_2}}((A \otimes_R R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} \text{qf}(R_{\mathfrak{m}}), (\sigma \otimes \text{id}_{R_{\mathfrak{m}}}) \otimes \text{id}_{\kappa(\alpha_2)}) \\ &= M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A \otimes_R \kappa(\alpha), \sigma \otimes \text{id}_{\kappa(\alpha)}). \end{aligned}$$

By property (3.2), we obtain $cs \in \text{Am}$. Therefore,

$$cs \otimes_R 1_{R_m} \in \text{Am} \otimes_R 1_{R_m} \subseteq (A \otimes_R R_m) m_2$$

(the inclusion follows from $am \otimes_R 1_{R_m} = (a \otimes_R 1_{R_m})(m \otimes_R 1_{R_m}) = (a \otimes_R 1_{R_m})(1 \otimes_R m)$) and thus $b = c \otimes_R \frac{1}{s} = (cs \otimes_R 1_{R_m}) \frac{1}{s^2} \in (A \otimes_R R_m) m_2$.

The statement about the maximality of α_2 follows from the fact that the homeomorphism in [15, Proposition 3.3.10] clearly preserves inclusions. ■

Lemma 3.16. *Let m be a maximal ideal of R and let $\alpha \in \text{Sper}^{\max} R$. Then there is $a \in \text{Sym}(A, \sigma)$ such that $a \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha))$ and $a \notin \text{Am}$.*

Proof. We assume that the conclusion does not hold, so that property (3.2) of Lemma 3.15 holds. We proceed in four steps.

Step 1. Take $a \in \text{Sym}(A, \sigma)$ such that $a \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha))$, cf. Lemma 3.13. Then $a \in \text{Am}$ by property (3.2). Furthermore, for every $r \in \text{Supp } \alpha$ we have $\text{sign}_{\alpha}^{\mathcal{M}} \langle a + r \rangle_{\sigma} = \text{sign}_{\alpha}^{\mathcal{M}} \langle a \rangle_{\sigma}$, cf. Definition 3.1 (since the first step in the computation is scalar extension to $\text{qf}(R/\text{Supp } \alpha)$). Thus, by hypothesis, $a + r \in \text{Am}$. Since $a \in \text{Am}$, we get that $\text{Supp } \alpha \subseteq \text{Am} \cap R = m$ (cf. [9, Corollary 7.1.2(1)] for the equality).

Step 2. We first apply Lemma 3.15 (1) and get that we can assume that $\text{Supp } \alpha = \{0\}$, and in particular that R is a domain. It is then possible to apply Lemma 3.15 (2) and we can also assume that R is a local domain with maximal ideal m .

Step 3. Since R is a local domain and $\text{Supp } \alpha = \{0\}$, the following holds:

For every $\frac{r_1}{s_1} \in \text{qf}(R)$ with $r_1, s_1 \in \alpha \setminus \{0\}$, there exists $r \in \alpha \cap R^{\times}$ such that $r \leq_{\alpha} \frac{r_1}{s_1}$.

Proof of this claim. By the description of α in Theorem 2.19, there is $r'_1 \in \alpha \cap R^{\times}$ such that $r'_1 \leq_{\alpha} r_1$, so that

$$\frac{r'_1}{s_1} \leq_{\alpha} \frac{r_1}{s_1}.$$

Observe that if there is $s'_1 \in \alpha \cap R^{\times}$ such that $s'_1 \geq_{\alpha} s_1$, then $\frac{r'_1}{s'_1} \leq_{\alpha} \frac{r_1}{s_1}$, and we can take $r := r'_1 s'_1{}^{-1}$.

However, since R is local, such an s'_1 exists: If s_1 is invertible, we take $s'_1 = s_1$. If s_1 is not invertible, then $s_1 \in m$. Therefore $1 + s_1 \notin m$, i.e., $1 + s_1 \in R^{\times}$, and of course $1 + s_1 \in \alpha$. We then take $s'_1 := 1 + s_1$. This proves the claim.

Step 4. We work in the central simple algebra with involution $(A(\alpha), \sigma(\alpha)) = (A \otimes_R \text{qf}(R), \sigma \otimes \text{id})$, and denote by $\bar{\alpha}$ the ordering induced by α on $\text{qf}(R)$. By Lemma 3.14 (2), there is $\frac{r_1}{s_1} \in \bar{\alpha} \setminus \{0\}$ such that $\langle a \otimes 1 - \frac{r_2}{s_2} \rangle_{\sigma(\alpha)}$ has maximal signature at $\bar{\alpha}$ for every $\frac{r_2}{s_2} \in \bar{\alpha}$ such that $\frac{r_2}{s_2} \leq_{\bar{\alpha}} \frac{r_1}{s_1}$. In other words, $a \otimes 1 - \frac{r_1}{r_2} \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha))$.

By Step 3, there is $r \in \alpha \cap R^{\times}$ such that $r \leq_{\bar{\alpha}} \frac{r_1}{s_1}$. In particular, we have

$$(a - r) \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}_{\bar{\alpha}}}(A(\alpha), \sigma(\alpha)).$$

Therefore, $a - r \in \text{Am}$ by property (3.2) and thus $r \in \text{Am}$. But this is impossible since Am is a proper ideal and r is invertible. ■

Lemma 3.17. *Assume that R is semilocal, let \mathfrak{m} be a maximal ideal of R , and let $\alpha \in \text{Sper}^{\max} R$. Then there is $b_{\mathfrak{m}} \in \mathcal{S}_{\alpha}(A, \sigma)$ such that $b_{\mathfrak{m}} + A\mathfrak{m} \in (A/A\mathfrak{m})^{\times}$.*

Proof. Let $a \in \text{Sym}(A, \sigma)$ be such that $a \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}}(A(\alpha), \sigma(\alpha))$ and $\pi_{A\mathfrak{m}}(a) \neq 0$, cf. Lemma 3.16. Then $D_{(A/A\mathfrak{m}, \bar{\sigma})}(k \times \langle \pi_{A\mathfrak{m}}(a) \rangle_{\bar{\sigma}}) \subseteq \mathcal{S}'_{\mathfrak{m}}$ for all $k \in \mathbb{N}$ by definition of $\mathcal{S}'_{\mathfrak{m}}$. Since $(A/A\mathfrak{m}, \bar{\sigma})$ is a central simple algebra with involution, [4, Lemma 2.4] applies and there is $\ell \in \mathbb{N}$ such that $\ell \times \langle \pi_{A\mathfrak{m}}(a) \rangle_{\bar{\sigma}}$ represents an invertible element b' . Since $b' \in \mathcal{S}'_{\mathfrak{m}}$ and $\pi_{A\mathfrak{m}}$ is surjective, we have

$$b' = \sum_{j=1}^{\ell} \bar{\sigma}(\pi_{A\mathfrak{m}}(x_j)) \pi_{A\mathfrak{m}}(a) \pi_{A\mathfrak{m}}(x_j),$$

with $x_j \in A$. Therefore, we take $b_{\mathfrak{m}} = \sum_{j=1}^{\ell} \sigma(x_j) a x_j$. ■

Proposition 3.18. *Assume that R is semilocal and let $\alpha \in \text{Sper}^{\max} R$. Then there are invertible elements in $\mathcal{S}_{\alpha}(A, \sigma)$. Furthermore, every invertible element $a \in \mathcal{S}_{\alpha}(A, \sigma)$ satisfies $\text{sign}_{\alpha}^{\mathcal{M}} \langle a \rangle_{\sigma} = m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$.*

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_{\ell}$ be the maximal ideals of R . Observe that for each $i \in \{1, \dots, \ell\}$ there is $b_i \in \mathcal{S}_{\alpha}(A, \sigma)$ such that $b_i + A\mathfrak{m}_i \in (A/A\mathfrak{m}_i)^{\times}$ by Lemma 3.17. By the Chinese remainder theorem, the canonical map $\xi: R \rightarrow R/\mathfrak{m}_1 \times \dots \times R/\mathfrak{m}_{\ell}$ is surjective. In particular, there are $r_1, \dots, r_{\ell} \in R$ such that $\xi(r_i) = (0, \dots, 0, 1, 0, \dots, 0)$, where the coordinate 1 is the one corresponding to the quotient R/\mathfrak{m}_i . Define

$$b := \sigma(r_1) b_1 r_1 + \dots + \sigma(r_{\ell}) b_{\ell} r_{\ell}.$$

Observe that $b \in \mathcal{S}_{\alpha}(A, \sigma)$. We check that b is invertible. By Proposition 2.10, it suffices to show that $b + A\mathfrak{m}$ is invertible in $A/A\mathfrak{m}$ for every maximal ideal \mathfrak{m} of R . Consider such an ideal \mathfrak{m}_i . By definition of r_i we have $r_i + \mathfrak{m}_i = 1 + \mathfrak{m}_i$ and $r_j + \mathfrak{m}_i = 0 + \mathfrak{m}_i$ for all $j \neq i$. Therefore $b + A\mathfrak{m}_i = b_i + A\mathfrak{m}_i$, which is invertible in $A/A\mathfrak{m}_i$.

We show that if $a \in \mathcal{S}_{\alpha}(A, \sigma)$ is invertible, then $\text{sign}_{\alpha}^{\mathcal{M}} \langle a \rangle_{\sigma} = m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$. Since $a \in \mathcal{S}_{\alpha}(A, \sigma)$, there are $a_1, \dots, a_k \in \text{Sym}(A, \sigma)$ such that $a \in D_{(A, \sigma)} \langle a_1, \dots, a_k \rangle_{\sigma}$ and $a_1 \otimes 1, \dots, a_k \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}}(A(\alpha), \sigma(\alpha))$. Since a is invertible, a standard argument gives $\langle a_1, \dots, a_k \rangle_{\sigma} \simeq \langle a \rangle_{\sigma} \perp h$ for some hermitian form h over (A, σ) . Extending the scalars to $\kappa(\alpha)$, we obtain $\langle a \otimes 1 \rangle_{\sigma(\alpha)} \perp h \otimes \kappa(\alpha) \simeq \langle a_1 \otimes 1, \dots, a_k \otimes 1 \rangle_{\sigma(\alpha)}$ over the central simple algebra with involution $(A(\alpha), \sigma(\alpha))$. Observe that $a_1 \otimes 1, \dots, a_k \otimes 1$ are invertible in $A(\alpha)$, and thus that $h \otimes \kappa(\alpha)$ is nonsingular. By [4, Lemma 2.2] there is $\ell \in \mathbb{N}$ such that $\ell \times (h \otimes \kappa(\alpha)) \simeq \langle c_1, \dots, c_t \rangle_{\sigma(\alpha)}$ for some $c_1, \dots, c_t \in \text{Sym}(A(\alpha)^{\times}, \sigma(\alpha))$ (they are invertible since $h \otimes \kappa(\alpha)$ is nonsingular). Therefore,

$$\ell \times \langle a \otimes 1 \rangle_{\sigma(\alpha)} \perp \langle c_1, \dots, c_t \rangle_{\sigma(\alpha)} \simeq \ell \times \langle a_1 \otimes 1, \dots, a_k \otimes 1 \rangle_{\sigma(\alpha)}.$$

Note that both forms are diagonal over $(A(\alpha), \sigma(\alpha))$, so that $\ell + t = \ell k$ for dimension reasons. Since $a_1 \otimes 1, \dots, a_k \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}}(A(\alpha), \sigma(\alpha))$, the form on the right-hand

side has the maximal signature that can be obtained by a nonsingular diagonal form of dimension ℓk over $(A(\alpha), \sigma(\alpha))$, namely $\ell k \cdot m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$. It is therefore the same for the form on the left-hand side, which implies that $a \otimes 1$ (and every c_i) belongs to $M_{\bar{\alpha}}^{\mathcal{M}}(A(\alpha), \sigma(\alpha))$, i.e., $\text{sign}_{\alpha}^{\mathcal{M}} \langle a \rangle_{\sigma} = m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$. ■

Since $m_{\alpha}(A, \sigma) \leq m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$ by the definition of signatures, the following corollary is an immediate consequence of Proposition 3.18.

Corollary 3.19. *Assume that R is semilocal and that $\alpha \in \text{Sper}^{\max} R$. Then*

- (1) $m_{\alpha}(A, \sigma) = m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha))$.
- (2) *If $a \in M_{\alpha}^{\mathcal{M}}(A, \sigma)$, then $a \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}}(A(\alpha), \sigma(\alpha))$.*

Note that if $\alpha \notin \text{Nil}[A, \sigma]$, then $m_{\bar{\alpha}}(A(\alpha), \sigma(\alpha)) = n_{\bar{\alpha}}$.

Proof. The final statement is the only one that still requires a proof and follows from [4, Proposition 6.7] since $\bar{\alpha} \notin \text{Nil}[A(\alpha), \sigma(\alpha)]$ by Lemma 3.7. ■

As already mentioned in Remark 3.2, we have $\text{sign}_{\alpha}^{\mathcal{M}} \langle a \rangle_{\sigma} \leq n_{\bar{\alpha}}$ for $a \in \text{Sym}(A, \sigma)$ by [3, Proposition 4.4]. It immediately follows from Corollary 3.19 that the definition of $m_{\alpha}(A, \sigma)$ could include non-invertible elements when R is semilocal.

Corollary 3.20. *Assume that R is semilocal and that $\alpha \in \text{Sper}^{\max} R$. Then*

$$m_{\alpha}(A, \sigma) = \max \{ \text{sign}_{\alpha}^{\mathcal{M}} \langle a \rangle_{\sigma} \mid a \in \text{Sym}(A, \sigma) \}. \quad \blacksquare$$

Remark 3.21. While an element of maximal signature in a central simple algebra with involution over a field is necessarily invertible (cf. Lemma 3.14), this may not be so for Azumaya algebras with involution, even already in the ring case. For example, let $(A, \sigma) = (\mathbb{Z}_{3\mathbb{Z}}, \text{id})$ (in particular, hermitian forms over (A, σ) are just bilinear forms over $\mathbb{Z}_{3\mathbb{Z}}$ and their signatures are the usual Sylvester signatures). The ring $\mathbb{Z}_{3\mathbb{Z}}$ has a unique ordering α_0 , and $\text{sign}_{\alpha_0} \langle 3 \rangle = 1 = m_{\alpha_0}(A, \sigma)$.

4. The involution trace pairing

Let R be a commutative ring (with $2 \in R^{\times}$), let (A, σ) be an Azumaya algebra with involution over R , let $S = Z(A)$ and let $\iota := \sigma|_S$. Note that A is Azumaya over S , but not necessarily Azumaya over R .

4.1. The involution trace form

We consider the reduced trace of A , $\text{Trd}_A: A \rightarrow S$, cf. [18, IV, Theorem 2], and recall that it is additive and S -linear. Furthermore, Trd_A commutes with scalar extensions of S since its computation does not depend on the choice of splitting ring, cf. [18, IV, Proposition 2.1]. In fact, Trd_A also commutes with scalar extensions of R (the case of interest to us) as the following computation shows.

Lemma 4.1. *Let R' be a commutative ring that contains R . Then for all $a \in A$,*

$$\mathrm{Trd}_A(a) \otimes_R 1_{R'} = \mathrm{Trd}_{A \otimes_R R'}(a \otimes_R 1_{R'}).$$

Proof. Observe that $A \otimes_R R' \cong A \otimes_S (S \otimes_R R')$, via $a \otimes_R r' \mapsto a \otimes_S 1 \otimes_R r'$. It follows that

$$\begin{aligned} \mathrm{Trd}_A(a) \otimes_R 1_{R'} &= \mathrm{Trd}_A(a) \otimes_S 1_S \otimes_R 1_{R'} \\ &= \mathrm{Trd}_{A \otimes_S S \otimes_R R'}(a \otimes_S 1_S \otimes_R 1_{R'}) \\ &= \mathrm{Trd}_{A \otimes_R R'}(a \otimes_R 1_{R'}). \end{aligned}$$

■

Lemma 4.2. *For all $a \in A$ we have $\mathrm{Trd}_A(\sigma(a)) = \iota(\mathrm{Trd}_A(a))$.*

Proof. If $\iota = \mathrm{id}_S$, i.e., $S = R$, the statement follows from [16, III, (8.1.1)]. We assume that $\iota \neq \mathrm{id}_S$, and first observe that

$$\begin{aligned} \mathrm{Trd}_A(\sigma(a)) &= \iota(\mathrm{Trd}_A(a)) \\ \Leftrightarrow \forall \mathfrak{p} \in \mathrm{Spec} R \quad \mathrm{Trd}_A(\sigma(a)) \otimes_R 1_{R_{\mathfrak{p}}} &= \iota(\mathrm{Trd}_A(a)) \otimes_R 1_{R_{\mathfrak{p}}} \\ \Leftrightarrow \forall \mathfrak{p} \in \mathrm{Spec} R \quad \mathrm{Trd}_{A \otimes_R R_{\mathfrak{p}}}(\sigma(a) \otimes 1_{R_{\mathfrak{p}}}) &= (\iota \otimes \mathrm{id})(\mathrm{Trd}_A(a) \otimes 1_{R_{\mathfrak{p}}}) \\ \Leftrightarrow \forall \mathfrak{p} \in \mathrm{Spec} R \quad \mathrm{Trd}_{A \otimes_R R_{\mathfrak{p}}}((\sigma \otimes \mathrm{id})(a \otimes 1_{R_{\mathfrak{p}}})) &= (\iota \otimes \mathrm{id})(\mathrm{Trd}_{A \otimes_R R_{\mathfrak{p}}}(a \otimes 1_{R_{\mathfrak{p}}})). \end{aligned}$$

Therefore, it suffices to prove the result for $(A \otimes_R R_{\mathfrak{p}}, \sigma \otimes \mathrm{id})$, and in particular it suffices to prove the statement of the lemma under the extra hypothesis that R is local. In this case, the arguments in [17, (2.15) and (2.16)] hold *mutatis mutandis* for the Azumaya algebra with involution (A, σ) over R . More precisely, since R is local, there is $\lambda \in S$ such that $\lambda^2 \in R^\times$, $\iota(\lambda) = -\lambda$ and $S = R \oplus \lambda R$, cf. Section 2.2. Then [17, (2.15)] holds for (A, σ) , i.e., the map

$$(A \otimes_R S, \sigma \otimes \mathrm{id}) \rightarrow (A \times A^{\mathrm{op}}, \varepsilon), \quad a \otimes s \mapsto (as, (\sigma(a)s)^{\mathrm{op}}),$$

where ε denotes the exchange involution, is an isomorphism of S -algebras with involution (replacing the element α in the proof of [17, (2.15)] by λ , and observing that $\lambda - \iota(\lambda) = 2\lambda \in S^\times$). The claimed equality becomes straightforward to verify after application of this isomorphism since the reduced trace is invariant under scalar extension. ■

The *involution trace form* of (A, σ) is the form

$$T_\sigma: A \times A \rightarrow S, \quad (x, y) \mapsto \mathrm{Trd}_A(\sigma(x)y).$$

By Lemma 4.2, T_σ is symmetric bilinear over R if $S = R$ and hermitian over (S, ι) otherwise.

Lemma 4.3. *The form T_σ is nonsingular, i.e., $(A, T_\sigma) \in \mathfrak{S}(S, \iota)$.*

Proof. Since S is a finitely generated R -module (cf. Definition 2.4 and Proposition 2.3), the form T_σ is nonsingular if and only if the form $T_\sigma \otimes_R R/\mathfrak{m}$ is nonsingular for all maximal ideals \mathfrak{m} of R , cf. [16, I, Lemma 7.1.3]. Since Trd_A commutes with scalar extension, $T_\sigma \otimes_R R/\mathfrak{m}$ is isometric to the involution trace form of the central simple algebra with involution $(A \otimes_R R/\mathfrak{m}, \sigma \otimes \text{id})$, which is nonsingular, cf. [17, Section 11]. ■

Lemma 4.4. *The sandwich map sw (cf. (2.3)) induces an isomorphism*

$$(A \otimes_S A^{\text{op}}, \sigma \otimes \sigma^{\text{op}}) \cong (\text{End}_S(A), \text{ad}_{T_\sigma})$$

of Azumaya algebras with involution over R .

Proof. By (2.3), the sandwich map is an isomorphism of S -algebras. We first show that it respects the involutions, i.e., $\text{ad}_{T_\sigma}(\text{sw}(a \otimes b^{\text{op}})) = \text{sw}(\sigma(a) \otimes \sigma^{\text{op}}(b^{\text{op}}))$ for all $a, b \in A$. With reference to (2.1) this follows from the straightforward computation

$$\begin{aligned} T_\sigma(x, \text{sw}(\sigma(a) \otimes \sigma^{\text{op}}(b^{\text{op}}))(y)) &= \text{Trd}_A(\sigma(x)\sigma(a)y\sigma(b)) \\ &= \text{Trd}_A(\sigma(b)\sigma(x)\sigma(a)y) \\ &= \text{Trd}_A(\sigma(axb)y) \\ &= T_\sigma(axb, y) \\ &= T_\sigma(\text{sw}(a \otimes b^{\text{op}})(x), y) \end{aligned}$$

which holds for all $x, y, a, b \in A$.

Finally, $(\text{End}_S(A), \text{ad}_{T_\sigma})$ is an Azumaya algebra with involution over R . Indeed, by Proposition 2.3, S is finite étale over R and A is Azumaya over S . Hence, $\text{End}_S(A)$ is Azumaya over S and in particular has centre S . Therefore, by Proposition 2.3, $\text{End}_S(A)$ is projective and separable over R . Clearly, ad_{T_σ} is R -linear. Thus we just have to show that if $s \in S \setminus R$, then $\text{ad}_{T_\sigma}(s \cdot \text{id}_A) \neq s \cdot \text{id}_A$. This can be checked by showing that if $\text{ad}_{T_\sigma}(s \cdot \text{id}_A) = s \cdot \text{id}_A$ for some $s \in S$, then $\sigma(s) = s$, using the nonsingularity of T_σ and the definition of ad_{T_σ} in a similar fashion to the computation above. ■

4.2. The Goldman element

We can view Trd_A as an element of $\text{End}_S(A)$. By the definition of the sandwich isomorphism (2.3), there is a unique element $g_A = \sum_i x_i \otimes y_i^{\text{op}}$ in $A \otimes_S A^{\text{op}}$ such that

$$\text{sw}(g_A)(a) = \sum_i x_i a y_i = \text{Trd}_A(a) \quad \text{for all } a \in A. \quad (4.1)$$

The element g_A is called the *Goldman element* of A .

Lemma 4.5. *The Goldman element g_A satisfies*

$$(\sigma \otimes \sigma^{\text{op}})(g_A) = g_A. \quad (4.2)$$

Proof. By Lemma 4.4 it suffices to show that $\text{ad}_{T_\sigma}(\text{Trd}_A) = \text{Trd}_A$ in $\text{End}_S(A)$ in order to prove the claim. Consider (2.1) with $h = T_\sigma$ and $f = \text{Trd}_A$. Using the properties of T_σ and Trd_A we have

$$\begin{aligned} T_\sigma(x, \text{ad}_{T_\sigma}(\text{Trd}_A)(y)) &= T_\sigma(\text{Trd}_A(x), y) \\ &= \iota(\text{Trd}_A(x))T_\sigma(1, y) \\ &= \text{Trd}_A(\sigma(x))T_\sigma(1, y) \\ &= T_\sigma(x, 1)\text{Trd}_A(y) \\ &= T_\sigma(x, \text{Trd}_A(y)) \end{aligned}$$

for all $x, y \in A$. Since T_σ is nonsingular, the claim follows. \blacksquare

We usually think of g_A as an element of $A \otimes_S A$ via the canonical S -module isomorphism $A \otimes_S A \rightarrow A \otimes_S A^{\text{op}}, a \otimes b \mapsto a \otimes b^{\text{op}}$ and write $g_A = \sum_i x_i \otimes y_i$, cf. [18, p. 112]. Since $\sigma \otimes \sigma$ and $\sigma \otimes \sigma^{\text{op}}$ correspond to each other as additive maps under this isomorphism, Lemma 4.5 yields

$$(\sigma \otimes \sigma)(g_A) = g_A. \quad (4.3)$$

in $A \otimes_S A$. Furthermore, we have

$$g_A^2 = 1, \quad (4.4)$$

$$g_A(a \otimes b) = (b \otimes a)g_A \quad \text{for all } a, b \in A, \quad (4.5)$$

cf. [18, IV, Proposition 4.1].

4.3. Module actions

We define tA to be the S -algebra given by the ring A equipped with the following left action by S :

$$S \times A \rightarrow A, \quad (s, a) \mapsto s \cdot^t a := at(s).$$

We denote this action of S on A with the symbol \cdot^t in order to distinguish it from the product in A of elements of S and A . Note that $({}^tA, \sigma)$ is an (S, ι) -algebra with involution as presented in Section 2.1, and that if $(S, \iota) = (R, \text{id})$, then ${}^tA = A$.

We can view A as a left $A \otimes_S {}^tA$ -module via the *twisted* sandwich action:

$$a \otimes b \cdot_{\text{ts}} x := ax\sigma(b) \quad (4.6)$$

for all $a, b, x \in A$ (it is necessary to use tA in the tensor product instead of A , in order for the action to be well defined, which is the motivation for introducing tA).

Lemma 4.6. *The twisted sandwich map*

$$A \otimes_S {}^tA \rightarrow \text{End}_S(A), \quad a \otimes b \mapsto [x \mapsto ax\sigma(b)]$$

induces an isomorphism

$$(A \otimes_S {}^tA, \sigma \otimes \sigma) \cong (\text{End}_S(A), \text{ad}_{T_\sigma})$$

of Azumaya algebras with involution over R .

Proof. The map $\sigma: A^{\text{op}} \rightarrow {}^t A$ is an isomorphism of S -algebras and yields an isomorphism $(A \otimes_S A^{\text{op}}, \sigma \otimes \sigma^{\text{op}}) \cong (A \otimes_S {}^t A, \sigma \otimes \sigma)$ of Azumaya algebras with involution over R . The result then follows from Lemma 4.4. ■

For a right A -module M we denote by ${}^t M$ the right ${}^t A$ -module with the same elements as M , with multiplication ${}^t M \times {}^t A \rightarrow {}^t M$ the same as the multiplication $M \times A \rightarrow M$ and with left action by S given by

$$S \times {}^t M \rightarrow {}^t M, \quad (s, m) \mapsto s \cdot {}^t m := m \iota(s).$$

If $(M, h) \in \mathfrak{S}\text{erm}^\varepsilon(A, \sigma)$, then h is $\iota(\varepsilon)$ -hermitian on ${}^t M$, and we denote it by ${}^t h$.

If M_1 and M_2 are right A -modules a direct verification shows that $M_1 \otimes_S {}^t M_2$ is a right $A \otimes_S {}^t A$ -module with multiplication induced by

$$(M_1 \otimes_S {}^t M_2) \times (A \otimes_S {}^t A) \rightarrow M_1 \otimes_S {}^t M_2, \quad (m_1 \otimes m_2) \cdot (a \otimes b) := m_1 a \otimes m_2 b.$$

4.4. The involution trace pairing

Let $(M_1, h_1) \in \mathfrak{S}\text{erm}^{\varepsilon_1}(A, \sigma)$ and $(M_2, h_2) \in \mathfrak{S}\text{erm}^{\varepsilon_2}(A, \sigma)$, and consider the involution trace form $(A, T_\sigma) \in \mathfrak{S}(S, \iota)$. Using the twisted sandwich action of $A \otimes_S {}^t A$ on A (cf. (4.6)), we can define the S -module $(M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A$, which carries the form

$$T_\sigma \bullet (h_1 \otimes {}^t h_2) \in \mathfrak{S}\text{erm}^{\varepsilon_1 \iota(\varepsilon_2)}(S, \iota),$$

where \bullet denotes the product of forms from [16, I, (8.1), (8.2)]. In other words,

$$T_\sigma \bullet (h_1 \otimes {}^t h_2)(m_1 \otimes m_2 \otimes a, m'_1 \otimes m'_2 \otimes a') := T_\sigma(a, h_1(m_1, m'_1) \otimes {}^t h_2(m_2, m'_2) \cdot_{\text{ts}} a')$$

for all $m_1, m'_1 \in M_1$, $m_2, m'_2 \in {}^t M_2$ and $a, a' \in A$. (Note that we do not indicate all parentheses in long tensor products of elements, in order not to overload the notation.)

For this product to be well defined, A needs to be an $(A \otimes_S {}^t A)$ - S bimodule and the form (A, T_σ) needs to “admit” $A \otimes_S {}^t A$, which means that the equality

$$T_\sigma(\sigma(x) \otimes \sigma(y) \cdot_{\text{ts}} a, b) = T_\sigma(a, x \otimes y \cdot_{\text{ts}} b)$$

must hold for all $x, y, a, b \in A$, but this follows from Lemma 4.6.

In this way we obtain the pairing

$$\begin{aligned} * : \mathfrak{S}\text{erm}^{\varepsilon_1}(A, \sigma) \times \mathfrak{S}\text{erm}^{\varepsilon_2}(A, \sigma) &\rightarrow \mathfrak{S}\text{erm}^{\varepsilon_1 \iota(\varepsilon_2)}(S, \iota), \\ (M_1, h_1) * (M_2, h_2) &:= ((M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A, T_\sigma \bullet (h_1 \otimes {}^t h_2)). \end{aligned} \tag{4.7}$$

Expanding the definition of the pairing and simply writing $h_1 * h_2$, we see that

$$\begin{aligned} h_1 * h_2(m_1 \otimes m_2 \otimes a, m'_1 \otimes m'_2 \otimes a') &= T_\sigma(a, h_1(m_1, m'_1) \otimes {}^t h_2(m_2, m'_2) \cdot_{\text{ts}} a') \\ &= \text{Trd}_A(\sigma(a) h_1(m_1, m'_1) a' \sigma(h_2(m_2, m'_2))) \\ &= \text{Trd}_A(h_1(m_1 a, m'_1 a') \sigma(h_2(m_2, m'_2))). \end{aligned} \tag{4.8}$$

Lemma 4.7. *The Azumaya algebras with involution $(A \otimes_S {}^t A, \sigma \otimes \sigma)$ and (S, ι) are Morita equivalent via*

$$\mathfrak{S}\mathrm{erm}^\varepsilon(A \otimes_S {}^t A, \sigma \otimes \sigma) \rightarrow \mathfrak{S}\mathrm{erm}^\varepsilon(S, \iota), \quad \varphi \mapsto T_\sigma \bullet \varphi.$$

Proof. This follows from Lemma 4.6 and [16, I, Theorem 9.3.5]. ■

Corollary 4.8. *The following properties hold:*

- (1) *The pairing $*$ preserves orthogonal sums in each component.*
- (2) *If h_1 and h_2 are nonsingular, then $h_1 * h_2$ is nonsingular.*
- (3) *If $h_1 \simeq h'_1$ and $h_2 \simeq h'_2$, then $h_1 * h_2 \simeq h'_1 * h'_2$.*

Proof. Let $h_i \in \mathfrak{S}\mathrm{erm}^{\varepsilon_i}(A, \sigma)$ for $i = 1, 2$, then $h_1 * h_2 = T_\sigma \bullet (h_1 \otimes {}^t h_2)$ and the three statements follow from Lemma 4.7, [16, I, Theorem 9.3.5] and standard properties of the tensor product of forms. ■

Theorem 4.9. *Let $(M_i, h_i) \in \mathfrak{S}\mathrm{erm}^{\varepsilon_i}(A, \sigma)$ for $i = 1, 2, 3$. Then*

$$(h_1 * h_2) \otimes_S h_3 \simeq (h_3 * h_2) \otimes_S h_1.$$

Proof. We are grateful to the first referee for suggesting this proof, which is significantly shorter and more conceptual than our original one.

Consider the $\varepsilon_1 \iota(\varepsilon_2) \varepsilon_3$ -hermitian forms $(M, h) := (M_1 \otimes_S {}^t M_2 \otimes_S M_3, h_1 \otimes {}^t h_2 \otimes h_3)$ and $(M', h') := (M_3 \otimes_S {}^t M_2 \otimes_S M_1, h_3 \otimes {}^t h_2 \otimes h_1)$ over $(A \otimes_S {}^t A \otimes_S A, \sigma^{\otimes 3})$. Let $g'_A \in A \otimes_S {}^t A \otimes_S A$ be the image of the Goldman element $g_A \in A \otimes_S A$ under the natural map $a \otimes b \mapsto a \otimes 1 \otimes b$. Using the properties of g'_A , induced by those of g_A (cf. Section 4.2), the following computation shows that (M, h) and (M', h') are isometric via the isomorphism of right $A \otimes_S {}^t A \otimes_S A$ -modules $M \rightarrow M'$, defined by $m_1 \otimes m_2 \otimes m_3 \mapsto (m_3 \otimes m_2 \otimes m_1)g'_A$:

$$\begin{aligned} & h'((m_3 \otimes m_2 \otimes m_1)g'_A, (m'_3 \otimes m'_2 \otimes m'_1)g'_A) \\ &= \sigma^{\otimes 3}(g'_A)h'(m_3 \otimes m_2 \otimes m_1, m'_3 \otimes m'_2 \otimes m'_1)g'_A \\ &= g'_A(h_3(m_3, m'_3) \otimes {}^t h_2(m_2, m'_2) \otimes h_1(m_1, m'_1))g'_A \\ &= h_1(m_1, m'_1) \otimes {}^t h_2(m_2, m'_2) \otimes h_3(m_3, m'_3) \\ &= h(m_1 \otimes m_2 \otimes m_3, m'_1 \otimes m'_2 \otimes m'_3). \end{aligned}$$

Since $(A, \sigma) \cong (\mathrm{End}_A(A), \mathrm{ad}_{(1)_\sigma})$, it follows from Lemmas 4.6 and 2.2 that

$$\begin{aligned} (A \otimes_S {}^t A \otimes_S A, \sigma^{\otimes 3}) &\cong (\mathrm{End}_S(A) \otimes_S A, \mathrm{ad}_{T_\sigma} \otimes \sigma) \\ &\cong (\mathrm{End}_S(A) \otimes_S \mathrm{End}_A(A), \mathrm{ad}_{T_\sigma} \otimes \mathrm{ad}_{(1)_\sigma}) \\ &\cong (\mathrm{End}_{S \otimes_S A}(A \otimes_S A), \mathrm{ad}_{T_\sigma \otimes (1)_\sigma}), \end{aligned}$$

which yields the Morita equivalence

$$\mathfrak{S}\mathrm{erm}^\varepsilon(A \otimes_S {}^t A \otimes_S A, \sigma^{\otimes 3}) \rightarrow \mathfrak{S}\mathrm{erm}^\varepsilon(S \otimes_S A, \iota \otimes \sigma), \quad \varphi \mapsto (T_\sigma \otimes \langle 1 \rangle_\sigma) \bullet \varphi \quad (4.9)$$

by [16, I, Theorem 9.3.5]. The isomorphism of right $A \otimes_S A$ -modules

$$\begin{aligned} & ((M_1 \otimes_S {}^t M_2) \otimes_S M_3) \otimes_{(A \otimes_S {}^t A) \otimes_S A} (A \otimes_S A) \\ & \rightarrow ((M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A) \otimes_S (M_3 \otimes_A A), \\ & m_1 \otimes m_2 \otimes m_3 \otimes a \otimes b \mapsto m_1 \otimes m_2 \otimes a \otimes m_3 \otimes b, \end{aligned}$$

followed by the isomorphism of right A -modules $M_3 \otimes_A A \rightarrow M_3, m_3 \otimes b \mapsto m_3 b$, then yield the isometries

$$(T_\sigma \otimes \langle 1 \rangle_\sigma) \bullet (h_1 \otimes {}^t h_2 \otimes h_3) \simeq T_\sigma \bullet (h_1 \otimes {}^t h_2) \otimes \langle 1 \rangle_\sigma \bullet h_3 \simeq (h_1 * h_2) \otimes h_3.$$

A similar argument shows that $(T_\sigma \otimes \langle 1 \rangle_\sigma) \bullet (h_3 \otimes {}^t h_2 \otimes h_1) \simeq (h_3 * h_2) \otimes h_1$. The result then follows from the isometry $(M, h) \simeq (M', h')$ since (4.9) preserves isometries. ■

Remark 4.10. The pairing $*$ was introduced and studied in detail for ε -hermitian forms over central simple algebras with involution by Garrel in [10]. (A similar construction for quaternion algebras had already been considered by Lewis [20], using the norm form instead of the involution trace form of the quaternion conjugation, cf. [10, Remark 4.4].) In our presentation we stayed close to Garrel's approach via hermitian Morita theory. We are grateful to the second referee for suggesting an alternative approach via the S -linear isomorphism

$$M_1 \otimes_A {}^\sigma M_2 \rightarrow (M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A, \quad m_1 \otimes m_2 \mapsto (m_1 \otimes m_2) \otimes 1$$

(where ${}^\sigma M_2$ is the left A -module obtained by twisting the right A -module structure of M_2 by σ) with inverse

$$\begin{aligned} & (M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A \rightarrow M_1 \otimes_A {}^\sigma M_2, \\ & (m_1 \otimes m_2) \otimes a \mapsto m_1 a \otimes m_2 = m_1 \otimes m_2 \sigma(a), \end{aligned}$$

from which the pairing $*$ can be defined directly as

$$h_1 * h_2(m_1 \otimes m_2, m'_1 \otimes m'_2) := \text{Trd}_A(h_1(m_1, m'_1), \sigma(h_2(m_2, m'_2))).$$

We finish this section with a number of results for later use. We first consider [10, Proposition 4.9] in our context.

Lemma 4.11. *If $b, c \in \text{Sym}(A, \sigma)$, then $\langle b \rangle_\sigma * \langle c \rangle_\sigma \simeq \varphi_{b,c}$, where $(A, \varphi_{b,c}) \in \mathfrak{S}(S, \iota)$ is given by*

$$\varphi_{b,c}: A \times A \rightarrow S, \quad (x, y) \mapsto \text{Trd}_A(\sigma(x)byc).$$

Proof. The form $\langle b \rangle_\sigma * \langle c \rangle_\sigma$ is defined on the S -module $(A \otimes_S {}^t A) \otimes_{A \otimes_S {}^t A} A$. Since A is a left $A \otimes_S {}^t A$ -module, the left action of $A \otimes_S {}^t A$ induces an isomorphism of left $A \otimes_S {}^t A$ -modules (and thus of S -modules):

$$f: (A \otimes_S {}^t A) \otimes_{A \otimes_S {}^t A} A \rightarrow A, \quad x \otimes y \otimes a \mapsto (x \otimes y) \cdot a = x a \sigma(y).$$

Using (4.8), we verify that it is the required isometry:

$$\begin{aligned}
& \varphi_{b,c}(f(x_1 \otimes y_1 \otimes a_1), f(x_2 \otimes y_2 \otimes a_2)) \\
&= \varphi_{b,c}(x_1 a_1 \sigma(y_1), x_2 a_2 \sigma(y_2)) \\
&= \text{Trd}_A(y_1 \sigma(a_1) \sigma(x_1) b x_2 a_2 \sigma(y_2) c) \\
&= \text{Trd}_A(\sigma(a_1) \sigma(x_1) b x_2 a_2 \sigma(y_2) c y_1) \\
&= \text{Trd}_A(\sigma(x_1 a_1) b x_2 a_2 \sigma(\sigma(y_1) c y_2)) \\
&= \text{Trd}_A(\langle b \rangle_\sigma(x_1 a_1, x_2 a_2) \sigma(\langle c \rangle_\sigma(y_1, y_2))) \\
&= \langle b \rangle_\sigma * \langle c \rangle_\sigma(x_1 \otimes y_1 \otimes a_1, x_2 \otimes y_2 \otimes a_2). \quad \blacksquare
\end{aligned}$$

Next we show that $*$ is well behaved under scalar extensions, and start with the following lemma (for which we could not find a reference).

Lemma 4.12. *Let Λ be a commutative ring, and let B and C be Λ -algebras. Let M_1 be a right B -module and M_2 a left B -module. Then*

$$\begin{aligned}
(M_1 \otimes_B M_2) \otimes_\Lambda C &\rightarrow (M_1 \otimes_\Lambda C) \otimes_{B \otimes_\Lambda C} (M_2 \otimes_\Lambda C), \\
(m_1 \otimes m_2) \otimes c &\mapsto (m_1 \otimes 1) \otimes (m_2 \otimes c)
\end{aligned}$$

is an isomorphism of right C -modules.

Proof. Let f be the map defined in the statement of the lemma and let

$$\begin{aligned}
g: (M_1 \otimes_\Lambda C) \otimes_{B \otimes_\Lambda C} (M_2 \otimes_\Lambda C) &\rightarrow (M_1 \otimes_B M_2) \otimes_\Lambda C, \\
(m_1 \otimes c_1) \otimes (m_2 \otimes c_2) &\mapsto (m_1 \otimes m_2) \otimes c_1 c_2.
\end{aligned}$$

A standard (but lengthy) verification shows that f and g are well defined and additive, are inverses of each other, and that f is right C -linear. \blacksquare

Lemma 4.13. *Let T be a commutative R -algebra. Then*

$$(h_1 * h_2) \otimes_R T \simeq (h_1 \otimes_R T) * (h_2 \otimes_R T).$$

Proof. The form $(h_1 * h_2) \otimes_R T$ is defined on $((M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A) \otimes_R T$. Using Lemma 4.12 twice we have

$$\begin{aligned}
& ((M_1 \otimes_S {}^t M_2) \otimes_{A \otimes_S {}^t A} A) \otimes_R T \\
& \cong ((M_1 \otimes_S {}^t M_2) \otimes_R T) \otimes_{(A \otimes_S {}^t A) \otimes_R T} (A \otimes_R T) \\
& \cong ((M_1 \otimes_R T) \otimes_{S \otimes_R T} ({}^t M_2 \otimes_R T)) \otimes_{(A \otimes_R T) \otimes_{S \otimes_R T} ({}^t A \otimes_R T)} (A \otimes_R T), \quad (4.10)
\end{aligned}$$

and the successive isomorphisms are given by

$$\begin{aligned}
[(m_1 \otimes m_2) \otimes a] \otimes t &\mapsto [(m_1 \otimes m_2) \otimes 1] \otimes (a \otimes t) \\
&\mapsto [(m_1 \otimes 1) \otimes (m_2 \otimes 1)] \otimes (a \otimes t).
\end{aligned}$$

We denote the composition of these two isomorphisms by ξ .

By definition, $(h_1 * h_2) \otimes_R T = (T_\sigma \bullet (h_1 \otimes_S {}^t h_2)) \otimes_R T$, while

$$(h_1 \otimes_R T) * (h_2 \otimes_R T) = T_{\sigma \otimes \text{id}_T} \bullet ((h_1 \otimes_R T) \otimes_{S \otimes_R T} ({}^t h_2 \otimes_R T))$$

and is defined on the module (4.10) (using that $Z(A \otimes_R T) = Z(A) \otimes_R T$, cf. Proposition 2.7). We check that ξ is an isometry:

$$\begin{aligned} & T_{\sigma \otimes \text{id}_T} \bullet ((h_1 \otimes_R T) \otimes_{S \otimes_R T} ({}^t h_2 \otimes_R T)) (\xi(m_1 \otimes m_2 \otimes a \otimes t), \xi(m'_1 \otimes m'_2 \otimes a' \otimes t')) \\ &= \text{Trd}_{A \otimes T} ((h_1 \otimes T)((m_1 \otimes 1)(a \otimes t), (m'_1 \otimes 1)(a' \otimes t')) \\ &\quad \cdot (\sigma \otimes \text{id}_T)((h_2 \otimes T)(m_2 \otimes 1, m'_2 \otimes 1))) \\ &= \text{Trd}_{A \otimes T} ((h_1 \otimes T)(m_1 a \otimes t, m'_1 a' \otimes t')(\sigma \otimes \text{id}_T)((h_2 \otimes T)(m_2 \otimes 1, m'_2 \otimes 1))) \\ &= \text{Trd}_{A \otimes T} ((h_1(m_1 a, m'_1 a') \otimes t t')(\sigma \otimes \text{id}_T)(h_2(m_2, m'_2) \otimes 1)) \\ &= \text{Trd}_{A \otimes T} (h_1(m_1 a, m'_1 a') \sigma(h_2(m_2, m'_2)) \otimes t t') \\ &= \text{Trd}_A (h_1(m_1 a, m'_1 a') \sigma(h_2(m_2, m'_2))) \otimes t t' \\ &= ((T_\sigma \bullet (h_1 \otimes {}^t h_2)) \otimes_R T)((m_1 \otimes m_2 \otimes a) \otimes t, (m'_1 \otimes m'_2 \otimes a') \otimes t'), \end{aligned}$$

where we used that Trd_A commutes with scalar extension in the penultimate step. \blacksquare

Remark 4.14. Recall that when S is not connected, the Witt groups $W^\varepsilon(A, \sigma)$ and $W^\varepsilon(S, \iota)$ are trivial, cf. Remark 2.16. Therefore the construction of the pairing $*$ is not interesting in this case.

5. Pairings and PSD quadratic forms

Throughout this section we assume that the commutative ring R (with $2 \in R^\times$) is semi-local. Let (A, σ) be an Azumaya algebra with involution over R , $S = Z(A)$ and $\iota = \sigma|_S$.

Let (M, h) be a hermitian form over (S, ι) . Then $h(x, x) \in \text{Sym}(S, \iota) = R$ for all $x \in M$. Therefore we say that h is positive semidefinite (resp. negative semidefinite) at $\alpha \in \text{Sper } R$ if $h(x, x) \in \alpha$ (resp. $h(x, x) \in -\alpha$) for all $x \in M$. We use the standard abbreviations PSD and NSD.

Lemma 5.1. *Let (M, h) be a hermitian form over (S, ι) and let $\alpha \in \text{Sper } R$. Then h is PSD at α if and only if $h \otimes \kappa(\alpha)$ is PSD at the ordering $\bar{\alpha}$ induced by α on $\kappa(\alpha)$.*

Proof. We use the description of $M \otimes_R \kappa(\alpha)$ given in Remark 2.17.

“ \Rightarrow ”: Let $m \otimes \frac{1}{b} \in M \otimes_R \kappa(\alpha)$, where $m \in M$ and $b \in R \setminus \text{Supp}(\alpha)$. Then

$$h \otimes \kappa(\alpha) \left(m \otimes \frac{1}{b}, m \otimes \frac{1}{b} \right) = \overline{h(m, m)} \left(\frac{1}{b} \right)^2,$$

which is in $\bar{\alpha}$ by hypothesis.

“ \Leftarrow ”: Let $m \in M$. We have $h \otimes \kappa(\alpha)(m \otimes 1, m \otimes 1) = \overline{h(m, m)}$, which belongs to $\bar{\alpha}$, and thus $h(m, m) \in \alpha$. \blacksquare

Remark 5.2. Let F be a real closed field and let P be the unique ordering on F . Recall from Remark 3.6 and the beginning of Section 3 that if $(A, \sigma) = (M_n(D), \vartheta^t)$ with $(D, \vartheta) \in \{(F, \text{id}), (F(\sqrt{-1}), \gamma), ((-1, -1)_F, \gamma)\}$, where γ denotes conjugation, resp. quaternion conjugation, and $a \in \text{Sym}(A, \sigma)$, then $\text{sign}_P^{\mathcal{M}_P} \langle a \rangle_\sigma$ is plus or minus the standard Sylvester signature of a at P .

Indeed, if a is a matrix in $\text{Sym}(M_n(D), \vartheta^t)$ such that $\vartheta^t(a) = a$, then under the hermitian Morita equivalence between $\mathfrak{S}\text{erm}^\varepsilon(M_n(D), \vartheta^t)$ and $\mathfrak{S}\text{erm}^\varepsilon(D, \vartheta)$ in [4, (2.1)], the one-dimensional hermitian form $\langle a \rangle_{\vartheta^t}$ over $(M_n(D), \vartheta^t)$ corresponds to the hermitian form over (D, ϑ) whose Gram matrix is a , and the signature of $\langle a \rangle_{\vartheta^t}$ is defined to be the signature of the matrix a . The definition of signature of hermitian forms allows the use of a different Morita equivalence \mathcal{M}_P than this particular one, which may result in a change of sign.

In particular, there exists $\varepsilon \in \{-1, 1\}$ (which depends only on \mathcal{M}_P) such that for all $a \in \text{Sym}(M_n(D), \vartheta^t)$, if $\text{sign}_P^{\mathcal{M}_P} \langle a \rangle_{\vartheta^t}$ is maximal (among signatures of hermitian forms of the form $\langle b \rangle_{\vartheta^t}$), then εa is a PSD matrix.

Lemma 5.3. *Let F be a field and (B, τ) a central simple F -algebra with involution. Let $P \in X_F \setminus \text{Nil}[B, \tau]$. There is $\delta \in \{-1, 1\}$ such that for every $b, c \in M_P^{\mathcal{M}_P}(B, \tau)$ and every $x \in B$, $\text{Trd}_B(\tau(x)bx c) \in \delta \cdot P$.*

Proof. By [4, Proposition 6.7 and equation (6.1)], we have

$$m_P(B, \tau) = m_{P'}(B \otimes_F F_P, \tau \otimes \text{id}),$$

where P' denotes the unique ordering on F_P . Hence, $x \in M_P^{\mathcal{M}_P}(B, \tau)$ implies

$$x \otimes 1 \in M_{P'}^{\mathcal{M}_{P'}}(B \otimes_F F_P, \tau \otimes \text{id}).$$

Therefore, we can assume that F is real closed with unique ordering P , that $(B, \tau) = (M_n(D), \text{Int}(a) \circ \vartheta^t)$ for some $a \in M_n(D)^\times$ with ϑ of the same kind as τ and

$$(D, \vartheta) \in \{(F, \text{id}), (F(\sqrt{-1}), \gamma), ((-1, -1)_F, \gamma)\},$$

where γ denotes conjugation, resp. quaternion conjugation (cf. [3, first page of Section 2.3]). Note that $\vartheta(a)^t = \delta a$ for some $\delta \in \{-1, 1\}$ by [17, Propositions 2.7 and 2.18]. Furthermore, the algebra (B, τ) is formally real in the sense of [4, Definition 3.2] since $P \notin \text{Nil}[B, \tau]$ (apply [4, Proposition 6.6], where $\tilde{X}_F := X_F \setminus \text{Nil}[B, \tau]$), and it follows that we can choose the involution ϑ such that $\delta = 1$ by [4, Corollary 3.8].

With reference to [1, Remark 3.13], observe that

$$a^{-1} \langle b \rangle_\tau = \langle a^{-1} b \rangle_{\text{Int}(a^{-1}) \circ \tau} = \langle a^{-1} b \rangle_{\vartheta^t},$$

and that there is a Morita equivalence \mathcal{M}'_P such that

$$m_P(B, \tau) = \text{sign}_P^{\mathcal{M}_P} \langle b \rangle_\tau = \text{sign}_P^{\mathcal{M}'_P} a^{-1} \langle b \rangle_\tau = \text{sign}_P^{\mathcal{M}'_P} \langle a^{-1} b \rangle_{\vartheta^t}.$$

Similarly we have $\text{sign}_P^{\mathcal{M}'_P} \langle a^{-1}c \rangle_{\vartheta^t} = m_P(B, \tau)$. By Remark 5.2 there exists $\varepsilon \in \{-1, 1\}$ such that $\varepsilon a^{-1}b$ and $\varepsilon a^{-1}c$ are both PSD with respect to P . Write $b' := \varepsilon a^{-1}b$ and $c' := \varepsilon a^{-1}c$. The matrix $\vartheta(a)^t c' a$ is then PSD over (D, ϑ) , and thus is a hermitian square in $(M_n(D), \vartheta^t)$, i.e., there exists a matrix $c'' \in M_n(D)$ such that $\vartheta(a)^t c' a = \vartheta(c'')^t c''$ (this is a classical consequence of the principal axis theorem, which also holds for quaternion matrices by [29, Corollary 6.2], cf. [4, Appendix A]). It follows that

$$\begin{aligned} \text{Trd}_B(\tau(x)bx c) &= \text{Trd}_B((\text{Int}(a) \circ \vartheta^t)(x)(a\varepsilon b')x(a\varepsilon c')) \\ &= \text{Trd}_B(a\vartheta(x)^t a^{-1}ab'x a c') \\ &= \text{Trd}_B(a\vartheta(x)^t b'x a c') \\ &= \delta \text{Trd}_B(\vartheta(x)^t b'x \vartheta(a)^t c' a) \\ &= \delta \text{Trd}_B(\vartheta(x)^t b'x \vartheta(c'')^t c'') \\ &= \delta \text{Trd}_B(c''\vartheta(x)^t b'x \vartheta(c'')^t), \end{aligned}$$

which belongs to δP since b' is PSD. \blacksquare

Proposition 5.4. *Let $\alpha \in \text{Sper } R \setminus \text{Nil}[A, \sigma]$. Then there is $\delta \in \{-1, 1\}$ such that for every $b, c \in M_\alpha^{\mathcal{M}}(A, \sigma)$, the hermitian form $\delta(\langle b \rangle_\sigma * \langle c \rangle_\sigma)$ is nonsingular and PSD with respect to α .*

Proof. The forms $\langle b \rangle_\sigma$ and $\langle c \rangle_\sigma$ are nonsingular since b and c are invertible, and thus $\langle b \rangle_\sigma * \langle c \rangle_\sigma$ is nonsingular by Corollary 4.8. By Lemma 5.1, it suffices to show that the form $(\delta \langle b \rangle_\sigma * \langle c \rangle_\sigma) \otimes \kappa(\alpha)$ is PSD with respect to $\bar{\alpha}$. Using now that

$$(\langle b \rangle_\sigma * \langle c \rangle_\sigma) \otimes \kappa(\alpha) \simeq \langle b \otimes 1 \rangle_{\sigma \otimes \text{id}} * \langle c \otimes 1 \rangle_{\sigma \otimes \text{id}}$$

by Lemma 4.13, and that $b \otimes 1, c \otimes 1 \in M_{\bar{\alpha}}^{\mathcal{M}}(A(\alpha), \sigma(\alpha))$ by Corollary 3.19, we can assume that (A, σ) is a central simple algebra with involution over the field $\kappa(\alpha)$ with ordering $\bar{\alpha}$ that is non-nil by Lemma 3.7. The result then follows from Lemma 5.3 since $\langle b \rangle_\sigma * \langle c \rangle_\sigma$ is isometric to the hermitian form $\varphi_{b,c}$, cf. Lemma 4.11. \blacksquare

Lemma 5.5. *Let $\alpha \in \text{Sper } R \setminus \text{Nil}[A, \sigma]$, and let $b, c \in M_\alpha^{\mathcal{M}}(A, \sigma)$. Then there are nonsingular hermitian forms φ_1 and φ_2 over (S, ι) that are PSD at α such that $\varphi_1 \otimes_S \langle b \rangle_\sigma \simeq \varphi_2 \otimes_S \langle c \rangle_\sigma$.*

Proof. Let $\delta \in \{-1, 1\}$ be as given by Proposition 5.4. Then the hermitian forms $\delta(\langle c \rangle_\sigma * \langle b \rangle_\sigma)$ and $\delta(\langle b \rangle_\sigma * \langle c \rangle_\sigma)$ over (S, ι) are PSD with respect to α . Furthermore, they are nonsingular since b and c are invertible and by Corollary 4.8. By Theorem 4.9 we have

$$(\langle c \rangle_\sigma * \langle b \rangle_\sigma) \otimes_S \langle b \rangle_\sigma \simeq (\langle b \rangle_\sigma * \langle c \rangle_\sigma) \otimes_S \langle c \rangle_\sigma,$$

hence

$$\delta(\langle c \rangle_\sigma * \langle b \rangle_\sigma) \otimes_S \langle b \rangle_\sigma \simeq \delta(\langle b \rangle_\sigma * \langle c \rangle_\sigma) \otimes_S \langle c \rangle_\sigma,$$

and we conclude with Proposition 5.4. \blacksquare

6. Sylvester's law of inertia and Pfister's local-global principle

Let (A, σ) be an Azumaya algebra with involution over R , $S = Z(A)$ and $\iota = \sigma|_S$. In the case of central simple algebras with involution over fields, a weaker version of the following result appears in [4, Theorem 8.9].

Theorem 6.1 (Sylvester's law of inertia). *Assume that R is semilocal connected and let $\alpha \in \text{Sper } R \setminus \text{Nil}[A, \sigma]$. Then S is connected and, letting $t := \text{rank}_S A$, we have:*

- (1) *Let h be a nonsingular hermitian form over (A, σ) . For every $c \in M_\alpha^{\mathcal{M}}(A, \sigma)$ there are $w_1, \dots, w_t, u_1, \dots, u_r, v_1, \dots, v_s \in \alpha \cap R^\times$ such that*

$$\langle w_1, \dots, w_t \rangle \otimes_R h \simeq \langle u_1, \dots, u_r \rangle \otimes_R \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_s \rangle \otimes_R \langle c \rangle_\sigma.$$

- (2) *Assume $\langle a_1, \dots, a_r \rangle_\sigma \perp \langle -b_1, \dots, -b_s \rangle_\sigma \simeq \langle a'_1, \dots, a'_p \rangle_\sigma \perp \langle -b'_1, \dots, -b'_q \rangle_\sigma$ with $a_1, \dots, a_r, b_1, \dots, b_s, a'_1, \dots, a'_p, b'_1, \dots, b'_q \in M_\alpha^{\mathcal{M}}(A, \sigma)$. Then $r = p$ and $s = q$.*

Proof. We first observe that S is connected by Remark 3.9. By Lemma 2.14 every nonsingular hermitian form over (S, ι) is diagonalizable. By Proposition 2.3, A is Azumaya over S , hence a projective S -module and so $\text{rank}_S A$ is defined, and is constant since S is connected, cf. [26, p. 12].

(1) By Theorem 4.9 we have $(\langle c \rangle_\sigma * \langle c \rangle_\sigma) \otimes_S h \simeq (h * \langle c \rangle_\sigma) \otimes_S \langle c \rangle_\sigma$. Since $h * \langle c \rangle_\sigma$ is nonsingular by Corollary 4.8 and diagonalizable, we can write

$$h * \langle c \rangle_\sigma \simeq \langle u_1, \dots, u_r \rangle_\iota \perp \langle -v_1, \dots, -v_s \rangle_\iota,$$

with $u_1, \dots, u_r, v_1, \dots, v_s \in \alpha \cap R^\times$. Therefore

$$\begin{aligned} (\langle c \rangle_\sigma * \langle c \rangle_\sigma) \otimes_S h &\simeq (h * \langle c \rangle_\sigma) \otimes_S \langle c \rangle_\sigma \\ &\simeq \langle u_1, \dots, u_r \rangle_\iota \otimes_S \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_s \rangle_\iota \otimes_S \langle c \rangle_\sigma. \end{aligned}$$

By Proposition 5.4, the nonsingular hermitian form $\langle c \rangle_\sigma * \langle c \rangle_\sigma$ over (S, ι) is PSD or NSD with respect to α . Up to replacing it by its opposite, we can assume it is PSD with respect to α . As observed above, it is diagonalizable, and by Lemma 4.11 it is defined on A and therefore has dimension $\text{rank}_S A = t$. Hence there are $w_1, \dots, w_t \in \alpha \cap R^\times$ such that $\langle c \rangle_\sigma * \langle c \rangle_\sigma \simeq \langle w_1, \dots, w_t \rangle_\iota$, and thus

$$\langle w_1, \dots, w_t \rangle_\iota \otimes_S h \simeq \langle u_1, \dots, u_r \rangle_\iota \otimes_S \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_s \rangle_\iota \otimes_S \langle c \rangle_\sigma.$$

The result now follows from Lemma 2.1.

(2) For dimension reasons (after localisation, since A is a projective R -module) we have $r + s = p + q$. The result will follow if we show $r - s = p - q$, i.e., $r + q = p + s$. We have the following equality in the Witt group $W(A, \sigma)$:

$$\langle a_1, \dots, a_r, b'_1, \dots, b'_q \rangle_\sigma = \langle a'_1, \dots, a'_p, b_1, \dots, b_s \rangle_\sigma,$$

which implies that

$$\langle a_1, \dots, a_r, b'_1, \dots, b'_q \rangle_\sigma \perp \varphi \simeq \langle a'_1, \dots, a'_p, b_1, \dots, b_s \rangle_\sigma \perp \psi,$$

where φ and ψ are hyperbolic forms over (A, σ) . Taking signatures on both sides yields

$$(r + q)m_\alpha(A, \sigma) = (p + s)m_\alpha(A, \sigma)$$

since hyperbolic forms have signature zero. The result follows. \blacksquare

For $r_1, \dots, r_\ell \in R$, we use the notation $\langle\langle r_1, \dots, r_\ell \rangle\rangle$ to denote the Pfister form

$$\langle 1, r_1 \rangle \otimes_R \cdots \otimes_R \langle 1, r_\ell \rangle.$$

Corollary 6.2. *Assume that R is semilocal connected.*

- (1) *Assume that σ is of orthogonal or symplectic type (so that $(S, \iota) = (R, \text{id})$). Let $\alpha \in \text{Nil}[A, \sigma]$ and let h be a nonsingular hermitian form over (A, σ) . Then there is a nonsingular quadratic form q over R of dimension $\text{rank}_R A$ that is PSD at α and such that $q \otimes_R h$ is hyperbolic.*
- (2) *Let $\alpha \in \text{Sper } R \setminus \text{Nil}[A, \sigma]$, let $a, b \in M_\alpha^{\text{M}}(A, \sigma)$, and let $t := \text{rank}_S A$. Then there are $\ell \in \mathbb{N}$ and $r_1, \dots, r_\ell, w_1, \dots, w_t \in \alpha \cap R^\times$ such that*

$$(\langle w_1, \dots, w_t \rangle \otimes_R \langle\langle r_1, \dots, r_\ell \rangle\rangle) \otimes_R \langle a, -b \rangle_\sigma$$

is hyperbolic.

Proof. (1) By Lemma 3.11 there exists a skew-symmetric element $a \in A^\times$. Let $\tau = \text{Int}(a) \circ \sigma$ and note that τ is orthogonal if σ is symplectic and vice versa (indeed, by Proposition 2.12 it suffices to check it for the central simple $\kappa(\alpha)$ -algebra with involution $(A(\alpha), \sigma(\alpha))$, where it is true by [17, Proposition 2.7]). It follows from Remark 3.6 and Lemma 3.7 (1) \Leftrightarrow (2) that $\alpha \in \text{Sper}(R) \setminus \text{Nil}[A, \tau]$. Scaling h by a gives the nonsingular skew-hermitian form $h' := ah$ over (A, τ) .

Let $c \in M_\alpha^{\text{M}}(A, \tau)$. Then the form $\langle c \rangle_\tau$ is nonsingular, and so the pairing $\langle c \rangle_\tau * \langle c \rangle_\tau$ is also nonsingular by Corollary 4.8. By Proposition 5.4, there exists $\delta \in \{-1, 1\}$ such that the form $\delta(\langle c \rangle_\tau * \langle c \rangle_\tau)$ is PSD at α .

We now consider the form $\delta(\langle c \rangle_\tau * \langle c \rangle_\tau) \otimes_R h'$ which is isometric to $\delta(h' * \langle c \rangle_\tau) \otimes_R \langle c \rangle_\tau$ by Theorem 4.9. But $h' * \langle c \rangle_\tau$ is a pairing of a skew-hermitian and a hermitian form over (A, τ) , and so is skew-symmetric over $(S, \iota) = (R, \text{id})$ by (4.7). Hence, $h' * \langle c \rangle_\tau$ is hyperbolic by [16, I, Corollary 4.1.2] since every projective R -module is free by [11].

Therefore, letting $q := \delta(\langle c \rangle_\tau * \langle c \rangle_\tau)$, the form $q \otimes_R h'$ is hyperbolic and since scaling by a^{-1} commutes with tensoring by q , we obtain that $q \otimes_R h$ is hyperbolic by [16, I, Theorem 9.3.5] applied to the scaling-by- a^{-1} Morita equivalence. Note that q is diagonalizable since R is semilocal and A is projective over R , and that the dimension of q is $\text{rank}_R A$, cf. Lemma 4.11.

(2) Observe that $\text{rank}_S A$ is constant by the first paragraph of the proof of Theorem 6.1. Let $c \in M_\alpha^{\mathcal{M}}(A, \sigma)$. By Theorem 6.1, there are $w_1, \dots, w_t, u_1, \dots, u_r, v_1, \dots, v_s \in \alpha \cap R^\times$ such that

$$\langle w_1, \dots, w_t \rangle \otimes_R \langle a, -b \rangle_\sigma \simeq \langle u_1, \dots, u_r \rangle \otimes_R \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_s \rangle \otimes_R \langle c \rangle_\sigma.$$

The signature at α of the left-hand side is 0, so we must have $r = s$. Therefore,

$$\begin{aligned} \langle w_1, \dots, w_t \rangle \otimes_R \langle a, -b \rangle_\sigma &\simeq \langle u_1, \dots, u_r \rangle \otimes_R \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_r \rangle \otimes_R \langle c \rangle_\sigma \\ &\simeq \langle \bar{u} \rangle \otimes_R \langle c \rangle_\sigma \perp \langle -\bar{v} \rangle \otimes_R \langle c \rangle_\sigma \simeq \langle \bar{u}, -\bar{v} \rangle \otimes_R \langle c \rangle_\sigma, \end{aligned}$$

where we write \bar{u} for u_1, \dots, u_r and similarly for \bar{v} . We use the notation $\langle \bar{u}, \bar{v} \rangle$ for the Pfister form $\langle u_1, \dots, u_r, v_1, \dots, v_s \rangle$. The form $\langle \bar{u}, \bar{v} \rangle \otimes_R \langle \bar{u}, -\bar{v} \rangle$ is hyperbolic since

$$\begin{aligned} \langle \bar{u}, \bar{v} \rangle \otimes_R \langle \bar{u}, -\bar{v} \rangle &\simeq \langle \bar{u}, \bar{v} \rangle \otimes_R \langle \bar{u} \rangle \perp \langle \bar{u}, \bar{v} \rangle \otimes_R \langle -\bar{v} \rangle \\ &\simeq \bigoplus_{i=1}^r \underbrace{\langle \bar{u}, \bar{v} \rangle \otimes_R \langle u_i \rangle}_{\langle \bar{u}, \bar{v} \rangle} \perp \bigoplus_{i=1}^r - \underbrace{\langle \bar{u}, \bar{v} \rangle \otimes_R \langle v_i \rangle}_{\langle \bar{u}, \bar{v} \rangle} \\ &\simeq r \times \langle \bar{u}, \bar{v} \rangle \perp -r \times \langle \bar{u}, \bar{v} \rangle, \end{aligned}$$

where the final equality holds since Pfister forms over R are round (cf. [5, Corollary 2.16] which uses the fact that the hypothesis $2 \in R^\times$ ensures that $|R/\mathfrak{m}| > 2$ for every maximal ideal \mathfrak{m} of R). Therefore $(\langle w_1, \dots, w_t \rangle \otimes_R \langle \bar{u}, \bar{v} \rangle) \otimes_R \langle a, -b \rangle_\sigma$ is hyperbolic, proving the result. ■

The main idea of the proof of the next result comes from Marshall's proof of Pfister's local-global principle in [23, Theorem 4.12].

Proposition 6.3. *Let R be semilocal connected and let $t_0 := \text{rank}_R A$. Let h be a nonsingular hermitian form over (A, σ) such that for every $n \in \mathbb{N}$, the form $2^n t_0^2 \times h$ is not hyperbolic. Then there is $\alpha \in \text{Sper}^{\max} R$ such that $\text{sign}_\alpha^{\mathcal{M}} h \neq 0$.*

Proof. Observe that if $t = \text{rank}_S A$ is defined, then t_0 is equal to t (if $R = S$) or $2t$ (if $R \neq S$). We identify nonsingular hermitian forms with their classes in the Witt group $W(A, \sigma)$, considered as a $W(R)$ -module. Note that a nonsingular hermitian form is hyperbolic if and only if its class is zero since we assume that $2 \in A^\times$ (and therefore metabolic forms are hyperbolic).

For the convenience of the reader we first present the ideas of the three main steps of the proof before giving the full details:

- (1) We define a maximal non-empty set \mathcal{S} of (nonsingular) Pfister forms over R such that $p \cdot h \neq 0$ for every $p \in \mathcal{S}$, and such that \mathcal{S} is closed under products. This final property produces an ideal $J := \bigcup_{p \in \mathcal{S}} \text{Ann}_{W(A, \sigma)}(p)$, and $h \notin J$ by construction (this is linked to the notion of Pfister quotient in [23, Chapter 4.7]).

The actual construction in the proof below is slightly different (a factor t_0^2 appears for technical reasons), and the ideal is called $J_{\mathcal{S}}$.

- (2) The maximality of \mathcal{S} ensures that the set α_0 of all elements of R^\times represented by these Pfister forms is “almost” an ordering, more precisely: $\alpha_0 = \alpha \cap R^\times$ for some $\alpha \in \text{Sper}^{\max} R$.
- (3) The final step of the proof consists in checking that $\text{sign}_\alpha^{\mathcal{M}} \subseteq J$, proving the result since $h \notin J$.

We now proceed with the proof. Let $\mathcal{S} \subseteq W(R)$ be a maximal subset of Pfister forms such that:

- (a) $\mathcal{S} \cdot \mathcal{S} \subseteq \mathcal{S}$;
- (b) for every $n \in \mathbb{N}$, $2^n \times \langle 1 \rangle \in \mathcal{S}$;
- (c) for every $p \in \mathcal{S}$, $(t_0^2 \times p) \cdot h \neq 0$.

Observe that \mathcal{S} exists since the set $\{2^n \times \langle 1 \rangle \mid n \in \mathbb{N}\}$ satisfies all these conditions. Define

$$\alpha_0 := \{r \in R^\times \mid \exists p \in \mathcal{S}, \langle r \rangle \cdot p = p\}.$$

Clearly $\alpha_0 \cdot \alpha_0 \subseteq \alpha_0$. We first prove the following four properties:

- (P1) $\alpha_0 \cup -\alpha_0 = R^\times$;
- (P2) $\alpha_0 \cap -\alpha_0 = \emptyset$;
- (P3) for every $u \in \alpha_0$, $\langle 1, u \rangle \in \mathcal{S}$;
- (P4) for every $k \in \mathbb{N}$ and all $u_1, \dots, u_k \in \alpha_0$, $\langle\langle u_1, \dots, u_k \rangle\rangle \in \mathcal{S}$.

Proof of (P1). For $x \in R^\times$ we define $\mathcal{S}_x := \mathcal{S} \cup \langle 1, x \rangle \mathcal{S}$. Obviously $\mathcal{S} \subseteq \mathcal{S}_x$. In particular, \mathcal{S}_x satisfies property (b). We check the non-obvious case of property (a): If $p, q \in \mathcal{S}$, then $\langle 1, x \rangle p \langle 1, x \rangle q = 2 \langle 1, x \rangle pq$, which belongs to \mathcal{S}_x by properties (a) and (b) of \mathcal{S} .

Let $x \in R^\times$. If $x \notin \alpha_0$ then $\langle 1, x \rangle \notin \mathcal{S}$ (otherwise, since $\langle x \rangle \langle 1, x \rangle = \langle 1, x \rangle$ we would get $x \in \alpha_0$), so $\mathcal{S} \subsetneq \mathcal{S}_x$.

So if we assume that $x \notin \alpha_0$ and $-x \notin \alpha_0$ we obtain $\mathcal{S} \subsetneq \mathcal{S}_x$ and $\mathcal{S} \subsetneq \mathcal{S}_{-x}$. Since both \mathcal{S}_x and \mathcal{S}_{-x} satisfy properties (a) and (b), by maximality of \mathcal{S} we get that (c) does not hold for either of \mathcal{S}_x and \mathcal{S}_{-x} : There are $p, q \in \mathcal{S}$ such that $t_0^2 \times \langle 1, x \rangle ph = 0$ and $t_0^2 \times \langle 1, -x \rangle qh = 0$. Therefore $t_0^2 \times \langle 1, x \rangle pqh = 0$ and $t_0^2 \times \langle 1, -x \rangle pqh = 0$. Adding both we obtain $t_0^2 \times 2pqh = 0$ in $W(A, \sigma)$, a contradiction since $2pq \in \mathcal{S}$. End of the proof of (P1).

Proof of (P2). Assume that there is $x \in \alpha_0 \cap -\alpha_0$. Then there are $p, q \in \mathcal{S}$ such that $\langle x \rangle p = p$ and $\langle -x \rangle q = q$. Therefore $\langle x \rangle pq = pq$ and $\langle -x \rangle pq = pq$. Adding both, we get $0 = 2pq$ and thus $2pq \cdot h = 0$, a contradiction since $2pq \in \mathcal{S}$. End of the proof of (P2).

Proof of (P3). Let $u \in \alpha_0$ and let $p_0 \in \mathcal{S}$ be such that $\langle u \rangle \cdot p_0 = p_0$. Consider, as in the proof of (P1), $\mathcal{S}_u := \mathcal{S} \cup \langle 1, u \rangle \mathcal{S}$. As seen in the proof of (P1), \mathcal{S}_u satisfies properties (a) and (b). We check property (c): Assume $t_0^2 \times \langle 1, u \rangle ph = 0$ for some $p \in \mathcal{S}$. Then $t_0^2 \times \langle 1, u \rangle p_0 ph = 0$, i.e., $t_0^2 \times 2p_0 ph = 0$, which is impossible since $2p_0 p \in \mathcal{S}$. Since \mathcal{S}_u satisfies properties (a), (b) and (c), and contains \mathcal{S} , we must have $\mathcal{S} = \mathcal{S}_u$ by maximality of \mathcal{S} . Therefore $\langle 1, u \rangle \in \mathcal{S}$. End of the proof of (P3).

(P4) is a direct consequence of (P3) and property (a). This finishes the proof of (P1)–(P4).

Next, we define

$$J_{\mathcal{S}} := \{\psi \in W(A, \sigma) \mid \exists p \in \mathcal{S}, (t_0^2 \times p) \cdot \psi = 0\}.$$

Clearly, $h \notin J_{\mathcal{S}}$ by property (c).

Consider the map

$$\chi: R^\times \rightarrow \{-1, 1\}, \quad \chi(r) := \begin{cases} 1 & \text{if } r \in \alpha_0, \\ -1 & \text{if } r \in -\alpha_0. \end{cases}$$

Then χ is a signature of R . Indeed, it is a character on R^\times since $\alpha_0 \cdot \alpha_0 \subseteq \alpha_0$ and by properties (P1) and (P2). Furthermore, by [14, top of p. 88], χ will then be a signature of R if it satisfies $\chi(-1) = -1$ (which is true), as well as $\chi(r_1^2 f_1 + \cdots + r_k^2 f_k) = 1$ whenever $f_1, \dots, f_k \in R^\times$ with $\chi(f_1) = \cdots = \chi(f_k) = 1$ and $r_1, \dots, r_k \in R$ are such that $z := r_1^2 f_1 + \cdots + r_k^2 f_k \in R^\times$. We check this: Since $f_1, \dots, f_k \in \alpha_0$, we have by (P4) that the Pfister form $\langle\langle f_1, \dots, f_k \rangle\rangle$ is in \mathcal{S} . Clearly, $z \in D_R \langle\langle f_1, \dots, f_k \rangle\rangle$ and by [5, p. 94, Theorem 2.1] (which applies since the hypothesis $2 \in R^\times$ ensures that every quotient of R by a proper ideal has at least 3 elements) we have $\langle z \rangle \langle\langle f_1, \dots, f_k \rangle\rangle = \langle\langle f_1, \dots, f_k \rangle\rangle$. Thus $z \in \alpha_0$ and $\chi(z) = 1$.

By the correspondence between $\text{Sign } R$ and $\text{Sper}^{\max} R$ (cf. Section 2.4) there is $\alpha \in \text{Sper}^{\max} R$ such that $\chi = \text{sign}_\alpha$, and thus $\alpha_0 = \alpha \cap R^\times$.

Claim 1. If σ is of unitary type, then $\alpha \not\subseteq \text{Nil}[A, \sigma]$.

Proof of Claim 1. Observe first that for every $q \in \mathcal{S}$, $q \otimes_R \langle 1 \rangle_\iota$ is not hyperbolic (if it were, then $(q \otimes_R \langle 1 \rangle_\iota) \otimes_S h$ would be hyperbolic, and thus, by Lemma 2.1, $(q \otimes_R \langle 1 \rangle_\iota) \otimes_S h \simeq q \otimes_R h$ would also be hyperbolic, contradicting the definition of \mathcal{S}).

Assume that $\alpha \in \text{Nil}[A, \sigma] = \text{Nil}[S, \iota] = \bar{H}(d)$ for some $d \in R^\times$ (by Lemma 3.7 and Corollary 3.8). Then $d \in \alpha \cap R^\times = \alpha_0$ and $\text{sign}^{\mathfrak{M}} \langle 1, d \rangle \otimes_R \langle 1 \rangle_\iota = 0$ on $\text{Sper } R$. (Indeed, the signature is zero on $\text{Nil}[A, \sigma]$ by definition, and the signature of $\langle 1, d \rangle$ is zero on $\text{Sper } R \setminus \text{Nil}[A, \sigma]$ by definition of d and since $d \notin \text{Supp } \alpha$.) Note that $\langle 1, d \rangle \in \mathcal{S}$ by (P3). Since R is connected, S is quadratic étale over R and ι is the standard involution on S by Proposition 2.12, and we may use Lemma 3.4. Therefore,

$$\text{sign } \text{Tr}_{S/R} (\langle 1, d \rangle \otimes_R \langle 1 \rangle_\iota) = 0$$

on $\text{Sper } R$. Since $\text{Tr}_{S/R} (\langle 1, d \rangle \otimes_R \langle 1 \rangle_\iota)$ is nonsingular (cf. Section 2.2), Pfister's local-global principle for semilocal rings (cf. [22, p. 194] or [5, Theorem 7.16]) applies and there is $k \in \mathbb{N}$ such that $2^k \times \text{Tr}_{S/R} (\langle 1, d \rangle \otimes_R \langle 1 \rangle_\iota)$ is hyperbolic. Applying [8, Corollary 8.3] yields that $2^k \times \langle 1, d \rangle \otimes_R \langle 1 \rangle_\iota$ is hyperbolic, a contradiction (as observed above) since $2^k \times \langle 1, d \rangle$ is in \mathcal{S} . End of the proof of Claim 1.

Claim 2. $\ker \text{sign}_\alpha^{\mathfrak{M}} \subseteq J_{\mathcal{S}}$.

Proof of Claim 2. Let $\psi \in \ker \text{sign}_\alpha^{\mathcal{M}}$. We consider two cases.

- (1) $\alpha \in \text{Nil}[A, \sigma]$. By Claim 1 we know that σ must be of orthogonal or symplectic type. In particular, $S = R$ and $t_0 = t$. By Corollary 6.2 (1), there is a nonsingular diagonal quadratic form $\langle u_1, \dots, u_{t_0} \rangle$ of dimension t_0 with coefficients in α (and thus in $\alpha \cap R^\times = \alpha_0$) such that $\langle u_1, \dots, u_{t_0} \rangle \psi$ is hyperbolic. Multiplying by $\langle\langle u_1, \dots, u_{t_0} \rangle\rangle$ and using that $\langle u_i \rangle \langle\langle u_1, \dots, u_{t_0} \rangle\rangle = \langle\langle u_1, \dots, u_{t_0} \rangle\rangle$ (cf. [5, Corollary 2.16] which uses the fact that the hypothesis $2 \in R^\times$ ensures that $|R/\mathfrak{m}| > 2$ for every maximal ideal \mathfrak{m} of R), we obtain $t_0 \times \langle\langle u_1, \dots, u_{t_0} \rangle\rangle \psi = 0$. By property (P4) the form $\langle\langle u_1, \dots, u_{t_0} \rangle\rangle$ is in \mathcal{S} and thus $\psi \in J_{\mathcal{S}}$.
- (2) $\alpha \notin \text{Nil}[A, \sigma]$. By Theorem 6.1, $t = \text{rank}_S A$ exists and we have

$$\langle w_1, \dots, w_t \rangle \psi \simeq \langle u_1, \dots, u_r \rangle \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_s \rangle \langle c \rangle_\sigma,$$

for some $w_1, \dots, w_t, u_1, \dots, u_r, v_1, \dots, v_s \in \alpha \cap R^\times$ and $c \in \text{Sym}(A^\times, \sigma)$ such that $\text{sign}_\alpha^{\mathcal{M}} \langle c \rangle_\sigma = m_\alpha(A, \sigma)$. Multiplying both sides by $\langle\langle w_1, \dots, w_t \rangle\rangle$ and then by 2, the left-hand side becomes first $t \times \langle\langle w_1, \dots, w_t \rangle\rangle \psi$ (using that $\langle w_i \rangle \langle\langle w_1, \dots, w_t \rangle\rangle = \langle\langle w_1, \dots, w_t \rangle\rangle$) and then $t_0 \times \langle\langle w_1, \dots, w_t \rangle\rangle \psi$, while the right-hand side still retains the same shape (up to taking larger values for r and s , and different elements $u_i, v_j \in \alpha \cap R^\times$). We thus have

$$t_0 \times \langle\langle w_1, \dots, w_t \rangle\rangle \psi \simeq \langle u_1, \dots, u_r \rangle \langle c \rangle_\sigma \perp \langle -v_1, \dots, -v_s \rangle \langle c \rangle_\sigma,$$

where $u_1, \dots, u_r, v_1, \dots, v_s \in \alpha \cap R^\times$.

Since $\text{sign}_\alpha^{\mathcal{M}} \psi = 0$ and $\text{sign}_\alpha^{\mathcal{M}} \langle u_i \rangle \langle c \rangle_\sigma = \text{sign}_\alpha^{\mathcal{M}} \langle v_j \rangle \langle c \rangle_\sigma = m_\alpha(A, \sigma)$ for $i = 1, \dots, r$ and $j = 1, \dots, s$, we must have $r = s$. In particular, we can pair each $u_i c$ with the corresponding $-v_i c$, so that

$$t_0 \times \langle\langle w_1, \dots, w_t \rangle\rangle \psi \simeq \bigoplus_{i=1}^r \langle u_i c, -v_i c \rangle_\sigma. \quad (6.1)$$

Fact. For each $i = 1, \dots, r$ there is $p_i \in \mathcal{S}$ such that $t_0 \times p_i \cdot \langle u_i c, -v_i c \rangle_\sigma = 0$.

Proof of the fact. By Corollary 6.2 (2), there are $z_1, \dots, z_t, r_1, \dots, r_\ell \in \alpha \cap R^\times = \alpha_0$ such that $\langle z_1, \dots, z_t \rangle \langle\langle r_1, \dots, r_\ell \rangle\rangle \langle u_i c, -v_i c \rangle_\sigma = 0$. Multiplying by the Pfister form $\langle\langle z_1, \dots, z_t \rangle\rangle$, we obtain

$$(t \times \langle\langle z_1, \dots, z_t, r_1, \dots, r_\ell \rangle\rangle) \langle u_i c, -v_i c \rangle_\sigma = 0,$$

and thus $(t_0 \times \langle\langle z_1, \dots, z_t, r_1, \dots, r_\ell \rangle\rangle) \langle u_i c, -v_i c \rangle_\sigma = 0$. The fact follows since the form $\langle\langle z_1, \dots, z_t, r_1, \dots, r_\ell \rangle\rangle$ is in \mathcal{S} by property (P4). End of the proof of the fact. Multiplying (6.1) by $t_0 \times p_1 \cdots p_r$ gives $t_0^2 \times (p_1 \cdots p_r \langle\langle w_1, \dots, w_t \rangle\rangle) \cdot \psi = 0$, proving that ψ is in $J_{\mathcal{S}}$ since $p_1 \cdots p_r \langle\langle w_1, \dots, w_t \rangle\rangle \in \mathcal{S}$ (by properties (P4) and (a)). End of the proof of Claim 2.

The conclusion is now clear since $h \notin J_{\mathcal{S}}$. ■

Proposition 6.4. *Let R be semilocal connected. The torsion in $W(A, \sigma)$ is 2-primary.*

Proof. By [6, Theorem 8.7 and Remark 8.8] there exists a connected finite étale R -algebra R_1 of odd rank, and an Azumaya algebra with involution (A_1, σ_1) over R_1 such that $A \otimes_R R_1$ and A_1 are Brauer equivalent over $S_1 := S \otimes_R R_1$, σ and σ_1 are both unitary, or are both non-unitary, and such that at least one of the following holds:

- (1) $Z(A_1) \cong R_1 \times R_1$;
- (2) $\deg A_1 = 1$;
- (3) The index and degree of A_1 are equal and divide the index of A . Moreover, $\deg A_1$ is a power of 2 and there exist $u, v \in A_1^\times$ such that $u^2 \in R_1^\times$, $\sigma_1(u) = -u$, $\sigma_1(v) = -v$ and $uv = -vu$.

By [6, Corollary 7.4], the canonical map of Witt groups

$$W(A, \sigma) \rightarrow W(A \otimes_R R_1, \sigma \otimes \text{id}_{R_1})$$

is injective and thus it suffices to show that the torsion in $W(A \otimes_R R_1, \sigma \otimes \text{id}_{R_1})$ is 2-primary.

If S_1 is not connected, then $W(A \otimes_R R_1, \sigma \otimes \text{id}_{R_1}) = 0$ (cf. Remark 2.16) and we can conclude. Thus we may assume that S_1 is connected, and in particular that we are not in case (1) above. By Theorem 2.15, we have a Witt group isomorphism

$$W(A \otimes_R R_1, \sigma \otimes \text{id}_{R_1}) \cong W^\delta(A_1, \sigma_1)$$

for some $\delta \in \{-1, 1\}$, where we may take $\delta = 1$ if σ and σ_1 are unitary, observing that σ and $\sigma \otimes \text{id}_{R_1}$ are of the same type. We now examine the remaining relevant cases from [6, Theorem 8.7], as listed in (2) and (3) above:

- (2) $\deg A_1 = 1$, i.e., $A_1 = S_1$: Assume first that σ_1 is not unitary. By Proposition 2.12 we then have $A_1 = S_1 = R_1$ and $\sigma_1 = \text{id}_{S_1}$. In this case $W^{-1}(R_1, \text{id}) = 0$ by [16, I, Corollary 4.1.2] (whose hypotheses are satisfied since R_1 is connected and also semilocal by [16, VI, Proposition 1.1.1]), while the torsion in $W(R_1, \text{id})$ is 2-primary by [5, Chapter V, Theorem 6.6].

On the other hand, if σ_1 is unitary (so that we may take $\delta = 1$), then S_1 is a quadratic étale R_1 -algebra and σ_1 is the standard involution. By [8, Corollary 8.3], the map $h \mapsto \text{Tr}_{S_1/R_1} \circ h$ is an injection from $W(S_1, \sigma_1)$ into $W(R_1, \text{id})$, which has 2-primary torsion as observed above.

- (3) $\deg A_1$ is a power of 2 and there exists $u \in A_1^\times$ such that $\sigma_1(u) = -u$. In particular, $\text{rank}_{S_1} A_1$ and hence $\text{rank}_{R_1} A_1$ are powers of 2. We consider two cases.

If $\delta = 1$, we conclude with Proposition 6.3: If h is torsion in $W(A_1, \sigma_1)$, then h has zero signature at every ordering of R_1 and by Proposition 6.3, there is $n \in \mathbb{N}$ such that $2^n \times h$ is hyperbolic.

If $\delta = -1$, we have

$$W^{-1}(A_1, \sigma_1) \cong W(A_1, \text{Int}(u) \circ \sigma_1)$$

(by Morita equivalence, more precisely the μ -conjugation equivalence of categories in [8, Section 2.7]), and we conclude again with Proposition 6.3 applied to $(A_1, \text{Int}(u) \circ \sigma_1)$. ■

Remark 6.5. If R is semilocal with only k maximal ideals, then any expression of R as a product $R_1 \times \cdots \times R_n$ of rings must be such that $n \leq k$. Therefore there is such an expression of R as a product $R_1 \times \cdots \times R_t$ that cannot be further decomposed as a product and thus where each R_i is connected.

Theorem 6.6 (Pfister's local-global principle). *Let R be semilocal, and recall that we assume $2 \in R^\times$. Let (M, h) be a nonsingular hermitian form over (A, σ) . The following statements are equivalent:*

- (1) $\text{sign}_\alpha^{\mathcal{M}} h = 0$ for every $\alpha \in \text{Sper } R$.
- (2) $\text{sign}_\alpha^{\mathcal{M}} h = 0$ for every $\alpha \in \text{Sper}^{\max} R$.
- (3) *There exists $n \in \mathbb{N} \cup \{0\}$ such that $2^n \times h$ is hyperbolic.*

In particular, the torsion in $W(A, \sigma)$ is 2-primary.

Proof. Observe that the final statement clearly follows from the equivalence of (1) and (3), since a torsion form has zero signature at every ordering.

Clearly (1) implies (2), and (3) implies (1), so we only need to show that (2) implies (3).

Following Remark 6.5, we may assume that $R = R_1 \times \cdots \times R_t$ with R_1, \dots, R_t connected semilocal rings. Writing $e_1 = (1, 0, \dots, 0), \dots, e_t = (0, \dots, 0, 1)$ in R , we have

$$(A, \sigma) \cong (Ae_1, \sigma|_{Ae_1}) \times \cdots \times (Ae_t, \sigma|_{Ae_t}).$$

Furthermore, we can identify M with $\bigoplus_{i=1}^t Me_i$, and we consider $h_i := h|_{Me_i}$ as a hermitian form over $(Ae_i, \sigma|_{Ae_i})$ for $i = 1, \dots, t$. A direct verification shows that:

- h is nonsingular if and only if each h_i is nonsingular for $i = 1, \dots, t$ (using for instance that h is nonsingular if and only if for every maximal ideal \mathfrak{m} of R the form $h \otimes_R R/\mathfrak{m}$ is nonsingular, cf. [16, I, Lemma 7.1.3]).
- If h_i is hyperbolic for $i = 1, \dots, t$, then h is hyperbolic. Indeed: We can write $Me_i = L_i \oplus P_i$ with $h_i(L_i, L_i) = 0$ and $h_i(P_i, P_i) = 0$ (cf. [8, Section 2.2]). So $M = \bigoplus_{i=1}^t L_i \oplus P_i \cong (\bigoplus_{i=1}^t L_i) \oplus (\bigoplus_{i=1}^t P_i)$. Let $L = \bigoplus_{i=1}^t L_i$ and $P = \bigoplus_{i=1}^t P_i$. We check that $h(L, L) = 0$; the proof of $h(P, P) = 0$ is similar. It suffices to show that $h(\ell_i, \ell'_j) = 0$ for each $\ell_i \in L_i$ and $\ell'_j \in L_j$. If $i \neq j$ then $h(\ell_i, \ell'_j) = 0$ and if $i = j$ then $h(\ell_i, \ell'_i) = h_i(\ell_i, \ell'_i) = 0$.

Every $\alpha \in \text{Sper}^{\max} R_i$ can be seen as an element α' in $\text{Sper}^{\max} R$, and a direct verification of the definition of signature shows that $\text{sign}_{\alpha'}^{\mathcal{M}} h = \text{sign}_{\alpha'}^{\mathcal{M}} h_i$. Therefore, (2) gives that $\text{sign}_{\alpha'}^{\mathcal{M}} h_i = 0$ for every $\alpha \in \text{Sper}^{\max} R_i$ and every $i = 1, \dots, t$. By Propositions 6.3 and 6.4 we have that for every $i = 1, \dots, t$ there exists $n_i \in \mathbb{N}$ such that $2^{n_i} \times h_i$ is hyperbolic. Thus, letting $n = n_1 + \cdots + n_t$, it follows that $2^n \times h_i$ is hyperbolic for $i = 1, \dots, t$ and hence that $2^n \times h$ is hyperbolic. ■

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