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Umemura quadric fibrations and maximal subgroups of $\mathrm{Cr}_n(\mathbb{C})$

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Abstract. We study the equivariant geometry of special quadric fibrations, called Umemura quadric fibrations, as well as the maximality of their automorphism groups inside $\mathrm{Cr}_n(\mathbb{C})$. We produce infinite families of pairwise non-conjugate maximal connected algebraic subgroups of $\mathrm{Cr}_n(\mathbb{C})$.

1. Introduction

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

The *Cremona group* $\mathrm{Cr}_n(\mathbf{k})$ is the group of birational transformations of the projective space $\mathbb{P}^n$ over a field $\mathbf{k}$. A classical problem in the theory of Cremona groups is the classification of their *maximal connected algebraic subgroups*, that is, subgroups acting rationally on $\mathbb{P}^n$ that are maximal with this property (see [8, Definition 1.1] for the precise definition of a rational action). While maximal connected algebraic subgroups and their classification are objects of interest by themselves, they also admit a nice geometric description: they correspond to automorphism groups of "highly symmetric" rational varieties.

Enriques [9] classified maximal connected algebraic subgroups of $\mathrm{Cr}_2(\mathbf{k})$ and showed that they are conjugate to $\mathrm{Aut}^\circ(S)$, where $S = \mathbb{P}^2$ or $\mathbb{F}_n$ with $n \neq 1$. In dimension 3 a similar classification was obtained by Umemura in a series of four papers [19–22].

Blanc, Fanelli and Terpereau in [2, 3] used an approach based on the Minimal Model Program MMP to recover most of the classification of Umemura. More precisely, if G is a connected subgroup of $\mathrm{Cr}_n(\mathbf{k})$ acting rationally on $\mathbb{P}^n$, then by Weil's regularization theorem there is a rational variety Z such that G acts regularly on Z. After running an MMP on Z, we obtain a Mori fiber space $X \to S$ with X rational, together with a regular action of G on X.

The group G is maximal if for every G-equivariant Sarkisov program from X/S to another Mori fiber space Y/T, the group $\mathrm{Aut}^\circ(Y)$ coincides with G (see [11]). The groups appearing in the classification of maximal subgroups of $\mathrm{Cr}_n(\mathbf{k})$ are thus automorphism groups of Mori fiber spaces.

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In total contrast to the case of dimension 2 and 3, where we have a full classification, and apart from some reduction results in dimension 4 (see [4]) there is no classification theory of maximal connected algebraic subgroups of $\mathrm{Cr}_n(\mathbf{k})$ for $n \geq 4$. In this paper we aim at initiating a study of maximal subgroups in higher dimensions.

It is worth noting that, while in dimension 2 and 3 as a byproduct of the classification every subgroup is contained in a maximal one, this is not true in dimension $n \geq 4$ by [10, 16].

In this note, we produce many examples of pairwise non-conjugated maximal subgroups of $\mathrm{Cr}_n(\mathbf{k})$ for $n \geq 3$. More specifically, we study a certain class of n-dimensional quadric fibrations, called *Umemura quadric fibrations*, and give necessary and sufficient criteria for when their automorphism groups are maximal in $\mathrm{Cr}_n(\mathbf{k})$.

Denote by $\mathcal{E}_a$ the vector bundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ over $\mathbb{P}^1$, let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$. Then the Umemura quadric fibration associated to g is the following divisor inside $\mathbb{P}(\mathcal{E}_a)$:

$$\mathcal{Q}_g := \left\{x_1^2 - x_0x_2 + x_3^2 + \cdots + x_{n-1}^2 + g(t_0, t_1)x_n^2 = 0\right\} \subset \mathbb{P}(\mathcal{E}_a).$$

See Definition 3.1 for more details. The restriction $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ of the projective bundle structure is a Mori fiber space. If g has more than two roots, then $\mathrm{Aut}^\circ(\mathcal{Q}_g) \cong \mathrm{SO}_n(\mathbf{k})$ (see Proposition 3.8 for a complete discussion of the automorphism group).

Automorphism groups of Umemura quadric fibrations occupy a remarkable place in the theory of maximal subgroups in dimension 3. They appear in the classification of Umemura and Blanc–Fanelli–Terpereau and are the only ones lying in continuous families. In dimension 4, they serve as natural candidates for producing maximal subgroups from the point of view of [4]. Moreover, finite subgroups of them have been utilized by Krylov to produce surprising results in the theory of finite subgroups of the Cremona group $\mathrm{Cr}_3(\mathbf{k})$ [18].

Our main result is the following.

Theorem 4.7. *Let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Write $g = f^2h$, where $f, h \in \mathbf{k}[t_0, t_1]$ are homogeneous polynomials with h being square-free. Then* $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ *is conjugated to a subgroup of* $\mathrm{Aut}^\circ(\mathcal{Q}_h)$.

Moreover, if h and h' are two square-free polynomials, we have:

1. $\mathrm{Aut}^\circ(\mathcal{Q}_h)$ *is a maximal connected algebraic subgroup of* $\mathrm{Cr}_n(\mathbf{k})$ *if and only if h is constant or has at least 4 roots;*
2. $\mathrm{Aut}^\circ(\mathcal{Q}_h)$ *and* $\mathrm{Aut}^\circ(\mathcal{Q}_{h'})$ *are conjugate if and only if $h(t_0, t_1) = h'(\alpha(t_0, t_1))$, with $\alpha \in \mathrm{PGL}_2(\mathbf{k})$.*

The outline of the paper is as follows: in Section 2, we collect some preliminary results that will be used throughout the paper; in Section 3, we introduce Umemura quadric fibrations, compute their automorphism groups and analyze their equivariant geometry; finally, in Section 4, we study birational relations among Umemura quadric fibrations and determine maximality of their automorphism groups in $\mathrm{Cr}_n(\mathbf{k})$.

2. Preliminary results

This section contains some preliminary definitions and results on the geometry of birational maps of varieties with terminal singularities. We refer to [17] for the basic notions of the MMP.

2.1. Extremal divisorial contractions

Definition 2.1. Let Y be a variety with terminal singularities and $\Gamma \subset Y$ an irreducible subvariety of codimension at least 2. An *extremal divisorial contraction* is a birational morphism

$$f\colon E \subset X \longrightarrow \Gamma \subset Y$$

such that:

- X is $\mathbb{Q}$-factorial and has terminal singularities;
- $f|_{X\setminus E}$ an isomorphism and E a prime divisor;
- $-K_X$ is f-ample;
- $\rho(X/Y) = 1$.

Typical examples of extremal divisorial contractions are blowups as well as an infinite family of weighted blowups. However Remark 4.4 shows these are not the only examples, even when the center is a smooth point. The situation is different if the center is an orbit of codimension 2 and f is equivariant with respect to some group G, as in this case by [5, Proposition 2.4] we have only blowups.

Example 2.2. Consider the standard $(1,\dots,1,b)$-weighted blowup of $0 \in \mathbb{A}^{n+1}_{x_i,t}$

$$\mathrm{Proj}\Big(\bigoplus_{k\geq 0} \mathcal{I}_k\Big) \longrightarrow \mathbb{A}^{n+1},$$

where $\mathcal{I}_k = (\{x_0^{m_0}\cdot\ldots\cdot x_{n-1}^{m_{n-1}}\cdot t^{m_n} \mid m_0+\cdots+m_{n-1}+bm_n \geq k\})$. This can also be described as

$$\begin{aligned} X := \mathbb{A}^{n+1}/\mathbb{G}_m &\longrightarrow \mathbb{A}^n \\ (u : x_0 : \cdots : x_{n-1} : t) &\longmapsto (ux_0,\dots,ux_{n-1},u^b t), \end{aligned}$$

where $\mathbb{G}_m$ acts linearly with weights $(-1,1,\dots,1,b)$. We demonstrate how to extract the valuation of $E=\{u=0\}$ using the *tower construction* [14, Construction 3.1].

Denote by U the open subset $\{x_0 = 1\} \subset X$. Since $E \mapsto 0 \in \mathbb{A}^n$, the first step of the tower construction is the blowup of $0 \in \mathbb{A}^n$, locally described by

$$\begin{aligned} V_1 = \mathbb{A}^{n+1}_{v_1,x_i,t} &\longrightarrow \mathbb{A}^{n+1} \\ (v_1,x_1,\dots,x_{n-1},t) &\longmapsto (v_1,v_1x_1,\dots,v_1x_{n-1},v_1t) \\ E_1 := \{v_1=0\} &\longmapsto 0. \end{aligned}$$

The induced birational map between V_1 and U is given by

$$(u, x_1, \dots, x_{n-1}, t) \longmapsto (u, x_1, \dots, x_{n-1}, ut)$$

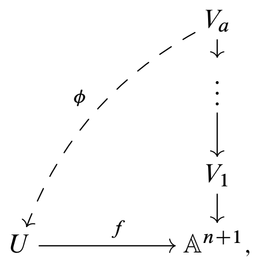
and $E \mapsto \Gamma_1 := \{v_1 = t = 0\}$. The second step is the blowup of V_1 along Γ_1, locally described by

$$\begin{aligned} V_2 = \mathbb{A}^{n+1}_{v_2, x_i, t} &\longrightarrow V_1 \\ (v_2, x_1, \dots, x_{n-1}, t) &\longmapsto (v_2, x_1, \dots, x_{n-1}, v_2 t) \\ E_2 := \{v_2\} &\longmapsto \Gamma_1. \end{aligned}$$

Again, the induced birational map between U and V_2 is given by

$$(u, x_1, \dots, x_{n-1}, t) \longmapsto (u, x_1, \dots, x_{n-1}, ut).$$

Continuing like that, we get the diagram



where, for $1 \leq i < b$, $E_{i+1} \subset V_{i+1} \to \Gamma_i \subset V_i$ is the blowup of V_i along Γ_i, with Γ_i being the intersection of E_i with the strict transform of $\{t = 0\} \subset \mathbb{A}^{n+1}$.

We will return to this example again in Proposition 4.3.

2.2. Action on a product of quadrics

Let $Q_n \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree 2. In this subsection we recall some basic facts on the action of $\mathrm{Aut}^\circ(Q_n)$ on $Q_n \times Q_n$.

Lemma 2.3. *Let $n \geq 3$ be an integer. Let $Q_n \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree 2. Let $x \in Q_n$ be a point and G_x the stabiliser of x in $\mathrm{Aut}^\circ(Q_n)$. Let H be the cone over a quadric of dimension $n-2$ obtained as intersection of Q_n with the projective tangent space in x. The orbits of G_x on Q_n are*

- *the point $\{x\}$;*
- *the $\mathbb{A}^1$-bundle $H \setminus \{x\}$;*
- *the open set $Q_n \setminus H$.*

Proof. Without loss of generality we may assume that

$$Q_n = \{x_0 x_1 + x_2^2 + \cdots + x_{n+1}^2 = 0\} \quad \text{and} \quad x = (1 : 0 : \cdots : 0).$$

Consider the birational map

$$\begin{aligned} Q_n &\dashrightarrow \mathbb{P}^n \\ (x_0 : x_1 : \cdots : x_{n+1}) &\longmapsto (x_1 : \cdots : x_{n+1}) \\ \left(\frac{x_2^2 + \cdots + x_{n+1}^2}{x_1} : x_1 : \cdots : x_{n+1}\right) &\longleftarrow (x_1 : \cdots : x_{n+1}), \end{aligned}$$

with its resolution $(p, q)\colon W \to Q_n \times \mathbb{P}^n$, where:

(1) $p\colon W \to Q_n$ is the blowup along x;

(2) $q\colon W \to \mathbb{P}^n$ is the contraction of the strict transform of H.

By (1) $\mathrm{Aut}^\circ(W) \cong G_x$. Moreover, q coincides with the blowup of $\mathbb{P}^n$ along $Q_{n-2} := \{x_1 = x_2^2 + \cdots + x_{n+1}^2 = 0\}$, and thus

$$pG_x p^{-1} = \mathrm{Aut}^\circ(W) = q\mathrm{Aut}^\circ(\mathbb{P}^n; Q_{n-2})q^{-1}.$$

The group $\mathrm{Aut}^\circ(\mathbb{P}^n; Q_{n-2})$ consists of matrices of the form

$$\left(\begin{array}{c|c} \alpha & 0\cdots 0 \\ \hline b & B \end{array}\right)$$

where $\alpha \neq 0$ and $B^\tau B = \lambda I_{n-2}$. The orbits of $\mathrm{Aut}^\circ(\mathbb{P}^n; Q_{n-2})$ are Q_{n-2}, $\{x_1 = 0\} \setminus Q_{n-2}$ and $\mathbb{P}^n \setminus \{x_1 = 0\}$. The orbits of $\mathrm{Aut}^\circ(W)$ are the intersection Z of the exceptional divisor E of q and the strict transform P of $\{x_1 = 0\}$, the complements $P \setminus Z$ and $E \setminus Z$ and $W \setminus (E \cup P)$. Finally, the orbits of G_x in Q_x are the images of the orbits of $\mathrm{Aut}^\circ(W)$ on W. Therefore the claim follows. ∎

The case $n = 2$ is similar, yet slightly different.

Lemma 2.4. *Let $Q_2 \subseteq \mathbb{P}^3$ be a smooth hypersurface of degree 2. Let $x \in Q_2$ be a point and G_x the stabiliser of x in $\mathrm{Aut}^\circ(Q_2)$. The intersection of Q_2 with the projective tangent space at x is the union of two lines l_1, l_2. The orbits of G_x on Q_n are*

- *the point $\{x\}$;*
- *the lines $l_1 \setminus \{x\}$ and $l_2 \setminus \{x\}$;*
- *the open set $Q_2 \setminus (l_1 \cup l_2)$.*

Proof. The proof is *mutatis mutandis* the one of Lemma 2.3, the *caveat* being that in this case $Q_{n-2} = Q_0$ is the union of 2 distinct points P_1, P_2. Consequently, the group $\mathrm{Aut}(\mathbb{P}^2; Q_0)$ is not connected. Restricting to the connected component containing the identity we get the claimed orbits. ∎

Lemma 2.5. *Let $n \geq 2$ be an integer. Let $Q_n \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree 2. Let $G = \mathrm{Aut}^\circ(Q_n)$ and consider the diagonal action of G on $Q_n \times Q_n$. Let $\Delta \subseteq Q_n \times Q_n$ be the diagonal and*

$$T := (Q_n \times Q_n) \cap TQ_n \subset \mathbb{P}^{n+1} \times \mathbb{P}^{n+1},$$

where TQ_n denotes the projectivized tangent bundle of Q_n.

- *If $n \geq 3$, the orbits of the action of G on $Q_n \times Q_n$ are: $\Delta, T \setminus \Delta, Q_n \times Q_n \setminus T$;*
- *If $n = 2$, $T = T_1 \cup T_2$ and the orbits of the action of G on $Q_n \times Q_n$ are: $\Delta, T_1 \setminus \Delta$, $T_2 \setminus \Delta, Q_n \times Q_n \setminus T$.*

Proof. Since the action of G on Q_n is transitive, Δ is an orbit.

We show the statement for $n \geq 3$. Let $(x, y), (x', y') \in T$. Let $g \in G$ be such that $g(x') = x$. Then $g(y') \in T_{gx'} Q_n$. By Lemma 2.3 there is $h \in G_{g(x)}$ such that $h(g(y')) = y$, and so $(hg) \cdot (x, y) = (x', y')$. The proof that $Q_n \times Q_n \setminus T$ is an orbit is similar.

We now treat the case $n = 2$. By Lemma 2.4, for any $x \in Q_n$, the fiber over x of the projection $T \to Q_n$ to the first factor has two irreducible components, namely the two 1-dimensional orbits of G_x Let $(x, y) \in T$, $x' \in Q_2$ and $h \in G$ such that $h(x) = x'$. Then for every $g \in G$ with $g(x) = x'$, $h(y)$ and $g(y)$ lie in the same irreducible component: indeed $hg^{-1} \in G_{x'}$ and $hg^{-1}(g(y)) = h(y)$. Thus the orbit of (x, y) is a proper subset of T, that has the same dimension as its complement. If we denote by T_1 the closure of $G \cdot (x, y)$ and T_2 the closure of its complement we have $T = T_1 \cup T_2$. The rest of the proof is verbatim the proof of the higher-dimensional case of the previous step. ■

2.3. A useful lemma

We end the section with the following lemma, which is well known to experts. We give the proof for the readers convenience.

Lemma 2.6. *Let Z be a smooth projective variety and $\mathcal{L}_i$ for $i = 1, 2$ line bundles on Z such that $\mathcal{L}_1 \not\equiv \mathcal{L}_2$. Denote by P the projectivization $\mathbb{P}_Z(\mathcal{L}_1 \oplus \mathcal{L}_2)$ together with the induced morphism $p \colon P \to Z$ and by Z_i the sections of p induced by $\mathcal{L}_i \to \mathcal{L}_1 \oplus \mathcal{L}_2$. Then every section of p meets either Z_1 or Z_2.*

If moreover Z has Picard rank one and $\mathcal{L}_1^\vee \otimes \mathcal{L}_2$ is anti-ample, then

$$\mathrm{NE}(P) = \mathbb{R}_+[f] + i_*\big(\mathrm{NE}(Z_2)\big)$$

where f is a fiber of p and $i \colon Z_2 \to P$ is the immersion.

Proof. Let $\widetilde{Z}$ be a section of p. There are divisors D_1, D_2 on Z such that

$$\widetilde{Z} \sim Z_1 + p^* D_1, \quad Z_2 \sim Z_1 + p^* D_2.$$

Assume that $\widetilde{Z}$ is disjoint from $Z_1 \cup Z_2$. Then

$$0 = Z_2|_{\widetilde{Z}} = Z_1|_{\widetilde{Z}} + p^* D_2|_{\widetilde{Z}} = p^* D_2|_{\widetilde{Z}}.$$

Thus $D_2 \sim 0$, implying that Z_1 and Z_2 are linearly equivalent. But this would imply that the linear equivalence classes of $\mathcal{O}_P(1)|_{Z_1}$ and $\mathcal{O}_P(1)|_{Z_2}$ are the same. Indeed, let D be a divisor such that $\mathcal{O}_P(1) \sim \mathcal{O}_P(Z_1 + p^* D)$. Thus $\mathcal{O}_P(1) \sim \mathcal{O}_P(Z_1 + p^* D) \sim \mathcal{O}_P(Z_2 + p^* D)$. Taking restrictions to Z_1 and to Z_2, we get $\mathcal{O}_P(1)|_{Z_1} \sim \mathcal{O}_P(p^* D)|_{Z_1}$ and $\mathcal{O}_P(1)|_{Z_2} \sim \mathcal{O}_P(p^* D)|_{Z_2}$. This is a contradiction as $\mathcal{O}_P(1)|_{Z_i} \sim \mathcal{L}_i$ and $\mathcal{L}_1 \not\equiv \mathcal{L}_2$.

For the second part of the statement, we have $\rho(P) = 2$ and so $\mathrm{NE}(P)$ is a cone with 2 extremal rays, one generated by the class f of a fiber of p. Note that we have

$$\mathcal{O}_P(Z_2) = p^*(\mathcal{L}_1^\vee) \otimes \mathcal{O}_P(1).$$

For any curve $C \subset Z_2$, we have

$$\begin{aligned} Z_2 \cdot C &= p^*(\mathcal{L}_1^\vee) \cdot C + \mathcal{O}_P(1) \cdot C = \mathcal{L}_1^\vee \cdot p_*C + \mathcal{O}_P(1)|_{Z_2} \cdot C \\ &= (\mathcal{L}_1^\vee \otimes \mathcal{L}_2) \cdot p_*C < 0. \end{aligned} \tag{1}$$

Thus $[C]$ cannot be in the interior of $\mathrm{NE}(P)$ since it has negative intersection with an effective divisor. Therefore, $i_*(\mathrm{NE}(Z_2))$ is the second extremal ray. ■

3. Umemura quadric fibrations

In this section, we introduce Umemura quadric fibrations and study their basic properties.

3.1. Definition and basic properties

Let $\mathbf{a} = (a_0, \dots, a_n)$ with $a_0, \dots, a_n \in \mathbb{Z}$. Denote by $\mathcal{E}_\mathbf{a}$ the vector bundle $\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^1}(-a_i)$ over $\mathbb{P}^1$. Then the projective bundle $\mathbb{P}(\mathcal{E}_\mathbf{a})$ can be described as the geometric quotient of $(\mathbb{A}^{n+1} \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$ by $\mathbb{G}_m^2$ with the action given by:

$$\begin{aligned} \mathbb{G}_m^2 \times \left(\mathbb{A}^{n+1} \setminus \{0\}\right) \times \left(\mathbb{A}^2 \setminus \{0\}\right) &\longrightarrow \left(\mathbb{A}^{k+1} \setminus \{0\}\right) \times \left(\mathbb{A}^2 \setminus \{0\}\right) \\ (\lambda, \mu), (x_0, x_1, \dots, x_n; t_0, t_1) &\longmapsto (\lambda\mu^{-a_0}x_0, \lambda\mu^{-a_1}x_1, \dots, \lambda\mu^{-a_n}x_n; \mu t_0, \mu t_1). \end{aligned}$$

In the special case when $\mathbf{a} = (0, 0, \dots, a)$, we will simply denote $\mathcal{E}_\mathbf{a}$ by $\mathcal{E}_a$.

Definition 3.1. Let $n \geq 3$, $a \in \mathbb{N}$ and let $g \in \mathbf{k}[t_0, t_1]_{2a}$ be a homogeneous polynomial of degree $2a$. We define the *Umemura quadric fibration* associated to g as

$$\mathcal{Q}_g := \left\{x_1^2 - x_0x_2 + x_3^2 + \dots + x_{n-1}^2 + g(t_0, t_1)x_n^2 = 0\right\} \subset \mathbb{P}(\mathcal{E}_a).$$

We will denote by $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the projection to $\mathbb{P}^1$.

The choice of a non-diagonal equation is to highlight the existence of a section (see [6]).

Remark 3.2. $\mathcal{Q}_g$ is rational: indeed, in the open subset $\{x_2 = 1\}$, we may solve

$$x_1^2 - x_0x_2 + x_3^2 + \dots + g(t_0, t_1)x_n^2 = 0 \quad \text{for } x_0;$$

therefore, the projection $(x_0 : \dots : x_n; t_0, t_1) \mapsto (x_1 : \dots : x_n; t_0, t_1)$ gives us a birational map to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$, the latter being rational.

The following lemma and corollary show that Umemura quadric fibrations appear naturally as standard birational models of quadric fibrations.

Lemma 3.3. *Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$ and let $Q \subseteq \mathbb{P}^n_{\mathbb{K}}$ be a smooth quadric, $n \geq 3$. Assume moreover that $Q(\mathbb{K}) \neq \emptyset$. Then up to a change of coordinates Q is given by the equation*

$$x_1^2 - x_0x_2 + \sum_{i=3}^{k-1} x_i^2 + \sum_{i=k}^{n} \mu_i x_i^2 = 0$$

and μ_i satisfying the condition that there exist no $a_k, \dots, a_n \in \mathbb{K}$ such that $\sum_{i=k}^n a_i^2 \mu_i$ is a non-zero square in $\mathbb{K}$.

Proof. Since $Q(\mathbb{K}) \neq 0$, up to a change of coordinates, we may assume that $(1 : 0 : \dots : 0) \in Q(\mathbb{K})$. Then $Q \cap \{x_3 = x_4 = \dots = x_n = 0\}$ is a plane quadric containing $(1 : 0 : 0)$ and, again up to change of coordinates, we may assume Q is given by the equation

$$x_1^2 - x_0x_2 + \sum_{i=3}^{n} l_i x_i = 0 \qquad (*)$$

with $l_i \in \mathbb{K}[x_0, \dots, x_n]_1$. Let M be the matrix of the quadratic form $(*)$. The top-left 3×3 block is given by the coefficients of $x_1^2 - x_0x_2$. After two changes of coordinates of the form $x_2 = x_2 - \lambda(x_3, \dots, x_n)$ and $x_1 = x_1 - \mu(x_3, \dots, x_n)$, $x_0 = x_0 - \nu(x_3, \dots, x_n)$ for some $\lambda, \mu, \nu \in \mathbb{K}[x_0, \dots, x_n]_1$, we may assume that there is $q \in \mathbb{K}[x_3, \dots, x_n]_2$ such that Q is given by the equation $x_1^2 - x_0x_2 + q = 0$. Thus we diagonalize q and we get the desired form. ∎

Proposition 3.4. *Let $\pi\colon X \to \mathbb{P}^1$ be a Mori fiber space where the generic fiber $X_{k(t)}$ is isomorphic to a smooth quadric hypersurface $\mathbb{P}^n_{k(t)}$. Let $G = \mathrm{Aut}_{\boldsymbol{k}}(X)_{\mathbb{P}^1}$. Then X is G-equivariantly birational to a hypersurface*

$$\begin{aligned} \mathcal{Q}_{\mathbf{g}} &:= \{x_1^2 - x_0x_2 + x_3^2 + \dots + x_k^2 + g_1 x_{k+1}^2 + \dots + g_l x_n^2 = 0\} \\ &\subset \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}^{\oplus k+1} \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-a_i)\right) \end{aligned}$$

where $a_i \in \mathbb{N}$ and $g_i \in \boldsymbol{k}[t_0, t_1]_{2a_i}$ are homogeneous polynomials of degree $2a_i$.

Proof. By Lemma 3.3 the generic fiber of π is of the form

$$x_1^2 - x_0x_2 + \sum_{i=3}^{k-1} x_i^2 + \sum_{i=k}^{n} \mu_i x_i^2 = 0,$$

where $\mu_i = \frac{r_i}{s_i} \in \mathbf{k}(t)$ for $i = k, \dots, n$.

Denote by H the closure of the subset $\{x_0 = 0\} \subset \pi^{-1}(U)$, where U is the locus of $\mathbb{P}^1$ where the s_i do not vanish. Since $\pi_*\mathcal{O}_X(H)$ is a locally free sheaf of rank $n + 1$ (it is a torsion free sheaf over $\mathbb{P}^1$), it is a direct sum of line bundles

$$\mathcal{E}_{\mathrm{b}} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(b_i).$$

Consider the rational map

$$\begin{array}{ccc} X & \overset{\chi}{\dashrightarrow} & Y \subset \mathbb{P}\big(\pi_*(\mathcal{O}_X(H))\big) \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1, & \end{array}$$

with Y the image of X. Note that, since $\rho(X/\mathbb{P}^1)=1$, $\mathcal{O}_X(H)$ is G-invariant and so χ is G-equivariant. Over U the sections $\{x_0,\dots,x_{k-1},s_kx_k,\dots,s_nx_n\}$ generate $H^0(\pi^{-1}U,\mathcal{E}_{\mathrm{b}})$. Thus χ is locally given by

$$\begin{aligned} \pi^{-1}(U) &\longrightarrow U\times\mathbb{P}^n \\ (x_0:\cdots x_n;t_0,t_1) &\longmapsto (x_0:\cdots:x_{k-1}:s_kx_k:\cdots:s_nx_n). \end{aligned}$$

Taking $g_i = r_is_i \in \mathbf{k}[t_0,t_1]_{2a_i}$ we recover the claimed equation and weights. ■

The following lemma is a computation of the Picard group and Mori cone of $\mathcal{Q}_g$. We omit the proof as it follows the same lines as [3, Lemma 4.4.3].

Lemma 3.5. *Let $g\in\boldsymbol{k}[t_0,t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi:\mathcal{Q}_g\to\mathbb{P}^1$ the associated Umemura quadric fibration. Let F be a fiber of π, $H=\{x_n=0\}$, e a curve in a fiber of π and σ the section of π defined as $\sigma=\{(1:0:\cdots:0;t_0:t_1)\,|\,(t_0:t_1)\in\mathbb{P}^1\}$. Then*

(1) $\mathrm{Pic}(\mathcal{Q}_g)=\mathbb{Z}[F]+\mathbb{Z}[H]$.

(2) $\mathrm{NE}(\mathcal{Q}_g)=\mathbb{R}_+[e]+\mathbb{R}_+[\sigma]$ *and curves with class in* $\mathbb{R}_+[\sigma]$ *cover the divisor* H.

(3) $K_{\mathcal{Q}_g}\equiv-(n-2)H+(a-2)F$.

(4) *The intersection numbers with the canonical divisor of* $\mathcal{Q}_g$ *are*

$$K_{\mathcal{Q}_g}\cdot e = n-1 \quad \textit{and} \quad K_{\mathcal{Q}_g}\cdot\sigma = a-2.$$

Proposition 3.6. *Let $g\in\boldsymbol{k}[t_0,t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi:\mathcal{Q}_g\to\mathbb{P}^1$ the associated Umemura quadric fibration.*

(1) *The singular locus of* $\mathcal{Q}_g$ *is the discrete set*

$$\mathrm{Sing}(\mathcal{Q}_g)=\big\{(0:\cdots:0:1;t_0,t_1)\mid(t_0,t_1)\ \textit{is a multiple root of}\ g=0\big\}.$$

(2) $\mathcal{Q}_g$ *has terminal singularities and is* $\mathbb{Q}$*-factorial.*

(3) $\pi:\mathcal{Q}_g\to\mathbb{P}^1$ *is a Mori fiber space.*

Proof. Item (1) follows from the Jacobian criterion. Terminality and $\mathbb{Q}$-factoriality follow from [15, Section 1.42] and [13, XI, Corollaire 3.14].

For (3), the variety $\mathcal{Q}_g$ is terminal and $\mathbb{Q}$-factorial by (2). Moreover, $\rho(\mathcal{Q}_g)=2$ by Lemma 3.5 and therefore $\rho(\mathcal{Q}_g/\mathbb{P}^1)=1$. Finally, let F denote a general fiber of $\mathcal{Q}_g\to\mathbb{P}^1$. Then F is isomorphic to a smooth quadric $Q_{n-1}\subset\mathbb{P}^n$ and thus $-K_{\mathcal{Q}_g}|_F=-K_F$ is ample, the equality obtained by adjunction formula. ■

3.2. Automorphism group

In this subsection, we will compute the automorphism group of $\mathcal{Q}_g$. We first begin by analyzing the automorphism group of the ambient space $\mathbb{P}(\mathcal{E}_a)$.

Lemma 3.7. *Let $a > 0$ and write $\mathcal{E}_a$ for the vector bundle*

$$\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^1}(-a).$$

Then $\mathrm{Aut}(\mathbb{P}(\mathcal{E}_a))_{\mathbb{P}^1}$ *equals*

$$\left\{ \left(\begin{array}{c|c} M & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline f_0 \cdots f_{n-1} & 1 \end{array} \right) \in \mathrm{PGL}_{n+1}\big(\boldsymbol{k}[t_0,t_1]\big) \;\middle|\; \begin{array}{l} M \in \mathrm{GL}_n(\boldsymbol{k}), \\ f_i \in \boldsymbol{k}[t_0,t_1]_a \end{array} \right\} \rtimes \mathbb{G}_m/\mu_a,$$

where μ_a denotes the group of a-th roots of unity, with the first factor acting on the coordinates x_i and the second on the t_i diagonally.

Proof. The statement follows from [12, Proposition 2] and the discussion that follows that result, and some easy observations on $\mathrm{Aut}(\mathcal{E}_a)$. See also [1, Section 6.1] for a sample computation in dimension 2. ■

Proposition 3.8. *Let $g \in \boldsymbol{k}[t_0,t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi : \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Then $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1} = \mathrm{SO}_n(\boldsymbol{k})$.*

Moreover, we have

$$\mathrm{Aut}^\circ(\mathcal{Q}_g) = \begin{cases} \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}, & \text{if } g \text{ has more than 2 roots;} \\ \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1} \rtimes \mathbb{G}_m, & \text{if } g \text{ has exactly 2 roots;} \\ \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1} \times \mathbb{G}_a, & \text{if } g \text{ has exactly 1 root;} \\ \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1} \times \mathrm{PGL}_2, & \text{if } g \text{ is constant.} \end{cases}$$

In particular, H acts trivially on $\mathbb{P}^1$ if g has more than 2 roots, and with an open orbit, otherwise.

Proof. By Lemma 3.5, we have $\rho(\mathcal{Q}_g/\mathbb{P}^1) = 1$. Moreover, the restriction map

$$H^0\big(\mathbb{P}(\mathcal{E}_a), \mathcal{O}(1)\big) \longrightarrow H^0\big(\mathcal{Q}_g, \mathcal{O}(1)|_{\mathcal{Q}_g}\big)$$

is surjective, and so the embedding $\mathcal{Q}_g \to \mathbb{P}(\mathcal{E}_a)$ is given by a complete linear system over $\mathbb{P}^1$. It is therefore $\mathrm{Aut}^\circ(\mathcal{Q}_g)$-equivariant. In particular, $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ coincides with the stabilizer of $\mathcal{Q}_g$ in $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_a))_{\mathbb{P}^1}$. If $\alpha \in \mathrm{Aut}(\mathbb{P}(\mathcal{E}_a))_{\mathbb{P}^1}$, by Lemma 3.7 there is $M \in \mathrm{GL}_n(\mathbf{k})$, there are $f_i \in \mathbf{k}[t_0,t_1]_a$ such that

$$\alpha = \left(\begin{array}{c|c} M & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline f_0 \cdots f_{n-1} & 1 \end{array} \right).$$

If α stabilizes $\mathcal{Q}_g$, we get: $f_i = 0, i = 0, \dots, n-1$, $M \in \mathrm{O}_n(\mathbf{k})$ and the action on the t_i is trivial. If moreover $\alpha \in \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_a))_{\mathbb{P}^1}$ then $M \in \mathrm{SO}_n(\mathbf{k})$.

For the second part, consider the short exact sequence

$$0 \longrightarrow \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1} \longrightarrow \mathrm{Aut}^\circ(\mathcal{Q}_g) \longrightarrow H \longrightarrow 0, \qquad (\dagger)$$

where H is the image of the homomorphism $\mathrm{Aut}^\circ(\mathcal{Q}_g) \to \mathrm{PGL}_2$ induced by Blanchard's lemma. Note that $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ must permute the singular fibers, therefore H is a connected group permuting the roots of g. Thus, in the four cases of the Proposition, we have that $H \le G \le \mathrm{PGL}_2$, where $G = 0, \mathbb{G}_m, \mathbb{G}_a$ and PGL_2 respectively.

If g has exactly two roots then, up to change of coordinates, $g = t_0^{a_0} t_1^{a_1}$ with $0 < a_0 \le a_1 \le 2a$ and $a_0 + a_1 = 2a$. In this case we have the $\mathbb{G}_m$-action

$$\lambda \cdot (x_0 : \cdots : x_n; t_0, t_1) \longmapsto (x_0 : \cdots : \lambda^{-a_0} x_n; t_0, \lambda^2 t_1),$$

which shows that $H = G$, and also provides a section to ($\dagger$).

In a similar fashion, if g has 1 root or is constant, we can write a G-action on $\mathcal{Q}_g$, which furthermore commutes with the action of $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$, showing that the product is direct. ∎

Remark 3.9. In Proposition 3.8 the automorphism group only depends on the number of roots without multiplicity. Nevertheless, we do not assume that the roots of g are of multiplicity one, as multiple roots naturally appear when performing the Sarkisov program, see Example 4.1.

We are now ready to compute the orbits of $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ on $\mathcal{Q}_g$.

Lemma 3.10. *Let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi : \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Set $H_n = \{x_n = 0\}$. Then $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ acts on $\mathcal{Q}_g$ with the following orbits:*

(1) *if $t \in \mathbb{P}^1$ is not a root of g, then we have the orbits $\Gamma_t := \pi^{-1}(t) \cap H_n$ and its complement $\pi^{-1}(t) \setminus \Gamma_t$;*

(2) *if $t \in \mathbb{P}^1$ is a root of g, then we have the orbits $p = (0 : \cdots : 0 : 1; t)$, $\Gamma_t := \pi^{-1}(t) \cap H_n$ and their complement $\pi^{-1}(t) \setminus (\Gamma_t \sqcup \{p\})$.*

Proof. The specific description of the orbits follows from the explicit action given in Lemma 3.7. ∎

Remark 3.11. When g has more than 2 roots, Lemma 3.10 gives a description of the orbits of $\mathrm{Aut}^\circ(\mathcal{Q}_g)$. This follows from Proposition 3.8, since in that case $\mathrm{Aut}^\circ(\mathcal{Q}_g) = \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$.

3.3. A structural result for equivariant birational maps to $\mathcal{Q}_g$

The next two subsections consist of a collection of technical computations, that culminate in results essential to the proof of Proposition 4.3. For the reader uninterested in the technical details, we now briefly describe the results necessary to skip ahead directly to Section 4. In what follows G denotes the group $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$.

In Section 3.4, we produce a "minimal" G-equivariant log-resolution $X_m \to \mathcal{Q}_g$ of the pair $(\mathcal{Q}_g, F)$ where F is a fiber of $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$. More specifically, in Corollary 3.13, we show that it is obtained by repeatedly blowing up the unique singular point P in the fiber over $\pi(P)$. In Proposition 3.15, we describe the relative Mori cone $\mathrm{NE}(X_m/\mathcal{Q}_g)$. In Corollary 3.17, we describe the irreducible components of the preimage of F in X_m.

In Section 3.5, we study G-equivariant birational morphisms $X \to X_m$ centered over P. In Lemma 3.18, we show that all orbits over P have codimension at most 2 and so, by [5, Proposition 2.4], $X \to X_m$ is a composition of smooth blowups. More specifically, Lemma 3.18 shows that the dual graph of the fiber over $\pi(P)$ is a chain. We end the subsection with Proposition 3.22, by computing the relative cone of curves $\mathrm{NE}(X/Q_g)$, as well as some intersection numbers of its extremal rays with K_X.

3.4. An equivariant resolution

In this subsection, we compute an explicit $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$-equivariant resolution of singularities for $\mathcal{Q}_g$. We also we compute the orbits of the action on the resolution.

A very useful feature of the projective bundle $\mathbb{P}(\mathcal{E}_a)$ is that it admits an open covering by affine spaces $U_{i,j} := \{x_i = t_j = 1\} \cong \mathbb{A}^{n+1}$, for $i = 0, \ldots, n$, $j = 0, 1$. The following lemma is a local calculation on the chart $U_{n,1}$.

Lemma 3.12. *Let $\gamma(t) \in \boldsymbol{k}[t]$ and consider the hypersurface*

$$U_0 := \left\{x_1^2 - x_0x_2 + x_3^2 + \cdots + x_{n-1}^2 + t^k\gamma(t) = 0\right\} \subset \mathbb{A}^{n+1}_{x_i,t}$$

with $k \geq 0$ and $\gamma(0) \neq 0$. Then U_0 is singular at the origin if and only if $k \geq 2$.

Suppose $k \geq 2$, let $f\colon X \to \mathbb{A}^{n+1}$ be the blowup of the origin, $\widetilde{U}_0$ the strict transform of U_0 and E the exceptional divisor of $\widetilde{U}_0 \to U_0$.

(1) *There exists an open neighborhood $V \cong \mathbb{A}^{n+1}$ of X, intersecting E such that*

$$U_1 := \widetilde{U}_0 \cap V = \left\{x_1^2 - x_0x_2 + x_3^2 + \cdots + x_{n-1}^2 + t^{k-2}\gamma(t) = 0\right\}$$

(2) *V contains all the singular points of $\widetilde{U}_0$ over the origin.*

(3) *Let $\pi\colon U_0 \to \mathbb{A}^1$ be the restriction of the projection $\mathbb{A}^{n+1} \to \mathbb{A}^1$ onto the last factor. Let F be the fiber of π over $0 \in \mathbb{A}^1$ and $\widetilde{F}$ its strict transform in $\widetilde{U}_0$. Then*
   - (a) *$\widetilde{F}$ is an $\mathbb{A}^1$-bundle over a smooth quadric in $\mathbb{P}^{n-1}$; the intersection $\widetilde{F} \cap E$ is a section of the $\mathbb{A}^1$-bundle.*
   - (b) *If $k = 2$ the divisor E is a smooth quadric in $\mathbb{P}^n$.*
   - (c) *If $k \geq 3$ the divisor E is a cone over a smooth quadric in $\mathbb{P}^{n-1}$. Let e be a generator of the ruling in E. Then $e \cdot \widetilde{F}|_E = 1$ and $\widetilde{F} \cap E$ does not contain the singular point of E.*

(4) *Finally, we have*

$$K_{\widetilde{U}_0} = f^*K_{U_0} + (n-2)E \quad \textit{and} \quad f^*F = \widetilde{F} + E.$$

Proof. The first claim is a straightforward application of the Jacobian criterion: all partial derivatives vanish at the origin if and only if $k \geq 2$.

We now proceed to the calculations on the blowup. The blowup of $\mathbb{A}^n$ at the origin may be described as

$$\left\{\big((x_0,\dots,x_{n-1},t),(y_0:\dots:y_{n-1}:s)\big)\in\mathbb{A}^{n+1}\times\mathbb{P}^n \mid \operatorname{rank}\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} & t \\ y_0 & y_1 & \cdots & y_{n-1} & s\end{pmatrix}=1\right\}.$$

This is covered by the open subsets $V_i := \{y_i = 1\}$, $i = 0,\dots,n-1$ and $V_s := \{s = 1\}$, all isomorphic to the affine space $\mathbb{A}^{n+1}$. Set $V = V_s$.

The strict transform $\widetilde{U}_0$ is given by $\{y_1^2 - y_0y_2 + y_3^2 + \dots + y_{n-1}^2 + t^{k-2}\gamma(t) = 0\}$ in V, proving (1). A local calculation in the open sets V_i, $i = 0,\dots,n-1$ reveals that $\widetilde{U}_0$ has no singular points over the origin there, proving (2).

We now prove (3). The fiber F is an affine quadric cone, the blowup of the vertex is a desingularization and $\widetilde{F}$ is an $\mathbb{A}^1$-bundle. Moreover, $\widetilde{F} \cap E$ is the preimage of the vertex in $\widetilde{F}$ and is thus a section of the $\mathbb{A}^1$-bundle. This proves (a). Let $\mathbb{E} \cong \mathbb{P}^n$ be the exceptional divisor of the blowup of $\mathbb{A}^{n+1}$ at the origin, with coordinates $(y_0 : \dots : y_{n-1} : s)$. Then equation of E in $\mathbb{E}$ is

$$\begin{aligned} y_1^2 - y_0y_2 + y_3^2 + \dots + y_{n-1}^2 + s^2\gamma(0) &= 0 \quad \text{if } k = 2,\\ y_1^2 - y_0y_2 + y_3^2 + \dots + y_{n-1}^2 &= 0 \quad \text{if } k \geq 3.\end{aligned}$$

Let $\mathbb{F}$ be the strict transform of the fiber of $\mathbb{A}^{n+1} \to \mathbb{A}^1$ over $0 \in \mathbb{A}^1$. Then

$$e \cdot \widetilde{F}|_E = e \cdot \mathbb{F}|_{\mathbb{E}} = 1.$$

The intersection $\widetilde{F} \cap E$ is cut out by the equation $s = 0$ in E, proving that it does not contain the vertex $(0 : \dots : 0 : 1)$ and concluding the proof of (b) and (c).

The final claim on the pullback of the canonical divisor follows from the adjunction formula. As for the pullback of F, we have $f^*F = \widetilde{F} + aE$, for some $a \geq 0$. Moreover, if e is as in (3c), we have

$$0 = f^*F \cdot e = \widetilde{F} \cdot e + aE \cdot e = 1 - a$$

and the claim follows. ∎

Corollary 3.13. *Let $g \in \boldsymbol{k}[t_0,t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Over every singular point $P \in \mathrm{Sing}(\mathcal{Q}_g)$, there exists a log resolution of $(\mathcal{Q}_g, F)$*

$$X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_0 := \mathcal{Q}_g$$

obtained by repeatedly blowing up the unique singular point over P (locally described in Lemma 3.12), where F is the fiber over the point $\pi(P) \in \mathbb{P}^1$. In particular, it is $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$-equivariant.

Proof. The existence of the resolution follows from Lemma 3.12 (1) and (2). Indeed notice that after i successive blowups $X_i \to \cdots \to X_0 = \mathcal{Q}_g$, X_i is locally, around its unique singular point p_i over p, given by

$$\{x_1^2 - x_0x_2 + x_3^2 + \cdots + x_{n-1}^2 + t^{k-2i}\gamma(t) = 0\}.$$

Moreover, by Lemma 3.12 (3), p_i is the vertex of the quadric cone $E_i := \operatorname{Exc}(X_i \to X_{i-1})$. Thus the strict transform of E_i in any further blowup is smooth. After $m = \lceil \frac{k}{2} \rceil$ blowups X_m is smooth and the preimage of F is a union of smooth prime divisors meeting transversally.

As for its equivariance, the action of $\operatorname{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ on $\operatorname{Sing}(\mathcal{Q}_g)$ is trivial, since the former is connected and the latter discrete. This proves the equivariance of the first $l = \lfloor \frac{k}{2} \rfloor$ blowups. If k is not even, the action of $\operatorname{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ on X_l fixes the quadric cone E_l and thus its vertex. ■

Notation 3.14. Let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi : \mathcal{Q}_g \to \mathbb{P}^1$ the associated n-dimensional Umemura quadric fibration. Let $P \in \mathcal{Q}_g$ be a singular point, let F be the fiber over the point $\pi(P) \in \mathbb{P}^1$ and let $f : X_m \to \mathcal{Q}_g$ be the log resolution of Corollary 3.13. We write $f = f_m \circ \cdots \circ f_1$ as decomposition of blowups. Denote by

- E_i the strict transform in X_m of the exceptional divisor of f_i,
- $\bar{E}_i$ the pullback of the exceptional divisor of f_i,
- $e_i \subseteq E_i$ for $i = 1, \ldots, m-1$ the generator of the ruling of E_i,
- $e_0 \subseteq \widetilde{F}$ be the generator of the ruling,
- e_m the generator of $\operatorname{NE}(E_m)$ (note that E_m is either isomorphic to $\mathbb{P}^{n-1}$ or to Q_{n-1} by Corollary 3.17).

Proposition 3.15. *Notation as in 3.14. Then*

(1) $\operatorname{NE}(X_m/\mathcal{Q}_g) = \sum_{i=1}^m \mathbb{R}_+[e_i]$.

(2) *The intersections with the canonical divisor of the resolution of singularities are computed by*

$$K_{X_m} \cdot e_i = \begin{cases} -(n-2), & i = m \text{ if } m \text{ is even}, \\ -(n-1), & i = m \text{ if } m \text{ is odd}, \\ 0, & i = 1, \ldots, m-1, \\ -1, & i = 0. \end{cases}$$

Proof. We prove the statement if $n \geq 5$, the proof in the case $n = 4$ being similar. We prove by induction on k that $\operatorname{NE}(X_k/\mathcal{Q}_g) = \sum_{i=1}^k \mathbb{R}_+[e_i]$. By a slight abuse of notation we denote by e_i the push-forward on X_k of $e_i \subseteq X_m$. If $k = 0$, we have $X_k = \mathcal{Q}_g$ and the claim is true. Assume that $\operatorname{NE}(X_k/\mathcal{Q}_g) = \sum_{i=1}^k \mathbb{R}_+[e_i]$ and let $X_{k+1} \to X_k$ be the blowup along the singular point.

Let C be an irreducible curve in $\mathrm{Exc}(X_{k+1} \to \mathcal{Q}_g)$. Let i be an integer such that $C \subseteq E_i$. We prove by induction on $k - i + 1$ that C is numerically equivalent to a positive combination of $e_i, \ldots, e_{k+1}$. If $i = k + 1$, since $\rho(E_{k+1}) = 1$ the curve C is numerically equivalent to a positive multiple of e_{k+1}. If $i < k + 1$, then E_i is a $\mathbb{P}^1$-bundle over a smooth quadric of dimension $n - 2$. The Mori cone of E_i can be written as $\mathbb{R}_+[e_i] + \mathbb{R}_+[\gamma_i]$ where γ_i is contained in $E_i \cap E_{i+1}$. Indeed $E_i \cap E_{i+1}$ is the exceptional divisor of $f_{i+1}|_{E_i}$ and the curves which span it are thus extremal.

Thus there are $a, b \geq 0$ such that $C \equiv a_i e_i + b\gamma_i$. The curve γ_i is contained in E_{i+1}, thus, by inductive hypothesis, there are $a_{i+1}, \ldots, a_{k+1}$ such that

$$\gamma_i \equiv a_{i+1}e_{i+1} \cdots + a_{k+1}e_{k+1}.$$

Part (1) follows.

Lemma 3.12 (4) implies that

$$f^*F = \widetilde{F} + E_1 + \cdots + E_{m-1} + rE_m,$$

where $r = 1$ if m is even and 2 otherwise. For $1 \leq i \leq m - 1$ we have

$$0 = f^*F \cdot e_i = E_{i-1} \cdot e_i + E_i \cdot e_i + E_{i+1} \cdot e_i \implies E_i \cdot e_i = -2.$$

Similarly we get $\widetilde{F} \cdot e_0 = E_m \cdot e_m = -1$.

Again by Lemma 3.12 (4) we get that

$$K_{X_m} = f^*K_{\mathcal{Q}_g} + \sum_{i=1}^{m-1} i(n-2)E_i + \big(s(m-1)(n-2) + t\big)E_m,$$

were $(s,t) = (1, n-2)$ if m is even and $(2, n-1)$ otherwise. Part (2) follows by computing the intersections using the formulas above. ∎

3.5. Equivariant geometry of $\mathcal{Q}_g$

Let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi: \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Let $P \in \mathcal{Q}_g$ be a singular point, let F be the fiber over the point $\pi(P) \in \mathbb{P}^1$. In Corollary 3.13 we provided an explicit $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$-equivariant log resolution $f: X_m \to \mathcal{Q}_g$ of $(\mathcal{Q}_g, F)$. In this section we describe the action of $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ on X_m over F as well as its action on higher models. We also compute some intersection numbers on these higher models.

Lemma 3.16. *Let $\gamma \in \mathbf{k}[t]$ and consider the hypersurface*

$$U_0 = \left\{x_1^2 - x_0x_2 + x_3^2 + \cdots + x_{n-1}^2 + t^k\gamma(t) = 0\right\} \subset \mathbb{A}^n_x \times \mathbb{A}^1_t$$

with $k \geq 2$ and $\gamma(0) \neq 0$. Consider the group $G \cong \mathrm{SO}_n(\mathbf{k})$ acting on $\mathbb{A}^n_x$ by preserving quadratic form

$$x_1^2 - x_0x_2 + x_3^2 + \cdots + x_{n-1}^2.$$

Let E_0 be the G-invariant subset $\{t = 0\} \cap U_0$.

Let $f\colon \widetilde{U}_0 \to U_0$ *be the blowup along the origin,* $\widetilde{E}_0$ *the strict transform of* E_0 *and* E_1 *the exceptional divisor. Then* f *is* G*-equivariant and*

(1) *if* $k = 1$*,* $E_1 \cong \mathbb{P}^{n-1}$*; the* G*-orbits contained in* E_1 *are the smooth* $(n-2)$*-dimensional quadric* $E_1 \cap \widetilde{E}_0$ *and its complement;*

(2) *if* $k = 2$*,* E_1 *is an* $(n-1)$*-dimensional smooth quadric; the* G*-orbits contained in* E_1 *are the hyperplane section* $E_1 \cap \widetilde{E}_0$ *and its complement;*

(3) *if* $k \geq 3$*, then* E_1 *is a quadric cone with vertex* P_1*; the* G*-orbits contained in* E_1 *are* P_1*, the base of the cone* $E_1 \cap \widetilde{E}_0$ *and their complement.*

Proof. The action of G on E_0 is transitive, thus E_0 is an orbit. The same is true for its strict transform $\widetilde{E}_0$ and thus its intersection $E_1 \cap \widetilde{E}_0$ with E_1. The rest is a local calculation: using the notation used in the first part of the proof of Lemma 3.12, the complement of $E_1 \cap \widetilde{E}_0$ is $E_1 \cap V_s$; the description of the action of G on the coordinates $y_0, \dots, y_{n-1}$, s can be deduced by its action on $x_0, \dots, x_{n-1}, t$ together with the equations $x_i = t y_i$.

For the second part we choose H to be the subgroup of G acting via

$$\alpha_\lambda \cdot (x_0, x_1, x_2, x_3, \dots, x_{n-1}, t) \longmapsto (\lambda x_0, x_1, \lambda^{-1} x_2, x_3, \dots, x_{n-1}, t),$$

with $\lambda \in \mathbf{k}^*$. Using the notation of Lemma 3.12 we take $V = V_0 := \{y_0 = 1\}$. There the exceptional divisor is given by $E_1 = \{h := x_0 = 0\}$ and the rest follows. ∎

Corollary 3.17. *Let* $g \in \mathbf{k}[t_0, t_1]$ *be a homogeneous polynomial of degree* $2a$ *and* $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ *the associated Umemura quadric fibration. Let* $P \in \mathcal{Q}_g$ *be the singular point of a singular fiber* F*. Let*

$$X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_0 := \mathcal{Q}_g$$

be the log-resolution of $(\mathcal{Q}_g, F)$ *of Corollary 3.13, with exceptional divisors* $E_i := \mathrm{Exc}(X_i \to X_{i-1})$ *and* $E_0 := F$*. We have that*

(1) *for* $i < m$ *the divisor* E_i *is a* $\mathbb{P}^1$*-bundle over a quadric* Q *of dimension* $n-2$*, not isomorphic to the product* $\mathbb{P}^1 \times Q$*;*

(2) *for* $0 < i < m$ *the action of* $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ *on* E_i *has exactly three orbits: the* 2 *disjoint sections* $E_i \cap E_{i\pm1}$ *and their complement.*

(3) E_m *is isomorphic to* $\mathbb{P}^{n-1}$ *if* m *is odd and to a quadric hypersurface of dimension* $n-1$ *if* m *is even;*

(4) *the action of* $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ *on* E_m *has exactly two orbits:* $E_m \cap E_{m-1}$ *and its complement.*

Proof. Assume that $i < m$. Lemma 3.16 implies that the exceptional divisor $E_i \subseteq X_i$ of f_i is a cone over a quadric $Q = Q_{n-2}$ of dimension $n-2$. Thus, its strict transforms in X_j for $j > i$ are $\mathbb{P}^1$-bundles over Q, isomorphic to $\mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_Q(-1))$ The computation of the orbits follows readily by Lemma 3.16. ∎

We now proceed to studying the action of $\mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ on higher models of X_m.

Lemma 3.18. *Let $g \in \boldsymbol{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated n-dimensional Umemura quadric fibration. Let $P \in \mathcal{Q}_g$ be the singular point of a singular fiber F. Denote by $G = \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ and let $f\colon X_m \to \mathcal{Q}_g$ be the log-resolution of $(\mathcal{Q}_g, F)$ of Corollary 3.13.*

Let

$$X_{m+\ell} \xrightarrow{f_{m+\ell}} \cdots \xrightarrow{f_{m+2}} X_{m+1} \xrightarrow{f_{m+1}} X_m$$

be a sequence of smooth G-equivariant blowups over P. Let Z_i be the center of f_i, E_i the strict transform of the exceptional divisor of f_i for $i = 1, \ldots, m+\ell$ and E_0 the strict transform of F. Then for every $j = 1 \cdots \ell$ there are integers $0 \leq h_j < k_j < m+j$ with the following properties:

(1) *$Z_{m+j} = E_{h_j} \cap E_{k_j}$, it is isomorphic to a quadric of dimension $n-2$, and G acts transitively on it;*

(2) *for every $j > 0$ the divisor E_{m+j} is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}(E_{h_j}) \oplus \mathcal{O}(E_{k_j}))$ over Z_{m+j}, with $E_{h_j}|_{Z_{m+j}} \not\equiv E_{k_j}|_{Z_{m+j}}$. In particular, E_{m+j} is not a product;*

(3) *the action of G on E_{m+j} has exactly three orbits: two disjoint sections of the $\mathbb{P}^1$-bundle corresponding to the injections*

$$\mathcal{O}(E_{h_j}) \hookrightarrow \mathcal{O}(E_{h_j}) \oplus \mathcal{O}(E_{k_j}) \quad \textit{and} \quad \mathcal{O}(E_{k_j}) \hookrightarrow \mathcal{O}(E_{h_j}) \oplus \mathcal{O}(E_{k_j})$$

and their complement in E_j.

Proof. We prove by induction on ℓ the following four statements:

$(1)_\ell$ on $X_{m+\ell-1}$ there exist integers h_ℓ, k_ℓ such that $Z_{m+\ell} = E_{h_\ell} \cap E_{k_\ell}$, $Z_{m+\ell}$ is isomorphic to a quadric of dimension $n-2$, and G acts transitively on it;

$(2)_\ell$ if $\ell > 1$ the divisor $E_{m+\ell}$ on $X_{m+\ell}$ is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}(E_{h_\ell}) \oplus \mathcal{O}(E_{k_\ell}))$ over $Z_{m+\ell}$, with $E_h|_{Z_{m+\ell}} \not\equiv E_k|_{Z_{m+\ell}}$. In particular, $E_{m+\ell}$ is not a product;

$(3)_\ell$ if $\ell > 1$ the action of G on $E_{m+\ell}$ in $X_{m+\ell}$ has exactly three orbits: two disjoint sections of the $\mathbb{P}^1$-bundle corresponding to the injections

$$\mathcal{O}(E_{h_\ell}) \hookrightarrow \mathcal{O}(E_{h_\ell}) \oplus \mathcal{O}(E_{k_\ell}) \quad \text{and} \quad \mathcal{O}(E_{k_\ell}) \hookrightarrow \mathcal{O}(E_{h_\ell}) \oplus \mathcal{O}(E_{k_\ell})$$

and their complement in $E_{m+\ell}$;

$(4)_\ell$ For every $h, k \in \{1, \ldots, m+\ell\}$ such that $E_h \cap E_k = Z \neq \emptyset$, either $\mathcal{O}_Z(E_h)$ is ample and $\mathcal{O}_Z(E_k)$ is anti-ample or vice versa.

Step 1. Let $i \leq m$, and let $K_i = E_i \cap E_{i-1}$. Let $f\colon X_m \to \mathcal{Q}_g$ be the log resolution of Corollary 3.13. We assume for simplicity that m is even, the other case being analogous. The restriction $E_i|_{E_{i-1}}$ is the exceptional divisor of the blowup f_i. Thus $E_i|_{K_i} = (E_i|_{E_{i-1}})|_{K_i}$ is anti-ample. Since $f^*F = \widetilde{F} + \sum E_j$, we have $0 = f^*F|_{K_i} = E_i|_{K_i} + E_{i-1}|_{K_i}$. Therefore $E_{i-1}|_{K_i}$ is ample.

If $\ell = 1$, the statements $(1)_1$ and $(4)_1$ follow from Corollary 3.17 and Step 1.

We assume then that $\ell > 1$ and that the four statements are true for $j < \ell$.

Step 2: proof of $(1)_\ell$ *and* $(2)_\ell$. By Corollary 3.17 and $(3)_j$, for every $a < \ell$ the orbits of codimension at least 2 in $E_a \subseteq X_{m+\ell-1}$ are quadrics of the form $E_a \cap E_b$. This implies $(1)_\ell$: the center $Z_{m+\ell}$ of $f_{m+\ell}$ is of the form $E_k \cap E_h$ for $h, k < m + \ell$. It also implies that the normal bundle of $Z_{m+\ell}$ in $X_{m+\ell}$ satisfies

$$N_{Z_{m+\ell}/X_{m+\ell}} = \mathcal{O}(E_h) \oplus \mathcal{O}(E_k)$$

and that $Z_{m+\ell}$ is disjoint from all the other exceptional divisors.

By Step 1 and $(4)_{\ell-1}$ without loss of generality $E_h|_{Z_{m+\ell}}$ is ample and $E_k|_{Z_{m+\ell}}$ is antiample. This implies $(2)_\ell$.

Step 3: proof of $(3)_\ell$. The restriction $f_{m+\ell}\colon E_{m+\ell} \to Z_{m+\ell}$ gives the $\mathbb{P}^1$-bundle structure. Let $E_h, E_k \subseteq X_{m+\ell-1}$ be such that $Z_{m+\ell} = E_k \cap E_h$. By abuse of notation we denote again by E_h, E_k their strict transform in $X_{m+\ell}$. By the equivariance of all the morphisms involved, the intersections $E_h \cap E_{m+\ell}$ and $E_k \cap E_{m+\ell}$ are G-invariant.

We now show that there are no other closed orbits. Suppose by contraposition that $\widetilde{Z} \subseteq E_{m+\ell}$ is a closed orbit distinct from $E_h \cap E_{m+\ell}$ and $E_k \cap E_{m+\ell}$. By $(1)_\ell$ G acts transitively on $Z_{m+\ell}$ and, since $f_{m+\ell}$ is G-equivariant, $\widetilde{Z}$ surjects onto $Z_{m+\ell}$ and the restriction $f_{m+\ell}\colon \widetilde{Z} \to Z_{m+\ell}$ is also G-equivariant. The variety $\widetilde{Z}$ being an orbit, the restriction is étale. Since $\widetilde{Z}, Z_{m+\ell}$ are Fano, the restriction of $f_{m+\ell}$ is an isomorphism [7, Corollary 4.18 (b)]. Thus $\widetilde{Z}$ is a section of the $\mathbb{P}^1$-bundle. Then we get a contradiction with Lemma 2.6.

Step 4: proof of $(4)_\ell$. Let $h, k \in \{1, \dots, m+\ell\}$. If $h \neq m+\ell$ and $k \neq m+\ell$, then $f_{m+\ell}$ is an isomorphism in a neighborhood of $E_h \cap E_k$ and the claim follows from $(4)_{\ell-1}$.

Assume now that $h = m+\ell$, set $Z = E_{m+\ell} \cap E_k$. In what follows we will denote by E_i the strict transform of the exceptional divisor of f_i in both $X_{m+\ell}$ and $X_{m+\ell-1}$. Notice that there is j such that $Z_{m+\ell} = E_i \cap E_k$ in $X_{m+\ell-1}$. Set $g = f_{m+\ell} \circ \cdots \circ f_1$. Then there are positive integers c_i such that $g^* F = \sum c_i E_i$. We notice that $f_{m+\ell}^*(E_i + E_k) = E_i + E_k + 2E_{m+\ell}$ and $f_{m+\ell}^*(E_k) = E_k + E_{m+\ell}$, proving that $c_{m+\ell} \geq c_k + 1$.

By $(4)_{\ell-1}$, the restriction $E_k|_{Z_{m+\ell}}$ is $\pm$ample, thus

$$f_{m+\ell}^*(E_k|_{Z_{m+\ell}}) = f_{m+\ell}^*(E_k)|_Z = E_k|_Z + E_{m+\ell}|_Z$$

is $\pm$ample. Moreover,

$$\begin{aligned} 0 \sim g^* F|_{Z_{m+\ell}} &= c_{m+\ell} E_{m+\ell}|_Z + c_k E_k|_Z \\ &= (c_{m+\ell} - c_k) E_{m+\ell}|_Z + c_k (E_k|_Z + E_{m+\ell}|_Z) \end{aligned}$$

which implies that $E_{m+\ell}|_Z$ is $\mp$ample and in turn that $E_k|_Z$ is $\pm$ample.

This concludes the proof of $(4)_\ell$. ∎

Lemma 3.19. *Let* $g \in \boldsymbol{k}[t_0, t_1]$ *be a homogeneous polynomial of degree* $2a$ *and* $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ *the associated Umemura quadric fibration. Let* $P \in \mathcal{Q}_g$ *be the singular point of a singular fiber* F. *Denote by* $G = \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ *and let* $f\colon X_m \to \mathcal{Q}_g$ *be the log-resolution of* $(\mathcal{Q}_g, F)$ *of Corollary* 3.13.

Let

$$X_{m+\ell} \xrightarrow{f_{m+\ell}} \cdots \xrightarrow{f_{m+2}} X_{m+1} \xrightarrow{f_{m+1}} X_m$$

be a sequence of smooth G-equivariant blowups over P.

Let $Z \subseteq \mathrm{Exc}(f_1 \circ \cdots \circ f_{m+\ell})$ be a codimension 2 orbit. Then $\mathrm{Exc}(f_1 \circ \cdots \circ f_{m+\ell}) \setminus Z$ has two connected components $\mathcal{C}_1, \mathcal{C}_2$. Assume that $E_m \cap \mathcal{C}_1 \neq \emptyset$. Then there are two integers N, $S \in \{1, \ldots, m+\ell\}$ such that:

- *$Z = E_N \cap E_S$, $E_N \cap \mathcal{C}_1 \neq \emptyset$, $E_S \cap \mathcal{C}_2 \neq \emptyset$;*
- *$\mathcal{O}_Z(E_S)$ is ample and $\mathcal{O}_Z(E_N)$ is anti-ample.*

Proof. We prove this by induction on ℓ. First suppose that $\ell = 0$. Then by the construction of the resolution of Corollary 3.13 $E_N = E_k$ and $E_S = E_{k-1}$. But E_k is the exceptional divisor of f_k and so $E_k|_Z$ is anti-ample. Moreover, by Lemma 3.12 (4), we have $f^*F = \sum_{i=1}^{k} c_i E_i$, where $c_i = 2$, if $i = m$ and m is odd, and 1 otherwise. Restricting to Z we get $E_{i-1}|_Z = -c_i E_i|_Z$, which is ample.

We now suppose that the statement is true for all $j < \ell$, and consider the blowup $f_{m+\ell}\colon X_{m+\ell} \to X_{m+\ell-1}$ with center $Z_{m+\ell} = E_N \cap E_S$. Then we only need to show the statement for the two centers $Z_N := E_N \cap E_{m+\ell}$ and $Z_S := E_S \cap E_{m+\ell}$. Denote by f_N the restriction $f_{m+\ell}|_{Z_N}\colon Z_N \to Z_{m+\ell}$, which is an isomorphism. By the inductive hypothesis we have that

$$f_N^*(E_S) = \cancelto{0}{E_S|_{Z_N}} + E_{m+l}|_{Z_N}$$

is ample. On the other hand

$$f_N^*(E_N) = E_N|_{Z_N} + E_{m+l}|_{Z_N}$$

is anti-ample, which implies that $E_N|_{Z_N}$ is anti-ample. Restricting the class of the pullback of the fiber to Z_N we get that $E_{m+\ell}|_{Z_N} = -E_N|_{Z_N}$, and is thus ample. The proof for Z_S is analogous. ∎

Remark 3.20. In the setup of Lemma 3.18, note that E_m is isomorphic to $\mathbb{P}^{n-1}$ if m is odd and to a quadric hypersurface of dimension $n-1$ if m is even: indeed, for every j the morphism f_{m+j} is the blowup of a smooth codimension 2 center either contained in or disjoint from E_m. Thus f_{m+j} induces an isomorphism on E_m.

Notation 3.21. Let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Let $P \in \mathcal{Q}_g$ be the singular point of a singular fiber F. Denote by $G = \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$ and let $f\colon X_m \to \mathcal{Q}_g$ be the log-resolution of $(\mathcal{Q}_g, F)$ of Corollary 3.13.

Let

$$X_{m+\ell} \xrightarrow{f_{m+\ell}} \cdots \xrightarrow{f_{m+2}} X_{m+1} \xrightarrow{f_{m+1}} X_m$$

be a sequence of smooth G-equivariant blowups over P. Denote by

- E_i the strict transform in X_m of the exceptional divisor of f_i,

- $e_i \subseteq E_i$ for $i \neq m$ the generator of the ruling of E_i
- $e_0 \subseteq \widetilde{F}$ be the generator of the ruling
- e_m the generator of $\mathrm{NE}(E_m)$ (recall that E_m is either isomorphic to $\mathbb{P}^{n-1}$ or to Q_{n-1} by Remark 3.20).

Proposition 3.22. *Notation as in 3.21. Assume that $\ell > 0$. Assume moreover that for every $j > 1$ the center of f_{m+j} lies in E_{m+j-1}. Then*

(1) $\mathrm{NE}(X_{m+\ell}/\mathcal{Q}_g) = \sum_{i=1}^{m} \mathbb{R}_+[e_i]$.

(2) *The intersections with the canonical divisor of $X_{m+\ell}$ are*

$$K_{X_{m+\ell}} \cdot e_i \begin{cases} \geq -(n-2), & i = m \text{ if } m \text{ is even}, \\ \geq -(n-1), & i = m \text{ if } m \text{ is odd}, \\ \geq 0, & i \neq m, m+\ell, \\ = -1, & i = m+\ell. \end{cases}$$

Proof. Assume that $n \geq 5$, the case $n = 4$ being analogous. Let C be an irreducible curve contained in the exceptional locus of $X_{m+\ell} \to \mathcal{Q}_g$ and k be such that $C \subset E_k$. If $k = m$ then $\rho(E_m) = 1$ and by Corollary 3.17, and $C \equiv a_m e_m$. Otherwise by Corollary 3.17 and Lemma 3.18 (2) we have $\mathrm{NE}(E_k) = \mathbb{R}_+[e_k] + \mathbb{R}_+[\gamma_k]$, where γ_i is a line in a smooth quadric of dimension $n-2$ of the form $E_i \cap E_k$. Thus it is enough to prove the statement for γ_k.

Corollary 3.13 and Lemma 3.18 imply that each exceptional divisor with $i \neq m, 1$ meets exactly two other exceptional divisors. Let $(F_j)_{j=1}^{m+\ell}$ be a relabeling of $(E_j)_{j=1}^{m+\ell}$ so that, for each j, F_j meet exactly F_{j-1} and F_{j+1}. We will prove by induction on i that

$$\gamma_{m+\ell-i} \equiv \sum_{m+\ell-i+1}^{m+\ell} a_i e_i,$$

with $a_i \geq 0$.

The base case $i = 0$ is trivial since $F_{m+e} = E_m$ and $\mathrm{NE}(F_{m+e}) = \mathbb{R}_+[e_{m+\ell}]$. Suppose that the statement holds for all $0 \leq i \leq n$. By Lemmas 3.19 and 2.6,

$$\gamma_{m+\ell-n} \subset F_{m+\ell-n} \cap F_{m+\ell-n+1}.$$

In particular, $\gamma_{m+\ell-n} \subset F_{m+\ell-n+1}$ and so there are positive numbers α, β such that

$$\gamma_{m+\ell-n} \equiv \alpha e_{m+\ell-n+1} + \beta \gamma_{m+\ell-n+1}.$$

By the inductive hypothesis $\gamma_{m+\ell-n+1}$ is a positive linear combination of the e_i, with $i > m+\ell-n+1$ and so we conclude (1).

We now prove (2) by induction on ℓ. The base case $\ell = 0$ follows from Proposition 3.15. Suppose that the statement holds for all $\ell < j$ and consider a G-equivariant blowup $f_{m+j}\colon X_{m+j} \to X_{m+j-1}$. Lemma 3.18 implies that the center Z_{m+j} is of the form $E_{h_j} \cap E_{k_j}$. We then have $K_{X_{m+j}} = f_{m+j}^* K_{X_{m+j-1}} + E_{m+j}$, thus

$$K_{X_{m+j}} \cdot e_i = \begin{cases} K_{X_{m+j-1}} \cdot e_i, & i \neq m+j, h_j, k_j, \\ K_{X_{m+j-1}} \cdot e_i + 1, & i = h_j, k_j, \\ -1, & i = m+j \end{cases}$$

and we conclude by the inductive hypothesis. ∎

4. Maximality of $\mathrm{Aut}^\circ(\mathcal{Q}_g)$

In this section we study maximality of $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ in various cases using the theory of the equivariant Sarkisov program.

We begin with two fundamental examples.

Example 4.1. Let $h \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial. The $\mathrm{Aut}^\circ(\mathcal{Q}_{ht_0^2})$-equivariant birational map

$$\begin{aligned} \phi\colon \mathcal{Q}_{ht_0^2} &\dashrightarrow \mathcal{Q}_h \\ (x_0 : \cdots : x_n; t_0 : t_1) &\longmapsto (x_0 : \cdots : t_0 x_n; t_0 : t_1). \end{aligned}$$

conjugates $\mathrm{Aut}^\circ(\mathcal{Q}_{ht_0^2})$ into $\mathrm{Aut}^\circ(\mathcal{Q}_h)$.

More specifically ϕ is a Sarkisov link factorizing as

$$\begin{array}{ccccc} & & X & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ \mathcal{Q}_{ht_0^2} & & \overset{\phi}{\dashrightarrow} & & \mathcal{Q}_h, \end{array}$$

where p is the blowup of the point $(0 : \cdots : 0 : 1; 0 : 1)$ and q is the blowup of $\{x_n = t_0 = 0\}$.

Example 4.2. Let $g = t_0^{a_0} t_1^{a_1}$ with $a_0 + a_1$ even. The $\mathrm{Aut}^\circ(\mathcal{Q}_g)$-equivariant morphism

$$\begin{aligned} p\colon \mathcal{Q}_g &\longrightarrow Q^n \subset \mathbb{P}^{n+1} \\ (x_0 : \cdots : x_n; t_0 : t_1) &\longmapsto (x_0 : \cdots : x_{n-1} : x_n t_0^{a_0} : x_n t_1^{a_1}). \end{aligned}$$

conjugates $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ into $\mathrm{Aut}^\circ(Q^n)$ where Q^n is the smooth n-dimensional quadric $\{y_1^2 - y_0 y_2 + y_3^2 + \cdots + y_{n-1}^2 + y_n y_{n+1} = 0\} \subset \mathbb{P}^{n+1}$.

More specifically, it is a Sarkisov link contracting $\{x_n = 0\}$ to $\Pi = \{y_n = y_{n+1} = 0\}$, and so p conjugates $\mathrm{Aut}^\circ(\mathcal{Q}_{t_0 t_1})$ into $\mathrm{Aut}^\circ(Q^n; \Pi) \subsetneq \mathrm{Aut}^\circ(Q^n)$.

Proposition 4.3. *Let $g \in \mathbf{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Assume that $a \geq 2$ and let $P \in \mathcal{Q}_g$ be a singular point of a fiber F.*

Let $f\colon E \subset X \to P \in \mathcal{Q}_g$ be a G-equivariant extremal divisorial contraction, where $G := \mathrm{Aut}^\circ(\mathcal{Q}_g)_{\mathbb{P}^1}$. Then, after a change of coordinates, f is the restriction of a standard $(1, \ldots, 1, b)$-blowup (see Example 2.2).

Proof. Let $X_m \to Q_g$ be the log resolution of (Q_g, F) of Corollary 3.13. If the valuation induced by E is not divisorial on X_m, then let $W := X_{m+\ell} \to X_m \to Q_g$ be the G-equivariant extraction of the valuation of E obtained via [14, Construction 3.1]. We thus get a birational map

$$\phi\colon W \dashrightarrow X$$

that is a contraction and such that $\phi(E_{m+\ell}) = E$.

Step 1. The map ϕ is a morphism. The map ϕ restricts to a G-equivariant birational map $\phi\colon E_{m+\ell} \dashrightarrow E$. The indeterminacy locus of the restriction of ϕ is both G-invariant and of codimension at least 2. All the G-orbits have codimension at most one in $E_{m+\ell}$, thus the restriction $\phi\colon E_{m+\ell} \to E$ is a morphism. Thus the closed orbits of G in E are images of the orbits of G in $E_{m+\ell}$ and are either points, quadrics of dimension $n-2$ or $\mathbb{P}^1$ (the latter case occurs only if $n = 4$).

Let $(p, q)\colon \widehat{W} \to W \times X$ be a G-equivariant resolution of the indeterminacies of ϕ. We can moreover assume that p is the composition of smooth blowups of smooth centers. Therefore, by Corollary 3.17 and Lemma 3.18, all the p-exceptional divisors are $\mathbb{P}^1$-bundles over smooth quadrics of dimension $n-2$. If ϕ is not a morphism, then there is a curve $C \subseteq \widehat{W}$ such that $p(C)$ is a point and $q(C)$ is a curve. The curve C is contained in the exceptional locus of p. Let $\widehat{E}$ be an irreducible component of $\mathrm{Exc}(p)$ such that $C \subseteq \widehat{E}$. Since $p(C)$ is a point, C is a fiber of the ruling defined by $p|_{\widehat{E}}$. We set $\hat{p}\colon \widehat{E} \to Q$ the ruling defined by the restriction of p. The group G acts on $\widehat{E}$ with at least two orbits, because it preserves the intersection of $\widehat{E}$ with the other components of $\mathrm{Exc}(\widehat{W} \to \mathcal{Q}_g)$. We set G_Q the kernel of the composition $G \to \mathrm{Aut}^\circ(\widehat{E}) \to \mathrm{Aut}^\circ(Q)$, where the last map is given by the Blanchard lemma. Then G_Q acts on the fibres of $\hat{p}$ with at least a fixed point, corresponding to the intersection of $\widehat{E}$ with the other components of $\mathrm{Exc}(\widehat{W} \to \mathcal{Q}_g)$.

Let us consider now the restriction $q\colon \widehat{E} \to E$. Then q is G-equivariant and $q(\widehat{E})$ is a G-stable irreducible closed set in E. It cannot be a point, as $C \subseteq \widehat{E}$ and $q(C)$ is a curve. We assume then that $q(\widehat{E}) = \mathbb{P}^1$. But then we must have $\widehat{E} = \mathbb{P}^1 \times Q$, contradicting Lemma 3.18 (2).

Assume now that $q(\widehat{E})$ is a quadric Q of dimension $n-2$. Thus the two restrictions yield an $\mathrm{Aut}^\circ(Q)$-equivariant morphism $(p, q)\colon \widehat{E} \to Q \times Q$ which is generically finite onto its image. The image is a G-stable subvariety of dimension $n-1$.

Assume that $n - 2 \geq 3$. Then by Lemma 2.5 the group $\mathrm{Aut}^\circ(Q)$ has no invariant subvariety in $Q \times Q$ of dimension $n-1$, this is a contradiction.

Assume that $n = 4$. Then the image of $\widehat{E}$ is one of the two varieties T_i from Lemma 2.5. But $\mathrm{Aut}^\circ(Q)$ preserves one section of $T_i \to Q$ and two sections of $\widehat{E} \to Q$ by Lemma 3.18 and this is a contradiction.

Step 2. The support of $\sum_{i=1}^{m+\ell-1} E_i$ is connected. Assume otherwise and let $\mathcal{C}_1, \mathcal{C}_2$ be the two connected components with $E_m \subseteq \mathcal{C}_1$. Thus, the morphism ϕ factors as

$$W \xrightarrow{\phi_2} W_2 \xrightarrow{\phi_1} X,$$

where $\mathrm{Exc}(\phi_i) = \mathcal{C}_i$.

Let $Z_i = \phi_2(\mathcal{C}_i)$ for $i = 1, 2$. Then away from Z_1 (resp. Z_2) W_2 is isomorphic to a neighborhood of W (resp. X), and so W_2 has terminal singularities. The relative Kodaira dimension of W over X, and therefore of W over W_2 is $-\infty$. Thus W_2 is the output of any MMP on W, relative over W_2. This contradicts Proposition 3.22: indeed, by Proposition 3.22 (1) the first extremal contraction from W is a contraction of a ray $\mathbb{R}_+[e_i]$ with $i \neq m, m+\ell$. By Proposition 3.22 (2) those rays are not K_W-negative.

Step 3. Finalizing the proof. By the previous step the strict transform of E in $X_{m+\ell}$ is either E_m, and in this case $\ell = 0$, or the unique divisor meeting the strict transform of F. In the first case, we conclude as in Step 2: the morphism ϕ is an MMP over X, but the rays $\mathbb{R}_+[e_i]$ for $i \neq m$ are not K_W-negative. In the second case, the only possibility is that for every j the morphism f_{m+j} is the blowup of $E_1 \cap \widetilde{F}$ if $j = 1$ and of $E_{m+j-1} \cap \widetilde{F}$ if $j > 0$, where $\widetilde{F}$ is the strict transform of F in any of the X_{m+j}. We are therefore in the setting of Example 2.2. ∎

Remark 4.4. We notice that, if $b > 1$ and P is a smooth point of $\mathcal{Q}_g$, the extremal divisorial contraction of Proposition 4.3 is an example of extremal divisorial contraction of a divisor to a smooth point which is not a weighted blowup. Indeed, the exceptional divisor is a cone over a quadric and not a weighted projective space.

Corollary 4.5. *Let $\mathcal{Q}_g$ be an Umemura quadric fibration with and $f: E \subset X \to P \in \mathcal{Q}_g$ be an extremal divisorial contraction, where P is point of a singular fiber F. Up to a change of coordinates we may assume that F is the fiber over $(0:1)$ and $g = t_0^k g'$, with $k \geq 1$ and $g'(0,1) \neq 0$.*

Then $\mathrm{NE}(X/\mathbb{P}^1) = \mathbb{R}_+[e] + \mathbb{R}_+[\tilde{l}_0]$ where $e \subseteq E$ and $\tilde{l}_0$ is the strict transform of a line $l_0 \subseteq F$. Moreover, $K_X \cdot l_0 < 0$ if and only if $k \geq 2$ and f is the blowup of $\mathcal{Q}_g$ along P.

Proof. By Proposition 4.3 we may assume that f is the restriction of a standard weighted blowup of the ambient space $\mathbb{P}(\mathcal{E}_a)$ with weights $(1, \dots, 1, b)$. In that case, using the adjunction formula, we obtain

$$
\begin{aligned}
-K_X &= f^*(-K_{\mathcal{Q}_g}) - \big(n - 1 + b - \min\{k, 2\}\big)E,\\
f^*F &= \widetilde{F} + bE
\end{aligned}
$$

where $\widetilde{F}$ is the strict transform of F. Let l_0 be a ruling of F and $\tilde{l}_0$ its strict transform in X. We first prove that $\mathrm{NE}(X/\mathbb{P}^1) = \mathbb{R}_+[e] + \mathbb{R}_+[\tilde{l}_0]$, where $e \subseteq E$. The ray $\mathbb{R}_+[e]$ is extremal and K_X-negative. By the discussion in Example 2.2, the variety $\widetilde{F}$ is a $\mathbb{P}^1$-bundle over a quadric of dimension $n-2$, and $\mathrm{NE}(\widetilde{F}) = \mathbb{R}_+[e] + \mathbb{R}_+[\tilde{l}_0]$. The intersection $\widetilde{F} \cdot \tilde{l}_0 = -b$ can be easily computed. Assume that we can write $\tilde{l}_0 \equiv \alpha e + \beta C$ with C another curve and $\alpha, \beta \geq 0$. Then $\widetilde{F} \cdot C < 0$ and $C \equiv \alpha' e + \beta' \tilde{l}_0$ with $\alpha', \beta' \geq 0$. We get $(1 - \beta\beta')\tilde{l}_0 \equiv (\alpha + \alpha'\beta)e$. Intersecting with an ample divisor, we get $1 - \beta\beta' \geq 0$ and $\alpha + \alpha'\beta \geq 0$. Intersecting with $\widetilde{F}$ we get $1 - \beta\beta' \leq 0$. We conclude that $\alpha = 0$ and $C \equiv \tilde{l}_0$, proving that $\mathbb{R}_+[\tilde{l}_0]$ is also extremal.

Finally,

$$-K_X \cdot \tilde{l}_0 = -K_{\mathcal{Q}_g} \cdot l_0 - \big(n - 1 + b - \min\{k, 2\}\big) = \min\{k, 2\} - b.$$

The previous quantity being positive if and only if $k \geq 2$ and $b = 1$, i.e., f is the blowup of $\mathcal{Q}_g$ along P. ∎

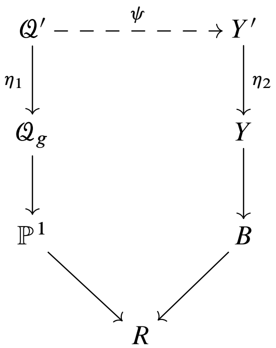
Lemma 4.6. *Let $g \in \boldsymbol{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Assume that $a \geq 2$ and g has more than 2 roots. Suppose that $\mathcal{Q}_g \dashrightarrow Y$ is an $\mathrm{Aut}^\circ(\mathcal{Q}_g)$-equivariant Sarkisov link to a Mori fiber space Y/B.*

Then $Y/B = \mathcal{Q}_h/\mathbb{P}^1$ with

$$h = l^2 g \quad \textit{or} \quad g = l^2 h,$$

for some linear polynomial $l \in \boldsymbol{k}[t_0, t_1]_1$.

Proof. Assume that Y/B is not isomorphic to $\mathcal{Q}_g/\mathbb{P}^1$. Let



be a Sarkisov link from $\mathcal{Q}_g$ to Y. We first prove that η_1 cannot be an isomorphism. Assume by contradiction that it is one. By Lemma 3.5 the extremal rays of $\mathrm{NE}(\mathcal{Q}_g)$ are the extremal ray inducing π and $\mathbb{R}_+[\sigma]$ that spans a divisor and where $K_{\mathcal{Q}_g} \cdot \sigma \geq 0$. Therefore $\mathbb{R}_+[\sigma]$ cannot be contracted giving rise to a divisorial contraction nor an isomorphism in codimension 1.

Thus $\eta_1\colon E \subset X \to Z \subset \mathcal{Q}_g$ is an $\mathrm{Aut}^\circ(\mathcal{Q}_g)$-equivariant extremal divisorial contraction. The center Z is an orbit and thus, by Lemma 3.10 and Remark 3.11, we have either

(1) $Z = H_n \cap F$ for some fiber F or

(2) Z is the singular point P of a singular fiber.

In the first case, Z has codimension 2 in $\mathcal{Q}_g$ and thus, by [5, Proposition 2.4], $X \to \mathcal{Q}_g$ is the blowup of Z. The resulting link is the inverse of the one in Example 4.1, whose target is $\mathcal{Q}_h$ with $h = gl^2$, for some linear polynomial $l \in \mathbf{k}[t_0, t_1]$.

In the latter case, by Proposition 4.3, the morphism η_1 is the restriction of a standard weighted blowup of the ambient space $\mathbb{P}(\mathcal{E}_a)$ with weights $(1, \ldots, 1, b)$. If $b \geq 2$

or Z is not a singular point of $\mathcal{Q}_g$ Corollary 4.5 implies that the extremal ray $\mathbb{R}_+[\tilde{l}_0]$ of $\mathrm{NE}(X/\mathbb{P}^1)$, not corresponding to η_1, is K_X-non-negative and span a subset of codimension 1. Therefore $\mathbb{R}_+[\tilde{l}_0]$ cannot be contracted giving rise to a divisorial contraction nor an isomorphism in codimension 1, contradicting the existence of the link. In the case that Z is a singular point of $\mathcal{Q}_g$ and $b = 1$ the resulting link is the one of Example 4.1, whose target is $\mathcal{Q}_h$ with $g = hl^2$, for some linear polynomial $l \in \mathbf{k}[t_0, t_1]$. ∎

We are now ready to prove the main result of our paper.

Theorem 4.7. *Let $g \in \boldsymbol{k}[t_0, t_1]$ be a homogeneous polynomial of degree $2a$ and $\pi\colon \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Write $g = f^2 h$, where $f, h \in \boldsymbol{k}[t_0, t_1]$ are homogeneous polynomials with h being square-free. Then* $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ *is conjugated to a subgroup of* $\mathrm{Aut}^\circ(\mathcal{Q}_h)$.

Moreover, if h and h' are two square-free polynomials, we have:

(1) $\mathrm{Aut}^\circ(\mathcal{Q}_h)$ *is a maximal connected algebraic subgroup of* $\mathrm{Cr}_n(\boldsymbol{k})$ *if and only if h is constant or has at least 4 roots;*

(2) $\mathrm{Aut}^\circ(\mathcal{Q}_h)$ *and* $\mathrm{Aut}^\circ(\mathcal{Q}_{h'})$ *are conjugate if and only if $h(t_0, t_1) = h'(\alpha(t_0, t_1))$, with $\alpha \in \mathrm{PGL}_2(\boldsymbol{k})$.*

Proof. The first claim follows by repeatedly applying the link in Example 4.1 to clear all square terms.

Suppose now that h is square-free and let $G := \mathrm{Aut}^\circ(\mathcal{Q}_h)$. If h is constant, then $\mathcal{Q}_h$ is isomorphic to the product $Q^{n-1} \times \mathbb{P}^1$. Then $G = \mathrm{Aut}^\circ(Q_{n-1}) \times \mathrm{Aut}^\circ(\mathbb{P}^1)$ acting factorwise, thus the action is transitive. This implies that there are no G-equivariant Sarkisov links, and so G is maximal by [11, Corollary 1.3].

If h has exactly two roots then, up to a change of coordinates, $h = t_0 t_1$. Example 4.2 shows that G is conjugate to a strict subgroup of $\mathrm{PSO}_{n+1}(\mathbf{k})$.

Finally, suppose that h has strictly more than 2 roots. Then, by Proposition 3.8,

$$\mathrm{Aut}^\circ(\mathcal{Q}_h) = \mathrm{Aut}^\circ(\mathcal{Q}_h)_{\mathbb{P}^1} = \mathrm{SO}_n(\mathbf{k}).$$

Successive applications of Lemma 4.6 show that if $\mathcal{Q}_h$ is G-equivariantly birational to an Mfs X/B, then $X/B = \mathcal{Q}_{hf^2}$. Since hf^2 has strictly more than 2 roots too, $\mathrm{Aut}^\circ(\mathcal{Q}_{hf^2}) = \mathrm{SO}_n(\mathbf{k})$. Thus G is maximal by [11, Corollary 1.3]. This concludes (1).

Finally, let h and h' be two square-free polynomials such that $\mathrm{Aut}^\circ(\mathcal{Q}_h)$ and $\mathrm{Aut}^\circ(\mathcal{Q}_{h'})$ are conjugate. Then there exists a birational map $\phi\colon \mathcal{Q}_h \dashrightarrow \mathcal{Q}_{h'}$. By Lemma 4.6, ϕ has to be an isomorphism of Mori fiber spaces, i.e., an isomorphism fitting in a diagram

$$\begin{array}{ccc} \mathcal{Q}_h & \xrightarrow{\ \phi\ } & \mathcal{Q}_{h'} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \mathbb{P}^1 & \xrightarrow[\ \alpha\ ]{} & \mathbb{P}^1, \end{array}$$

where α is an isomorphism. Since ϕ has to send singular fibers of π to singular fibers of π', and by Lemma 3.6 (1), we have $h(t_0, t_1) = h'(\alpha(t_0, t_1))$. Conversely, if $h(t_0, t_1) =$

$h'(\alpha(t_0, t_1))$ for some $\alpha \in \mathrm{PGL}_2(\mathbf{k})$ then we have the Mori fiber space isomorphism

$$\begin{array}{ccc} \mathcal{Q}_h & \xrightarrow{\phi} & \mathcal{Q}_{h'} \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{P}^1 & \xrightarrow[\alpha]{} & \mathbb{P}^1 \end{array}$$

that conjugates $\mathrm{Aut}^\circ(\mathcal{Q}_h)$ into $\mathrm{Aut}^\circ(\mathcal{Q}_{h'})$. This concludes the proof of (2). ■

Finally, we can deduce the following characterization of the maximality of $\mathrm{Aut}^\circ(\mathcal{Q}_g)$.

Corollary 4.8. *Let $g \in \boldsymbol{k}[t_0, t_1]$ be a homogeneous polynomial of even degree and $\pi : \mathcal{Q}_g \to \mathbb{P}^1$ the associated Umemura quadric fibration. Write $g = f^2 h$ where $f, h \in \boldsymbol{k}[t_0, t_1]$ are homogeneous polynomials with h being square-free. Then $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ is a maximal connected algebraic subgroup of $\mathrm{Cr}_n(\boldsymbol{k})$ if and only if either f and h are constant or h has at least 4 roots.*

Remark 4.9. When g has 2 roots, one can actually prove a more precise result: $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ is contained in a unique maximal connected algebraic subgroup M of $\mathrm{Cr}_n(\mathbf{k})$; namely $M = \mathrm{PSO}_{n+1}(\mathbf{k})$ with the conjugation being given by Example 4.2.

Indeed, using the description of $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ of Proposition 3.8, we can compute the orbits, and deduce that all equivariant Sarkisov links from $\mathcal{Q}_g$ are of the two forms of Examples 4.1 and 4.2. It then suffices to notice that the links of the two examples commute, i.e., if $g = t_0^{a_0} t_1^{a_1}$ and $g' = t_0^{b_0} t_1^{b_1}$, with $b_i = a_i \pm 2k_i$, then the diagram

$$\begin{array}{ccccc} \mathcal{Q}_g & & \dashrightarrow & & \mathcal{Q}_{g'} \\ & \searrow & & \swarrow & \\ & & Q^n & & \end{array}$$

is commutative.

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